Income Growth Over the Lifetime Versus Between Generations (Modigliani (1986), Carroll and Summers (1991)). This question concerns the effects on aggregate saving of income growth over the lifetime versus income growth between generations. Consider an overlapping generations economy in which each individual lives for two periods. Population is constant, so that the population growth factor between generations is $\Xi = 1$; normalize population itself to $\mathbf{N} = 1$ per generation. The individuals' noncapital incomes in each period are exogenous. The first period noncapital income of an individual born at time t is $y_{t,1}$, and the second-period noncapital income of the same individual is $y_{t+1,2} = \mathsf{X} y_{t,1}$ where X can be greater than or less than one. The consumer solves the optimization problem:

$$
\begin{aligned}
\max \quad & \log(c_{t,1}) + \beta \log(c_{t+1,2}) \\
& \text{s.t.} \\
c_{t+1,2} &= (y_{t,1} - c_{t,1})\mathsf{R} + y_{t+1,2}.
\end{aligned}
$$

Finally, between generations the first period noncapital incomes grow by a factor $\mathrm{G} = (1 + \mathrm{g})$ so that:

$$
y_{t+1,1} = \mathrm{G} y_{t,1}
$$

For the purposes of this question, consider this to be an open economy so that the aggregate interest rate R and noncapital incomes y are fixed (that is, don't try to derive their values from an aggregate production function).

1. How does an increase in the growth rate of noncapital income over the lifetime, X, affect the assets accumulated by young households? Explain.

   Answer:

$$
\begin{aligned}
c_{t,1} &= \frac{y_{t,1} + y_{t+1,2}/\mathsf{R}}{1+\beta} \\
&= y_{t,1}\left[\frac{1 + \mathsf{X}/\mathsf{R}}{1+\beta}\right] \\
a_{t,1} &= y_{t,1} - c_{t,1} \\
&= y_{t,1}\left[1 - \frac{1 + \mathsf{X}/\mathsf{R}}{1+\beta}\right] \\
&= y_{t,1}\left[\frac{\beta - \mathsf{X}/\mathsf{R}}{1+\beta}\right]
\end{aligned}
$$

   So an increase in X reduces the assets accumulated by young households. Higher X reduces the need for asset accumulation to provide for consumption in the second period of life, because with higher X you will be richer in the second period of life anyway.

2. Calculate the amount of aggregate capital accumulation[1] as $\Delta K_{t+1} = K_{t+1} - K_t$ as a function of $y_{t,1}$, and then calculate the aggregate rate of capital accumulation compared to aggregate income as $\sigma_t = \Delta K_{t+1}/Y_t = \Delta K_{t+1}/(y_{t,1}+y_{t,2})$. How is the aggregate capital accumulation rate related to the growth rate of income between generations, G? How and why does the answer depend on the relationship between β and X/R?

Answer:

The aggregate capital stock in period $t+1$ is given by the assets of the young in period t:

$$K_{t+1} = \mathbf{N}a_{t,1} = a_{t,1} \tag{1}$$

Aggregate capital accumulation is given by the change in the aggregate capital stock:

$$\Delta K_{t+1} = K_{t+1} - K_t \tag{2}$$

Using the expression for $a_{t,1}$ derived above, this is:

$$\Delta K_{t+1} = [y_{t,1} - y_{t,1-1}] \left(\frac{\beta - \mathsf{X}/\mathsf{R}}{1+\beta} \right) \tag{3}$$

If $y_{t,1} = \mathrm{G}y_{t,1-1}$ this is:

$$\Delta K_{t+1} \quad = \quad (\mathrm{G} - 1)y_{t,1-1} \left[\frac{\beta - \mathsf{X}/\mathsf{R}}{1+\beta} \right]$$

Aggregate income is equal to the sum of the income of the young and the income of the old,

$$\begin{aligned} Y_t &= y_{t,1} + y_{t,2} \\ &= \mathrm{G}y_{t,1-1} + \mathsf{X}y_{t,1-1} \\ &= y_{t,1-1}[\mathrm{G} + \mathsf{X}] \end{aligned}$$

