Supplementary Material: Nonparametric Empirical Bayes Estimation and Testing for Sparse and Heteroscedastic Signals

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A Simulation

The goal of the simulation is to investigate the adaptivity of DNP and SNP to different levels of signal strength, sparsity level, and heteroscedasticity. To be specific, $y_i|\mu_i \sim \mathcal{N}(\mu_i, \sigma_i)$ where μ_i's and σ_i's are generated as follows:

$$\mu_i \overset{i.i.d}{\sim} w\delta_0 + (1-w)N(V,1), \quad \sigma_i^2 \overset{i.i.d}{\sim} U(0.5,u) \tag{1}$$

where $\omega \in \{0.55, 0.65, \ldots, 0.95\}$, $V \in \{1, 1.5, 2, 2.5, 3\}$ and $u \in \{1, 1.5, 2, 2.5\}$. Note that $V \in \{1, 1.5, 2\}$ For each setting, we set $n = 1000$ and report the above-mentioned metrics across 100 Monte Carlo repetitions.

We compare the following metrics:

1. Relative mean Squared Error (MSE) of posterior mean $n^{-1}\sum_{i=1}^{n}(\hat{\mu}_i - \mu_i)^2$;
2. Relative mean Squared Error (MSE) of posterior mode;
3. Bias of sparsity estimation $\text{Bias}_{\hat{\omega}} = \hat{\omega} - \omega$;
4. Credible interval coverage $n^{-1}\sum_{i=1}^{n}\mathbf{1}\{\mu \in \widehat{CI}\}$;
5. Empirical FDR controlling FDR at different nominal levels;
6. Empirical power controlling FDR at 0.05 level.

Note that the relative ratio uses SNP as the base, i.e., the relative ratio is the ratio of the metric for any competing estimator to that of SNP. If the ratio is larger than 1, SNP performs better than the competing estimator.

Figure A1 compares different methods in terms of different metrics with $u = 1.5$ and $V = 2$ (This is same as Figure 2 in the main manuscript. We added the adaptive threshold of Stephens [2017] (ash) per Reviewer S253 suggested). As suggested by other reviewers, we provide the tables corresponding to Figure A1. Table A1 to A9 shows the MSE of posterior mean, the relative MSE of posterior mean, the MSE of posterior mode, the relative MSE of posterior mode, the bias of the sparsity estimate, the coverage and average length of credible interval, the empirical FDR with $\omega = 0.95$ and the empirical power controlling FDR level at 0.05.

Figure A2 to A9 show more detailed comparisons varying u and V. In Section D, we provide all the information in the table format with the best performers set in bold type.

Figure A2 reports the relative MSE of posterior mean varying the signal strength V, sparsity level w and variance heterogeneity u. The columns correspond to different signal strength V and the rows are across different variance heterogeneity u. For each plot, the x-axis is the sparsity level w_0 and the y-axis the ratio of the MSE of competing estimator to the MSE of SNP. We compare with 1) the

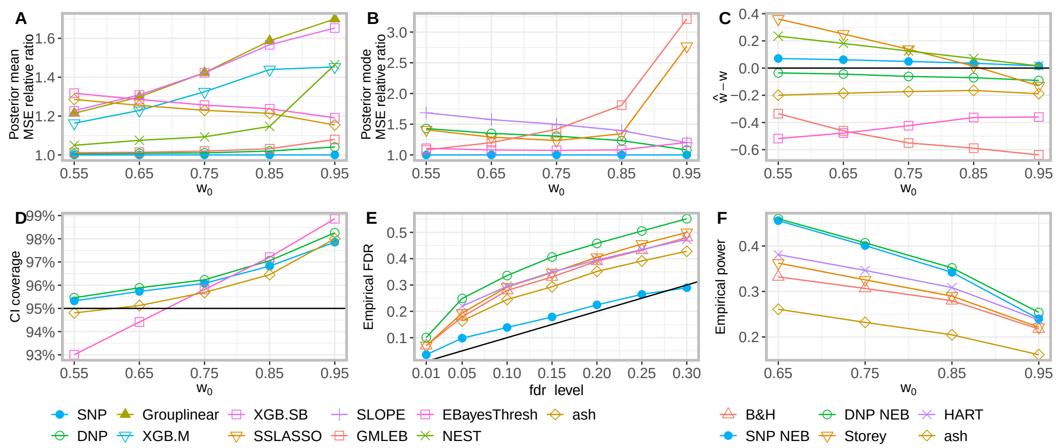


Figure A1: A) Relative mean Squared Error (MSE) of posterior mean $n^{-1}\sum_{i=1}^{n}(\hat{\mu}_i - \mu_i)^2$; B) Relative mean Squared Error (MSE) of posterior mode; C) Bias of sparsity estimation $\text{Bias}_{\hat{\omega}} = \hat{\omega} - \omega$; D) Credible interval coverage; E) Empirical FDR; F) Empirical power controlling FDR at 0.05 level. Note that the relative ratio uses SNP as the base – if the ratio is larger than 1, SNP performs better than the competing estimator.

Table A1: Posterior mean MSE with $u = 1.5$ and $V = 2$

ω	SNP	DNP	Grouplinear	XGB.M	XGB.SB	GMLEB	EBayesThresh	NEST	ash
0.55	**0.509**	0.511	0.618	0.591	0.624	0.514	0.67	0.534	0.654
0.65	**0.456**	0.459	0.591	0.56	0.596	0.462	0.585	0.49	0.572
0.75	**0.38**	0.384	0.541	0.504	0.541	0.388	0.478	0.415	0.467
0.85	**0.276**	0.281	0.438	0.397	0.432	0.284	0.341	0.316	0.335
0.95	**0.133**	0.139	0.227	0.194	0.22	0.144	0.159	0.195	0.154

Table A2: Posterior mean MSE Relative ratio with $u = 1.5$ and $V = 2$

ω	SNP	DNP	Grouplinear	XGB.M	XGB.SB	GMLEB	EBayesThresh	NEST	ash
0.55	**1**	1.005	1.215	1.163	1.228	1.012	1.318	1.05	1.286
0.65	**1**	1.008	1.298	1.229	1.309	1.014	1.285	1.075	1.255
0.75	**1**	1.011	1.424	1.325	1.423	1.02	1.257	1.093	1.23
0.85	**1**	1.02	1.587	1.439	1.567	1.031	1.237	1.147	1.214
0.95	**1**	1.041	1.7	1.453	1.653	1.079	1.191	1.464	1.154

Table A3: Posterior mode MSE with $u = 1.5$ and $V = 2$

ω	SNP	DNP	SSLASSO	SLOPE	GMLEB	EBayesThresh
0.55	**0.536**	0.767	0.756	0.904	0.58	0.594
0.65	**0.448**	0.605	0.577	0.705	0.537	0.484
0.75	**0.336**	0.439	0.414	0.503	0.476	0.361
0.85	**0.217**	0.267	0.291	0.302	0.392	0.234
0.95	**0.084**	0.091	0.232	0.1	0.269	0.101

parametric empirical Bayes mean estimator (EBayesThresh) with Laplace as the slab of Johnstone and Silverman [2004]; 2) the generalized maximum likelihood Empirical Bayes estimator (GMLEB) of Jiang et al. [2020] using Koenker and Gu [2017]; 3) the group linear estimator by Weinstein et al. [2018]; 4) the semi-parametric monotonically constrained SURE estimator (XKB.SB) and the parametric SURE estimator (XKB.M) from Xie et al. [2012]; 5) the Nonparametric Empirical Bayes Structural Tweedie (NEST) by Banerjee et al. [2020]; 6) adaptive thresholding by Stephens [2017]. Note that NEST is designed for unknown variance. We can see two clusters in terms of performance, one of the parametric methods and one of the nonparametric. The nonparametric methods perform better than the parametric counterparts. When the signal V is strong, the advantages

Table A4: Posterior mode MSE Relative ratio with $u = 1.5$ and $V = 2$

ω	SNP	DNP	SSLASSO	SLOPE	GMLEB	EBayesThresh
0.55	**1**	1.43	1.409	1.686	1.081	1.107
0.65	**1**	1.351	1.289	1.575	1.2	1.081
0.75	**1**	1.307	1.234	1.499	1.419	1.074
0.85	**1**	1.233	1.345	1.396	1.809	1.082
0.95	**1**	1.081	2.769	1.199	3.213	1.203

Table A5: Bias of sparsity with $u = 1.5$ and $V = 2$

ω	SNP	DNP	SSLASSO	GMLEB	EBayesThresh	HART	ash
0.55	0.07	**-0.036**	0.361	-0.335	-0.518	0.235	-0.199
0.65	0.061	**-0.044**	0.251	-0.461	-0.478	0.181	-0.185
0.75	**0.049**	-0.061	0.138	-0.55	-0.424	0.126	-0.173
0.85	0.036	-0.071	**0.015**	-0.589	-0.363	0.071	-0.164
0.95	**0.016**	-0.091	-0.13	-0.638	-0.36	**0.016**	-0.189

Table A6: Coverage of credible interval with $u = 1.5$ and $V = 2$

ω	SNP	DNP	EBayesThresh	ash
0.55	0.95	0.96	0.93	0.95
0.65	0.96	0.96	0.94	0.95
0.75	0.96	0.96	0.96	0.96
0.85	0.97	0.97	0.97	0.96
0.95	0.98	0.98	0.99	0.98

Table A7: Length of credible interval with $u = 1.5$ and $V = 2$

ω	SNP	DNP	EBayesThresh	ash
0.55	2.428	**2.414**	2.957	2.971
0.65	2.266	**2.235**	2.85	2.757
0.75	**2.034**	2.044	2.698	2.493
0.85	**1.724**	1.73	2.49	2.146
0.95	**1.289**	**1.289**	2.307	1.668

Table A8: Empirical FDR with $u = 1.5$ and $V = 2$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART	ash
	0.01	0.07	0.04	0.10	0.07	0.09	0.10
	0.05	0.18	0.10	0.25	0.19	0.22	0.16
	0.10	0.28	0.14	0.34	0.29	0.30	0.24
0.95	0.15	0.33	0.18	0.41	0.35	0.35	0.29
	0.20	0.39	0.22	0.46	0.41	0.40	0.35
	0.25	0.43	0.26	0.50	0.46	0.44	0.39
	0.30	0.48	0.29	0.55	0.50	0.47	0.43

of the nonparametric methods are even larger. In general, SNP and DNP perform better than the others. The advantages is more pronounced as the sparsity level w_0 increases since the other methods are not specially designed for sparse data. The closest competitor with SNP and DNP is GMLEB which adapts g-modeling as well but does not use an EM algorithm. NEST is also comparable when the heterogeneity u is relatively small.

Figure A3 reports the performance of the posterior mode estimators varying the signal strength V, sparsity level w and variance heterogeneity u. We compare 1) the parametric empirical Bayes

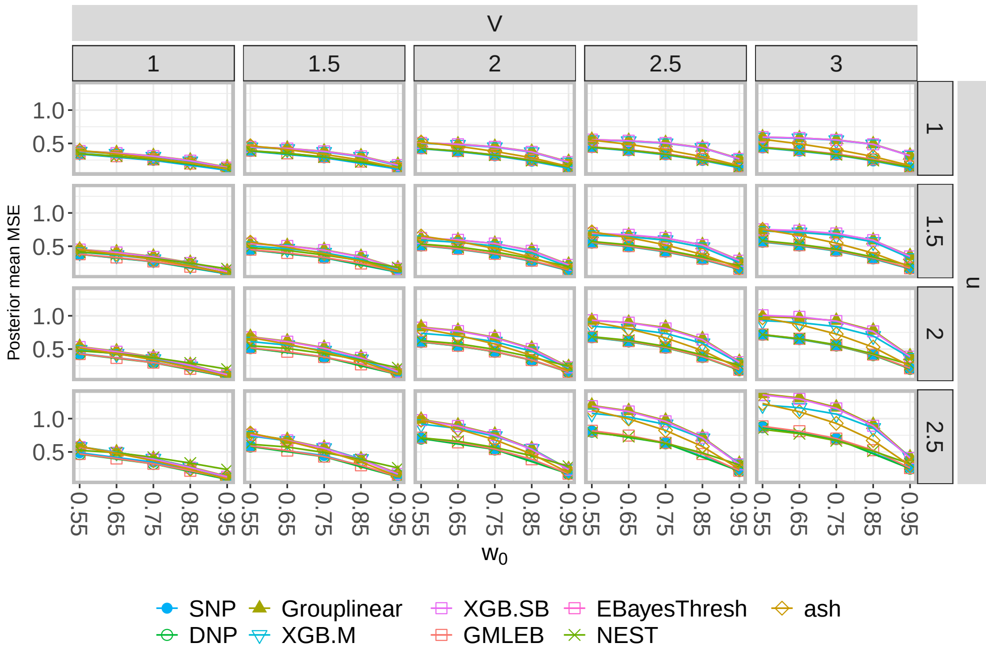


Figure A2: Posterior mean MSE relative ratio.

