Methoden der Algorithmik

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January 22, 2019

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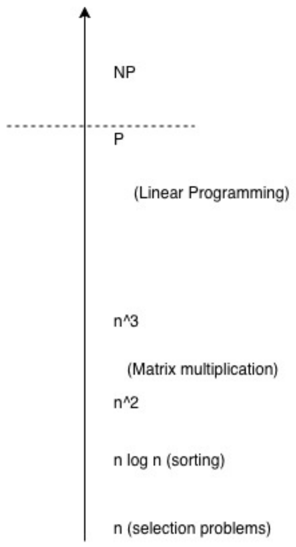


0.1 Complexity Analysis of Algorithms: time/space

0.1.1 Bubble-Sort

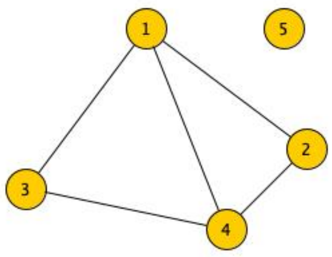
Array: after (i+1)-step, maximum in A[1...n-i-1]

```
Bubblesort(A[1..n])
    for(i=1...n)
        for(j=1...n-i)
            if(A[j+1]<A[j])
                swap(A[j+1],A[j])
```

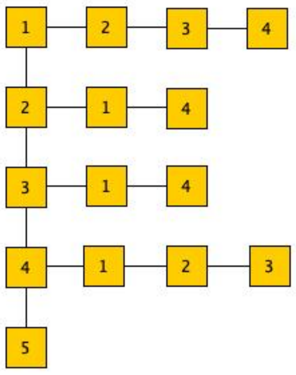
Complexity: $\mathcal{O}(n^2)$

0.2 Space complexity analysis

Given Graph:



0.2.1 Adjacency list



Graph has n vertices, m edges. Space requirement: $\mathcal{O}(n \cdot m) \Rightarrow$ Overestimation, holds for full graph, correct answer is $\mathcal{O}(n+m)$.

0.2.2 Adjacency matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Entry A[i,j] = 1, if $(i,j) \in E$. On diagonal we see self loops (1). Simple graph has 0s on diagonal. Space requirements: $\mathcal{O}(n^2)$, cause matrix is $n \times n$ big.

0.3 Lower bounds

Input Array A[1..n]

Output An Array in sorted form

$\exists$ algorithm that sorts in $\mathcal{O}(n \log n)$ time.

Question Can we do better? $\Leftrightarrow \exists$ algorithm to solve the problem in $o(n \log n)$

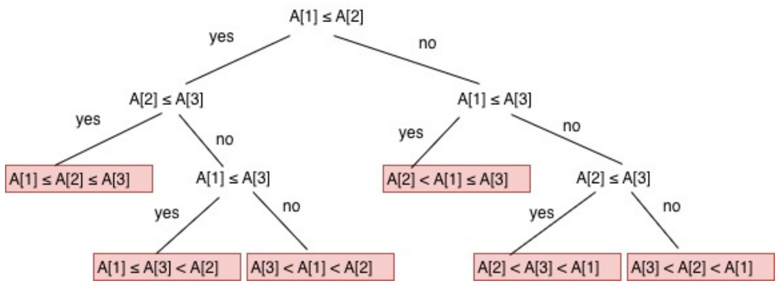
Remember:

- $f = \mathcal{O}(g)$ upper bound $\Leftrightarrow \exists c > 0 \exists x_0 > 0 : f(x) \leq c \cdot g(x) \forall x \geq x_0$
- $f = \Omega(g)$ lower bound $\Leftrightarrow \geq$
- $f = o(g)$ strict upper bound $\Leftrightarrow <$
- $f = \Theta(g) \Leftrightarrow f = \mathcal{O}(g) \wedge f = \Omega(g)$

Answer: No! Sorting is in $\Omega(n \log n)$. We can simulate any algorithm that sorts an array using comparisons with a decision tree.

0.3.1 Example

A[1],A[2],A[3].



$\Rightarrow$ We have n numbers, so $\#leaves = n!(= \#permutations) \Rightarrow$ height of decision tree $\geq \log(n!) = \log(1 \cdot 2 \cdot ... \cdot n/2) \cdot ...) \geq \log(n/2)^{n/2} = \mathcal{O}(n \log n) \Rightarrow$ we need at least $\log(n!)$ comparisons.

0.4 Mergesort

```
MergeSort(A,l,r)
    if(l = r)
        return A[l]
    m = roundDown((l+r)/2)
    A1 = Mergesort(A,l,m)
    A2 = Mergesort(A, m+l, r)
    B = merge(A1,A2) // O(r-l)
    return B
```

$$T(n) = \begin{cases} T(\lfloor \frac{n}{2} \rfloor) + T\left(\lceil \frac{n}{2} \rceil\right) + cn & n > 1 \\ 1 & n = 1 \end{cases}$$

$$\begin{aligned} n = 2^k \\ T(n) &= 2T(\frac{n}{2}) + cn \\ &= 2(2T(\frac{n}{2^2}) + c\frac{n}{2}) + cn \\ &= ... \\ &= 2^k T(\frac{n}{2^k}) + k \cdot c \cdot n = 2^{\log n} T(\frac{n}{2^{\log n}}) + c \cdot \log n \\ &= \mathcal{O}(n \log n) \end{aligned}$$

With that we have proven, that sorting is in $\Theta(n \log n)$.

0.5 Convex hull (in computational complexity)

Input Points in $\mathbb{R}^2$: $p_1 = (x_1, y_1), ...$

Output find a polygon (the smallest one) containing $p_1, ...p_n$
$\exists$ many algorithms that solve convex hull in $\mathcal{O}(n \log n)$ time! But: Can we do better = Can we solve the problem in $o(n \log n)$?

Idea CH $\leq o(n \log n)$ sorting (CH is reduced to sorting in the less than $\mathcal{O}(n \log n)$)
Instance: $x_1, x_2, ..., x_n$ to be sorted. $\Rightarrow$ CH-instance $(x_1, x_1^2), (x_2, x_2^2), ..., (x_n, x_n^2)$ (all of them are in CH). The order that appear on CH is a sorting of them (CH-instances drawn look like an exponential function, if sorted by first component). ¡br¿ if CH $= o(n \log n) \Rightarrow$ Sorting $= o(n \log n) \Rightarrow$ that doesn't work with $\Omega(n \log n)$.

0.6 Example Recursions

$$\begin{aligned}
T(n) &= 2T(\sqrt{n}) + \log n & &|m = \log n \\
T(2^m) &= 2T(2^{m/2}) + m & &|S(m) = T(2^m) \\
S(m) &= 2S\left(\frac{m}{2}\right) + m & &|\text{from before} \\
&\in \mathcal{O}(m \cdot \log m) = \mathcal{O}(\log n \cdot \log\log n)
\end{aligned}$$

$$\begin{aligned}
T(n) &= 3T\left(\frac{n}{4}\right) + n \\
&= 3\left(3T\left(\frac{n}{16}\right) + \frac{n}{4}\right) + n \\
&= 3^2 T\left(\frac{n}{4^2}\right) + \frac{3}{4}n + n \\
&= 3^2 T\left(3T\left(\frac{n}{4^3}\right) + \frac{n}{4^3}\right) + \frac{3}{4}n + n \\
&= 3^3 T\left(\frac{n}{4^3}\right) + \frac{3^2}{4^2}n + \frac{3}{4}n + n \\
&= c \cdot 3^{\log_4(n)} + n \cdot \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{4}\right)^i & &|\sum_{i=0}^{\infty} q^i \overset{(q<1)}{\to} \frac{1}{1-q} \\
&\leq n^{\log_4 3} + n \cdot \frac{1}{1 - \frac{3}{4}} & &|a^{\log_b n} = n^{\log_b a}; n^1 > n^{\log_4 3} \\
&= o(n) + 4 \cdot n = \mathcal{O}(n)
\end{aligned}$$

0.7 Logarithmic rules

1. $\log a^b = b \log c$
2. $\log(a \cdot b) = \log a + \log b$
3. $x = b^{\log_b x}$
4. $\log_a x = \frac{\log_b x}{\log_b a}$

Proof $3^{\log_4 n} = n^{\log_4 3}$

$$\begin{aligned}
&3^{\log_4 n} = n^{\log_4 3} \\
&\Leftrightarrow \log_4 n = \log_3(n^{\log_4 3}) & &|\text{rule 1} \\
&\Leftrightarrow \log_4 n = \log_4 3 \cdot \log_3 n \\
&\Leftrightarrow \frac{\log_4 n}{\log_4 3} = \log_3 n \\
&\Rightarrow \text{true with rule 4}
\end{aligned}$$

□

0.8 One more recursion example

Given: A, B, $n \times n$ matrices. $A \times B = C$

Take every matrices and split it into partitions (for example quarters: $\begin{pmatrix} Q1 & Q2 \\ Q3 & Q4 \end{pmatrix}$) but you loose some parts in C. But do it more often and combine solutions. For example you have A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 and we have the Partitions C_1, C_2, C_3, C_4 in our solutions matrices. We do it like we had no partitions, so we need to multiply and add the correct Partitions of A and B:

- $C_1 = A_1 \cdot B_1 + A_2 \cdot B_3$
- $C_2 = A_1 \cdot B_2 + A_2 \cdot B_4$
- $C_3 = A_3 \cdot B_1 + A_4 \cdot B_3$
- $C_4 = A_3 \cdot B_2 + A_4 \cdot B_4$

That leads to the recursion formula (*irgendwas bisschen komsich aber naja passt so in etwa*):

$$\begin{aligned}
T(n) &= 8 \cdot T\left(\frac{n}{2}\right) + \mathcal{O}(n^2) \\
&= 8 \cdot \left(8 \cdot T\left(\frac{n}{2^2}\right) + c\frac{n^2}{2^2}\right) + c \cdot \frac{n^2}{2} \\
&= 8^2 \cdot T\left(\frac{n}{2^2}\right) + c \cdot \frac{8}{2^2} n^2 + c \cdot n^2 \\
&= 8^{\log n} + c \cdot n^2 \cdot \sum_{i=0}^{\log n - 1} \left(\frac{8^i}{2^{2i}}\right) & \mid \sum_{i=0}^{\log n - 1} \left(\frac{8^i}{2^{2i}}\right) = \sum_{i=0}^{\log n - 1} 2^i = 2^{\log n} - 1 \to n \\
&= n^{\log_2 8} + c \cdot n^2 \cdot n \\
&= n^3 + c \cdot n^3 = \mathcal{O}(n^3)
\end{aligned}$$

Other way for matricemultiplication:
$P_1 = A_1 \cdot (B_2 - B_4)$, $P_2 = B_4 \cdot (A_1 - A_2)$,...,$P_7 = (A_1 - A_3) \cdot (B_1 + B_2)$
$C_1 = P_5 + P_4 - P_2 + P_6$, $C_2 = ...,...$
Recursion formula: $T(n) = 7 \cdot T\left(\frac{n}{2}\right) + \mathcal{O}(n^2)$, so we only have $n^{\log_2 7} \approx n^{2.8} < n^3$. This is called STRASSEN - schema

0.9 Analyse complexity/costs with probabilities (average analysis)

```
find_min(Array A)
    min = A[0] //c1
    for(i=1,...n-1)
        if(A[i] < min) //c2
        min = A[i] // c3
    return(min)
```

Total costs: $c = c_1 + (n-1) \cdot c_2 + (n-1) \cdot c_3$

Get probabilities for doing c_3:

$$P(A[1] < A[0]) = \frac{1}{2}$$

$$P(A[2] < min(A[0], A[1])) = \frac{1}{3}$$

$$P(A[i] < min(A[0], ..., A[i-1])) = \frac{1}{i+1}$$

Now we can calculate the costs with these probabilities:

$$X_i = \begin{cases} 1 & if(A[i-1] < min(...)) \\ \\ 0 & otherwise \end{cases}$$

$$c = c_1 + (n-1) \cdot c_2 + \sum_{i=1}^{n-1} X_i$$

$$E[X_i] = P(X_i = 1) = \frac{1}{i+1}$$

$$E\left[\sum_{i=1}^{n-1} X_i\right] = \sum_{i=1}^{n-1} E[X_i] = \sum_{i=1}^{n-1} \frac{1}{i+1} = \log n$$

0.10 Armortised Analysis

Stack S with `PUSH(x)`$(\mathcal{O}(1))$ and `POP()`$(\mathcal{O}(1))$ and `MULTI-POP(k)` $(\mathcal{O}(min(|S|, k)))$ operation.
n operations: $\mathcal{O}(n^2)$ because worst operation (‘MULTI-POP(k)‘) is linear. That’s pessimistic.

0.10.1 Accountingmethod

cost	operation
2 coins	`push(x)`
0 coins	`pop()`
0 coins	`multi-pop(k)`

$\Rightarrow n$ opertions cost $\leq 2n \Rightarrow \mathcal{O}(n)$

if `pop()` or `multi-pop(k)` are applied, at least 1 or k Elements are on the stack $\Rightarrow k$ coins are on the stack.

0.10.2 Potentialmethod

”function that describes the bank/the benefit one get doing cheap operations in order to do bad operations”

- D Data structure
- i step
- Φ real value ≥ 0
- $\hat{c_i}$ amortized cost
- c_i cost of operation
- $D_0 \Rightarrow \Phi(D_0) = 0$
- $D_i(i > 0) \Rightarrow \Phi(D_i) \geq i$

$$\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1})$$

- `push(x)` amortized: $1 + |S| + 1 - |S| = 2$
- `pop()` amortized: $1 + |S| - 1 - |S| = 0$
- `multi-pop(k)` amortized: $min(|S|, k) + |S| - min(|S|, k) - |S| = 0$

$$\sum_{i=1}^{n} \hat{c_i} = \sum_{i=1}^{n} (c_i + \Phi(D_i) - \Phi(D_{i-1}) = \sum_{i=1}^{n} c_i + \sum_{i=1}^{n} (\Phi(D_i) - \Phi(D_{i-1})) = \sum_{i=1}^{n} c_i + \Phi(D_n)[\geq 0] - \Phi(D_0)[= 0]$$

1 MST

cost/weight function $w : E \to \mathbb{N}/\mathbb{R}$

1.1 Red rule

$\forall$ cycle colour red the most expensive edge. $\Rightarrow$ the graph without all the red edges is a MST.

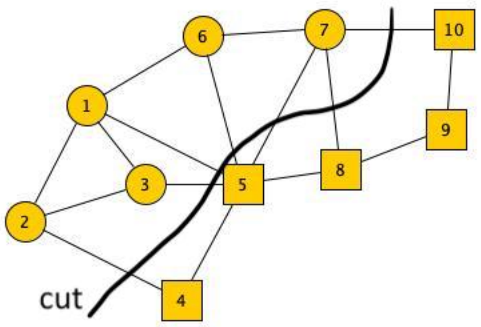
Search cycle; $m = \#edges, n = \#vertices, m \geq n-1 \Rightarrow \mathcal{O}(n)$

Selecting max: $\mathcal{O}(\#\text{vertices in the cycle}) = \mathcal{O}(n)$

How many times do one needs to search in the graph for a cycle:
$\mathcal{O}(m-n) = \mathcal{O}(m)$ (until the graph is tree)

$$\Rightarrow \mathcal{O}(n) \cdot \mathcal{O}(m) = \mathcal{O}(m \cdot n) = \mathcal{O}(n^3)$$

1.2 Cut-concept



- separate the vertices into two sets $V = X \cup X'$, where X, X' are non trivial sets.
- all the edges that cut the partition/cross the cut
- $\exists$ MST that contains the edge with the minimum weight.

1.3 Blue rule

$\forall$ cut, colour blue the cheapest edge.

1.4 Invariant

$\exists MST(G)T : B \leq T, R \cap T = \emptyset$
B = set of all the blue coloured edges
R= set of all the red coloured edges

Proof base case: $B = \emptyset, R = \emptyset$ holds, because there exists a MST because of all the trees in G, just take the minimum one. Therefore $B \leq T\checkmark, R \cap T = \emptyset\checkmark$

inductive step:
RED RULE: edge e
in the previous step e is not coloured

1. edge is not contained in T $\Rightarrow$ Red rule holds

2. edge is contained in T $\Rightarrow$ walk along the cycle and take a edge into T that $w(e') \leq (e) \Rightarrow$ holds

BLUE RULE: edge e
was not coloured in the previous step

1. e is part of T $\Rightarrow$ Blue rule holds

2. e is not part of T; $e = (u, v)$, (u, v) were connected with a path and $u \in X$ and $v \in X'$ for every transition along the path $(v' \to v'') = e'$. $W(e') \geq w(e)$ so it is possible to just take e and leave our e'.

COMPLETENESS Suppose there is a situation where no rule can be applied to an uncolored edge e
blue edges: there is no cycle $\Rightarrow$ blue edges form a forest

1. e connects two vertices in the same tree $\Rightarrow$ RED RULE can be applied ($\Rightarrow$ colour it red)

2. e connects two different trees $\Rightarrow$ look at the cut of the two different treefs (B) $\Rightarrow$ BLUE RULE can be applied

There are 7 algorithms that deal with the problem.

