QM 2020 Fall Homework

HW0

1-4

Any complex number $z = x + iy$ and its complex conjugate $\bar{z} = x - iy$ can be written as

$z = x + iy = |z|(\cos\varphi + i\sin\varphi) = re^{i\varphi}$

$\bar{z} = x - iy = |z|(\cos\varphi - i\sin\varphi) = re^{-i\varphi}$

$r = |z| = \sqrt{x^2 + y^2}$ is the magnitude of z

$\varphi = \arg z = \operatorname{atan} 2(y, x)$

In particular, $\forall k \in \mathbb{Z},\ e^{i(2k)\pi} = 1$

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1. a) $9\sqrt{2}\exp\left(\frac{\pi}{4}i\right)$ b) $\sqrt{2}\exp\left(-\frac{\pi}{4}i\right)$

2. a) $3i$ b) $-i$ c) $\exp(4\pi)$

3. a) $(1+2i)^{\frac{3}{2}} + \exp(3-4i)$ b) $\frac{1}{8\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}}\frac{r}{a_0}\exp\left(\frac{-r}{2a_0}\right)\sin(\theta)\exp(-i\varphi)$

4. $7^{\frac{1}{5}}e^{\frac{2k}{5}\pi i}, k = 0, 1, 2, 3, 4$

5-6

Chain rule in Lagrange's notation: $\forall x$, let $F(x) = f(g(x))$, then $F'(x) = f'(g(x))g'(x)$

Product rule $(f \cdot g)' = f' \cdot g + f \cdot g'$

Counterpart of product rule --- **Integration by parts**: $\int udv = uv - \int vdu$

Even and odd functions

If $f(-x) = f(x)$, then f is an even function of x

$\int_{-a}^{+a} f(x)dx = 2\int_0^a f(x)dx$ for $f(x)$ even

If $g(-x) = -g(x)$, then g is an even function of x

$\int_{-a}^{+a} g(x)dx = 0$ for $g(x)$ odd

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5. a) $[\ln(x) + 1]x^x$ b) $\frac{a+2a\cos(ax)}{(\cos(ax)+2)^2}$ c) $zx^{(z-1)} + z^x \ln z$

6. a) $xe^x - e^x + C$ b) 0

7-9

Let V be an inner-product space.

Definition Nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$ form an orthogonal set if they are orthogonal to each other: $\left\langle \mathbf{v}_i, \mathbf{v}_j \right\rangle = 0$ for $i \neq j$.

If, in addition, all vectors are of unit norm, $\|\mathbf{v}_i\| = 1$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is called an orthonormal set.

Theorem Any orthogonal set is linearly independent.

The Gram-Schmidt orthogonalization process

Let V be a vector space with an inner product. Suppose $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is a basis for V.

Let $\mathbf{v}_1 = \mathbf{x}_1$

$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1$

$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2$

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$\mathbf{v}_n = \mathbf{x}_n - \frac{\langle \mathbf{x}_n, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \cdots - \frac{\langle \mathbf{x}_n, \mathbf{v}_{n-1} \rangle}{\langle \mathbf{v}_{n-1}, \mathbf{v}_{n-1} \rangle} \mathbf{v}_{n-1}$

Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal basis for V.

Normalization

Let V be a vector space with an inner product. Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal basis for V. Let $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \ldots, \mathbf{w}_n = \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}$

Then $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ is an orthonormal basis for V.

Some Properties of Determinant

If the columns (or rows) are linearly dependent then $\det(A) = 0$

Proof: let $A_n = \alpha_1 A_1 + \cdots + \alpha_{n-1} A_{n-1}$

$\det A = \det\begin{bmatrix} A_1 & \ldots & A_{n-1} & \sum_{i=1}^{n-1} \alpha_i A_i \end{bmatrix} = \sum_{i=1}^{n-1} \alpha_i \det\begin{bmatrix} A_1 & \ldots & A_{n-1} & A_i \end{bmatrix} = 0$

which used the property that exchanging two columns or rows changes the sign of the determinant.

