Saving and Growth in the Ramsey (1928) Model. Consider the following Ramsey/Cass-Koopmans growth model with labor-augmenting technological progress at rate ϕ:

$$\max \int_0^\infty \mathrm{u}(C_t)e^{-\vartheta t}dt \tag{1}$$

$$\begin{aligned} &\text{s.t.} \\ \dot{K}_t &= Y_t - C_t - \delta K_t \\ Y_t &= K_t^\alpha (\mathrm{Z}_t L_t)^{1-\alpha} \\ \frac{\dot{\mathrm{Z}}_t}{\mathrm{Z}_t} &= \phi \\ \mathrm{u}(C) &= \frac{C^{1-\rho}}{1-\rho} \end{aligned} \tag{2}$$

and suppose there is no population growth so that we can normalize the labor force to be $L_t = 1 \; \forall \; t$. Finally, normalize the initial level of productivity to $\mathrm{Z}_0 = 1$.

1. Rewrite the problem in per-efficiency-unit terms, designating small-letter variables as the corresponding capital-letter divided by labor productivity, e.g. $c_t = C_t/\mathrm{Z}_t$, and show that in such an economy, the steady-state will be given by the point where $\dot{K}/K = \dot{C}/C = \dot{Y}/Y = \phi$. (Hint: Begin by rewriting the utility function in terms of consumption per efficiency unit and the growth in productivity since time 0.) Use the first order condition for consumption per efficiency unit in the rewritten problem to show that the steady-state normalized capital stock will be given by

$$k = \left(\frac{\alpha}{\rho\phi + \vartheta + \delta}\right)^{\frac{1}{1-\alpha}} \tag{3}$$

2. Briefly discuss the effect of labor productivity growth on the *absolute* marginal product of capital in this model, and discuss the implications of this finding for the return to saving (the effective interest rate). (Hint: 'absolute' means don't normalize by Z).

3. Using a phase diagram, analyze the dynamic effects of increasing the degree of impatience ϑ for an economy which starts off in period 0 in the steady-state corresponding to the old, lower ϑ. Then make a graph showing the path of the aggregate saving rate over time. *Explain both graphs in intuitive terms; be sure, in particular, to explain why the saving rate returns to its steady-state over time.* Show and discuss how the graphs would change with a higher or lower coefficient of relative risk aversion.

4. Now analyze the effects of an unexpected permanent increase in ϕ, the rate of labor-augmenting technological progress. Again make a phase diagram, and compare the effects of ϕ on the $\dot{c} = 0$ and $\dot{k} = 0$ loci with the effects of ϑ, and explain the reason for any similarities or differences.

Make a graph showing the path of the aggregate saving rate over time after the economy switches into the fast-growth regime. *Explain the reason both graphs look the way they do in intuitive terms.* Discuss what determines whether the saving rate rises or falls in the instant when consumers learn that growth has increased. (Hint: Whether the new steady-state saving rate is higher or lower depends on parameter values, as shown in last year's exam. You can make whatever assumption you wish about whether the new steady-state saving rate is higher or lower, but given your assumption about the new steady-state you can draw the path of saving over time.)

References

Ramsey, Frank (1928): "A Mathematical Theory of Saving," *Economic Journal*, 38(152), 543–559.