Dynamic Inefficiency and the Capital Share Coefficient in an OLG Model.

Consider a Diamond (1965) OLG economy like the one in the handout `OLGModel` and the notebook `DiamondOLG`, assuming logarithmic utility and a Cobb-Douglas aggregate production function,

$$\boldsymbol{Y} = \mathrm{F}(\boldsymbol{K}, \boldsymbol{P}\mathbf{N}) \tag{1}$$

where $\boldsymbol{P}$ is a measure of labor productivity that grows according to

$$\boldsymbol{P}_{\tau+1} \quad = \quad \mathrm{G}\boldsymbol{P}_{\tau}. \tag{2}$$

Population growth is zero ($\Xi = 1$; for convenience normalize the population at $\mathbf{N}_\tau = 1 \; \forall \; \tau$), and until date t productivity growth has occurred at the rate $\mathrm{g} > 0$ (equivalently, $1 + \mathrm{g} = \mathrm{G} \geq 1$) forever. Under these assumptions, it can be shown that the dynamic process for aggregate $k \equiv \boldsymbol{K}/\boldsymbol{P}\mathbf{N}$ is

$$k_{\tau+1} \quad = \quad \left(\frac{(1-\alpha)\beta}{\mathrm{G}_{\tau+1}(1+\beta)}\right) k_\tau^\alpha \tag{3}$$

1. Derive the steady-state level of k_τ that the economy will have achieved by date t if the rate of productivity growth has always been $\mathrm{G}_\tau = \mathrm{G} \; \forall \; \tau$.

   Answer:

   In the steady state $k_{\tau+1} = k_\tau = \bar{k}$. Substituting into equation (3):

$$\bar{k} \quad = \quad \bar{k}^\alpha \left[\frac{(1-\alpha)\beta}{\mathrm{G}(1+\beta)}\right] \tag{4}$$

$$\bar{k}^{1-\alpha} \quad = \quad \left[\frac{(1-\alpha)\beta}{\mathrm{G}(1+\beta)}\right] \tag{5}$$

$$\bar{k} \quad = \quad \left[\frac{(1-\alpha)\beta}{\mathrm{G}(1+\beta)}\right]^{1/(1-\alpha)} \tag{6}$$

   Now suppose that, after an eternity of remaining in the steady state, all of a sudden at the beginning of period t everybody learns that henceforth and forever more, the exponent on capital in the production function will change to $\hat{\alpha} > \alpha$.

2. Define the new steady-state as $\bar{\hat{k}}$. Will this be larger or smaller than the original steady state $\bar{k}$? *Explain your answer.*

   Answer:

   Note first that for $0 < \alpha < 1$, increasing α makes $1-\alpha$ a smaller number and therefore makes $1/(1-\alpha)$ a larger number. So, whether $\bar{\hat{k}}$ rises when this happens depends on whether the term in [] in (6) is greater or less than 1.

   But both $(1-\alpha)$ and $\frac{\beta}{1+\beta}$ are less than one and we assume $\mathsf{G} \geq 1$, so the term in brackets is certainly less than one. Exponentiating it to a larger value will make the result smaller.

The reason is that, with logarithmic utility, the increase in the marginal product of capital (and hence the incentive to save) has a net zero effect on consumption, as the income and substitution effects offset each other. However, as labor's share has dropped, the reduction in wages leads to lower income for the young. The old spend all their income, so all aggregate saving comes from the young. With lower saving by the young and unchanged saving by the old, the total amount of saving declines, resulting in a lower capital to output ratio.

3. Next, use a diagram to show how the $k_{\tau+1}(k_\tau)$ curve changes when the new α takes effect, and show the dynamic adjustment process for the capital stock toward its new steady-state, assuming that the economy was at its original steady state leading up to period t.

   Answer:

   Defining the original steady-state capital stock as $\bar{k}$ and the new steady-state capital stock as $\hat{\bar{k}}$, the convergence process looks as indicated in the figure in `OLGModel`.

4. Define an index of aggregate consumption per efficiency unit of labor in period τ as $\chi_\tau = c_{1,\tau} + c_{2,\tau}/\mathsf{G}$, and derive a formula for the sustainable level of χ associated with a given level of k.

   Answer:

$$
\begin{aligned}
K_{\tau+1} &= K_\tau + K_\tau^\alpha P_\tau^{1-\alpha} - C_{1,\tau} - C_{2,\tau} && (7)\\
\left(\frac{K_{\tau+1}}{P_\tau}\right) &= k_\tau + k_\tau^\alpha - c_{1,\tau} - \frac{c_{2,\tau}P_{\tau-1}}{P_\tau} && (8)\\
\left(\frac{K_{\tau+1}}{P_{\tau+1}}\frac{P_{\tau+1}}{P_\tau}\right) &= k_\tau + k_\tau^\alpha - c_{1,\tau} - c_{2,\tau}/\mathsf{G} && (9)\\
k_{\tau+1}\mathsf{G} &= k_\tau + k_\tau^\alpha + \chi_\tau && (10)
\end{aligned}
$$

   The sustainable level of χ is the level $\bar{\chi}$ such that $k_{\tau+1} = k_\tau = \bar{k}$:

$$
\begin{aligned}
(1+\mathsf{g})\bar{k} &= \bar{k} + \bar{k}^\alpha - \bar{\chi} && (11)\\
\bar{\chi} &= \bar{k}^\alpha - \mathsf{g}\bar{k}. && (12)
\end{aligned}
$$

5. Derive the conditions under which a marginal increase in α will result in an increase in the steady-state level of χ, and explain in words why this result holds.

   Answer:

   First, if $\bar{k}$ is the steady state level of capital, (12) implies

$$
\begin{aligned}
\bar{\chi} &= f(\bar{k}) - \mathsf{g}\bar{k} && (13)\\
\frac{d\bar{\chi}}{d\alpha} &= \left(\frac{\partial f(\bar{k})}{\partial k} - \mathsf{g}\right)\frac{d\bar{k}}{d\alpha} && (14)
\end{aligned}
$$

$$\frac{d\bar{\chi}}{d\alpha} = (\bar{\mathsf{r}} - \mathsf{g}) \frac{d\bar{k}}{d\alpha}. \tag{15}$$

We know from the question above that if $\mathsf{G} \geq 1$, then $\frac{d\bar{k}}{d\alpha}$ is always strictly negative. Hence, the sign of the total derivative depends on whether or not $\bar{\mathsf{r}} < \mathsf{g}$.

If $\bar{\mathsf{r}} < \mathsf{g}$ holds, in which case the economy is dynamically inefficient, then the reduction in $\bar{k}$ that accompanies a rise in α (which we derived above) is a Pareto improvement and the steady state level of consumption rises. If the economy was *not* dynamically inefficient, then a reduction in the steady state level of capital will result in a reduction in the steady state level of consumption. For more intuition on dynamic inefficiency see the `OLGModel`.

References

Diamond, Peter A. (1965): "National Debt in a Neoclassical Growth Model," *American Economic Review*, 55, 1126–1150, http://www.jstor.org/stable/1809231.