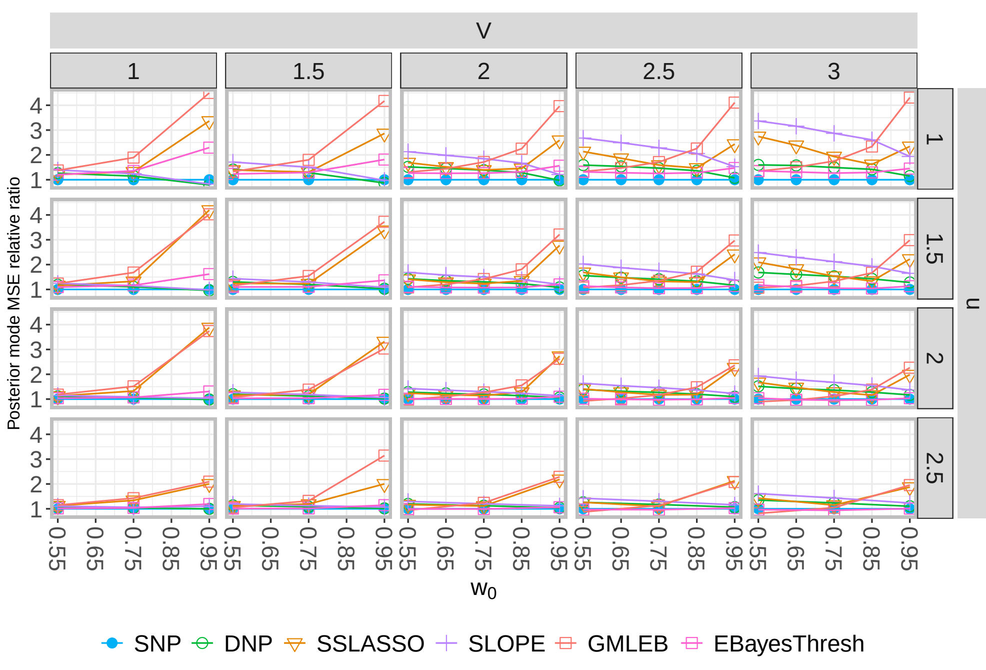


Figure A3: Posterior mode MSE relative ratio.

Table A9: Emprical power with $u = 1.5$ and $FDR = 0.05$

V	ω	BH	SNP NEB	DNP NEB	Storey	HART	ash
2	0.65	0.332	0.456	**0.46**	0.362	0.381	0.261
	0.75	0.307	0.401	**0.407**	0.326	0.346	0.232
	0.85	0.28	0.342	**0.352**	0.29	0.309	0.205
	0.95	0.218	0.24	**0.254**	0.221	0.238	0.161

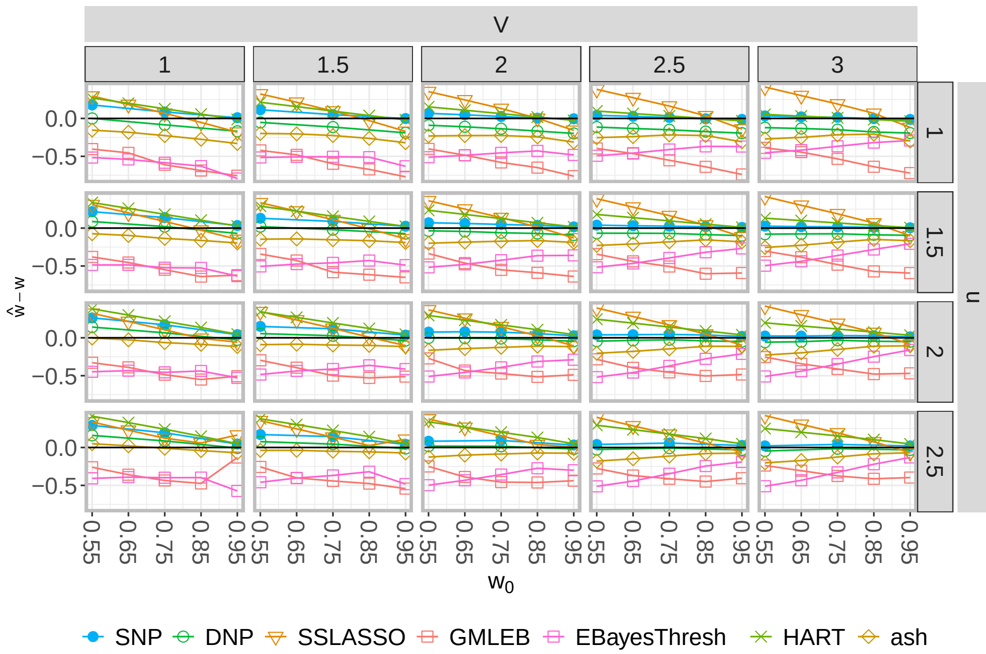


Figure A4: Bias of the sparsity.

mode estimator (EBayesThresh) with Laplace as the slab of Johnstone and Silverman [2004]; 2) the SLOPE estimator of Bogdan et al. [2011], Su et al. [2016] with $q = 0.1$; 3) the two-step Spike-and-Slab LASSO estimator of Rocková [2018]; 4) GMLEB posterior mode estimator. SNP is robust across different settings; while DNP can be less competitive since it tends to underestimate the sparsity. Comparing SNP and DNP shows the advantages of the Laplacian spike capturing sparsity. Surprisingly, our closest competitor in posterior mean estimator, GMLEB, is quite off using its posterior mode estimator. This may due to the convex relaxation Koenker and Gu [2017].

Figure A4 shows the performance of sparsity estimation $\hat{\omega}$ varying the signal strength V, sparsity level w and variance heterogeneity u. We can only compare methods mentioned above that are able to estimate sparsity. Unfortunately, all the f-modeling approaches cannot provide sparsity estimation. Therefore, we only compare DNP and SNP with GMLEB, SSLASSO, EBayesThresh, HART and ash. Note that HART uses Jin and Cai [2007] with a theoretical null $N(0, 1)$ to estimate the sparsity. SNP and DNP estimate the sparsity level with high accuracy, while DNP underestimates the sparsity when the heterogeneity u is low compared to SNP. GMLEB, EBayesThresh and ash underestimate the sparsity level, while HART and SSLASSO tend to overestimate. The overestimating behavior of SSLASSO agrees with Theorem 4.2 of Rocková [2018].

As a bonus of the Bayesian mechanism, we are able to provide uncertainty quantification in addition to point estimate. We construct the 95% equal-tailed credible interval from the posterior distribution. Figure A5 shows the empirical coverage and the average width of the credible interval across w and V, comparing with EBayesThresh and ash. Most credible intervals are overshoot while the widths of the interval are acceptable. The credible intervals are below nominal coverage when signal is weak and the heterogeneity of noise is large. In general, the average length of the credible interval by ash

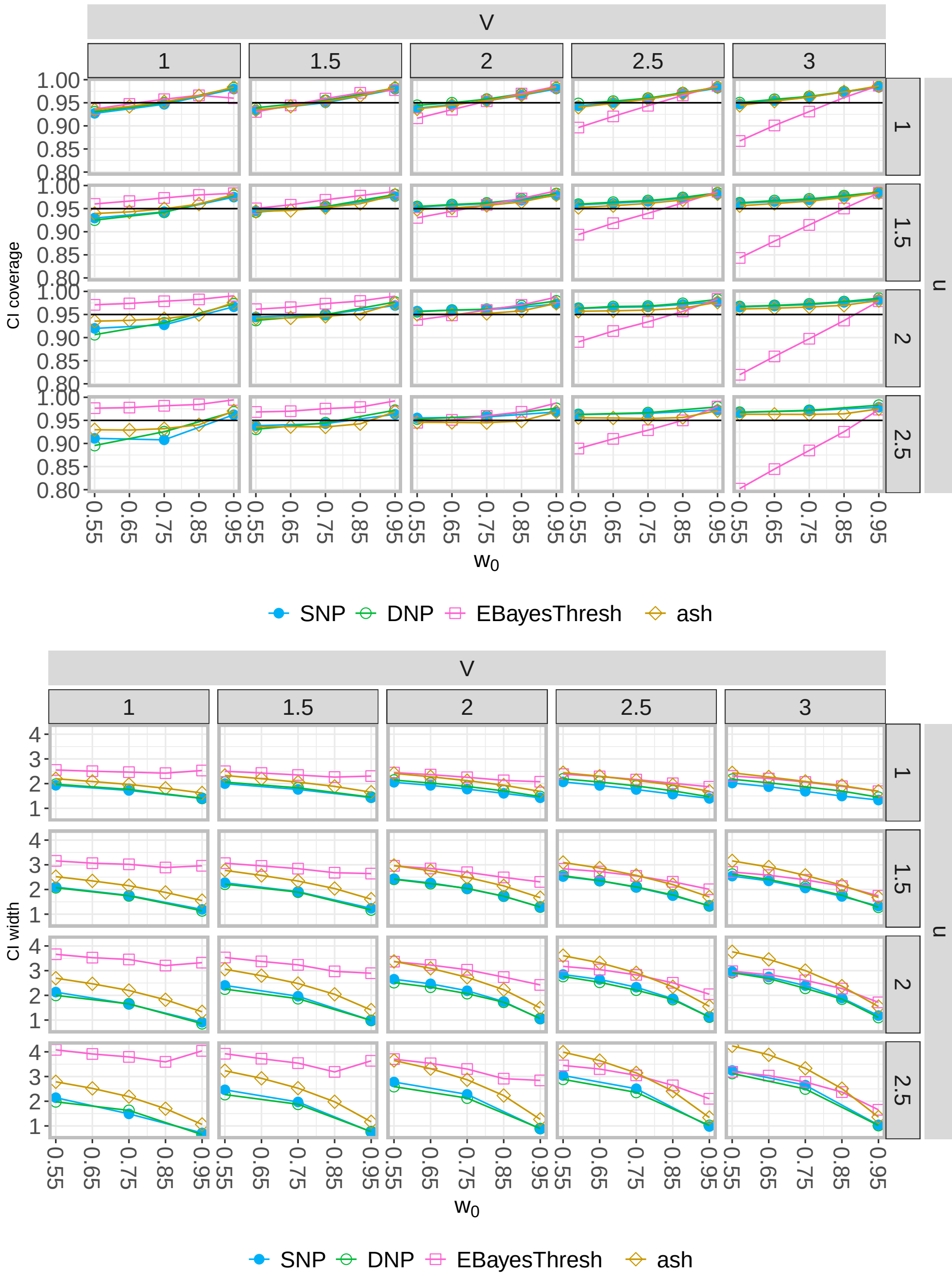


Figure A5: Coverage and width of the credible interval.

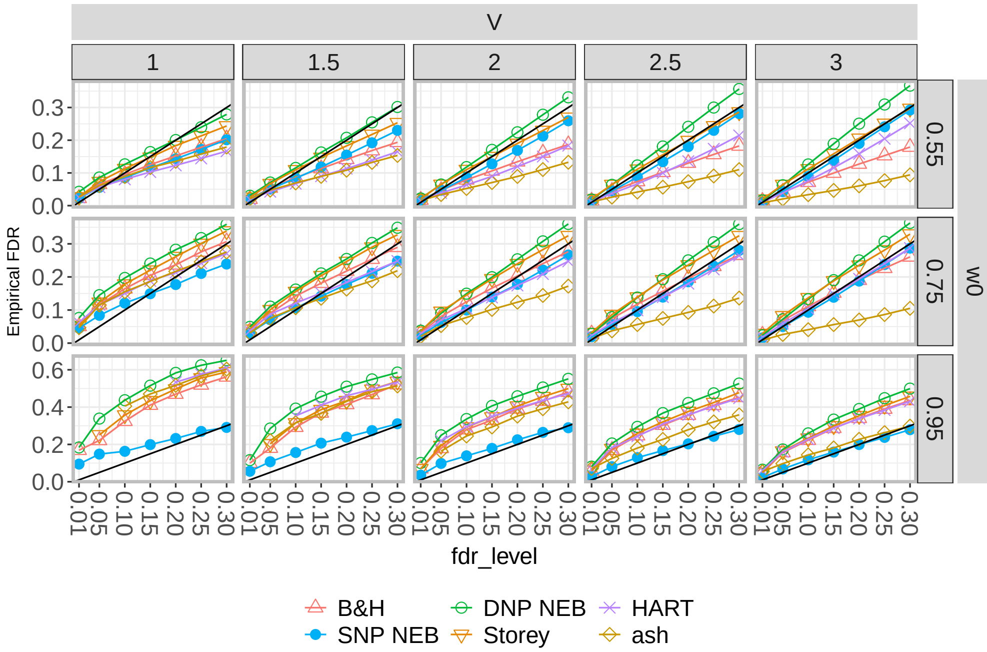


Figure A6: FDR control at heterogeneity level $u = 1.5$.

is wider than that of DNP and SNP. The credible interval by EBayesThresh is below the nominal coverage, especially with high signal and heterogeneity levels and is wider than the others.

