Midterm Exam

Intertemporal Choice

Fall, 2025

Answers

You are expected to answer all parts of all questions. If you cannot solve part of a question, *do not give up*. The exam is written so that you should be able to answer later parts even if you are stumped by earlier parts.

Write all answers on the exam itself; if you run out of room, use the back of the previous page.

Part I: Long Question

1. Consider a consumer with quadratic utility $\mathrm{u}(c) = -(1/2)(c - \cancel{c})^2$ for whom the interest and discount factors satisfy $\mathsf{R}\beta = 1$. The consumer wishes to maximize the discounted sum of intertemporally separable utility from consumption

$$\sum_{s=t}^{\infty} \mathbb{E}_t \left[\beta^{s-t}\mathrm{u}(c_s)\right] \tag{1}$$

subject to a budget constraint

$$m_{t+1} \;=\; (m_t - c_t)\mathsf{R} + y_{t+1}$$

where m is market resources, y is labor income, and $\mathsf{R} = 1 + \mathsf{r}$ is the interest factor.

a) Write the problem in Bellman equation form, and show how to use the Envelope condition and the first order condition to obtain the Euler equation for consumption.

Answer:

Bellman:

$$\mathrm{v}(m_t) \;=\; \max_{c_t} \mathrm{u}(c_t) + \beta\mathbb{E}_t[\mathrm{v}(m_{t+1})] \tag{2}$$

Envelope:

$$\mathrm{v}'_{t+1}(m_{t+1}) \;=\; \mathrm{u}'(c_{t+1}) \tag{3}$$

FOC:

$$\mathrm{u}'(c_t) \;=\; \mathbb{E}_t[\mathrm{v}'_{t+1}(m_{t+1})] \tag{4}$$

Combo: Substitute (3) into (4).

b) Show that the Euler equation $\mathrm{u}'(c_t) = \mathsf{R}\beta\mathbb{E}_t[\mathrm{u}'(c_{t+1})]$ implies that the level of consumption follows a random walk,

$$\Delta c_{t+1} \;=\; \epsilon_{t+1}$$

where $\mathbb{E}_t[\epsilon_{t+1}] = 0$.

Answer:

$$
\begin{aligned}
\mathrm{u}'(c) &= -(c - \cent) \\
\mathrm{u}'(c_t) &= \mathsf{R}\beta\mathbb{E}_t[\mathrm{u}'(c_{t+1})] \\
-(c_t - \cent) &= \mathbb{E}_t[-(c_{t+1} - \cent)] \\
c_t &= \mathbb{E}_t[c_{t+1}] \\
c_{t+1} &= c_t + \epsilon_{t+1}
\end{aligned}
$$

Now assume that income is subject to transitory and permanent shocks:

$$
\begin{aligned}
y_{t+1} &= p_{t+1} + \theta_{t+1} \\
p_{t+1} &= p_t \quad + \psi_{t+1}
\end{aligned}
$$

where the shocks are mutually and serially uncorrelated ($\mathbb{E}_t[\psi_{t+n}] = \mathbb{E}_t[\theta_{t+n}] = \mathbb{E}_t[\psi_{t+m}\theta_{t+n}] = 0 \ \forall \ m, n > 0$).

c) Defining end-of-period human wealth as the PDV of expected future income,

$$
\mathfrak{h}_t = \mathsf{R}^{-1}\mathbb{E}_t\left[\sum_{s=t+1}^{\infty} \mathsf{R}^{t+1-s} y_s\right]
$$

show that

$$
\mathfrak{h}_t = p_t/r
$$

Answer:

$$
\begin{aligned}
\mathfrak{h}_t &= \mathsf{R}^{-1}\mathbb{E}_t[p_t + \mathsf{R}^{-1}(p_t + \psi_{t+1}) + \mathsf{R}^{-2}(p_t + \psi_{t+1} + \psi_{t+2}) + ...] \\
&= \mathsf{R}^{-1}\left(\frac{p_t}{1 - \mathsf{R}^{-1}}\right) \\
&= \left(\frac{p_t}{\mathsf{R} - 1}\right) = p_t/\mathsf{r}
\end{aligned}
$$

Define 'balances' b as the amount of resources the consumer has in hand before receiving income in the current period:

$$
b_{t+1} = (b_t + y_t - c_t)\mathsf{R}
$$

and define total wealth o as the sum of asset balances b, current income y, and human wealth $\mathfrak{h}$,

$$
o_{t+1} = b_{t+1} + y_{t+1} + \mathfrak{h}_{t+1},
$$

d) Use the intertemporal budget constraint to show that

$$c_t \;=\; p_t + (b_t + \theta_t)(\mathsf{r}/\mathsf{R}) \tag{5}$$

Explain why this equation implies a marginal propensity to consume out of shocks to permanent income ψ_t of 1, and out of transitory income θ of (r/R).