Thus the aggregate capital accumulation rate σ_t is

$$\sigma_t \quad = \quad \left(\frac{\mathrm{G} - 1}{\mathrm{G} + \mathsf{X}} \right) \left[\frac{\beta - \mathsf{X}/\mathsf{R}}{1+\beta} \right]$$

How does the capital accumulation rate vary with G?

$$\begin{aligned} \frac{d\sigma_t}{d\mathrm{G}} &= \frac{(\mathrm{G}+\mathsf{X}) - (\mathrm{G}-1)}{(\mathrm{G}+\mathsf{X})^2} \left[\frac{\beta - \mathsf{X}/\mathsf{R}}{1+\beta} \right] \\ &= \frac{1+\mathsf{X}}{(\mathrm{G}+\mathsf{X})^2} \left[\frac{\beta - \mathsf{X}/\mathsf{R}}{1+\beta} \right] \end{aligned}$$

[1]Another term often used for this is 'aggregate saving'; but economists frequently confuse the proper uses of the two words 'saving' and 'savings'. Saving is an ongoing activity, that you are doing over time. 'Savings' is the amount of wealth you have accumulated as a result of your past saving. To avoid such confusion, wherever possible I substitute another term for either word

The sign of this relationship depends on whether $\beta > \mathsf{X}/\mathsf{R}$ or, equivalently, whether $\beta\mathsf{R} > \mathsf{X}$. Recall that $\beta\mathsf{R}$ is the growth rate of consumption over the lifetime and X is the growth rate of income. If the growth rate of consumption is greater than the growth rate of income, this means that the level of consumption in the first period of life must be below the level of income - that is, young people accumulate assets. The parameter G determines how much richer the young are than the old. If the young accumulate assets, then making them richer than the old increases the aggregate capital accumulation rate. If the young decumulate assets, then making them richer than the old reduces the aggregate capital accumulation rate.

3. Thus far in the problem, we have assumed that X is independent of G; that is, we have assumed that the rate at which income grows during your lifetime is unrelated to the rate at which each generation's income exceeds the income of the previous generation. Now assume that $\mathsf{X} = \gamma \mathrm{G}$ for some constant γ. The idea is that some part of the growth of your income over your lifetime is attributable to the aggregate productivity growth that occurs over that time. γ is the rate at which your income would grow (or, more likely, shrink) if there were no productivity growth. Also assume (for simplicity) that $\beta = 1/\mathsf{R}$ (qualitative results are the same when $\beta \neq 1/\mathsf{R}$, but the analysis is messier). Now when does an increase in G increase or reduce the aggregate capital accumulation rate?

   Answer:

   Since $\mathsf{X} = \gamma \mathrm{G}$ and $\beta = 1/\mathsf{R}$, we have $\mathsf{X}/\mathsf{R} = \gamma \mathrm{G}\beta$, which can be substituted into (4):

$$
\begin{aligned}
\sigma_t &= \left(\frac{\mathrm{G}-1}{\mathrm{G}+\gamma\mathrm{G}}\right)\left[\frac{\beta-\gamma\mathrm{G}\beta}{1+\beta}\right] \\
&= \left(\frac{\mathrm{G}-1}{\mathrm{G}}\right)(1-\gamma\mathrm{G})\left[\frac{\beta}{(1+\gamma)(1+\beta)}\right] \\
&= \left(\frac{\mathrm{G}-1+\gamma\mathrm{G}-\gamma\mathrm{G}^2}{\mathrm{G}}\right)\left[\frac{\beta}{(1+\gamma)(1+\beta)}\right] \\
&= (1-1/\mathrm{G}+\gamma-\gamma\mathrm{G})\left[\frac{\beta}{(1+\gamma)(1+\beta)}\right]
\end{aligned}
$$

   So the derivative of the capital accumulation rate with respect to G is

$$
\frac{d\sigma_t}{d\mathrm{G}} = \left((1/\mathrm{G}^2)-\gamma\right)\left[\frac{\beta}{(1+\gamma)(1+\beta)}\right]
$$

   which will be positive only if $1/\mathrm{G}^2 > \gamma$.