Figure A6, A7, A8 and A9 show the performance of the multiple testing procedure. Each plot is at different signal strength V and different heterogeneity level u or sparsity level w. The x-axis is the FDR control level (α) and the y-axis is the bias of the average of the empirical ratio V/R and the empirical power across 100 runs. We compare SNP-OPT and DNP-OPT with the original linear step-up Benjamini and Hochberg [1995] procedure, the pFDR of Storey [2002], the adaptive threshold (ash) of Stephens [2017], and the HART procedure of Fu et al. [2020].

Figure A6 shows the empirical FDR varying the signal strength V and sparsity level w when the heterogeneity $u = 1.5$ across different nominal levels; while Figure A7 fixes the sparsity level $w = 0.95$ and varies the signal strength V and the heterogeneity level. SNP-OPT controls false discovery rate at the desired nominal level α, while others are overconfident and reject too many hypotheses in most of the cases. When the sparsity level is low, B&H procedure is conservative as expected. Storey's procedure is quite robust since it estimates the error rate of a predetermined rejection region (other than the sparse setting when $w = 0.95$). It is instructive to compare DNP-OPT and SNP-OPT to see the benefits of the Laplacian spike. DNP-OPT tends to reject too many hypotheses, overshooting the nominal level α. This is expected since DNP underestimates the sparsity level as in Figure A4. As a result, the posterior probability of being zero $\hat{p}_i(y_i)$ is underestimated. Therefore, following our NEB-OPT procedure, DNP-OPT is over-confident in rejecting hypotheses.

Similarly, Figure A8 shows the empirical power by fixing the heterogeneity level at $u = 1.5$ and varying the signal strength V and sparsity level w across different nominal FDR levels; while Figure A9 fixes the sparsity level $w = 0.95$ and varies the signal strength V and the heterogeneity level. SNP-OPT controls FDR at nominal levels and at the same time increases power in most settings.

This concludes the simulation study. The nonparametric mixture prior is robust and versatile across various sparsity levels and signal strengths, especially for SNP. The multi-directional shrinkage property is particular desirable in the sparse set up where the noises are shrunk towards zero while the signals towards their corresponding centers. The multi-directional effect also reflects in the adaptive thresholding for the posterior mode estimator. In addition to point estimate, uncertainty quantification is readily provided from the Bayesian mechanism. The equal-tailed credible intervals show coverage at nominal level and are of reasonable widths.

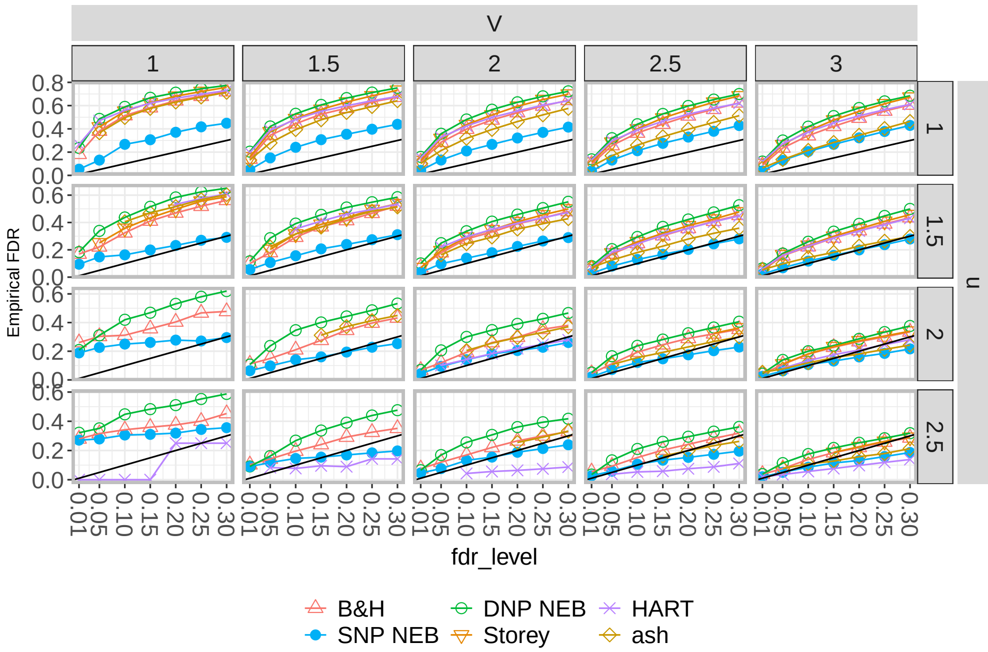


Figure A7: FDR control at sparsity level $w = 0.95$.

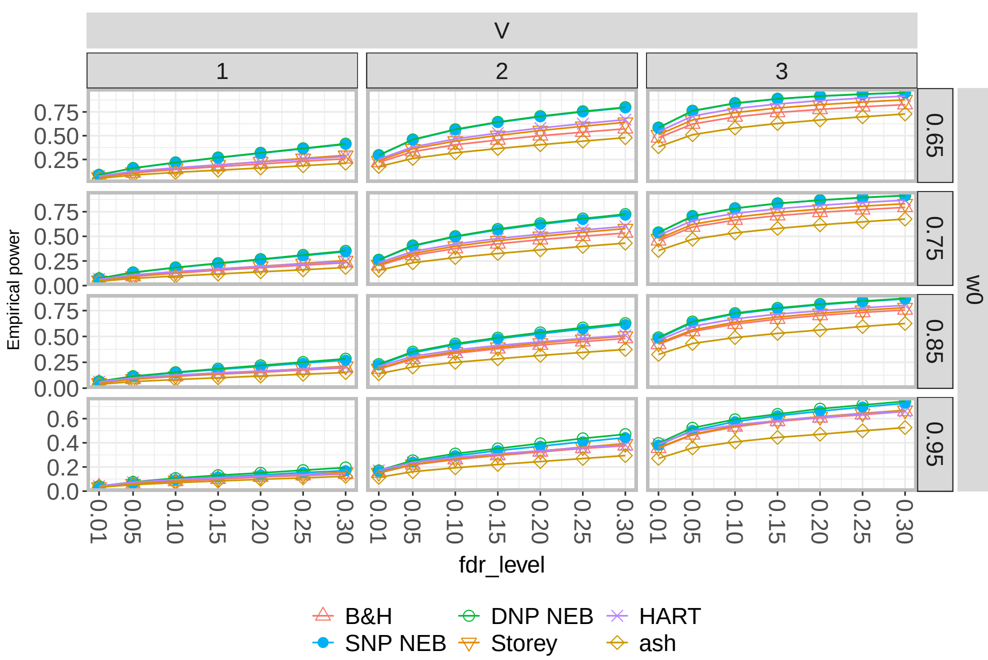


Figure A8: Empirical power at heterogeneity level $u = 1.5$.

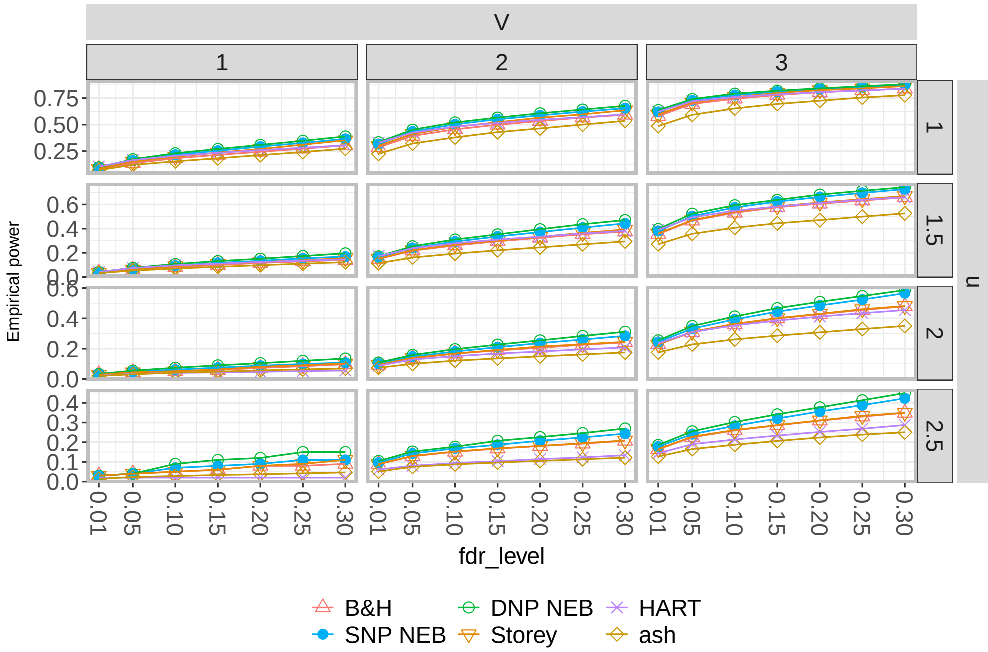


Figure A9: Empirical power at sparsity level $w = 0.95$.

With a focus on sparse data, it is desirable to have an accurate sparsity estimate. SNP is able to estimate the sparsity level by nature, and it estimates the sparsity well. It had been troubling to the authors that DNP tends to underestimates the sparsity due to the dispersion around zero. The underestimate of DNP also compromises the performance of the multiple testing procedures. Remedies such as a wider gap between zero and the grid points around zero are not adaptive and rather artificial. SNP, on the other hand, replaces the point-mass at zero with a Laplacian spike to handle the sparsity. The adaptivity of the spike component is particularly attractive.

The proposed multiple testing procedures, NEB-OPT, control FDR at nominal levels in different settings and achieve higher power.

B Gene expression data

We first compare different methods as in Figure 3 of the main manuscript but in separate plots for clearer comparison. Figure A10 compares DNP and SNP with the classic the BH and Storey's procedure. Figure A11 compares SNP with the state-of-the-art HART and its jackknifed variant.

As argued in Efron [2004], a small deviation from the theoretical null $N(0, 1)$ will distort the FDR analysis, resulting in too many inappropriate rejections as in Figure 3. In order to compare our methods and other methods thoroughly, we adopt the empirical null approach in Efron [2004] to first estimate the empirical null. The estimated empirical null turns out to be $N(0, 1.09^2)$. We then obtain the new p-value by converting the z-value as $p_i' = 2\Phi'(-|Z_i|)$ where Φ' is the CDF of a $N(0, 1.09^2)$ variable. We can see that comparing Figure A12a and A12b, the histogram of p-value estimated from the empirical null is closer to uniform compared to that estimated from the theoretical null $N(0, 1)$.

Storey	HART	DNP-OPT	SNP-OPT
0.93	0.99	0.91	0.96

Table A10: Sparsity level estimation

We are now ready to apply BH, Storey, HART as well as our SNP-OPT and DNP-OPT to the microarray data. All procedures target to control FDR at level 0.05. For SNP-OPT and DNP-OPT, we simply plug in the differential difference and the pooled estimate of the standard deviation to

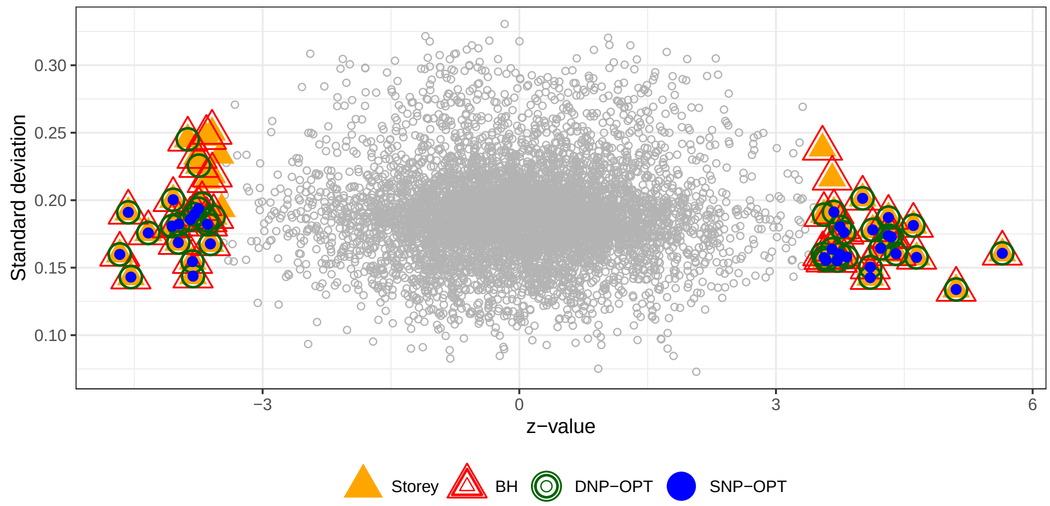


Figure A10: The scatter plot of Z vs σ. The red triangles ($\triangle$) label the 51 discoveries by the BH procedure. The green circle ($\bigcirc$) labels the 44 discoveries by DNP-OPT. The blue solid circle ($\bullet$) label the 37 discoveries by SNP-OPT. The yellow solid triangle ($\blacktriangle$) label the 53 discoveries by Storey's procedure. All the procedures target FDR level at 0.05.