Answer:

$$\begin{aligned}
\sum_{s=t}^{\infty} \mathbb{E}_t[\mathsf{R}^{t-s} c_s] &= b_t + y_t + \mathfrak{h}_t \\
c_t(\mathsf{R}/\mathsf{r}) &= b_t + \theta_t + p_t(1 + 1/r) \\
c_t &= p_t + (b_t + \theta_t)(\mathsf{r}/\mathsf{R})
\end{aligned}$$

MPC of one because $p_t = p_{t-1} + \psi_t$ and the MPC out of shocks to permanent income is $dc_t/d\psi_t$. The MPC of (r/R) out of transitory shocks is a direct implication of (5).

Part II: Short Question

1. **Small Open Economy Models of A Shock to Capital.** In the spring of 2011, Japan was hit by a terrible earthquake, tsunami, and resulting nuclear accident that destroyed a substantial amount of capital. This question asks what effect different models might lead you to expect for consumption spending in response to this disaster. Assume that before the shock, Japan was at its steady-state level of the capital/output ratio, and that the shock is perceived as a one-time destruction of a fixed amount of aggregate capital. Describe the dynamics of consumption per capita under the following models:

   a) A Diamond 2-Period OLG model

   Answer:

   > Consumption drops discretely at the time of the shock, then gradually increases over time back toward its original value

   b) A Hall Random Walk model

   Answer:

   > Consumption drops discretely at the time of the shock, and then permanently remains at the lower level forever

   c) A model with habit formation in aggregate consumption

   Answer:

   > Consumption drops a bit in the first period and gradually subsides toward a lower steady-state level

   d) A model where the reason aggregate consumption is normally sluggish is 'sticky information' (hint: you do not need to assume that information is *always* equally sticky in every period)

   Answer:

   > Taking the hint, recall that the idea of this model is that people *usually, on average* do not quickly notice macroeconomic events that will ultimately affect their consumption. But it is hard to imagine that there are any Japanese people who have not noticed the earthquake, tsunami, and nuclear calamity. So the spirit of this model would lead you to expect that consumption in this instance will behave as it would in the rational Hall model: An immediate permanent downward drop.

Part III

Predictable and Unpredictable Changes in Consumption in a Perfect Foresight Model.

In representative agent (RA) macroeconomic models, a key channel by which the central bank's control of interest rates affects aggregate demand is through its effect on consumption. For a small open economy, consumption behavior of the RA can be well approximated by the solution for a perfect foresight consumer, which `PerfForesightCRRA`[1] shows can be written approximately as:

$$\boldsymbol{c}_t \approx \boldsymbol{p}_t\left(1 - \text{Þ}_\gamma/(\mathsf{r}-\gamma)\right) - \text{Þ}_\mathsf{r}\boldsymbol{b}_t \tag{6}$$

$$\mathrm{c}_t \approx \left(1 - \text{Þ}_\gamma/(\mathsf{r}-\gamma)\right) - \text{Þ}_\mathsf{r} b_t \tag{7}$$

where

$$\text{Þ}_\gamma \equiv \rho^{-1}(\mathsf{r}-\vartheta) - \gamma$$

is the 'growth patience' rate while the 'return patience' rate is

$$\text{Þ}_\mathsf{r} \equiv \rho^{-1}(\mathsf{r}-\vartheta) - \mathsf{r}.$$

1. a) Briefly:

   i. Explain why we must assume that $\text{Þ}_\mathsf{r} < 0$ ("return patience") and $\text{Þ}_\gamma < 0$ ("growth patience").

   Answer:

   We need $\text{Þ}_\mathsf{r} < 0$ to ensure the consumer is 'return patient' – willing to save when returns are high. This requires $\rho^{-1}(\mathsf{r}-\vartheta) < \mathsf{r}$, or equivalently $\vartheta > \mathsf{r}(1-\rho)$. Similarly, $\text{Þ}_\gamma < 0$ (growth patience) ensures $\rho^{-1}(\mathsf{r}-\vartheta) < \mathsf{r}-\gamma$. Both conditions prevent infinite consumption or wealth accumulation and ensure convergence to a steady state. (See `PerfForesightCRRA` for detailed derivation.)

   ii. Use these formulae to describe the three channels by which a change in r should affect the level of $\boldsymbol{c}$: (i) the income effect, (ii) the intertemporal substitution effect, and (iii) the human wealth effect. For each channel, explain which term in the consumption function captures it and whether it increases or decreases consumption when r rises.