4. Empirical evidence shows that the ratio of the income of the old (people aged 55-85) to income of the young (people aged 25-55) is about 0.7 in both the US and in Japan. From the late 1940s to the late 1980s, Japan's economic growth rate was

about 4 percent per year in per-capita terms. Over the same period income growth in the US was about 1 percent per capita. Japan's aggregate capital accumulation rate was also much higher than the US capital accumulation rate during this period. Discuss whether the overlapping generations model can explain Japan's high capital accumulation rate as being the result of its rapid growth rate (continue to assume $\beta = 1/\mathsf{R}$). (Hint: start by figuring out the OLG model's implications for the ratio of the income of the old to income of the young.)

Answer:

(The figures below show some evidence supporting the view that the ratio of consumption of the old to that of the young is not very different across countries.)

$$\begin{aligned} y_{t,2} &= \mathsf{X} y_{t,1-1} \\ &= \mathsf{X} y_{t,1}/\mathrm{G} \\ \frac{y_{t,2}}{y_{t,1}} &= \mathsf{X}/\mathrm{G} \end{aligned}$$

The fact that the income of the old is about 0.7 times the income of the young implies that $\mathsf{X} = 0.7\mathrm{G}$ in both countries. The US per capita growth rate of 1 percent per year implies an increase in per-capita income between generations of $\mathrm{G} = 1.01^{30} \approx 1.35$. The Japanese per capita growth rate translates to $\mathrm{G} = 1.04^{30} \approx 3.24$.

$$\begin{aligned} \sigma_{\mathrm{US}} &= (1 - 1/1.35 + 0.7 - 0.7 * 1.35) \left[\frac{\beta}{(1.7)(1+\beta)} \right] \\ &= 0.014 \left[\frac{\beta}{(1.7)(1+\beta)} \right] \\ \sigma_{\mathrm{Japan}} &= (1 - 1/3.24 + 0.7 - 0.7 * 3.24) \left[\frac{\beta}{(1.7)(1+\beta)} \right] \\ &= -0.87 \left[\frac{\beta}{(1.7)(1+\beta)} \right] \end{aligned}$$

Thus the Japanese capital accumulation rate should actually be strongly negative, and certainly much lower than the US capital accumulation rate.[2] You can also see that this must be true because from equation (4) if $\gamma = 0.7$ the derivative of the capital accumulation rate with respect to G is negative for any G such that

$$1/\mathrm{G}^2 \;<\; 0.7$$

[2]You may wonder how the model can imply a negative steady-state capital accumulation rate forever. The reason this does not violate the economy's Intertemporal Budget Constraint is that the economy is growing so fast that it can always repay its borrowing in period t with some of its much greater period $t+1$ income.

$$\begin{aligned} 1/0.7 &< G^2 \\ 1.19 &< G. \end{aligned}$$

Since we calculated G = 1.34 for the US and G = 3.24 for Japan, as we increase G from the US value to the Japanese value the capital accumulation rate must be steadily falling.

The reason the model implies a negative capital accumulation rate for Japan is simple: the young Japanese consumers know that their incomes will be much higher when they are old than when they are young, so they borrow large amounts while young and repay those loans when old.

Thus, the OLG model is not a good candidate for explaining why Japan's capital accumulation rate has been so high during its period of rapid growth.