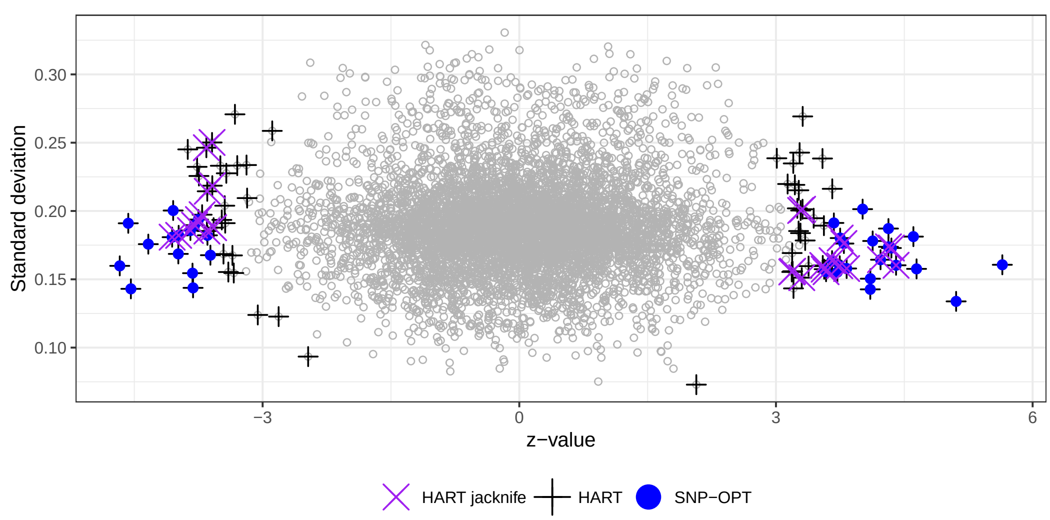
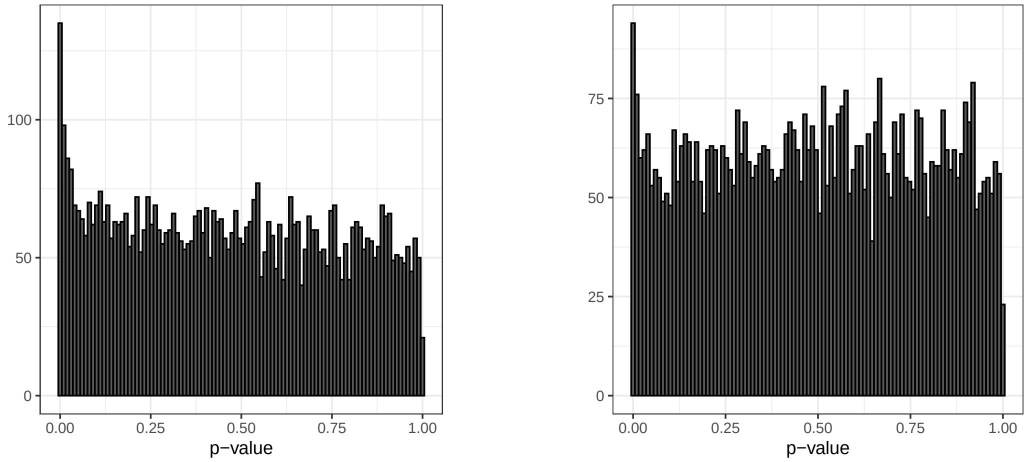


Figure A11: The scatter plot of Z vs σ. The black plus ($+$) labels the 89 discoveries by HART. The purple cross ($\times$) labels the 29 discoveries by HART jackknifed procedure. The blue solid circle ($\bullet$) label the 37 discoveries by SNP-OPT. All the procedures target FDR level at 0.05.



(a) Histogram of the unadjusted p-value (b) Histogram of the adjusted p-value

Figure A12: Figure (A) shows the histogram of the unadjusted p-value. Figure (B) shows the histogram of the adjusted p-value p_i''s estimated from the empirical null $N(0, 1.09^2)$.

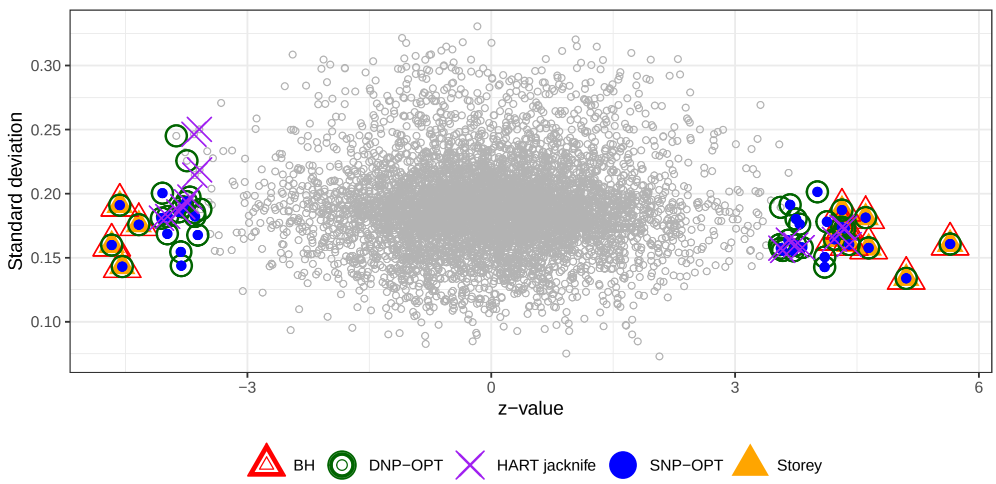


Figure A13: The scatter plot of Z vs σ. The red triangles ($\triangle$) label the 13 discoveries by the BH procedure. The green circle ($\bigcirc$) labels the 44 discoveries by DNP-OPT. The purple cross ($\times$) labels the 19 discoveries by HART jackknifed procedure. The blue solid circle ($\bullet$) label the 37 discoveries by SNP-OPT. The yellow solid triangle ($\blacktriangle$) label the 13 discoveries by Storey's procedure. All the procedures target FDR level at 0.05.

our procedures. For the others, we will use the empirical null and the adjusted p-values p_i'. Both BH and Storey use the p-value p_i' estimated from the empirical null. HART estimates the sparsity level using Jin-Cai's method with the empirical null $N(0, 1.09^2)$, following the procedure as in Fu et al. [2020], and adopts a jacknifed procedure to estimate the marginal density. Since SNP also provides an estimate to the sparsity, we can plug in to the HART procedure as an alternative approach to estimate the non-null proportion.

BH	Storey	HART jk	HART jk + SNP	DNP-OPT	SNP-OPT	SNP Mode
13	13	19	29	44	37	59
(0.22%)	(0.22%)	(0.31%)	(0.48%)	(0.73%)	(0.61%)	(0.98%)

Table A11: Number and proportion of discoveries controlling FDR level at 0.05

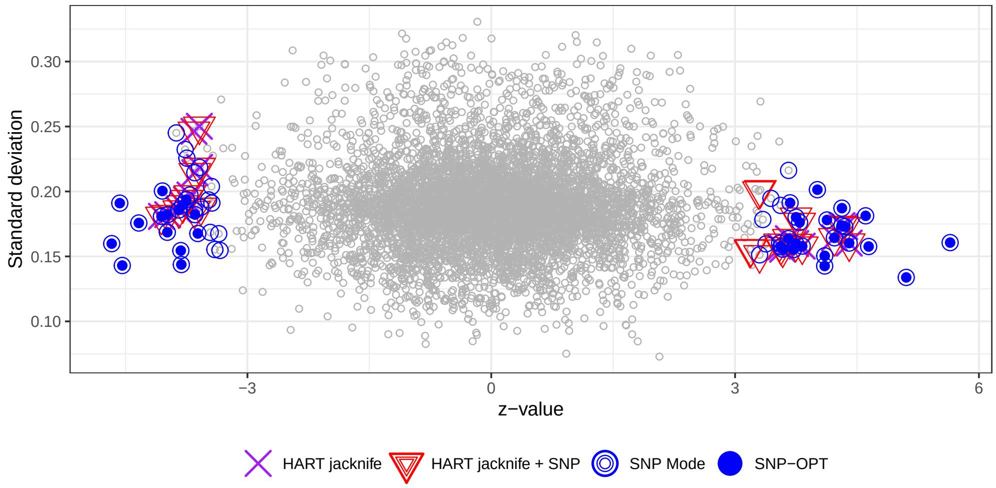


Figure A14: The scatter plot of Z vs σ. The purple cross ($\times$) labels the 19 discoveries by HART jackknifed procedure. The red triangle (∇) label the 29 discoveries by HART with SNP sparsity plug in. The blue circle ($\bigcirc$) labels the 59 non-zero estimate from the posterior mode estimator of SNP. The blue solid circle ($\bullet$) label the 37 discoveries by SNP-OPT. All the multiple testing procedures target FDR level at 0.05.

Table A11 shows the number of discoveries and Figure A13 shows the discoveries on the Z vs σ scatter plot controlling FDR at 0.05 level. SNP-OPT and DNP-OPT reject more hypotheses than other methods. SNP-OPT rejects 37 hypotheses and DNP-OPT rejects 44. The BH procedure is the most conservative as expected, claiming 13 discoveries. Note that if we use the unadjusted p-value from the theoretical null, the BH procedure yield 51 discoveries, many of which could be over-confident false rejections as we see in the simulation. The Storey procedure obtains similar results. The HART procedure yields 19 discoveries while the HART variant using the sparsity estimated by SNP yields 29 discoveries.

Since we do not know the ground truth, we again cannot claim much. However, it is insightful to compare the rejection regions as in Figure A13. The reject regions for SNP-OPT and DNP-OPT depend on both Z and σ. The dependency is more obvious for SNP-OPT – SNP-OPT does not reject hypothesis that corresponds to large σ_i.

Storey	HART	DNP-OPT	SNP-OPT
0.93	0.99	0.91	0.96

Table A12: Sparsity level estimation

Table A12 shows the sparsity estimation. Storey's procedure estimates the sparsity at 0.93 while the Jin-Cai procedure [Jin and Cai, 2007] used by HART estimates the sparsity at 0.99. SNP-OPT, which demonstrates the accurate sparsity estimation in the simulation, estimates 0.96, in between the two mentioned.

We also compare the posterior mode estimators. The posterior mode estimator of SNP produces 59 non-zero estimate although it does not provide guarantee for FDR control. On the other hand, SLOPE rejects 0 hypothesis when controlling FDR at 0.05 level.

Lastly, Figure A15 shows the heatmap of the discovery genes by SNP-OPT vs non-discovery genes among the cancer patients and healthy subjects. Comparing the differential expression of the significant genes (discoveries) on the left and the randomly picked from the non-discoveries on the right, we see that the difference in expression level among the the cancer patients and healthy subjects is distinguishable from the significant genes, while the difference in the non-discovery group is less obvious.

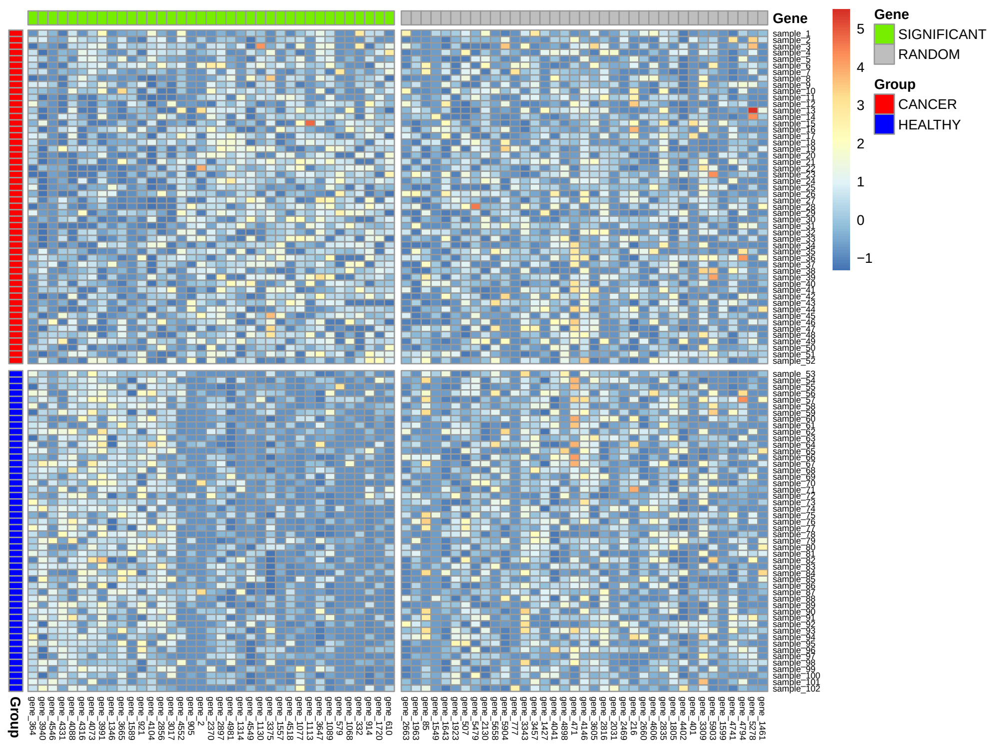


Figure A15: The heatmap of the discovery genes by SNP-OPT vs non-discovery genes among the cancer patients and healthy subjects. The two vertical panels are the significant group (discoveries) with the header labelled in green and the randomly selected group from the non-discoveries with the header labelled in grey. The two horizontal panels are the cancer patients with the side bar labelled in red and the healthy group with the side bar lablled in blue.

