The Equity Premium Puzzle and the Riskfree Rate

Mehra and Prescott (1985) consider a representative agent solving the joint consumption and portfolio allocation problem:

$$\begin{aligned} \mathrm{v}(m_t) &= \max_{\{c_t,\varsigma_t\}} \quad \mathrm{u}(c_t) + \mathbb{E}_t\left[\sum_{n=1}^{\infty}\beta^n \mathrm{u}(c_{t+n})\right] \\ &\text{s.t.} \\ m_{t+1} &= (m_t - c_t)\mathbb{R}_{t+1} + y_{t+1} \\ \mathbb{R}_{t+1} &= \varsigma_t \mathbf{R}_{t+1} + (1-\varsigma_t)\mathsf{R} \end{aligned}$$

where R denotes the return on a perfectly riskless asset and $\mathbf{R}_{t+1}$ denotes the return on equities (the risky asset) held between periods t and $t+1$; ς_t is the share of end-of-period savings invested in the risky asset; $\mathbb{R}_{t+1}$ is the portfolio-weighted rate of return; and y_{t+1} is noncapital income in period $t+1$.

As usual, the objective can be rewritten in recursive form:

$$\mathrm{v}(m_t) = \max_{\{c_t,\varsigma_t\}} \quad \mathrm{u}(c_t) + \beta\,\mathbb{E}_t\left[\mathrm{v}\left(\underbrace{[\varsigma_t\mathbf{R}_{t+1} + (1-\varsigma_t)\mathsf{R}]}_{\mathbb{R}_{t+1}}(m_t - c_t) + y_{t+1}\right)\right] \tag{1}$$

The first order condition with respect to c_t is

$$\mathrm{u}'(c_t) = \beta\,\mathbb{E}_t[\mathbb{R}_{t+1}\mathrm{v}'(m_{t+1})]. \tag{2}$$

and, taking m and c as given, the FOC with respect to ς_t is

$$\begin{aligned} \mathbb{E}_t[(\mathbf{R}_{t+1} - \mathsf{R})\mathrm{v}'(m_{t+1})(m_t - c_t)] &= 0 \\ \mathbb{E}_t[(\mathbf{R}_{t+1} - \mathsf{R})\mathrm{v}'(m_{t+1}) \qquad\qquad] &= 0. \end{aligned} \tag{3}$$

But the usual logic of the `Envelope` theorem tells us that

$$\mathrm{u}'(c_{t+1}) = \mathrm{v}'(m_{t+1}), \tag{4}$$

so, substituting (4) into (2) and (3), the above FOC's reduce to

$$\mathrm{u}'(c_t) = \mathbb{E}_t\left[\beta\mathbb{R}_{t+1}\mathrm{u}'(c_{t+1})\right], \tag{5}$$

and

$$\mathbb{E}_t[(\mathbf{R}_{t+1} - \mathsf{R})\mathrm{u}'(c_{t+1})] = 0. \tag{6}$$

Now assume CRRA utility, $\mathrm{u}(c) = c^{1-\rho}/(1-\rho)$ and divide both sides by $c_t^{-\rho}$ to get

$$\mathbb{E}_t[(c_{t+1}/c_t)^{-\rho}(\mathbf{R}_{t+1} - \mathsf{R})] = 0. \tag{7}$$

and the consumption ratio can of course be rewritten as

$$c_{t+1}/c_t = c_t(1 + \Delta c_{t+1}/c_t) \tag{8}$$

Now use the fact TaylorOne:

- If z is small, $(1+z)^{\lambda} \approx 1+\lambda z$

to approximate equation (7)

$$\mathbb{E}_t[(1-\rho\Delta\log c_{t+1})(\mathbf{R}_{t+1}-\mathsf{R})] \approx 0. \tag{9}$$

Using one more fact,

- $\mathbb{E}[xy] = \mathbb{E}[x]\,\mathbb{E}[y] + \text{cov}(x,y)$,

we get

$$(1-\rho\,\mathbb{E}_t[\Delta\log c_{t+1}])(\mathsf{R}-\mathbb{E}_t[\mathbf{R}_{t+1}]) + \text{cov}_t(-\rho\Delta\log c_{t+1}, -\mathbf{R}_{t+1}) \approx 0 \tag{10}$$

or

$$\begin{aligned} \mathbb{E}_t[\mathbf{R}_{t+1}] - \mathsf{R} &\approx \frac{\rho\text{cov}_t(\Delta\log c_{t+1}, \mathbf{R}_{t+1})}{1-\rho\,\mathbb{E}_t[\Delta\log c_{t+1}]} \\ &\approx \rho\text{cov}_t(\Delta\log c_{t+1}, \mathbf{R}_{t+1}) \end{aligned} \tag{11}$$

where the last approximation holds because $\mathbb{E}_t[\Delta\log c_{t+1}]$ is small. (See the appendix for a derivation of the portfolio share when next period's consumption function is known).