Figure 4a

Predicted C By Age In Steady State in the LC Model
Across Countries With Different Rates of Growth of Income

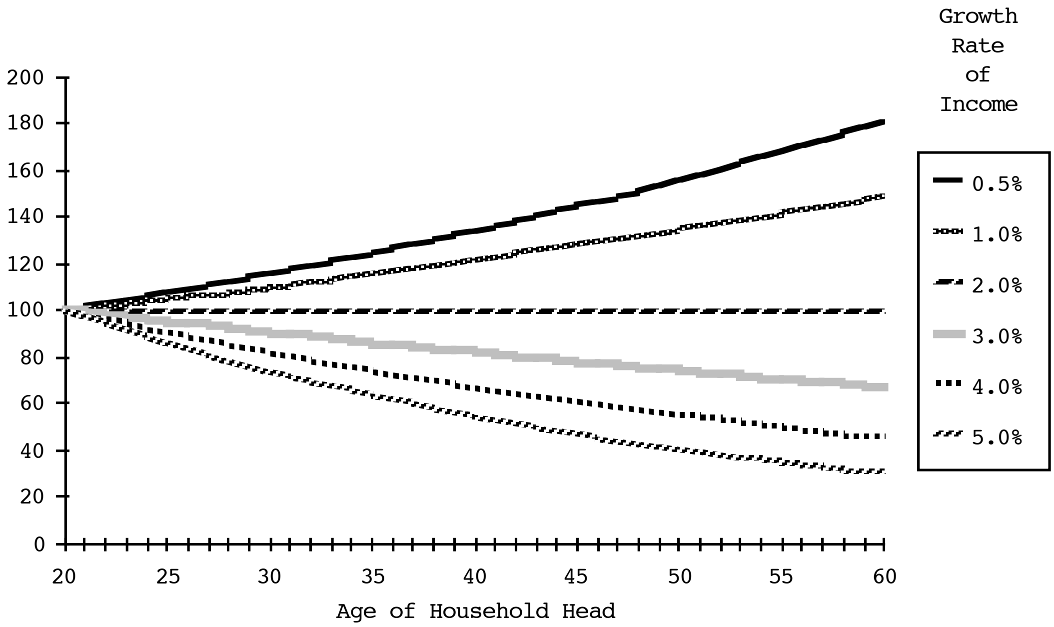
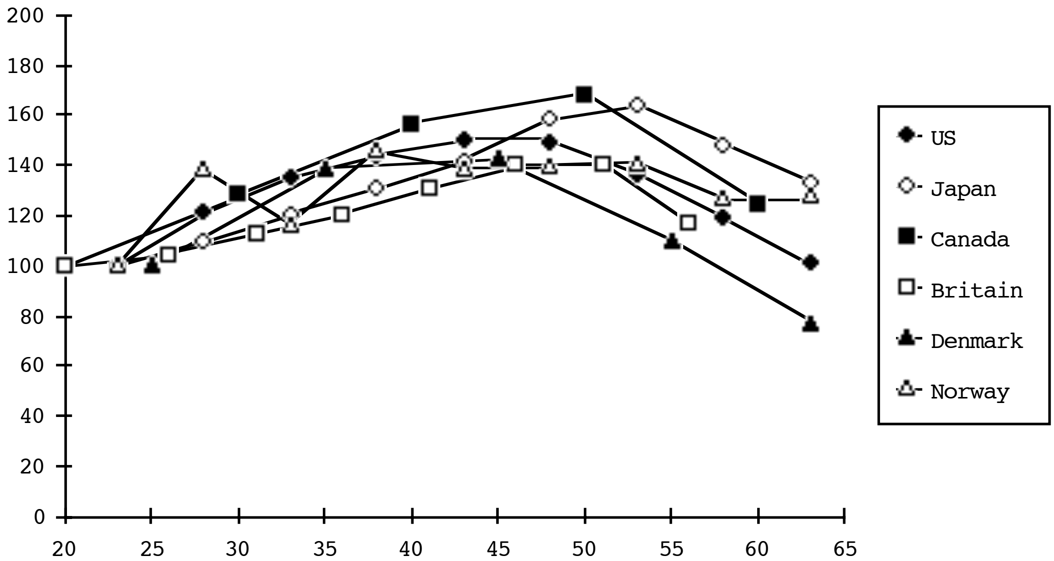


Figure 4b

Age/Consumption Cross-Section Data
for the US, Canada, Japan, Britain, Denmark, and Norway



References

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