MATH M118

Finite Mathematics

Spring 2025
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Preface

These notes started mostly as a transcription of an edition Diane Hampshire's lecture notes for Finite Mathematics taught in the Spring of 2017 at Indiana University Kokomo. It has gained a few additions since the first draft typeset in the Fall semester of 2018.

The Spring 2023 edition, on which this current edition is built, expanded some exposition on certain combinatorial examples, as well as included some new examples.

Note that Section 1.1.2 is included only for those students with a curiosity exceeding the course's expectations.

1 Sets and Counting

1.1 Sets

Definition 1.1. A **set** is a collection of objects. The objects of a set are called the **elements**.

Note. Sometimes we call an element of a set a *member* of the set.

We can define a set by listing all of its elements between curly braces; e.g.

$$\{\text{Sun., Mon., Tues., Wed., Thurs., Fri., Sat.}\}$$

We can also define a set by giving a description of its elements using *set builder* notation; e.g.

$$\{x : x \text{ is a day of the week}\}$$

Notation. We write $x \in A$ to mean that "x is an element of the set A". We write $x \notin B$ to mean that "x is not an element of the set B".

Example 1.1. $a \in \{a, b, c\}$

Example 1.2. $d \notin \{a, b, c\}$

Definition 1.2. The number of elements in a set is called the **cardinality**, denoted by $n(A)$, where A is a set.

Example 1.3. $n(\{x : x \text{ is a day of the week}\}) = 7$

Example 1.4. $n(\{a, b, c\}) = 3$

Definition 1.3. If two sets A and B have the same elements, then A **equals** B.

Example 1.5. If

$$D = \{\text{Sun., Mon., Tues., Wed., Thurs., Fri., Sat.}\}$$

and

$$W = \{x : x \text{ is a day of the week}\},$$

then $D = W$.

Definition 1.4. A set A is a **subset** of another set B, denoted by $A \subseteq B$, if every element of A is also an element of B.

Example 1.6. Let $A = \{a, b, c\}$, $B = \{a, b, c, d, e, f, g\}$, and $C = \{a, z\}$. Then

- $A \subseteq B$
- $C \nsubseteq B$

Definition 1.5. A set A is a **proper subset** of B, denoted $A \subset B$, if a set A is a subset of B but $A \neq B$.

Example 1.7. Let $A = \{e, a, r\}$ and $B = \{y, e, a, r\}$. Then $A \subset B$.

Example 1.8. Let $X = \{e, a, t\}$ and $Y = \{a, t, e\}$. Then $X \subseteq Y$.

Note. $A \subseteq A$ for any set A.

Definition 1.6. The **empty set** (or **null set**) is the set that contains no elements, denoted $\emptyset$ or $\{\}$.

Note. $n(\emptyset) = 0$.

Note. $\emptyset \subseteq A$ for any set A.

Example 1.9. List the subsets of $\{a, b, c\}$.

$$\begin{gathered}\emptyset, \\ \{a\}, \{b\}, \{c\}, \\ \{a, b\}, \{a, c\}, \{b, c\}, \\ \{a, b, c\}\end{gathered}$$

1.1.1 Set Algebra

Definition 1.7. If A and B are sets, the **intersection** of A and B, denoted by $A \cap B$, is the set of elements that belong to both A <u>and</u> B.

Example 1.10. Let $A = \{a, b, c, d\}$ and $B = \{f, a, c, e\}$. Then $A \cap B = \{a, c\}$.

Note. Intersection $\cap$ like an "N": A a<u>n</u>d B

Definition 1.8. If A and B are sets, the **union** of A and B, denoted by $A \cup B$, is the set of all elements that belong to either A <u>or</u> B (or both).

Example 1.11. Let $A = \{a, b, c, d\}$ and $B = \{f, a, c, e\}$. Then $A \cup B = \{a, b, c, d, e, f\}$

Note. Union $\cup$ like a "U" for union.

Example 1.12. Let $A = \{a, b, c, d\}$ and $B = \{e, f, g\}$. Then $A \cap B = \emptyset$ (A and B are **disjoint**). Also, $A \cup B = \{a, b, c, d, e, f, g\}$.

Definition 1.9. In any particular problem or scenario, there is usually an understood set U, which we call the **universal set**, consisting of all things being considered. Then any other set we discuss will be a subset of U.

Definition 1.10. If A is a set, the **complement** of A, denoted by A', is the set of elements in the universal set U that are not in A.

Example 1.13. Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{0, 2, 4, 6, 8\}$, $B = \{3, 6, 9\}$, and $C = \{5, 7, 9\}$. Then

- $A' = \{1, 3, 5, 7, 9\}$
- $B' = \{0, 1, 2, 4, 5, 7, 8\}$
- $A' \cup B' = \{0, 1, 2, 3, 4, 5, 7, 8, 9\}$
- $B \cup C = \{3, 5, 6, 7, 9\}$
- $A' \cap (B \cup C) = \{3, 5, 7, 9\}$
- $A \cap B = \{6\}$

- $(A \cap B)' = \{0, 1, 2, 3, 4, 5, 7, 8, 9\}$ (notice a similarity to $A' \cup B'$?)
- $A \cap B \cap C = \emptyset$

Note. For any set A,

- $A \cap A' = \emptyset$
- $A \cup A' = U$

Note. Think of

- $\cup$ the cup that holds everything
- $\cap$ cap the common

We end this section with a comment on language.

Translating natural language to formal mathematical symbols can be challenging. Here are some examples to hopefully clarify some issues with phrasing.

Example 1.14. Let U be the universal set of all university students, A be the set of all student athletes, and M be the set of all math majors. Describe the set $A \cup M$.

- *A correct description.* $A \cup M$ is the set of all students (x) that are athletes $(x \in A)$ or math majors $(x \in M)$.
- *An incorrect description.* The set of all students that are athletes or the set of all students that are math majors.

The reason the second description is incorrect will hopefully be made clear in the next example.

Example 1.15. In a ping pong league, there are a few teams. Two of teams are the Angry Avocets and the Pitiless Peregrines. (Since teams have open registrations, people can be members of more than one team.) Consider two scenarios.

- For an upcoming international competition, you are going to form a larger team consisting of the members of the Angry Avocets and the Pitiless Peregrines. The resulting team is the union of the teams.
- A new member wants to join one team and, due to regional restrictions, they can either join the Angry Avocets or the Pitiless Peregrines. In this scenario, there is no union operation; this is representing a choice between two teams.

1.1.2 Words of Caution

One may be tempted to think that any description of things yields a set. For example, one can define G to be the set of all geese that have existed, exist, and will ever exist. Unfortunately, we have to exercise some amount of care when defining sets if we want to avoid contradictions.

The Barber Paradox. In the small town of Russellville, there is a barber who shaves the beards all of those, and only those, who do not shave themselves[1]. We can express this using sets with

$$S = \{P : P \text{ does not shave himself}\}.$$

Let U be the set of all Russellville residents that shave their beards. Can the barber shave himself?

Russell's Paradox. This is similar to the Barber Paradox but much more formal. Consider a potential set definition

$$R = \{x : x \notin x\},$$

the set of all sets which don't contain themselves.

As an example to see that there might be something to this kind of property, consider

$$A = \{x : x \text{ is not an orange}\}.$$

The collection A surely isn't an orange so $A \in A$.

Now we return to R. If R is itself a set, then we have

- $R \in R \implies R \notin R$
- $R \notin R \implies R \in R$

In either scenario, we have a contradiction. Therefore, R cannot be a set.

1.2 The Inclusion-Exclusion Principle

Inclusion-Exclusion Principle. If A and B are sets, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Equivalently,

$$n(A \cup B) + n(A \cap B) = n(A) + n(B).$$

If we assume, in addition, that A and B are disjoint (i.e., $A \cap B = \emptyset$), then $n(A \cup B) = n(A) + n(B)$ since $n(A \cap B) = 0$.

Example 1.16. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, f, g\}$. Then

- $n(A \cup B) = 7 \quad \longleftarrow \quad A \cup B = \{a, b, c, d, e, f, g\}$
- $n(A \cap B) = 2 \quad \longleftarrow \quad A \cap B = \{a, b\}$
- $n(A) = 5$
- $n(B) = 4$

$$\begin{array}{ccccccc} n(A \cup B) & = & n(A) & + & n(B) & - & n(A \cap B) \\ 7 & = & 5 & + & 4 & - & 2 \end{array}$$

Example 1.17. Given $n(A \cup B) = 40$, $n(A) = 28$, and $n(A \cap B) = 10$, find $n(B)$.

[1] so everyone's beard gets shaved

From the equation $n(B) = n(A \cup B) + n(A \cap B) - n(A)$ we compute that

$$\begin{aligned} n(B) &= 40 + 10 - 28 \\ &= 22. \text{ //} \end{aligned}$$

Example 1.18. If 80 pigs fly, 60 pigs talk, and 52 pigs fly and talk, determine

(a) how many pigs either fly or talk (or do both) and

(b) how many pigs fly but don't talk.

Let F be the set of pigs that fly and T be the set of pigs that talk. We want to find $n(F \cup T)$ and we know that

- $n(F) = 80$
- $n(T) = 60$
- $n(F \cap T) = 52$.

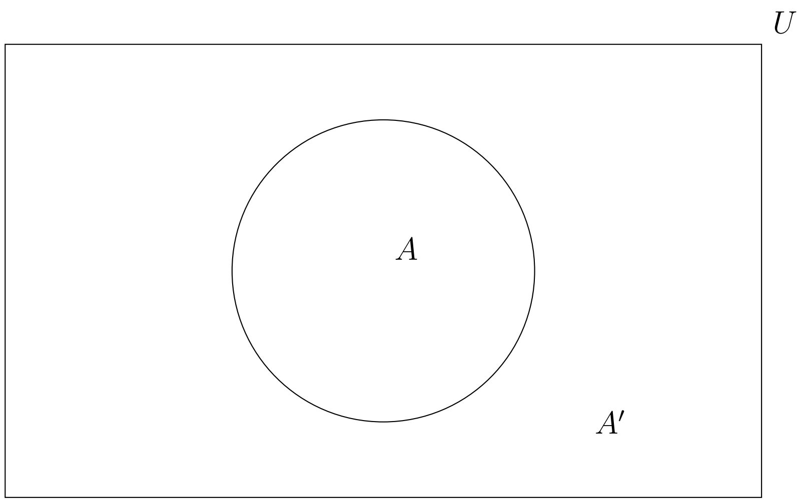
So,

$$\begin{aligned} n(F \cup T) &= n(F) + n(T) - n(F \cap T) \\ &= 80 + 60 - 52 \\ &= 88. \text{ //} \end{aligned}$$

Though there are 60 pigs that talk, only 52 of those talking pigs also fly. To find how many pigs fly but don't talk, we need only count the number of pigs that fly and, of those, take away the pigs that also talk. In sets, we are interested in $F \cap T'$ and we compute

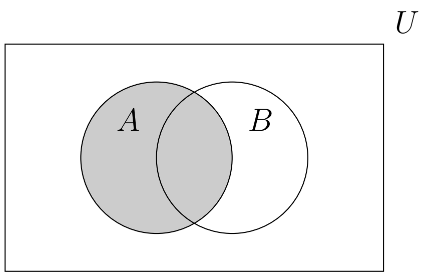
$$n(F \cap T') = n(F) - n(F \cap T) = 80 - 52 = 28. \text{ //}$$

Venn diagrams are often used to help visualize sets. For example, we can draw

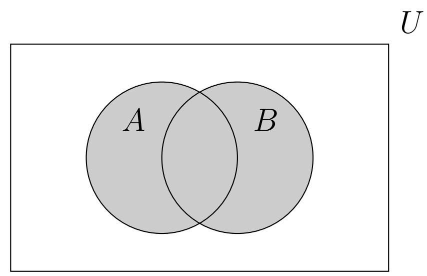


which graphically represents the universal set U, the set A, and the set A'.

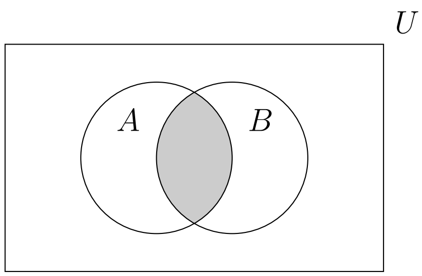
If we have two sets A and B, we can use shading to represent particular regions:



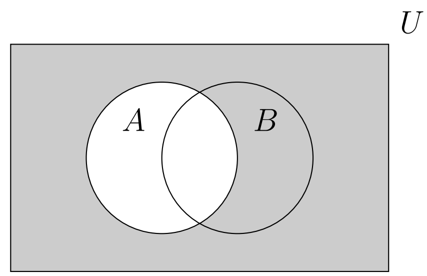
A is shaded



$A \cup B$ is shaded



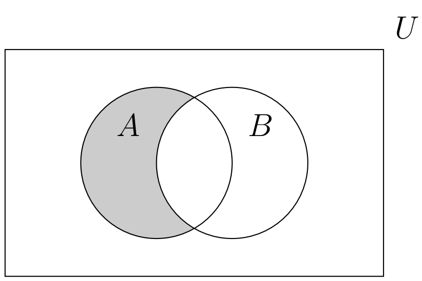
$A \cap B$ is shaded



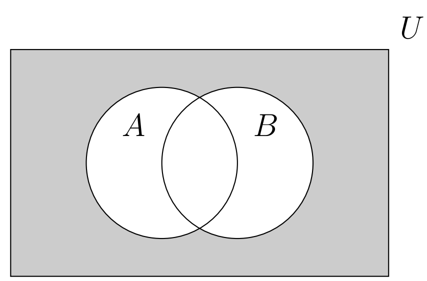
A' is shaded

Example 1.19. Draw Venn diagrams with two sets A and B and shade the regions

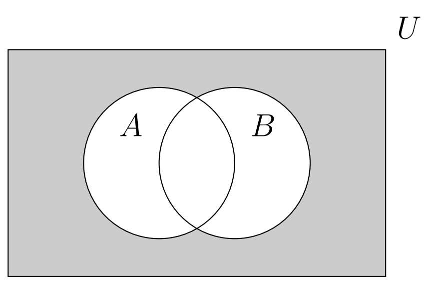
- $A \cap B'$
- $A' \cap B'$
- $(A \cup B)'$



$A \cap B'$ is shaded



$A' \cap B'$ is shaded



$(A \cup B)'$ is shaded

De Morgan's Laws. For sets A and B,

- $(A \cup B)' = A' \cap B'$
- $(A \cap B)' = A' \cup B'$

1.3 Venn Diagrams and Counting

Example 1.20. Let U be the collection of all 120 people who participated in a survey. In this survey,

- 75 people said they liked apples (set A),
- 60 said they liked bananas (set B), and
- 42 said they liked both.

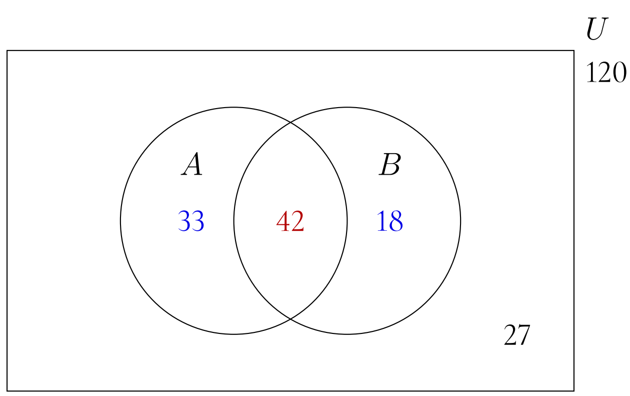
Draw a Venn diagram describing the survey and determine

(a) how many only like apples,

(b) how many only like bananas,

(c) how many like apples or bananas, and

(d) how many don't like apples or bananas.



(a) To find the number of people that only liked apples, we take the number of people that liked apples and subtract the number of people that also like bananas:

$$n(A) - n(A \cap B) = 75 - 42 = 33.$$

(b) Similarly, to find the number of people that only liked bananas, we take the number of people that liked bananas and subtract the number of people that also like apples:

$$n(B) - n(A \cap B) = 60 - 42 = 18.$$

(c) Those that liked apples or bananas are represented by the set $A \cup B$. So here we could apply the inclusion-exclusion principle:

$$n(A \cup B) = 75 + 60 - 42 = 93.$$

Alternatively, we could sum up the number of people who only like apples, the number of people who only like bananas, and those that like both:

$$33 + 18 + 42 = 93.$$

(d) To count those who don't like either, we look at everyone and subtract those that like apples or bananas: $120 - 93 = 27$.

Example 1.21. In a survey of 80 gym-goers,

38	jog (set J)
46	swim (set S)
28	ride bike (set B)
24	jog and swim
18	jog and bike
16	swim and bike
11	do all three

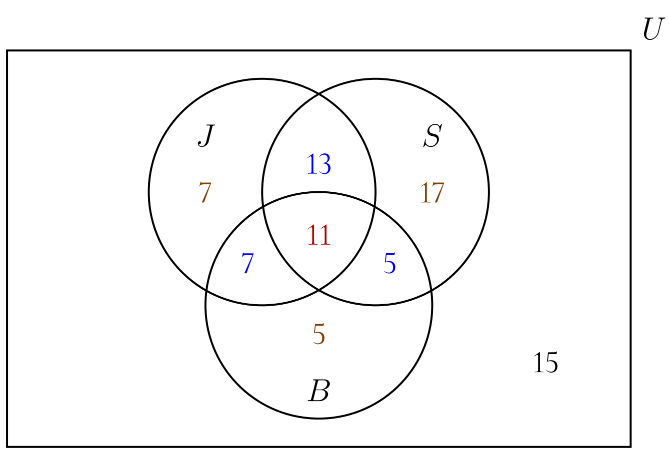
Draw a Venn diagram to describe the information and determine the following:

(a) How many do none of these?

(b) How many do exactly two of these activities?

(c) How many do at least two of these activities?

(d) How many do at most two of these activities?



- To make sure we keep of the regions distinct, we start by writing 11 in the intersection of J, S, and B, the number of people that do all three activities.

- To fill in the other intersections (e.g. those in $J \cap S$ but that aren't in $J \cap S \cap B$), we count the number in the intersection and take away those already accounted for that do all three. For example, the number of people that job and swim but don't bike is $24 - 11 = 13$.
- To find the remaining portion of each single activity (e.g. those who jog), we look at the number of people who do the activity and subtract those already accounted for in the subregions. For example, for the remaining people that job, we calculate

$$38 - (13 + 11 + 7) = 7.$$

(a) To find the number that do none of these activities, we take the total number of gym-goers, 80, and subtract those that partake in activities:

$$80 - (17 + 7 + 5 + 13 + 5 + 7 + 11) = 15.$$

(b) Those that do exactly two are colored in blue so we simply add them up:

$$13 + 5 + 7 = 25.$$

(c) Those that do at least two activities either do two activities or three activities. So we can add the number of those that do three to those that do exactly two:

$$25 + 11 = 36.$$

(d) To count the number of people that do at most two activities, we need to account for those that

- do no activities,
- do only one activity, and
- do exactly two activities.

So we calculate

$$\underset{\text{none}}{15} + \underset{\text{one}}{(17 + 7 + 5)} + \underset{\text{two}}{25} = 69$$

Alternatively, we can count all gym-goers and take away those that do three activities:

$$80 - 11 = 69.$$

Example 1.22. In a survey of 82 investors, 55 invest in mutual funds, 42 invest in bonds, 13 invest *only* in stocks, 27 invest in stocks and mutual funds, 35 invest in stocks and bonds, 3 invest in none of these three, and 26 invest in all three. Draw a Venn diagram to describe the information and determine the following:

(a) How many invest in mutual funds or bonds?

(b) How many invest in both mutual funds and bonds, but not all three?

(c) How many invest in stocks?

(d) How many invest in exactly one of these?

(e) How many invest in at least two of these?

(f) How many invest only in bonds?

Let M be the set of those that invest in mutual funds, B be the set of those that invest in bonds, and S be the set of those that invest in stocks. We can go ahead and fill in the values 13 for those that invest only in stocks, 3 for those investing in none of the three, and 26 for those investing in all three.

From here, we note that 27 invest in stocks and mutual funds, so we write the value $27 - 26 = 1$ into the corresponding region. Similarly, we use the value 35 that invest in stocks in bonds to fill $35 - 26 = 9$ in the corresponding region.

Since we weren't explicitly given the number of investors that invest in both mutual funds and bonds, we need to somehow find it. By the Inclusion-Exclusion Principle, we know that

$$n(M \cup B) = n(M) + n(B) - n(M \cap B).$$

To find $n(M \cup B)$, we can observe that there are $13 + 3 = 16$ investors that are not in $M \cup B$. Then $n(M \cup B) = 82 - 16 = 66$. So

$$\begin{aligned} n(M \cup B) = n(M) + n(B) - n(M \cap B) &\to 66 = 55 + 42 - n(M \cap B) \\ &\to n(M \cap B) = 31. \end{aligned}$$

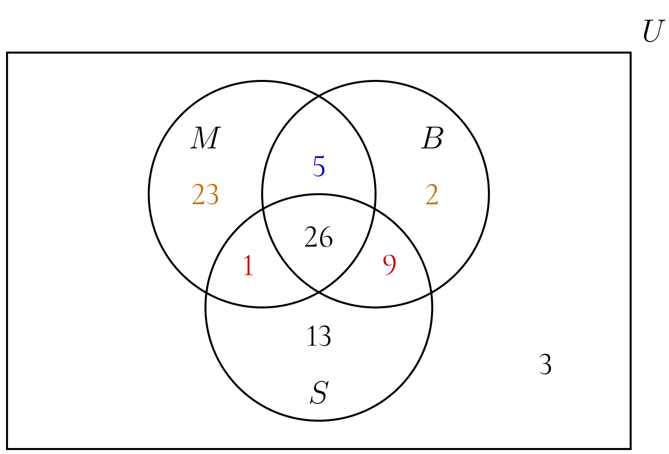
So then we can fill in $31 - 26 = 5$ into the corresponding region.

For the remaining two regions, we compute

$$55 - (5 + 26 + 1) = 23$$

and

$$42 - (5 + 26 + 9) = 2$$



Now we answer the questions.

(a) The number of investors that invest in mutual funds or bonds is $n(M \cup B) = 66$.

(b) The number that invest in both mutual funds and bonds, but not all three, is 5.

(c) The number that invest in stocks is $26 + 1 + 9 + 13 = 49$.

(d) The number that invest in exactly one of these is $23 + 2 + 13 = 38$.

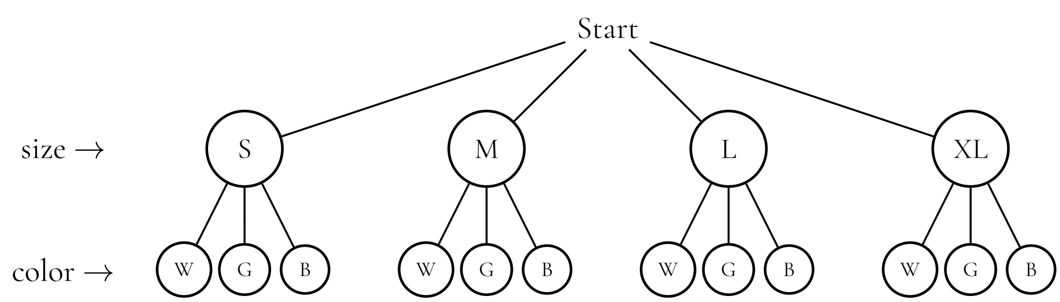
(e) The number that invest in at least two of these is $5 + 1 + 9 + 26 = 41$.