1.5 Kruskal

Sort the edges in non-decreasing order. Consider the edge next in the order (u, v).
if u and v belong to the same subtree $\Rightarrow (u, v)$ red
if u and v connect two different subtrees $\Rightarrow (u, v)$ blue
$\mathcal{O}(m \cdot \log m)$

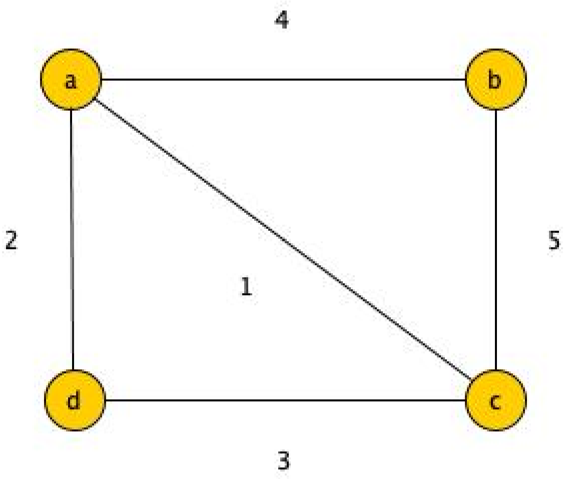
2 Minimum Cut Problem

Input a graph $G = (V, E, w), W : E \to \mathbb{R}^+ [s \neq t, s, t, \in V]$

Output a partition of V into S,T (T = V S) s.t.

(i) $S \neq \emptyset \wedge T \neq \emptyset [s \in S, t \in T]$

(ii) $w(S,T) = \sum_{u \in S, v \in T} w(u,v)$ is minimum over all partitions



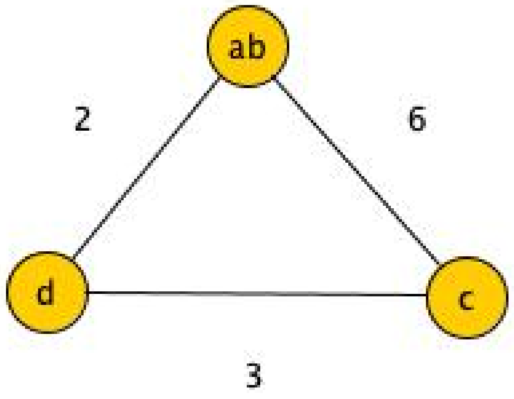
S	$w(S,T)$
a	7
b	9
c	9
d	**5**
a,b	8
b,c	8
c,d	8
d,a	8

(Max-Cut Problem is NP complete)

Observation MinCut $\overset{n^2}{\leq}$ Min(s-t)-Cut $\to$ can we do better: $\wr(n^2)$

```
MinCut(G)
min <- infty
for each s in V
    for each t in V: s != T // two fors -> n^2
        (S,T) <- Min-s-t-Cut(s,t)
        if(S,T) is lighter than min // in Term of weights
            min <- (S,T)
```

Merging a and b in example



Edge ab - c is sum of weight (5 and 1)

2.0.1 Operation: Merge of two verties

Given a graph $G = (V, E, w)$ & $s, t \in V$, $s \neq t$ the Graph $G\ \{s, t\} = (V', E', w')$ obtained by merging s, t.

$$V' = V\ \{s, t\} \cup \{X_{s,t}\}$$

$$E' = E\ \{(u, v) \in E : u, v \in \{s, t\}\} \cup \{(X_{s,t}, v) : (s, v) \in E \vee (t, v) \in E\}$$

$$w'(X_{st}, v) = \begin{cases} w(s,v) & (s,v) \in E\&(t,v) \notin E \\ w(t,v) & (s,v) \notin E\&(t,v) \in E \\ w(s,v) + w(t,v) & (s,v) \in E\&(t,v) \in E \\ 0 & otherwise \end{cases}$$

2.1 Theorem 1

Let $s, t \in V$ of a graph $G = (V, E)$ & let $G\ \{s, t\}$ be the graph obtained by G by merging s & t.
Then:

$$min - cut(G) = min\{min - (s, t) - cut(G), minCut(G\ \{s, t\})\}$$

Proof Let (S, T) be a min cut of G
$\Rightarrow s, t \in \vee \{s, t\} \in T$ [first part]
or
$s \in S$ & $t \in T \vee t \in S$ & $s \in T$ [second part]

□

```
n = |V(G)|
if(n=2) return ({v1},{v2})
else let s,t in V, s != t
    (S,T) <- min-(s,t)-cut for G exists algorithm
    (S',T') <- MinGut(G \ {s,t}) // recursion
    return min{(S,T),(S',T')}
```

Theorem 1 $\Rightarrow$ Correctnes

Complexity $\approx (n-1) \cdot c$, where c min-(s,t)-cut(G)

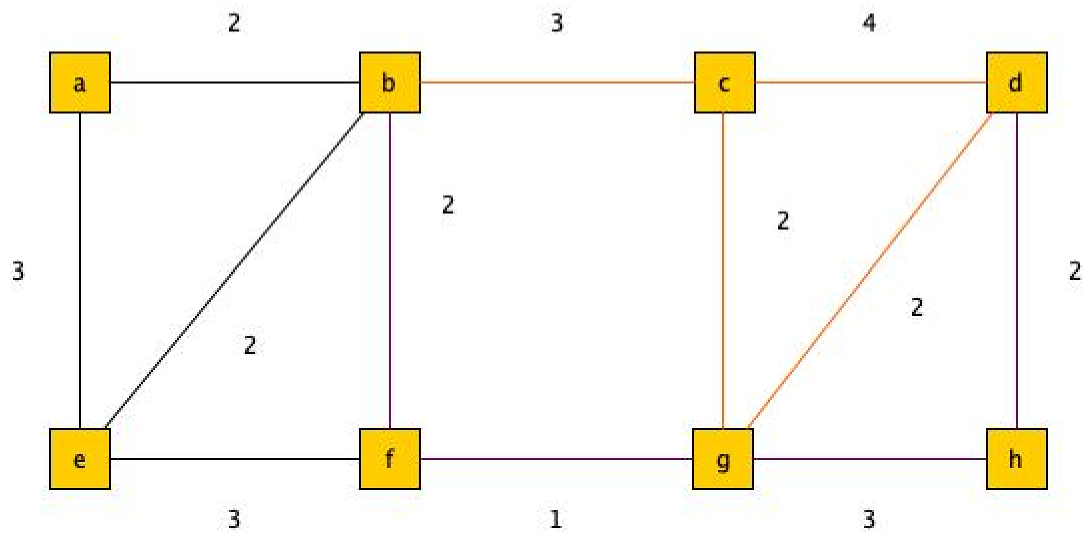
Observation in linear time reduceable $\mathcal{O}(n)$

2.2 Min-(s,t)-Cur Problem (Stoer & Wagner)

Idea

- finds an ordering of the vertices $v_1, .., v_n$, $n = |V(G)|$: $(\{v_1, v_2, ...v_{n-1}\}, \{v_n\})$ is a min (v_{n-1}, v_n)-cut
- Let $S_i = \{v_1, ..., v_i\}$ starting from an arbitrary vertex v_1. Then v_{i+1} is selected $s,t.$ $w(v_{i+1}, S_i) \geq w(v, S_i) \forall v \in V\ \ S_i$

2.2.1 Example



i	v	S
1	b	b
2	c	b,c
3	d	b,c,d
4	g	b,c,d,g
5	h	b,c,d,g,h
6	f	b,c,d,g,h,f
7	e	b,c,d,g,h,f,e

$\Rightarrow$ Stoer & Wagner $(\{a\}, \{b, c, d, g, h, f, e\})$ must be a min (a,e)-cut.

```
MinCutPhase(G)
    v1 = any vertex of G
    S = {v1}
    for i = 2 ... n do
        vi <- w({bi},S) >= w({v}, S), v not in V \ S
```

```
      S = untion(S, {vi}) // ordered set
  return ({vn},{v1,...,n(n-1)})
```

$$\forall v \in V\ S : key(v) = \sum_{v' \in S} w(v, v')$$

v_{i+1} in the vertex with max key

2.2.2 Fibonacci Heaps

$\mathcal{O}(Extract - Max) = \mathcal{O}(\log n)$

UpdateKey $= \mathcal{O}(1)$

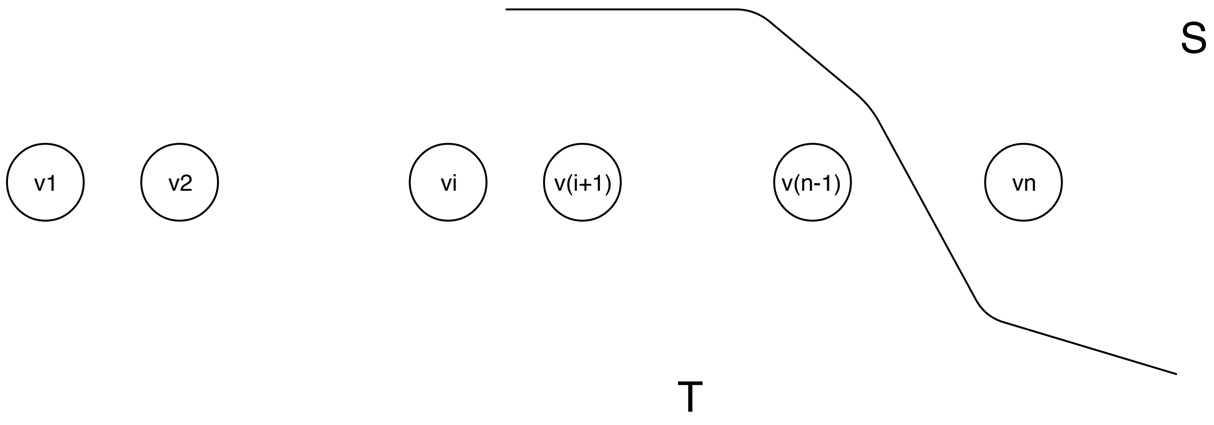
in total: $|V|\times$ Extract Max and $|E|\times$ UpdateKey $\Rightarrow \mathcal{O}(|V| \log |V| + |E|)$

2.3 Theroem 2

$(T, s) = (\{v_1, ..., v_{n-1}\}, \{v_n\})$ is a min $(v_{n-1},\ v_n)$-cut

Proof Let (S, T) be any (v_{n-1}, v_n) cut. We prove:

$$w(\{v_1, ...mv_{n-1}\}, \{v_n\}) \leq w(S, T)$$



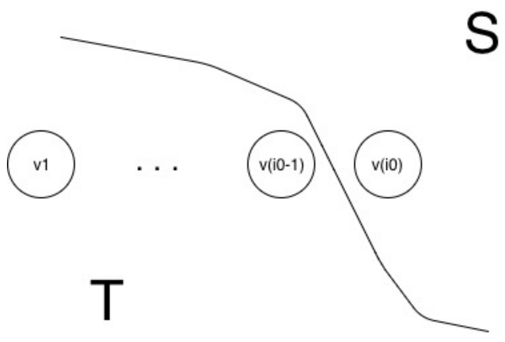
Critical: v_i is critical iff $\Leftrightarrow (v_i \in S \& v_{i-1} \in T)$ or $(v_i \in T \& v_{i-1} \in S) \Rightarrow v_n \in S \& v_{n-1} \in T$

$A_i := w(\{v_1, .., v_{i-1}\}, \{v_i\}) \leq w(\{v_1, ..., v_i\} \cap S, \{v_1, ..., v_i\} \cap T) =: B_i$

$\forall critical vertex v_i \Rightarrow$ Theorem is proved

By induction on the number of critical vertices

- for the critical vertiex v_{i_0} :



$\Rightarrow A_{i_0} \leq B_{i_0} \checkmark$

- $A_i \leq B_i$ holds up to the critical vertex v_i [Induction hypothesis]
- let v_j be the next critical vertex

$$\begin{aligned} A_j &= w(\{v_1, ..., v_{j-1}\}, \{v_j\}) \\ &= w(\{v_1, ..., v_{i-1}\}, \{v_i\}) + w(\{v_1, ..., v_{j-1}\}, \{v_j\}) \\ &\overset{Stoer+Wagner}{\leq} w(\{v_1, ..., v_{i-1}\}, \{v_i\}) + w(\{v_1, ..., v_{i-1}\}, \{v_i\}) \\ &\overset{IH}{\leq} w(\{v_1, ..., v_i\} \cap S, \{v_1, ..., v_i\} \cap T) + w(\{v_1, ..., v_{j-1}\}, \{v_j\}) \\ &\leq w(\{v_1, ..., v_j\} \cap S, \{v_1, ..., v_j\} \cap T) \end{aligned}$$

□

Claim $w(\{v_1, ..., v_{j-1}\}, \{v_j\}) \leq w(\{v_1, ..., v_{j-1}\} \cap S, \{v_1, ..., v_j\} \cap T)$

Proof

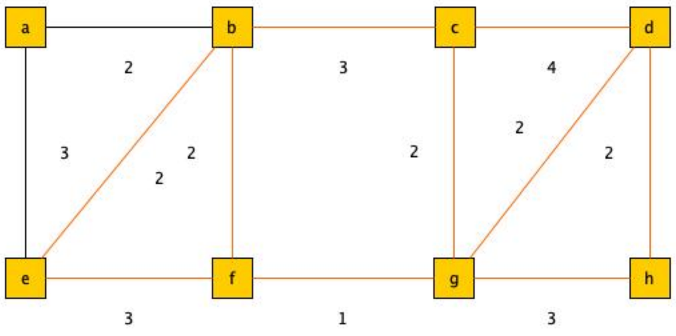
$$\begin{aligned} w(\{v_1, ..., v_{j-1}\}, \{v_j\}) &= w(\{v_1, ..., v_{i-1}\}, \{v_j\}) + w(\{v_i, ..., v_j\}, \{v_j\}) \\ &\overset{Stoer+Wagner}{\leq} w(\{v_1, ..., v_{i-1}\}, \{v_i\}) + w(\{v_i, ..., v_j\}, \{v_j\}) \\ &\overset{I.H.}{\leq} w(\{v_1, ..., v_i\} \cap S, \{v_1, ..., v_i\} \cap T) + w(\{v_i, ..., v_j\}, \{v_j\}) \\ &\overset{\text{By the fact that vj is critical}}{\leq} w(\{v_1, ..., v_j\} \cap S, \{v_1, ..., v_j\} \cap T) \end{aligned}$$

Code

```
MinCut(G)
    n=|V(G)|
    if n = 2 return ({v1},{v2})
    else
        ({v1,...,v(n-1)}{vn}) <- MinCutPhase(G) // V logV+E (Stoer Wagner)
        (S,T) <- MinGut(G\{s,t}) // O(V) (first part in complexity)
        return the lighter of (S,T) and ({v1,...,v(n-1)}{vn})
```

Complexity $\mathcal{O}(V(V \log V + E))$

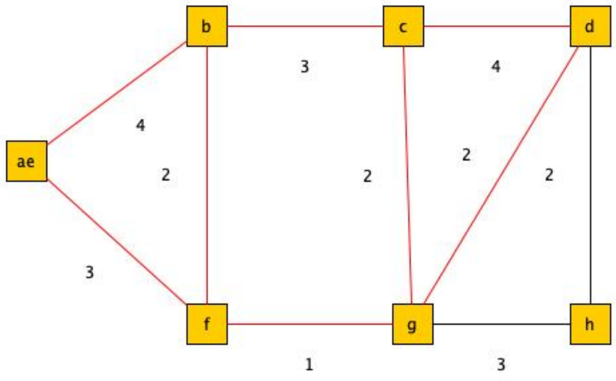
2.3.1 Example



$v_1 = \text{b}$
$v_2 = \text{c}$
$v_3 = \text{d}$
$v_4 = \text{g}$
$v_5 = \text{h}$
$v_6 = \text{f}$
$v_7 = \text{e}$
—
$v_8 = \text{a}$

$\Rightarrow w(\{v_1, ..., v_7\}, \{v_8\}) = 5$

$\Rightarrow$ merge a and e



$v_1 = \text{b}$
$v_2 = \text{ae}$
$v_3 = \text{f}$
$v_4 = \text{c}$
$v_5 = \text{d}$
—
$v_6 = \text{g}$

$\Rightarrow w(\{v_1, v_2, ..., v_6\},^{\{} v_7\}) = 5$

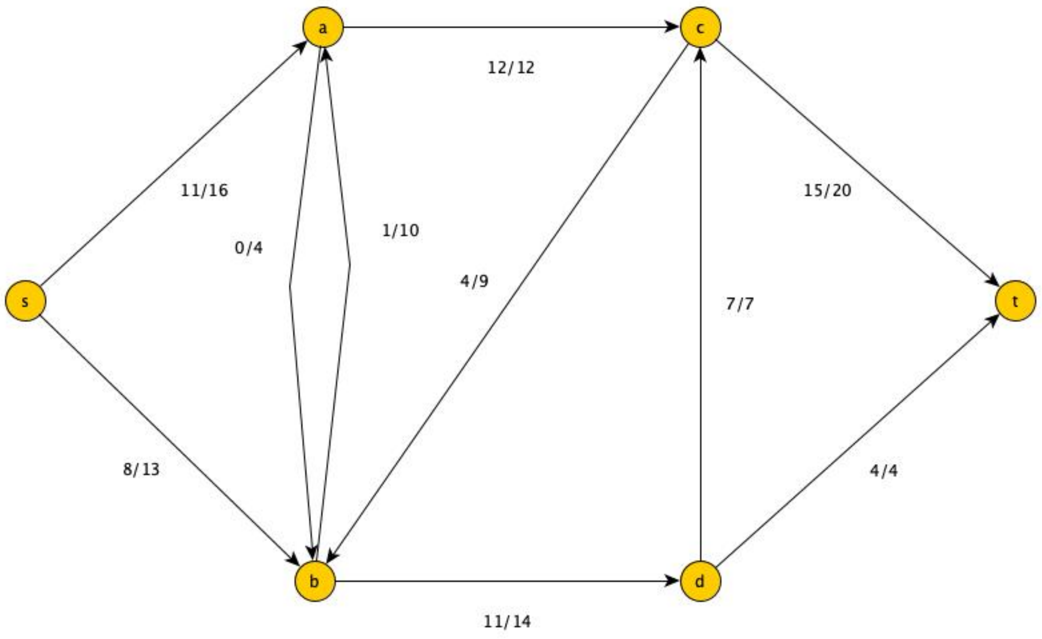
$\Rightarrow$ merge h and g

go on like that.

