Portafolio Media-Varianza

December 28, 2022

Temas

Espacio de Activos de MV.

Problema de Optimización I

Separación en dos fondos

Solución analítica y numérica.

Frontera de Media-Varianza I

Extensiones

Problema de Optimización II

Problema de Optimización III

Frontera de Media-Varianza II

Capital Market Line

Riesgo-Retorno

	Mean	Standard Deviation
Treasury Bills	3.5%	3.1%
Treasury Bonds	6.0%	9.9%
Corporate Bonds	6.3%	8.5%
Large-Cap Stocks*	12.0%	20.0%
Small-Cap Stocks**	16.3%	31.8%

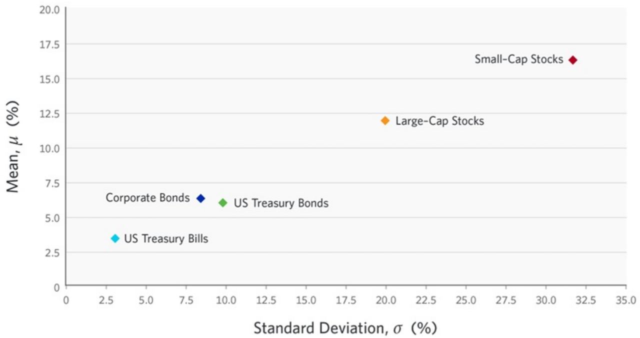
* Large-cap stocks represent large companies, specifically those that compose the S&P 500 Index.

** Small-cap stocks consist of approximately the smallest 5% (by market capitalization) of the universe of public companies traded on US exchanges over the period.

Source: Author's calculations based on data from *Ibbotson SBBI Classic Yearbook*, 2015 edition.

Timothy A. Luehrman, *Core Reading: Risk and Return 1: Stock Returns and Diversification*, HBP No. 5220 (Boston: Harvard Business School Publishing, 2017).

Mapa Riesgo-Retorno



Timothy A. Luehrman, *Core Reading: Risk and Return 1: Stock Returns and Diversification*, HBP No. 5220 (Boston: Harvard Business School Publishing, 2017).

Problema de optimización: definiciones

- Vector de retornos R (Nx1).
  $R = [r_1, \dots, r_N]$.
- Matriz Varianza-covarianza.

$$\Omega = \begin{pmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,N} \\ \sigma_{1,2} & \sigma_2^2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1,N} & \cdots & \cdots & \sigma_N^2 \end{pmatrix}$$

  - Simetrica.
  - NxN
- Vector de pesos w
  $w = [w_1, \dots, w_N]$.

Problema de optimización I

$$min_w \ \sigma_p^2 := \frac{1}{2} w' \Omega w \text{ s.t. } w' E[R] = \mu ; w' \mathbf{1} = 1$$

$$\mathcal{L} = \frac{1}{2} w' \Omega w + \underbrace{\gamma}_{\text{scalar}} (\mu - w' E[R]) + \underbrace{\lambda}_{\text{scalar}} (1 - w' \mathbf{1})$$

Condiciones de primer orden:

1. $\frac{\partial \mathcal{L}}{\partial w} = 0 \Leftrightarrow \Omega w = \gamma E[R] + \lambda \mathbf{1}$.
2. $\frac{\partial \mathcal{L}}{\partial \gamma} = 0 \Leftrightarrow w' E[R] = \mu$.
3. $\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Leftrightarrow w' \mathbf{1} = 1$.
4. $w^* = \gamma \Omega^{-1} E[R] + \lambda \Omega^{-1} \mathbf{1}$

Teorema de separación en dos fondos

$$w^* = \gamma \Omega^{-1} E[R] + \lambda \Omega^{-1} \mathbf{1}$$

1. Maximizar *trade-off* riesgo retorno.
2. Minimizar riesgo.
3. Portafolio óptimo:
   combinación linear 1 y 2.
   Frontera de media-varianza
   $w^* = f(E[R], \Omega, \gamma, \lambda) = f(E[R], \Omega)$

Frontera de media varianza I

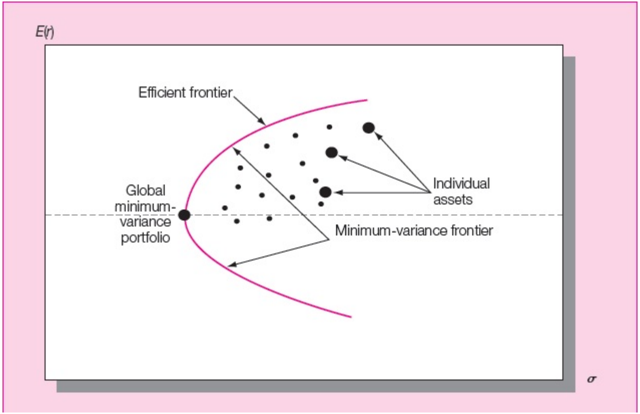
Visualizar portafolios óptimos
en espacio de activos.

$$
\begin{aligned}
\sigma_p^2 &= w^{*\prime}\Omega w^* = w^{*\prime}\Omega(\gamma\Omega^{-1}E[R] + \lambda\Omega^{-1}\mathbf{1}) \\
&= \gamma w^{*\prime}E[R] + \lambda w^{*\prime}\mathbf{1} = \gamma\mu + \lambda \\
&= (\frac{A\mu - B}{D})\mu + (\frac{C - B\mu}{D}) \\
&= \frac{A\mu^2 - 2B\mu + C}{D}
\end{aligned}
$$

- parabola invertida (σ^2 x-axis).
- *trade-off* riesgo-retorno.

