Growth Under Total Utility Maximization. In the continuous-time ?/Cass-Koopmans model presented in class, the social planner's objective was to maximize

$$\int_0^\infty \mathrm{u}(c_t)e^{-\vartheta t}dt \tag{1}$$

where $c_t = C_t/\mathrm{L}_t$ is per-capita consumption. Thus, the social planner is assumed to care about per-capita consumption. (For short, refer to this as the "per-capita" model.) An alternative would be to assume that the social planner's utility is given by total utility of all households in the population. In that case, the maximand would be

$$\int_0^\infty \mathrm{L}_t\mathrm{u}(c_t)e^{-\vartheta t}dt. \tag{2}$$

Answer the following questions about the optimum if this equation characterizes the social planner's maximand. Assume that the aggregate budget constraint is $Y_t = \mathrm{F}(K_t, \mathrm{L}_t) = \dot{K}_t + C_t + \delta K_t$, population growth is constant at $\dot{\mathrm{L}}_t/\mathrm{L}_t = \xi$, the CRS production function is $\mathrm{F}(K, \mathrm{L}) = K^\alpha \mathrm{L}^{1-\alpha}$, and there is no productivity growth.

1. Rewrite the aggregate budget constraint in per-capita terms, and then write down the maximization problem and the first order conditions for the optimum.

   Answer:

   We showed in RamseyCassKoopmans that the budget constraint can be rewritten as

   $$\dot{k}_t = \mathrm{f}(k_t) - c_t - (\xi + \phi)k_t \tag{3}$$

   so the maximization problem is

   $$\int_0^\infty \mathrm{L}_t\mathrm{u}(c_t)e^{-\vartheta t}dt \tag{4}$$

   subject to (3). The present value Hamiltonian is

   $$\mathcal{H}_t = \mathrm{L}_t\mathrm{u}(c_t) + (\mathrm{f}(k_t) - c_t - (\xi + \phi)k_t)\lambda_t. \tag{5}$$

   The optimality conditions are

   $$\begin{aligned} \partial H_t/\partial c_t &= 0 && (6) \\ \mathrm{L}_t\mathrm{u}'(c_t) &= \lambda_t, && (7) \end{aligned}$$

   $$\begin{aligned} \dot{\lambda}_t &= \vartheta\lambda_t - \partial H_t/\partial k_t && (8) \\ &= \vartheta\lambda_t - \lambda_t(\mathrm{f}'(k_t) - (\xi + \phi)) && (9) \\ &= -\lambda_t(\mathrm{f}'(k_t) - \vartheta - (\xi + \phi)), && (10) \end{aligned}$$

   the DBC, and the TVC,

   $$\lim_{t\to\infty} e^{-\vartheta t}\lambda_t k_t = 0 \tag{11}$$

2. Assume CRRA utility $\mathrm{u}(c) = c^{1-\rho}/(1-\rho)$. Derive the optimal growth rate of per-capita consumption, and explain the intuition for why it's different from the per-capita model.

   Answer:

   From (7) we have

$$\begin{aligned}
\dot{\lambda}_t &= \left(\frac{d(\mathrm{L}_t \mathrm{u}'(c_t))}{dt}\right) && (12)\\
&= \mathrm{L}_t \mathrm{u}''(c_t)\dot{c}_t + \dot{\mathrm{L}}_t \mathrm{u}'(c_t) && (13)
\end{aligned}$$

   Substitute (7) and (13) into (10) to obtain

$$\mathrm{L}_t \mathrm{u}''(c_t)\dot{c}_t + \dot{\mathrm{L}}_t \mathrm{u}'(c_t) = \mathrm{L}_t \mathrm{u}'(c_t)[\vartheta + (\xi + \phi) - \mathrm{f}'(k_t)]. \quad (14)$$

   Dividing both sides by L_t,

$$\begin{aligned}
\mathrm{u}''(c_t)\dot{c}_t + (\xi + \phi)\mathrm{u}'(c_t) &= \mathrm{u}'(c_t)[\vartheta + (\xi + \phi) - \mathrm{f}'(k_t)]. && (15)\\
\dot{c}_t &= \left(\frac{-\mathrm{u}'(c_t)}{\mathrm{u}''(c_t)}\right)(\mathrm{f}'(k_t) - \vartheta) && (16)\\
\dot{c}_t/c_t &= \underbrace{\left(\frac{-\mathrm{u}'(c_t)}{c_t \mathrm{u}''(c_t)}\right)}_{=\rho^{-1}}(\mathrm{f}'(k_t) - \vartheta). && (17)
\end{aligned}$$

   So at a given level of the capital stock, n is no longer subtracted off, so the growth rate of per-capita consumption is higher in this model than in the per-capita model. The intuition is that when utility is proportional to the number of households, if there is positive population growth it makes sense to postpone consumption (thus achieving higher consumption growth) into the future when the utility from per-capita consumption will be multiplied by a larger number of households, generating larger total utility.

3. Solve for the golden rule and modified golden rule levels of capital and consumption. How do they compare to the per-capita model?

   Answer:

   The golden rule level of the capital stock is the level that maximizes per capita consumption. This depends solely on the budget constraint. Since the budget constraint was not changed, the golden rule level of the capital stock remains the level where $\mathrm{f}'(k_t) = (\xi + \phi)$ as before.

   The modified golden rule for this model is the point at which $\dot{c}/c = 0$, i.e. where $\mathrm{f}'(k_t) = \vartheta$. Since the modified golden rule in the standard model was where $\mathrm{f}'(k_t) = \vartheta + (\xi + \phi)$, clearly the equilibrium marginal product of capital is lower here, so the equilibrium capital stock is higher. The reason capital is greater is the same as the reason consumption growth is faster: It is more important to save for the future when the marginal utility of future consumption is multiplied by a factor L_t that is growing over time.