A segmented and observable Yield Curve

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May 10, 2019

Abstract

We implement a segmented three factor Nelson-Siegel model for the yield curve using daily observable bond prices and short term inter-bank rates for Colombia. The flexible estimation for each segment (short, medium, and long) provides an improvement over the classical Nelson-Siegel approach in particular in terms of in-sample performance. A segmented term structure model based on observable bond prices provides a tool closer to the needs of practitioners in terms of reproducing the market quotes and allowing for independent local shock in the different segments of the curve.
Keywords: Term structure, Nelson-Siegel, Preferred habitat theory.

1 Introduction

The term structure of interest rates is the relationship between interest rates or bond yields and different terms or maturities. The term structure of interest rates is also known as a yield curve, and it plays a central role economic and financial analysis. For example, the term structure reflects expectations of market participants about future changes in interest rates and their assessment of monetary policy conditions. A large part of finance literature and applications by practitioners, regarding the term structure of non defaultable securities, is concerned with using available security prices to estimate the fair market prices of other non-observable securities. As mentioned by [ERS17] *"this is important because fixed-income securities and their derivatives trade only occasionally, and so must be priced based on other securities that do trade. A typical part of estimating the price at which a bond would trade involves decomposing its price into term and risk premiums. This analysis formally constrains the yield curve to be arbitrage-free"*.

The Nelson-Siegel model [NS87] is a statistical approach that provides a parsimony specification to capture the differences in rates along the curve (for different maturities). Its implementation in one or two stages allows to recover the temporal variation of the factors maintaining the factor loading's constant over time. The specification of the model and the estimation methods provide a simple implementation, which is why it also turns out to be a successful model outside academia.

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Although the Nelson-Siegel model is not arbitrage-free, [CDR11] propose a representation of the model that is arbitrage-free. However, it remains unclear if no-arbitrage restrictions improve statistical validity and therefore some empirical applications shown that different variations of the Nelson-Siegel model provide a better in-sample and out-of-samples fit of the yields than the class of affine term structure models. [FW18] provides a non-parametric alternative to estimate the discount curve using market quotes that has maximal smoothness.

One of the important drawbacks of the academic literature is that the estimation of the term structure is performed on synthetic interpolated zero coupon yields. For example, one of the most important sources of US yields are based on interpolated data from [GSW07]. Working with interpolated data it is not clear exactly what is the forecasting error since there is no direct observable variable, it is preferable to work with data directly form the market. In addition, creating a long time series of yields requires strong assumptions on matching bonds across time, without understanding the implications on the outcome of interest, a point forecast or a the level of uncertainty for risk management purpose. Only recently, have some authors proposed methodologies to work directly on the bond price data. [ACR19] use an arbitrage-free Nelson-Siegel model and propose a one-step estimation approach that uses the directly observable bond prices in a non-linear state space model. They compare this approach to the traditional two-step procedure that estimates these types of models on the synthetic interpolated yields. Using simulations and Canadian bonds they find negligible errors in the synthetic interpolated yields can lead to parameter instability of the models. Additionally they find superior forecasting accuracy from the one-step approach.

Another regularity in modeling the term structure of interest rates is the use of a similar functional form and constant parameters to describe the entire term structure. However, preferred habitat theory of the term structure [MS66] advocates that local shocks may influence interest rates for each maturity. Empirical evidence related to this theory reveals that U.S. Treasury bonds' supply and demand shocks have non negligible effects on yield spreads, term structure movements, and bond risk premium. In an attempt to formalize the preferred habitat theory, [VV09] propose an equilibrium model in which demand directly influences and determines all yields in the term structure, in a dynamic way. According to this theory, the equilibrium yield rate for each term is determined by the demand and supply forces for that market, in other words, the preferences of investors on securities at that point in the curve. Investors can substitute preferences over terms that are not available in the market for a near term but available in the market.[DK13] note that investors act as arbitrators, guaranteeing the relationship between the demand for the securities and the returns along the curve; and on the other hand they guarantee that the curve is smooth, meaning that the yields for close periods are similar.