(f) The number that invest in only bonds is 2.

1.4 The Multiplication Principle

Example 1.23. Fresh Appeal offers polo shirts in sizes S, M, L, and XL and in colors white, green, and black. How many choices of polo are there?

The following **tree diagram** is helpful:



In the end, we see that there are

$$\underset{\text{sizes}}{4} \cdot \underset{\text{colors}}{3} = 12$$

different choices:

	white	**green**	**black**
small	small white	small green	small black
medium	medium white	medium green	medium black
large	large white	large green	large black
extra large	extra large white	extra large green	extra large black

The Multiplication Principle. Suppose a task consists of two choices. If choice 1 can be made in m ways and choice 2 can be made in n ways, then the complete task can be completed in $m \cdot n$ ways.

The Generalized Multiplication Principle. Suppose a task consists of k choices. If choice j can be made in m_j ways, then the complete task can be completed in

$$m_1 \cdot m_2 \cdots m_k$$

ways.

Example 1.24. At Party Bonanza, balloons come in 4 colors, 2 sizes, and 7 shapes. How many choices for a balloon are there?

There are

$$\underset{\text{color}}{\underline{4}} \cdot \underset{\text{size}}{\underline{2}} \cdot \underset{\text{shape}}{\underline{7}} = 56$$

different options.

Definition 1.11. By a **word** we mean any string of letters; e.g., *abc* is a word of length 3.

Example 1.25. How many four-letter words are there?

There are

$$\underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26} = 26^4 = 456\,976$$

different words of length 4.

Example 1.26. How many four-letter words can be made without repeating any letters?

There are

$$\underline{26} \cdot \underline{25} \cdot \underline{24} \cdot \underline{23} = 358\,800$$

four-letter words without any repeated letters.

Example 1.27. How many four-letter words are there that start with a "k" or "w"?

There are

$$\underline{2} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26} = 2 \cdot 26^3 = 35\,152$$

different words that start with a "k" or "w".

Example 1.28. How many four-letter words are there that start with a "k" or "w" and don't allow for repeated letters?

There are

$$\underline{2} \cdot \underline{25} \cdot \underline{24} \cdot \underline{23} = 27\,600$$

different words that start with a "k" or "w" and that don't repeat any letters.

Example 1.29. How many four-letter words are there that don't have repeated letters and begin with a "w" or end with a "z"?

Let A be the set of four-letter words without repetition of letters that start with "w" and B be the set of four-letter words without repetition of letters that end with "z". Then we can use the inclusion-exclusion principle

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

to count what we want. Note that

$$n(A) = \underline{1} \cdot \underline{25} \cdot \underline{24} \cdot \underline{23} = 13\,800,$$

$$n(B) = \underline{25} \cdot \underline{24} \cdot \underline{23} \cdot \underline{1} = 13\,800,$$

and

$$n(A \cap B) = \underline{1} \cdot \underline{24} \cdot \underline{23} \cdot \underline{1} = 552.$$

Finally, we obtain

$$n(A \cup B) = 13\,800 + 13\,800 - 552 = 27\,048.$$

Example 1.30. How many four-letter words have repeated letters?

We know from Example 1.25 that there are 456 976 four-letter words and we know from Example 1.26 that there are 358 800 four-letter words without any repeated letters. So there must be

$$456\,976 - 358\,800 = 98\,176$$

four-letter words with a repeated letter.

Example 1.31. In how many ways can a 5-question True/False exam be completed?

Each of the five questions has two possible responses so there are

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$$

ways to complete the exam.

Example 1.32. If questions can be left blank, in how many ways can a 5-question True/False exam be completed?

Now each of the questions has 3 possible responses: True, False, or blank. So there are

$$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^5 = 243$$

ways to complete the exam.

Example 1.33. Three couples go on a movie date. In how many ways can they be seated in a row of six so that each couple is seated together?

There are

$$\underline{6} \cdot \underline{1} \cdot \underline{4} \cdot \underline{1} \cdot \underline{2} \cdot \underline{1} = 48$$

ways for them to be arranged in the desired manner.

1.5 Permutations and Combinations

In this section, the **factorial** function will be critical. Recall that, for a positive integer n,

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n,$$

the product of all positive integers $\leqslant n$. The quantity $n!$ is referred to as "n factorial".

Example 1.34. There are four members of a relay team. How many ways can the runners be lined up?

There are

$$4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$$

ways to line up the four runners.

1.5.1 Permutations

Definition 1.12. An ordered arrangement where no item is used more than once is called a **permutation**.

Remark. In the definition of permutations, *order* and *no repetition* are vital.

Example 1.35. Eight suspects are to be lined up by the police. In how many ways can they line up all 8 suspects?

There are
$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8! = 40\,320$$
ways to line up the 8 suspects.

In general, given n items, there are $n!$ ways to order those items. From this, we see the convention that $0! = 1$ since there is exactly one way to line up no items.

Example 1.36. There are 10 members on a track team. In how many ways can we make a 4-person relay team?

Here, since we only need four people, we count
$$\underset{1^{\text{st}}}{\underline{10}} \cdot \underset{2^{\text{nd}}}{\underline{9}} \cdot \underset{3^{\text{rd}}}{\underline{8}} \cdot \underset{4^{\text{th}}}{\underline{7}} = 5\,040$$
ways to line them up as a relay team.

Permutations. The number of permutations of n objects taken r at a time is
$$P(n, r) := \frac{n!}{(n-r)!}.$$

Example 1.37. How many three-letter words without repeated letters can be formed from the set $\{a, b, c, d\}$?

There are
$$P(4, 3) = \frac{4!}{1!} = 24$$
three-letter words that can be made.

1.5.2 Combinations

Example 1.38. How many 3-person committees can be formed from a set of four people $\{a, b, c, d\}$?

Consider the list of 24 three-letter words that can be made:

abc	abd	acd	bcd
acb	adb	adc	bdc
bac	bad	cad	cbd
bca	bda	cda	cdb
cab	dab	dac	dbc
cba	dba	dca	dcb

Since order doesn't matter in the making of a committee, there are

$$\frac{24}{6} = \frac{P(4,3)}{3!} = 4$$

possible three-person committees.

Definition 1.13. An un-ordered arrangement where no item is used more than once is called a **combination**.

> **Combinations.** The number of combinations of n objects taken r at a time is
>
> $$C(n,r) := \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}.$$

Example 1.39. How many ways can 5 people be assigned to seats in a 12-seat row?

We can approach this problem in two ways.

1. We can list the people and choose their seats which gives

$$\begin{array}{rccccccccccc} \text{\# of seats:} & \underline{12} & \cdot & \underline{11} & \cdot & \underline{10} & \cdot & \underline{9} & \cdot & \underline{8} & = & 95\,040 \\ \text{person:} & 1^{\text{st}} & & 2^{\text{nd}} & & 3^{\text{rd}} & & 4^{\text{th}} & & 5^{\text{th}} & & \end{array}$$

   Notice that this is $P(12,5)$.

2. We can also split it up into two tasks:

   - choose 5 seats and then
   - arrange the 5 people.

   There are $C(12,5) = 792$ ways to choose 5 seats of the 12. Then there are 5! many ways to arrange the 5 people. So there should be

$$792 \cdot 5! = 95\,040$$

   ways to seat these 5 people in a 12-seat row.

Example 1.40. A hand in the game of Poker consists of 5 cards dealt from a deck of 52 cards. How many possible hands are there?

There are

$$C(52,5) = 2\,598\,960$$

possible hands.

Example 1.41. Suppose 5 men and 7 women have submitted applications to 4 different positions at The Academy of Swole. In how many ways can The Academy choose

(a) 2 men and 2 women?

(b) 4 of them without regard to gender?

(c) at least 3 women?

- We can choose 2 men in $C(5,2) = 10$ ways and we can choose 2 women in $C(7,2) = 21$ ways. So we can choose 2 men and 2 women in

$$10 \cdot 21 = 210$$

  ways.

- If we ignore the genders, there are $C(12,4) = 495$ ways to select 4 candidates.

- If we wish to pick at least 3 women, there are two possibilities:

  ◇ 3 women and one man

  ◇ 4 women and no men

  There are

$$\underset{\text{women}}{C(7,3)} \quad \cdot \quad \underset{\text{men}}{C(5,1)} \quad = \quad 35 \cdot 5 = 175$$

  ways to pick 3 women and a man. There are

$$\underset{\text{women}}{C(7,4)} \quad \cdot \quad \underset{\text{men}}{C(5,0)} \quad = \quad 35 \cdot 1 = 35$$

  ways to pick 4 women and no men.

  Now, these aren't two steps in one process but two separate processes. So the number of ways to pick

  ◇ 3 women and a man or

  ◇ 4 women and no men

  is $175 + 35 = 210$.

Example 1.42. A team of six people is going to line up for a photograph. Two of the six team members are seniors. In how many ways can the team line up so that the seniors are next to each other?

To aid in this explanation, label the team members $P1$, $P2$, $P3$, $P4$, $S1$, and $S2$ where $S1$ and $S2$ are the seniors. We want to think of the seniors as a unit in the grouping. Two examples of arrangements meeting the requirements are

$$\boxed{P2} \quad \boxed{P4} \quad \boxed{S2 \qquad S1} \quad \boxed{P3} \quad \boxed{P1}$$

and

$$\boxed{P3} \quad \boxed{P1} \quad \boxed{P2} \quad \boxed{S1 \qquad S2} \quad \boxed{P4}$$

We can build any valid arrangement in two steps:

1. First, arrange the five "units" where the units are $P1$, $P2$, $P3$, $P4$, and the seniors.
2. Then arrange the seniors.

There are 5! ways to arrange five units and 2! ways to arrange the seniors. So there are

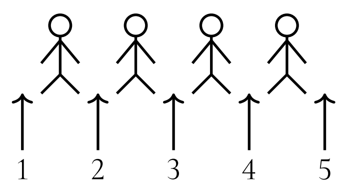
$$5! \cdot 2! = 240$$

different ways for them to line up for a photo where the seniors are standing next to each other.

We can also address this example in another way. There are 4 non-seniors. We can

- start by arranging the 4 non-seniors,
- then choose where the seniors will stand, and
- finally arrange the seniors.

To get a visual, suppose we have an arrangement of the 4 non-seniors. Then there are 5 spots where the seniors could go, as seen in the following diagram:



So, by way of the multiplication principle, there are

$$4! \cdot 5 \cdot 2! = 240$$

different ways for them to line up for a photo where the seniors are standing next to each other.

Example 1.43. A club has 24 members. The club is to elect a captain, a captain's assistant, and a supporting team of 6 members. How many different assignments are there?

We investigate to different ways to count this, both of which use the Multiplication Principle.

In one way, we can

- count the number of assignments of two ordered members (one is captain and the other is captain's assistant) from the club, and
- then count the number of unordered groups of 6 from the remaining $24 - 2 = 22$ members.

So then the number of assignments to these roles are

$$P(24, 2) \cdot C(22, 6) = 41\,186\,376.$$

In another way, we can

- count the number of unordered groups of $6 + 2 = 8$ members and
- then count the number of ways to choose 2 of those 8 to make an ordered pair.

So the number of assignments to these roles are

$$C(24, 8) \cdot P(8, 2) = 41\,186\,376.$$

Example 1.44. You are to receive a package of 50 light bulbs, 6 of which are defective. You will randomly pick four light bulbs.

(a) How many samples are possible?

(b) How many samples have 2 defective bulbs?

(c) How many samples consist of only good bulbs?

(d) How many samples have at least one defective bulb?

- There are
$$C(50, 4) = 230\,300$$
samples.

- Since there are 6 defective, there are 44 good bulbs. There are
$$C(44, 2) = 946$$
ways to get 2 good bulbs and
$$C(6, 2) = 15$$
ways to get 2 defective bulbs. Then there are
$$946 \cdot 15 = 14\,190$$
ways to get a sample of 4 where 2 bulbs are defective.

- There are
$$C(44, 4) = 135\,751$$
samples without any defective bulbs.

- There are
    - ◇ $C(6, 1) \cdot C(44, 3) = 6 \cdot 13\,244 = 79\,464$ ways to get one defective bulb and 3 good bulbs
    - ◇ $C(6, 2) \cdot C(44, 2) = 15 \cdot 946 = 14\,190$ ways to get two defective bulbs and 2 good bulbs
    - ◇ $C(6, 3) \cdot C(44, 1) = 20 \cdot 44 = 880$ ways to get three defective bulbs and 1 good bulb
    - ◇ $C(6, 4) \cdot C(44, 0) = 15 \cdot 1 = 15$ ways to get four defective bulbs and no good bulbs

so there are

$$79\,464 + 14\,190 + 880 + 15 = 94\,549$$

samples with at least one defective bulb.

Alternatively, we know that there are $230\,300$ total samples and $135\,751$ with only good bulbs. So there are $230\,300 - 135\,751 = 94\,549$ samples with at least one defective bulb.

Example 1.45. An urn contains three gold, five silver, and nine bronze coins. Three coins are randomly selected.

(a) How many samples are there?

(b) How many samples consist of only gold coins?

(c) How many samples consist of one coin of each color?

(d) How many samples consist of two bronze coins?

(e) How many samples contain at least one gold coin?

First, note that the urn contains $3 + 5 + 9 = 17$ coins, in total.

- So there are $C(17, 3) = 680$ samples, in total.
- To get only gold coins, we need our sample of three to come from the group of three gold coins. There is $C(3, 3) = 1$ sample consisting of only gold coins.
- To get one coin of each color, we need one gold, one silver and one bronze. So there are

  $$C(3, 1) \cdot C(5, 1) \cdot C(9, 1) = 135$$

  samples that have one coin of each color.
- To have a sample of three coins with exactly two bronze coins, we will also need one other coin which is not bronze. Notice that there are $3 + 5 = 8$ coins that are not bronze. So there are

  $$C(9, 2) \cdot C(8, 1) = 288$$

  samples with exactly two bronze coins.
- To get at least one gold coin, we need to have either one gold coin, two gold coins, or three gold coins. However, it is easier to count the opposite; that is, to count the number of samples that have no gold coins. Observe that there are $5 + 9 = 14$ coins that aren't gold. So there are $C(14, 3) = 364$ samples that have no gold coins. Since there are 680 samples total, we see that there are $680 - 364 = 316$ samples with at least one gold coin.

Example 1.46. Toss a fair coin 10 times and note the sequence of heads and tails.

(a) How many outcomes are possible?

(b) How many outcomes have exactly two heads?

(c) How many outcomes have at most two heads?

(d) How many outcomes have at least two heads?

- Consider the 10 blank slots:

$$\underset{1}{\underline{\qquad}}\quad \underset{2}{\underline{\qquad}}\quad \underset{3}{\underline{\qquad}}\quad \underset{4}{\underline{\qquad}}\quad \underset{5}{\underline{\qquad}}\quad \underset{6}{\underline{\qquad}}\quad \underset{7}{\underline{\qquad}}\quad \underset{8}{\underline{\qquad}}\quad \underset{9}{\underline{\qquad}}\quad \underset{10}{\underline{\qquad}}$$

  For each slot, it can either be a heads or tails; e.g.

$$\underline{\text{H}}\ \ \underline{\text{H}}\ \ \underline{\text{H}}\ \ \underline{\text{T}}\ \ \underline{\text{H}}\ \ \underline{\text{T}}\ \ \underline{\text{T}}\ \ \underline{\text{T}}\ \ \underline{\text{H}}\ \ \underline{\text{T}}$$

  So there are $2^{10} = 1024$ different possible sequences.

- The outcomes with exactly 2 heads can be determined by their place in the sequence. Here, order doesn't matter since the two heads occurring are indistinguishable. For example, consider the sequences with colored heads:

$$\underline{\text{T}}\ \ \underline{\text{T}}\ \ \underline{\text{H}}\ \ \underline{\text{T}}\ \ \underline{\text{H}}\ \ \underline{\text{T}}\ \ \underline{\text{T}}\ \ \underline{\text{T}}\ \ \underline{\text{T}}\ \ \underline{\text{T}}$$

  and

$$\underline{\text{T}}\ \ \underline{\text{T}}\ \ \underline{\text{H}}\ \ \underline{\text{T}}\ \ \underline{\text{H}}\ \ \underline{\text{T}}\ \ \underline{\text{T}}\ \ \underline{\text{T}}\ \ \underline{\text{T}}\ \ \underline{\text{T}}$$

  Without coloration, the two sequences are not distinguishable from each other. There are $C(10, 2) = 45$ ways to select two spots to place heads in from the 10 available.

- To get at most 2 heads, we can get
  - no heads,
  - 1 head, or
  - 2 heads.

  Hence, we count that there are

$$C(10, 0) + C(10, 1) + C(10, 2) = 1 + 10 + 45 = 56$$

  ways to get at most two heads.

- With similar reasoning, we see that there should be

$$C(10, 2) + C(10, 3) + C(10, 4) + C(10, 5) + \cdots + C(10, 9) + C(10, 10)$$

  ways to get at least two heads. But as before, we can see that there are

$$\underset{\text{total}}{\underline{1024}} \quad - \quad \underset{\text{at most 1H}}{\underline{(C(10, 0) + C(10, 1))}} \quad = \quad 1013$$

  ways to get at least 2 heads.

Remark. Again we see that it is sometimes easier to count the elements in the complement of a set and then subtract from the total number of possibilities.

Example 1.47. At Glacial Surprise, you can get an ice cream bowl in one of 4 flavors and choose from any of the 12 toppings.

(a) How many possible ice cream bowls are there?

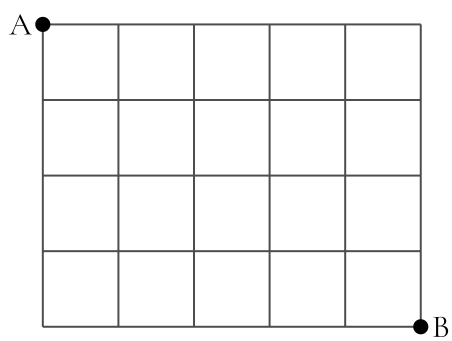
(b) How many possible ice cream bowls are there with exactly 2 toppings?

- We can build an ice cream bowl by first choosing the flavor and then going through each option of topping and deciding whether we want that topping or not. So, for each topping, the possibilities can be represented by a "Yes" or "No". That gives us 2 choices for each topping. Therefore, there are
$$4 \cdot 2^{12} = 16\,384$$
different ice cream bowls possible.

- If we wish to choose exactly 2 toppings, there are
$$4 \cdot C(12, 2) = 264$$
options.

Example 1.48. In general, a set with n elements has 2^n subsets. This can be seen by listing out the n elements in the set and noting that a subset is essentially assigning a "Yes" or "No" to each element; Y to those in the set and N to those not.

Example 1.49. Consider the following arrangement of city blocks.



Ernest Hemingway is at Fun Palace at point A but needs to get to his flat at point B. How many ways can he get from point A to point B if he only moves to the South or to the East at each block?

Observe that Ernest must go East five times and and South four times. He can choose these in any order he desires and he will have to make $5 + 4 = 9$ decisions. Of those 9 decisions, his path is determined by the placement of the choices of East so he can make it from A to B in

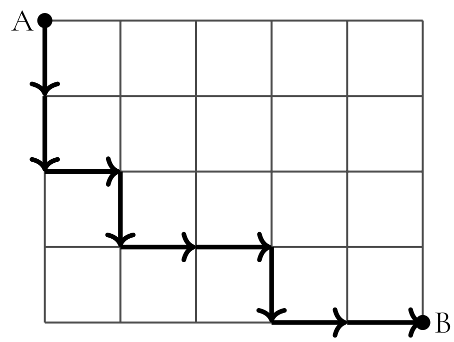
$$C(9, 5) = 126$$

ways.

To really spell this out, imagine we are filling up 9 slots

$$\underset{1}{\_\_}\quad\underset{2}{\_\_}\quad\underset{3}{\_\_}\quad\underset{4}{\_\_}\quad\underset{5}{\_\_}\quad\underset{6}{\_\_}\quad\underset{7}{\_\_}\quad\underset{8}{\_\_}\quad\underset{9}{\_\_}$$

with five Es and four Ss. For example, the sequence SSESEESEE is the path where Ernest goes

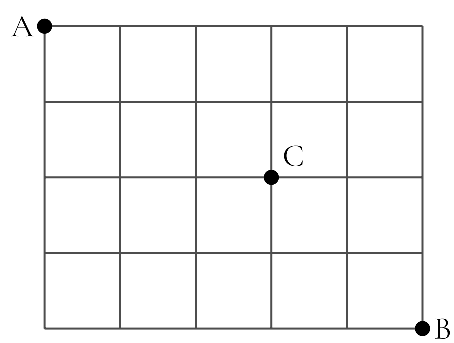


So, of the 9 slots, we first choose 5 of them to be filled with Es. This can be done in $C(9,5) = 126$ ways. After we choose five slots, there are only 4 open slots remaining which we fill with Ss. This can be done in $C(4,4) = 1$ way. So there are

$$C(9,5) \cdot C(4,4) = 126$$

distinct paths that Ernest can take, just as we saw above.

Example 1.50. Ernest just remembered that he has to pick up Gertrude Stein on his way back home. Gertrude is at point C in the city block arrangement that follows.



If they can only travel to the South or to the East at each block, how many ways can Ernest make it from point A to point B passing through C?

Since Ernest has to first make it to point C, we can break this into two parts:

1. Travel from A to C and then
2. travel from C to B.

- To get from A to C, Ernest needs to go East three times and South twice.

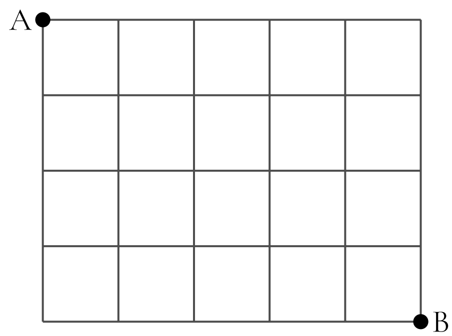
- To get from C to B, Ernest needs to go East twice and South twice.

Both of these scenarios are similar to Example 1.49. So there are

- $C(5,3) \cdot C(2,2) = 10$ ways to get from A to C and
- $C(4,2) \cdot C(2,2) = 6$ ways to get from C to B.

Finally, we see that there are $10 \cdot 6 = 60$ ways to get from A to B passing through C on the way.

Example 1.51. In this scenario, Agnes Denes is at Pyramid Point at point A and needs to get back to her studio at point B.



If Agnes travels South twice in a row, the Earth evaporates. In how many ways can she make it from point A to point B only going South or East at each block without ever going South twice in a row?

