# Fourier Analysis

Any periodic function $f(x)$ with period in $-L \leq x \leq L$ can be written in the form,

$$f(x) = a_0 + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n \cos\left(\frac{m\pi x}{L}\right)$$

<u>Theorem</u>: We have that sine and cosine functions form an orthonormal set, that is to say,

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } n \neq m \\ 2L & \text{if } m = n \end{cases},$$

Same for cosine funtions,

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0 \Rightarrow \begin{cases} \text{any} \\ m, n \end{cases}$$

Proof orthogonality:

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = \frac{1}{2}\left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right]$$

$$\int_{-L}^{L} \frac{1}{2}\left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right] dx$$

$$\frac{1}{2}\left\{\frac{L}{(n-m)\pi} \sin\left(\frac{(n-m)\pi x}{L}\right) - \frac{L}{(n+m)\pi} \sin\left(\frac{(n+m)\pi x}{L}\right)\right\}\Bigg|_{-L}^{L}$$

$$= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\left[2\sin((n-m)\pi)\right] - \frac{L}{(n+m)\pi}\left[2\sin((n+m)\pi)\right]\right]$$

If $n \neq m$

$$\sin((n-m)\pi) = 0 \qquad \sin((n+m)\pi) = 0$$

$\Rightarrow I = 0$

If $n=m$

$$\frac{L}{(n-m)\pi}\sin((n-m)\pi) = \frac{L}{(n-m)\pi}\left((n-m)\pi - \frac{1}{3!}[(n-m)\pi]^3 + \frac{1}{5!}[(n-m)\pi]^5 + \cdots +\right.$$

$$= L$$

Then,

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \begin{cases} 0 & \text{if } n \neq m \\ L & \text{if } n = m \end{cases}$$

By the same argument,

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = \begin{cases} 0 & \text{if } n \neq m \\ L & \text{if } n = m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = 0 \text{ for any } n \text{ and } m$$

The coeficients of the Fourier Series are,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\text{sen}\left(\frac{n\pi x}{L}\right)dx$$

It can be easily verified from orthogonality.

<u>Example:</u>

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le L \\ -1 & \text{if } -L \le x \le 0 \end{cases} \qquad \text{periodic}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{0}(-1)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$+ \frac{1}{L}\int_{0}^{L}(1)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\left(\frac{L}{n\pi}\right)\left(+\cos\frac{n\pi x}{L}\right)\Bigg|_{-L}^{0} + \frac{1}{L}\left(\frac{L}{n\pi}\right)\left(-\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{0}^{L}$$

$$= \frac{1}{n\pi}\Big[1 - \cos(n\pi)\Big] + \frac{1}{n\pi}\Big[-\cos(n\pi) - (-1)\Big] \qquad \cos(-n\pi)$$

$$b_n = \frac{2}{n\pi}\Big[1 - \cos(n\pi)\Big] = \frac{2}{n\pi}\Big[1 - (-1)^n\Big]$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \rightarrow$$

product of an even function with an odd function. is an even function, integral of an odd function is 0.

$a_n = 0$

$$\Rightarrow \quad f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\Big[1 - (-1)^n\Big]\sin\left(\frac{n\pi x}{L}\right)$$

Complex form of the Fourier Series:

We have,
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

But
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\Rightarrow \quad f(x) = a_0 + \frac{1}{2}\sum_{n=1}^{\infty} a_n \left(e^{in\pi x/L} + e^{-in\pi x/L}\right) + \frac{1}{2}\sum_{n=1}^{\infty} -ib_n\left(e^{in\pi x/L} - e^{-in\pi x/L}\right)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - ib_n)\, e^{in\pi x/L} + \sum_{n=1}^{\infty} \frac{1}{2}(a_n + ib_n)\, e^{-in\pi x/L}$$

$\Rightarrow$
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}$$

where,
$$c_n = \begin{cases} a_n - i\, b_n & n < 0 \\ a_0 & n = 0 \\ a_n + i\, b_n & n > 0 \end{cases}$$

The coefficients can be found from orthogonality,
$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{in\pi x/L}$$

References

Haberman, Richard. Applied partial differential equations with Fourier series and boundary value problems. Pearson Higher Ed, 2012.