Inspired by the preferred habitat theory, and more specifically by its recent formalization by [VV09], [AAK+18] propose a class of models that separate the yield curve into segments which present their own local shocks, but which are simultaneously interconnected, composing the whole yield curve. The main objective of the family of segmented models is to first, partition the segments of the curve in such a way that the dynamics of each segment can be determined by maturities that are represented in that segment; and second, the implementation of the segments along the curve must be globally consistent and smooth. This is achieved by ensuring that the rates of return are similar for the terms that connect the segments and therefore are close to each other. This is not equivalent to imposing non-arbitration restrictions.

In this paper we model and forecast the Colombian term structure of interest rates using approximate yields that are estimated directly from the bonds, but are not interpolated, and we use a segmented Nelson-Siegel three factor model as proposed by [AAK+18]. We estimate the model using daily data for each year from January 2013 to September 2018. We provide independent results for each year in the sample in order to avoid strong assumptions on the equivalent maturities across the bonds. Although we could think that this is a drawback in terms of forecasting it is important to note that the cross sectional estimation in the Nelson-Siegel three factor model provides a dimension reduction approach from the m maturities to the three factors and hence we can use these estimated factors to create out-of-sample forecast from one year to the next for any desired maturity. Our results show that the segmented model with smoothing restrictions has an in-sample and out-of-sample performances that is superior to non-segmented Nelson-Siegel model. In addition the out-of-sample performance of the segmented model, has a similar performance to the random walk model for short term horizons (one and five days)[1]. However, for monthly forecast the random walk model has a better performance. The remainder of the paper is structured as follows. Section 2 gives a brief introduction to the methodology as proposed by [AAK+18] for the segmented term structure model. Section 3 provides a description of the Colombian bond data. Section 4 presents the estimation results and the forecasting exercise. Section 5 concludes.

2 Methodology

The segmented term structure model is proposed by [AAK+18]. The authors propose a general framework that can be applied to any parametric model using exponential splines. In particular, we choose a three factor Nelson-Siegel type model. The segmented model provides estimates for the factor that are specific and independently estimated for each segment of the yield curve. In the traditional Nelson-Siegel model the factor are estimated using simultaneously all of the maturities. This is an important drawback because, for example, small changes in the short part of the curve could affect the long part of the curve. In addition, the methodology provides an approach to estimate latent yields at the knots in which the term structure is partitioned.

For the exercise we require N maturities in order to compose the yield curve $\tau_i \quad i \in [1, N]$, where i's represents each of the observed elements in the vector of maturities. Then, we need to define in an exogenous way the latent yields and set of k elements from the $\tau's$ vector in which we define the external and the internal knots; we denote $\phi = \{\tau_1, ..., \tau_k\}$. Note that the internal knot are imposed by the

[1]The random walk model has always been a strong benchmark model in out-of-sample forecasting of interest rates

researcher and denote the maturity in the curve that is in between two segments. On the other hand, the remaining elements of τ are treated as observed yields and recognized as $\tilde{\tau} = \{\tau_2, ..., \tau_{k-1}\}$, which are assumed to be measured with error in contrast with the latent yield that do not contain error.

The term structure representation of observed and latent yields that make up the yield curve are given by the following two equations,

$$y_t(\tilde{\tau}) = W(\tilde{\tau})B_t + \epsilon_t(\tilde{\tau})$$
$$y_t(\phi) = W(\phi)B_t,$$

where the matrix of factors loadings $(W(\tau_i) = \{1, g_i(\tau_i), h_i(\tau_i)\})$ is time invariant and only depends on maturity and segments (1). On the other hand, the vector $(B_t = \{a_t^i, b_t^i, c_t^i\})$ represents the factors, that vary over time and also depend on the segmentation i. These independent segments provide enough flexibility so that the model is consistent with preferred habitat theory. The Nelson-Siegel model provides the functional form for the factor loadings, see the appendix 6. As the methodology is based on splines, the process requires to have some constrains on the above equations that creates smoothness across the segments. This smoothness is consistent with the role of active arbitrageurs that ensure no-arbitrage conditions along the yield curve. The following equations contain the segmented model, but in addition introduces the smoothness restriction.