Agnes needs to travel East 5 times and South 4 times. So, like in Example 1.49, we can think of this as building a sequence of nine letters, five of which are Es and four of which are Ss so that S does not occur more than once at a time. Consider the diagram:

$$\underset{1}{\underline{\hspace{1.5em}}}\ \mathrm{E}\ \underset{2}{\underline{\hspace{1.5em}}}\ \mathrm{E}\ \underset{3}{\underline{\hspace{1.5em}}}\ \mathrm{E}\ \underset{4}{\underline{\hspace{1.5em}}}\ \mathrm{E}\ \underset{5}{\underline{\hspace{1.5em}}}\ \mathrm{E}\ \underset{6}{\underline{\hspace{1.5em}}}$$

Of the 6 spots available, Agnes needs to choose 4 of them to go South. So there are

$$C(6,4) = 15$$

ways for her to get from A to B in the desired way.

Example 1.52. A chess player is going to play 16 games in an upcoming tournament. In how many different ways can she have nine wins, five ties, and two losses?

Consider 16 slots for the games.

$$\underset{1}{\underline{\hspace{1.5em}}}\ \underset{2}{\underline{\hspace{1.5em}}}\ \underset{3}{\underline{\hspace{1.5em}}}\ \underset{4}{\underline{\hspace{1.5em}}}\ \underset{5}{\underline{\hspace{1.5em}}}\ \underset{6}{\underline{\hspace{1.5em}}}\ \underset{7}{\underline{\hspace{1.5em}}}\ \underset{8}{\underline{\hspace{1.5em}}}\ \underset{9}{\underline{\hspace{1.5em}}}\ \underset{10}{\underline{\hspace{1.5em}}}\ \underset{11}{\underline{\hspace{1.5em}}}\ \underset{12}{\underline{\hspace{1.5em}}}\ \underset{13}{\underline{\hspace{1.5em}}}\ \underset{14}{\underline{\hspace{1.5em}}}\ \underset{15}{\underline{\hspace{1.5em}}}\ \underset{16}{\underline{\hspace{1.5em}}}$$

We can conceptualize the arrangement of wins, ties, and losses as arrangements of the letters W, T, and L in these 16 slots.

- There are $C(16, 9) = 11\,440$ ways to select 9 spots to place Ws.
- After placing the nine Ws, there are $16 - 9 = 7$ open spots. There are $C(7, 5) = 21$ ways to select 5 of those spots to place Ts.
- There are only 2 spots open and so there is $C(2, 2) = 1$ way to fill these two spots with Ls.

Hence, there are

$$C(16, 9) \cdot C(7, 5) \cdot C(2, 2) = 240\,240$$

ways for her 16 games to result in nine wins, five ties, and two losses.

1.5.3 More on the Multiplication Principle

An important aspect of the Multiplication Principle is that the results of subsequent tasks create distinct outcomes. We will elaborate on this by example.

Example 1.53. In attempting to solve the question "How many four-letter words have at least two w's?" Stefi wrote the following:

> I use the Multiplication Principle.
>
> - First, choose two of the letter locations to put w in: $C(4, 2)$.
> - Of the two remaining letter locations, we can choose any two letters to go in those spaces: 26^2.
>
> So there are
>
> $$C(4, 2) \cdot 26^2 = 4\,056$$
>
> words with at least two w's.

Explain why Stefi is wrong and explain the correct answer to her.

Firstly, Stefi's method counts the word $wwww$ $C(4, 2) = 6$ times when it should only be counted once. To see this, note that the first step tells us to place two w's in two of the four slots; for example, suppose we start with

$$\underline{\quad}\ \underline{w}\ \underline{\quad}\ \underline{w}.$$

Then in the second step, we get to place whatever letters we want in the remaining two slots. In particular, this allows for the choice of putting w in both remaining slots to obtain the word $wwww$.

However, if we had started with

$$\underline{w}\ \underline{w}\ \underline{\quad}\ \underline{\quad},$$

we have a different initial choice, so this is a separate "branch" from that of the previous choice. Then if we fill the remaining spots with w, we have the word $wwww$ again, and so this method counts $wwww$ more than once.

To count the number of four-letter words, we can count

- the number of four-letter words with exactly two w's,

- the number of four-letter words with exactly three w's, and
- the number of four-letter words with exactly four w's.

Since each of these collections of words are disjoint from the others, our final answer is the sum of each of those numbers.

The number of four-letter words with exactly two w's is

$$C(4,2) \cdot 25^2 = 3\,750.$$

The number of four-letter words with exactly three w's is

$$C(4,3) \cdot 25 = 100.$$

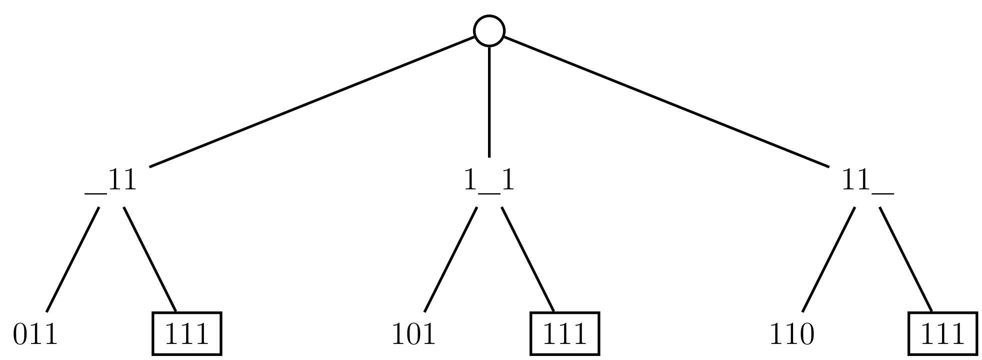
The number of four-letter words with exactly four 2's is

$$C(4,4) = 1.$$

In the end, the number of four-letter words with at least two w's is

$$3\,750 + 100 + 1 = 3\,851.$$

To explain the original issue with a smaller example with which we can display the entire tree, suppose we wish to count the number of three-digit binary strings with at least two 1s; that is, sequences of three digits from 0 or 1 that have at least two 1s. If we try Stefi's method which is to choose two spots for 1s and fill in the remaining spot with any of the two digits, we would have the following tree diagram:



So we see the result of $C(3,2) \cdot 2 = 6$ where

- $C(3,2) = 3$ is the number of ways to choose two of the three slots for 1s and
- 2 is the number of choices for the remaining slot.

However, we've over-counted the sequence 111.

As we can see, there are only 4 distinct three-digit binary strings with at least two 1s. There are $C(3,2) = 3$ with exactly two 1s and $C(3,3) = 1$ with exactly three 1s. Therefore, there are $C(3,2) + C(3,3) = 4$ with at least two 1s.

2 Probability

2.1 Sample Spaces and Events

Definition 2.1. A **probability** expresses the long-term likelihood of an event occurring and is a number between 0 and 1, inclusive.

Remark. 0 corresponds to 0% and 1 corresponds to 100%.

Example 2.1. Something that has a probability of $0.9 = 90\% = \dfrac{9}{10}$ is very likely to happen. Something that has a probability of $0.5 = 50\% = \dfrac{5}{10} = \dfrac{1}{2}$ is just as likely to occur as not to occur.

Definition 2.2. We offer the following terms.

- **Experiment** — an activity with an observable result
- **Trial** — each repetition of an experiment
- **Outcome** — each possible result of an experiment
- **Sample Space** — the set of all possible outcomes of an experiment
- **Event** — a subset of the sample space (i.e., a collection of outcomes)

Example 2.2. An experiment consists of rolling a 6-sided die. The sample space is $\{1, 2, 3, 4, 5, 6\}$. The event of rolling an odd number can be expressed in set notation: $\{1, 3, 5\}$. Notice that $\{1, 3, 5\} \subseteq \{1, 2, 3, 4, 5, 6\}$.

Example 2.3. An experiment consists of picking 3 balls from a bowl containing red, green, blue, and yellow balls and observing the number of balls of each color in the selection.

(a) Find the cardinality of the sample space S.

(b) Let E be the event of picking at least one red ball. Find the cardinality of E.

(c) Let F be the event of picking at least one purple ball. Find the cardinality of F.

First, note that we are not given the total number of balls in the bowl and that we are just observing the number of balls of each color in the selection. This makes this a different kind of example than Examples 1.44 and 1.45. The possible scenarios are as follows:

- 3 balls of one color.
- 2 balls of one color and 1 ball of another color.
- 3 different colored balls.

Now we apply our counting techniques.

- Notice that there are

- ◇ $C(4,1) = 4$ ways to choose one color (a one-color selection of 3 balls)
- ◇ $C(4,2) = 6$ ways to pick two colors and 2 ways to choose which color appears twice in the selection. So there are $6 \cdot 2 = 12$ ways in which we can select two colors, one of which appears twice in the selection.

  Alternatively, there are $C(4,1) = 4$ ways to choose the color which appears twice and $C(3,1) = 3$ ways to choose the other color. So there are $4 \cdot 3 = 12$ ways to make this selection of one color appearing twice and another color appearing once.
- ◇ $C(4,3) = 4$ ways to pick out three colors (a three-color selection of 3 balls).

So $n(S) = 20$.

We can also visualize the sample space as follows:

$$
\begin{array}{lllll}
S = \{ & RRR, & GGG, & BBB, & YYY, \\
 & RRG, & RRB, & RRY, & \\
 & GGR, & GGB, & GGY, & \\
 & BBR, & BBG, & BBY, & \\
 & YYR, & YYG, & YYB, & \\
 & RGB, & RGY, & RBY, & GBY \quad \}
\end{array}
$$

There is another, more efficient, way[2] to count this sample space. Since we are only interested in the number of each color in the selection, we want to count the number of three-letter words with Rs, Gs, Bs, and Ys, in that order. This is where the role of separators come in to play. There are three separators that we need since we have four different letters. So we have $3 + 3 = 6$ slots (3 for the three letters and 3 separators) to place the separators. So there are $C(6,3) = 20$ ways to do this. We give a few examples to explain how the separators determine the selections.

$$\underline{\ \ }\ \underline{|}\ \underline{\ \ }\ \underline{|}\ \underline{|}\ \underline{\ \ } \longrightarrow \underline{\text{R}}\ \underline{|}\ \underline{\text{G}}\ \underline{|}\ \underline{|}\ \underline{\text{Y}}$$

$$\underline{|}\ \underline{\ \ }\ \underline{\ \ }\ \underline{|}\ \underline{\ \ }\ \underline{|} \longrightarrow \underline{|}\ \underline{\text{G}}\ \underline{\text{G}}\ \underline{|}\ \underline{\text{B}}\ \underline{|}$$

$$\underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{|}\ \underline{|}\ \underline{|} \longrightarrow \underline{\text{R}}\ \underline{\text{R}}\ \underline{\text{R}}\ \underline{|}\ \underline{|}\ \underline{|}$$

- To pick no red balls, notice that there are
  - ◇ $C(3,1) = 3$ ways to choose one color which is not red
  - ◇ $C(3,2) = 3$ ways to pick two colors that are not red and 2 ways to choose which color appears twice in the selection. So there are $3 \cdot 2 = 6$ ways in which we can select two non-red colors, one of which appears twice in the selection.

    Alternatively, there are $C(3,1) = 3$ ways to choose a non-red color which appears twice. Then there are $C(2,1) = 2$ ways to choose another non-red color to appear once. This leaves us with $3 \cdot 2 = 6$ possible ways to assign a color to appear twice and another color to appear once, neither of which is red.

[2] known as the method of *stars and bars*

◇ $C(3,3) = 1$ way to pick out three non-red colors

So there are $3 + 6 + 1 = 10$ ways to pick no red balls. Hence, there are $20 - 10 = 10$ ways to pick at least one red ball. We can also refer to the sample space and pick out the outcomes where a red appears:

$$E = \{RRR, RRG, RRB, RRY, GGR, BBR, YYR, RGB, RGY, RBY\}.$$

We can also count the number of outcomes using the method of separators discussed above. To have no reds, a separator must go in the first entry. Of the five remaining entries, we have two more separators to place. So there are $C(5,2) = 10$ samples with no reds.

- Since there are no purple balls, there are 0 ways to pick at least one purple ball. More explicitly, $n(F) = 0$.

Definition 2.3. We say that $\emptyset$ is an **impossible event** since it never occurs. We say that the whole sample space S is a **certain event** because any trial of the experiment must product an outcome in the sample space.

Example 2.4. In Example 2.3 (c) the event F is an impossible event.

Remark. Let E and F be events.

- The event where either E or F occurs is $E \cup F$.
- The event where both E and F occur is $E \cap F$.
- The event where E does not occur is E'.

Example 2.5. An experiment consists of rolling a 6-sided die twice.

(a) Describe the sample space.

(b) Write the event $A =$ "the sum is greater than 9" in set notation.

(c) Write the event $B =$ "the numbers on both rolls are equal" in set notation.

(d) Write the event $C =$ "the sum is 7" in set notation.

(e) List the elements of $A \cap B$ and $A \cup B$.

(f) Find $n((A \cup B)')$.

(g) Investigate $A \cap C$ and $B \cap C$.

The sample space is represented with

$$
\begin{array}{lllllll}
S = \{ & (1,1), & (2,1), & (3,1), & (4,1), & (5,1), & (6,1), \\
& (1,2), & (2,2), & (3,2), & (4,2), & (5,2), & (6,2), \\
& (1,3), & (2,3), & (3,3), & (4,3), & (5,3), & (6,3), \\
& (1,4), & (2,4), & (3,4), & (4,4), & (5,4), & (6,4), \\
& (1,5), & (2,5), & (3,5), & (4,5), & (5,5), & (6,5), \\
& (1,6), & (2,6), & (3,6), & (4,6), & (5,6), & (6,6) \quad \}
\end{array}
$$

Then observe that

- $A = \{(6,4),(5,5),(4,6),(6,5),(5,6),(6,6)\}$
- $B = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$
- $C = \{(6,1),(5,2),(4,3),(3,4),(2,5),(1,6)\}$
- Notice that $A \cap B = \{(5,5),(6,6)\}$ and

$$A \cup B = \{(1,1),(2,2),(3,3),(4,4),(6,4),(5,5),(4,6),(6,5),(5,6),(6,6)\}.$$

- First, note that $n(S) = 36$ and $n(A \cup B) = 10$. Then $n((A \cup B)') = 36 - 10 = 26$.
- Since A and C have no elements in common, we have that $A \cap C = \emptyset$. Similarly, $B \cap C = \emptyset$.

Definition 2.4. Let E and F be events in a sample space. Then E and F are said to be **mutually exclusive** if their intersection is empty; i.e., $E \cap F = \emptyset$.

Remark. Mutually exclusive events can't occur at the same time.

Example 2.6. As in Example 2.3, three balls are selected from a box containing red, green, blue, and yellow balls. Let A be the event of selecting balls of the same color and B be the event of selecting balls, all of which have a different color. Show that the events A and B are mutually exclusive.

Note that

$$A = \{ \; RRR, \; GGG, \; BBB, \; YYY \; \}$$

and

$$B = \{ \; RGB, \; RGY, \; RBY, \; GBY \; \}$$

which shows that $A \cap B = \emptyset$.

Note. A standard deck of cards contains 52 cards. There are

- 13 ranks, in ascending order: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- and 4 suits: $\heartsuit, \clubsuit, \spadesuit, \diamondsuit$

The cards in the ranks J, Q, and K are called face cards. Each suit contains the 13 ranks. The suits clubs ♣ and spades ♠ are colored black. The suits hearts ♡ and diamonds ♢ are colored red.

Example 2.7. Select one card from a deck of 52. Let

- A be the event that the card drawn is a K,
- B be the event that the card drawn is a Q,
- C be the event of drawing a heart,
- D be the event of drawing a diamond, and
- E be the event of drawing a face card.

Show that

(a) A and B are mutually exclusive,

(b) A and C are not mutually exclusive,

(c) D and E are not mutually exclusive, and

(d) C and D are mutually exclusive.

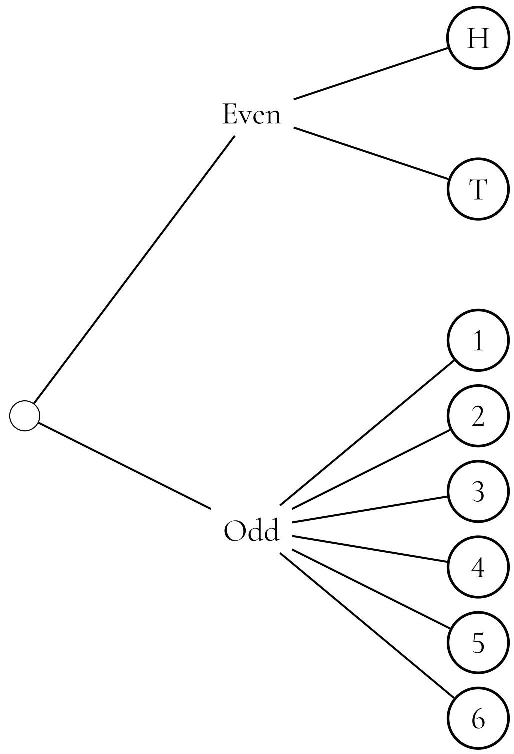
- K and Q are different cards so $A \cap B = \emptyset$.
- There is a K of hearts so $A \cap C \neq \emptyset$.
- There are three face cards in the suit of diamonds so $D \cap E \neq \emptyset$.
- The diamond and heart suits are different suits so $C \cap D = \emptyset$.

——————————— — — — ———————————

In general, we can use tree diagrams to help us investigate outcomes or events that are dependent upon a prior outcome or event.

Example 2.8. An experiment consists of rolling a 6-sided die and then another activity depending on the result. If the result of the roll is even, then a coin is flipped. If the result of the roll is odd, then the die is rolled again. Draw a tree diagram describing the sample space and then write the sample space using set notation.

The first activity in this experiment breaks up into two mutually exclusive events: either we get an even number of an odd number. Then, depending on the outcome, we do something else. A corresponding tree diagram is



Now, we will set up our sample space using ordered pairs where the first entry is the result of the first roll and the second entry is the result of the second activity. Since the Even group stands for the outcomes 2, 4, and 6 and the Odd group stands for outcomes 1, 3, and 5, we see that our sample space is

$$\begin{array}{lllllll} \{ & (2,\mathrm{H}), & (2,\mathrm{T}), & (4,\mathrm{H}), & (4,\mathrm{T}), & (6,\mathrm{H}), & (6,\mathrm{T}), \\ & (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ & (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), \\ & (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \quad \}. \end{array}$$

2.2 Assigning and Calculating Probabilities

Definition 2.5. A **probability distribution** for a sample space $S = \{a_1, a_2, \ldots, a_n\}$ assigns a probability to each outcome so that the sum of the probabilities equals 1. If the probability distribution assigns the same probability to each outcome, then we say that all outcomes are **equally probable** or **equally likely**.

Example 2.9. Toss a fair coin. Then

outcome	probability
H	1/2
T	1/2

is a table expressing the probability distribution. Notice that both outcomes are equally likely.

Definition 2.6. An **experimental probability** is calculated after conducting many trials and examining the data. The probability calculated is sometimes called the **relative frequency**.

Example 2.10. Three hundred freshmen were asked how many colleges they had applied to and the data is summarized below:

# of colleges	frequency	probability		(decimal)
1	51	51/300	=	0.17
2	36	36/300	=	0.12
3	42	42/300	=	0.14
4	48	48/300	=	0.16
5+	123	123/300	=	0.41

Fundamental Properties of Probability. For a sample space $\{a_1, a_2, \ldots, a_n\}$, suppose the probability distribution is $\{p_1, p_2, \ldots, p_n\}$ where p_j is the probability of a_j.

(P1) The probability of any individual outcome is between 0 and 1, inclusive; i.e., $0 \leqslant p_j \leqslant 1$.

(P2) $p_1 + p_2 + \cdots + p_n = 1$

(P3) The probability of an event E, denoted $\Pr(E)$, is equal to the sum of the probabilities of the outcomes in E.

Remark. Property (P3) is known as the **Addition Principle** for probability.

Theorem 2.7. If S is a sample space where all outcomes are equally likely, then for any event E,

$$\Pr(E) = \frac{n(E)}{n(S)} = \frac{\text{number of outcomes in the event } E}{\text{number of possible outcomes}}.$$

Example 2.11. An experiment consists of rolling two 6-sided dice. Find the probability that the sum of the two rolls is 7.

In Example 2.5(d), we showed that there are 6 outcomes so that the sum of the two rolls is 7. We also saw that there are 36 total possile outcomes. Therefore, the probability that the sum of the two rolls is 7 is $\dfrac{6}{36} = \dfrac{1}{6}$.

Inclusion-Exclusion Principle. For any two events E and F,

$$\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F).$$

Equivalently,

$$\Pr(E \cup F) + \Pr(E \cap F) = \Pr(E) + \Pr(F).$$

If E and F are mutually exclusive events, then $\Pr(E \cup F) = \Pr(E) + \Pr(F)$ since $E \cap F = \emptyset$ and $\Pr(\emptyset) = 0$.

Example 2.12. If a random card is drawn from a standard deck of cards, find the probability that the card drawn is a Queen or a red card.

Note that there are

- 52 cards, total
- 4 Queens
- 26 red cards
- 2 red Queens.

So,

$$\begin{aligned} \Pr(\text{Q or red}) &= \Pr(\text{Q}) + \Pr(\text{red}) - \Pr(\text{Q and red}) \\ &= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} \\ &= \frac{7}{13}. \end{aligned}$$

Example 2.13. Suppose $\Pr(E) = 0.5$, $\Pr(F) = 0.6$, and $\Pr(E \cup F) = 0.9$. Find $\Pr(E \cap F)$.

By the Inclusion-Exclusion Principle,

$$\begin{array}{rcl} \Pr(E \cup F) + \Pr(E \cap F) &=& \Pr(E) + \Pr(F) \\ 0.9 + \Pr(E \cap F) &=& 0.5 + 0.6 = 1.1 \\ \hline \Pr(E \cap F) &=& 0.2 \end{array}$$

Example 2.14. A fair coin is tossed 4 times and the number of heads is noted. Describe the sample space and find the probability of getting at most one head.

There are $2^4 = 16$ different sequences. We then have

$$\begin{array}{lllll} \{ & \text{TTTT}, & \text{HTTT}, & \text{THTT}, & \text{TTHT}, \\ & \text{TTTH}, & \text{HHTT}, & \text{HTHT}, & \text{HTTH}, \\ & \text{THHT}, & \text{THTH}, & \text{TTHH}, & \text{HHHT}, \\ & \text{HHTH}, & \text{HTHH}, & \text{THHH}, & \text{HHHH} \quad \} \end{array}$$

as our sample space. There are $C(4,0) = 1$ ways to get no heads and $C(4,1) = 4$ ways to get exactly one head. Then there are $1 + 4 = 5$ may ways to get at most one head. Hence, the probability of getting at most one head is

$$\Pr(\text{at most 1H}) = \frac{5}{16}.$$

We could also see this by counting the sequences with at most one head appearing in the list of all possible sequences above.

Example 2.15. An urn contains 6 white balls and 5 red balls. A sample of four balls is taken at random. Calculate

(a) the probability of getting only white balls,

(b) the probability of getting 2 white and 2 red balls,

(c) the probability of getting at least one red ball.