3 The Max Flow problem

Input

- A directed weighted graph (network) $D = (V, E, c)$
- $c : E \to \mathbb{R}^+$ capacity
- $s, t \in V : s \neq t \& indegree(s) = 0 \& outdegree(t) = 0$
- s: source, t: target



Syntax: x/y: x is the flow, y is capacity. —f— = 19
Informal: The max quantity of items that can be send from s to b without deviating the capacities & without storing anything in between.

Output $f : E \to \mathbb{R}^+$: flow function

(i) $0 \leq f(u, v) \leq c(u, v) \forall (u, v) \in E$

(ii) $\sum_{(u,v)\in E} f(u, v) = \sum_{(v,u)\in E} f(v, u) \forall v \in V \ \{s, t\}$

(iii) $\sum_{(s,v)\in E} f(s, v)$ is maximized $(s, v) \in E$ over all flow functions

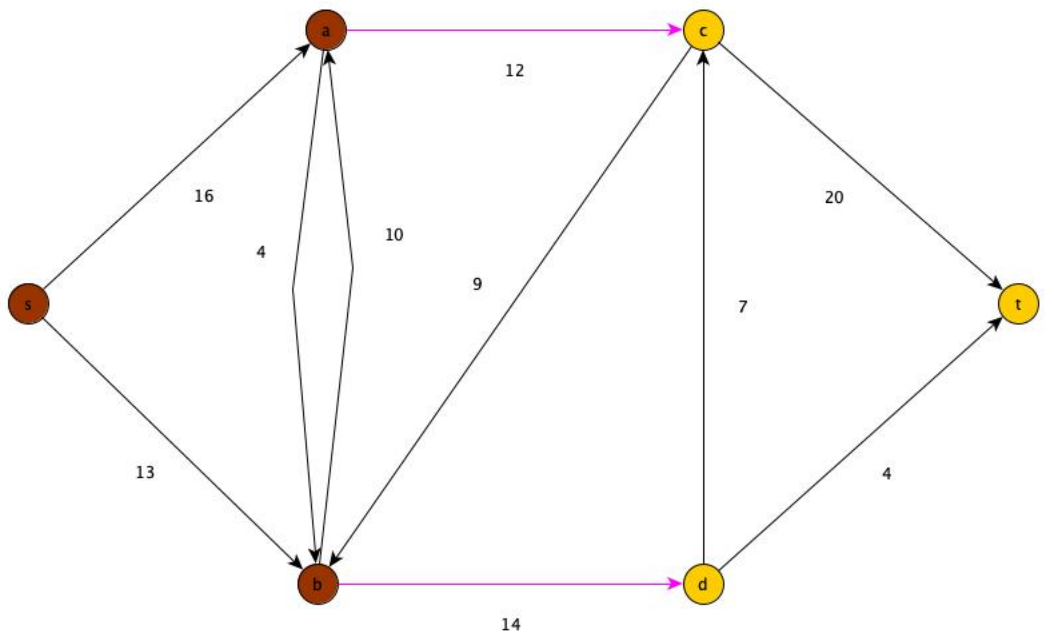
(i) called **capacity constraint**

(ii) called **flow consertation**

(iii) **called max flow** denoted by $|f^*|$

inflow (flow coming in) and outflow (flow going out) on v

3.1 Example



Cut brown (S) and yellow (T), so $c(S,T) = 26(= 12 + 14)$ and $|l| \leq c(S,T) \forall f$

3.2 Lemma

$$\forall \text{flow of D} \& \forall cut(S,T) : s \in S \& t \in T : |f| \leq c(S,T)$$

Proof

$$|f| = \sum_{(s,v)\in E} f(s,v) - \sum_{(v,s)\in E} f(v,s) + \underbrace{\left(\sum_{(u,v)\in E; u\in S\ \{s\}} f(u,v) - \sum_{(u,v)\in E; u\in S\ \{s\}} f(v,u) \right)}_{[0 \text{ by (ii)}]}$$

$$= \sum_{u\in S} \left(\sum_{(u,v)\in E} f(u,v) - \sum_{(u,v)\in E} f(v,u) \right)$$

$$= \underbrace{\left(\sum_{(u,v)\in E/u,v\in S} f(u,v) - \sum_{(u,v)\in E/u,v\in S} f(v,u) \right)}_{[0 \text{ by (ii)}]} + \left(\sum_{(u,v)\in E/u\in S, v\in T} f(u,v) - \sum_{(u,v)\in E/u\in S, v\in T} f(v,u) \right)$$

$$= \sum_{(u,v)\in E/u\in S, v\in T} f(u,v) - \underbrace{\sum_{(u,v)\in E/u\in S, v\in T} f(v,u)}_{[\text{this part} \geq 0 \text{ by (i) so cut out}]}$$

$$\leq \sum_{(u,v)\in E/u\in S, v\in T} f(u,v) \overset{(i)}{\leq} \sum_{(u,v)\in E/u\in S, v\in T} c(u,v) = c(S,T)$$

3.3 Corolary 1

$|f^*| \leq c(S,T) \forall (S,T)$ being an (s,t)-cut

3.4 Corolary 2

If there exists a flow f and (s,t)-cut (S,T):

$$|f| = c(S,T) = \sum_{(u,v)\in E, s\in S, v\in T} c(u,v)$$

$$\Rightarrow |f^*| = |f| it's optimal s \in S, v \in T$$

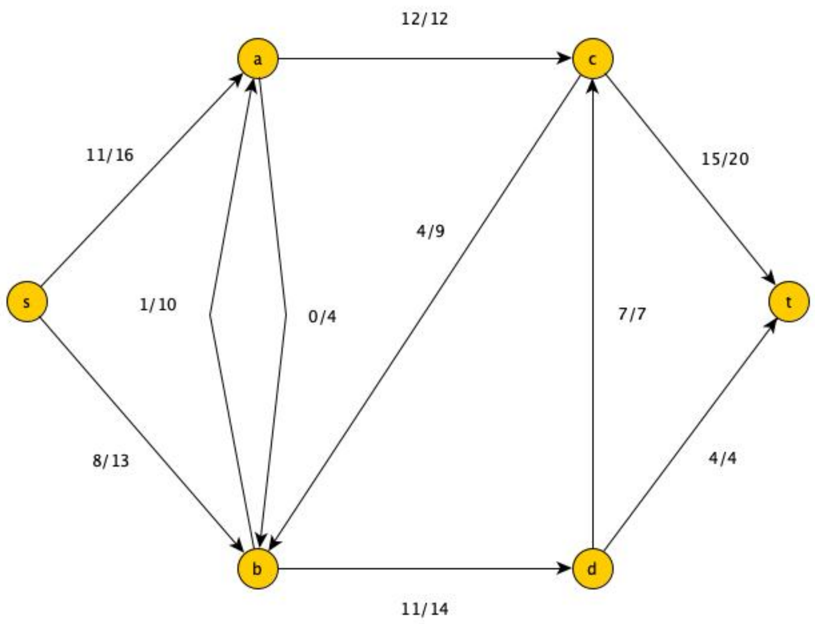
3.5 Max-flow/min-cut Theorem:

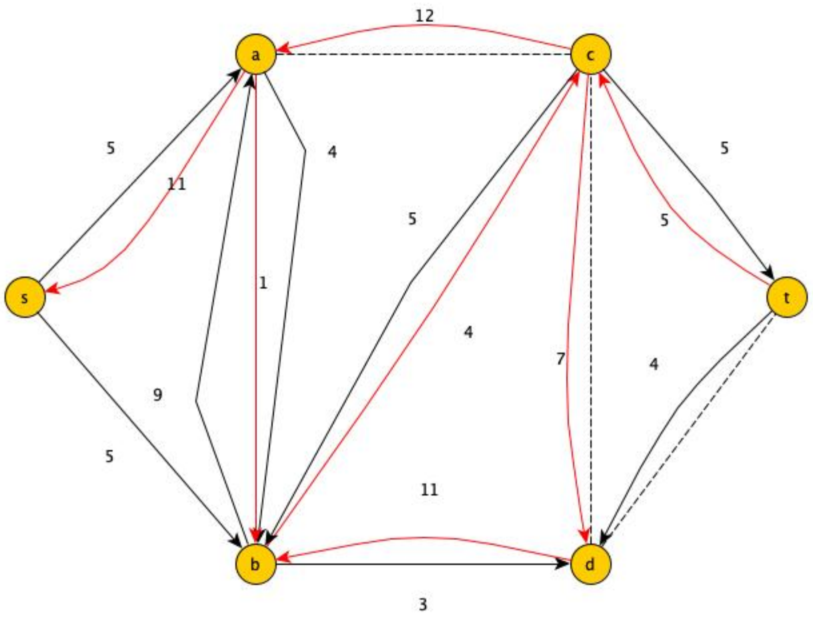
$$|f^*| is max \Leftrightarrow \exists (s,t) - cut(S,T) : |f| = c(S,T)$$

3.6 Augmenting Path

(s,t)-Path in D align which we can push flow (and augment the flow)

```
Ford-Fulkerson (G,s,t)
    initialize f to 0
    While(esists augmenting path P from s to t)
        Augment f by the bottleneck of P
    return f
```



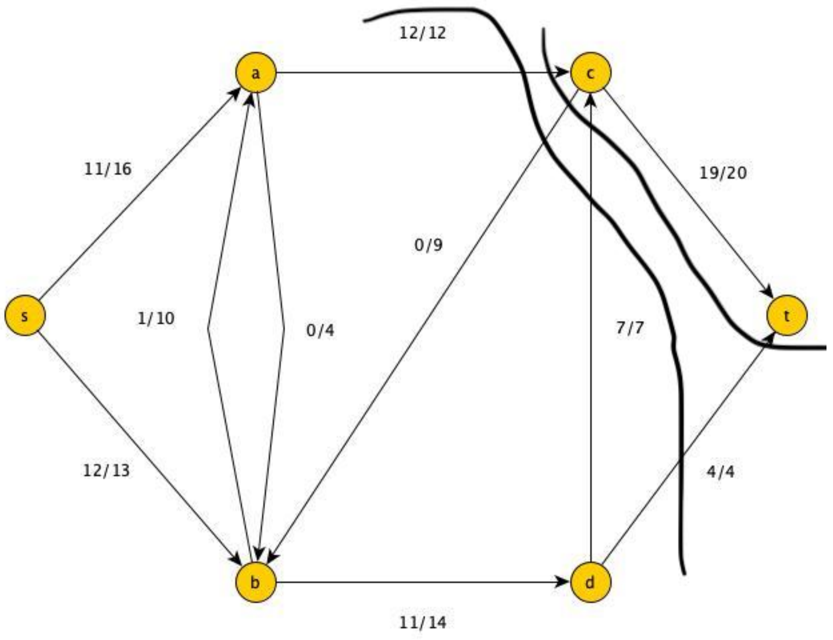


3.7 Residual Network

Describes for every edge (u, v) ∈E the amount of flow that can be further pushed along (u, v).

$$c_f = (V_f, E_f) : V_f = V, E_f = \{(u, v) : c_f(u, v) > 0\},$$

$$c_f \text{ residual capacity} \rightarrow c_f = \begin{cases} c(u,v) - f(u,v) & (u,v) \in E \\ f(v,u) & (v,u) \in E \\ 0 & othehrwise \end{cases}$$



path: (s,b,c,t), $|f| = 24 > 19, c(S,T) = 24 \Rightarrow f$optimal

Question Is the algorithm correct?

Augmenting Path Theorem:

f^* is max flow $\Leftrightarrow \nexists$ augmenting path inG_f^*

Max-Flow Min-Cut Theorem:
f^* is max flow $\Leftrightarrow \exists (s,t)-cut(S_0,T_0), |f^*| = c(S_0,T_0)$

3.7.1 Theorem

(i)(ii)(iii) are equivalent:

(i) f^* is max flow

(ii) $\nexists$ augmenting path in G_f^*

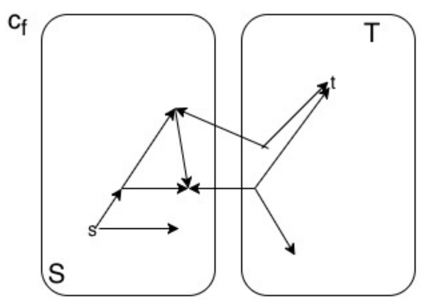
(iii) $\exists (s,t)-cut(S_0,T_0)$: $|f^*| = c(S_0,T_0)$

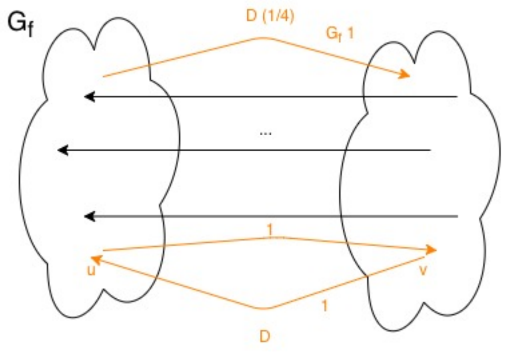
(i) $\Rightarrow$ (ii)$\Leftrightarrow \neg$ (ii) $\Rightarrow \neg$(i)
$\neg$(ii) $\Rightarrow \exists$ aug. path $\Rightarrow$ aug. the path $\Rightarrow f^*$ was not max

$\square$

(ii) $\Rightarrow$ (iii)

Example





S: vertices that are reachable by s in G_f
T = V S
(ii) $\Rightarrow T \neq \emptyset (t \in T), (S \neq \emptyset, T \neq \emptyset)$
$|f^*| = ... = \sum_{u,v \in E, u,v \in S} f(u,v) - \sum_{u,v \in E, v,u \in S} f(v,u) + \sum_{u,v \in E, u \in S, v \in T} f(u,v) - \sum_{u,v \in E, u \in S, v \in T} f(v,u)$
G_p s.a.
if$(u,v) \in E \Rightarrow f(u,v) = c(u,v)$ full
if$(v,u) \in E \Rightarrow f(u,v) = 0$ empty

$\square$

(iii) ⇒ (i)
see last lecture, follows from the fact that

$$\forall f \forall (s,t) - cut(S,T) : |f| \leq c(S,T)$$

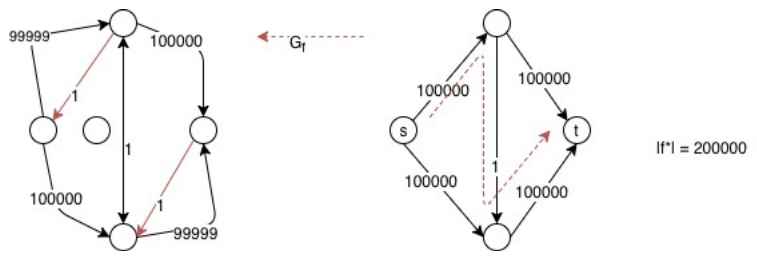
If capacities are integers, then $|f^*|$ is also an integer
theorem (iii) $|f^*|$ is sum of integers

```
FordFulkerson(D,s,t)
    for each edge (u,v)\in E // O(|E|)
        f[u,v] <- 0
        flow[u,v] <- 0
    compute G_f // O(|V|+|E|)
    While(exists aug path P from s to t in G_f) // P * O(|E|) oder so aehnlich
        x = min{c_f(u,v): (u,v) in P}
        for earch edge (u,v) in P # O(|E|)
            if(u,v) in E f[u,v] = f[u,v] + x
            if(v,u) in E f[v,u] = f[v,u] - x
            update G_f // O(|V|)
```

if capacities are not integers, then FordFulkersion algorithm may not terminate
Assumption: all capacities are integers ⇒ $|f^*|$ is integer

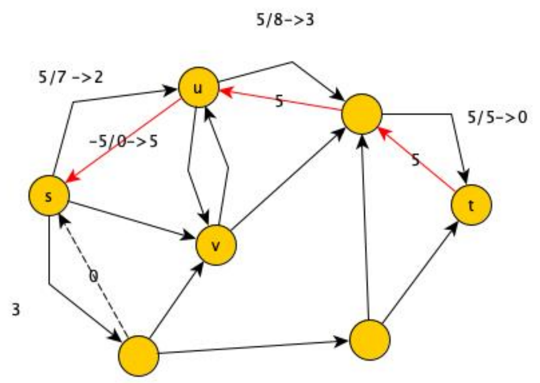
At each interation the flow is augmenting by at least 1 unit (the augmenting path goes through an edge (u,v): $c_f(u,v) = 1$)
⇒ While-loop terminates after $|f^*|$ steps
⇒ Complexity: $\mathcal{O}(|f^*| \cdot |E|)$

1. $\mathcal{O}(|f^*| \cdot |E|)$ depends on the size of the output
2. ∃ instances that require $\mathcal{O}(|f^*| \cdot |E|)$



Can't pass more then max flow on edge. So on more paths the sum still has to be smaller than the max flow on each edge. To calculate max flow efficiently you need to find the min cut.