$$y_t(\tau) = W(\tau)B_t, \quad s.t \ \ R(\phi)B = 0$$

The smoothness restrictions create an equality constrained optimization problem. These constraint guarantee that the parametric functional forms of each segment and their first and second derivatives have the same value at each internal knot[2] However, since we have equality constraints in the optimization problem we can transform the constrain problem into an unconstrained representation. The unconstrained problem has an additional advantage since we are able to reduce the number of parameters to estimate. 6 provides the details and the steps to have an optimization problem without restrictions,

$$y_t(\tilde{\tau}) = \pi(\tilde{\tau}, \phi)y_t(\phi) + \epsilon(\tilde{\tau}),$$

The factor loading's from the unconstrained model, $(B_t = \{a_t^i, b_t^i, c_t^i\})$, can be estimated with OLS.

3 Bond price data

We obtain daily data on bond prices from January 2013 to September 2018, provided by Precia S.A.[3] These are sovereign bonds denominated in local currency that are trading in the secondary market[4]. Although we have six years of daily data we use the bond prices data for each year so we use specific maturities that are trading over the course of a year. We want to avoid pairing bonds across time. In addition, we

[2]Details of how to build and introduce the matrix R are provided in 6.

[3]We thank Precia S.A. for providing a sample of historical bond prices.

[4]The Ministry of Finance of Colombia currently issued two types of bonds in the local market: local currency denominates or inflation indexed bonds.

are only using bonds with maturity above or equal to one year. That is we disregard any bond that in the current year have a time to maturity smaller than 12 months. For example for the year 2014 we have bonds with a time to maturity (in years) equal to $1.8, 1.9, 2.5, 4.8, 4.9, 5.7, 6.6, 8.3, 10.6, 12.7, 14.3$ at the beginning of the year (figure 2).

Since we want to avoid any complex pre-processing of the bond price data in order to obtain the yields we estimate a quick approximation of the yield to maturity that most importantly does not involve any type of interpolation,

$$y(\tau) = \frac{\frac{\text{Coupon/Interest Payment(Par Value - Bond Price)}}{\text{Years to maturity}}}{\frac{\text{(Par Value + Bond Price)}}{2}}$$

Because the issuance of short term bond by the government is not stable over time, in order to model the short part of the curve we use the inter-bank rate index. The inter-bank rate in Colombia (IBR) is the reference short term rates in the wholesale money market[5]. The index currently provides daily rates for the overnight, one month, three months and six months. Although, it is important to note that the risk factors associated to the sovereign bonds and the inter-bank rate differ, that is the latter are not risk free and contain a credit risk premium [6]

4 Empirical Application

For the empirical application we consider three segments representing the short, medium and long term part of the yield curve and we estimate the model using the sample (2013-2018) but using the observed yields during each year. On average we have 14 observable yields, the minimum number is 12 in 2016 and the maximum is 17 in 2013. The first knot in the curve is the overnight rate and the interior knots are 1.6 and 8.

With the three segments and three factor in the Nelson-Siegel model we have a total of 9 factors to estimate ($B_t = \{a_t^{ST}, b_t^{ST}, c_t^{ST}, a_t^{MT}, b_t^{MT}, c_t^{MT}, a_t^{LT}, b_t^{LT}, c_t^{LT}\}$).

We estimate the classical three factor Nelson-Siegel model ($NS3$), and following [AAK+18] we estimate three segmented models. The first we denote as a segmented model but without imposing the smoothness constraints (NS3_S), the second is the weekly segmented model with the smoothness constraints (NS3_W_S) and finally the strongly segmented model with the smoothness constraints (NS3_S_S)[7]. The only difference in our application is that we do not fine-tune the parameters that controls the degree of loading segmentation, we set that parameter to 0.5.