First, note that there are

$$C(11,4) = 330$$

different possible samples. Of these, there are

$$C(6,4) = 15$$

that consist only of white balls. So the probability of getting only white balls is

$$\Pr(\text{only W}) = \frac{15}{330} = \frac{1}{22}.$$

Now, there are

$$C(6,2) \cdot C(5,2) = 15 \cdot 10 = 150$$

samples with exactly two white and two red balls. So the probability of getting a sample with two white and two red balls is

$$\Pr(\text{2W and 2R}) = \frac{150}{330} = \frac{5}{11}.$$

To calculate the probability of getting at least one red ball, we can first consider the event of getting no red balls. Note that, in this context, a sample fails to have at least one red when it contains only white balls. Of the 330 total samples, we found 15 to consist only of white balls. So there are $330 - 15 = 315$ samples with at least one red ball. Thus, the probability of getting a sample with at least one red ball is

$$\Pr(\text{at least 1R}) = \frac{315}{330} = \frac{21}{22}.$$

Note. For any event E, $\Pr(E) + \Pr(E') = 1$ and, equivalently, $\Pr(E) = 1 - \Pr(E')$. So

$$\Pr(\text{at least one}) = 1 - \Pr(\text{none}).$$

Example 2.16. A bag contains 9 tomatoes, one of which is rotten. A sample of three is chosen. What is the probability the sample contains the rotten tomato?

First, note that there are $C(9,3) = 84$ total possible samples. There is $C(1,1) = 1$ way to choose the rotten tomato and $C(8,2) = 28$ ways to choose two more tomatoes from the remaining 8. So the probability that a random sample of three tomatoes contains the rotten tomato is

$$\Pr(\text{contains rotten}) = \frac{28}{84} = \frac{1}{3}.$$

Example 2.17. A coin is tossed 4 times and the sequence of results is recorded. What is the probability that there are more heads than tails?

To ensure more heads than tails, we need either 3 heads or 4 heads in the sequence. There are $C(4,3) = 4$ ways to get 3 heads and $C(4,4) = 1$ way to get 4 heads. So there are $4 + 1 = 5$ ways that result in more heads than tails. Since there are $2^4 = 16$ total possible sequences,

$$\Pr(\text{more H than T}) = \frac{5}{16}.$$

Example 2.18. A 6-sided die is rolled 4 times and the sequence of results is recorded.

(a) What is the probability that 4 different numbers are rolled?

(b) What is the probability that exactly two 3's are rolled?

(c) What is the probability that no 6's appear?

(d) What is the probability that all the numbers appearing are odd?

First, note that there are

$$6^4 = 1296$$

different possible sequences of rolls. There are

$$P(6,4) = 6 \cdot 5 \cdot 4 \cdot 3 = 360$$

ways to get four different numbers rolled in sequence. Hence,

$$\Pr(\text{all different}) = \frac{360}{1296} = \frac{5}{18}.$$

There are $C(4,2) = 6$ ways for exactly two 3's to appear. Once those 3's are placed, there are $5 \cdot 5 = 25$ ways to assign values other than 3 to the remaining two spots. So there are $6 \cdot 25 = 150$ ways to get exactly two 3's which means

$$\Pr(\text{exactly two 3's}) = \frac{150}{1296} = \frac{25}{216}.$$

There are

$$5^4 = 625$$

sequences with no 6's appearing so

$$\Pr(\text{no 6's}) = \frac{625}{1296}.$$

There are 3 odd numbers between 1 and 6: 1, 3, and 5. So there are

$$3^4 = 81$$

sequences with only odd numbers appearing. Therefore,

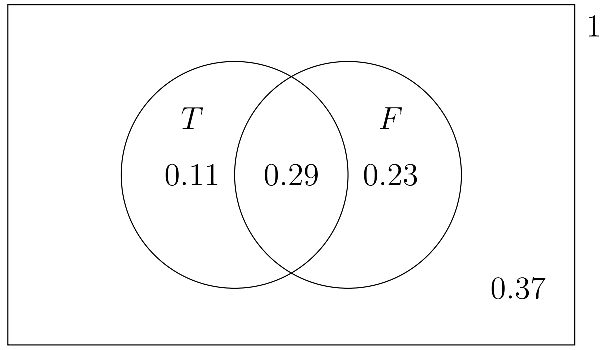
$$\Pr(\text{all odd}) = \frac{81}{1296} = \frac{1}{16}.$$

Example 2.19. Suppose 40% of pigs can talk, 52% of pigs can fly, and 29% of pigs can both fly and talk. Draw a Venn diagram to describe the information, label each region, and determine the probability that a randomly selected pig

(a) can fly or talk,

(b) can fly but not talk, and

(c) can neither fly nor talk.



From this we see that

- $\Pr(T \cup F) = 0.11 + 0.29 + 0.23 = 0.63$,
- $\Pr(F \cap T') = 0.23$, and
- $\Pr((T \cup F)') = 0.37$.

Example 2.20. A coin is tossed 12 times. Find

(a) the probability there are exactly 3 heads,

(b) the probability there are at least 2 heads, and

(c) the probability there are 4 heads or 7 heads.

First, observe that there are $2^{12} = 4096$ different possibilities. Of those, $C(12, 3) = 220$ have exactly 3 heads. So

$$\Pr(3\text{H}) = \frac{220}{4096} = \frac{55}{1024}.$$

To compute the number of sequences where at least 2 heads appear, we note that

- there are $C(12, 0) = 1$ ways for no heads to appear and
- there are $C(12, 1) = 12$ ways for one head to appear.

Then, we see that there are $4096 - (1 + 12) = 4083$ sequences with at least two heads. Thus,

$$\Pr(\text{at least 2H}) = \frac{4083}{4096}.$$

Lastly, we note that there are

- $C(12, 4) = 495$ ways to get exactly 4 heads and
- $C(12, 7) = 792$ ways to get exactly 7 heads.

Hence,

$$\Pr(4\text{H or 7H}) = \frac{495 + 792}{4096} = \frac{1287}{4096}.$$

Example 2.21. Find the probability that, in a class of 35 students, at least two students share the same birthday.

For simplicity's sake, we will not count February 29th and we assume that all other days in the year are equally likely to be a birthday. Now, imagine every student's name listed in a roster along with the day and month of their birthday. There are 365^{35} different possible sequences of birthdays.

Since we are interested in the event of at least two students having the same birthday, it may be easier to find out how many ways all students can have different birthdays. Indeed, there are $P(365, 35)$ ways in which all 35 students have a different birthday. What we get in the end is that the probability that at least two students share a birthday is

$$\frac{365^{35} - P(365,35)}{365^{35}} = 1 - \frac{P(365,35)}{365^{35}} \approx 81.438\%$$

Example 2.22. A True/False test has 10 questions. What is the probability that a student who randomly guesses each problem gets 7 or more questions correct?

There are

- $C(10, 7) = 120$ ways to get exactly 7 right,
- $C(10, 8) = 45$ ways to get exactly 8 right,
- $C(10, 9) = 10$ ways to get exactly 9 right, and
- $C(10, 10) = 1$ ways to get them all right.

There are $2^{10} = 1024$ different ways to answer all of the questions on the test. Therefore, the probability this guessing student gets at least 7 questions correct is

$$\frac{120 + 45 + 10 + 1}{1024} = \frac{11}{64}.$$

Example 2.23. Refer back to Example 1.42 where a photograph is going to be taken of a team. The team has six members, two of which are seniors. If the team lines up to take the picture randomly, what is the probability the seniors are standing next to each other in the photograph?

Since there are six members, total, they can all line up in $6! = 720$ different ways. Of these arrangements, 240 of them have the seniors next to each other, as we calculated in Example 1.42. So the probability that the seniors are standing next to each other in the photo is

$$\frac{240}{720} = \frac{1}{3}.$$

Example 2.24. Refer back to Example 1.45 where we had an urn that contains three gold, five silver, and nine bronze coins. A random sample of three coins is taken out.

(a) What is the probability that the sample consists of only gold coins?

(b) What is the probability that the sample consists of one coin of each color?

(c) What is the probability that the sample consists of two bronze coins?

(d) What is the probability that the sample contains at least one gold coin?

Relying on the work we did in Example 1.45, we can compute

- the probability that the sample consists of only gold coins to be $\frac{1}{680}$.
- the probability that the sample consists of one coin of each color to be $\frac{135}{680} = \frac{27}{136}$.
- the probability that the sample consists of two bronze coins to be $\frac{288}{680} = \frac{36}{85}$.
- the probability that the sample contains at least one gold coin to be $\frac{316}{680} = \frac{79}{170}$.

2.3 Conditional Probability

A conditional probability is essentially one relating two events E and F by finding the probability that E occurs given that F has already occurred. By assuming that F has already occurred, we can think of limiting the sample space to compute the newly framed probability of E.

Notation. We write $\Pr(E|F)$ to denote the probability of E given F.

Example 2.25. Find the probability of randomly drawing a Jack given that you have drawn a face card.

There are 12 face cards and 4 of them are Jacks. Let J be the event of drawing a Jack and F be the event of drawing a face card. Since we are given that we've drawn one of the 13 faces cards and 4 of those are Jacks, we intuit that the conditional probability should be

$$\Pr(J|F) = \frac{4}{12} = \frac{1}{3}.$$

We formalize this with the next definition.

Definition 2.8. In general, the **conditional probability** of E given F, where $\Pr(F) \neq 0$, is defined to be

$$\Pr(E|F) = \frac{\Pr(E \cap F)}{\Pr(F)}.$$

Referring back to Example 2.25, we see that

$$\Pr(J|F) = \frac{\Pr(J \cap F)}{\Pr(F)} = \frac{4}{52} \cdot \frac{52}{12} = \frac{1}{3},$$

consistent with our informal calculations.

Example 2.26. Of the 26 students in a speech class at Academico, 12 are Canadian, 17 are freshmen, and 5 are neither Canadian nor freshmen. Use a Venn diagram for cardinalities to find

(a) the probability a randomly selected student is Canadian and a freshman

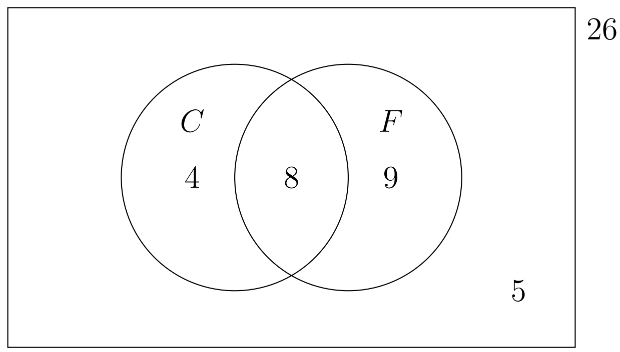
(b) the probability a randomly selected student is Canadian given that they are a freshman

(c) the probability a randomly selected student is a freshman given that they are Canadian

Let C be the set of Canadian students and F be the set of freshmen in this speech class. Notice that $n(C \cup F) = 26 - 5 = 21$. Then

$$\begin{array}{rcl} n(C \cup F) + n(C \cap F) & = & n(C) + n(F) \\ 21 + n(C \cap F) & = & 12 + 17 \\ \hline n(C \cap F) & = & 8 \end{array}$$

which allows us to fill in the Venn diagram:



So the probability that a randomly selected student is a Canadian and a freshmen is

$$\Pr(C \cap F) = \frac{8}{26} = \frac{4}{13}.$$

By the definition of conditional probability, the probability that a randomly selected student is Canadian given that they are a freshman is

$$\Pr(C|F) = \frac{\Pr(C \cap F)}{\Pr(F)} = \frac{4}{13} \cdot \frac{26}{17} = \frac{8}{17}.$$

Similarly, the probability that a randomly selected student is a freshman given that they are Canadian is

$$\Pr(F|C) = \frac{\Pr(F \cap C)}{\Pr(C)} = \frac{4}{13} \cdot \frac{26}{12} = \frac{2}{3}.$$

Remark. In general, if S is a sample space where all outcomes are equally likely and $n(F) \neq 0$,

$$\begin{array}{rcl} \Pr(E|F) & = & \dfrac{\Pr(E \cap F)}{\Pr(F)} \\ & = & \Pr(E \cap F) \cdot \dfrac{1}{\Pr(F)} \\ & = & \dfrac{n(E \cap F)}{n(S)} \cdot \dfrac{n(S)}{n(F)} \\ & = & \dfrac{n(E \cap F)}{n(F)}. \end{array}$$

Example 2.27. We roll a 6-sided die twice.

(a) What is the probability the sum of the rolls is 7 given that one of the rolls is a 3?

(b) What is the probability that one of the rolls was a 3 given that the sum of the rolls is 7?

Let A be the event that the sum of the rolls is 7 and B be the event that one of the rolls is a 3. Then we list

- $A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$
- $B = \{(1,3),(2,3),(3,3),(4,3),(5,3),(6,3),(3,1),(3,2),(3,4),(3,5),(3,6)\}$
- $A \cap B = \{(3,4),(4,3)\}$

Since all outcomes, of the 36 possible outcomes, are equally likely, we see that

- $\Pr(A|B) = \dfrac{2}{11}$
- $\Pr(B|A) = \dfrac{2}{6} = \dfrac{1}{3}$

Example 2.28. Two cards are drawn at random, in sequence, from a standard deck of 52 cards. What is the probability that first card is a face card and the second card is an Ace?

Let F be the event that the first card drawn is a face card and A be the event that the second card drawn is an Ace. Notice that we can calculate $\Pr(F) = \frac{12}{52} = \frac{3}{13}$ and

$$\Pr(A|F) = \frac{4}{51}.$$

We want to find $\Pr(A \cap F)$ and, by definition of conditional probability,

$$\Pr(A|F) = \frac{\Pr(A \cap F)}{\Pr(F)}.$$

Multiplying by $\Pr(F)$ yields

$$\Pr(A \cap F) = \Pr(F) \cdot \Pr(A|F) = \frac{3}{13} \cdot \frac{4}{51} = \frac{4}{221}.$$

Product Rule. If $\Pr(F) \neq 0$, then

$$\Pr(E \cap F) = \Pr(F) \cdot \Pr(E|F).$$

Example 2.29. WoodyMammoth has recently outsourced some of their guitar manufacturing to Nevada and 30% of all WoodyMammoth guitars are made at the Nevada branch. The quality control team has determined that 5% of guitars made in the Nevada branch are defective. What is the probability that a randomly purchased WoodyMammoth guitar was made in Nevada and is defective?

Let N be the event that a guitar is made in Nevada and D be the event that a guitar is defective. We want to find $\Pr(N \cap D)$. The information given is $\Pr(N) = 0.3$ and $\Pr(D|N) = 0.05$. By the Product Rule,

$$\Pr(N \cap D) = \Pr(N) \cdot \Pr(D|N) = (0.3)(0.05) = 0.015$$

so the probability that a randomly purchased WoodyMammoth guitar was made in Nevada and is defective is 1.5%.

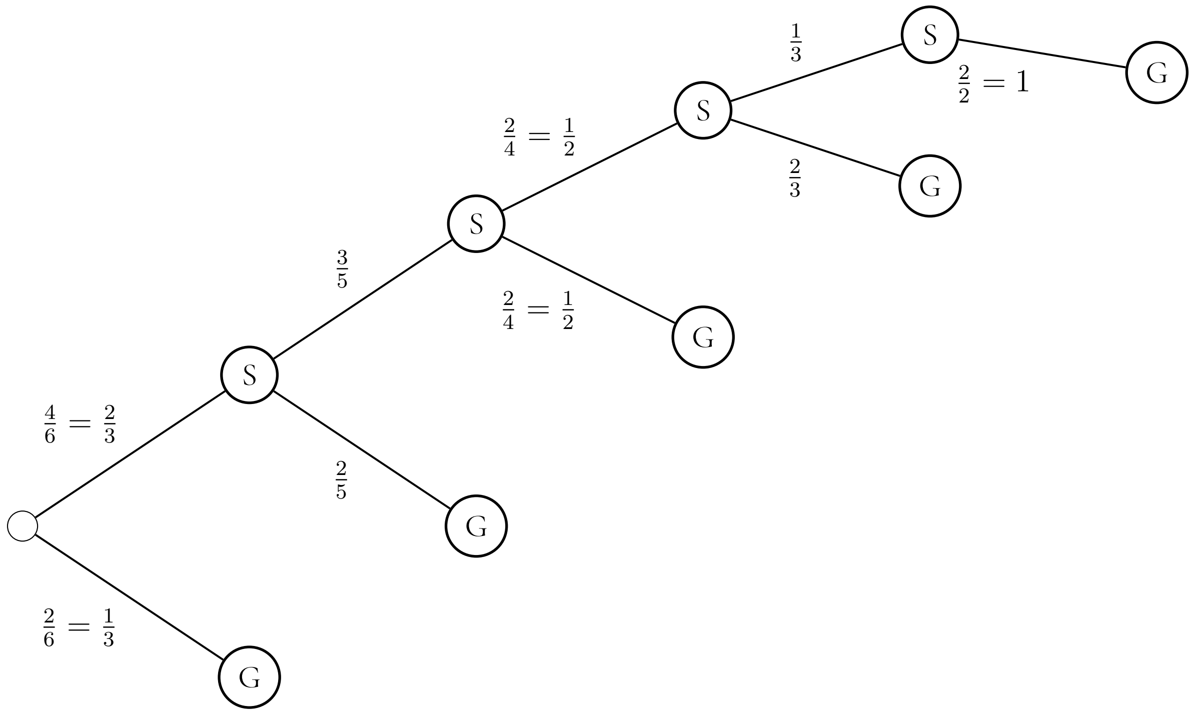
We can use tree diagrams to help us evaluate probabilities that occur as sequences of events.

Example 2.30. In the distant land of Moneralando, people only use coins as the currency. Coins are either silver or gold. Sofia is randomly reaching into her coin bag which contains four silver coins and two gold coins. She draws a coin out of the bag one at a time, without replacement, until she draws a gold coin. Use a tree diagram to investigate the corresponding probabilities.

In words, we describe the possibilities.

- In the first draw, she can either draw a silver coin or a gold coin. If she draws a gold coin, she is done.
- If she drew a silver coin on the first draw, she can draw another silver coin or she can draw a gold coin. If she draws a gold coin, she is done.
- If she's drawn two silver coins, she can draw another silver coin or she can draw a gold coin. If she draws a gold coin, she is done.
- If she's drawn three silver coins, she can still draw another silver coin or she can draw a gold coin. If she draws a gold coin, she's done.
- If she's already drawn four silver coins, there are no silver coins left so she has to draw a gold coin.

Using G to stand for drawing a gold coin and S for drawing a silver coin, the corresponding tree diagram is

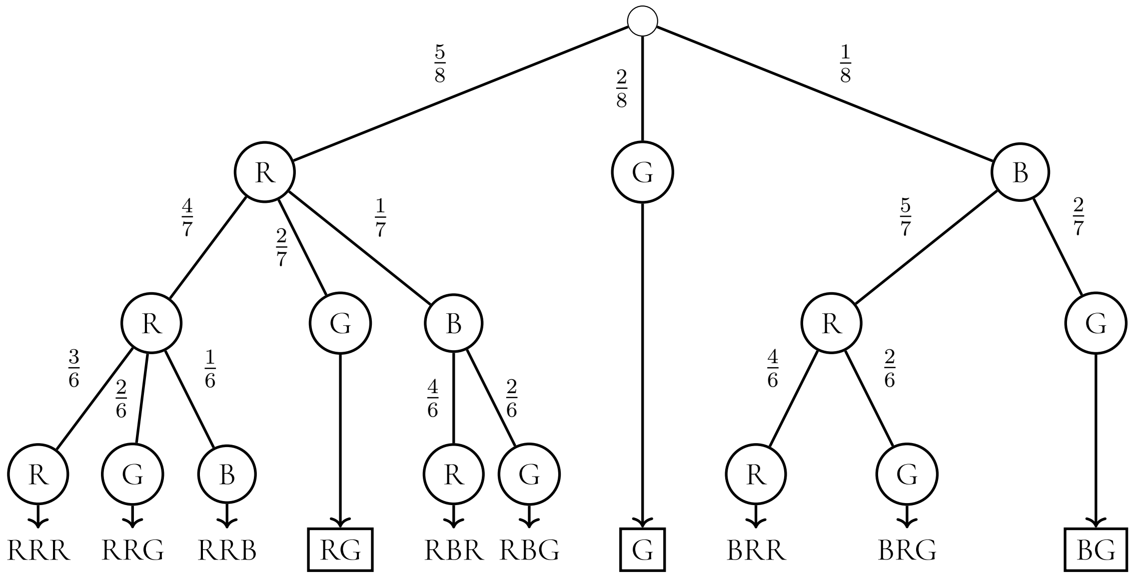


Notice that the fractions on each branch are conditional probabilities. Extending the Product Rule for conditional probability in the right way (this means we multiply throw branches of the tree), we can compute the probability that Sofia draws the gold coin on the

- first draw to be $\frac{1}{3}$.
- second draw to be $\frac{2}{3} \cdot \frac{2}{5} = \frac{4}{15}$.
- third draw to be $\frac{2}{3} \cdot \frac{3}{5} \cdot \frac{1}{2} = \frac{3}{15}$.
- fourth draw to be $\frac{2}{3} \cdot \frac{3}{5} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{15}$.
- fifth draw to be $\frac{2}{3} \cdot \frac{3}{5} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot 1 = \frac{1}{15}$.

Example 2.31. An urn contains 5 red, 2 green, and 1 blue marbles. You reach in to randomly draw one marble at a time, without replacement, until you've drawn one green marble or three marbles. Draw a tree diagram that describes the outcomes of this experiment and find the probability that at most two marbles are selected.

We start with a tree diagram. The outcomes where at most two marbles are selected are boxed.



Using the conditional probabilities along the branches, we find the probability that at most two marbles are selected to be

$$\frac{5}{8}\cdot\frac{2}{7}+\frac{2}{8}+\frac{1}{8}\cdot\frac{2}{7}=\frac{13}{28}.$$

Example 2.32. A lawsuit is being raised against the University of Alexandria for gender bias in their graduate school acceptance rates. The plaintiff claims that males are favored against females. As the evidence, they cite that, in the most recent year, about 41% of the 1384 male applicants were accepted whereas only about 33% of the 339 female applicants were accepted. The university's legal counsel pulled the admissions data which show the proportions of applicants and accepted applications across the three departments at the University of Alexandria: History, English, and Math. The data are as follows:

	Male		Female	
	Applicants	Accepted	Applicants	Accepted
History	389	136	109	44
English	742	408	73	44
Math	253	25	157	24
Total	1384	569	339	112

Randomly select one of the applicants to the university and let A be the event that applicant was accepted, let M be the event that applicant identifies as male, let F be the event that applicant identifies as female, and let T be the event that the applicant applied to the Math program.

(a) Find $\Pr(A|M)$.

(b) Find $\Pr(A|F)$.

(c) Find $\Pr(A\cap T|M)$.

(d) Find $\Pr(A\cap T|F)$.

(e) Calculate the acceptance rates by gender in each of the programs and compare your findings to the plaintiff's claim. Is there evidence of gender bias in favor of males in the acceptance data?

We can compute the conditional probabilities based on the table: $\Pr(A|M) = \dfrac{569}{1384} \approx 41.11\%$, $\Pr(A|F) = \dfrac{112}{339} \approx 33.04\%$. $\Pr(A \cap T|M) = \dfrac{25}{1384} \approx 1.81\%$, and $\Pr(A \cap T|F) = \dfrac{24}{339} \approx 7.08\%$.