   Answer:

   The three channels are:

   (1) **Income effect** ($\mathsf{r}b$ term): Higher interest rates increase income from existing assets, raising consumption.

[1] Notation: Bank balances are $\boldsymbol{b}$, noncapital income is $\boldsymbol{p}_t$, relative risk aversion is ρ, the permanent rate of income growth is γ, etc.

(2) **Intertemporal substitution effect** (through $\rho^{-1}\mathsf{r}$ in þ_{r} and þ_{γ}): Higher interest rates make future consumption cheaper relative to current consumption, inducing consumers to postpone consumption, reducing $\boldsymbol{c}$ today.

(3) **Human wealth effect** (denominator $\mathsf{r}-\gamma$ term): Higher interest rates reduce the present value of future labor income, lowering consumption.

(See `PerfForesightCRRA` for full derivation.)

b) Discuss why, for quantitatively plausible calibrations in which r and ϑ are roughly the same size, this model implies that the effects of interest rates on consumption should be very large. (Hint: The calculations are easier if you assume that γ is 0.) Take the derivative $\frac{dc}{d\mathsf{r}}$ with $\gamma = 0$.

Answer:

Taking the derivative of consumption with respect to r with $\gamma = 0$:

$$\frac{dc}{d\mathsf{r}} \approx \underbrace{-\boldsymbol{p}\frac{\rho^{-1}\vartheta}{\mathsf{r}^2}}_{\text{human wealth effect}} - \underbrace{\rho^{-1}\boldsymbol{b}}_{\text{substitution effect}} + \underbrace{\mathsf{r}\boldsymbol{b}}_{\text{income effect}}$$

The **human wealth effect** (first term) is enormous because r^2 appears in the denominator: when r and ϑ are both around 0.03, the term $\vartheta/\mathsf{r}^2 \approx 0.03/0.0009 = 33$, amplified by $\boldsymbol{p}$.

The **substitution effect** (second term) is also large: with $\rho = 2$, the intertemporal elasticity of substitution is $1/\rho = 0.5$, much larger than empirical estimates (which are close to zero or even negative).

The **income effect** depends on wealth $\boldsymbol{b}$ and could be positive (for savers), negative (for borrowers), or small (for hand-to-mouth consumers).

Overall, the model predicts consumption should be highly sensitive to interest rates, contradicting empirical evidence of low interest rate sensitivity.

Consider now an economy in which consumers always behave as though they *believe* that interest rates will remain constant at the current level. But, every now and then a completely unexpected shock occurs that changes the interest rate; subsequently, consumers believe interest rates will remain forever at their new level. (An n 'MIT shock'). Mathematically,

$$\mathbb{E}_t[\mathsf{r}_{t+n}] = \mathsf{r}_t \;\forall\; n > 0$$

2. Assume that leading up to time t, interest rates had been constant at some level $\underline{\mathsf{r}}$. At date t there is an MIT shock that causes interest rates to go up permanently to $\bar{\mathsf{r}} > \underline{\mathsf{r}}$. Explain why this model implies the equation that Hall (1988) estimated:

$$\Delta \log c_{t+1} \;\approx\; \rho^{-1}(\mathbb{E}_t[\mathsf{r}_{t+1}] - \nu) + \epsilon_{t+1} \tag{8}$$

for $\mathbb{E}_t[\epsilon_{t+1}] = 0$ if our expectations of interest rates come from surveys of consumers.

(Hint: Consider separately the 'normal' periods in which there is no shock to interest rates, and the 'rare' periods in which there are such shocks. In normal periods, what does the Euler equation predict for $\Delta \log c_{t+1}$? In shock periods, why is there a deviation $\epsilon_{t+1} \neq 0$? Why does rational expectations imply $\mathbb{E}_t[\epsilon_{t+1}] = 0$ even though shocks occur?)

Answer:

The equation holds because of the dichotomy between normal and shock periods:

Normal periods (no shock): When there's no surprise change in interest rates, the standard consumption Euler equation applies: $\Delta \log c_{t+1} \approx \rho^{-1}(\mathbb{E}_t[\mathsf{r}_{t+1}] - \nu)$. Here $\epsilon_{t+1} = 0$ because expectations are realized.

MIT shock periods: When an unanticipated shock changes interest rates, there's a deviation $\epsilon_{t+1} \neq 0$ from the expected consumption growth. The shock causes a level shift in consumption (through the three channels discussed above), creating an unexpected component in consumption growth.

Zero-mean forecast errors: Because these MIT shocks are completely unexpected by assumption, $\mathbb{E}_t[\epsilon_{t+1}] = 0$ even though realizations can be nonzero when shocks occur. This is consistent with rational expectations: consumers cannot systematically predict the shocks.

Therefore, the equation holds with zero-mean forecast errors, as Hall estimated.

