Risky Portfolio Shares in a Buffer Stock Model.

Consider a consumer who ordinarily behaves according to the tractable buffer stock model discussed in class. At the end of period t, after the consumption decision has been made, this consumer is offered a one-time-only, never-to-be-repeated opportunity to invest in a risky asset. The returns on the asset will be revealed at the beginning of period $t+1$, and thereafter the consumer will behave again according to the optimal solution to the buffer stock model.

The risky asset's return factor $\mathbf{R}_{t+1}$ will take one of two values:

$$\mathbf{R}_{t+1} = \begin{cases} \bar{\mathbf{R}} = \left(\frac{\mathsf{R}\boldsymbol{\varphi}}{1-\wp}\right) & \text{with probability } \not\wp = (1-\wp) \\ \underline{\mathbf{R}} = 0 & \text{with probability } \wp \end{cases} \tag{1}$$

where $\boldsymbol{\varphi} > 1$ is the 'risky return premium' that compensates the consumer for the fact that the asset is riskier than the safe asset which earns return factor R.

Call the share that the consumer invests in the risky asset ς_t. Thus, the consumer arrives in period $t+1$ with bank balances of

$$\begin{aligned} b'_{t+1} &= a_t(\varsigma_t \mathbf{R}_{t+1} + (1-\varsigma_t)\mathsf{R}) && (2) \\ &= a_t\left(\mathsf{R} + \varsigma_t(\mathbf{R}_{t+1} - \mathsf{R})\right) && (3) \\ &= a_t\mathsf{R}\left(1 + \varsigma_t(\mathbf{R}_{t+1}/\mathsf{R} - 1)\right) && (4) \\ &= b_{t+1}\left(1 + \varsigma_t\phi_{t+1}\right) && (5) \end{aligned}$$

where $b_{t+1} = \mathsf{R}a_t$ is the amount of bank balances the consumer would have had in the absence of any investment in the risky asset, and $\phi_{t+1} \equiv (\mathbf{R}_{t+1}/\mathsf{R} - 1)$ is called the 'excess return' on the risky asset.

So, the excess return takes on two possible values:

$$\phi_{t+1} = \begin{cases} \bar{\phi} \equiv \bar{\mathbf{R}}\not\wp^{-1} & \text{with probability } \not\wp = (1-\wp) \\ \underline{\phi} \equiv -1 & \text{with probability } \wp \end{cases} \tag{6}$$

which correspondingly imply two possible realizations of b'_{t+1}:

$$b'_{t+1} = \begin{cases} \bar{b}_{t+1} \equiv b_{t+1}(1+\varsigma_t\bar{\phi}) & \text{with probability } \not\wp = (1-\wp) \\ \underline{b}_{t+1} \equiv b_{t+1}(1-\varsigma_t) & \text{with probability } \wp \end{cases} \tag{7}$$

1. Explain why the consumer's problem is solved by
$$\max_{\varsigma_t} \wp \left(\mathrm{v}^u(\underline{b}_{t+1})\mho + \mathrm{v}^e(\underline{b}_{t+1}+1)\not\mho\right) + \not\wp \left(\mathrm{v}^u(\bar{b}_{t+1})\mho + \mathrm{v}^e(\bar{b}_{t+1}+1)\not\mho\right) \tag{8}$$

   Answer:

   The consumer's goal is to
   $$\max_{\varsigma_t} \Gamma^{1-\rho}\,\mathbb{E}_t[\mathrm{v}(m_{t+1})] \tag{9}$$
   but since Γ is a constant this is equivalent to
   $$\max_{\varsigma_t} \mathbb{E}_t[\mathrm{v}(m_{t+1})] \tag{10}$$
   and (8) simply writes out explicitly the elements of that expected value, equal to the probability-weighted outcomes that the consumer will experience in each of the four possible conditions: Risky asset bust, unemployed; risky asset bust, employed; risky asset boom, unemployed; risky asset boom, employed.