To address the other two programs, we let H be the event that the randomly selected applicant applied to the history program and E be the event that the randomly selected applicant applied to the English program. We then compute $\Pr(A|M \cap H) = \dfrac{136}{389} \approx 34.96\%$, $\Pr(A|F \cap H) = \dfrac{44}{109} \approx 40.37\%$, $\Pr(A|M \cap E) = \dfrac{408}{742} \approx 54.99\%$, $\Pr(A|F \cap E) = \dfrac{44}{73} \approx 60.27\%$, $\Pr(A|M \cap T) = \dfrac{25}{253} \approx 9.88\%$, and $\Pr(A|F \cap T) = \dfrac{24}{157} \approx 15.29\%$.

So, when we restrict our attention to each of the individual programs, we see that the applicants that identified as women actually had a higher relative acceptance rate, despite the computations ignoring the individual programs. Therefore, the plaintiff's claim doesn't seem justified.

Example 2.32 is an instance of Simpson's Paradox.

2.4 Independent Events

We introduce the concept of independence with an example.

Example 2.33. Flip a fair coin twice. Let A be the event that the first toss results in a heads and B be the event that the second toss results in a heads. Find $\Pr(A)$, $\Pr(A|B)$, $\Pr(B)$, and $\Pr(B|A)$.

We can do this by listing out the sample space:

$$\{\text{TT}, \text{HT}, \text{TH}, \text{HH}\}.$$

Note that $\Pr(A) = \frac{2}{4} = \frac{1}{2}$ and $\Pr(B) = \frac{2}{4} = \frac{1}{2}$.

Now, since $\Pr(A \cap B) = \frac{1}{4}$, we calculate

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1}{4} \cdot \frac{2}{1} = \frac{1}{2}$$

and

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{1}{2}.$$

What we see here is that $\Pr(A) = \Pr(A|B)$ and $\Pr(B) = \Pr(B|A)$.

In this example, we conceptually see that A is *independent* of B in the sense that the occurrence of B does not affect the probability of A; i.e., $\Pr(A) = \Pr(A|B)$. In general, if A and B are events so that $\Pr(B) \neq 0$ and $\Pr(A) = \Pr(A|B)$, we have

$$\Pr(A) = \Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \implies \Pr(A \cap B) = \Pr(A) \cdot \Pr(B).$$

This is what drives the following definition.

Definition 2.9. Two events E and F are said to be **independent events** if

$$\Pr(E \cap F) = \Pr(E) \cdot \Pr(F).$$

Example 2.34. Suppose there are two entrance exams that need to be taken when applying to Mathemagix University. To be accepted, you need only pass one of the two exams. There is a 40% chance you pass Exam I, a 30% chance you pass Exam II, and both events are independent. What is the probability you get accepted?

There are $2 \cdot 2 = 4$ different outcomes since there are two tests and we can pass or fail each test. The four different outcomes are

- fail I and fail II
- pass I and fail II
- fail I and pass II
- pass I and pass II

The outcomes we are interested in are

- pass I and fail II
- fail I and pass II
- pass I and pass II

so we calculate the probability:

$$(0.4)(0.7) + (0.6)(0.3) + (0.4)(0.3) = 0.58$$

Hence, there is a 58% you get accepted.

We could also approach this using the Inclusion-Exclusion Principle. Observe that

$$\Pr(\text{pass Exam I or pass Exam II}) = 0.4 + 0.3 - (0.4)(0.3) = 0.58.$$

This uses the independence of the two events to compute the probability that you pass both Exam I and Exam II.

Example 2.35. Roll a 6-sided die twice and let E be the event that the first roll is a 6 and F be the event that the second roll is a 4. Show that E and F are independent events.

We need to show that $\Pr(E \cap F) = \Pr(E) \cdot \Pr(F)$. Firstly,

$$\Pr(E \cap F) = \frac{1}{36}$$

since the only outcome in $E \cap F$ is $(6, 4)$.

Since

$$E = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

and

$$F = \{(1,4), (2,4), (3,4), (4,4), (5,4), (6,4)\}$$

we see that $\Pr(E) = \frac{1}{6}$ and $\Pr(F) = \frac{1}{6}$.

Thus,

$$\Pr(E) \cdot \Pr(F) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = \Pr(E \cap F)$$

which establishes that E and F are independent.

Example 2.36. Two tests are used to determine whether or not someone has been contaminated with nanomachines. Test I is correct 70% of the time and Test II is correct 80% of the time. At least one of them is correct 99% of the time. Determine whether or not the correct identification by these tests are independent.

Let A be the event that Test I is correct and B be the event that Test II is correct. By the Inclusion-Exclusion Principle,

$$\begin{array}{ccccccc} \Pr(A \cup B) & = & \Pr(A) & + & \Pr(B) & - & \Pr(A \cap B) \\ \hline 0.99 & = & 0.7 & + & 0.8 & - & \Pr(A \cap B) \end{array}$$

which provides

$$\Pr(A \cap B) = 0.51$$

Now, notice that

$$\Pr(A) \cdot \Pr(B) = (0.7)(0.8) = 0.56 \neq 0.51 = \Pr(A \cap B)$$

so the correct identification by these tests are not independent; i.e., they are dependent.

Example 2.37. A new alien life detection device has three components: A, B, and C. Component A has a 3% failure rate, component B has a 5% failure rate, and component C has a 12% failure rate. The failure of each component is independent of the others.

(a) Find the probability that the device has at least one failed component.

(b) A repairs specialist has received an alien life detection device which has at least one failed component. Find the probability that *only* component C has failed.

Notice that the opposite event of at least one component failing is the event that no components fail. To compute this, we use independence:

$$\Pr(\text{no failures}) = (0.97)(0.95)(0.88) = 0.81092$$

which provides

$$\Pr(\text{at least one fails}) = 1 - 0.81092 = 0.18908 = 18.908\%$$

For the conditional probability, notice that

$$\Pr(\text{C failed} \cap \text{at least one component failed}) = \Pr(\text{C failed}) = (0.97)(0.95)(0.12).$$

So

$$\Pr(\text{C failed}|\text{at least one component failed}) = \frac{(0.97)(0.95)(0.12)}{1-(0.97)(0.95)(0.88)} \approx 0.58483.$$

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Example 2.38. Suppose E and F are mutually exclusive with $\Pr(E) \neq 0$ and $\Pr(F) \neq 0$. Determine whether or not E and F are independent.

Since E and F are mutually exclusive,

$$\Pr(E \cap F) = \Pr(\emptyset) = 0 \neq \Pr(E) \cdot \Pr(F).$$

That is, E and F are dependent.

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Another way to think about it is that, assuming that E and F are mutually exclusive, then the occurrence of E makes the occurrence of F *impossible*. Since the possibility of F occurring at all is affected by the occurrence at all, E and F cannot be independent.

2.5 Bayes' Theorem

We'll build up to Bayes' Theorem by first revisiting tree diagrams and using them to evaluate certain probabilities.

Example 2.39. Two stones are drawn from a cup containing 2 white and 8 black stones.

(a) What is the probability of drawing a white stone on the second draw?

(b) What is the probability of drawing a black stone on the second draw?

We are interested in the second stone being white but this depends on the first stone drawn. There are two possibilities for the first stone: it can be white or black. Let W_2 be the event that the second stone drawn is white, W_1 be the event that the first stone drawn is white, and B be the event that the first stone drawn is black. Let's investigate both cases.

- Assuming the first stone drawn is white, we compute

$$\Pr(W_1 \cap W_2) = \Pr(W_1) \cdot \Pr(W_2|W_1) = \frac{2}{10} \cdot \frac{1}{9} = \frac{1}{45}.$$

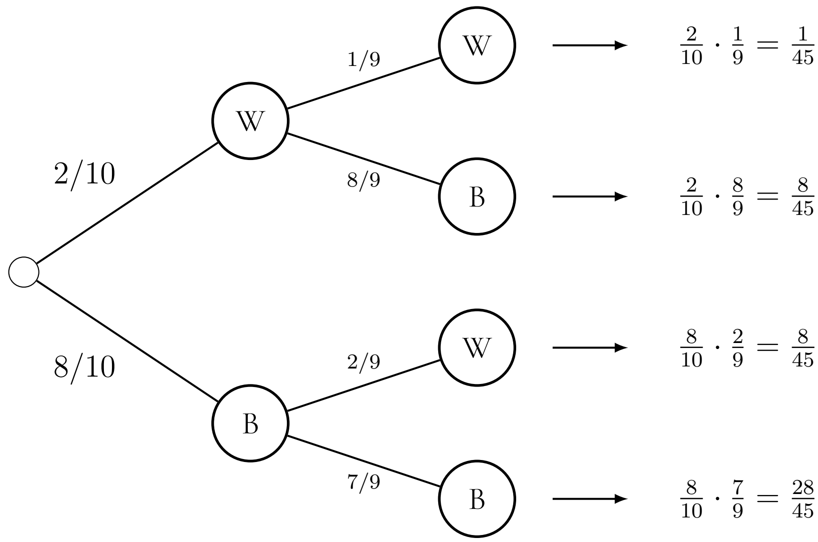
- Now, assuming the first stone drawn is black, we compute

$$\Pr(B \cap W_2) = \Pr(B) \cdot \Pr(W_2|B) = \frac{8}{10} \cdot \frac{2}{9} = \frac{8}{45}.$$

Since these two cases are separate and capture all possibilities where the second draw results in a white stone, we see

$$\Pr(W_2) = \frac{1}{45} + \frac{8}{45} = \frac{1}{5}.$$

Now, we can capture this process using trees.



From the tree, we can can compute the probability that the second stone drawn is black:

$$\frac{8}{45} + \frac{28}{45} = \frac{4}{5}.$$

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Example 2.40. Two stones are drawn from a cup containing 2 white and 8 black stones. What is the probability that the first stone was black given that the second stone is white?

Note that

$$\Pr(1^{\text{st}}\ \text{B}|2^{\text{nd}}\ \text{W}) = \frac{\Pr(1^{\text{st}}\ \text{B} \cap 2^{\text{nd}}\ \text{W})}{\Pr(2^{\text{nd}}\ \text{W})}$$

which we compute by referring to the tree in Example 2.39:

$$\Pr(1^{\text{st}}\ \text{B}|2^{\text{nd}}\ \text{W}) = \frac{\Pr(1^{\text{st}}\ \text{B} \cap 2^{\text{nd}}\ \text{W})}{\Pr(2^{\text{nd}}\ \text{W})} = \frac{\frac{8}{45}}{\frac{1}{45} + \frac{8}{45}} = \frac{8}{9}.$$

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In general, we have

Theorem 2.10 (Bayes' Theorem)**.** Given a sample space S, mutually exclusive events $F_1, F_2, \ldots, F_n$ so that $S = F_1 \cup F_2 \cup \cdots \cup F_n$, and an event E with $\Pr(E) \neq 0$,

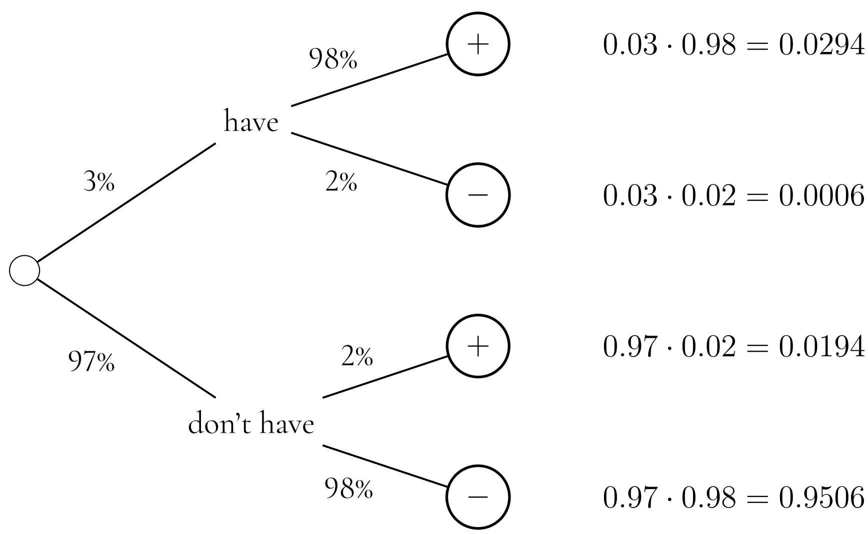
$$\Pr(F_j|E) = \frac{\Pr(F_j) \cdot \Pr(E|F_j)}{\Pr(F_1) \cdot \Pr(E|F_1) + \Pr(F_2) \cdot \Pr(E|F_2) + \cdots + \Pr(F_n) \cdot \Pr(E|F_n)}.$$

Bayes' Theorem is often used to study probabilities involving medical screening tests. For example, a test could claim that you haven't used steroids when you actually have used them. This would be known as a *false negative*. The following chart outlines the possibilities.

	test reads positive	**test reads negative**
actually positive	correctly identified	false negative
actually negative	false positive	correctly identified

Example 2.41. We have discovered that aliens have implanted nanomachines in 3% of the human population and a new test has an accuracy rate of 98%. That is, it correctly identifies a person with nanomachines 98% of the time and it correctly identifies someone without nanomachines 98% of the time. If you have tested positive for nanomachines, find the probability that you actually have nanomachines.

Using a tree diagram:



From here, we see that $0.0294 + 0.0194 = 0.0488$ tested positive, 0.0294 of those are correctly identified so

$$\frac{0.0294}{0.0488} \approx 0.6025$$

would actually have nanomachines given that they tested positive for them.

Now, to help make this clearer, imagine that the population is 10 000. Then 300 have nanomachines. Of those 300 with nanomachines, 98% of them will test positively; i.e., 294 of them will be correctly identified. Of those 9700 without nanomachines, 98% of them will test negatively; i.e., 9506 of them will be correctly identified. Let's organize this in a table:

	tested positive	**tested negative**
actually positive	294	6
actually negative	194	9506

So there are 488 people which test positive, 294 of those actually have it. Hence, the probability that you actually have nanomachines given you tested positive for them is $294/488 \approx 0.6025$.

Example 2.42. An alien life detection device company has six different manufacturing plants. Defect rates and proportions of production are given below.

plant	proportion	defect
A	0.35	0.0007
B	0.25	0.0013
C	0.15	0.0002
D	0.12	0.05
E	0.1	0.01
F	0.03	0.009

You receive one alien life detection device at random.

(a) What is the probability that the device you receive is free of defects?

(b) Assuming the device has a defect, what is the probability that it was manufactured at plant A?

(c) Assuming the device has a defect, which manufacturing plant most likely produced your device?

To find the probability that the device doesn't have a defect, we compute

$$\begin{aligned}\Pr(\text{no defect}) &= (0.35)(0.9993) + (0.25)(0.9987) + (0.15)(0.9998) \\ &\quad +(0.12)(0.95) + (0.1)(0.99) + (0.03)(0.991) \\ &= 0.99213\end{aligned}$$

To find the probability that plant A produced the defect given that the device has a defect, we let A be the event that plant A produced the device and B be the event that the device has a defect. From what we found above, we know that $\Pr(B) = 1 - 0.99213 = 0.00787$. Now, behold that

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B|A) = (0.35)(0.0007) = 0.000245$$

Therefore,

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{0.000245}{0.00787} \approx 3.113\%.$$

Consider the table of probabilities:

plant	Pr(plant produced def.)	Pr(plant|def.)
A	$(0.35)(0.0007)$	$\approx 3.113\%$
B	$(0.25)(0.0013)$	$\approx 4.130\%$
C	$(0.15)(0.0002)$	$\approx 0.381\%$
D	$(0.12)(0.05)$	$\approx 76.239\%$
E	$(0.1)(0.01)$	$\approx 12.706\%$
F	$(0.03)(0.009)$	$\approx 3.431\%$

Hence, assuming the device has a defect, the device was most likely produced by plant D.

Example 2.42 may lead one to think that the fact that plant D most likely produced the defect because of its overall higher defect rate. However, this is not generally enough information, as the next example will illustrate.

Example 2.43. A bottle manufacturing company has two plants: one in Houston and the other in some unspecified foreign country. The Houston plant makes 98% of all bottles. Quality control studies have demonstrated that the Houston plant has a defect rate of 1% and the defect rate from the outsourced manufacturing is 12%. Assuming that a randomly purchased bottle has a defect, which plant did it most likely come from?

The probability you get a defective bottle is $(0.98)(0.01) + (0.02)(0.12) = 0.0122$. Note that

$$\Pr(\text{Houston}|\text{defective}) = \frac{(0.98)(0.01)}{(0.98)(0.01) + (0.02)(0.12)} \approx 0.80328$$

and

$$\Pr(\text{abroad}|\text{defective}) = \frac{(0.02)(0.12)}{(0.98)(0.01) + (0.02)(0.12)} \approx 0.19672.$$

Therefore, even though the outsourced manufacturing has a larger defect rate, it's still more likely that a defective bottle came from the Houston plant.

2.6 Binomial Trials

Definition 2.11. A **random variable** is any rule X which assigns numerical values to each outcome of a sample space S.

Example 2.44. Toss a coin 15 times and let X be the number of heads appearing. Then X is a random variable.

Definition 2.12. A **binomial experiment** has the following properties:

- The number of trials in the experiment is a fixed number n.
- There are two outcomes: Success or Failure.
- The probability of success in each trial is the same, p. Since there are only two outcomes, the probability of failure is $q = 1 - p$.
- The trials are independent of each other.

Remark. For a binomial experiment, we usually let X be the random variable giving the number of successes.

Example 2.45. Toss a coin 12 times and consider heads the "success" case. Then let X be the random variable which counts the number of successes. We write $X = 4$ to mean four heads appeared.

Notation. For a binomial experiment, we write $\Pr(X = k)$ to denote the probability that $X = k$; i.e., that there are k successes.

Formula. Given a binomial experiment with n trials where p is the probability of success, $q = 1 - p$ is the probability of failure, and X is the number of successes, the probability of k successes is

$$\Pr(X = k) = C(n, k)p^k q^{n-k}.$$

Example 2.46. When a pair of 6-sided dice is rolled, the probability of obtains a sum of 7 is $\frac{1}{6}$. Suppose a pair of dice is rolled 25 times. Find the probability that the sum is 7 eight times.

This is a binomial experiment where

- $n = 25$,
- success is the sum being 7,
- $p = \frac{1}{6}$,
- $q = \frac{5}{6}$, and
- $k = 8$.

By the formula,

$$\Pr(X = 8) = C(25, 8) \cdot \left(\frac{1}{6}\right)^8 \cdot \left(\frac{5}{6}\right)^{17} \approx 2.902\%.$$

Note. Notice that, doing Example 2.46 combinatorially, we would count that there are

- $C(25, 8)$ to choose the rolls which sum to 7,
- 6 ways that a pair of dice add up to 7,
- 6^8 ways that eight pairs add up to 7,
- $36 - 6 = 30$ ways that a pair of dice don't add up to 7,
- 30^{17} ways that 17 pairs don't add up to 7.

Then we would calculate the probability that eight pairs add up to 7 to be

$$\frac{C(25, 8) \cdot 6^8 \cdot 30^{17}}{36^{25}} \approx 2.902\%.$$

Example 2.47. A coin is weighted so that when it is flipped, there is a 70% chance it ends up tails. If we toss this coin 9 times, find

(a) the probability that we end up with exactly 6 tails,

(b) the probability that we end up with at most 6 tails, and

(c) the probability that we end up with at least 7 tails.

Let X be the number of tails in the sequence of tosses. We can compute

$$\Pr(X = 6) = C(9, 6) \cdot (0.7)^6 \cdot (0.3)^3 \approx 0.26683.$$

For the probability there are at most 6 tails, we compute

$$\begin{aligned}
\Pr(X \leqslant 6) &= \Pr(X=0) \\
&\quad + \Pr(X=1) \\
&\quad + \Pr(X=2) \\
&\quad + \Pr(X=3) \\
&\quad + \Pr(X=4) \\
&\quad + \Pr(X=5) \\
&\quad + \Pr(X=6) \\
&= C(9,0)\cdot(0.7)^0\cdot(0.3)^9 \\
&\quad + C(9,1)\cdot(0.7)^1\cdot(0.3)^8 \\
&\quad + C(9,2)\cdot(0.7)^2\cdot(0.3)^7 \\
&\quad + C(9,3)\cdot(0.7)^3\cdot(0.3)^6 \\
&\quad + C(9,4)\cdot(0.7)^4\cdot(0.3)^5 \\
&\quad + C(9,5)\cdot(0.7)^5\cdot(0.3)^4 \\
&\quad + C(9,6)\cdot(0.7)^6\cdot(0.3)^3 \\
&\approx 0.53717.
\end{aligned}$$

To find the probability there are at least 7 tails, we could compute a sum similar to the one above; we can also note that the opposite of "at least 7" is "at most 6." This can be visualized as follows:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \boxed{7 \quad 8 \quad 9}$$

Therefore,

$$\Pr(X \geqslant 7) = 1 - \Pr(X \leqslant 6) \approx 0.46283.$$

2.7 Expected Value

Conceptually, the *expected value* of a random value is the value which we can expect *on average*. In general, the expected value doesn't have to be a value that the random variable can possibly assume.

Definition 2.13. For a random variable X, the **expected value** of X is defined to be

$$E(X) = x_1p_1 + x_2p_2 + \cdots + x_np_n$$

where $x_1, x_2, \ldots, x_n$ are the values of X and $p_j = \Pr(X = x_j)$ for $1 \leqslant j \leqslant n$.

Example 2.48. A 6-sided die is rolled. Find the expected value.

Check the table:

outcome, x	$\Pr(X = x)$	$x \cdot \Pr(X = x)$
1	$\frac{1}{6}$	$\frac{1}{6}$
2	$\frac{1}{6}$	$\frac{2}{6}$
3	$\frac{1}{6}$	$\frac{3}{6}$
4	$\frac{1}{6}$	$\frac{4}{6}$
5	$\frac{1}{6}$	$\frac{5}{6}$
6	$\frac{1}{6}$	$\frac{6}{6}$

which shows that

$$E(X) = \frac{21}{6} = 3.5$$

Of course, we can never roll a 3.5 but expected value doesn't have to be possible outcomes.

Example 2.49. Roll a 6-sided die twice and let X be the sum of the rolls. Find the expected value of X.

Recall that

x	$f(x)$	x	$f(x)$
2	$1/36$	8	$5/36$
3	$2/36$	9	$4/36$
4	$3/36$	10	$3/36$
5	$4/36$	11	$2/36$
6	$5/36$	12	$1/36$
7	$6/36$		

which shows that $E(X) = 7$. Coincidentally, in this case, this is also the most likely outcome.

Example 2.50. You pay \$1 to draw a card from a standard deck of 52 cards. If you draw

- an Ace, you win \$9,
- a face card, you win \$2, and
- otherwise, you win nothing.

If X is your net earning, find the expected value of X. Is this game worth playing?

Incorporating the cost, notice that

$$E(X) = 8 \cdot \frac{4}{52} + 1 \cdot \frac{12}{52} + (-1) \cdot \frac{36}{52} = \frac{2}{13} \approx \$0.15.$$

So, on average, we should expect to win 15 cents. As this is a positive value, this is a good game to play.