Table 1 presents the in-sample fit using the root mean square error for the three models in 2014 [8] The results indicate that segmented models that impose smoothness conditions that guarantee continuity across the yield curve have a better fit

[5] We obtain information on the inter-bank reference rates from the Banco de la Republica de Colombia (the central bank)

[6] We do not make any adjustment or assumptions to compensate for this risk premium in the inter-bank rates.

[7] The strongly segmented model is an extension that provides a more flexible specification of the factor loading's in each segment, this approach is explained in [AAK+18]

[8] The results for the other years are available upon request to the authors.

than the traditional Nelson-Siegel model. The reason behind the better fit is the possibility to accommodate the specific dynamics of each segment in the yield curve and hence validates empirically the preferred habitat theory.

We also perform an out-of-sample forecasting exercise for one, five (week), and 21 days (month). We use a rolling window of 126 days (approximately 6 months) and we obtain forecast for the years 2014 to 2018. Our forecasting setup is different, because we use the information of the previous year up to the last observed data to generate forecast of the year of interest and then update the information. It is important to remember that since we are not pairing bond across years we do not observe the same maturities between the years. However, this is not a problem to make out-of-sample forecast of the data using information from the previous year, the reason is that the main estimation outcome of the Nelson-Siegel model are the time varying factors. In other words, we take advantage of dimension reduction mechanism behind these parametric term structure models. Suppose that in the year 2013 we had 17 observed maturities and we estimate using the classical Nelson-Siegel or the segmented version of the model 3, 5 or nine factors (depending on the model). Then we can use an autoregressive of order one model to estimate and forecast these factors. To update the information as we roll the window onto 2014 and we estimate using the cross-section of maturities the factors and feed-in the information on to the time series of factors and re-estimate the autoregressive models and forecast up until the end of the evaluation window. For every year we have around 240 out-of-sample forecast for the different horizons that represent on average the number of trading days in the year.
For the out-of-sample forecast we compare the performance of three models base on the RMSE: the classical Nelson-Siegel (NS), the random walk (RW) and the strongly segmented Nelson-Siegel (NS_S_S). The average errors are presented in basis points for the year 2014 [9]. The results indicate that the segmented model provides smaller errors for most observed maturities for one day and weekly forecast. However, for the monthly forecast horizon it is not clear whether the segmented model performs better than a random walk. As mentioned previously both the in-sample and out-of-sample results does show a better performance of segmentation in the family of parametric term structure models based on Nelson-Siegel. Our results confirm the results of [AAK+18] but are based on real market quotes rather than using synthetically interpolated yields. For the different years we find that the previous results holds systematically, but not with respect to the advantage over the random walk model; that is, in some years we find that the random walk model out-performs the segmented models.

5 Conclusions

Term structure models are important for the pricing of instruments that are part of trading and reporting activities in financial markets. The financial literature has benefited from the research on term structure models performed from a monetary policy perspective. However, central banks are mainly focused on the changes in the general shape of the yield curve because this is informative as to the expectations

[9] The results for the other years are available upon request to the authors.