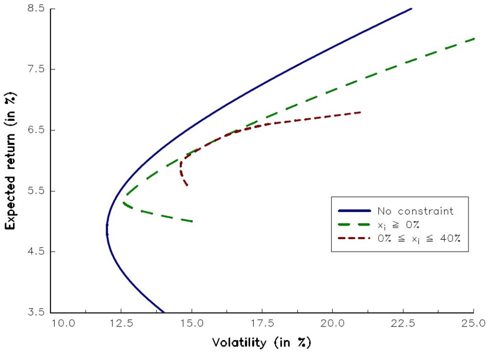
$$
\frac{d\sigma_p^2}{d\mu} = \gamma = \frac{A\mu - B}{D}
$$

Frontera de media varianza I



Frontera media varianza restringida

Figure: The efficient frontier with some weight constraints



- Posiciones largas.
- Evitar concentración.

Problema op. II: Maximizar trade-off riesgo-retorno

Maximizar función de utilidad de media varianza.

$$max_w w'E[R] - \frac{\gamma}{2} w'\Omega w \text{ s.t. } w'\mathbf{1} = 1$$

$$\mathcal{L} = w'E[R] - \frac{\gamma}{2} w'\Omega w + \lambda(w'\mathbf{1} - 1)$$

Condiciones de primer orden:

1. $\frac{\partial \mathcal{L}}{\partial w} = 0 \Leftrightarrow \frac{1}{\gamma}\Omega^{-1}(E[R] + \lambda \mathbf{1}) = w^*$.
2. $\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Leftrightarrow w'\mathbf{1} = 1$.

$$w^* = \frac{\Omega^{-1}\mathbf{1}}{\mathbf{1}'\Omega^{-1}\mathbf{1}} + \frac{1}{\gamma} \frac{(\mathbf{1}'\Omega^{-1}\mathbf{1})\Omega^{-1}E[R] - (\mathbf{1}'\Omega^{-1}E[R])(\Omega^{-1}\mathbf{1})}{\mathbf{1}'\Omega^{-1}\mathbf{1}}$$

Refinar resultado Op. problema III: activo sin riesgo

$$min_w \frac{1}{2} w' \Omega w \text{ s.t. } w'(E[R] - R_f \mathbf{1}) = \mu - R_f$$

$$L : \frac{1}{2} w' \Omega w + \gamma[\mu - R_f - w'[E[R]] - R_f \mathbf{1}]$$

Condiciones de primer orden:

1. $\frac{dL}{dw} : \Omega w = \gamma[E[R] - R_f \mathbf{1}] \Leftrightarrow w^* = \gamma \Omega^{-1}[E[R] - R_f \mathbf{1}]$.
2. $\frac{dL}{d\gamma} : w'[E[R] - R_f \mathbf{1}] = \mu - R_f$.

$$w^{Tang} = \frac{(\mu - R_f)\Omega^{-1}[E[R] - R_f \mathbf{1}]}{\underbrace{[E[R] - R_f \mathbf{1}]' \Omega^{-1}[E[R] - R_f \mathbf{1}]}_{\text{F}}}$$

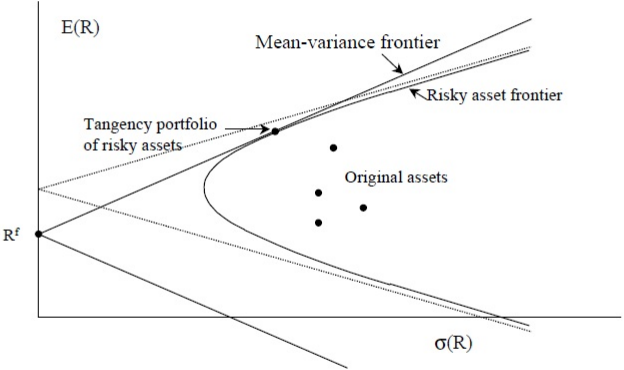
Frontera de media varianza II

Visualizar portafolios óptimos
en espacio de activos.

$$
\begin{aligned}
\sigma_p^2 &= w^{*\prime}\Omega w^* = w^{*\prime}\Omega[\gamma\Omega^{-1}[E[R] - R_f\mathbf{1}]] \\
&= \gamma\Omega\Omega^{-1}w^{*\prime}[E[R] - R_f\mathbf{1}] = \gamma(\mu - R_f) \\
&= \frac{(\mu - R_f)^2}{F}
\end{aligned}
$$

- $(\sigma_p = \frac{\mu - R_f}{\sqrt{F}})$.
- recta, $\mu = R_f + \sqrt{F}\sigma_p$.

Frontera de media varianza II



Capital Market Line