We can also compute the expected value of the reward first and then incorporate the cost afterwards to obtain the same result. Indeed,

$$9 \cdot \frac{4}{52} + 2 \cdot \frac{12}{52} + 0 \cdot \frac{36}{52} \approx 1.15.$$

Then, we incorporate the cost of the game to compute the expected value of the net earnings: $1.15 - 1 = 0.15$.

3 Matrices

3.1 Systems of Linear Equations

Notice that, for constant values a, b, and c, $ax + by = c$, as long as $b \neq 0$, can be written as

$$y = \frac{-a}{b} \cdot x + \frac{c}{b},$$

the familiar form of a line. For this reason, we say that the equation $ax + by = c$ is linear. For three variables, a linear equation will be one of the form

$$ax + by + cz = d.$$

Definition 3.1. A **system of linear equations** is any collection of linear equations sharing the same variables.

Example 3.1. The following is a system of equations:

$$\begin{cases} 5x & - & 3y & - & 6z & = & -4 \\ 3x & + & y & - & 5z & = & 5 \\ 4x & - & 2y & + & z & = & -13 \end{cases}$$

We'll be interested in finding solution sets to these equations. First, let's consider some 2-dimensional examples by graphing.

Example 3.2. The system

$$\begin{cases} -6x & + & 3y & = & 1 \\ 2x & - & y & = & 2 \end{cases}$$

has no solutions. This is seen by graphing the two lines and noting that they are parallel and not equal.

Example 3.3. The system

$$\begin{cases} 6x & + & y & = & 1 \\ 2x & - & 3y & = & 7 \end{cases}$$

has exactly one solution. This is seen by graphing the two lines and noting that they intersect at one point.

Example 3.4. The system

$$\begin{cases} -6x & + & 3y & = & -9 \\ 2x & - & y & = & 3 \end{cases}$$

has infinitely many solutions. This is seen by graphing and noting that they are the same line.

When solving systems of linear equations, we want to get them into **diagonal form** which is

$$\begin{cases} x & & & = & a \\ & y & & = & b \\ & & z & = & c \end{cases}$$

We can do this using the **Gauss-Jordan elimination method** which will involve matrices and a combination of *elementary row operations* and *pivots*.

Remark. Think of the elementary row operations as the "legal moves" and the Gauss-Jordan elimination method as a particular "strategy" to attain a specific result.

> **Elementary Row Operations.** For systems of equations, the elementary row operations are:
>
> 1. Interchange any two equations.
> 2. Multiply an equation by a non-zero number.
> 3. Change an equation by adding to it a multiple of another equation.

Remark. Row operation 2 is used to get *the one* and row operation 3 is to get *the zeroes*. We will discuss this more when we get to pivots.

Example 3.5. Here, we will give examples of each elementary row operation.

1. This is an example of interchanging two equations along with the appropriate notation:

$$\left\{\begin{array}{rcrcrcr} 5x & - & 3y & - & 6z & = & -4 \\ 3x & + & y & - & 5z & = & 5 \\ 4x & - & 2y & + & z & = & -13 \end{array}\right. \xrightarrow{R_1 \leftrightarrow R_2} \left\{\begin{array}{rcrcrcr} 3x & + & y & - & 5z & = & 5 \\ 5x & - & 3y & - & 6z & = & -4 \\ 4x & - & 2y & + & z & = & -13 \end{array}\right.$$

2. This is an example of multiplying an equation by a non-zero number with the appropriate notation:

$$\left\{\begin{array}{rcrcrcr} 5x & - & 3y & - & 6z & = & -4 \\ 3x & + & y & - & 5z & = & 5 \\ 4x & - & 2y & + & z & = & -13 \end{array}\right. \xrightarrow{2R_3} \left\{\begin{array}{rcrcrcr} 5x & - & 3y & - & 6z & = & -4 \\ 3x & + & y & - & 5z & = & 5 \\ 8x & - & 4y & + & 2z & = & -26 \end{array}\right.$$

3. This is an example of changing an equation by adding a multiple of another row to it with the appropriate notation:

$$\left\{\begin{array}{rcrcrcr} 5x & - & 3y & - & 6z & = & -4 \\ 3x & + & y & - & 5z & = & 5 \\ 4x & - & 2y & + & z & = & -13 \end{array}\right. \xrightarrow{R_2 - 2R_3} \left\{\begin{array}{rcrcrcr} 5x & - & 3y & - & 6z & = & -4 \\ -5x & + & 5y & - & 7z & = & 31 \\ 4x & - & 2y & + & z & = & -13 \end{array}\right.$$

Example 3.6. We will often be performing elementary row operations multiple times. For example,

$$\left\{\begin{array}{rcrcrcr} 5x & - & 3y & - & 6z & = & -4 \\ 3x & + & y & - & 5z & = & 5 \\ 4x & - & 2y & + & z & = & -13 \end{array}\right. \xrightarrow{R_2 - 2R_3} \left\{\begin{array}{rcrcrcr} 5x & - & 3y & - & 6z & = & -4 \\ -5x & + & 5y & - & 7z & = & 31 \\ 4x & - & 2y & + & z & = & -13 \end{array}\right.$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left\{\begin{array}{rcrcrcr} -5x & + & 5y & - & 7z & = & 31 \\ 5x & - & 3y & - & 6z & = & -4 \\ 4x & - & 2y & + & z & = & -13 \end{array}\right.$$

$$\xrightarrow{R_2 + R_1} \left\{\begin{array}{rcrcrcr} -5x & + & 5y & - & 7z & = & 31 \\ & & 2y & - & 13z & = & 27 \\ 4x & - & 2y & + & z & = & -13 \end{array}\right.$$

Definition 3.2. For a system of equations

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

the corresponding **augmented matrix** is

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

Column 1 consists of the coefficients for x, column 2 the coefficients for y, and column 3 the coefficients for z. The vertical bar is used as a reminder that column 4 consists of the constant values.

Using the augmented matrix, we can symbolically simplify the elementary row operations.

Elementary Row Operations. For matrices, the elementary row operations are:

1. Interchange any two rows.
2. Multiply a row by a non-zero number.
3. Change a row by adding to it a multiple of another row.

Example 3.7. Repeat Example 3.6 using the augmented matrices.

$$\left[\begin{array}{rrr|r} 5 & -3 & -6 & -4 \\ 3 & 1 & -5 & 5 \\ 4 & -2 & 1 & -13 \end{array}\right] \xrightarrow{R_2-2R_3} \left[\begin{array}{rrr|r} 5 & -3 & -6 & -4 \\ -5 & 5 & -7 & 31 \\ 4 & -2 & 1 & -13 \end{array}\right]$$

$$\xrightarrow{R_1\leftrightarrow R_2} \left[\begin{array}{rrr|r} -5 & 5 & -7 & 31 \\ 5 & -3 & -6 & -4 \\ 4 & -2 & 1 & -13 \end{array}\right]$$

$$\xrightarrow{R_2+R_1} \left[\begin{array}{rrr|r} -5 & 5 & -7 & 31 \\ 0 & 2 & -13 & 27 \\ 4 & -2 & 1 & -13 \end{array}\right]$$

Pivot Method. To pivot a matrix about a given non-zero entry,

1. Multiply the row by the reciprocal of the entry (to make the given entry a one).
2. Transform all other entries of the same column to zero (by using the one in the pivot entry).

Example 3.8. Pivot around the circled entry in

$$\begin{cases} 4x - 2y + 3z = 4 \\ 8x - 3y + 5z = 7 \\ 7x - 2y + 4z = 6 \end{cases}$$

Observe:

$$\left\{\begin{array}{rcrcrcr} 4x & - & 2y & + & 3z & = & 4 \\ 8x & - & 3y & + & 5z & = & 7 \\ 7x & - & 2y & + & 4z & = & 6 \end{array}\right. \xrightarrow{\frac{1}{4}R_1} \left\{\begin{array}{rcrcrcr} x & - & \frac{1}{2}\cdot y & + & \frac{3}{4}\cdot z & = & 1 \\ 8x & - & 3y & + & 5z & = & 7 \\ 7x & - & 2y & + & 4z & = & 6 \end{array}\right.$$

$$\xrightarrow{R_2-8R_1} \left\{\begin{array}{rcrcrcr} x & - & \frac{1}{2}\cdot y & + & \frac{3}{4}\cdot z & = & 1 \\ & & y & - & z & = & -1 \\ 7x & - & 2y & + & 4z & = & 6 \end{array}\right.$$

$$\xrightarrow{R_3-7R_1} \left\{\begin{array}{rcrcrcr} x & - & \frac{1}{2}\cdot y & + & \frac{3}{4}\cdot z & = & 1 \\ & & y & - & z & = & -1 \\ & & \frac{3}{2}\cdot y & - & \frac{5}{4}\cdot z & = & -1 \end{array}\right.$$

We will see how this method of pivoting looks in the context of augmented matrices when we learn about the Gauss-Jordan elimination method.

3.1.1 The Gauss-Jordan Elimination Method

Generally, the Gauss-Jordan elimination method on an augmented matrix

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

follows a particular recipe:

1. If $a_1 = 0$, swap R_1 with another row below it where $a_j \neq 0$.

2. With $a_1 \neq 0$, perhaps after relabeling, pivot the matrix around a_1. This results in a new augmented matrix of the form
$$\left[\begin{array}{ccc|c} 1 & b_1^* & c_1^* & d_1^* \\ 0 & b_2^* & c_2^* & d_2^* \\ 0 & b_3^* & c_3^* & d_3^* \end{array}\right]$$

3. If $b_2^* = 0$, swap R_2 with a row below it where $b_j^* \neq 0$. Then, up to relabeling, pivot the matrix about b_2^* which results in a matrix of the form
$$\left[\begin{array}{ccc|c} 1 & 0 & c_1^{**} & d_1^{**} \\ 0 & 1 & c_2^{**} & d_2^{**} \\ 0 & 0 & c_3^{**} & d_3^{**} \end{array}\right]$$

4. As long as $c_3^{**} \neq 0$, pivot the matrix about c_3^{**} which results in a matrix of the form
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & d_1^{***} \\ 0 & 1 & 0 & d_2^{***} \\ 0 & 0 & 1 & d_3^{***} \end{array}\right]$$

If we successfully end with a matrix of the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

then the solution to the original system is $x = a$, $y = b$, and $z = c$.

Note. Generally, if we end up with a matrix that looks like

$$\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ 0 & 0 & 0 & 0 \end{array}\right]$$

during the course of applying the Gauss-Jordan elimination method, then the original system has **infinitely many solutions**. In this scenario, we can describe the solution set as is done in Example 3.10.

Note. Generally, if we end up with a matrix that looks like

$$\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ 0 & 0 & 0 & t \end{array}\right]$$

where $t \neq 0$ during the course of applying the Gauss-Jordan elimination method, then the original system has **no solutions**.

Example 3.9. Solve the system

$$\begin{cases} 4x - 2y + 3z = 4 \\ 8x - 3y + 5z = 7 \\ 7x - 2y + 4z = 6 \end{cases}$$

using the Gauss-Jordan elimination method and augmented matrices.

Recall that the goal of the Gauss-Jordan elimination method is to apply elementary row operations in a formulaic way to arrive at a diagonal matrix. Let's see how it's done:

$$\left[\begin{array}{ccc|c} 4 & -2 & 3 & 4 \\ 8 & -3 & 5 & 7 \\ 7 & -2 & 4 & 6 \end{array}\right] \xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{ccc|c} 1 & -1/2 & 3/4 & 1 \\ 8 & -3 & 5 & 7 \\ 7 & -2 & 4 & 6 \end{array}\right]$$

$$\xrightarrow{R_2-8R_1} \left[\begin{array}{ccc|c} 1 & -1/2 & 3/4 & 1 \\ 0 & 1 & -1 & -1 \\ 7 & -2 & 4 & 6 \end{array}\right]$$

$$\xrightarrow{R_3-7R_1} \left[\begin{array}{ccc|c} 1 & -1/2 & 3/4 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 3/2 & -5/4 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 3/4 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 3/2 & -5/4 & -1 \end{array}\right] \xrightarrow{R_1+\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1/4 & 1/2 \\ 0 & 1 & -1 & -1 \\ 0 & 3/2 & -5/4 & -1 \end{array}\right]$$

$$\xrightarrow{R_3-\frac{3}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1/4 & 1/2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1/4 & 1/2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/4 & 1/2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1/4 & 1/2 \end{array}\right] \xrightarrow{4R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1/4 & 1/2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_1-\frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

This completes the Gauss-Jordan elimination method. If we rewrite the final augmented matrix as a system of equations, we have

$$\begin{cases} x & & & = & 0 \\ & y & & = & 1 \\ & & z & = & 2 \end{cases}$$

and we see that our solution is $x = 0, y = 1, z = 2$. //

Example 3.10. Solve

$$\begin{cases} x & - & 3y & + & z & = & 5 \\ -2x & + & 7y & - & 6z & = & -9 \\ x & - & 2y & - & 3z & = & 6 \end{cases}$$

Using the Gauss-Jordan elimination method,

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ -2 & 7 & -6 & -9 \\ 1 & -2 & -3 & 6 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 1 & -4 & 1 \\ 1 & -2 & -3 & 6 \end{array}\right]$$

$$\xrightarrow{R_3-R_1} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 1 & -4 & 1 \\ 0 & 1 & -4 & 1 \end{array}\right]$$

$$\xrightarrow{R_1+3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -11 & 8 \\ 0 & 1 & -4 & 1 \\ 0 & 1 & -4 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccr|c} 1 & 0 & -11 & 8 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since $0 = 0$, there are <u>infinitely many solutions</u>. To see what those solutions are, we can rewrite the final matrix as a system of two equations:

$$\begin{cases} x & - & 11z & = & 8 \\ y & - & 4z & = & 1 \end{cases}$$

which gives us the solution set

$$x = 11z + 8, \qquad y = 4z + 1, \qquad z = \text{any real number.}$$ //

Example 3.11. Solve

$$\begin{cases} x & + & 2y & + & 3z & = & 1 \\ 4x & + & 5y & + & 6z & = & -9 \\ x & + & 2y & + & 3z & = & 6 \end{cases}$$

By the Gauss-Jordan elimination method,

$$\left[\begin{array}{ccc|r} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & -9 \\ 1 & 2 & 3 & 6 \end{array}\right] \xrightarrow{R_2 - 4R_1} \left[\begin{array}{crr|r} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -13 \\ 1 & 2 & 3 & 6 \end{array}\right]$$

$$\xrightarrow{R_3 - R_1} \left[\begin{array}{crr|r} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -13 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

but $0 \neq 5$. Therefore, this system has <u>no solutions</u>. //

Finally, we will see an example where interchanging rows is relevant.

Example 3.12. Solve

$$\begin{cases} & & 3y & + & 3z & = & 6 \\ x & - & 5y & + & z & = & 1 \\ & & & & 3z & = & 9 \end{cases}$$

By the Gauss-Jordan elimination method,

$$\left[\begin{array}{crc|c} 0 & 3 & 3 & 6 \\ 1 & -5 & 1 & 1 \\ 0 & 0 & 3 & 9 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{crc|c} 1 & -5 & 1 & 1 \\ 0 & 3 & 3 & 6 \\ 0 & 0 & 3 & 9 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3} \cdot R_2} \left[\begin{array}{crc|c} 1 & -5 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right]$$

$$\xrightarrow{R_1+5R_2} \left[\begin{array}{ccc|c} 1 & 0 & 6 & 11 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}\cdot R_3} \left[\begin{array}{ccc|c} 1 & 0 & 6 & 11 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_1-6R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_2-R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

so the solution is $x = -7$, $y = -1$, $z = 3$. //

3.2 Arithmetic with Matrices

Definition 3.3. A **matrix** is a rectangular array of numbers. The **size** (or **dimensions**) of a matrix are written in row-column format; i.e., an $n \times m$ matrix has n rows and m columns.

Example 3.13. Here we list some examples of matrices.

(a) A 2×2 matrix:

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

(b) A 2×3 matrix:

$$\begin{bmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix}$$

(c) A 3×2 matrix:

$$\begin{bmatrix} 0 & 2 \\ 1 & 3 \\ 4 & 5 \end{bmatrix}$$

(d) A 3×4 matrix:

$$\begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(e) A 1×1 matrix:

$$\begin{bmatrix} 8 \end{bmatrix}$$

(f) A 4×1 matrix:

$$\begin{bmatrix} 5 \\ 10 \\ 15 \\ 20 \end{bmatrix}$$

(g) A 1×3 matrix:

$$\begin{bmatrix} 3 & 6 & 12 \end{bmatrix}$$

Definition 3.4. A **square** matrix is any matrix with the same number of rows as columns; e.g. Example 3.13 (a) and (e). A **column** matrix is any matrix with only one column; e.g. Example 3.13 (f). A **row** matrix is any matrix with only one row; e.g. Example 3.13 (g).

Notation. For an $n \times m$ matrix A, we refer to the entry in row j, column k as a_{jk}.

Example 3.14. For

$$A = \begin{bmatrix} 12 & 2 & 10 & -7 \\ 9 & 7 & 5 & 3 \\ 0 & 1 & -3 & -2 \end{bmatrix}$$

$a_{11} = 12$, $a_{21} = 9$, $a_{32} = 1$, and $a_{23} = 5$. For the complete list,

$$\begin{array}{llll} a_{11} = 12, & a_{12} = 2, & a_{13} = 10, & a_{14} = -7, \\ a_{21} = 9, & a_{22} = 7, & a_{23} = 5, & a_{24} = 3, \\ a_{31} = 0, & a_{32} = 1, & a_{33} = -3, & a_{34} = -2. \end{array}$$

Definition 3.5. We say that two matrices A and B are **equal** if they have the same dimensions and the same entries.

Example 3.15. Given that

$$\begin{bmatrix} x & 2 \\ 3 & x+y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

find x and y.

Immediately, $x = 1$. Then $1 + y = 4$ which means $y = 3$. //

3.2.1 Matrix Addition and Subtraction

Note. To add or subtract matrices, they must be of the same dimensions.

So, to add an $n \times m$ matrix A to another $n \times m$ matrix B, we add the entries to each other; that is, for $C = A + B$, $c_{jk} = a_{jk} + b_{jk}$.

Example 3.16.

$$\begin{bmatrix} 12 & 2 & 10 & -7 \\ 9 & 7 & 5 & 3 \\ 0 & 1 & -3 & -2 \end{bmatrix} + \begin{bmatrix} -3 & 3 & 4 & 7 \\ 2 & 10 & -7 & 3 \\ 3 & -2 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 5 & 14 & 0 \\ 11 & 17 & -2 & 6 \\ 3 & -1 & 2 & 1 \end{bmatrix}$$

Similarly, to subtract an $n \times m$ matrix B from another $n \times m$ matrix A, we subtract the entries from each other; that is, for $C = A - B$, $c_{jk} = a_{jk} - b_{jk}$.

Example 3.17.

$$\begin{bmatrix} 12 & 2 & 10 & -7 \\ 9 & 7 & 5 & 3 \\ 0 & 1 & -3 & -2 \end{bmatrix} - \begin{bmatrix} -3 & 3 & 4 & 7 \\ 2 & 10 & -7 & 3 \\ 3 & -2 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 15 & -1 & 6 & -14 \\ 7 & -3 & 12 & 0 \\ -3 & 3 & -8 & -5 \end{bmatrix}$$

3.2.2 Scalar Multiplication

We can multiply any real number to a matrix. That is, for an $n \times m$ matrix A and number b, the product bA is an $n \times m$ matrix obtained by multiplying each entry in A by b.

Example 3.18.

$$3 \cdot \begin{bmatrix} 12 & 2 & 10 & -7 \\ 9 & 7 & 5 & 3 \\ 0 & 1 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 36 & 6 & 30 & -21 \\ 27 & 21 & 15 & 9 \\ 0 & 3 & -9 & -6 \end{bmatrix}$$

3.2.3 Matrix Multiplication

One may suspect that we multiply similarly sized matrices entry by entry, but this isn't how we define matrix multiplication.

Note. Let A be an $n \times m$ matrix and B be a $\ell \times k$ matrix. Then we can form the product matrix $A \cdot B$ only if $m = \ell$.

Definition 3.6. Now, given an $n \times m$ matrix A and an $m \times \ell$ matrix B, the resulting product matrix $A \times B = C$ is an $n \times \ell$ matrix where the entries are determined by

$$c_{jk} = a_{j1}b_{1k} + a_{j2}b_{2k} + a_{j3}b_{3k} + \cdots + a_{jm}b_{mk}$$

for $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant \ell$. Notice here that we use the j^{th} row of A and the k^{th} column of B for our calculation.

Example 3.19. Work out the matrix product $A \cdot B$ where

$$A = \begin{bmatrix} 12 & 2 & 10 & -7 \\ 9 & 7 & 5 & 3 \\ 0 & 1 & -3 & -2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ -1 & 5 \\ 0 & 1 \end{bmatrix}$$

entry by entry.

Since A is a 3×4 and B is a 4×2, the resulting product matrix $A \cdot B$ will be a 3×2.

- To get the entry in row 1, column 1 of $A \cdot B$, we use row 1 of A and column 1 of B:

$$\begin{bmatrix} \color{red}{12} & \color{red}{2} & \color{red}{10} & \color{red}{-7} \\ 9 & 7 & 5 & 3 \\ 0 & 1 & -3 & -2 \end{bmatrix} \cdot \begin{bmatrix} \color{blue}{1} & 0 \\ \color{blue}{3} & 2 \\ \color{blue}{-1} & 5 \\ \color{blue}{0} & 1 \end{bmatrix}$$

$$\color{red}{12} \cdot \color{blue}{1} + \color{red}{2} \cdot \color{blue}{3} + \color{red}{10} \cdot (\color{blue}{-1}) + (\color{red}{-7}) \cdot \color{blue}{0} = 8.$$

So

$$A \cdot B = \begin{bmatrix} 8 & ? \\ ? & ? \\ ? & ? \end{bmatrix}$$

- To get the entry in row 1, column 2 of $A \cdot B$, we use row 1 of A and column 2 of B:

$$\begin{bmatrix} 12 & 2 & 10 & -7 \\ 9 & 7 & 5 & 3 \\ 0 & 1 & -3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ -1 & 5 \\ 0 & 1 \end{bmatrix}$$

$$12 \cdot 0 + 2 \cdot 2 + 10 \cdot 5 + (-7) \cdot 1 = 47.$$

So

$$A \cdot B = \begin{bmatrix} 8 & 47 \\ ? & ? \\ ? & ? \end{bmatrix}$$

- To get the entry in row 2, column 1 of $A \cdot B$, we use row 2 of A and column 1 of B:

$$\begin{bmatrix} 12 & 2 & 10 & -7 \\ 9 & 7 & 5 & 3 \\ 0 & 1 & -3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ -1 & 5 \\ 0 & 1 \end{bmatrix}$$

$$9 \cdot 1 + 7 \cdot 3 + 5 \cdot (-1) + 3 \cdot 0 = 25.$$

So

$$A \cdot B = \begin{bmatrix} 8 & 47 \\ 25 & ? \\ ? & ? \end{bmatrix}$$

- To get the entry in row 2, column 2 of $A \cdot B$, we use row 2 of A and column 2 of B:

$$\begin{bmatrix} 12 & 2 & 10 & -7 \\ 9 & 7 & 5 & 3 \\ 0 & 1 & -3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ -1 & 5 \\ 0 & 1 \end{bmatrix}$$

$$9 \cdot 0 + 7 \cdot 2 + 5 \cdot 5 + 3 \cdot 1 = 42.$$

So

$$A \cdot B = \begin{bmatrix} 8 & 47 \\ 25 & 42 \\ ? & ? \end{bmatrix}$$

- To get the entry in row 3, column 1 of $A \cdot B$, we use row 3 of A and column 1 of B:

$$\begin{bmatrix} 12 & 2 & 10 & -7 \\ 9 & 7 & 5 & 3 \\ 0 & 1 & -3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ -1 & 5 \\ 0 & 1 \end{bmatrix}$$

$$0 \cdot 1 + 1 \cdot 3 + (-3)(-1) + (-2) \cdot 0 = 6.$$

So

$$A \cdot B = \begin{bmatrix} 8 & 47 \\ 25 & 42 \\ 6 & ? \end{bmatrix}$$

- To get the entry in row 3, column 2 of $A \cdot B$, we use row 3 of A and column 2 of B:

$$\begin{bmatrix} 12 & 2 & 10 & -7 \\ 9 & 7 & 5 & 3 \\ 0 & 1 & -3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ -1 & 5 \\ 0 & 1 \end{bmatrix}$$

$$0 \cdot 0 + 1 \cdot 2 + (-3) \cdot 5 + (-2) \cdot 1 = -15.$$

So

$$A \cdot B = \begin{bmatrix} 8 & 47 \\ 25 & 42 \\ 6 & -15 \end{bmatrix}$$

3.3 Inverse Matrices

With addition, subtraction, and multiplication defined for particular matrix pairs, it may seem natural to have a kind of division. Here, we will develop the notion of inverse matrices by first discovering *identity* matrices.