regarding monetary policy. For this use case, using synthetic interpolated yields as a starting point in the model seems to be the norm and also performing simultaneous estimation of the parameters or functional forms of interest using all of the maturities. On the other hand, practitioner and investors are more interested in the performance of the model in terms of the observed market quotes. In addition are not comfortable, for example, when small changes in short part yield curve have an important impact on the long part and they are certain that these changes are in part driven by the model. From an statistical point one would prefer models that are locally robust to changes other parts of the curve and that are able to accommodate idiosyncratic shocks locally. The parametric terms structure model proposed by [AAK+18] provides a simple approach that splits the yield curve into segments and at the same time guarantee the smoothness and consistency across the curve.
Our paper provides an example on how to accomplish the segmentation of the curve and a proposal for using the observable yield during the year for in-sample and out-of-sample predictions. The out-of-sample predictions are not affected by using different maturities of the same bonds without the need to make assumptions on the matching of the bonds requires for a long historical sample. Out point of view is that this is not necessary because of the dimension reductions that is performed on this type of factor model. Since the factors are the time-varying element of interest we can use them to connect the data generating process when the sample is estimated year-by-year rather than forcefully building a long historical sample.
There is room for improvement on the performance of the in- and out-of-sample forecasting exercise that can be explored using fine tuning. For example, the researcher could identify which is the optimal split for the segments (re-define the internal knots) and also change the values of the parameters that controls how strong is the segmentation across the yield curve. In other words, how important are idiosyncratic local shocks versus shocks that affect the entire curve.

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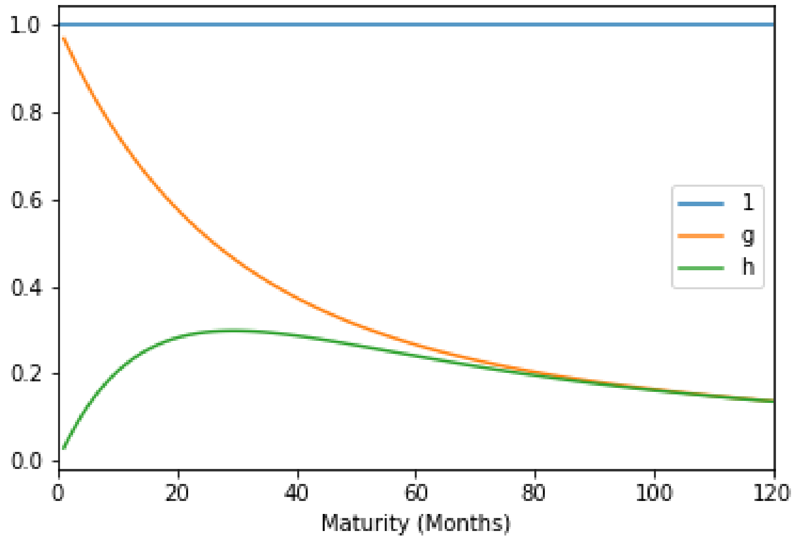

Figure 1: Factor loadings

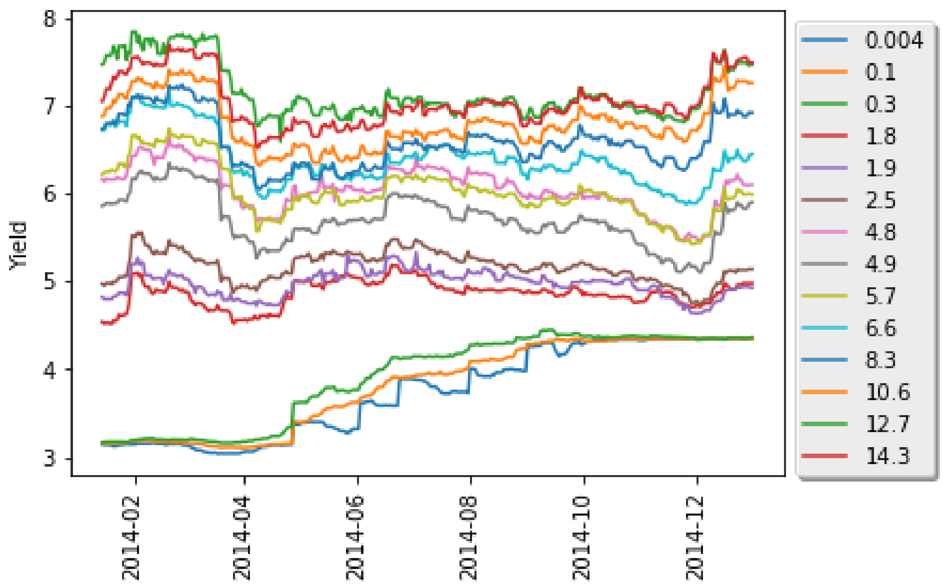

Figure 2: Observed yields from the Precia S.A. and IBR providesd by the central bank in 2014