Lagrange multipliers

If $F(x, y)$ is a (sufficiently smooth) function in two variables and $g(x, y)$ is another function in two variables, and we define $H(x, y, z) := F(x, y) + zg(x, y)$, and (a, b) is a relative extremum of F subject to $g(x, y) = 0$, then there is some value $z = \lambda$ such that $\left.\frac{\partial H}{\partial x}\right|_{(a,b,\lambda)} = \left.\frac{\partial H}{\partial y}\right|_{(a,b,\lambda)} = \left.\frac{\partial H}{\partial z}\right|_{(a,b,\lambda)} = 0.$

$g(x, y)$ can be seen as a single constraint on $F(x, y)$. This idea can be generalized to multi-variables and multi-constraints.

7. a) linearly dependent; $dim(V) = 1; \begin{pmatrix} \frac{-14}{\sqrt{205}} \\ \frac{3}{\sqrt{205}} \end{pmatrix}$ b) linearly independent; $dim(V) = 2; \begin{pmatrix} \frac{-14}{\sqrt{205}} \\ \frac{3}{\sqrt{205}} \end{pmatrix}, \begin{pmatrix} \frac{-3}{\sqrt{205}} \\ \frac{-14}{\sqrt{205}} \end{pmatrix}$

c) linearly independent; $dim(V) = 2; \begin{pmatrix} \frac{-14}{\sqrt{205}} \\ \frac{3}{\sqrt{205}} \end{pmatrix}, \begin{pmatrix} \frac{3}{\sqrt{205}} \\ \frac{14}{\sqrt{205}} \end{pmatrix}$

8. a) 0 b) 0 c) 0

9. a) $8\sqrt{2}$ b) 16

HW1

1-5

$\lim_{x\to 0} e^x = 1 + x$

$\lim_{x\to\infty} (e^x - 1) = e^x$

Photoelectric equation: $hv = W + \frac{mV^2}{2},\ \ W: workfunction\ \ \ V: speed$

de Broglie wavelength: $\lambda = \frac{h}{p} = \frac{h}{mv}$

$\hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$

The product of standard deviations of two properties of a quantum-mechanical system whose state function is Ψ must satisfy the inequality:

$$\sigma(A)\sigma(B) \equiv \Delta A \Delta B \geq \frac{1}{2}\left|\int \Psi^*[\hat{A}, \hat{B}]\Psi d\tau\right|$$

$$\Delta x \Delta p_x \geq \frac{1}{2}\left|\int \Psi^* \left[\hat{x}, \hat{p}_x\right] \Psi d\tau\right| = \frac{1}{2}\left|\int \Psi^* i\hbar \Psi d\tau\right| = \frac{1}{2}\hbar |i| \left|\int \Psi^* \Psi d\tau\right|$$

$$\sigma(x)\sigma(p_x) \equiv \Delta x \Delta p_x \geq \frac{1}{2}\hbar$$

$$\sigma(x)\sigma(p_z) \equiv \Delta x \Delta p_z \geq 0$$

Bohr's equation for H-like atom: $v = \frac{m_e q_e^4}{8\varepsilon_0^2 h^3}\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$

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1. If either $\lim_{x\to 0} E_v(T) \neq \frac{8\pi v^2}{c^3}kT$ (Rayleigh-Jeans law) or $\lim_{v\to\infty} E_v(T) \neq \frac{8\pi h v^3}{c^3}e^{\frac{hv}{kT}}$ (Wien's law), then the sleepy Plank made a mistake.

a) No. b) No. c) No.

2. According to the above Photoelectric equation, a) frequency.

3. a) $1.89 \times 10^{-33}\mathrm{m}$; b) $3.98 \times 10^{-10}\mathrm{m}$; c) $6.31 \times 10^{-38}\mathrm{m}$

4. a) No. ; b) No.; c) No. (See above inequality relationship)

5. a) According to above Bohr's equation for H-like atom, emission frequency will change due to double mass of electron, and some changed emission frequency will overlap with normal hydrogen under certain $\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$ value.

b) Yes. We can measure it under $\sqrt{2}\varepsilon_0$.