Idea Choose a path from s to t. So how much flow can be passed through that path? ⇒ smallest max flow on an individual edge on the path. Push this amount through the path. (At least on edge (with the smallest max flow) is now full). So it translates to the following graph with the capacities $(u,v) \Rightarrow c(u,v) - f(u,v)$. Non existing edges have capacity 0 and get - the flow the edge in the other direction has. (red in graph):



Continue with other paths. With the backwards edges the algorithm can tell that the push over this edge was a bad decision by wanting to push over this edge. So we leave the algorithm the freedom to backtrack.
Maximum complexity: $\mathcal{O}(|E| \cdot f^*)$.

3.8 EdmondKarp

Search for the shortest path in the Graph ($\#edges$). Augmenting path s to t is the shortest from s to t. $\Delta_f(v)$ = min number of edges in G_f from s to v. We only need to prove the complexity. (Correctness follows from `FordFulkerson`)
$\mathcal{O}(|V| \cdot |E|^2)$ complexity on `EdmondKarp`. Wen want to prove that we have $\mathcal{O}(|V| \cdot |E|)$ steps (cause shortest path is $\mathcal{O}(|E|)$.

Proof At each step there is a bottleneck edge. We want to show that every edge becomes bottleneck once. Because we have more than $|E|$ steps some need to be bottleneck twice. So we prove:

$$\forall (u,v) : (u,v) \text{ is a BN at most } \mathcal{O}(|V|) \text{ times}$$

We need to state on more property: One edge on the residual network goes to zero (bottleneck). $\Delta_f(w)$ = |shortest path from s to w in G_f| By changing flow (with the algorithm) we change the network (add and drop edges). By adding an edge there may be a shorter path from s to w.
We have flow f and after a step we have f' with $\forall w : \Delta_f(w) \leq \Delta_{f'}(w)$. We will prove this first by looking at the contradiction: $\Delta_f(w) > \Delta_{f'}(w)$.

A: $\Delta_f(w) > \Delta_{f'}(w)$ [Hy]

$$\underbrace{s - \ldots - u - w}_{\text{shortest path}}$$

B: $\Delta_f(u) = \Delta_f(w) - 1$

C: $\Delta_f(u) \leq \Delta_{f'}(u)$

So:

$$\Delta_f(w) = \Delta_f(u) + 1 \leq \Delta_{f'}(u) + 1 = \Delta_{f'}(w) \Leftrightarrow \Delta_f(w) \leq \Delta_{f'}(w) \lightning \text{to [Hy]}$$

So now we know $\Delta_f(w) \leq \Delta_{f'}(w)$. So now we prove

$$\forall (u,v) : (u,v) \text{ is a BN at most } \mathcal{O}(|V|) \text{ times}$$

We can say $\Delta_f(w) \leq |V|$, so $|V|$ is the upper bound. We have a network with a bottleneck path. At step i the edge (u,w) is the bottleneck so it doesn't exist in step $i+1$. But it exists from w to u. On step j it can become bottleneck again but there has to be a step k between $i+1$ and j where the edge (w,u) needs to be bottleneck ($\Rightarrow$ on the shortest path from s to t). So we have the shortest paths:

$$(i): s - ... - u - w - ... - t \Rightarrow \Delta_{f_i}(u) = \Delta_{f_i}(w) - 1$$

$$(k): s - ... - w - u - ... - t \Rightarrow \Delta_{f_k}(u) = \Delta_{f_k}(w) + 1$$

$$\Delta_{f_i}(w) \leq \Delta_{f_k}(w)$$

$$\Delta_{f_i}(u) = \Delta_{f_i}(w) - 1 \leq \Delta_{f_k}(w) - 1 = \Delta_{f_k}(u) - 1 - 1$$

Augmentation of at least -2.
Because of the upper bound an edge can be bottleneck not more than $\frac{|V|}{2}$ times. $\Rightarrow \mathcal{O}(\frac{|V|}{2} \cdot |E|)$
$\Rightarrow \mathcal{O}(|V| \cdot |E|^2)$ complexity.

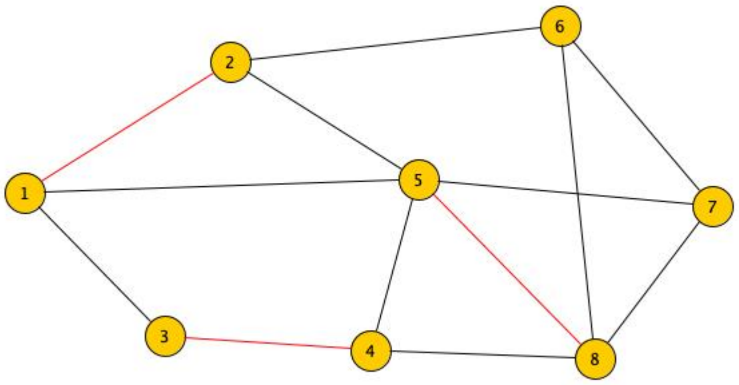
□

$|E|$ can me much larger than $|V|$. There is an algorithm that solves Max-Flow in $\mathcal{O}(|V|^2 \cdot |E|)$: `PUSH-RELABEL` or `TARJAN-GOLDBERG`. There exist also implementations where you have $\mathcal{O}(|V|^3)$ or $\mathcal{O}(|V|^2 \cdot \sqrt{|E|})$.
Residual Network is same to EdomdKarp. Difference: you try to do things ore local. On EdmondKarp you always search from s to $t \Rightarrow |E|^2$.
Locally you don't know if the flow will reach t, so the vertex tries to push the flow to following vertices. If you are lucky this system brings as much flow as possible to t.
Each vertex has a label, that may change while the algorithm runs (`relabel`). s will have label $|v|$ and t label $|0|$ all the time. With the algorithm the other labels can change. A vertex can push flow to other vertices if

1. There is an edge where you can push (with residual capacity)
2. The vertex that pushes has a higher label than the vertex that is receiving.

Each step we take one vertex and try to push. So you push to a vertex with lower label and update the capacity on this edge. This you do edge by edge. It can appear that a vertex needs to push flow but has only edges to vertices with higher labels. So the flow gets stuck. So you relabel: You look at the vertices that the vertex could push to and increase the pushing vertex-label to min(neighbours labels) + 1. You start with every label (except s) 0. Then you start pushing from s. You won't be able to push more flow than the outgoing capacities on s. That is the first upper bound. If somewhere you get stuck and can't go on, try to push it a little bit back.
It could happen that s can push as much flow as all the outgoing capacities. So all neighbours of s are now "active". All these vertices has label 0 now. That means they can't push. Relabel one of them to 1. This one tries now to send his received flow. Maybe it has not enough outgoing capacity, so there will be some flow stuck un this vertex, but you don't need to push it back, because s has no capacity left. So the vertex stays active. At some point you give that vertex the label $|v| + 1$ to send the flow, that is too much, back to s to say s that this is too much. That happens at the closing part of the algorithm.

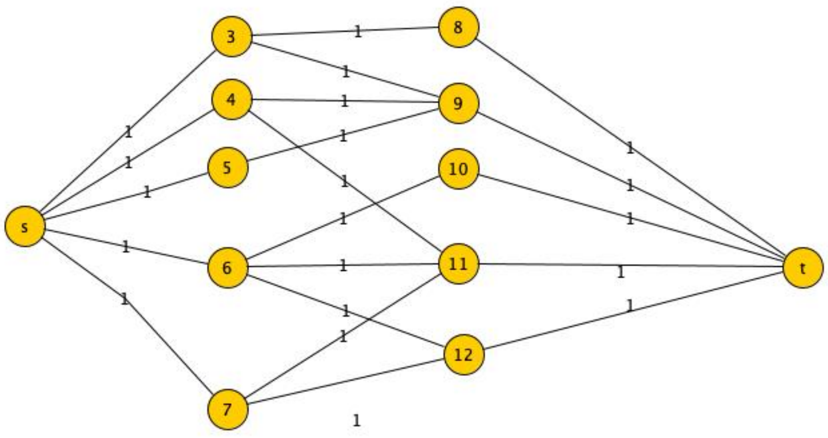
3.9 Matching



Matching On the chosen set of edges there is no common vertex (equivalent to independent set on vertices).

Maximal Matching There is no edge that can be added without violating the matching constraint. So on the example above only edge $(6, 7)$ could be added to get the maximum matching.

Maximum Matching The maximum of the maximal matchings.
The concept of Max Flow and Maximum Matching is the same.
Don't go greedy on Maximum Matching. Usually you want the get the Maximum Matching.
We want to find the Maximum Matching in a bipartite graph. Give every edge the capacity 1. Add s at one side with edge (1) to every vertex on that side and equivalent t to the other side. SO at least one edge per vertex on the connecting edges can be taken. SO you search the maximum flow in the graph.



So to proof:

1. If a flow $|f^*|$ exists $\Rightarrow$ matching exists
2. a max matching exists $\Rightarrow$ a flow $|f^*|$ exists.

4 Randomized Algorithms

Random Algorithm In the algorithm you compute something random. So on the same input you don't have to get the same output.

You can't say, every run gives the optimal solution. But you can get the probability with that you get the optimal solution.

opt. Result	exact Alg	rand Alg (2)	Approx (1)
100	100	100	92
100	100	8	93
100	100	90	99
100	100	99	100
100	100	100	91
100	100	100	92

(1) know that the approximation is very close
(2) expect to be very close in p of cases

4.1 Example

Two Polynomials:

$$P(x) = (x+1)(x-2)(x+3)(x-4)(x+5)(x-6)$$

$$Q(x) = x^6 - 7x^3 + 25$$

$$P(x) \overset{?}{\equiv} Q(x)$$

Check $x = 2$ (P will evaluate to 0), but

$$Q(2) = 2^6 - 7 \cdot 2^3 + 25 = 33 \neq 0$$

So we can say $P(x) \not\equiv Q(x)$ with $P = 1$.
If selected number right $\Rightarrow P(x) \not\equiv Q(x)$ with $P < 1$.
The random part is to choose $x = 2$.

$$P(x) = x^2 + 7x + 1$$

$$Q(x) = (x+2)^2$$

$$P(1) = 1 + 7 + 1 = 9 = (1+2)^2 = Q(1)$$

Choose x from a range of numbers (1...100).
Good numbers: $P(x)$ and $Q(x)$ evaluate to different results.
Bad numbers: They evaluate to the same result.
So how many bad numbers do we have?

$$F(X) = P(X) - Q(X)$$

If they are equivalent F would evaluate to 0 all the time. The bad numbers are those with $F = 0$ ($F(3) = P(3) - Q(3) = 0$). Number of bad numbers $\leq d$ where d is the degree of the polynomial. Get a very large range ($d \cdot 100$)so you have many good numbers. So the probability for a bad number is

$$P_{bad} = \frac{d}{100 \cdot d} = \frac{1}{100}$$

One would differentiate between the following two types of randomized algorithms

- Monte Carlo: takes polynomial time, gets the right/optimal solution with expected probability
- Las Vegas: sure that the right/optimal solution will be found, takes expected polynomial time

$$\text{Monte-Carlo} \overset{\text{repeat } \infty\text{-many times}}{\to} \text{Las Vegas}$$

Example from last time:
If $\exists x^* P(x^*) \neq Q(x^*) \Rightarrow P(x) \not\equiv Q(x)$
$P(x) = x^3 + 2x^2 - 3x, Q(x) = (x-1)(x^2+2) = x^3 + 2x - x^2 - 2$
$P(1) = 0 = Q(1) \Rightarrow$ don't know whether $P(x) \equiv Q(x)$
$P(2) = 10 \neq 6 = Q(2) \Rightarrow P(x) \not\equiv Q(x)$
Correct answer with some probability.
Extract number out of a certain range $[0, 100]$. How many good numbers (evaluate to different solutions) and bad (to the same result) are there?
Look at: $F(x) = P(x) - Q(x)$
$F(x) = 0 \Leftrightarrow P(x) \equiv Q(x)$
Roots of $F(x)$: "bad numbers" only a few where $F(x) = 0$

Question: how many "bad numbers" are there?
$\Rightarrow$ suppose $deg(Q) = deg(P) = d \Rightarrow d$ bad numbers at most $(\leq d)$
Instead of fixed range choose $[0, 100 \cdot d]$
$\Rightarrow$ good numbers are $1 - \frac{d}{100d} = 99\%$
$\Rightarrow$ increase the range proportionally to d (Monte-Carlo-Algorithm)
To increase the probability

- increase range
- repeat the algorithm several times

Probability after 2 or k solutions respectively

$$\frac{1}{100} \cdot \frac{1}{100} = \frac{1}{100^2} \Rightarrow \frac{1}{100^k}$$

$1 - \frac{1}{100^k}$ can make solution arbitrarily close to 1.
Also a correct answer is given in one of cases with probability 1 - the "no"-answer.
repetition-probabilities for bad numbers

$$\underbrace{\frac{d}{100d} \cdot \frac{d-1}{100d-1} \cdot \frac{d-2}{100d-1}}_{\substack{\to \\ \text{prob. gets better/prob. decreases}(\Rightarrow \text{no independent repetitions})}}$$

if $k > d \Rightarrow \frac{0}{100d-k}$
if $K = d+1$ repetitions the correct solution is found but $\mathcal{O}(d) \cdot d + 1 = \mathcal{O}(d^2)$

Complexity $\mathcal{O}(d \cdot k)$ ($d \to$ per step, $k \to$ number of repetitions).
Good is $k << d \Rightarrow$ Complexity stays $\mathcal{O}(d)$ randomization is done to get an efficient algorithm.

More general formulation conditional probability
two events: w_1, w_2
independence: $w_1 \Rightarrow Pr(w_1), w_2 \Rightarrow Pr(w_2)$
non-independence: $w_1 \Rightarrow Pr(w_1), w_2 \Rightarrow Pr(w_1|w_2)$
if independence is given $\Rightarrow Pr(w_2) = Pr(w_2|w_1)$
if dependence is given $\Rightarrow Pr(w_2|w_1) = \frac{Pr(w_2 \cap w_1)}{Pr(w_1)}$
events w_1, w_2:

$$\begin{aligned} P(w_2|w_1) &= \frac{P(w_2 \cap w_1)}{P(w_1)} \\ P(w_1 \cap w_2) &= P(w_2|w_1) \cdot P(w_1) \\ P(w_1 \cap w_2) &= P(w_2) \cdot P(w_1) && |\text{if independent} \end{aligned}$$

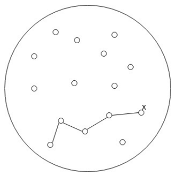
let $w_i :=$ get bad numbers at step/repetition i

$$P(w_1 \cap w_2 \cap ... \cap w_k) = ?$$

correct $W' = 1 - P(w_1 \cap w_2 \cap ... \cap w_k)$

$$\begin{aligned} P(w_1 \cap w_2 \cap ... \cap w_k) &= P(w_k|w_{k-1} \cap ... \cap w_1) \cdot P(w_1 \cap ... \cap w_{k-1}) \\ &= P(w_k|w_{k-1} \cap ... \cap w_1) P(w_{k-1}|w_{k-1} \cap ... \cap w_1) \cdot P(w_1 \cap ... \cap w_{k-2}) \\ &= \left(\prod_{i=2}^{k} (P|w_1 \cap ... \cap w_{i-1}) \right) \cdot P(w_1) \end{aligned}$$

4.2 Random Walk

Search space  optimal Solution
do from every point a random choice and walk down the chosen path

$$-n - ... - \underbrace{\overset{you}{0}}_{0.5 \leftarrow \rightarrow 0.5} - \overset{\times}{n}$$

Probability of doing a step with $\frac{1}{p}$ and $1 - \frac{1}{p}$ in the other direction. Want to estimate $\#steps$ to reach n. w : event *you* reaches $\times$, $Pr(w) = ?$
Z : random variable $= \#steps$ before reaching $\times$
Try to compute the expected value

$$\begin{aligned} E[Z] &= ? \\ E[Z] &= \sum_i (i \cdot P(Z = i)) && |i \in \{\text{values } Z \text{ can take}\} \end{aligned}$$

Example dice
$Z =$ numbers on dice

$$\begin{aligned} E(Z) &= \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 \\ &= 1 \cdot P(Z = 1) + 2 \cdot P(Z = 2) + ... + 6 \cdot P(Z = 6) \\ &= 3.5 \end{aligned}$$

4.2.1 Random Walker

$0 - - - - - - - - -(i-1) - \overset{\leftarrow\ \rightarrow}{i} - (i+1) - - - - - - - - - - - n - \rightarrow$
$Pr(i \rightarrow i+1) = \frac{1}{2}$, $Pr(i \rightarrow i-1) = \frac{1}{2}$
Initially is at position 0. $\rightarrow$ What is the expected number of steps that needs to reach n?
X_i the random variable that corresponds to the number of moves that needs to reach i. Find $E[X_n]$. Sei $T(i) = E(X_i)$

$$\begin{aligned}
t(0) &= 0\\
t(1) &= 1 + \frac{1}{2}\cdot t(0) + \frac{1}{2}\cdot t(2)\\
&...\\
t(i) &= 1 + \frac{1}{2}\cdot t(i-1) + \frac{1}{2}\cdot t(i+1)\\
&...\\
t(n-1) &= 1 + \frac{1}{2}\cdot t(n-2) + \frac{1}{2}\cdot t(n)\\
t(n) &= 1 + \frac{1}{2}\cdot t(n-1) && |unique
\end{aligned}$$

$$\left.\begin{aligned}
t(n-1) = 1 + \frac{1}{2}\cdot t(n-2) + \frac{1}{2}(1 + \cdot t(n-1))\\
\Leftrightarrow \frac{1}{2}t(n-1) = \frac{3}{2} + \frac{1}{2}t(n-2)\\
\Leftrightarrow t(n-1) = 3 + t(n-2)\\
\\
t(n-2) = 1 + \frac{1}{2}t(n-3) + \frac{1}{2}(3 + t(n-2))\\
\Leftrightarrow \frac{1}{2}t(n-2) = \frac{5}{2} + \frac{1}{2}t(n-3)\\
\Leftrightarrow t(n-2) = 5 + t(n-3)\\
...\\
t(n-i) = 1 + 2i + t(n-i-1)\\
t(1) = 1 + 2(n-1) + t(0) = 1 + 2(n-1)
\end{aligned}\right\}\text{Backward-Propagation}$$

$$\left.\begin{aligned}
t(1) = 1 + 2(n-1)\\
t(2) = 1 + 2(n-1) + 1 + 2(n-1)\\
= 3 + 2((n-1) + (n-2))\\
...\\
t(n) = n + 2((n-1) + (n-2) + ... + 0)\\
= n + 2\frac{(n-1)n}{2}\\
= n + n^2 - n = n^2
\end{aligned}\right\}\text{Forward-Propagation}$$

4.2.2 2-SAT

Input A logical formula Φ with n variables and m clauses each with 2 literals

Example $\Phi = (x_1 \vee \overline{x_2}) \wedge (\overline{x_2} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_4})$

Output An assignment of the truth values of each variable of Φ that make Φ satisfiable.