τ	NS3	NS3_S	NS3_W_S	NS3_S_S
0.004	128.1	788.1	63.3	57.0
0.1	95.5	750.8	34.8	23.3
0.3	105.3	705.6	83.6	66.3
1.8	145.7	590.6	84.2	82.9
1.9	198.9	586.5	127.2	35.6
2.5	147.3	499.5	51.9	36.4
4.8	172.9	390.5	136.1	217.9
4.9	201.3	295.1	235.6	153.6
5.7	106.7	259.0	103.4	55.0
6.6	69.3	238.9	108.8	99.0
8.3	181.1	23.4	64.2	172.6
10.6	176.1	85.2	210.1	86.6
12.7	124.4	109.6	74.8	171.5
14.3	87.2	47.8	82.3	27.2

Note: errors are reported in basis points.

Table 1: In-sample root-mean-square error (RMSE) between modeled and observed yields

τ	NS	RW	NS_S_S
0.004	157.2	155.4	**75.5**
0.1	114.4	114.8	**37.1**
0.3	142.8	144.6	**85.2**
1.8	195.0	192.8	**94.7**
1.9	247.7	246.3	**53.6**
2.5	189.0	187.6	**56.6**
4.8	**184.8**	186.8	227.8
4.9	216.3	213.1	**171.3**
5.7	122.6	119.2	**73.5**
6.6	92.0	**91.9**	123.3
8.3	211.4	211.9	**194.9**
10.6	210.3	209.4	**114.1**
12.7	**156.9**	158.2	200.2
14.3	107.3	106.4	**60.2**

Note: errors are reported in basis points.

Table 2: Out-of-sample root-mean-square error (RMSE) between modeled and observed yields for one day (2014).

τ	NS	RW	NS_S_S
0.004	155.6	146.3	**88.2**
0.1	106.5	104.7	**60.4**
0.3	136.6	140.8	**100.7**
1.8	223.3	211.5	**121.0**
1.9	263.9	258.5	**86.5**
2.5	214.0	204.3	**95.9**
4.8	**195.3**	202.4	237.3
4.9	248.1	232.0	**200.8**
5.7	160.6	142.0	**114.8**
6.6	129.3	**128.0**	157.2
8.3	247.0	247.5	**235.7**
10.6	243.3	240.7	**164.4**
12.7	**177.8**	182.0	217.5
14.3	143.5	140.5	**126.2**

Note: errors are reported in basis points.

Table 3: Out-of-sample root-mean-square error (RMSE) between modeled and observed yields for one week (2014).

τ	NS	RW	NS_S_S
0.004	149.4	**127.3**	128.4
0.1	125.5	**99.5**	146.1
0.3	171.4	**155.8**	175.0
1.8	311.1	265.8	**184.4**
1.9	335.0	296.7	**152.8**
2.5	303.6	259.9	**184.1**
4.8	302.6	**293.4**	316.2
4.9	383.1	342.2	**339.9**
5.7	310.6	**269.5**	279.5
6.6	277.6	**264.0**	294.6
8.3	379.5	377.2	**370.4**
10.6	376.6	369.4	**308.8**
12.7	283.5	**277.1**	288.9
14.3	278.9	**256.7**	268.2

Note: errors are reported in basis points.

Table 4: Out-of-sample root-mean-square error (RMSE) between modeled and observed yields for one month (2014).