Definition 3.7. The **identity** matrix I_n is the $n \times n$ matrix with ones along the diagonal and zeroes elsewhere.

Example 3.20. Here are some identity matrices.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Fact. For any square $n \times n$ matrix A,

$$I_n \cdot A = A \cdot I_n = A.$$

If B is an $n \times m$ matrix, then

$$I_n \cdot B = B$$

and if C is an $m \times n$ matrix, then

$$C \cdot I_n = C.$$

Remark. The identity matrix I_n in the context of $n \times n$ matrices plays the same role as 1 does with regular number multiplication. Namely, for any number a, $a \cdot 1 = 1 \cdot a = a$.

For a non-zero real number a, we can write $a^{-1} = \dfrac{1}{a}$ and notice that

$$a \cdot a^{-1} = a^{-1} \cdot a = \frac{a}{a} = 1.$$

This motivates the notion of *inverse* matrix.

Definition 3.8. Given a square $n \times n$ matrix A, we say that the $n \times n$ matrix A^{-1}, if it exists, is the **inverse matrix** of A if

$$A \cdot A^{-1} = A^{-1} \cdot A = I_n.$$

Example 3.21. Verify that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is its own inverse.

Notice that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which establishes the inverse property. //

Example 3.22. Verify that

$$\begin{bmatrix} 2 & -1 \\ 7 & -4 \end{bmatrix}$$

and

$$\begin{bmatrix} 4 & -1 \\ 7 & -2 \end{bmatrix}$$

are inverses of each other.

Note that both

$$\begin{bmatrix} 2 & -1 \\ 7 & -4 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 \\ 7 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 4 & -1 \\ 7 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ 7 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which establishes the inverse property. //

3.3.1 The Gauss-Jordan Method for Finding Inverses

Given a square matrix, how might one determine if the inverse matrix exists? Moreover, if it exists, how might one find it? The Gauss-Jordan elimination method on a particular augmented matrix does just that.

To find the inverse of a square matrix A, complete the Gauss-Jordan elimination method on the augmented matrix

$$\left[\begin{array}{c|c} A & I \end{array}\right]$$

where I is the identity matrix corresponding to A. Then there are two cases.

- If the Gauss-Jordan method ends in diagonal form, i.e., we reach an augmented matrix of the form
$$\left[\begin{array}{c|c} I & B \end{array}\right]$$
then $B = A^{-1}$.

- If the Gauss-Jordan method does not result in diagonal form, the matrix A does not have an inverse.

Example 3.23. Determine whether or not the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 4 \\ 1 & 3 & 4 \end{bmatrix}$$

has an inverse. If it has an inverse, write the inverse.

We use the Gauss-Jordan method:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}\cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1 & -1/2 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 + 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -5/2 & 3 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1 & -1/2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -5/2 & 3 \\ 0 & 1 & 0 & 2 & 3/2 & -2 \\ 0 & 0 & 1 & -1 & -1/2 & 1 \end{array}\right]$$

So the inverse matrix is

$$A^{-1} = \begin{bmatrix} -2 & -5/2 & 3 \\ 2 & 3/2 & -2 \\ -1 & -1/2 & 1 \end{bmatrix}$$

Example 3.24. Determine whether or not the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 2 & 3 & 6 \end{bmatrix}$$

has an inverse. If it has an inverse, write the inverse.

We use the Gauss-Jordan method:

$$\begin{aligned}
\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 5 & 0 & 1 & 0 \\ 2 & 3 & 6 & 0 & 0 & 1 \end{array}\right]
&\xrightarrow{R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & -1 & 1 & 0 \\ 2 & 3 & 6 & 0 & 0 & 1 \end{array}\right] \\
&\xrightarrow{R_3-2R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & -1 & 1 & 0 \\ 0 & 1 & 4 & -2 & 0 & 1 \end{array}\right] \\
&\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 2 & -1 & 0 \\ 0 & 1 & 4 & -1 & 1 & 0 \\ 0 & 1 & 4 & -2 & 0 & 1 \end{array}\right] \\
&\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 2 & -1 & 0 \\ 0 & 1 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]
\end{aligned}$$

Since the Gauss-Jordan method does not produce a diagonal matrix, A does not have an inverse matrix. ///

3.3.2 Using the Inverse Matrix to Solve Systems

We can use inverse matrices to solve systems of linear equations. First, note that a system

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

can be rewritten as a matrix equation. Specifically, let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and

$$D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix},$$

We will call A the **coefficient matrix**. Then notice that the matrix equation

$$AX = D$$

corresponds to the original system of equations.

If we are given the inverse matrix A^{-1}, then

$$\begin{aligned} AX &= D \\ A^{-1}AX &= A^{-1}D \\ IX &= A^{-1}D \\ X &= A^{-1}D \end{aligned}$$

That is, the solution is given by $X = A^{-1}D$.

Example 3.25. Given that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 5 & -2 & -9 \\ -2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

solve the system

$$\begin{cases} x + 2y + 3z = 1 \\ 2x + 5y + 3z = 5 \\ z = 2 \end{cases}$$

of linear equations.

First, rewrite the system as a matrix equation:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$$

Then multiply on the left by the inverse matrix to obtain

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 & -2 & -9 \\ -2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -23 \\ 9 \\ 2 \end{bmatrix}$$

So the solution is $x = -23$, $y = 9$, $z = 2$. //

3.4 Markov Processes

Definition 3.9. A **Markov process** is a sequence of experiments with a consistent set of outcomes in which the result of the next experiment depends only on the current result. The consistent outcomes are referred to as **states** and the current outcome of the experiment is called the **current state**.

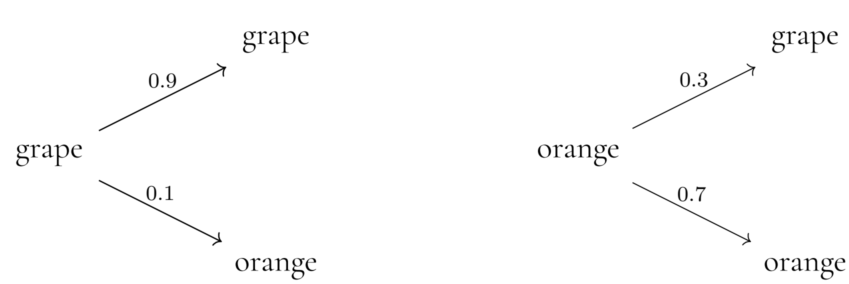
Example 3.26. In a small community, everyone gets together to drink flavored drinks at 8pm and they can choose either grape drink or orange drink. If someone drinks grape drink one evening, they have a 90% probability of choosing grape again the next evening. On the other hand, if someone drinks orange drink one evening, they have a 70% probability of choosing orange again the next evening. This is a Markov process where we have two states: grape and orange. //

In a Markov process, the transition between states can be organized using a tree diagram, a transition diagram, or a transition matrix.

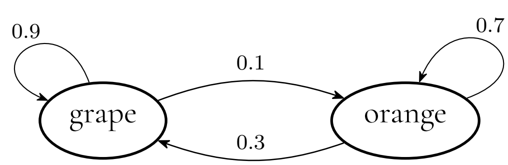
Example 3.27. In reference to Example 3.26, determine all of the relevant probabilities and write the corresponding tree diagram, transition diagram, and transition matrix.

Since 90% of those that choose grape one evening will choose grape the next, 10% of them will choose orange the next day. Similarly, 30% of those that choose orange one evening will choose grape the next.

Tree Diagram



Transition Diagram



Transition Matrix

In this context, let's abbreviate grape with G and orange with R.

$$\begin{array}{cc|c} & & \text{current} \\ & & \begin{matrix} G & R \end{matrix} \\ \hline \text{next} & \begin{matrix} G \\ R \end{matrix} & \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix} \end{array}$$

The ordering in the transition matrix is important. Notice that, if someone is currently at G, then there is a 0.1 probability that they switch to orange in the next state. This is indicated by the "current" and "next" labeling.

Definition 3.10. A square matrix A is said to be a **stochastic** matrix if all of its entries are non-negative and each column sums to 1.

Note. All transition matrices are stochastic.

Definition 3.11. Given an $n \times n$ stochastic matrix A which serves as the transition matrix for some Markov process, a **distribution** matrix is a column matrix with n rows so that the entries are all non-negative and they sum to 1. The distribution matrix is used to describe the outcomes of the experiment. Moreover, Given a distribution matrix X representing the current state, the next state is given by AX.

Notation. We will usually denote an *initial* distribution matrix by using a subscript of zero; e.g. X_0.

Example 3.28. With reference to Example 3.26, assuming that 50% of the community chose grape drink today, find the percentage of those choosing grape and orange drink tomorrow, and the next day.

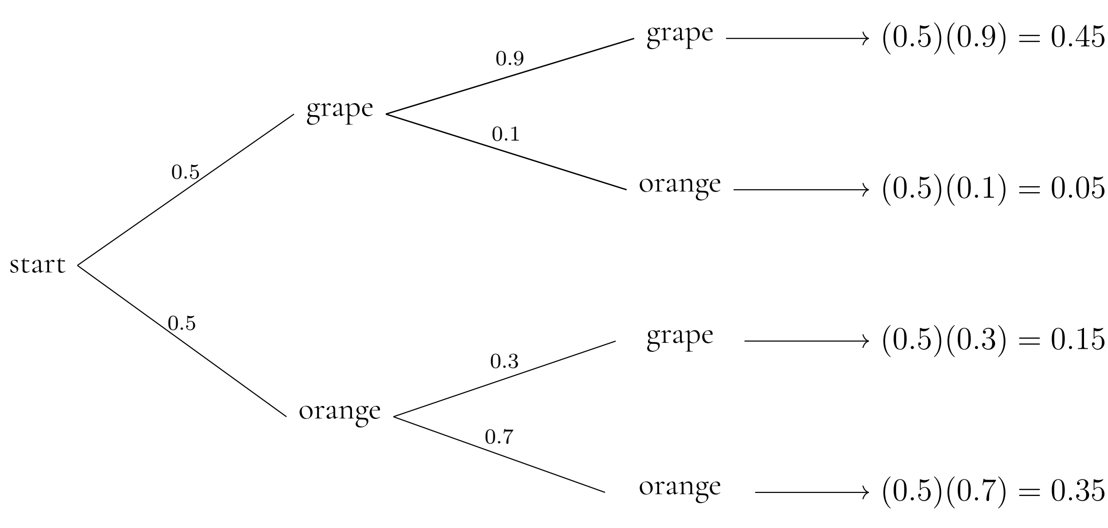
First, since we have two states, we write the initial distribution matrix as follows:

$$\begin{array}{c} G \\ R \end{array} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}_0$$

Tomorrow's distribution is given by

$$\begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}_1$$

That is, 60% will be drinking grape drink and 40% will be drinking orange drink. To see this in the context of a tree diagram, consider using the initial distribution along with the transition information:



- Adding the probabilities associated with grape, we have $0.45 + 0.15 = 0.6$.

- Adding the probabilities associated with orange, we have $0.05 + 0.35 = 0.4$.

The day after tomorrow's distribution is given by

$$\begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}_1 = \begin{bmatrix} 0.66 \\ 0.34 \end{bmatrix}_2$$

That is, 66% will be drinking grape drink and 34% will be drinking orange drink in two days. //

In general, given a transition matrix A and an initial distribution matrix X_0, we have

$$X_1 = AX_0$$

and

$$X_2 = AX_1 = A(AX_0) = A^2X_0.$$

Continuing in this way, we see that the n^{th} state is given by

$$X_n = A^nX_0.$$

Example 3.29. With reference to Example 3.28, find the distribution in a week and in 30 days.

To find the distribution a week from now, we compute

$$\begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}^{30} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}_0 = \begin{bmatrix} 0.743002 \\ 0.256998 \end{bmatrix}_7$$

To find the distribution in 30 days, we compute

$$\begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}^{30} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}_0 = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}_{30}$$

Example 3.30. For the transition matrix

$$A = \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}$$

consider three initial distributions:

$$X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_0 \qquad Y_0 = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}_0 \qquad Z_0 = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}_0$$

Find the 50^{th} generation for all three distributions.

Observe:

$$\begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}^{50} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_0 = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}_{50}$$

$$\begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}^{50} \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}_0 = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}_{50}$$

$$\begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}^{50} \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}_0 = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}_{50}$$

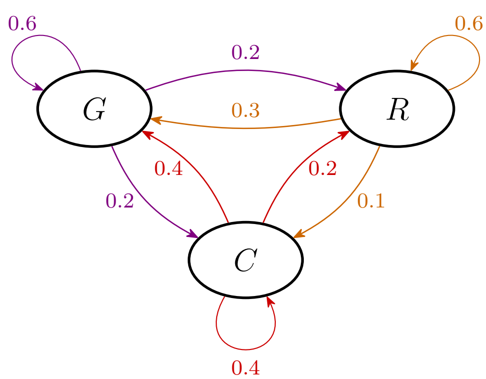
In fact, with reference to Example 3.30, notice that

$$\begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}_0 = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}_1$$

which is $Z_1 = AZ_0 = Z_0$. We will explore this kind of equality in more depth in Section 3.5.

Example 3.31. In a particular community, everyone gets together to drink flavored drinks at 5pm every day and they choose from grape, orange or cherry drink. Those who choose grape one day have a 20% probability of choosing orange the next day and a 20% probability of choosing cherry the next day. Those who choose orange one day have a 30% probability of choosing grape the next day and a 10% probability of choosing cherry the next day. Those who choose cherry one day have a 40% probability of choosing grape the next day and a 20% probability of choosing orange the next day. Draw the transition diagram and write the transition matrix for this Markov process. Then, given that 5%, 75%, and 20% of the community drank grape, orange, and cherry drink, respectively, find what the distribution will be in two weeks.

Let G stand for grape, R for orange, and C for cherry. The transition diagram is



The transition matrix is given by

$$\begin{array}{c|c} & \begin{array}{ccc} & \text{current} & \\ \hline G & R & C \end{array} \\ \text{next} \begin{array}{c} G \\ R \\ C \end{array} & \begin{bmatrix} 0.6 & 0.3 & 0.4 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.1 & 0.4 \end{bmatrix} \end{array}$$

Our initial distribution matrix is given by

$$\begin{array}{c} G \\ R \\ C \end{array} \begin{bmatrix} 0.05 \\ 0.75 \\ 0.2 \end{bmatrix}_0$$

Finally, the distribution in two weeks is given by

$$\begin{bmatrix} 0.6 & 0.3 & 0.4 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.1 & 0.4 \end{bmatrix}^{14} \begin{bmatrix} 0.05 \\ 0.75 \\ 0.2 \end{bmatrix}_0 = \begin{bmatrix} 0.458333 \\ 0.333334 \\ 0.208333 \end{bmatrix}_{14}$$

3.5 Regular Matrices and Stability

As suggested by Example 3.30, some transition matrices tend towards a unique distribution in the long run. In this section, we will investigate this long-term behavior.

Definition 3.12. For a stochastic matrix A, if A^n approaches some matrix B, then B is called the **stable matrix** for A. Given an initial distribution matrix X_0, if $X_n = A^n X_0$ approaches some matrix Y, then Y is called the **stable distribution** of A.

Definition 3.13. A stochastic matrix A is said to be **regular** if some power A^n has all positive entries.

Example 3.32. The matrix

$$\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$$

is a regular stochastic matrix. //

Example 3.33. Show that

$$A = \begin{bmatrix} 0 & 0.4 \\ 1 & 0.6 \end{bmatrix}$$

is a regular stochastic matrix.

Note that

$$A^2 = \begin{bmatrix} 0 & 0.4 \\ 1 & 0.6 \end{bmatrix}^2 = \begin{bmatrix} 0.4 & 0.24 \\ 0.6 & 0.76 \end{bmatrix}$$

which establishes that A is a regular stochastic matrix. //

Example 3.34. Show that

$$A = \begin{bmatrix} 1 & 0.3 \\ 0 & 0.7 \end{bmatrix}$$

is a stochastic matrix which is not regular.

First, note that

$$A^2 = \begin{bmatrix} 1 & 0.51 \\ 0 & 0.49 \end{bmatrix}$$

and

$$A^3 = \begin{bmatrix} 1 & 0.657 \\ 0 & 0.343 \end{bmatrix}$$

If

$$A^n = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$$

then

$$A^{n+1} = A^n \cdot A = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} 1 & 0.3 \\ 0 & 0.7 \end{bmatrix} = \begin{bmatrix} 1 & 0.3 + a(0.7) \\ 0 & b(0.7) \end{bmatrix}$$

Since this holds for any power n, A is not regular. //

Properties of regular stochastic matrices. Let A be a regular stochastic matrix. Then

1. A has a stable matrix.
2. For any initial distribution X_0, $A^n \cdot X_0$ approaches the stable distribution of A.
3. All columns of the stable matrix are the same and are equal to the stable distribution.
4. The stable distribution X of A can be found by solving the system of linear equations

$$\begin{cases} \text{sum of the entries of } X = 1 \\ AX = X \end{cases}$$

Example 3.35. For

$$A = \begin{bmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{bmatrix}$$

find the stable matrix and stable distribution.

Let

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

and notice that

$$AX = \begin{bmatrix} 0.9x + 0.4y \\ 0.1x + 0.6y \end{bmatrix}$$

Then consider the system

$$\begin{cases} x & + & y & = & 1 \\ 0.9x & + & 0.4y & = & x \\ 0.1x & + & 0.6y & = & y \end{cases}$$

We must put our variables together so we rewrite the second equation:

$$\begin{array}{rcrcr} 0.9x & + & 0.4y & = & x \\ -x & & & & -x \\ \hline -0.1x & + & 0.4y & = & 0 \end{array}$$

and the third equation:

$$\begin{array}{rcrcr} 0.1x & + & 0.6y & = & y \\ & & -y & & -y \\ \hline 0.1x & - & 0.4y & = & 0 \end{array}$$

Notice that both of the rewritten equations only differ by a multiplicative constant. So we really only need to solve

$$\begin{cases} x & + & y & = & 1 \\ 0.1x & - & 0.4y & = & 0 \end{cases}$$

Using the Gauss-Jordan elimination method, we obtain $x = 0.8$ and $y = 0.2$. Therefore, the stable matrix of A is

$$\begin{bmatrix} 0.8 & 0.8 \\ 0.2 & 0.2 \end{bmatrix}$$

and the stable distribution of A is

$$\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

Example 3.36. For

$$A = \begin{bmatrix} 0.7 & 0.1 & 0.25 \\ 0.2 & 0.8 & 0.55 \\ 0.1 & 0.1 & 0.2 \end{bmatrix}$$

find the stable matrix and stable distribution.

Let

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and consider the system

$$\begin{cases} x & + & y & + & z & = & 1 \\ 0.7x & + & 0.1y & + & 0.25z & = & x \\ 0.2x & + & 0.8y & + & 0.55z & = & y \\ 0.1x & + & 0.1y & + & 0.2z & = & z \end{cases}$$

which can be rewritten as

$$\begin{cases} x & + & y & + & z & = & 1 \\ -0.3x & + & 0.1y & + & 0.25z & = & 0 \\ 0.2x & - & 0.2y & + & 0.55z & = & 0 \\ 0.1x & + & 0.1y & - & 0.8z & = & 0 \end{cases}$$

By using the Gauss-Jordan elimination method, we arrive at the reduced system

$$\begin{cases} x & & & = & 7/24 \\ & y & & = & 43/72 \\ & & z & = & 1/9 \\ & & 0 & = & 0 \end{cases}$$

which provides

$$\begin{bmatrix} 7/24 & 7/24 & 7/24 \\ 43/72 & 43/72 & 43/72 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

as the stable matrix for A and

$$\begin{bmatrix} 7/24 \\ 43/72 \\ 1/9 \end{bmatrix}$$

as the stable distribution for A. //

4 Linear Programming

Linear programming is used to solve certain kinds of optimization problems. In particular, a **linear programming problem** consists of

1. a (linear) objective function to be maximized or minimized and
2. a system of inequalities (where each inequality is in terms of a linear function).

The inequalities are known as the **constraints**. Then the goal is to find the largest/smallest value of the objective function subject to the constraints.

Definition 4.1. In the context of linear programming problems, the set of points which satisfy all of the given inequalities is called the **feasible set**.

Example 4.1. Find the feasible set for the system of inequalities given below:

$$\begin{cases} 3x - 2y \leqslant 6 \\ 2x + 3y \leqslant 30 \\ x \geqslant 0, \ y \geqslant 0 \end{cases}$$

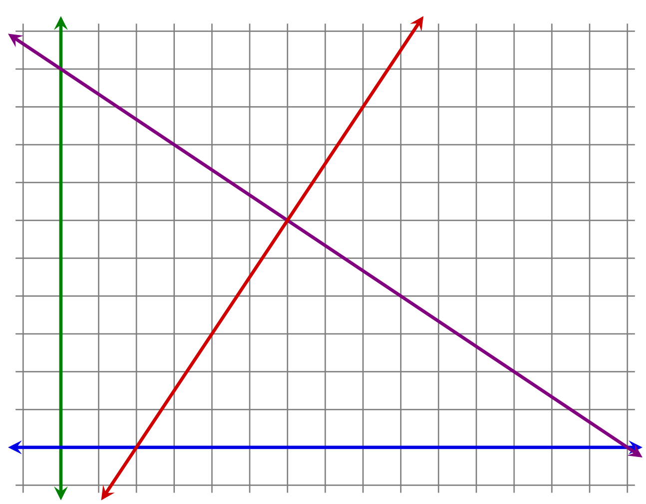
We start by graphing the lines

$$\begin{cases} 3x - 2y = 6 \\ 2x + 3y = 30 \\ x = 0, \ y = 0 \end{cases}$$

We could rewrite the lines in slope-intercept form, or we can simply find the intercepts.