C Proof

C.1 Proof of Theorem 1

Lemma 1. *Suppose we have an estimate of $\pi^{(t)}$ in round t. Define*

$$\pi_{j,i}^{(t)} = \frac{p(y_i \mid \tau_j)\, \pi_j^{(t)}}{\sum_{m=1}^{M} p(y_i \mid \tau_m)\, \pi_m^{(t)}}. \tag{2}$$

The EM algorithm update $\boldsymbol{\pi}$ by

$$\pi_j^{(t+1)} = \frac{1}{n} \sum_{i=1}^{n} \pi_{j,i}^{(t)}. \tag{3}$$

Proof. With no information on which μ_i each y_i conditions on, we introduce an unobserved indicator variable z_i that is k if $y_i | z_i \sim N(\tau_k, \sigma^2)$. Note that

$$z \sim Multinomial(M, \pi).$$

The full likelihood function then becomes

$$\begin{aligned} L(\mathbf{y} \mid \mathbf{z}, \pi) &= p\left(\mathbf{y} \mid \mathbf{z}\right) p\left(\mathbf{z} \mid \pi\right) \\ &= \prod_{i=1}^{n} \prod_{j=1}^{M} \left(p\left(y_i \mid \tau_j\right) \cdot \pi_j\right)^{\mathbf{1}_{\{z_i = j\}}} \end{aligned} \tag{4}$$

and thus the log-likelihood

$$\ell(\mathbf{y} \mid \mathbf{z}, \boldsymbol{\pi}) = \sum_{i=1}^{n} \sum_{j=1}^{M} \mathbf{1}_{\{z_i=j\}} \left(\log p\left(y_i \mid \tau_j\right) + \log \pi_j\right). \tag{5}$$

In the E-step, we take expected value of the above likelihood function over $\mathbf{z}$ conditional on $\mathbf{y}$ and $\pi^{(t)}$, i.e. $\mathbb{E}^{(t)}$ refers to averaging z over the distribution $\mathbb{P}(\mathbf{z} \mid \pi^{(t)}, \mathbf{y})$. Given the full log-likehood (5), the expected log posterior density is

$$\mathbb{E}^{(t)}_{Z|\pi^{(t)},y} \ell(\mathbf{z}, \pi \mid \mathbf{y}) = \sum_{i=1}^{n} \sum_{j=1}^{M} \mathbb{P}\left(Z_i = j | y_i, \pi^{(t)}\right) \left(\log p\left(y_i \mid \tau_j\right) + \log \pi_j\right). \tag{6}$$

For the M-step, we maximize (6) over $\boldsymbol{\pi}$. Let

$$\pi^{(t)}_{j,i} = \mathbb{P}(Z_i = j \mid Y_i = y_i, \pi^{(t)}) = \frac{p(y_i \mid \tau_j)\pi^{(t)}_j}{\sum_{m=1}^{M} f(y_i \mid \tau_m)\pi^{(t)}_m} \tag{7}$$

Since the only unknowns in (7) are π_j's with the constraint of $\sum \pi_j = 1$, it is a constrained optimization problem

$$\begin{aligned} &\underset{\pi}{\operatorname{argmax}} \sum_{i=1}^{n} \sum_{j=1}^{M} \mathbb{P}\left(Z_i = j | y_i, \pi^{(t)}\right) \left(\log p\left(y_i \mid \tau_j\right) + \log \pi_j\right) \\ &\text{subject to } \sum \pi_j = 1 \end{aligned} \tag{8}$$

By the method of Lagrange multipliers, we find the maximizer being

$$\pi^{(t+1)}_j = \frac{\sum_{i=1}^{n} \pi^{(t)}_{j,i}}{\sum_{k=1}^{M} \sum_{i=1}^{n} \pi^{(t)}_{k,i}} = \frac{1}{n} \sum_{i=1}^{n} \pi^{(t)}_{j,i}. \tag{9}$$

□

Proof of Theorem 1. The general consistency of the nonparametric maximum likelihood estimator was established in Kiefer and Wolfowitz [1956]. Our setup is a special case of Example 1 in Kiefer and Wolfowitz [1956], i.e., for any $p(y|\mu)$ in the exponential family, assuming if π_1 and π_2 are two different distributions on μ, then for at least one y, $m(y|\pi_1) \neq m(y|\pi_2)$ where $m(y|\pi)$ is the marginal density of y, i.e. $m(y|\pi) = \int p(y|\mu)\pi(\mu)d\mu$. □

It worths mentioning that the likelihood has a convex geometry. The functional of interest is

$$L(\pi) = \sum_{i=1}^{n} \Big\{ -\log \int p(y_i|\mu_i)\pi(\mu_i)d\mu_i \Big\}, \tag{10}$$

which is a sum of $-\log(\cdot)$ of a linear functional $\int p(y_i|\mu_i)\pi(\mu_i)d\mu_i$. Since $-\log(\cdot)$ is convex and non-decreasing, $L(\pi)$ is convex if the space of π is convex. This is also the case for the discretized version,

$$L(\pi) = \sum_{i=1}^{n} \Big\{ -\log \sum_{j} p(y_i|\tau_j)\pi_j \Big\}. \tag{11}$$

C.2 Proof of Proposition 1

Let a decision vector be $\mathbf{a} = (a_1, \cdots, a_m)$ where $a_i = 1$ if we reject the i-th hypothesis and $a_i = 0$ otherwise. A false discovery can be expressed as $a_i \mathbf{I}_{\mu_i=0}$ where $\mathbf{I}$ is an indicator function, and similarly a false non-discovery as $(1 - a_i)\mathbf{I}_{\mu_i \neq 0}$.

An optimal testing procedure can be constructed to minimize the objective function $\mathbb{E}(T)$, subject to $R = k$ for a positive integer number k. That is, given the number of total discoveries, we want to

Table A13: Posterior mean MSE with $u = 1$

V	ω	SNP	DNP	Grouplinear	XGB.M	XGB.SB	GMLEB	EBayesThresh	NEST	ash
1.0	0.55	**0.34**	0.347	0.388	0.378	0.385	0.348	0.398	0.346	0.393
	0.75	**0.246**	0.257	0.312	0.3	0.309	0.26	0.286	0.261	0.28
	0.95	**0.097**	0.116	0.144	0.126	0.138	0.119	0.121	0.124	0.115
1.5	0.55	**0.383**	0.389	0.45	0.441	0.448	0.392	0.475	0.389	0.465
	0.75	**0.285**	0.295	0.383	0.373	0.38	0.298	0.344	0.296	0.337
	0.95	**0.116**	0.137	0.181	0.166	0.175	0.135	0.144	0.137	0.136
2.0	0.55	**0.42**	0.427	0.508	0.501	0.508	0.43	0.537	0.424	0.519
	0.65	**0.376**	0.385	0.488	0.48	0.487	0.387	0.473	0.385	0.458
	0.75	**0.314**	0.326	0.451	0.443	0.448	0.33	0.391	0.324	0.38
	0.85	**0.233**	0.249	0.377	0.369	0.373	0.252	0.292	0.243	0.284
	0.95	**0.128**	0.148	0.226	0.214	0.22	0.149	0.162	0.147	0.155
2.5	0.55	**0.438**	0.447	0.556	0.55	0.557	0.449	0.577	0.441	0.55
	0.65	**0.39**	0.4	0.54	0.534	0.54	0.405	0.508	0.399	0.486
	0.75	**0.328**	0.343	0.508	0.502	0.506	0.345	0.419	0.336	0.403
	0.85	**0.245**	0.263	0.438	0.432	0.435	0.265	0.312	0.25	0.301
	0.95	**0.134**	0.155	0.273	0.265	0.268	0.157	0.173	0.151	0.165
3.0	0.55	**0.432**	0.443	0.594	0.589	0.595	0.446	0.595	0.437	0.562
	0.65	**0.383**	0.396	0.581	0.576	0.582	0.402	0.52	0.392	0.494
	0.75	**0.325**	0.339	0.554	0.549	0.553	0.342	0.428	0.331	0.409
	0.85	**0.237**	0.259	0.49	0.486	0.487	0.262	0.316	0.244	0.303
	0.95	**0.131**	0.156	0.32	0.315	0.315	0.157	0.174	0.147	0.168

minimize the averaged number of false non-discoveries. Correspondingly, the objective function can be written as

$$\min_{(a_1,\cdots,a_m)} \mathbb{E}\Big[\sum_{i=1}^{m}(1-a_i)\mathbf{I}_{\mu_i\neq 0}\Big] \quad \text{s.t.} \quad \sum_{i=1}^{m} a_i = k. \tag{12}$$

The expectation in (12) is over the distribution of $\mathbf{y}$ and the prior distribution of $\boldsymbol{\mu}$. It is equivalent to minimize the expectation of loss function conditional on $\mathbf{y}$, that is, minimizing

$$\min_{(a_1,\cdots,a_m)} L(\mathbf{a}) = \sum_{i=1}^{m}\Big[(1-a_i)P(\mu_i\neq 0|\mathbf{y})\Big] \tag{13}$$

$$\text{subject to} \quad \sum_{i=1}^{m} a_i = k.$$

After some algebra, (13) can be re-arranged as

$$L(\mathbf{a}) = \sum_{i=1}^{m}(1-P(\mu_i=0|\mathbf{y})) + \sum_{i=1}^{m} a_i[P(\mu_i=0|\mathbf{y})-1].$$

Note that $L(\mathbf{a})$ is increasing when the second term is increasing. Therefore, to minimize $L(\mathbf{a})$ subject to $\sum_{i=1}^{m} a_i = k$, for the smallest k values of $P(\mu_i = 0|\mathbf{y})$, we set the corresponding $a_i = 1$, and the rest of the a_i's are set as 0. The proof is now complete.

D Simulation tables

We provide all the information of Figure A2 to A9 in the table format for detailed comparison. In each row, the best performer is set in bold type. Note that we show the bias of sparsity estimation and we consider the method yielding the smallest absolute value as the best performer.