The Equity Premium Puzzle

Because this expression must hold at all t, we can check it empirically by calculating empirical estimates of the two components and assuming that the sample averages correspond to the representative agent's expectations. That is, if we have data for periods $1\ldots n$, we assume that the unconditional expectations correspond to the sample means, $\mathbb{E}[\mathbf{R}] = (1/n)\sum_{s=1}^{n}\mathbf{R}_s$; $\mathbb{E}[\Delta\log c] = (1/n)\sum_{s=1}^{n}\Delta\log c_s$; and $\text{cov}(\Delta\log c, \mathbf{R}) = (1/n)\sum_{s=1}^{n}(\Delta\log c_s - \mathbb{E}[\Delta\log c])(\mathbf{R}_s - \mathbb{E}[\mathbf{R}])$.

The equity premium puzzle is essentially that $\text{cov}(\Delta\log c, \mathbf{R})$ is very small (about 0.004) but $\mathbb{E}[\mathbf{R}] - \mathsf{R}$ is about 0.08 (stocks have earned real returns of about 8 percent more than riskless assets over the historical period), which means that the only way equation (11) can hold is if ρ is implausibly large (these values imply a value of $\rho = 20$).

How do we know what plausible values of ρ are? Consider the following. You must choose between a gamble in which you consume \$50,000 for the rest of your life with probability 0.5 and \$100,000 with probability 0.5, or consuming some amount X with certainty. The coefficient of relative risk aversion determines the X which would make you indifferent between consuming X or being exposed to the gamble. For example, if $\rho = 0$, then you have no risk aversion at all and you will be indifferent between \$75,000 with certainty and the 50/50 gamble with expected value of \$75,000. Here are the values of X associated with different values of ρ (table taken from Mankiw and Zeldes Mankiw and Zeldes (1989).)

ρ	X
1	70,711
3	63,246
5	58,565
10	53,991
20	51,858
30	51,209
∞	50,000

The Riskfree Rate Puzzle

Rewrite the consumption Euler equation (5) as

$$\mathrm{u}'(c_t) = \mathbb{E}_t\left[\beta(\mathsf{R} + \varsigma_t[\mathbf{R}_{t+1} - \mathsf{R}])\mathrm{u}'(c_{t+1})\right] \tag{12}$$

and note that from (7) we know that $\mathbb{E}_t[\beta\varsigma_t(\mathbf{R}_{t+1} - \mathsf{R})\mathrm{u}'(c_{t+1})] = 0$ so that (12) reduces to the ordinary Euler equation

$$\begin{aligned} \mathrm{u}'(c_t) &= \mathbb{E}_t[\beta\mathsf{R}\mathrm{u}'(c_{t+1})] \\ 1 &= \beta\mathsf{R}\,\mathbb{E}_t[(c_{t+1}/c_t)^{-\rho}]. \end{aligned} \tag{13}$$

Using the same 'facts' and approximations as above, we get the standard approximation to the Euler equation,

$$\Delta \log c_{t+1} \approx (1/\rho)(\mathsf{r} - \vartheta). \tag{14}$$

The 'riskfree rate puzzle' is that average consumption growth per capita has been about 1.5 percent (in the US in the postwar period) while real riskfree interest rates have been at most 1 percent. Even if we assume a time preference rate of $\vartheta = 0$ (no impatience at all, e.g. $\beta = 1$), the only way this equation can hold is if ρ is a very small number (maybe even less than one). Of course, this is precisely the opposite of the conclusion of the equity premium puzzle, which implies the ρ must be very large.

Appenix

The risky portfolio share when c_{t+1} is Known

In either the life cycle version of the model or an infinite horizon model that is being solved by time iteration, the consumption function in $t+1$ will be known.

Designating that consumption function as $\mathrm{c}_{t+1}(m)$, with derivative $\mathrm{c}'_{t+1}(m)$, we can derive an approximation to the optimal portfolio share as follows.