Example $x_1 = true; x_3 = false; x_4 = false, x_2 = true$

A randomized algorithm

```
Start with arbitrary assignment (e.g. xi = false forall i=1,...,n)
while(exists an unsatisfied clause ) [or tired flipping coins]
  c <- some unsatisfied clause
  choose a variable of c by flipping a coind
  change the value of the variable
```

Example $x_1 = x_2 = x_3 = x_4 = false$
$(x_1 \vee x_2)$ chosen $\rightarrow x_2 = true$
$(x_1 \vee \overline{x_2})$ chosen $\rightarrow x_2 = false$
...
$-----------------------------S------\rightarrow$
What is $Pr \rightarrow$?, What is $Pr \leftarrow$?
$S \leftarrow$ a (any) truth assignment of Φ
$A_j \leftarrow$ the assignment of values to the variables of Φ after the j-th iteration
$X_i \leftarrow$ the number of variables that have the same value in A_j and S.
If $X_j = n \Rightarrow S = A_j$
$\left.\begin{array}{r} S = x_1 = t, x_3 = t, x_4 = f, x_2 = t \\ A_0 = x_1 = x_2 = x_3 = x_4 = f \end{array}\right\} X_0 = 2$ x_0 : How many do the two sets have in common ($X_j =$ $(A_j \cap S)$)

Claim If Φ has a truth assignment S making it true, then in n^2 iterations the algorithm is expected to find S.

Proof $---------------------------S(X_j = n)------\rightarrow$
Wlog ("without loss of generality") $X_0 = 0$
What is the probability that at some iteration j moves closer/further away from S?
$Pr(X_{j+1} = k + 1 | X_{j=k}) \rightarrow$ moves closer
$Pr(X_{j+1} = k - 1 | X_{j=k}) \rightarrow$ moves away
$(x_k, x_\lambda) \rightarrow$ the unsatisfied clause of the j-th iteration
$\rightarrow$ *Observation:* (x_k, x_λ) *cannot have the same value as* S. *At least one of the* x_k, x_λ *must have a different value in* S_i, *otherwise* (x_k, x_λ) *satisfied*

- Both have different values: $P(closer) = 1$
- Exactly one has a different value: $P(closer) = \frac{1}{2} = P(away)$

Theorem If the algorithm makes sn^2 iterations and S is satisfiable, then $Pr(\text{not finding } S) \leq \frac{1}{2}$

Proof $X \leftarrow$ random variable that corresponds to the number of iterations to find $S \overset{\text{RandomWalker}}{\Rightarrow}$
$E(X) = n^2$
$Pr(\text{not finding S}) = Pr(X > 2n^2) \overset{MarkovInequality}{\leq} \frac{E[X]}{2n^2}) = \frac{1}{2}$

Markov Inequality $Pr(X \geq \alpha) \leq \frac{E(X)}{\alpha}$

Proof

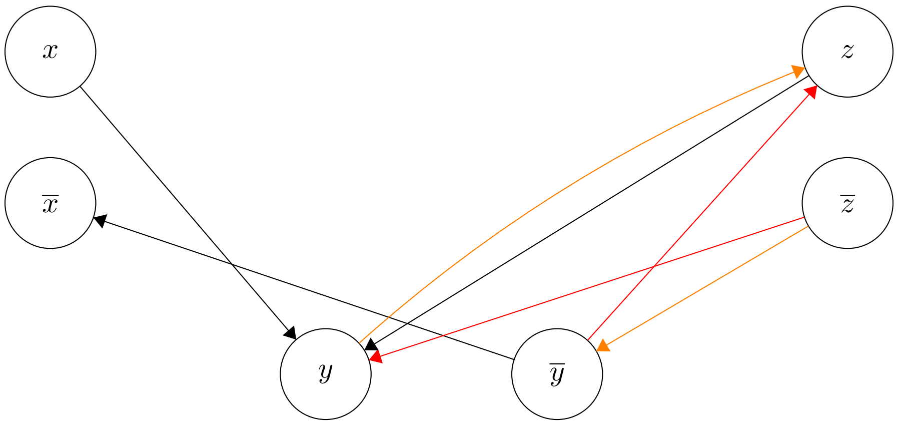
$$\begin{aligned} E(X) &= \sum_{i=0}^{\infty} Pr(X = i) \\ &= \underbrace{\sum_{i \geq \alpha} iPr(X = i)}_{\geq 0} + \underbrace{\sum_{i < \alpha} iPr(X = i)}_{\geq 0} \\ &\Rightarrow E(X) \geq \sum_{i \geq \alpha} iPr(X = i) \geq \alpha \sum_{I \geq \alpha} Pr(X = i) \end{aligned}$$

2-SAT is in P $\Phi = (\overline{x} \vee y) \wedge (\overline{y} \vee z) \wedge (z \vee y)$

$x \vee y \Leftrightarrow \overline{x} \Rightarrow y \Leftrightarrow \overline{y} \Rightarrow x$

Based Φ creates a graph G_Φ: $\forall$ variable $x \in \Phi$, G_ϕ has two vertices $x, \overline{x}$

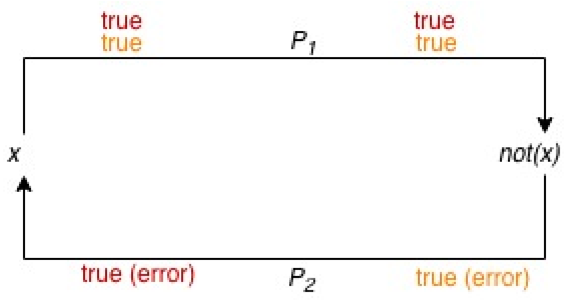
example G_Φ :



$\forall$ clause $(x \vee y)$ has two edges $\overline{x} \rightarrow y$, $\overline{y} \rightarrow x$

Theorem: Φ is unsatisfiable $\Leftrightarrow$ $\exists$ variable x in Φ:$x \overset{P_1}{\rightsquigarrow} \overline{x}$ and $\overline{x} \overset{P_2}{\rightsquigarrow} x$ in G_Φ.

Proof: $(\Leftrightarrow)$:



Assume Φ is unsatisfiable.
$(\Rightarrow)$: If there exsit paths P_1 and P_2 then Φ is unsatisfiable.
Assume Φ is satisfiable.

$x\ \overset{T}{x}\ \underbrace{\to\cdot\to\cdot\to}_{\tau}\underbrace{a...b}_{F}\to\overset{F}{\overline{x}}$

a true, b false, $\overline{a}\vee b$ =false $\lightning$.
String connected components:
If there is a component which contains x ad $\overline{x}$ then Φ is unsatisfiable.

4.3 Convex hull

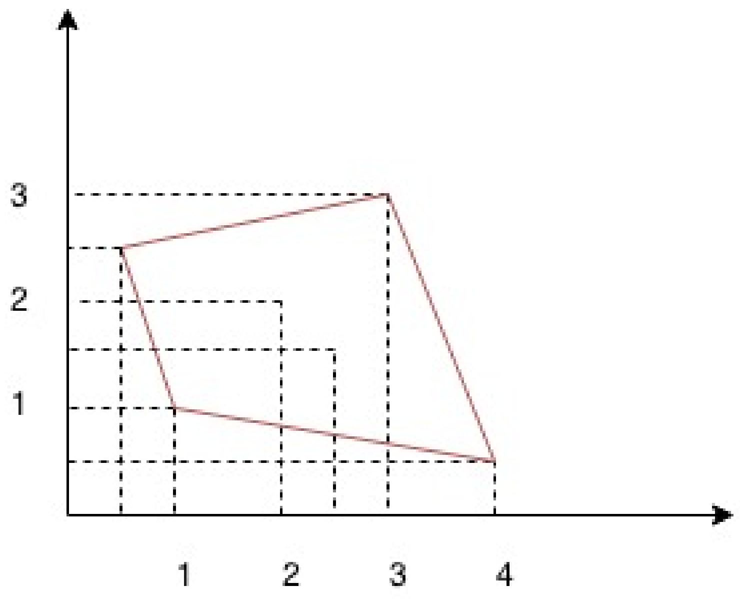
Input n points $P=\{p_1,p_2,...,p_n\}, p_i=(x,y)\in\mathbb{R}^2$. *in general position*

(i) no three points on a line

(ii) no two points on a vertical line $\Rightarrow x_i\neq x_j \forall i\neq j$

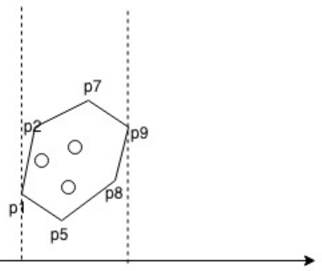
Output The smallest convex hull polygon Q that contains all points, i.e. $P\subseteq Q$
Q is convex $\Leftrightarrow \forall p,q\in Q:\overline{pq}\in Q\Leftrightarrow$ *all angels of Q are* $\leq 180°$

4.3.1 Example



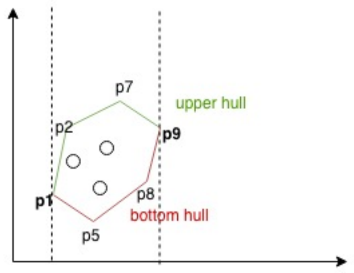
Recall Convex Hull $\in o(n\log n)$, Convex Hull $\geq_n$ Sorting

4.3.2 Convex hull representation



The points of CH in clockwise (ccw) order around the boundary. $\underbrace{p_1}_{leftmost}, p_2, p_7, p_9, p_8, p_5$

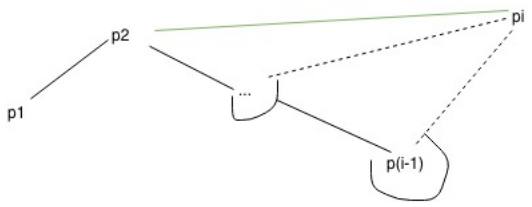
Observation The leftmost, rightmost, topmost, bottommost points belong to the convex hull.Â



4.3.3 Incremental algorithm

to compute upper (bottom) hull. (*incremental: processes on point at each step*). Points are sorted left to right!

Invariant For the first $i-1$ points, the upper hull has benn computed.



Angle at p_{i-1} too big, so pop p_{i-1} from stack. $p_{\dots}$ angle too big, so pop $p_{\dots}$ from stack. p_2 is ok so let it on the stack and push p_i.

Pseudo code

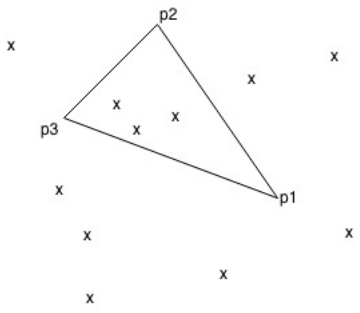
```
Sort the points L -> R: p1,...,pn // O(n log n)
push (p1, H)
push (p2, H)
for i=3 to n // O(n)
    while (|H| >= 2 & < (pi, first(H),second(H)) > 180grad)
        pop(H)
    push(p1, H)
```

Correctness $p_i \in$ Upper hull (rightmost); the removed point $\notin$ Upper hull

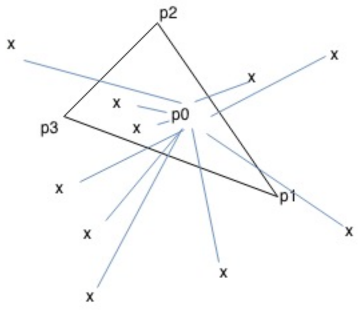
Complexity $\#push/pop \in \mathcal{O}(n)$ (can't pop more than we have pushed)

4.3.4 Randomised incremental Construction (RIC)

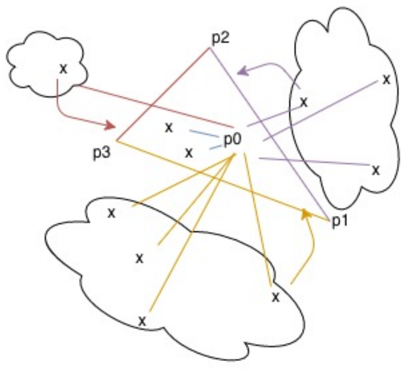
1. shuffle points: $P_1, ..., P_n$ (randomisation): $S_i = \{p_1, ..., p_i\}$



2. $conv(S_3) \leftarrow$ convex hull of p_1, p_2, p_3
3. $p_0 \leftarrow$ in $conv(S_3)$
4. Draw an arc from p_0 to $\forall p \in P \backslash S_3$



5. The conflict list of each edge e of $conv(S_3)$ consits of all points of $P \backslash S_3$ whose arc crosses e

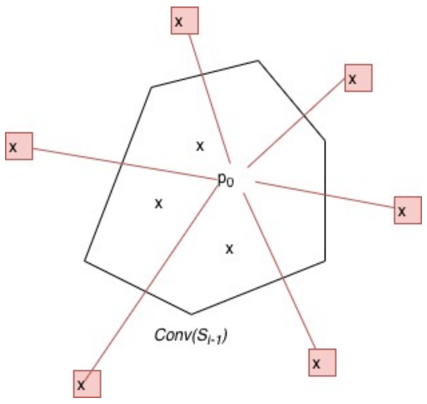


6. The points of $P \backslash S_3$ that are not in any conflict lists $\rightarrow$ inactive

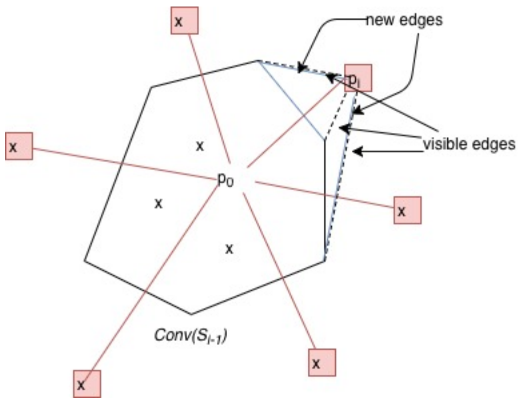
Invariant(s) for the first $i - 1$ points

- $conv(S_{i-1})$: convex hull of S_{i-1}

- each edge of $conv(S_{i-1})$ has a conflict list of active points



- each point in $P \backslash S_{i-1}$ that is active is associated with an edge of $conv(S_{i-1})$



```
for i=4 ...n
    if pi is active
        e <- associated with pi
        (S1) go left & right on conv(S_{i-1}) to find the visible edges from pi
        (S2) replace the visible edges with two new edges incident to pi
        (S3) "move" th epoints in the conflict lost of the visible edges to
          the new edge /filter for inactive/active
    else conv(S_i) = conv(S_{i-1})
```

Complexity Steps 1 (S1) & 2 (S2): $\mathcal{O}(n)$ in total. Each time we add two edges and to remove an edge it must have been added first.

$$\#removals \leq 2\#additions \in \mathcal{O}(n)$$

Step 3 (S3): $\mathcal{O}(n^2)$ is straight-forward.
Backward Analysis: $S_i \to S_{i-1}$

$$
\begin{aligned}
E(\#of PointerUpdates) &= \sum_{e\in conv(S_i)} (\text{size of conf. list of } e)\cdot \underbrace{Pr(e \text{ is removed})}_{\frac{2}{i}} \\
&= \frac{2}{i}\sum_{e\in conv(S_i)} (\text{size of conf. list of } e) \\
&= \frac{n}{i} \\
&\Rightarrow E(\text{overall pointer update}) = \sum_{i=1}^{n}\frac{n}{i} \\
&= n\cdot\sum_{i=1}^{n}\frac{1}{i} = \mathcal{O}(n\log n) && |\text{Harmonic sequence}
\end{aligned}
$$

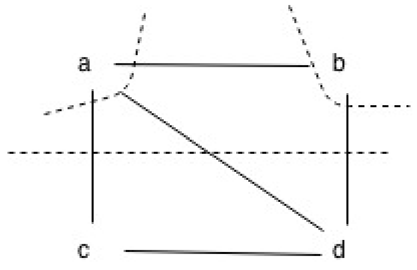
4.4 Unweighed Min-Cur Problem

Input a graph $G=(V,E)$

Output a partition of V into S,T:

- $S\cup T = V \& S\cap T=\emptyset$
- $S\neq\emptyset\neq T$
- $w(S,T)=\#edges \equiv |\{(u,v)\in E : u\in S \& v\in T\}|$ is minimum over all partitions of V.