Table A14: Posterior mean MSE Relative ratio with $u = 1$

V	ω	SNP	DNP	Grouplinear	XGB.M	XGB.SB	GMLEB	EBayesThresh	NEST	ash
	0.55	**1**	1.019	1.14	1.111	1.131	1.023	1.17	1.018	1.155
1.0	0.75	**1**	1.047	1.271	1.221	1.257	1.057	1.163	1.063	1.139
	0.95	**1**	1.2	1.489	1.303	1.423	1.23	1.254	1.285	1.188
	0.55	**1**	1.015	1.174	1.152	1.169	1.022	1.239	1.014	1.213
1.5	0.75	**1**	1.037	1.345	1.31	1.335	1.046	1.208	1.039	1.182
	0.95	**1**	1.184	1.565	1.433	1.511	1.167	1.24	1.18	1.178
	0.55	**1**	1.016	1.209	1.192	1.208	1.023	1.278	1.009	1.235
	0.65	**1**	1.022	1.297	1.276	1.293	1.028	1.256	1.023	1.218
2.0	0.75	**1**	1.038	1.433	1.408	1.426	1.049	1.244	1.032	1.209
	0.85	**1**	1.069	1.619	1.584	1.602	1.084	1.254	1.043	1.221
	0.95	**1**	1.153	1.764	1.67	1.718	1.162	1.266	1.148	1.21
	0.55	**1**	1.02	1.271	1.257	1.272	1.025	1.319	1.007	1.256
	0.65	**1**	1.027	1.386	1.37	1.386	1.039	1.302	1.023	1.248
2.5	0.75	**1**	1.045	1.547	1.528	1.542	1.051	1.277	1.025	1.228
	0.85	**1**	1.073	1.789	1.765	1.774	1.081	1.273	1.021	1.231
	0.95	**1**	1.157	2.037	1.976	1.998	1.17	1.288	1.13	1.235
	0.55	**1**	1.025	1.374	1.363	1.377	1.033	1.378	1.012	1.301
	0.65	**1**	1.033	1.517	1.504	1.519	1.048	1.357	1.023	1.289
3.0	0.75	**1**	1.043	1.703	1.687	1.699	1.052	1.317	1.017	1.257
	0.85	**1**	1.089	2.065	2.046	2.05	1.103	1.33	1.026	1.276
	0.95	**1**	1.189	2.438	2.403	2.403	1.199	1.329	1.123	1.279

Table A15: Posterior mode MSE with $u = 1$

V	ω	SNP	DNP	SSLASSO	SLOPE	GMLEB	EBayesThresh
	0.55	**0.376**	0.475	0.454	0.524	0.517	0.454
1.0	0.75	**0.234**	0.269	0.309	0.291	0.443	0.317
	0.95	0.07	**0.056**	0.237	0.058	0.316	0.161
	0.55	**0.408**	0.576	0.568	0.699	0.542	0.503
1.5	0.75	**0.258**	0.332	0.338	0.389	0.465	0.334
	0.95	0.08	**0.069**	0.228	0.078	0.332	0.144
	0.55	**0.425**	0.645	0.717	0.904	0.554	0.539
	0.65	**0.355**	0.516	0.539	0.705	0.512	0.447
2.0	0.75	**0.272**	0.378	0.38	0.503	0.467	0.344
	0.85	**0.181**	0.232	0.263	0.302	0.407	0.238
	0.95	0.083	**0.081**	0.215	0.1	0.329	0.13
	0.55	**0.419**	0.667	0.895	1.123	0.557	0.552
	0.65	**0.351**	0.54	0.656	0.875	0.517	0.451
2.5	0.75	**0.274**	0.401	0.436	0.625	0.468	0.343
	0.85	**0.183**	0.249	0.269	0.375	0.412	0.234
	0.95	**0.082**	0.088	0.199	0.125	0.335	0.12
	0.55	**0.4**	0.638	1.097	1.346	0.538	0.541
	0.65	**0.333**	0.524	0.793	1.049	0.5	0.437
3.0	0.75	**0.261**	0.39	0.505	0.749	0.457	0.331
	0.85	**0.173**	0.245	0.279	0.45	0.404	0.223
	0.95	**0.077**	0.089	0.181	0.15	0.333	0.112

Table A16: Posterior mode MSE Relative ratio with $u = 1$

V	ω	SNP	DNP	SSLASSO	SLOPE	GMLEB	EBayesThresh
	0.55	**1**	1.264	1.209	1.394	1.374	1.206
1.0	0.75	**1**	1.147	1.321	1.243	1.893	1.355
	0.95	1	**0.791**	3.366	0.825	4.496	2.293
	0.55	**1**	1.411	1.39	1.711	1.327	1.232
1.5	0.75	**1**	1.284	1.309	1.505	1.801	1.292
	0.95	1	**0.869**	2.864	0.975	4.173	1.813
	0.55	**1**	1.518	1.689	2.13	1.303	1.268
	0.65	**1**	1.453	1.517	1.984	1.442	1.259
2.0	0.75	**1**	1.388	1.396	1.848	1.714	1.264
	0.85	**1**	1.283	1.454	1.67	2.251	1.315
	0.95	1	**0.973**	2.59	1.21	3.965	1.565
	0.55	**1**	1.591	2.135	2.679	1.329	1.316
	0.65	**1**	1.537	1.866	2.491	1.472	1.283
2.5	0.75	**1**	1.466	1.592	2.284	1.711	1.253
	0.85	**1**	1.363	1.473	2.056	2.258	1.281
	0.95	**1**	1.076	2.436	1.528	4.107	1.473
	0.55	**1**	1.596	2.743	3.367	1.345	1.353
	0.65	**1**	1.575	2.386	3.156	1.504	1.313
3.0	0.75	**1**	1.492	1.935	2.869	1.752	1.267
	0.85	**1**	1.419	1.618	2.606	2.339	1.293
	0.95	**1**	1.149	2.342	1.936	4.31	1.45

Table A17: Bias of sparsity with $u = 1$

V	ω	SNP	DNP	SSLASSO	GMLEB	EBayesThresh	HART	ash
	0.55	0.178	**0**	0.294	-0.405	-0.516	0.269	-0.154
1.0	0.75	0.085	-0.084	**0.068**	-0.612	-0.585	0.127	-0.227
	0.95	**0.01**	-0.17	-0.181	-0.756	-0.793	-0.014	-0.331
	0.55	0.114	**-0.052**	0.323	-0.422	-0.515	0.213	-0.198
1.5	0.75	**0.05**	-0.114	0.097	-0.598	-0.506	0.097	-0.226
	0.95	**-0.002**	-0.19	-0.171	-0.77	-0.631	-0.02	-0.32
	0.55	**0.067**	-0.09	0.353	-0.402	-0.512	0.152	-0.232
	0.65	**0.048**	-0.11	0.244	-0.485	-0.487	0.108	-0.227
2.0	0.75	**0.025**	-0.136	0.131	-0.581	-0.454	0.063	-0.225
	0.85	0.003	-0.167	**0.002**	-0.649	-0.429	0.018	-0.244
	0.95	**-0.004**	-0.198	-0.157	-0.762	-0.482	-0.026	-0.312
	0.55	**0.04**	-0.114	0.383	-0.396	-0.493	0.097	-0.254
	0.65	**0.027**	-0.133	0.274	-0.464	-0.458	0.065	-0.24
2.5	0.75	**0.012**	-0.15	0.163	-0.552	-0.41	0.033	-0.218
	0.85	-0.005	-0.174	0.034	-0.639	-0.369	**0**	-0.226
	0.95	**-0.013**	-0.195	-0.14	-0.739	-0.371	-0.033	-0.309
	0.55	**0.032**	-0.122	0.415	-0.389	-0.454	0.053	-0.272
	0.65	**0.02**	-0.133	0.301	-0.444	-0.419	0.03	-0.25
3.0	0.75	0.014	-0.148	0.19	-0.534	-0.368	**0.008**	-0.216
	0.85	**-0.003**	-0.179	0.067	-0.637	-0.319	-0.015	-0.207
	0.95	**-0.013**	-0.194	-0.12	-0.722	-0.291	-0.037	-0.297

Table A18: Coverage of credible interval with $u = 1$

V	ω	SNP	DNP	EBayesThresh	ash
	0.55	0.93	0.93	0.94	0.94
1.0	0.75	0.95	0.95	0.96	0.95
	0.95	0.98	0.98	0.96	0.98
	0.55	0.93	0.94	0.93	0.93
1.5	0.75	0.95	0.96	0.96	0.95
	0.95	0.98	0.98	0.98	0.98
	0.55	0.94	0.94	0.92	0.94
	0.65	0.95	0.95	0.94	0.94
2.0	0.75	0.95	0.96	0.95	0.96
	0.85	0.97	0.97	0.97	0.97
	0.95	0.98	0.98	0.98	0.98
	0.55	0.94	0.95	0.90	0.94
	0.65	0.95	0.95	0.92	0.95
2.5	0.75	0.96	0.96	0.94	0.96
	0.85	0.97	0.97	0.97	0.97
	0.95	0.98	0.98	0.99	0.98
	0.55	0.95	0.95	0.87	0.94
	0.65	0.96	0.96	0.90	0.95
3.0	0.75	0.96	0.96	0.93	0.96
	0.85	0.97	0.97	0.96	0.97
	0.95	0.98	0.98	0.99	0.99

Table A19: Length of credible interval with $u = 1$

V	ω	SNP	DNP	EBayesThresh	ash
	0.55	**1.933**	1.975	2.553	2.198
1.0	0.75	**1.723**	1.766	2.464	1.966
	0.95	**1.4**	1.401	2.531	1.621
	0.55	**1.999**	2.062	2.5	2.33
1.5	0.75	**1.763**	1.828	2.349	2.06
	0.95	**1.433**	1.45	2.306	1.653
	0.55	**2.055**	2.144	2.446	2.413
	0.65	**1.928**	2.022	2.365	2.279
2.0	0.75	**1.778**	1.885	2.257	2.121
	0.85	**1.607**	1.705	2.132	1.936
	0.95	**1.426**	1.474	2.075	1.684
	0.55	**2.07**	2.192	2.384	2.441
	0.65	**1.925**	2.066	2.292	2.3
2.5	0.75	**1.761**	1.906	2.165	2.129
	0.85	**1.574**	1.709	2.014	1.945
	0.95	**1.401**	1.464	1.873	1.692
	0.55	**2.024**	2.177	2.311	2.436
	0.65	**1.876**	2.033	2.207	2.268
3.0	0.75	**1.687**	1.868	2.065	2.086
	0.85	**1.497**	1.699	1.893	1.917
	0.95	**1.334**	1.449	1.692	1.691

Table A20: Empirical FDR with $u = 1$ and $V = 1$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART	ash
	0.01	0.02	0.02	0.03	0.03	0.02	0.02
	0.05	0.06	0.05	0.08	0.08	0.05	0.05
	0.10	0.09	0.09	0.13	0.12	0.08	0.07
0.35	0.15	0.11	0.11	0.16	0.15	0.10	0.10
	0.20	0.13	0.14	0.20	0.18	0.12	0.12
	0.25	0.14	0.16	0.23	0.21	0.14	0.14
	0.30	0.16	0.19	0.26	0.23	0.16	0.16
	0.01	0.04	0.03	0.05	0.05	0.04	0.03
	0.05	0.10	0.08	0.12	0.12	0.09	0.08
	0.10	0.14	0.12	0.18	0.18	0.13	0.12
0.55	0.15	0.18	0.16	0.24	0.23	0.16	0.14
	0.20	0.22	0.20	0.28	0.28	0.20	0.18
	0.25	0.25	0.23	0.33	0.32	0.23	0.21
	0.30	0.27	0.26	0.37	0.35	0.26	0.24
	0.01	0.07	0.04	0.09	0.08	0.08	0.06
	0.05	0.16	0.12	0.20	0.19	0.16	0.13
	0.10	0.23	0.17	0.28	0.27	0.22	0.19
0.75	0.15	0.29	0.23	0.34	0.34	0.27	0.23
	0.20	0.34	0.27	0.40	0.39	0.32	0.28
	0.25	0.38	0.31	0.45	0.44	0.36	0.32
	0.30	0.42	0.35	0.49	0.49	0.40	0.36
	0.01	0.18	0.05	0.24	0.22	0.27	0.19
	0.05	0.38	0.13	0.49	0.42	0.47	0.37
	0.10	0.51	0.27	0.59	0.55	0.56	0.50
0.95	0.15	0.58	0.31	0.67	0.62	0.62	0.58
	0.20	0.64	0.37	0.71	0.68	0.66	0.63
	0.25	0.68	0.42	0.74	0.72	0.70	0.67
	0.30	0.72	0.45	0.77	0.76	0.73	0.71

