Rational Functions and Their Graphs

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November 1, 2018

Set up

College Algebra student/High School advanced student

Pre-requisites: know how to solve polynomials, operations with polynomials, synthetic division, graphing.

Introduction

Recall that a **rational number** is a number which can be expressed as the quotient (fraction p/q) of two integers.

$$\frac{1}{2}, \frac{2}{1}, \frac{7}{5}, \frac{2147487}{6700417}$$

Definition

Rational Functions are quotients of polynomial functions.
They have the general form of:

$$f(x) = \frac{p(x)}{q(x)},\ q(x) \neq 0.$$

$$\text{i.e. } \frac{x^2 + 7x + 9}{x(x-2)(x+5)}$$

Looking Forward to:

Solving a Function.

Domain and Range.

Difficulties with the previous two items.

Simplifying Rational Functions*.

Graphing a given function.

Using notation to express an answer.

Working backwards*.

Material

Rational Functions $f(x) = \frac{p(x)}{q(x)}$, where p and q are rational functions and $q \neq 0$. The domain of any rational function is the set of all real numbers except the x-values that make the denominator zero.

Example

$$f(x) = \frac{x}{x^2 - 9}$$

Denominator equals zero at: $-3, 3$ Domain: $f = \{x | x \neq -3, 3\}$

Domain: $f = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

Basics

$$f(x) = \frac{1}{x}$$

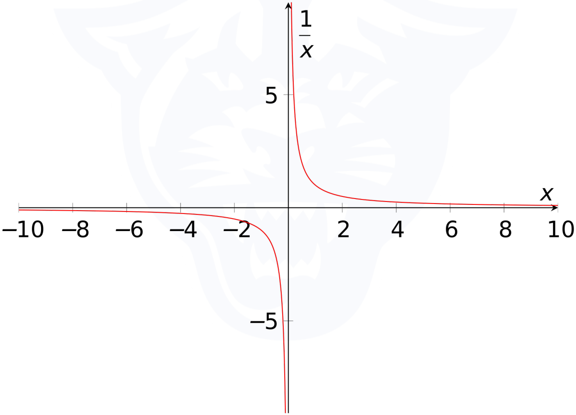
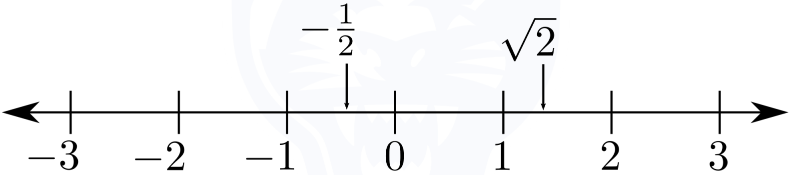


Figure: Basic Rational Function

New Approach

Approaching from the left $\rightarrow$



$\leftarrow$ Approaching from the right

$\longrightarrow$

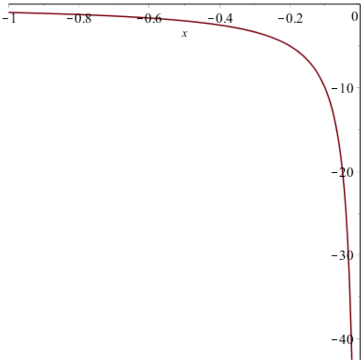


Figure: Function from x: -1 to -0.01

x	-1	-0.5	-0.1	-0.01
$f(x) = \frac{1}{x}$	-1	-2	-10	-100

"x approaches 0 from the left"

As $x \to 0^-$, $f(x) \to -\infty$

←

x	1	0.5	0.1	0.01
$f(x) = \frac{1}{x}$	1	2	10	100

"x approaches 0 from the right"

As $x \to 0^+$, $f(x) \to \infty$

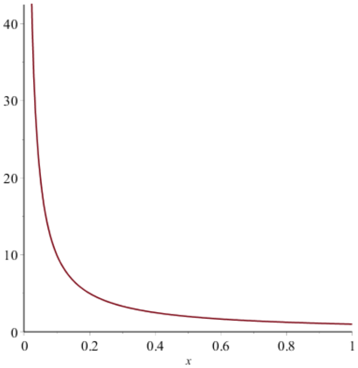


Figure: Function from x: 0.01 to 1

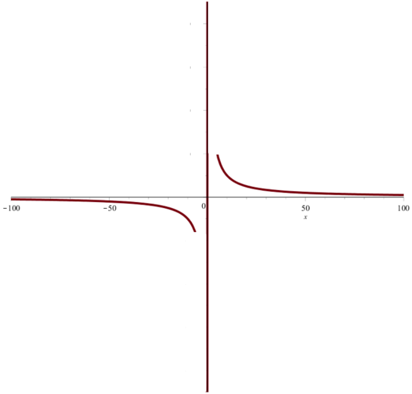


Figure: $f(x) = \dfrac{1}{x}$

$\leftarrow$

x	-1	-10	-100	-1000
$f(x)$	-1	-0.1	-0.01	-0.001

$\rightarrow$

x	1	10	100	1000
$f(x)$	1	0.1	0.01	0.001

QR-Arrow Notation

Symbol . . . . Meaning

$x \to a^+$ . . . x approaches a from the right.