- For $3x - 2y = 6$,
  - ◇ when $x = 0$, we see that $-2y = 6$ which means $y = -3$. This gives us the point $(0, -3)$.
  - ◇ when $y = 0$, we see that $3x = 6$ which means $x = 2$. This gives us the point $(2, 0)$.
- For $2x + 3y = 30$,
  - ◇ when $x = 0$, we see that $3y = 30$ which means $y = 10$. This gives us the point $(0, 10)$.
  - ◇ when $y = 0$, we see that $2x = 30$ which means $x = 15$. This gives us the point $(15, 0)$.

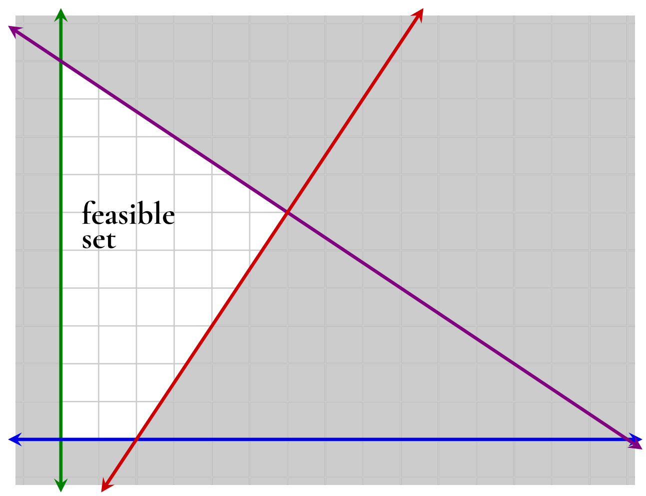
Now we graph.



By comparing the regions against the inequalities

$$\begin{cases} 3x - 2y \leqslant 6 \\ 2x + 3y \leqslant 30 \\ x \geqslant 0,\ y \geqslant 0 \end{cases} \xrightarrow[\text{form}]{\text{slope-intercept}} \begin{cases} y \geqslant \frac{3}{2} \cdot x - 6 \\ y \leqslant -\frac{2}{3} \cdot x + 10 \\ x \geqslant 0,\ y \geqslant 0 \end{cases}$$

we obtain



Note that we shade the regions that don't satisfy the inequalities to make the feasible set more clearly identifiable. //

To elaborate more on the feasible set, consider the point $(1, 3)$ which is in the feasible set in Example 4.1. Immediately, we see that $x = 1 \geqslant 0$ and $y = 3 \geqslant 0$. Also,

$$3x - 2y = 3 \cdot 1 - 2 \cdot 3 = -3 \leqslant 6$$

and

$$2x + 2y = 2 \cdot 1 + 3 \cdot 3 = 11 \leqslant 30.$$

That is, all four of the inequalities are satisfied.

On the other hand, if we consider any point outside of the feasible set, the inequalities won't all be satisfied. For example, consider $(5, 2)$. In this case, $x = 5 \geqslant 0$ and $y = 2 \geqslant 0$ still. Nevertheless,

$$3x - 2y = 3 \cdot 5 - 2 \cdot 2 = 11 > 6.$$

As we will see later, the *corner points* of the feasible set are of critical importance. These corner points are called **vertices**.

Example 4.2. In reference to Example 4.1, find the vertices of the feasible set.

By visual inspection of the graph of the feasible set, we anticipate four vertices. In fact, we can see that the vertices are the intersection points of the following pairs of lines:

$$\begin{cases} x = 0 \text{ and } y = 0 \\ y = 0 \text{ and } 3x - 2y = 6 \\ 3x - 2y = 6 \text{ and } 2x + 3y = 30 \\ x = 0 \text{ and } 2x + 3y = 30 \end{cases}$$

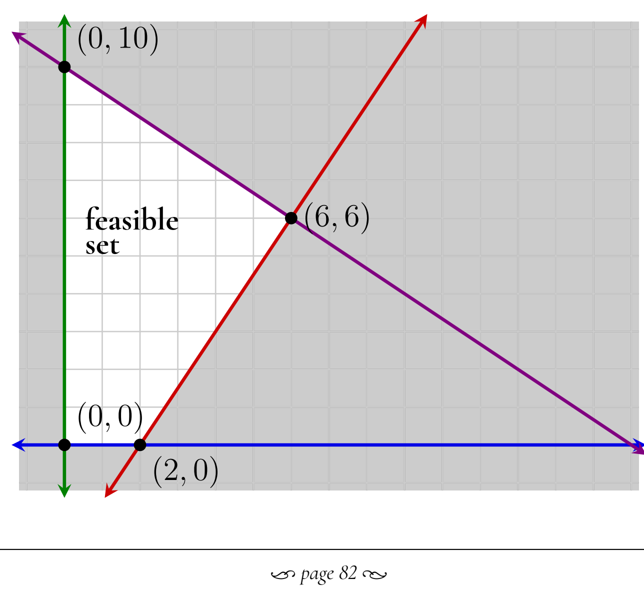
Now we find the intersection of the pairs of lines above.

- The intersection of $x = 0$ and $y = 0$ is the origin, a vertex.
- For For $3x - 2y = 6$ and $y = 0$, we solve
$$\left\{\begin{array}{rcrcr} 3x & - & 2y & = & 6 \\ & & y & = & 0 \end{array}\right. \xrightarrow{\text{rref}} \left\{\begin{array}{rcrcr} x & & & = & 2 \\ & & y & = & 0 \end{array}\right.$$
which gives us the vertex $(2, 0)$.
- For $3x - 2y = 6$ and $2x + 3y = 30$, we solve
$$\left\{\begin{array}{rcrcr} 3x & - & 2y & = & 6 \\ 2x & + & 3y & = & 30 \end{array}\right. \xrightarrow{\text{rref}} \left\{\begin{array}{rcrcr} x & & & = & 6 \\ & & y & = & 6 \end{array}\right.$$
which gives us the vertex $(6, 6)$.
- For $2x + 3y = 30$ and $x = 0$, we solve
$$\left\{\begin{array}{rcrcr} 2x & + & 3y & = & 30 \\ x & & & = & 0 \end{array}\right. \xrightarrow{\text{rref}} \left\{\begin{array}{rcrcr} x & & & = & 0 \\ & & y & = & 10 \end{array}\right.$$
which gives us the vertex $(0, 10)$.

Now, we could have also started finding the intersection points of other pairs of lines relevant to the problem. However, when we compare the resulting points with the feasible set, we see that they are not vertices of the feasible set.

- For $2x + 3y = 30$ and $y = 0$, we solve
$$\left\{\begin{array}{rcrcr} 2x & + & 3y & = & 30 \\ & & y & = & 0 \end{array}\right. \xrightarrow{\text{rref}} \left\{\begin{array}{rcrcr} x & & & = & 15 \\ & & y & = & 0 \end{array}\right.$$
which gives us the point $(15, 0)$. This is not a vertex since it isn't in the feasible set.
- For $3x - 2y = 6$ and $x = 0$, we solve
$$\left\{\begin{array}{rcrcr} 3x & - & 2y & = & 6 \\ x & & & = & 0 \end{array}\right. \xrightarrow{\text{rref}} \left\{\begin{array}{rcrcr} x & & & = & 0 \\ & & y & = & -3 \end{array}\right.$$
which gives us the point $(0, -3)$. This is not a vertex since it isn't in the feasible set.

So, the vertices here are $(0, 0)$, $(2, 0)$, $(6, 6)$, and $(0, 10)$.



Example 4.3. The following is an example of a linear programming problem. We wish to maximize $3x+5y-9$ subject to the constraints

$$\begin{cases} 3x-2y \leqslant 6 \\ 2x+3y \leqslant 30 \\ x \geqslant 0,\ y \geqslant 0 \end{cases}$$

The goal of the problem is to find a point (x, y) of the feasible set for which $3x+5y-9$ achieves a maximal value. At first blush, one may think we have to check all of the points in the feasible set, a daunting task depending on the size of the feasible set. In fact, we only have to check the vertices.

Fundamental Theorem of Linear Programming. The maximum (or minimum) value of the objective function is achieved at one of the vertices of the feasible set.

Definition 4.2. The vertex at which the maximum (or minimum) value of the objective function is obtained is called the **optimal point**. The maximum (or minimum) value itself may be referred to as the **optimal value**.

Example 4.4. Maximize $3x+5y-9$ subject to the constraints

$$\begin{cases} 3x-2y \leqslant 6 \\ 2x+3y \leqslant 30 \\ x \geqslant 0,\ y \geqslant 0 \end{cases}$$

By the Fundamental Theorem of Linear Programming and Example 4.2, we just need to calculate the objective function at each of the vertices:

(x,y)	$3x+5y-9$
$(0,0)$	-9
$(2,0)$	-3
$(6,6)$	39
$(0,10)$	41

Therefore, the optimal point is $(0, 10)$ and the optimal value is 41. //

The optimal point will depend on the objective function even under the same constraints.

Example 4.5. Maximize $7x+2y+1$ subject to the constraints

$$\begin{cases} 3x-2y \leqslant 6 \\ 2x+3y \leqslant 30 \\ x \geqslant 0,\ y \geqslant 0 \end{cases}$$

By the Fundamental Theorem of Linear Programming and Example 4.2, we just need to calculate the objective function at each of the vertices:

(x,y)	$7x+2y+1$
$(0,0)$	1
$(2,0)$	15
$(6,6)$	55
$(0,10)$	21

Therefore, the optimal point is $(6, 6)$ and the optimal value is 55. //

Solving Linear Programming Problems. In general, to solve a linear programming problem, one always follows the following strategy.

1. Identify variables and translate the information given into linear inequalities.
2. Determine the objective function and whether it is to be maximized or minimized.
3. Graph the feasible set and find the vertices.
4. Evaluate the objective function at each vertex to solve the problem.

Example 4.6. In the effort to get more vitamins, Yoshimi will supplement her diet with two vitamin supplements: MaxLife and PureHealth. MaxLife costs 10 cents per tablet and each tablet contains 9 mg of Vitamin C, 70 mg of Vitamin D, 1 mg of Vitamin B_6, and no Vitamin E. PureHealth costs 23 cents per tablet and each tablet contains 9 mg of Vitamin C, 280 mg of Vitamin D, no Vitamin B_6, and 10 mg of Vitamin E. Yoshimi wants at least 72 mg of Vitamin C, 980 mg of Vitamin D, 2 mg of Vitamin B_6, and 10 mg of Vitamin E per day with these supplements. How many tablets of each supplement should she take to minimize her cost?

First, let's organize the data:

	MaxLife	PureHealth
cost	10	23
C	9	9
D	70	280
B_6	1	0
E	0	10

Let x be the number of MaxLife and y be the number of PureHealth tablets she will take. To meet her plan, we need the following inequalities to hold:

$$\begin{cases} 9x + 9y \geqslant 72 \\ 70x + 280y \geqslant 980 \\ x \geqslant 2,\ 10y \geqslant 10 \end{cases}$$

Her cost is given by

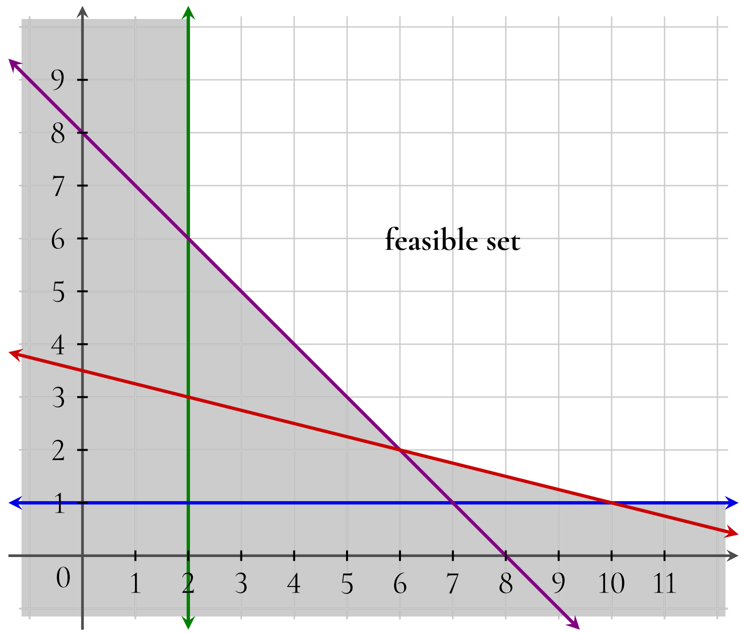
$$10x + 23y$$

which is the objective function which we wish to minimize. We can concisely write the problem as follows:

$$\begin{array}{rl} \text{objective:} & 10x + 23y \\ \text{constraints:} & \begin{cases} 9x + 9y \geqslant 72 \\ 70x + 280y \geqslant 980 \\ x \geqslant 2,\ 10y \geqslant 10 \end{cases} \end{array}$$

Before finding the vertices, let's graph the feasible set. If it's helpful, we can convert the constraints into inequalities in slope-intercept form.

$$\begin{cases} 9x + 9y \geqslant 72 \\ 70x + 280y \geqslant 980 \\ x \geqslant 2, \ 10y \geqslant 10 \end{cases} \longrightarrow \begin{cases} y \geqslant -x + 8 \\ y \geqslant -\frac{x}{4} + \frac{7}{2} \\ x \geqslant 2, \ y \geqslant 1 \end{cases}$$



Now, we can find the vertices with the help of the graph.

- For $9x + 9y = 72$ and $70x + 280y = 980$, we solve

$$\begin{cases} 9x + 9y = 72 \\ 70x + 280y = 980 \end{cases} \xrightarrow{\text{rref}} \begin{cases} x = 6 \\ y = 2 \end{cases}$$

  which gives us the point $(6, 2)$.

- For $9x + 9y = 72$ and $x = 2$, we solve

$$\begin{cases} 9x + 9y = 72 \\ x = 2 \end{cases} \xrightarrow{\text{rref}} \begin{cases} x = 2 \\ y = 6 \end{cases}$$

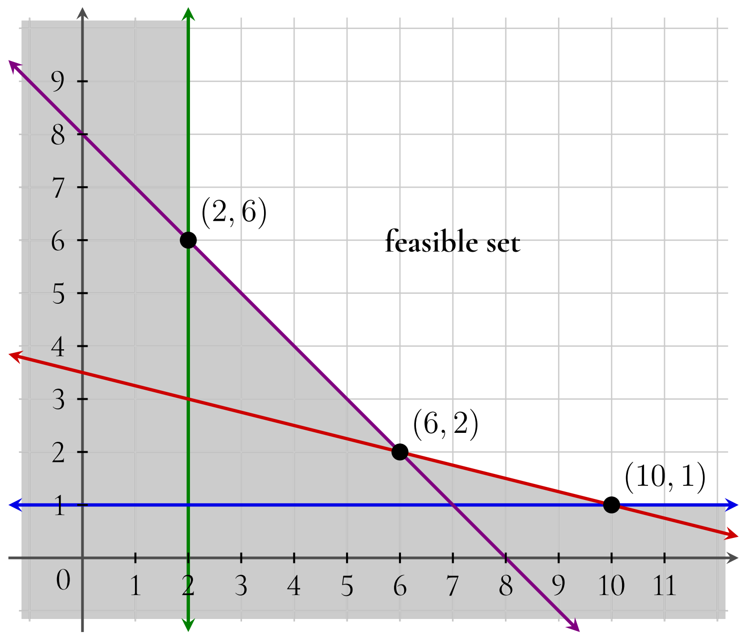
  which gives us the point $(2, 6)$.

- For $70x + 280y = 980$ and $10y = 10$, we solve

$$\begin{cases} 70x + 280y = 980 \\ 10y = 10 \end{cases} \xrightarrow{\text{rref}} \begin{cases} x = 10 \\ y = 1 \end{cases}$$

  which gives us the point $(10, 1)$.

Let's now graph the feasible set and label the vertices.



To find the optimal point, we evaluate the objective function at each of the vertices.

(x, y)	$10x + 23y$
$(2, 6)$	158
$(6, 2)$	106
$(10, 1)$	123

Therefore, since the optimal point is $(6, 2)$, Yoshimi should take 6 tablets of MaxLife and 2 tablets of PureHealth to minimize her cost. //

In some scenarios that seem to involve three variables, there is a relationship between the variables that can reduce it to a problem of two variables.

Example 4.7. Catalina has \$14 000 to invest into three types of stocks: low-risk, medium-risk, and high-risk. Her financial adviser has told her to follow the following principles:

- The amount invested in the low-risk stocks should be at most \$2 000 more than the amount invested in medium-risk stocks.
- At least \$10 000 should be invested in low- and medium-risk stocks.
- No more than \$9 000 should be invested in medium- and high-risk stocks.

Her financial adviser expects that the yields are going to be 2% for low-risk stocks, 3.4% for medium-risk stocks, and 7.3% for high-risk stocks. Assuming Catalina is going to invest all of the \$14 000, how much should she invest in each kind of stock to maximize her total expected yield?

Let x be the amount invested in low-risk stocks, y be the amount invested in medium-risk stocks, and z be the amount invested in high-risk stocks, all measured in thousands of dollars. Now, to see that this is really a problem of two variables, note that $x + y + z = 14$ which gives us that $z = 14 - x - y$.

We will be able to use this to translate the objective function and all of the constraints to expressions in terms of x and y, only.

The objective function is

$$\begin{aligned} 0.02x + 0.034y + 0.073z &= 0.02x + 0.034y + 0.073(14 - x - y) \\ &= 1.022 - 0.053x - 0.039y \end{aligned}$$

and we wish to maximize it.

Now we can find the constraints. Let's translate the three principles recommended by the financial adviser into linear inequalities involving the declared variables.

- $x \leqslant y + 2$ which we rewrite as $x - y \leqslant 2$.
- $10 \leqslant x + y$.
- $y + z \leqslant 9$ which we can rewrite as $y + (14 - x - y) \leqslant 9$ which we simplify to $5 \leqslant x$.

Now, we must also account for the fact that $0 \leqslant x \leqslant 14$, $0 \leqslant y \leqslant 14$, and $0 \leqslant z \leqslant 14$. To convert the inequality $0 \leqslant z \leqslant 14$ into an expression involving only x and y, we look at the following:

- If $z \geqslant 0$, then $14 - x - y \geqslant 0$ which is equivalent to $14 \geqslant x + y$.
- If $z \leqslant 14$, then $14 - x - y \leqslant 14$ which is equivalent to $0 \leqslant x + y$. Since both $x \geqslant 0$ and $y \geqslant 0$, the inequality $0 \leqslant x + y$ is redundant.

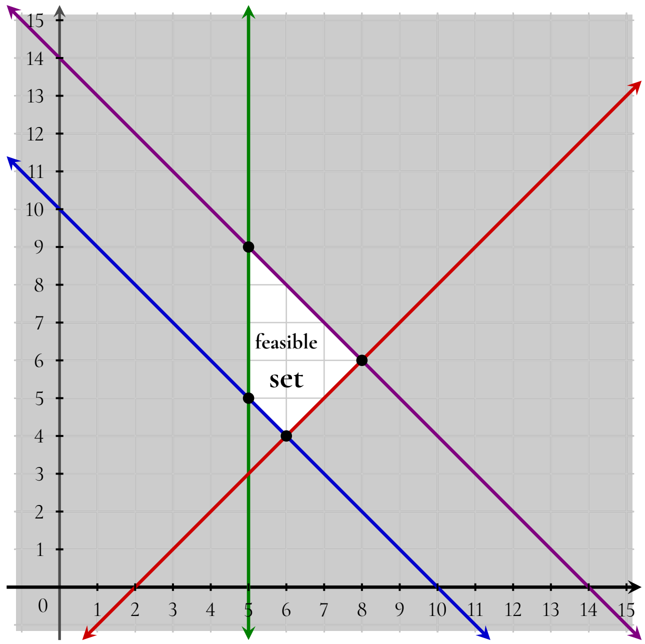
Let's now state the constraints, omitting only those which are redundant.

$$\begin{cases} \begin{array}{ccccc} x & - & y & \leqslant & 2 \\ x & + & y & \geqslant & 10 \\ & & x & \geqslant & 5 \\ x & + & y & \leqslant & 14 \\ & & y & \geqslant & 0 \end{array} \end{cases}$$

To aid in our graph, we can convert all of these inequalities to slope-intercept form

$$\begin{cases} \begin{array}{ccccc} x & - & y & \leqslant & 2 \\ x & + & y & \geqslant & 10 \\ & & x & \geqslant & 5 \\ x & + & y & \leqslant & 14 \\ & & y & \geqslant & 0 \end{array} \end{cases} \longrightarrow \begin{cases} \begin{array}{lll} y & \geqslant & x - 2 \\ y & \geqslant & -x + 10 \\ x & \geqslant & 5 \\ y & \leqslant & -x + 14 \\ y & \geqslant & 0 \end{array} \end{cases}$$

So the graph of the feasible set (with vertices emphasized) is



As usual, we can find the vertices by finding the intersection points of the corresponding pairs of lines.

- For $x = 5$ and $x + y = 14$, we can substitute $x = 5$ into $x + y = 14$ to obtain $y = 9$. This gives us the vertex $(5, 9)$.
- For $x = 5$ and $x + y = 10$, we can substitute $x = 5$ into $x + y = 10$ to obtain $y = 5$. This gives us the vertex $(5, 5)$.
- For $x - y = 2$ and $x + y = 10$, we solve

$$\left\{\begin{array}{ccccc} x & - & y & = & 2 \\ x & + & y & = & 10 \end{array}\right. \xrightarrow{\text{rref}} \left\{\begin{array}{cccc} x & & = & 6 \\ & y & = & 4 \end{array}\right.$$

  This gives us the vertex $(6, 4)$.
- For $x - y = 2$ and $x + y = 14$, we solve

$$\left\{\begin{array}{ccccc} x & - & y & = & 2 \\ x & + & y & = & 14 \end{array}\right. \xrightarrow{\text{rref}} \left\{\begin{array}{cccc} x & & = & 8 \\ & y & = & 6 \end{array}\right.$$

  This gives us the vertex $(8, 6)$.

Now we evaluate the objective function at each of these vertices.

(x, y)	$1.022 - 0.053x - 0.039y$
$(5, 9)$	0.406
$(5, 5)$	0.562
$(6, 4)$	0.548
$(8, 6)$	0.364

We see that the objective function obtains a maximum value of 0.562 at $(5, 5)$. Now, to answer the original question, we can use the equation $z = 14 - x - y$ again. In particular, notice that the optimal point is $(5, 5)$ which tells us that $x = 5$, $y = 5$, and $z = 4$. That is, to maximize her total expected yield, Catalina should invest \$5 000 in low-risk stocks, \$5 000 in medium-risk stocks, and \$4 000 in high-risk stocks. //