3. Given your analysis above, explain what forces will be the chief contributors to the magnitude of the ϵ shocks? (Note: The ϵ shock is the unexpected change in consumption when interest rates change unexpectedly.)

   a) Answer this question analytically first, under the convenient assumption that the time preference rate is $\nu = \underline{\mathsf{r}}$ and that leading up to t bank balances were constant at $b = 1$ and permanent income was $\boldsymbol{p}_t = 1$. (Hint: Compare what consumption would have been if rates had stayed at $\underline{\mathsf{r}}$ versus what it actually is after the shock to $\bar{\mathsf{r}}$. Use equation (6) in both cases and decompose the difference into the three channels.)

b) Now answer the question quantitatively, under the convenient assumption that $\underline{\mathsf{r}} = 0.02$ and $\bar{\mathsf{r}} = 0.04$ and $\rho = 2$ and $\gamma = 0$. (Hint: Substitute these numerical values into your answer from part (a) to calculate the size of each channel. Which channel dominates?)

c) After you have done the algebraic and quantitative analysis, draw a diagram showing the level of consumption from the period leading up to t through an interval after t. On the diagram, label changes that correspond to the combined income, substitution, and human wealth effects in period $t+1$, and to the intertemporal substitution channel of predictable consumption growth in periods before t and after $t+1$. (Hint: Before period t, consumption should be growing slowly; at $t+1$ there should be a sudden jump; after $t+1$, consumption resumes growing.)

Answer:

To find the magnitude of the ϵ shock, we need to compare what consumption would have been in the absence of the interest rate shock (the expected value $\mathbb{E}_t[c_{t+1}]$) with what it actually is after the shock occurs (the realized value c_{t+1}). The difference between these two is the unexpected component ϵ_{t+1}.

First, calculate the expected consumption path (no shock). Under the assumptions given, with $\nu = \underline{\mathsf{r}}$, $b = 1$, and $\boldsymbol{p}_t = 1$, the consumption function from equation (6) implies:

$$\mathbb{E}_t[c_{t+1}] = 1 + \underline{\mathsf{r}}$$

Next, calculate actual consumption after the MIT shock raises the interest rate to $\bar{\mathsf{r}}$. Using the same consumption function but with the new interest rate:

$$\begin{aligned} c_{t+1} &= 1 - \rho^{-1}(\bar{\mathsf{r}} - \nu)/\bar{\mathsf{r}} - (\rho^{-1}(\bar{\mathsf{r}} - \nu) - \bar{\mathsf{r}}) \\ c_{t+1} &= 1 - \bar{\mathsf{r}} + \rho^{-1}(\bar{\mathsf{r}} - \nu)/\bar{\mathsf{r}} - \rho^{-1}(\bar{\mathsf{r}} - \nu) \end{aligned}$$

The magnitude of the ϵ shock is the difference $c_{t+1} - \mathbb{E}_t[c_{t+1}]$, which we can decompose into the three channels by which the interest rate change affects consumption:

$$
\begin{aligned}
c_{t+1} - \mathbb{E}_t[c_{t+1}] &= \underbrace{\bar{\mathsf{r}} - \underline{\mathsf{r}}}_{\text{income effect}} - \underbrace{\rho^{-1}(\bar{\mathsf{r}} - \nu)/\bar{\mathsf{r}}}_{\text{human wealth}} - \underbrace{(\rho^{-1}(\bar{\mathsf{r}} - \nu))}_{\text{substitution}} \\
&= 0.02 - (1/2)(0.02/0.04) - (1/2)(0.02) \\
&= 0.02 - 0.25 - 0.01 \\
&= -0.24
\end{aligned}
$$

so the effects on consumption from the shock to interest rates come almost entirely from the human wealth effect.

But, after the *level* of consumption has adjusted downward in period $t+1$, it is still true that

$$
\mathbb{E}_{t+1}[\log c_{t+2}/c_{t+1}] \quad = \quad \rho^{-1}(\bar{\mathsf{r}} - \nu)
$$

The diagram should

- Show consumption being flat at $1 + \underline{\mathsf{r}}$ leading up to period t
- In period $t + 1$ there should be a big jump downward, of size 0.24, in the level of consumption, corresponding to the combined 'effects' of the interest rate shock on the level of consumption.
- In periods after $t+1$ there should be consumption growth at the rate $\rho^{-1}(\bar{\mathsf{r}} - \nu)$, corresponding to the intertemporal substitution channel.

References

Hall, Robert E. (1988): "Intertemporal Substitution in Consumption," *Journal of Political Economy*, XCVI, 339–357, Available at `http://www.stanford.edu/~rehall/Intertemporal-JPE-April-1988.pdf`.