First define $\bar{\mathbb{R}}_{t+1}(\varsigma) = \mathbb{E}_t[\mathbb{R}_{t+1}]$ and define consumption at the expectation of the portfolio return as

$$\begin{aligned}\bar{c}_{t+1} &\equiv \mathrm{c}_{t+1}(\bar{\mathbb{R}}_{t+1}a_t + y_{t+1})\\ \bar{c}'_{t+1} &\equiv \mathrm{c}'_{t+1}(\bar{\mathbb{R}}_{t+1}a_t + y_{t+1})\end{aligned}$$

For simplicity, we will henceforth assume that $y_{t+1} = 1$. Results below all go through for the case where $\mathbb{E}_t[y_{t+1}] = 1$.

Calling the realized equity premium $\varphi_{t+1} = (\mathbf{R}_{t+1} - \mathsf{R})$, note that for a portfolio share of ς the realized return premium will be $\mathbb{R}_{t+1} = (\mathbf{R}_{t+1} - \mathsf{R})\varsigma$. Now, use $\mathrm{u}'(\bullet) = \bullet^{-\rho}$ to rewrite the FOC for ς, equation (6):

$$\begin{aligned}\mathbb{E}_t[(\mathrm{c}_{t+1}(\mathbb{R}_{t+1}a_t + 1))^{-\rho}\varphi_{t+1}] &= 0\\ \mathbb{E}_t[(\mathrm{c}_{t+1}((\mathbb{R}_{t+1} - \bar{\mathbb{R}}_{t+1} + \bar{\mathbb{R}}_{t+1})a_t + 1))^{-\rho}\varphi_{t+1}] &= 0\\ \mathbb{E}_t[(\mathrm{c}_{t+1}((\bar{\mathbb{R}}_{t+1} + \varsigma\varphi_{t+1})a_t + 1))^{-\rho}\varphi_{t+1}] &= 0\end{aligned} \tag{15}$$

Now make a first order Taylor expansion of next period's consumption around $\bar{\mathrm{c}}_{t+1}$ (you could use a second order expansion for an even more accurate approximation), and then use `TaylorOne`:

$$\begin{aligned}\mathbb{E}_t[(\bar{\mathrm{c}}_{t+1} + \varsigma\varphi_{t+1}\bar{\mathrm{c}}'_{t+1}a_t)^{-\rho}\varphi_{t+1}] &\approx 0\\ (\bar{\mathrm{c}}_{t+1})^{-\rho}\,\mathbb{E}_t[(1 + \bar{\mathrm{c}}_{t+1}^{-1}(\varsigma\varphi_{t+1}\bar{\mathrm{c}}'_{t+1}a_t))^{-\rho}\varphi_{t+1}] &\approx 0\\ \mathbb{E}_t[(1 + (\varsigma\varphi_{t+1}\bar{\mathrm{c}}'_{t+1}a_t)\bar{\mathrm{c}}_{t+1}^{-1})^{-\rho}\varphi_{t+1}] &\approx 0\\ \mathbb{E}_t[(1-\rho(\varsigma\varphi_{t+1}\bar{\mathrm{c}}'_{t+1}a_t)\bar{\mathrm{c}}_{t+1}^{-1})\varphi_{t+1}] &\approx 0\end{aligned} \tag{16}$$

where the last step uses $(1+\epsilon)^{\delta} \approx (1+\delta\epsilon)$.

Now define the proportional MPC out of an additional unit of return as

$$\kappa = \bar{\mathrm{c}}'_{t+1}a_t\bar{\mathrm{c}}_{t+1}^{-1} \tag{17}$$

and calling $\varphi \equiv \mathbb{E}_t[\varphi_{t+1}]$ and $\mathbb{E}_t[(\varphi_{t+1} - \varphi)^2] = \sigma^2_{\varphi}$ (so $\mathbb{E}_t[\varphi^2_{t+1}] = \sigma^2_{\varphi} - \varphi^2$) substitute this into the foregoing to obtain

$$\begin{aligned}\mathbb{E}_t[(1-\rho\,(\kappa\varsigma\varphi_{t+1}))\varphi_{t+1}] &\approx 0\\ \varphi &\approx \rho\kappa\varsigma\,\mathbb{E}_t[\varphi^2_{t+1}]\\ \varphi &\approx \rho\kappa\varsigma(\sigma^2_{\varphi} - \varphi^2)\\ \left(\frac{\varphi}{\rho\kappa(\sigma^2_{\varphi} - \varphi^2)}\right) &\approx \varsigma\end{aligned} \tag{18}$$

References

Mankiw, N. Gregory, and Stephen P. Zeldes (1989): "The Consumption of Stockholders and Non Stockholders," *Journal of Financial Economics*, 15, 145–61.

Mehra, Rajnish, and Edward C. Prescott (1985): "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15, 145–61.