Example



S	T	$w(S,T)$
a	b,c,d	3
b	a,c,d	2
c	a,b,d	3
d	a,b,c	2
ab	c,d	3
...	...	...

4.4.1 Stoer & Wagner

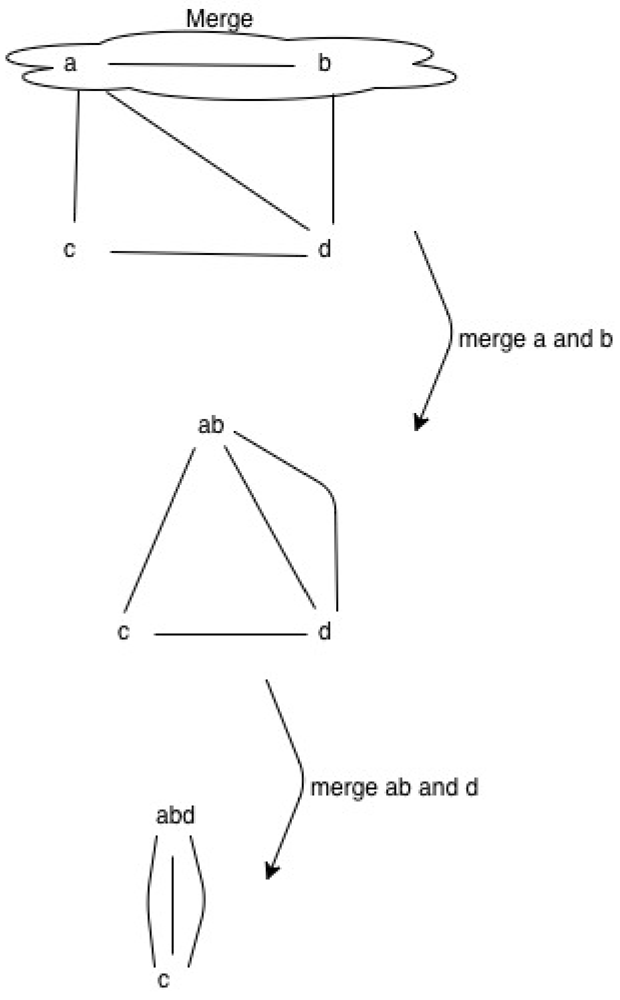
$\mathcal{O}(|V|(|V|\log|V|+|E|))$

Question Can we do better?

4.4.2 A randomized algorithm

```
Repeat n-2 times the following
  1) Pick an edge (u,v) uniformly at random
  2) Merge(*) the two endpoints of (u,v) into a supernode
G has only two supernodes: v1,v2
Return(S(v1),S(v2)) //S(vi) = {v in V: v was merged to Vi} i = 1,2
```

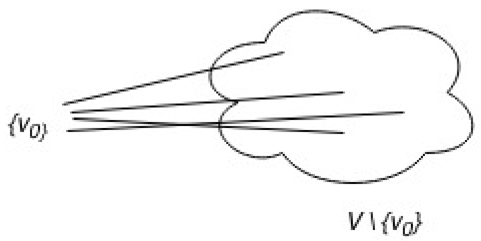
(*) Replace by supernode, Remove self-loops, keep parallel edges.



Idea: Unlikely that (u,v) is in $MinCut(S^*, T^*)$

Lemma 1 If (S^*, T^*) is a min-cut, then the $w(S^*, T^*) \leq \deg(v) \forall v \in V$

Proof Assume $\exists v_0 \in V : \deg(v_0) < w(S^*, T^*)$ (*1)
new graph:



Define cut $(A, B) : A = \{v_0\}, B = V \backslash \{v_0\}$
$w(A, B) = \deg(v_0) \overset{(*1)}{<} w(S^*, T^*) \lightning$

□

Corollary $|E| \geq \frac{n}{2} \underbrace{w(S^*, T^*)}_{minCut}$

Proof

$$\begin{aligned} 2 \cdot |E| &= \sum_{v \in V} \deg(v) \\ &\overset{\text{Lemma 1}}{\geq} \sum_{v \in V} w(S^*, T^*) \\ &= n \cdot w(S^*, T^*) \\ &\Rightarrow |E| \geq \frac{n}{2} w(S^*, T^*) \end{aligned}$$

□

Theorem The randomized algorithm return s a min-cut with probability $\geq \frac{2}{n^2}$

Proof $G_j \leftarrow$ the graph obtained after the j-th iteration with $n_j = n$
$A_j \leftarrow$ the event that no edge of (S^*, T^*) is selected at this iteration
$Pr(\text{algorithm return a min-cut}) = Pr(A_1 \cap A_2 \cap ... \cap A_{n-2}) = Pr(A_1) \cdot Pr(A_2|A_1) \cdots Pr(A_{n-2}|A_1 \cap ... \cap A_{n-2})$
$Pr(A_1) = 1 - Pr(\text{an edge of } (S^*, T^*) \text{ is selected in the first iteration}) = 1 - \frac{w(S^*,T^*)}{|E|}$ (Corollary 1)
$|E| \geq \frac{n}{2} w(S^*, T^*) \Leftrightarrow \frac{2}{n} \geq \frac{w(S^*,T^*)}{|E|} \Leftrightarrow 1 - \frac{w(S^*,T^*)}{|E|} \geq 1 - \frac{2}{n}$
$\Rightarrow 1 - \frac{w(S^*,T^*)}{|E|} \geq 1 - \frac{n}{2} = \frac{n-2}{n}$ independent of $|E|$
$Pr(A_2|A_1) = ... \geq \frac{n_1 - 2}{n_1} \overset{n_1 = n-1}{=} \frac{n-1-2}{n-1} = \frac{n-3}{n-1}$
$Pr(A_j|A_1 \cap ... \cap A_{j-1}) \geq \frac{n-j-2}{n-j}$
$Pr(\text{algorithm returns min-cut}) = \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot ... \frac{2}{4} \cdot \frac{1}{3} = \frac{2}{n(n-1)} \geq \frac{2}{n^2}$

□

Corollary 2 If we repeat the algorithm $n^2 \ln n$ times, then the probability of not reporting the min-cut is $\leq \frac{1}{n^2}$

Proof

$$
\begin{aligned}
Pr(\text{not reporting the min-cut}) &= (1-\frac{2}{n^2})^{n^2 \ln n} \\
&= \left[\left(1-\frac{2}{n^2}\right)^{\frac{1}{2}n^2}\right]^{2\ln n} \qquad |(1-\frac{1}{x})^x \le \frac{1}{e}\text{Monte Carlo} \\
&\le \left(\frac{1}{e}\right)^{2\ln n} \\
&= (e^{\ln n})^{-2} = n^{-2} = \frac{1}{n^2}
\end{aligned}
$$

□

Note: If we repeat randomized algorithm $n^2 \ln n = \Theta(n^2 \log n)$ then the $Pr(\text{not reporting a solution correct}) \le \frac{1}{n^2}$

Complexity $n^2 \ln n \cdot \underbrace{(n-2)}_{(n-2)\text{ edges for merge}} \cdot \underbrace{\mathcal{O}(n)}_{\text{do the merge}} = \mathcal{O}(n^4 \log n)$ (worse than S&W)

(Stoer & Wagner): $\mathcal{O}(n^2 \log n + n|E|)$

5 Approximated Algorithms

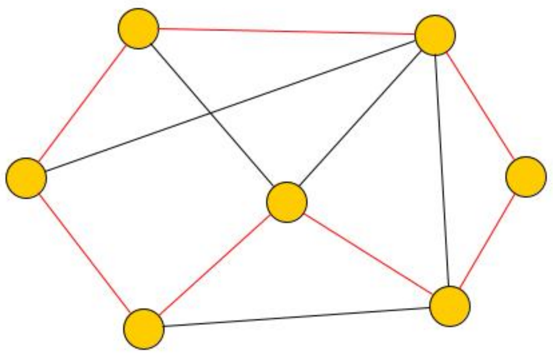
$APP(I)$, I input, solution of approximation algorithm
$OPT(I)$, optimal solution
We want:

$$APP(I) \overset{maximization\ problem}{\geq} \alpha OPT(I) + \beta \ \forall I$$

(for approximation problems we forget about decision problems since it doesn't make sense.)
If above holds then $APP(I)$ is called $\alpha-$approximation
Approximation schema: does not define a specific α, but defines a time-optimality trade-off (spend more time $\Rightarrow$ get a better solution).

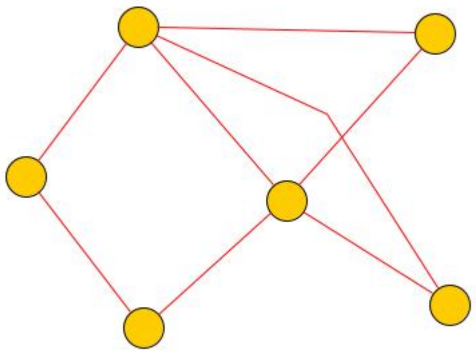
5.1 Hamiltion Cycle Problem

Is there a cycle which touches every vertex once?

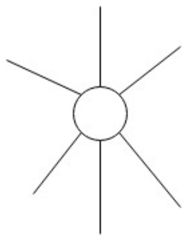


NP-Complete

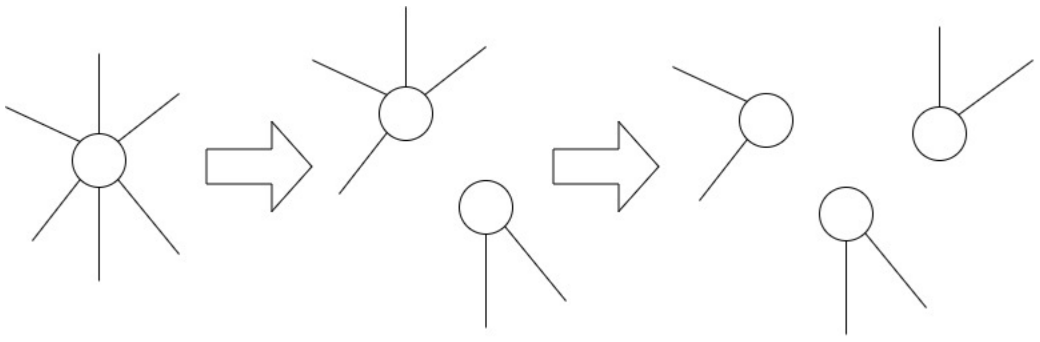
5.2 Eulerian Cycle Problem



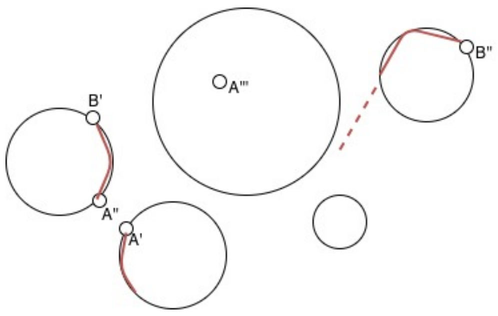
Cycle exists iff deg(v) is even $\forall v \in V$
 ($\Rightarrow$): easy, need to go through a vertex (in & out) $\rightarrow \circ \rightarrow$
($\Leftarrow$): suppose that every vertex has even degree $\Rightarrow$ eulerian cycle observation:



take vertex and any two edges and split it into



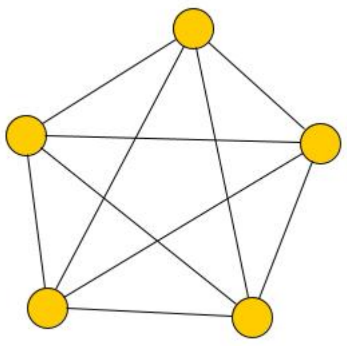
$\Rightarrow$ Graph consists of cycles



construct cycle by arbitrary selecting a node and go along the cycle. If there is a vertex on the way which is a copy of a vertex, go to the next cycle. When a cycle is closed (vertex visited twice) go back.

5.3 TSP

- complete graph K_n

$n = 5:$ 

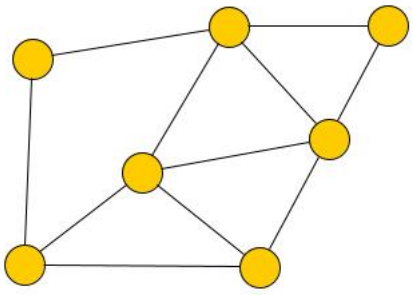
- weights on the edges
- need one with minimum weight
- NP complete

5.3.1 Reduction

There is no construct that can approximate TSP

$$\nexists \alpha : APP(I) \leq \alpha TSP(I) + \beta$$

"Inapproximability Result"
Suppose given a Graph G



Specific graph $G \to K_n$ and weights

- add missing edges with $n = |G|$ to get K_n
- weight of e is, with $G = (V, E), K_n = (V', E')$,

  1 if $e \in E \; \forall e \in G \Rightarrow w(e) = 1$
  α if $e \notin E \; \forall e \notin G \Rightarrow n \cdot \alpha$

$\Rightarrow$ starting at a problem/instance of the known problem

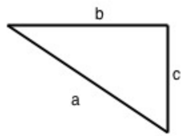
$\Rightarrow$ construct an instance of the problem you want to lower bound

Assume we have a $\alpha-$approximation of TSP

$\Rightarrow$ Hamiltonian Cycle could be solved in polynomial time

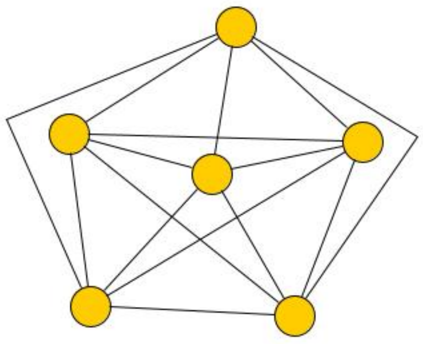
$\Rightarrow$ Hamiltonian Cycle $\in P$ ↯

Suppose $\exists$ Hamiltonian Cycle in G: $\Rightarrow \exists TSP$ with *cost* $= n \Rightarrow APP(I) \leq \alpha \cdot n$
Suppose $\nexists$ Hamiltonian Cycle in G: $\Rightarrow \forall TSP$ has *cost* $> \alpha n \Rightarrow APP(I) > \alpha \cdot n$
Metric TSP: weights satisfy the triangular inequality (also NP-complete!)