6 Appendix

1. Nelson-Siegel Loading's: the level, slope and curvature of the yield curve. The loading's below are used in order to build the matrix W used to represent the segmented curve.

$$1 \text{ (level)}$$
$$g(\tau) = \frac{1-e^{-\lambda\tau}}{\lambda\tau} \text{ (slope)}$$
$$h(\tau) = \frac{1-e^{\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \text{ (curvature)}$$

2. **R Matrix** In order to create the matrix R of restrictions we require smoothness conditions with the purpose to guarantee and arbitrageurs role who propagate all local shocks within each segment. Then, is necessary to know the first and second derivatives from the loading's that are placed inside the structure.

First Derivatives	***Second Derivatives***
$g'(\tau) = \frac{e^{-\lambda\tau}(1+\lambda\tau)}{\lambda\tau^2}$	$g''(\tau) = \frac{1}{\lambda\tau^3}[2 - e^{-\lambda\tau}((\lambda\tau)^2 + 2(\lambda\tau + 1))]$
$h'(\tau) = \frac{e^{-\lambda\tau}((\lambda\tau)^2+(\lambda\tau)+1)-1}{\lambda\tau^2}$	$h''(\tau) = \frac{1}{\lambda\tau^3}[2 - e^{-\lambda\tau}((\lambda\tau)^3 + (\lambda\tau)^2 + 2(\lambda\tau + 1))]$

R Matrix Construction

Note that all the j's yields within the same segment need to be governed by the same function ($W(\tau_j) = f_t^i(\tau_j)$). With the purpose too be more clear with the notation we will refer to ST, MT and LT the segments related to short, middle and long term.

- $f^{ST}(\tau_4) = f^{MT}(\tau_4), \quad f^{MT}(\tau_7) = f^{LT}(\tau_7)$
- $f'^{ST}(\tau_4) = f'^{MT}(\tau_4), \quad f'^{MT}(\tau_7) = f'^{LT}(\tau_7)$
- $f''^{ST}(\tau_4) = f''^{MT}(\tau_4), \quad f''^{MT}(\tau_7) = f''^{LT}(\tau_7)$
- $f_t^i(\tau_i) = y_t(\tau_i)$

The last condition will be important to measure the latent yields (Yields at knots) and transforming a constrain problem to a final unconstrained with a lower dimensional parameter space. Having mentioned all previous information, it is important to see the structural form of R and how to decomposed into a square invertible matrix R_1 and a complementary R_2 in order to reduce the dimensionality of parameters to be estimated.

$$X_i(\tau) = [1, g_i(\tau), h_i(\tau)] \quad X_i'(\tau) = [1, g_i'(\tau), h_i'(\tau)] \;\; ; X_i''(\tau) = [1, g_i''(\tau), h_i''(\tau)]$$

$$R = \begin{bmatrix} X_{ST}(\tau_{i-1}) & -X_{MT}(\tau_{i-1}) & 0_{1x3} \\ 0_{1x3} & X_{MT}(\tau_i) & -X_{LT}(\tau_i) \\ X'_{ST}(\tau_{i-1}) & -X'_{MT}(\tau_{i-1}) & 0_{1x3} \\ 0_{1x3} & X'_{MT}(\tau_i) & -X'_{LT}(\tau_i) \\ X''_{ST}(\tau_{i-1}) & -X''_{MT}(\tau_{i-1}) & 0_{1x3} \\ 0_{1x3} & X''_{MT}(\tau_i) & -X''_{LT}(\tau_i) \end{bmatrix}$$

$$R_1 = \begin{bmatrix} X_{ST}(\tau_{i-1}) & -X_{MT}(\tau_{i-1}) \\ 0_{1x3} & X_{MT}(\tau_i) \\ X'_{ST}(\tau_{i-1}) & -X'_{MT}(\tau_{i-1}) \\ 0_{1x3} & X'_{MT}(\tau_i) \\ X''_{ST}(\tau_{i-1}) & -X''_{MT}(\tau_{i-1}) \\ 0_{1x3} & X''_{MT}(\tau_i) \end{bmatrix} \quad R_2 = \begin{bmatrix} 0_{1x3} \\ -X_{LT}(\tau_i) \\ 0_{1x3} \\ -X'_{LT}(\tau_i) \\ 0_{1x3} \\ -X''_{LT}(\tau_i) \end{bmatrix},$$