Table A21: Empirical FDR with $u = 1$ and $V = 2$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART	ash
0.35	0.01	0.01	0.02	0.03	0.03	0.02	0.01
	0.05	0.04	0.06	0.09	0.08	0.05	0.03
	0.10	0.07	0.09	0.14	0.13	0.08	0.05
	0.15	0.09	0.13	0.20	0.17	0.11	0.07
	0.20	0.11	0.17	0.25	0.20	0.14	0.09
	0.25	0.13	0.21	0.30	0.24	0.17	0.11
	0.30	0.14	0.25	0.33	0.28	0.21	0.14
0.45	0.01	0.02	0.02	0.03	0.03	0.02	0.01
	0.05	0.06	0.06	0.10	0.09	0.06	0.04
	0.10	0.09	0.10	0.16	0.15	0.10	0.06
	0.15	0.12	0.14	0.21	0.20	0.13	0.08
	0.20	0.15	0.18	0.26	0.24	0.17	0.10
	0.25	0.17	0.22	0.32	0.28	0.20	0.12
	0.30	0.19	0.27	0.36	0.32	0.24	0.15
0.55	0.01	0.03	0.02	0.04	0.04	0.03	0.02
	0.05	0.07	0.07	0.11	0.11	0.07	0.05
	0.10	0.12	0.12	0.18	0.17	0.12	0.07
	0.15	0.15	0.16	0.24	0.23	0.16	0.10
	0.20	0.19	0.21	0.30	0.28	0.20	0.12
	0.25	0.22	0.25	0.35	0.32	0.24	0.14
	0.30	0.25	0.29	0.40	0.36	0.28	0.17
0.65	0.01	0.04	0.03	0.05	0.05	0.04	0.02
	0.05	0.09	0.08	0.12	0.13	0.09	0.05
	0.10	0.15	0.13	0.20	0.20	0.15	0.08
	0.15	0.19	0.18	0.27	0.27	0.20	0.11
	0.20	0.23	0.23	0.32	0.32	0.24	0.14
	0.25	0.27	0.27	0.38	0.37	0.28	0.17
	0.30	0.31	0.32	0.43	0.41	0.32	0.20
0.75	0.01	0.05	0.03	0.06	0.06	0.05	0.03
	0.05	0.12	0.10	0.15	0.15	0.12	0.07
	0.10	0.19	0.16	0.24	0.24	0.19	0.11
	0.15	0.24	0.21	0.31	0.30	0.24	0.14
	0.20	0.29	0.26	0.37	0.36	0.29	0.17
	0.25	0.33	0.30	0.42	0.42	0.34	0.21
	0.30	0.38	0.35	0.48	0.46	0.38	0.24
0.85	0.01	0.07	0.04	0.08	0.08	0.07	0.04
	0.05	0.17	0.12	0.20	0.20	0.17	0.10
	0.10	0.25	0.18	0.30	0.29	0.25	0.16
	0.15	0.31	0.24	0.38	0.37	0.31	0.21
	0.20	0.37	0.29	0.44	0.43	0.36	0.25
	0.25	0.42	0.34	0.50	0.49	0.42	0.29
	0.30	0.47	0.39	0.55	0.54	0.46	0.33
0.95	0.01	0.12	0.04	0.16	0.13	0.15	0.10
	0.05	0.28	0.13	0.36	0.32	0.33	0.21
	0.10	0.40	0.21	0.48	0.44	0.42	0.32
	0.15	0.48	0.27	0.56	0.52	0.50	0.40
	0.20	0.54	0.32	0.63	0.59	0.55	0.46
	0.25	0.60	0.37	0.68	0.65	0.60	0.52
	0.30	0.65	0.42	0.72	0.70	0.64	0.58

Table A22: Emprical power with $u = 1$ and $V = 1$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART
0.65	0.01	0.13	0.172	**0.176**	0.144	0.151
	0.05	0.214	0.29	**0.294**	0.241	0.243
	0.10	0.274	0.369	**0.372**	0.314	0.305
	0.15	0.319	0.427	**0.43**	0.37	0.351
	0.20	0.36	0.481	**0.484**	0.419	0.39
	0.25	0.396	0.529	**0.535**	0.466	0.426
	0.30	0.429	0.578	**0.584**	0.513	0.462
0.75	0.01	0.118	0.152	**0.157**	0.129	0.139
	0.05	0.196	0.258	**0.263**	0.22	0.222
	0.10	0.254	0.331	**0.336**	0.285	0.28
	0.15	0.297	0.388	**0.392**	0.338	0.322
	0.20	0.336	0.435	**0.442**	0.385	0.357
	0.25	0.371	0.482	**0.489**	0.429	0.39
	0.30	0.404	0.526	**0.536**	0.473	0.423
0.85	0.01	0.107	0.131	**0.137**	0.112	0.128
	0.05	0.179	0.222	**0.231**	0.194	0.203
	0.10	0.228	0.286	**0.295**	0.249	0.249
	0.15	0.268	0.337	**0.344**	0.298	0.287
	0.20	0.305	0.381	**0.388**	0.342	0.317
	0.25	0.339	0.418	**0.427**	0.38	0.347
	0.30	0.369	0.455	**0.465**	0.417	0.374
0.95	0.01	0.079	0.089	0.097	0.084	**0.105**
	0.05	0.137	0.161	**0.174**	0.152	0.166
	0.10	0.183	0.216	**0.23**	0.197	0.206
	0.15	0.215	0.254	**0.271**	0.236	0.234
	0.20	0.244	0.293	**0.309**	0.27	0.26
	0.25	0.273	0.328	**0.348**	0.311	0.283
	0.30	0.304	0.366	**0.388**	0.352	0.305

Table A23: Emprical power with $u = 1$ and $V = 2$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART
0.65	0.01	0.391	0.483	**0.488**	0.426	0.443
	0.05	0.521	0.641	**0.645**	0.569	0.585
	0.10	0.593	0.718	**0.723**	0.65	0.658
	0.15	0.64	0.769	**0.774**	0.701	0.705
	0.20	0.674	0.806	**0.812**	0.741	0.741
	0.25	0.705	0.837	**0.843**	0.774	0.771
	0.30	0.731	0.865	**0.872**	0.806	0.799
0.75	0.01	0.372	0.448	**0.453**	0.398	0.418
	0.05	0.5	0.597	**0.6**	0.534	0.55
	0.10	0.567	0.673	**0.676**	0.609	0.618
	0.15	0.615	0.719	**0.724**	0.659	0.661
	0.20	0.649	0.758	**0.763**	0.699	0.697
	0.25	0.679	0.79	**0.795**	0.733	0.725
	0.30	0.705	0.818	**0.822**	0.764	0.754
0.85	0.01	0.348	0.406	**0.416**	0.366	0.386
	0.05	0.463	0.54	**0.547**	0.488	0.502
	0.10	0.526	0.613	**0.62**	0.558	0.564
	0.15	0.573	0.662	**0.669**	0.61	0.607
	0.20	0.611	0.699	**0.706**	0.652	0.643
	0.25	0.643	0.73	**0.739**	0.685	0.672
	0.30	0.672	0.761	**0.769**	0.717	0.698
0.95	0.01	0.288	0.317	**0.334**	0.301	0.329
	0.05	0.394	0.436	**0.452**	0.412	0.425
	0.10	0.456	0.503	**0.521**	0.478	0.481
	0.15	0.499	0.551	**0.566**	0.523	0.516
	0.20	0.534	0.587	**0.607**	0.565	0.546
	0.25	0.567	0.623	**0.642**	0.597	0.57
	0.30	0.593	0.655	**0.677**	0.635	0.598

Table A24: Posterior mean MSE with $u = 1.5$

V	ω	SNP	DNP	Grouplinear	XGB.M	XGB.SB	GMLEB	EBayesThresh	NEST	ash
1.0	0.55	**0.379**	0.38	0.453	0.411	0.45	0.381	0.452	0.414	0.449
	0.75	**0.267**	0.27	0.347	0.306	0.34	0.272	0.307	0.32	0.303
	0.95	**0.084**	0.091	0.124	0.091	0.118	0.095	0.098	0.162	0.093
1.5	0.55	**0.446**	0.447	0.538	0.503	0.54	0.449	0.569	0.476	0.561
	0.75	**0.327**	0.329	0.447	0.404	0.442	0.332	0.397	0.369	0.391
	0.95	**0.109**	0.115	0.169	0.137	0.163	0.12	0.128	0.18	0.123
2.0	0.55	**0.509**	0.511	0.618	0.591	0.624	0.514	0.67	0.534	0.654
	0.65	**0.456**	0.459	0.591	0.56	0.596	0.462	0.585	0.49	0.572
	0.75	**0.38**	0.384	0.541	0.504	0.541	0.388	0.478	0.415	0.467
	0.85	**0.276**	0.281	0.438	0.397	0.432	0.284	0.341	0.316	0.335
	0.95	**0.133**	0.139	0.227	0.194	0.22	0.144	0.159	0.195	0.154
2.5	0.55	**0.551**	0.555	0.687	0.668	0.696	0.558	0.746	0.571	0.716
	0.65	**0.495**	0.499	0.664	0.642	0.672	0.503	0.655	0.522	0.632
	0.75	**0.416**	0.42	0.619	0.592	0.624	0.424	0.537	0.445	0.52
	0.85	**0.304**	0.31	0.52	0.486	0.518	0.314	0.386	0.336	0.376
	0.95	**0.152**	0.157	0.29	0.257	0.283	0.162	0.183	0.206	0.178
3.0	0.55	**0.564**	0.569	0.743	0.729	0.754	0.573	0.794	0.583	0.751
	0.65	**0.506**	0.511	0.724	0.708	0.733	0.516	0.696	0.529	0.663
	0.75	**0.428**	0.432	0.685	0.665	0.692	0.436	0.572	0.452	0.547
	0.85	**0.313**	0.319	0.591	0.565	0.591	0.324	0.412	0.339	0.397
	0.95	**0.16**	0.165	0.355	0.323	0.347	0.17	0.197	0.209	0.193

Table A25: Posterior mean MSE Relative ratio with $u = 1.5$

V	ω	SNP	DNP	Grouplinear	XGB.M	XGB.SB	GMLEB	EBayesThresh	NEST	ash
1.0	0.55	**1**	1.001	1.194	1.085	1.187	1.004	1.192	1.093	1.184
	0.75	**1**	1.01	1.297	1.143	1.273	1.017	1.149	1.197	1.133
	0.95	**1**	1.085	1.485	1.086	1.414	1.136	1.166	1.939	1.109
1.5	0.55	**1**	1.003	1.207	1.128	1.212	1.009	1.276	1.069	1.259
	0.75	**1**	1.009	1.369	1.237	1.353	1.017	1.215	1.13	1.196
	0.95	**1**	1.051	1.547	1.247	1.493	1.098	1.172	1.641	1.125
2.0	0.55	**1**	1.005	1.215	1.163	1.228	1.012	1.318	1.05	1.286
	0.65	**1**	1.008	1.298	1.229	1.309	1.014	1.285	1.075	1.255
	0.75	**1**	1.011	1.424	1.325	1.423	1.02	1.257	1.093	1.23
	0.85	**1**	1.02	1.587	1.439	1.567	1.031	1.237	1.147	1.214
	0.95	**1**	1.041	1.7	1.453	1.653	1.079	1.191	1.464	1.154
2.5	0.55	**1**	1.008	1.247	1.212	1.264	1.013	1.354	1.038	1.3
	0.65	**1**	1.009	1.342	1.297	1.358	1.015	1.324	1.055	1.276
	0.75	**1**	1.01	1.488	1.422	1.5	1.02	1.291	1.069	1.249
	0.85	**1**	1.019	1.712	1.6	1.704	1.033	1.272	1.104	1.237
	0.95	**1**	1.034	1.904	1.69	1.859	1.062	1.2	1.353	1.17
3.0	0.55	**1**	1.009	1.318	1.294	1.337	1.017	1.408	1.034	1.332
	0.65	**1**	1.011	1.431	1.4	1.449	1.02	1.376	1.046	1.31
	0.75	**1**	1.011	1.6	1.553	1.617	1.02	1.336	1.056	1.278
	0.85	**1**	1.018	1.886	1.804	1.886	1.035	1.313	1.081	1.267
	0.95	**1**	1.028	2.209	2.013	2.162	1.059	1.23	1.301	1.2

Table A26: Posterior mode MSE with $u = 1.5$

V	ω	SNP	DNP	SSLASSO	SLOPE	GMLEB	EBayesThresh
1.0	0.55	**0.421**	0.499	0.487	0.524	0.521	0.461
	0.75	**0.253**	0.28	0.336	0.291	0.425	0.294
	0.95	0.059	**0.057**	0.249	0.058	0.239	0.096
1.5	0.55	**0.487**	0.635	0.604	0.699	0.562	0.535
	0.75	**0.299**	0.359	0.369	0.389	0.462	0.331
	0.95	**0.072**	0.074	0.242	0.078	0.267	0.098
2.0	0.55	**0.536**	0.767	0.756	0.904	0.58	0.594
	0.65	**0.448**	0.605	0.577	0.705	0.537	0.484
	0.75	**0.336**	0.439	0.414	0.503	0.476	0.361
	0.85	**0.217**	0.267	0.291	0.302	0.392	0.234
	0.95	**0.084**	0.091	0.232	0.1	0.269	0.101
2.5	0.55	**0.554**	0.864	0.935	1.123	0.596	0.629
	0.65	**0.467**	0.689	0.695	0.875	0.547	0.505
	0.75	**0.356**	0.503	0.472	0.625	0.481	0.374
	0.85	**0.231**	0.307	0.299	0.375	0.396	0.244
	0.95	**0.09**	0.105	0.218	0.125	0.269	0.103
3.0	0.55	**0.542**	0.912	1.136	1.346	0.579	0.638
	0.65	**0.458**	0.734	0.833	1.049	0.524	0.503
	0.75	**0.354**	0.541	0.542	0.749	0.47	0.371
	0.85	**0.233**	0.332	0.311	0.45	0.388	0.242
	0.95	**0.091**	0.116	0.202	0.15	0.271	0.102

