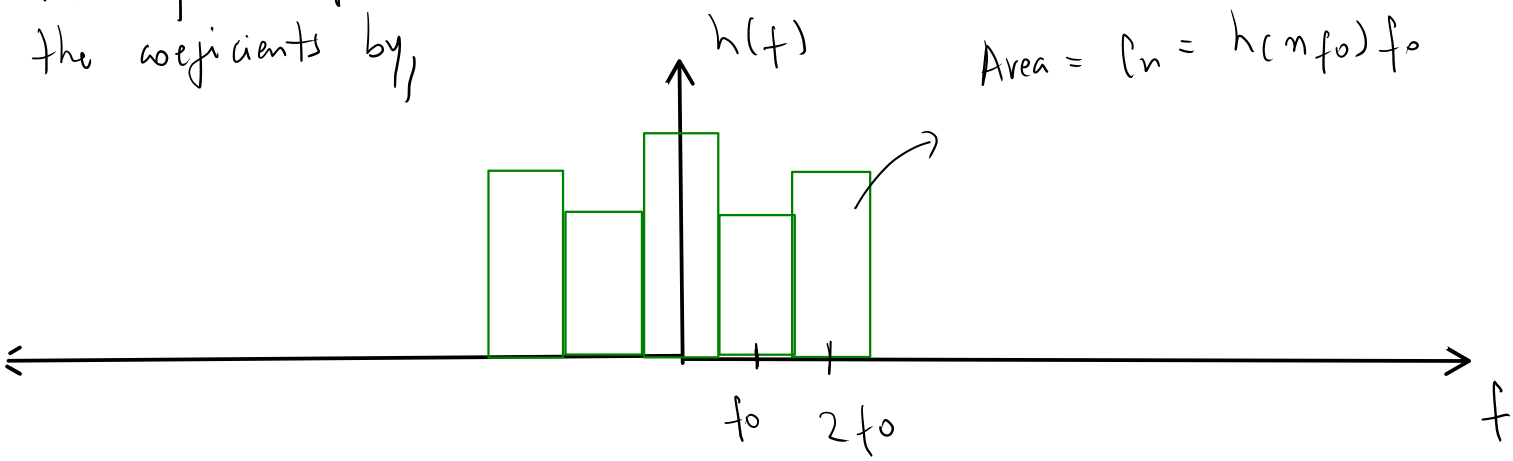
Discrete Fourier Transform

We can write any periodic function with period T, as,

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{-i2\pi nt/T} = \sum_{n=-\infty}^{\infty} C_n e^{-i2\pi n f_0 t}$$ with $f_0 = \frac{1}{T}$, the frequency.