$x \to a^-$ . . . x approaches a from the left.

$x \to \infty$ . . . x increases without a bound.

$x \to -\infty$ . . . x decreases without a bound.

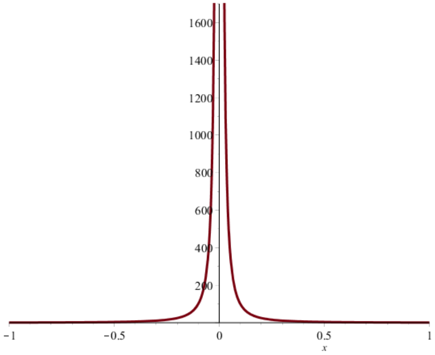


Figure: $f(x) = \dfrac{1}{x^2}$

This function has symmetry... which kind?
Think back to: Even, Odd or Neither.

What is the behavior of the line?

which direction? $\rightarrow, \leftarrow$?

We can observe the function approaching ∞.

It also approaches the value 0.

Will it cross the y-axis?

Will it ever be equal to $x = 0$?

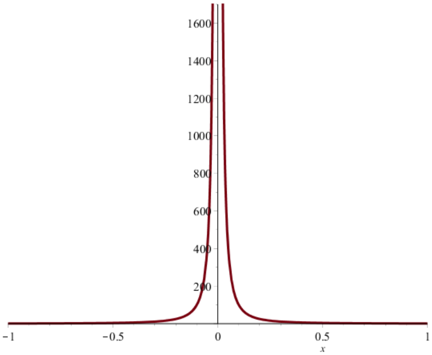


Figure: $f(x) = \dfrac{1}{x^2}$

Vertical Asymptote

Definition

The line $x = k$ is a vertical asymptote of the graph of a function f if $f(x)$ increases or decreases without bound as x approaches k

Example of Asymptote

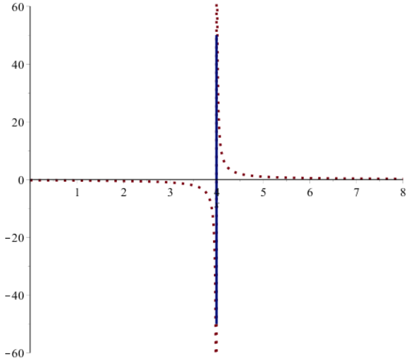


Figure: $r(x) = \frac{1}{x-4}$

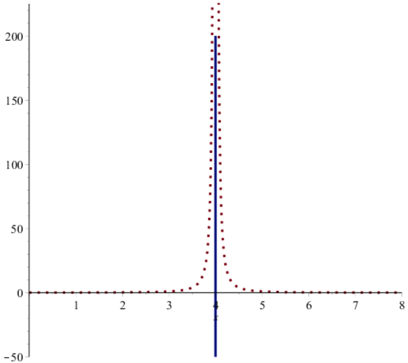


Figure: $g(x) = \frac{1}{(x-4)^2}$

How to spot Vertical Asymptotes . . .

. . . without a graph

Given a rational function, say $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ & $q(x)$ have no common factors and u is a zero of $q(x)$, then $x = u$ is a vertical asymptote of the graph of f

Factoring is usually helpful in identifying zeros of denominators and any common factors within the rational function.

Yes, there can be more than one vertical asymptote.

Yes, there can be a combination of asymptotes in one function/graph.

A function may be simpler* than it is originally presented.

Quick Recap

$$f(x) = \frac{x^2 - 25}{x - 5} = \frac{(x+5)(x-5)}{x-5} = x + 5,\ x \neq 5$$

But there's something new...

Holes

Informal Definition

Cancellation of terms in the rational function-between numerator & denominator-will be considered as missing values i.e. holes in the graph of the function.

$$f(x) = \frac{x^2 - 25}{x - 5} = \frac{(x+5)(x-5)}{x-5} = x + 5,\ x \neq 5$$

$x \neq 5$ will be considered a hole in the graph of $f(x)$

Horizontal Asymptote

Definition

The line $y = d$ is a horizontal asymptote of the graph of a function f if $f(x)$ approaches d as x increases or decreases without a bound.

Think back on the function $f(x) = \dfrac{1}{x}$

Many, but not all, rational functions have horizontal asymtotes.

Rational functions may have several vertical asymptotes

However, a rational function can have at most one horizontal asymptote.

Vertical asymptotes cannot be intersected.

However, horizontal asymptotes can be crossed.

How to determine a Horizontal Asymptote ...