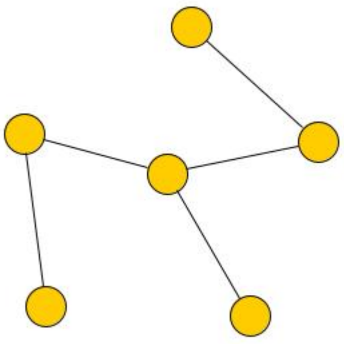


$a \leq b + c$

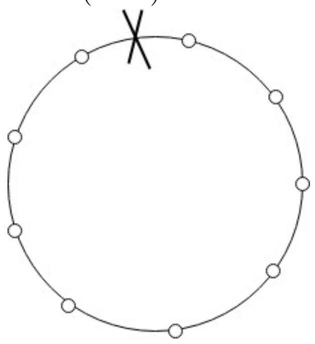
2-approx



$\overset{MST=:T}{\Rightarrow}$

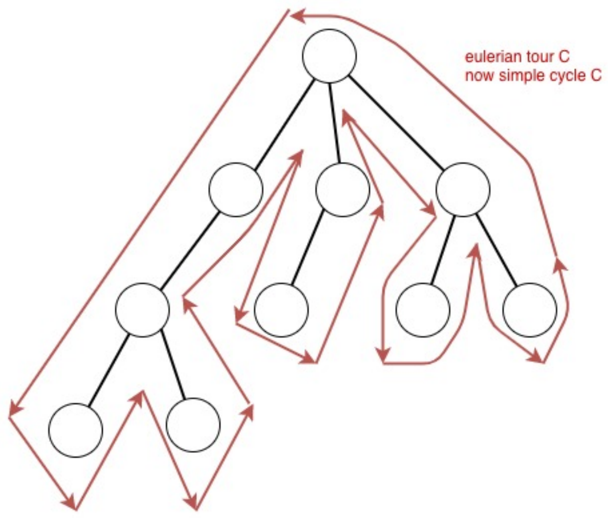


OPT $(=: \alpha)$



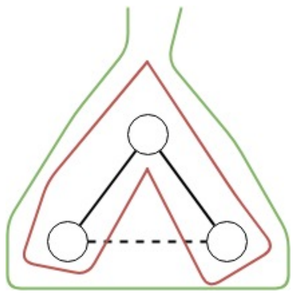
by removing one edge wer get a spanning tree

$cost(T) \leq cost(spanning - path\alpha) < OPT$



degree of v gets doubled $\forall v \in V$

$$\Rightarrow cost(c) = 2cost(T)$$



C' = Hamiltonian Simple Cycle

Introduce shortcuts

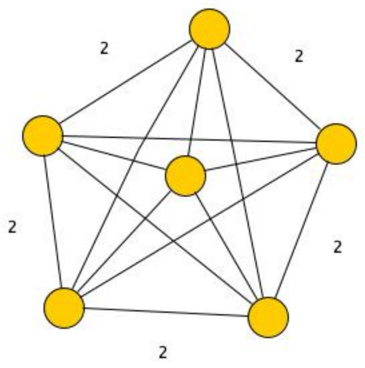
$cost(C') =?$

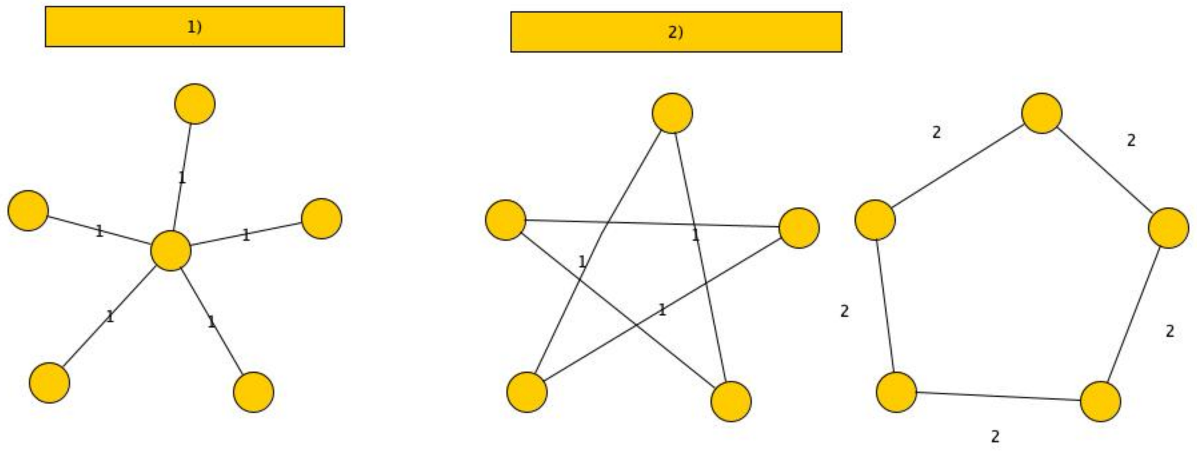
replace the two edges by near one ... $\leq / + /$(triangle inequality!)

$cost(C') \leq cost(C)$

$$cost(C') \leq cost(C) = 2cost(T) \leq 2 \cdot OPT$$

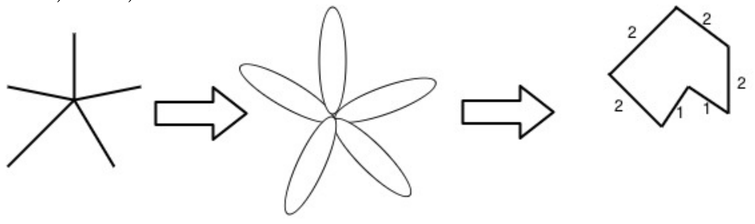
Is the analysis tight?





5.4 MST

either 1) or 2)

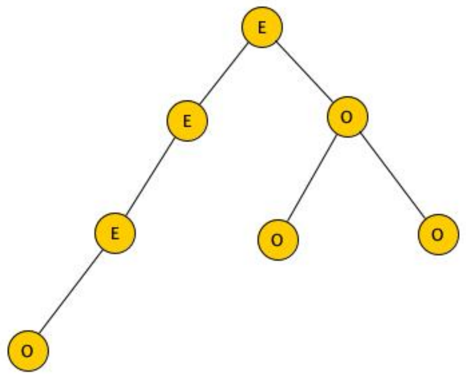


1): $2n-2\ cost$

5.5 Christofides

$K_n \overset{MST}{\rightarrow} T \rightarrow \underbrace{V_0}_{\text{Set of ODD-DEG Vertice}} \overset{MIN-COST-PERF.-MATCH}{\longrightarrow} M \rightarrow T \cup M = C \overset{SHORTCUT}{\rightarrow} \underbrace{C'}_{hamiltonian\ cycle}$

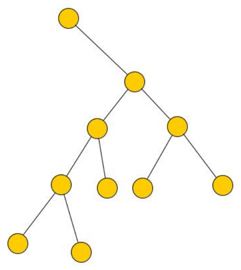
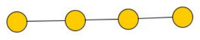
Where
K_n like before
T: (E=even, O=odd)



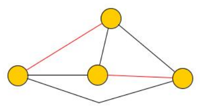
We can't have an odd number of vertices with an odd degree

$$\sum_{v \in T} \deg(v) = 2|E|$$

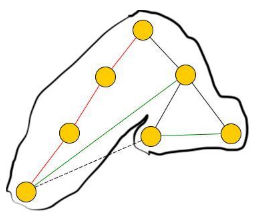
$$\underbrace{\sum_{v\ ODD} \deg(v)}_{EVEN} + \underbrace{\sum_{v\ EVEN} \deg(v)}_{EVEN} = \underbrace{2|E|}_{EVEN}$$

 All vertices have degree 3/1 ;  All vertices have degree 2 except 2

M:

 Perfect Matching (red) (even number of vertices $\to \exists$ perfect matching) with Min-Cost

$T \cup M$:

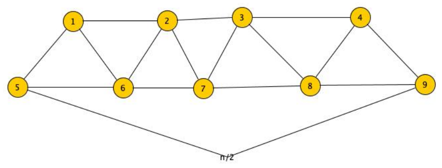
1. red (even vertices?), 2. green (perfect match): 

$$\begin{aligned}
cost(C') &\leq cost(C) && |\text{triangle inequality} \\
cost(C) &= cost(T) + cost(M) && |cost(T) < OPT, (*) \\
cost(C_o^*) &\leq OPT && |(*1) \\
(cost(M) \leq) cost(M^*) &\leq \frac{cost(C_0^*)}{2} && |cost(M) \leq \frac{OPT}{2} \\
cost(T) + cost(M) &\leq \frac{3}{2} OPT
\end{aligned}$$

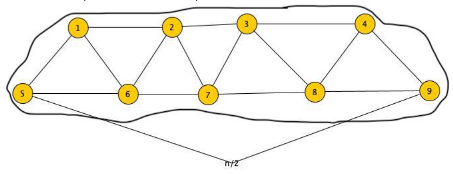
$$(*) C_0^* = \text{cycle derived from } C^* \text{ but only odd degree vertices}$$

$$(*1)\text{There are 2 perf. match. in } C_0^*, \text{take the minimal } M^*$$

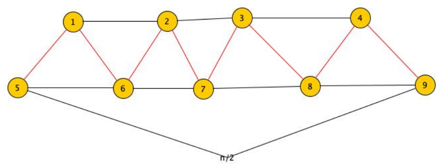
We have everything we want, but we don't know whether we can do better or not. The best bound known is $\frac{3}{2}OPT$. Example that gets $\frac{3}{2}OPT$: (all edges without cost have cost 1, missing edges are very very expensive)



OPT $(cost = n)$:



Consider the red MST is found:



,

the perfect matching would include the n/2 edge that leads to a cycle. The cost is $n-1$ for the tree and $\frac{n}{2}$ for the edge. So: $\frac{3}{2}n$
On the n/2 you can't put something greater than $\frac{n}{2}$ because of the triangle inequality.

5.6 PTAS: Polynomial Time Approximation Schema

Schema: Not one approximation but an algorithm with a parameter. Balance between complexity and accuracy.
$\varepsilon > 0$
Approx $\underbrace{(1+\varepsilon)}_{Minimization}$ or $\underbrace{(1-\varepsilon)}_{Maximazation}$
Time depends on $\frac{1}{\varepsilon}$ and n, possible approx.: $\underbrace{\mathcal{O}(n^{\frac{1}{\varepsilon}})}_{PTAS}$, much better: $\underbrace{\mathcal{O}(\frac{1}{\varepsilon}n)}_{F(ully)PTAS}$

5.6.1 Knapsack

Input $a_1, ..., a_n$ items, each item has a size $s(a_i)$ and a profit $P(a_i)$.
Bag B with a capacity $c(B)$

Output Set of item S that fits in the bag: $\sum_{a_i \in S} s(a_i) \leq c(B)$ and maximize $\sum_{a_i \in S} P(a_i)$
simpler: ($\notin NP$) raking functions of items
$A(i, P)$: smallest subset of the first i items whose profit is equal to P
Try to compote the first (specific value of a profit $\in \{1...\}$) i items Knapsack, continue to $i+1 \in \{1...n\}$
Example:

	$s(a_i)$	$P(a_i)$
a_1	3	20
a_2	5	32
a_3	4	40
a_4	1	28
a_5	2	20

$A(5, 100)$:
$a_2, a_3, a_4 \to P = 100, s = 10$
$a_1, a_2, a_4, a_5 \to P = 100, s = 11$
$A(5, 99) = \infty$

	$i \to$	1	2	3	...	n
	1	∞	∞			
	2	∞	∞			
	3	∞	∞			
	...	...	...			
20	...	3	3			$\leq B$
	...	...	∞			
32	...	...	5			
	...	...	∞			
52	...	...	85			
	...	...	∞			
	n	∞	...			
	$\uparrow$					$\uparrow$
	P					

$$A(i, P) = \begin{cases} s(a_i) + A(i-1, P - P(a_i)) & \textit{use } i \\ A(i-1, P) & \textit{don't use } i \end{cases}$$

stop at $A(n, n \cdot \underbrace{P_M}_{\textit{max profit of an item}}) \to$ Solution S cannot get better!
then read the table (last column)
complexity: $\#cells$, $\mathcal{O}(n \cdot n \cdot P_M) = \mathcal{O}(n^2 \cdot P_M)$
P_M is problem, because P_M is a number $\Rightarrow$ represented in logarithmic number of bits $\Rightarrow$ input log $\Rightarrow$ output exonential ($\Rightarrow \in NP$) in terms of input.

$$K = \frac{\varepsilon \cdot P_M}{n} \Rightarrow P^*(a_i) = \lfloor \frac{P(a_i)}{K} \rfloor$$

obtain complexity:

$$\mathcal{O}(n^2 \cdot P^*_M) = \mathcal{O}(n^2 \cdot \lfloor \frac{P_M}{\underbrace{\varepsilon \cdot P_M}_{n}} \rfloor) = \mathcal{O}(\frac{n}{\varepsilon})$$

S^* set computed on the star instance
know: $S^* = OPT^*$
compare to: $OPT = S_0$

$$\frac{P(a_i)}{K} - 1 \leq P^*(a_i) \leq \frac{P(a_i)}{K} (da \lfloor \rfloor)$$

$$P(a_i) - KP^*(a_i) \leq K \forall i$$
$$\overset{\sum_{S_0}}{\Rightarrow} \underbrace{\sum_{a_i \in S_0} (P(a_i) - KP^*(a_i))}_{P(S_0) = KP^*(S_0) \leq K_n}$$
$$\leq K|S_0| \leq K_n$$
$$P(S^*) \leq KP^*(S^*) \geq K \qquad \underbrace{P^*(S_0)}_{\text{just some solution in } P^* \text{ and } S^* = OPT} \qquad \geq P(S_0) - K_n \geq OPT - K_n$$
$$= OPT - \varepsilon P_M \overset{P_M \leq OPT}{\geq} OPT - \varepsilon OPT$$
$$= (1-\varepsilon) \cdot OPT$$

5.6.2 Bin Packing

Input $a_1, ..., a_n$ items with sizes $s(a_1), ..., s(a_n)$, $\forall i s(a_i) \leq 1$. No profits. We want to fit all items in a certain number of bins. We have maximal n bins (every item one bin), but want minimal number of bins. Bin-size is 1.

Output Configuration to fit $a_1, ..., a_n$ in minimal number of bins.
to be continued ...

5.6.3 2-Partition

Strong related to Bin Packing (pack items in two bags).

Input $i_1, ..., i_n$ Integers

Output Partition of $i_1, ..., i_n$ in two partitions I_1, I_2 with

$$\sum_{i \in I_1} i = \sum_{i \in I_2} i$$

Reduction between these two problems 2-Partition $\leq$ Bin Packing: Positive instance $i_1, ...i_n$ means they fit into 2 bins, else they don't.

Approximation The approximation is $\frac{3}{2}$ (positive instance). There is no $\alpha \leq \frac{3}{2}$.
PTAS: $(1+\varepsilon) \rightarrow$ PTAS doesn't exist (inaproximability) $\Rightarrow$ APTAS

5.7 APTAS

Asymptotic PTAS. Look at large instances $\Rightarrow \frac{n+1}{n} \frac{n(1+\frac{1}{n})}{n} \rightarrow 1$

$$ALG(I) \leq (1+\varepsilon)OPT + C$$

Add a constant C to handle extreme cases in small instances

5.7.1 Bin Packing (continuation)

Greedy approach

- a_1 in bin_1
- a_2: does it fit in bin_1? yes: put it in bin_1, else in bin_2 (then bin_1 is closed)
- a_3: does it fit in the current? ...

$\Rightarrow$ try to fit it into the current bin, if it doesn't fit, got to the next bin.

Optimal solution: all bins are completely full $\Rightarrow OPT\ bins \Rightarrow \sum i = OPT$
There could be *bin*s that are nearly empty, because the next item doesn't fit in the bin, so the bin stays nearly empty (even if there were other items that fit in).
Connect bin_1, bin_2 and bin_3, bin_4,.... For each pair the sum is > 1 and ≤ 2. $\Rightarrow$ Approximation with factor 2.

APTAS instance I: $1, 3, 3, 5, 6, 6, 8, 10, 12, 14$. Select $\varepsilon = 4$

$\underbrace{1,3,3}_{small(<4)} \ | \underbrace{5,6,6,8,10,12,14,16}_{large}$

$I_L = \underbrace{5,6,6}_{=\frac{|I_L|}{k}}, \underbrace{8,10,12}_{=\frac{|I_L|}{k}}, \underbrace{14,16}_{\leq\frac{|I_L|}{k}}$, bin capacity c, items $\geq \varepsilon \rightarrow \frac{c}{\varepsilon}$ items maximal

Â $I'_L : 6,6,6|12,12,12|\times$

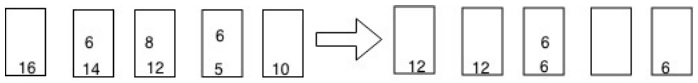
$$\Rightarrow ALG(I'_L) \overset{brute-force}{=} OPT(I'_L)$$

Now: $ALG(I'_L) \rightarrow ALG(I_L) \rightarrow ALG(I)$ and how does it change the relationship to $OPT(I_L), OPT(I)$

$ALG(I'_L) \rightarrow ALG(I_L)$: Split I'_L to the bins. Replace the Element by their real value (works because they just got greater in I'_L). The last group $\times$ just gets a bin for each item.

$$ALG(I_L) \leq ALG(I'_L) + \frac{|I_L|}{k}$$

$OPT(I_L) \rightarrow OPT(I'_L)$ The last element in the first I'_L group is smaller than the first in the next group in I_L. So every group in I'_L can be mapped to the next group in I_L. The last group of I_L remains . We have an optimal solution for I_L given. Cut the elements of the first group. replace group 2 elements of I_L by group 1 elements of I'_L. In the example: replace in OPT-solution: $8, 10, 12$ by $6, 6, 6$.



$$OPT(I'_L) \leq OPT(I_L)$$

$$ALG(I_L) \leq ALG(I'_L) + \frac{|I_L|}{k} = OPT(I'_L) + \frac{|I_L|}{k} \leq OPT(I_L) + \frac{|I_L|}{k}$$

Chosse $k = \frac{1}{\varepsilon^2}$. The following holds: $OPT(I_L) \geq \sum sizes \geq |I_L|\varepsilon$. So:

$$OPT(I_L) + \frac{|I_L|}{k} = OPT(I_L) + |I_L|\varepsilon^2 \leq OPT(I_L) + OPT(I_L) \cdot \varepsilon = (1+\varepsilon)OPT(I_L)$$

$$\Rightarrow ALG(I_L) \leq (1+\varepsilon)OPT(I_L)$$

Now we want to go back to the initial instance. So we need to put the small item somewhere. For that use the greedy approach and just fill the gaps.