By construction all rows of R_1 are linearly independent, then, the matrix is invertible with rank equal to M. As we have divided the matrix R into two sub-matrix, we also require the same process for the vector B of factors. Hence, the original constrain is re-expressed from $(R(\tilde{\phi}) = 0)$ to $(R_1(\tilde{\phi})\theta_1 + R_2(\tilde{\phi})\theta_2 = 0)$, where the vectors $\{\theta_1, \theta_2\}$ are adjusted to match the dimensionality of sub-matrix $\{R_1, R_2\}$.

$$F_t^i = \begin{pmatrix} a_t^i \\ b_t^i \\ c_t^i \end{pmatrix}, \theta_1 = \begin{pmatrix} a_t^{ST} \\ b_t^{ST} \\ c_t^{ST} \\ a_t^{MT} \\ b_t^{MT} \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} c_t^{MT} \\ a_t^{LT} \\ b_t^{LT} \\ c_t^{LT} \end{pmatrix}$$

Both $\{\theta_1, \theta_2\}$ are unobserved factors or parameters to be estimated, then, we can not know the value of one without the other. Below we show the process to capture the latent yields through the factors which will make our estimation procedure easier. First we need to state that θ_1 is the vector that shares a neighborhood with the square invertible matrix R_1 and can be represented as an algebraic product of the form:

$$\theta_1 = -R_1^{-1} R_2 \theta_2$$

3. ***Unconstrained Procedure***

Now we explains how to transform the constrained problem into an unconstrained one with lower dimensionality of parameters.

$$y_t(\tilde{\tau}) = W(\tilde{\tau})B_t + \epsilon_t(\tilde{\tau}), \quad B = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$
$$y_t(\tilde{\tau}) = w_1(\tilde{\tau})\theta_1 + w_2(\tilde{\tau})\theta_2 + \epsilon_t(\tilde{\tau})$$
$$y_t(\tilde{\tau}) = [w_2(\tilde{\tau}) - w_1(\tilde{\tau})R_1^{-1}R_2]\theta_2 + \epsilon_t(\tilde{\tau})$$
$$y_t(\tilde{\tau}) = Z(\tilde{\tau})\theta_2 + \epsilon_t(\tilde{\tau}), \quad dim(Z(\tilde{\tau})) = m_x(k+1)$$

We know from condition four that yields at knots are observed without error, however they are not force to be; If we choose a knot equal to maturity of an observed yield, we assume that the yield at that knot is observed with error.

$$y_t(\phi) = W(\phi)B, \;\; y_t(\phi) = w_1(\phi)\theta_1 + w_2(\phi)\theta_2, \;\; dim(y_t(\phi)) = (k+1)$$

The above equation allows us to introduce the remaining constrains (missing in R) in order to have a complete identification procedure. In all knots of the term structure we will require an exact fit compare to the observed curve.

$$y_t(\phi) = [w_2(\phi) - w_1(\phi)R_1^{-1}R_2]\theta_2 + \epsilon_t(\phi)$$
$$y_t(\phi) = Z(\phi)\theta_2 + \epsilon_t(\phi), \;\; dim(Z(\phi)) = m_x(k+1) \text{ invertible.}$$

If we define a matrix π, as a projection of the factor loadings of the observe yields onto the latent yields, then we can recover an estimate of the latent yields in the last expression.

$$\pi(\tilde{\tau}, \phi) = Z(\tilde{\tau})[Z(\phi)^{-1}],$$
$$y_t(\tilde{\tau}) = \pi(\tilde{\tau}, \phi)y_t(\phi) + \epsilon(\tilde{\tau}),$$
$$y_t(\phi) = Z(\phi)(Z(\tilde{\tau})'Z(\tilde{\tau}))^{-1}Z(\tilde{\tau}'y_t(\tilde{\tau})$$