Risky Portfolio Shares in a Buffer Stock Model.

Consider a consumer who ordinarily behaves according to the tractable buffer stock model discussed in class. At the end of period t, after the consumption decision has been made, this consumer is offered a one-time-only, never-to-be-repeated opportunity to invest in a risky asset. The returns on the asset will be revealed at the beginning of period $t + 1$, and thereafter the consumer will behave again according to the optimal solution to the buffer stock model.

The risky asset's return factor $\mathbf{R}_{t+1}$ will take one of two values:

$$\mathbf{R}_{t+1} = \begin{cases} \bar{\mathbf{R}} = \left(\frac{\mathsf{R}\boldsymbol{\varphi}}{1-\wp}\right) & \text{with probability } \not\wp = (1-\wp) \\ \underline{\mathbf{R}} = 0 & \text{with probability } \wp \end{cases} \tag{1}$$

where $\boldsymbol{\varphi} > 1$ is the 'risky return premium' that compensates the consumer for the fact that the asset is riskier than the safe asset which earns return factor R.

Call the share that the consumer invests in the risky asset ς_t. Thus, the consumer arrives in period $t + 1$ with bank balances of

$$\begin{aligned} b'_{t+1} &= a_t(\varsigma_t\mathbf{R}_{t+1} + (1-\varsigma_t)\mathsf{R}) && (2) \\ &= a_t\left(\mathsf{R} + \varsigma_t(\mathbf{R}_{t+1} - \mathsf{R})\right) && (3) \\ &= a_t\mathsf{R}\left(1 + \varsigma_t(\mathbf{R}_{t+1}/\mathsf{R} - 1)\right) && (4) \\ &= b_{t+1}\left(1 + \varsigma_t\phi_{t+1}\right) && (5) \end{aligned}$$

where $b_{t+1} = \mathsf{R}a_t$ is the amount of bank balances the consumer would have had in the absence of any investment in the risky asset, and $\phi_{t+1} \equiv (\mathbf{R}_{t+1}/\mathsf{R} - 1)$ is called the 'excess return' on the risky asset.

So, the excess return takes on two possible values:

$$\phi_{t+1} = \begin{cases} \bar{\phi} \equiv \bar{\mathbf{R}}\not\wp^{-1} & \text{with probability } \not\wp = (1-\wp) \\ \underline{\phi} \equiv -1 & \text{with probability } \wp \end{cases} \tag{6}$$

which correspondingly imply two possible realizations of b'_{t+1}:

$$b'_{t+1} = \begin{cases} \bar{b}_{t+1} \equiv b_{t+1}(1 + \varsigma_t\bar{\phi}) & \text{with probability } \not\wp = (1-\wp) \\ \underline{b}_{t+1} \equiv b_{t+1}(1 - \varsigma_t) & \text{with probability } \wp \end{cases} \tag{7}$$

1. Explain why the consumer's problem is solved by
$$\max_{\varsigma_t} \wp \left(\mathrm{v}^u(\underline{b}_{t+1})\mho + \mathrm{v}^e(\underline{b}_{t+1}+1)\cancel{\mho}\right) + \cancel{\wp} \left(\mathrm{v}^u(\bar{b}_{t+1})\mho + \mathrm{v}^e(\bar{b}_{t+1}+1)\cancel{\mho}\right) \tag{8}$$

2. Explain why the first order condition for this problem can be rewritten as
$$\mathbb{E}_t[\phi_{t+1}\mathrm{u}'(\mathrm{c}(m_{t+1}))] \;=\; 0 \tag{9}$$

3. Explain why (9) implies that this consumer will never choose to invest $\varsigma_t = 1$ in the risky asset.

4. Now define $m_{t+1} = b_{t+1} + y_{t+1}$ as the value that m_{t+1} would take in the absence of the opportunity to invest in the risky asset, so that $\grave{m}_{t+1} = m_{t+1} + b_{t+1}\phi_{t+1}\varsigma_t$ and explain how the consumption function can be approximated by

$$\mathrm{c}(\grave{m}_{t+1}) \approx \mathrm{c}(m_{t+1}) + \mathrm{c}'(m_{t+1})b_{t+1}\phi_{t+1}\varsigma_t \tag{10}$$

5. Use the approximation (10) to explain why the consumer will not choose $\varsigma_t = 0$ by showing that at $\varsigma_t = 0$ a small increase in the risky share of the portfolio would increase expected utility. (Hint: Use (10) in (9)). Explain the intuition behind this result. (Hint: If none of the portfolio is invested in the risky asset, what is the covariance between $u'(c_{t+1})$ and ϕ_{t+1}?)

6. Using the intuition developed by the foregoing, explain why it is plausible to suppose that an increase in the magnitude of the unemployment risk might reduce the extent to which the consumer would be willing to invest in the risky asset.

References