# Fourier Transform

We have that any periodic function can be written as

$$f(x) = a_0 + \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=0}^{\infty} b_n \operatorname{sen}\left(\frac{n\pi x}{L}\right)$$

In complex form we can write,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{-in\pi x/L} \qquad \text{where,} \quad C_n = \begin{cases} a_n - ib_n & n<0 \\ a_0 & n=0 \\ a_n + ib_n & n>0 \end{cases}$$

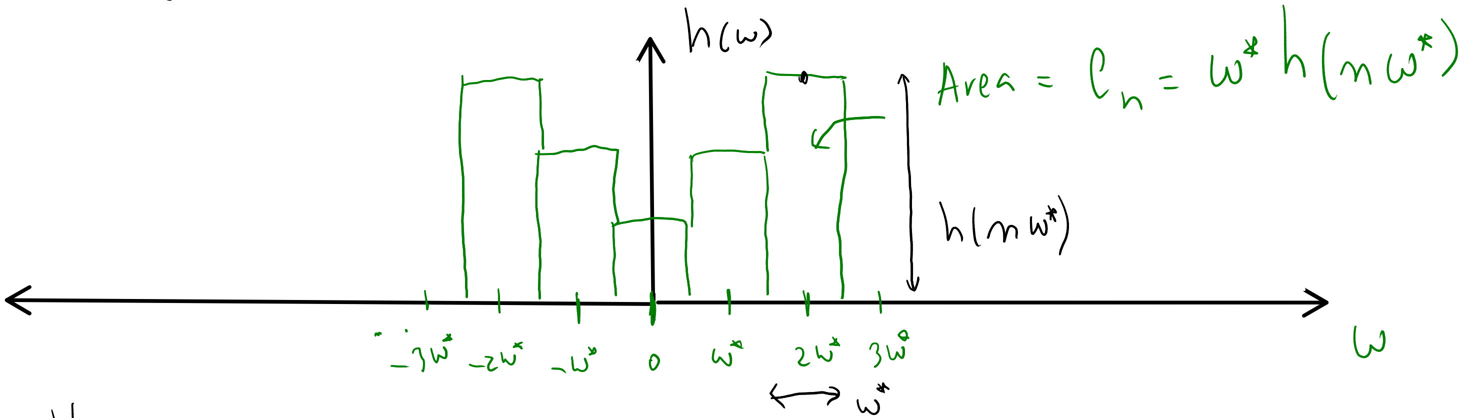
We can write it in terms of time,

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{-i(2\pi n t)/T} \quad , \ T \text{ the period}$$

then

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{-i(n\omega^* t)} \quad , \text{ where } \omega^* = \frac{2\pi}{T}$$

the $C_n$ terms can be drawn as




Hence,

$$f(t) = \sum_{n=-\infty}^{\infty} \omega^* h(n\omega^*) e^{-in\omega^* t}$$

We may set,

$$n\omega^* \to \omega$$
$$\omega^* \to \delta\omega$$

$$f(t) = \sum_{n=-\infty}^{\infty} h(\omega) e^{-i\omega t} \delta\omega$$

By taking a limit as $\Delta w \to 0$, the sum becomes an integral,

$$\boxed{f(t) = \int_{-\infty}^{\infty} h(w)\, e^{-iwt}\, dw}$$

this is the Fourier transform

Given the Fourier sum,

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{-in2\pi t/T}$$

the coefficients can be found by,

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{in2\pi t/T}\, dt$$

We have $C_n = h(w)\Delta w$ and $2\pi n/T = w$

$$\Rightarrow \quad h(w)\Delta w = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{iwt}\, dt$$

But $\Delta w = \frac{2\pi}{T}$, taking the limit $\Delta w \to 0$ $T \to \infty$

$$\Rightarrow \quad \boxed{h(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{iwt}\, dt}$$

this is the inverse Fourier transform,

Example:

Compute the Fourier transform of $f(t) = \cos(w_0 t)$, where $w_0$ is a constant frequency,

$$h(w) = \int_{-\infty}^{\infty} f(t)\, e^{iwt}\, dt = \int_{-\infty}^{\infty} \cos(w_0 t)\, e^{-iwt}\, dt$$

$$h(\omega) = \int_{-\infty}^{\infty} \cos(\omega_0 t)\left(\cos(\omega t) - i\sin(\omega t)\right) dt$$

From orthogonality of sin and cosine functions we have that, the function $h(\omega)$ is 0 everywhere except when $\omega = \omega_0$ or $\omega = -\omega_0$, where the value becomes $\infty$.

Then, $h(\omega) = \frac{1}{2}\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$

Therefore, the Fourier transform picks the frequency of the wave.

Now, compute the inverse Fourier transform of $h(\omega)$

$$f(t) = \int_{-\infty}^{\infty} h(\omega)\, e^{+i\omega t}\, d\omega = \int \frac{1}{2}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]\left[\cos(\omega t) + i\sin(\omega t)\right] d\omega$$

$$f(t) = \frac{1}{2}\left[\cos(\omega_0 t) + \cos(\omega_0 t)\right] + \frac{1}{2} i\left[\sin(\omega t) + \sin(-\omega t)\right]$$

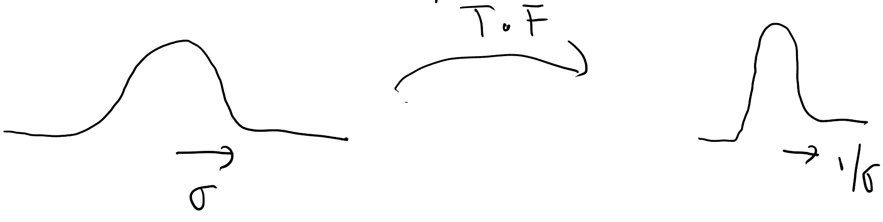
$$f(t) = \cos(\omega_0 t)$$

$$\cos(\omega_0 t) \underset{\text{Inverse Fourier transform.}}{\overset{\text{Fourier Transform}}{\rightleftarrows}} \frac{1}{2}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$

Example 2: Fourier transform of a Gaussian, $f(x) = e^{-\frac{x^2}{2\sigma}}$

$$h(k) = \int e^{-\frac{x^2}{2\sigma^2}} e^{-ikx}\, dx = \frac{\sigma}{\sqrt{2\pi}}\, e^{-\sigma^2 k^2/2} = \sqrt{\frac{\sigma}{2\pi}}\, e^{-k^2/(2/\sigma^2)}$$

the Fourier transform of a Gaussian is also a Gaussian




Uncertainty principle!