... without a graph

Let f be the rational function given by:

$$f(x) = \frac{k_n x^n + k_{n-1}x^{n-1} + \cdots + k_1 x + k_0}{p_m x^m + p_{m-1}x^{m-1} + \cdots + p_1 x + p_0},\ k_n \neq 0, p_m \neq 0$$

- if $n < m$, the x-axis, or $y = 0$ is the horizontal asymptote.
- if $n = m$, the line $y = \dfrac{k_n}{p_m}$ is the horizontal asymptote.
- if $n > m$, the graph of f has no horizontal asymptote.

How to Graph a Rational Function

Strategy

$$\text{Given } f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions with no common factors.

Continuation...

1. Determine the Symmetry of f
   $f(-x) = f(x)$ y-axis symmetry
   $f(-x) = -f(x)$ origin symmetry
2. Find the y-intercept (if any) by evaluating $f(0)$
3. Find the x-intercepts (if any) by solving the equation $p(x) = 0$
4. Find any vertical asymptote(s) by solving the equation $q(x) = 0$
5. Find the horizontal asymptote (if any) using the rule for determining the horizontal asymptote of a rational function.
6. Plot at least one point between and beyond each x-intercept and vertical asymptote
7. Use the information obtained previously to graph the function between and beyond the vertical asymptotes.

Oblique or Slant Asymptote

Definition

The graph of a rational function has a slant asymptote if the degree of the numerator is one more than the degree of the denominator.

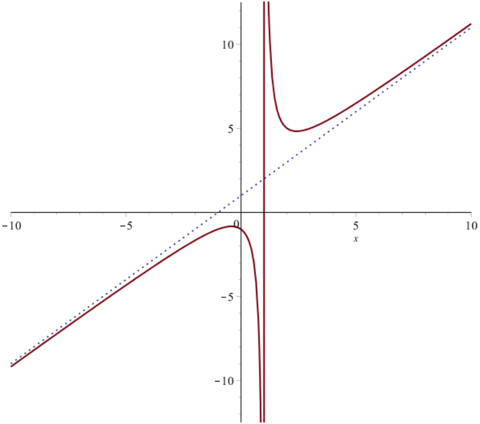
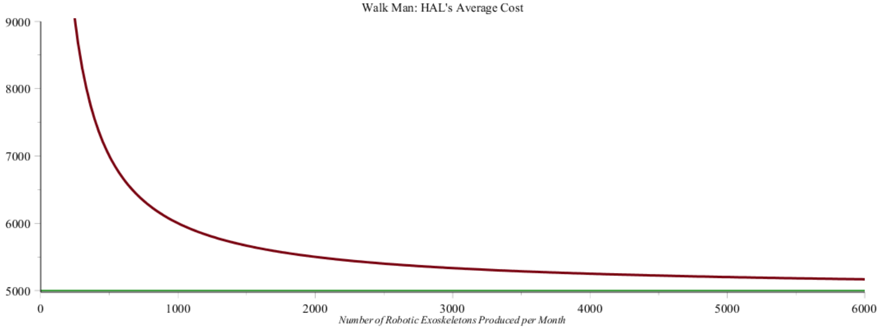


Figure: Slant/Oblique

HAL

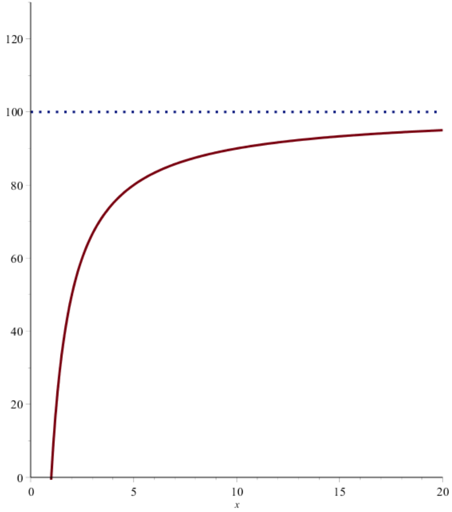

Walk Man: HAL's Average Cost
9000
8000
7000
6000
5000
0
1000
2000
3000
4000
5000
6000
Number of Robotic Exoskeletons Produced per Month

Hybrid Assistive Limb, or HAL. Robotic Exoskeleton which helps the elderly and disabled walk.

Cost ranging from \$14,000 and \$20,000. (*Source: sanlab.kz.tsukuba.ac.jp*)

Figure: HAL

Diseased



The Rational function $P(x) = \dfrac{100(x-1)}{x}$ model the percentage of smoking-related deaths among all deaths from a disease, $P(X)$, in terms of the disease's incidence ration. (`Source: Alexander M. Walker,`*`Observations and Inference`*`, Epidemiology Resources Inc.,1991`

Fin

Figure: Blitzer

Blitzer, Robert. *College Algebra Essentials*. Prentice Hall, 2010.