Following last lecture, we may interpret the coefficients by,

$$C_n = h(nf_0) f_0$$

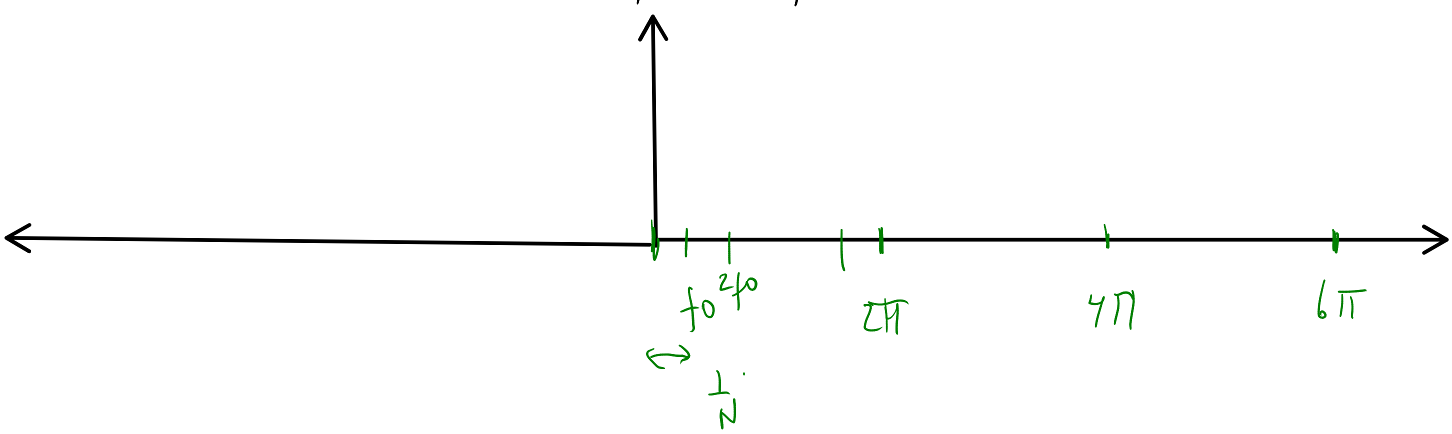


$$f(t) = \sum_{n=-\infty}^{\infty} h(nf_0) f_0 \, e^{-i2\pi n f_0 t}$$

Since, $e^{-i2\pi n f_0 t}$ repeats every 2π, we may take values for which

$$0 \leq 2\pi n f_0 \leq 2\pi$$

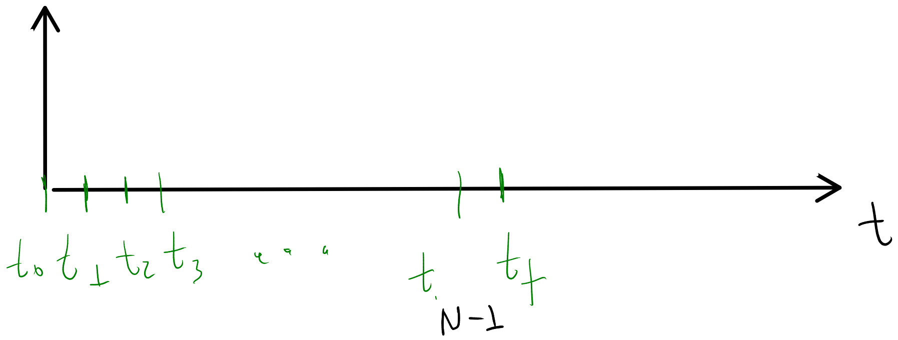
Hence, we may divide the frequency domain in parts of 2π and each 2π section by N frequencies



Hence $f_0 = \frac{1}{N}$ and $f = n f = \frac{n}{N}$

$$f(t) = \sum_{n=0}^{N-1} h(nf_0)\,\frac{1}{N}\, e^{-i\frac{2\pi n}{N}t} = \frac{1}{N}\sum_{n=0}^{N-1} x_n\, e^{-i\frac{2\pi n}{N}t}$$

The resulting function is in time domain, then we may divide the time domain to N intervals.



$$f(t_k) = \frac{1}{N}\sum_{n=0}^{\infty} x_n\, e^{-i2\pi n k t_0/N}$$

Hence,

$$\boxed{X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n\, e^{-i2\pi n k/N}}$$ ← Discrete Fourier transform.

where, we define $f(t_k) = X_k$ and we scale the total time t_f to the integer N, so that, $t_0 = 1$.

Also given the Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{-i2\pi f_0 t}$$

the coefficients are given by,

$$C_n = f_0 \int_{-\infty}^{\infty} f(t)\, e^{i2\pi n f_0 t}\, dt$$

Following the last part we may convert approximate the integral with a sum, so that,

$t = k t_0 \qquad dt = t_0 \qquad f(t) = f(k t_0) = X_k$

$$\int_{-\infty}^{\infty} \to \sum_{k=0}^{N-1}$$

$$x_n \frac{1}{N} = \frac{1}{N} \sum_{k=0}^{N-1} X_k \, e^{i 2\pi n k / N}$$

$$\boxed{x_n = \sum_{k=0}^{N-1} X_k \, e^{i 2\pi n k / N}}$$

← this is the inverse discrete Fourier tranform.

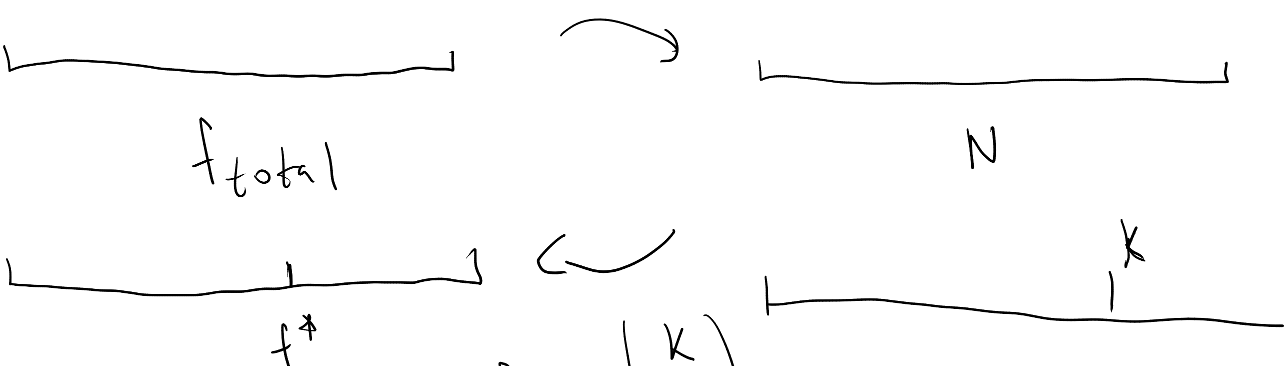
Hence $x_n \to f(t)$

$X_k \to h(f)$

$$\begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix} = e^{i2\pi} \begin{bmatrix} 0\cdot 0 & 0\cdot 1 & 0\cdot 2 \cdots & 0\cdot N-1 \\ 1\cdot 0 & & & \\ \vdots & & k\cdot n & \\ (N-1)\cdot 0 & \cdots & & (N-1)(N-1) \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

column, n,
row, k

Since f_{total} is scaled to N, to find the frequencies we have to do a regla de tres



$$f^* = f_{total}\left(\frac{k}{N}\right)$$