1. All small elements fit in the gaps $\Rightarrow ALG(I) = ALG(I_L)\checkmark$, we are ok
2. Otherwise: We need extra bin(s). If they don't fit create a new bin and try to fit them there. If this is full get a new one and so on. But there are still gaps in the bins before. There can't be any elements greater than ε (they are filled by $1-\varepsilon$ if the capacity is 1).

   $$(ALG(I) - 1) \cdot (1-\varepsilon) \leq \sum sizes \leq OPT(I)$$

   $$ALG(I) \leq \frac{OPT(I)}{1-\varepsilon} + 1$$

   (The 1 (=C) defines the APTAS)
   for $\varepsilon \leq \frac{1}{2} \rightarrow 1 + 2\varepsilon$

   $$ALG(I) \leq \frac{OPT(I)}{1-\varepsilon} + 1 \leq (1+2\varepsilon)OPT(I) + 1$$

6 Linear Programming

6.1 Problem

A factory produces tables and chairs

- 1 table requires 1 unit of metal and 3 units of wood
- 1 chair requires 2 units of metal and 1 unit of wood
- The factory has 600 units (900 units) of metal (wood)
- The profit of 1 table is 100€and 1 chair is 100€

Question How many tables/chairs to produce to maximize the profit?

$$x_1 : \#\textit{tables to produce} \Rightarrow x_1 \geq 0$$

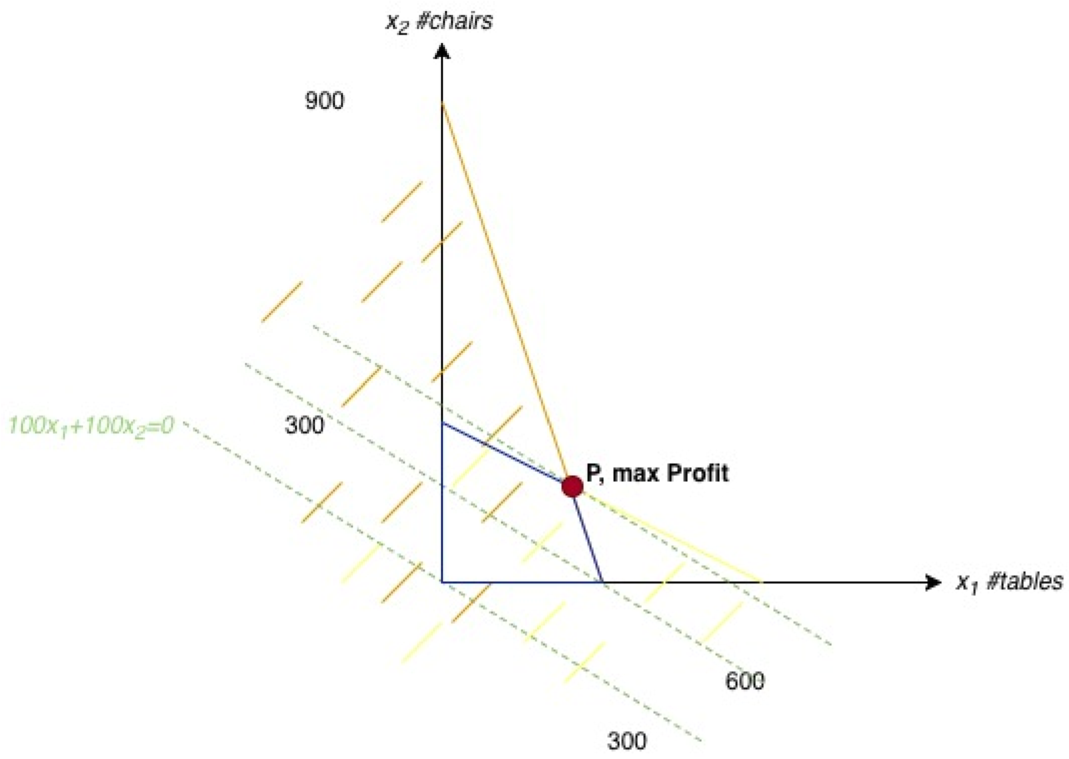
$$x_2 : \#\textit{chairs to produce} \Rightarrow x_2 \geq 0$$

Profit $100(x_1 + x_2) = 100x_1 + 100x_2$

Constraints $x_1 + 2x_2 \leq 600$ and $3x_1 + x_2 \leq 900$

Maximize $100x_1 + 100x_2$

Subject to $x_1 + 2x_2 \leq 600$ and $3x_1 + x_2 \leq 900$ and $x_1, x_2 \geq 0$



$$100x_1 + 100x_2 = 30000(x_1 = 0, x_2 = 300 \equiv x_1 = 900, x_2 = 0)$$

Observation As the line $100x_1 + 100x_2 = c$ moves toward P, the profit gets increased. $\Rightarrow P$ is optimal (intersection of $\overset{(1)}{x_1} + 2x_2 = 600$, $3x_1 + \overset{(2)}{x_2} = 900$)

(1) $\Rightarrow x_1 = 600 - 2x_2$

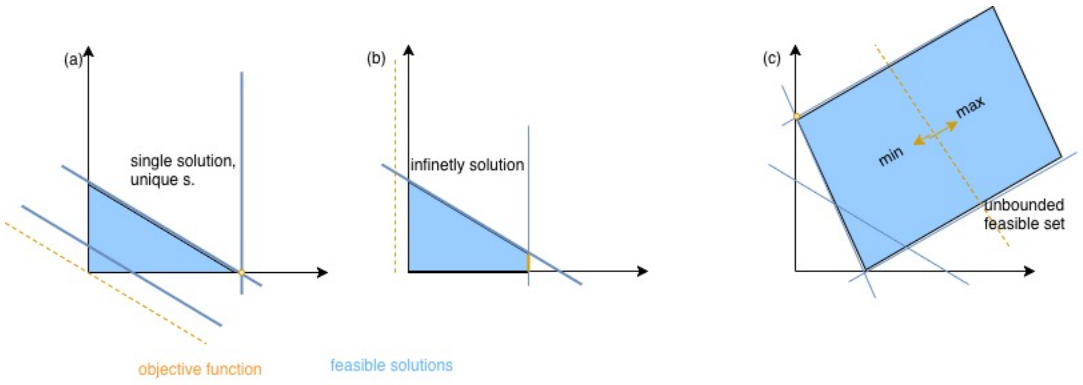
(2)

$$\begin{aligned} &\overset{(1)}{=} 2(600 - 2x_2) + x_2 = 900 \\ &1800 - 6x_2 + x_1 = 900 \\ &5x_2 = 900 \Rightarrow x_2 = 180 \qquad \Rightarrow x_1 = 240 \end{aligned}$$

Optimal $x_1 = 240, x_2 = 180$

Profit 42000 €

Different cases



(a),(b) $\Rightarrow$ bounded feasible solution $\Rightarrow \exists$ solution $\Rightarrow$ corner solution
(c) $\Rightarrow$ feasible (corner solution) or infeasible/unbounded

$$\text{(a),(b),(c)}\exists solution \Rightarrow \exists corner solution$$

(1) bounded feasible (a),(b)

(2) unbounded feasible (c) min

(3) unbounded infeasible (c) max

NOT ~~un~~bounded infeasible

6.1.1 Generalization

Maximize $c_1x_1 + c_2x_2 + ... + c_nx_n \rightarrow$ objective function (linear)

Minimize

Subject to constraints set

$$\begin{array}{r|c|c} & \text{one-of} & \\ a_{11}x_1 + a_{12}x_2 + ...a_{1n}x_n & & b_1 \\ & \leq & \\ a_{21}x_1 + a_{22}x_2 + ...a_{2n}x_n & & b_2 \\ & = & \\ ... & & ... \\ & \geq & \\ a_{m1}x_1 + a_{m2}x_2 + ...a_{mn}x_n & & b_m \end{array}$$

optional non-negativity constraint: $(x_1, x_2, ..., x_n \geq 0)$

6.2 Another example: Max-flow

Input directed graph $G = (V, E), s, t \in V$
$indegree(s) = outdegree(t) = 0$
capacities $c : E \to \mathbb{R}^+$

Output find $f : E \to \mathbb{R}^+$ such that:

(1) $\forall (u, v) \in E : 0 \leq f(u, v) \leq c(u, v)$

(2) $\sum_{(u,v)\in E} f(u.v) = \sum_{(v,u)\in E} f(u, v) \forall u \in V - s, t$

(3) $max \sum_{(s,v)\in E} f(s, v)$

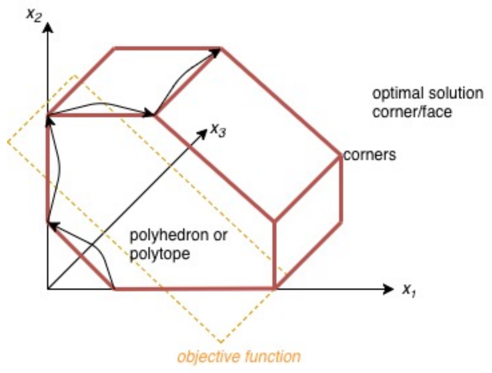
6.2.1 LP formulation

maximize $\sum_{(s,v)\in E} f(s, v)$ (3)
s.t.

Constraints $f(u, v) \leq c(u, v) \forall (u, v) \in E (\approx 1)$
$\sum_{(u,v)\in E} f(u.v) = \sum_{(v,u)\in E} f(u, v) \forall u \in V - s, t$ (2)

Negativity $f(u, v) \geq 0 \forall (u, v) \in E$

6.2.2 Geometric representation in higher dimensions



Simplex: Move from corner to corner

6.3 Standard form

Maximization $c_1x_1 + c_2x_2 + ... + c_nx_n$

Subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &\leq b_2 \\ &... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &\leq b_m \\ x_1, x_2, ..., x_n &\geq 0 \end{aligned}$$

Maximization C^Tx

Subject to

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned}$$

$$C = \begin{bmatrix} c_1 \\ c_2 \\ ... \\ c_n \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} c_1, ..., c_n \end{bmatrix}$$

$$x_1 = \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ ... \\ b_m \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & & & \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

First step of (pre-processing) simplex to bring the input LP to the standard form.
Why not in standard form?

(1) minimize $c_1x_1 + c_2x_2 + ... + c_nx_n \Leftrightarrow$ maximize $-c_1x_1 - c_2x_2 - ... - c_nx_n$

(2) Equality $a_{j1}x_1 + a_{j2}x_2 + ... + a_{jn}x_n = b_j$
$\Leftrightarrow a_{j1}x_1 + a_{j2}x_2 + ... + a_{jn}x_n \leq b_j$
$1(-)a_{j1}x_1 + (-)a_{j2}x_2 + (-)... + (-)a_{jn}x_n \geq (\leq)b_j$

(3) $\geq$ inequality $a_{j1}x_1 + a_{j2}x_2 + ... + a_{jn}x_n \geq b_j \Leftrightarrow -a_{j1}x_1 - a_{j2}x_2 - ... - a_{jn}x_n \leq b_j$

(4) Absence of non-negativity constraints.

Replace:
$x_j \rightarrow x'_j \& x''_j$
$x_j = x'_j - x''_j; x'_j, x''_j \geq 0$

Example

$$\begin{array}{ccccc} \text{minimize} & -2x_1 + 3x_2 & & \text{maximize} & 2x_1 - 3x'_2 + 3x''_2 \\ \text{s.t.} & x_1 + x_2 = 7 & & \text{s.t.} & x_1 + x'_2 - x''_2 \leq 7 \\ & x_1 - 2x_2 \leq 5 & \rightarrow & & -x_1 - x'_2 + x''_2 \leq -7 \\ & x_1 \geq 0 & & & x_1 - 2x''_2 + 2x''_2 \leq 5 \\ & & & & x_1, x'_2, x''_2 \geq 0 \end{array}$$

non-negativity for $x_1 = x'_2 - x''_2$

6.3.1 The simplex method (by an example)

$$\begin{aligned}
\text{maximize} & 5x_1 + 4x_2 + 3x_3 \\
\text{s.t.} & 2x_1 + 3x_2 + x_3 \leq 5 \\
\text{(in standard} & 4x_1 + x_2 + 2x_3 \leq 11 \\
\text{form)} & 3x_1 + 4x_1 + 2x_3 \leq 8 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

(1) Introduce a name for the objective function

$$J = 5x_1 + 4x_2 + 3x_3$$

(2) Introduce a slack (difference between right and left) variable for each constraint

$$x_4 = 5 - 2x_1 - 3x_2 - x_3; x_4 \geq 0$$

$$\text{right-left side} \geq 0 \Rightarrow \text{right-left side} = x_4, x_4 \geq 0$$

Standardform $\rightarrow$ slack form

$$\begin{aligned}
\text{maximize} J &= 5x_1 + 4x_2 + 3x_3 \\
x_4 &= 5 - 2x_1 - 3x_2 - x_3 (1) \\
x_5 &= 11 - 4x_1 - x_2 - 2x_3 (2) \\
x_6 &= 8 - 3x_1 - 4x_2 - 2x_3 (3) \\
x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}$$

non-zero variables, zero variables

Idea Start with a solution $(x_1, x_2, x_3, x_4, x_5, x_6)$, dinf another solution $(\overline{x_1}, \overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_5}, \overline{x_6})$ s.t, iterate $5x_1 + 4x_2 + 3x_2 \leq 5\overline{x_1} + 4\overline{x_2} + 3\overline{x_3}$

Problem Initial feasible solution to start the iteration?
In the example: Easy! $x_1 = x_2 = x_3 = 0.x_4 = 5, x_5 = 11, x_6 = 8, J = 0$

Assumption $b_1, b_2, ..., b_n \geq 0 \Rightarrow x_1 = x_2 = ... = x_n = 0$ is feasible
How can we find a better solution than $J = 0$?

Observation If we increase x_1 (or x_2 od x_3) (positive coefficient), then J will also increas, but we must ensure that $x_4, x_5, x_6 \geq 0$
Since $x_2 = x_3 = 0$

- $x_4 \geq 0 \Rightarrow 5 - 2x_1 \geq 0 \Leftrightarrow x_1 \leq \frac{5}{2} 2.5$
- $x_5 \geq 0 \Rightarrow 11 - 4x_1 \geq 0 \Leftrightarrow x_1 \leq \frac{11}{4} = 2.75$
- $x_6 \geq 0 \Rightarrow 8 - 3x_1 \geq 0 \Leftrightarrow x_1 \leq \frac{8}{3} = 2.66...$

New solution: $x_1 = \frac{5}{2}, x_2, x_3 = 0, x_4 = 0, x_5 = 1, x_6 = \frac{1}{2} J = \frac{25}{2} = 12.5$
How do we proceed?

Observation THe first step was easy beacuse all non-zero variables were expected as linear combination of zero variables (x_1 now got non zero).

- (1) $\Leftrightarrow 2x_1 = 5 - 3x_2 - x_3 - x_4 \Leftrightarrow x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{3}x_3 - \frac{1}{2}x_4$
- $J = 5(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{3}x_3 - \frac{1}{2}x_4) + 4x_2 + 3x_3 \Leftrightarrow J = \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4$
- $x_5 = 11 - 4(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{3}x_3 - \frac{1}{2}x_4) - x_2 - 2x_3 \Leftrightarrow x_5 = 1 + 5x_2 + 2x_4$
- $x_6 = 8 - 3(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{3}x_3 - \frac{1}{2}x_4) - 4x_2 - 2x_3 \Leftrightarrow x_6 = \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4$

maximize $J = \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4$

$$\begin{aligned} x_1 &= \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{3}x_3 - \frac{1}{2}x_4 \\ x_5 &= 1 + 5x_2 + 2x_4 \\ (*)x_6 &= \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\ &x_1, ..., x_6 \geq 0 . x_1, x_5, x_6 \text{ non zero variables} \end{aligned}$$

Observation If we increase x_2 or x_4, then J will decrease (bad idea), so we have to increase x_3 (positive coefficient)
Since $x_2 = x_4 = 0$:

- $x_1 \geq 0 \Leftrightarrow \frac{5}{2} - \frac{1}{2}x_3 \geq 0 \Leftrightarrow x_3 \leq 5$
- x_5 does not relate $x_3 \Rightarrow$ not constrained
- $x_6 \geq 0 \Leftrightarrow \frac{1}{2}x_3 \geq 0 \Leftrightarrow x_3 \leq 1$

New soluion: $x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 1, x_6 = 0$
maximize $J = 13 - 3x_2 - x_4 - x_6$

$$\begin{aligned} \text{s.t. } x_3 &= 1 + x_2 + 3x_4 - 2x_6 (*) \\ x_1 &= 2 - 2x_2 - 2x_4 + x_6 \\ x_5 &= 1 + 5x_2 + 2x_4 \\ &x_1, x_2, ..., x_6 \geq 0 \end{aligned}$$

If x_2, x_4, x_6 are increased, then J decreases $\Rightarrow J$ is optimal.

1. The last system is equivalent to the first
2. In the last system all variables must be non-zero (cant't decrease variable to < 0 e.g. $13 - 3 \cdot (-2) \lightning x_2 \geq 0$!)