2. Explain why the first order condition for this problem can be rewritten as
$$\mathbb{E}_t[\phi_{t+1}\mathrm{u}'(\mathrm{c}(m_{t+1}))] \;=\; 0 \tag{11}$$

   Answer:

   The FOC of
   $$\max_{\varsigma_t} \mathbb{E}_t[\mathrm{v}(a_t(\mathsf{R}+(\mathbf{R}_{t+1}-\mathsf{R})\varsigma_t)+y_{t+1})] \tag{12}$$
   is
   $$\mathbb{E}_t[\mathrm{v}'(m_{t+1})(\mathbf{R}_{t+1}-\mathsf{R})a_t] \;=\; 0 \tag{13}$$
   but since a_t is predetermined at the time that ς_t is chosen, and since $\mathrm{u}'(c_{t+1}) = \mathrm{v}'(m_{t+1})$, (13) is equivalent to (11) (dividing by R).

3. Explain why (11) implies that this consumer will never choose to invest $\varsigma_t = 1$ in the risky asset.

   Answer:

   If the consumer were to invest all his wealth in the risky asset and it returned $\mathbf{R}_{t+1} = 0$, and the consumer also became unemployed in that same period, his m_{t+1} would be zero forcing consumption to be zero and so his marginal utility would be infinity and so the expectation in (11) could not be satisfied for any $\wp$ and $\mho$ both > 0.

4. Now define $m_{t+1} = b_{t+1} + y_{t+1}$ as the value that m_{t+1} would take in the absence of the opportunity to invest in the risky asset, so that $\grave{m}_{t+1} = m_{t+1} + b_{t+1}\phi_{t+1}\varsigma_t$ and explain how the consumption function can be approximated by
$$\mathrm{c}(\grave{m}_{t+1}) \;\approx\; \mathrm{c}(m_{t+1}) + \mathrm{c}'(m_{t+1})b_{t+1}\phi_{t+1}\varsigma_t \tag{14}$$

Answer:

(14) is a first order Taylor expansion of $\mathrm{c}(\grave{m}_{t+1})$ around $\varsigma_t = 0$ using the definition of (5).

5. Use the approximation (14) to explain why the consumer will not choose $\varsigma_t = 0$ by showing that at $\varsigma_t = 0$ a small increase in the risky share of the portfolio would increase expected utility. (Hint: Use (14) in (11)). Explain the intuition behind this result. (Hint: If none of the portfolio is invested in the risky asset, what is the covariance between $\mathrm{u}'(c_{t+1})$ and ϕ_{t+1}?)

   Answer:

$$\mathbb{E}_t[\phi_{t+1}\left(\mathrm{u}'(\mathrm{c}(m_{t+1})) + \mathrm{u}''(\mathrm{c}(m_{t+1}))\mathrm{c}'(m_{t+1})b_{t+1}\phi_{t+1}\varsigma_t\right)] \approx 0$$

   Following up on the hint, if none of the portfolio is invested in the risky asset, then the covariance between the risky asset's returns and consumption must be zero, because the second term in (5) is zero. But then for $a_t > 0$, since $\mathbb{E}_t[\phi_{t+1}] > 0$ and $\mathbb{E}_t[\mathrm{u}'(\mathrm{c}(m_{t+1}))] > 0$ and the covariance is zero, it must be the case that the expectation on the LHS of is a strictly positive number.

   Recall the logic of the C-CAPM model: any asset whose covariance with consumption is zero, and which earns a positive expected return premium, must be marginally preferable to the riskless asset. So the consumer will want to increase his share in the risky asset from zero to some positive amount.

6. Using the intuition developed by the foregoing, explain why it is plausible to suppose that an increase in the magnitude of the unemployment risk might reduce the extent to which the consumer would be willing to invest in the risky asset.

   Answer:

   The consumer's problem balances the appeal of the excess return from in the risky asset (which is irresistible for a consumer who has nothing invested in it) against the fear of the consequences of investing too much in the risky asset and experiencing the calamity of a simultaneous unemployment event and stock market collapse. If the likelihood of the unemployment calamity increases, it is plausible that the fear of the simultaneous calamity will become more powerful. Hence, an increase in unemployment risk can reduce the consumer's willingness to invest in the risky asset.

References