Table A27: Posterior mode MSE Relative ratio with $u = 1.5$

V	ω	SNP	DNP	SSLASSO	SLOPE	GMLEB	EBayesThresh
1.0	0.55	**1**	1.186	1.157	1.244	1.237	1.095
	0.75	**1**	1.105	1.326	1.151	1.68	1.164
	0.95	1	**0.958**	4.196	0.979	4.025	1.617
1.5	0.55	**1**	1.304	1.24	1.435	1.154	1.099
	0.75	**1**	1.2	1.231	1.298	1.541	1.105
	0.95	**1**	1.029	3.381	1.082	3.724	1.367
2.0	0.55	**1**	1.43	1.409	1.686	1.081	1.107
	0.65	**1**	1.351	1.289	1.575	1.2	1.081
	0.75	**1**	1.307	1.234	1.499	1.419	1.074
	0.85	**1**	1.233	1.345	1.396	1.809	1.082
	0.95	**1**	1.081	2.769	1.199	3.213	1.203
2.5	0.55	**1**	1.559	1.687	2.025	1.075	1.135
	0.65	**1**	1.476	1.49	1.875	1.172	1.083
	0.75	**1**	1.411	1.325	1.754	1.35	1.049
	0.85	**1**	1.327	1.296	1.625	1.712	1.054
	0.95	**1**	1.167	2.414	1.38	2.971	1.139
3.0	0.55	**1**	1.683	2.096	2.484	1.069	1.176
	0.65	**1**	1.604	1.82	2.293	1.145	1.1
	0.75	**1**	1.529	1.532	2.119	1.33	1.048
	0.85	**1**	1.425	1.335	1.932	1.668	1.038
	0.95	**1**	1.28	2.222	1.648	2.986	1.129

Table A28: Bias of sparsity with $u = 1.5$

V	ω	SNP	DNP	SSLASSO	GMLEB	EBayesThresh	HART	ash
	0.55	0.217	0.087	0.31	-0.38	-0.486	0.337	**-0.074**
1.0	0.75	0.139	**0.02**	0.088	-0.549	-0.523	0.182	-0.136
	0.95	0.034	-0.07	-0.149	-0.621	-0.633	**0.028**	-0.196
	0.55	0.13	**0.02**	0.334	-0.342	-0.505	0.29	-0.147
1.5	0.75	0.089	**-0.021**	0.112	-0.582	-0.458	0.156	-0.154
	0.95	0.024	-0.073	-0.141	-0.65	-0.489	**0.022**	-0.19
	0.55	0.07	**-0.036**	0.361	-0.335	-0.518	0.235	-0.199
	0.65	0.061	**-0.044**	0.251	-0.461	-0.478	0.181	-0.185
2.0	0.75	**0.049**	-0.061	0.138	-0.55	-0.424	0.126	-0.173
	0.85	0.036	-0.071	**0.015**	-0.589	-0.363	0.071	-0.164
	0.95	**0.016**	-0.091	-0.13	-0.638	-0.36	**0.016**	-0.189
	0.55	**0.039**	-0.067	0.388	-0.344	-0.517	0.18	-0.228
	0.65	**0.031**	-0.068	0.277	-0.439	-0.466	0.138	-0.209
2.5	0.75	**0.025**	-0.069	0.165	-0.507	-0.394	0.095	-0.181
	0.85	**0.019**	-0.085	0.041	-0.602	-0.318	0.052	-0.155
	0.95	**0.008**	-0.099	-0.116	-0.592	-0.268	0.01	-0.181
	0.55	**0.026**	-0.08	0.416	-0.303	-0.495	0.131	-0.256
	0.65	**0.019**	-0.076	0.301	-0.384	-0.438	0.099	-0.226
3.0	0.75	**0.017**	-0.075	0.189	-0.482	-0.362	0.068	-0.185
	0.85	**0.011**	-0.088	0.067	-0.571	-0.282	0.036	-0.149
	0.95	**0.004**	-0.095	-0.101	-0.594	-0.207	0.005	-0.166

Table A29: Coverage of credible interval with $u = 1.5$

V	ω	SNP	DNP	EBayesThresh	ash
	0.55	0.93	0.92	0.96	0.94
1.0	0.75	0.94	0.94	0.97	0.95
	0.95	0.98	0.98	0.98	0.98
	0.55	0.94	0.94	0.95	0.94
1.5	0.75	0.95	0.95	0.97	0.95
	0.95	0.98	0.98	0.99	0.98
	0.55	0.95	0.96	0.93	0.95
	0.65	0.96	0.96	0.94	0.95
2.0	0.75	0.96	0.96	0.96	0.96
	0.85	0.97	0.97	0.97	0.96
	0.95	0.98	0.98	0.99	0.98
	0.55	0.96	0.96	0.89	0.95
	0.65	0.96	0.96	0.92	0.96
2.5	0.75	0.97	0.97	0.94	0.96
	0.85	0.97	0.97	0.96	0.97
	0.95	0.98	0.98	0.99	0.98
	0.55	0.96	0.96	0.84	0.96
	0.65	0.96	0.97	0.88	0.96
3.0	0.75	0.97	0.97	0.92	0.97
	0.85	0.98	0.98	0.95	0.97
	0.95	0.98	0.99	0.98	0.98

Table A30: Length of credible interval with $u = 1.5$

V	ω	SNP	DNP	EBayesThresh	ash
1.0	0.55	2.098	**2.07**	3.158	2.52
	0.75	1.759	**1.741**	3.019	2.153
	0.95	1.193	**1.135**	2.96	1.551
1.5	0.55	2.273	**2.215**	3.066	2.775
	0.75	1.912	**1.892**	2.844	2.342
	0.95	1.236	**1.171**	2.644	1.608
2.0	0.55	2.428	**2.414**	2.957	2.971
	0.65	2.266	**2.235**	2.85	2.757
	0.75	**2.034**	2.044	2.698	2.493
	0.85	**1.724**	1.73	2.49	2.146
	0.95	**1.289**	**1.289**	2.307	1.668
2.5	0.55	**2.524**	2.558	2.839	3.094
	0.65	**2.352**	2.357	2.724	2.869
	0.75	**2.087**	2.104	2.55	2.572
	0.85	**1.749**	1.779	2.313	2.198
	0.95	1.337	**1.336**	2.008	1.698
3.0	0.55	**2.542**	2.62	2.709	3.158
	0.65	**2.349**	2.403	2.58	2.91
	0.75	**2.062**	2.109	2.391	2.573
	0.85	**1.719**	1.765	2.137	2.17
	0.95	1.328	**1.28**	1.745	1.68

Table A31: Empirical FDR with $u = 1.5$ and $V = 1$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART	ash
0.35	0.01	0.02	0.02	0.03	0.02	0.01	0.02
	0.05	0.04	0.04	0.06	0.05	0.03	0.04
	0.10	0.06	0.06	0.09	0.08	0.05	0.06
	0.15	0.08	0.09	0.12	0.10	0.06	0.07
	0.20	0.10	0.11	0.15	0.13	0.07	0.09
	0.25	0.11	0.13	0.18	0.15	0.09	0.11
	0.30	0.13	0.16	0.22	0.17	0.10	0.13
0.55	0.01	0.02	0.02	0.04	0.03	0.03	0.03
	0.05	0.06	0.06	0.09	0.07	0.06	0.06
	0.10	0.10	0.09	0.13	0.11	0.08	0.09
	0.15	0.12	0.12	0.16	0.15	0.10	0.12
	0.20	0.15	0.14	0.20	0.18	0.12	0.14
	0.25	0.18	0.17	0.24	0.21	0.14	0.16
	0.30	0.20	0.20	0.28	0.24	0.17	0.18
0.75	0.01	0.05	0.04	0.08	0.05	0.06	0.05
	0.05	0.12	0.08	0.14	0.12	0.12	0.12
	0.10	0.16	0.12	0.20	0.18	0.15	0.16
	0.15	0.20	0.15	0.24	0.22	0.19	0.19
	0.20	0.24	0.18	0.28	0.26	0.21	0.22
	0.25	0.28	0.21	0.32	0.30	0.24	0.25
	0.30	0.31	0.24	0.36	0.34	0.27	0.28
0.95	0.01	0.17	0.09	0.18	0.15	0.20	0.17
	0.05	0.22	0.15	0.34	0.25	0.33	0.33
	0.10	0.32	0.16	0.44	0.36	0.43	0.41
	0.15	0.41	0.20	0.52	0.43	0.50	0.47
	0.20	0.47	0.23	0.58	0.50	0.53	0.52
	0.25	0.52	0.27	0.62	0.56	0.58	0.57
	0.30	0.56	0.29	0.65	0.59	0.61	0.60

Table A32: Empirical FDR with $u = 1.5$ and $V = 2$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART	ash
	0.01	0.01	0.01	0.02	0.01	0.01	0.01
	0.05	0.03	0.04	0.06	0.05	0.02	0.02
	0.10	0.05	0.08	0.12	0.08	0.04	0.04
0.35	0.15	0.07	0.12	0.17	0.12	0.06	0.06
	0.20	0.08	0.16	0.22	0.15	0.08	0.08
	0.25	0.10	0.20	0.28	0.18	0.11	0.10
	0.30	0.12	0.25	0.32	0.21	0.14	0.13
	0.01	0.01	0.01	0.02	0.02	0.01	0.01
	0.05	0.04	0.04	0.06	0.05	0.03	0.03
	0.10	0.06	0.08	0.11	0.10	0.05	0.04
0.45	0.15	0.09	0.12	0.16	0.13	0.07	0.06
	0.20	0.10	0.16	0.22	0.17	0.10	0.08
	0.25	0.13	0.21	0.27	0.21	0.13	0.10
	0.30	0.15	0.25	0.33	0.24	0.16	0.12
	0.01	0.02	0.02	0.02	0.02	0.01	0.01
	0.05	0.05	0.05	0.06	0.06	0.04	0.03
	0.10	0.08	0.09	0.12	0.11	0.06	0.05
0.55	0.15	0.11	0.13	0.17	0.15	0.09	0.07
	0.20	0.13	0.17	0.22	0.19	0.12	0.09
	0.25	0.16	0.21	0.28	0.23	0.15	0.11
	0.30	0.19	0.26	0.33	0.27	0.18	0.13
	0.01	0.02	0.02	0.03	0.03	0.02	0.02
	0.05	0.06	0.05	0.07	0.08	0.05	0.04
	0.10	0.10	0.09	0.13	0.12	0.08	0.06
0.65	0.15	0.13	0.13	0.18	0.17	0.11	0.08
	0.20	0.16	0.17	0.23	0.21	0.14	0.10
	0.25	0.20	0.22	0.29	0.26	0.17	0.12
	0.30	0.23	0.26	0.34	0.29	0.21	0.14
	0.01	0.03	0.02	0.04	0.04	0.03	0.02
	0.05	0.08	0.06	0.09	0.10	0.07	0.05
	0.10	0.12	0.10	0.15	0.14	0.10	0.08
0.75	0.15	0.17	0.14	0.20	0.19	0.14	0.10
	0.20	0.20	0.18	0.25	0.24	0.17	0.12
	0.25	0.24	0.22	0.31	0.28	0.21	0.14
	0.30	0.28	0.27	0.36	0.32	0.25	0.17
	0.01	0.05	0.03	0.05	0.05	0.05	0.03
	0.05	0.10	0.07	0.13	0.12	0.10	0.07
	0.10	0.16	0.12	0.19	0.18	0.15	0.11
0.85	0.15	0.21	0.15	0.25	0.23	0.19	0.14
	0.20	0.26	0.19	0.30	0.28	0.23	0.17
	0.25	0.30	0.23	0.35	0.33	0.27	0.21
	0.30	0.34	0.27	0.40	0.38	0.31	0.23
	0.01	0.07	0.04	0.10	0.07	0.09	0.10
	0.05	0.18	0.10	0.25	0.19	0.22	0.16
	0.10	0.28	0.14	0.34	0.29	0.30	0.24
0.95	0.15	0.33	0.18	0.41	0.35	0.35	0.29
	0.20	0.39	0.22	0.46	0.41	0.40	0.35
	0.25	0.43	0.26	0.50	0.46	0.44	0.39
	0.30	0.48	0.29	0.55	0.50	0.47	0.43

Table A33: Emprical power with $u = 1.5$ and $V = 1$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART
	0.01	0.06	0.087	**0.088**	0.065	0.074
	0.05	0.108	0.158	**0.16**	0.118	0.125
	0.10	0.145	0.216	**0.219**	0.159	0.162
0.65	0.15	0.174	0.267	**0.271**	0.192	0.193
	0.20	0.202	0.315	**0.319**	0.227	0.223
	0.25	0.231	0.362	**0.367**	0.261	0.252
	0.30	0.258	0.409	**0.416**	0.294	0.282
	0.01	0.052	0.072	**0.074**	0.054	0.064
	0.05	0.092	0.131	**0.134**	0.097	0.108
	0.10	0.126	0.181	**0.184**	0.136	0.142
0.75	0.15	0.156	0.223	**0.227**	0.168	0.169
	0.20	0.182	0.263	**0.268**	0.196	0.195
	0.25	0.207	0.303	**0.311**	0.225	0.219
	0.30	0.232	0.344	**0.352**	0.255	0.243
	0.01	0.049	0.062	**0.065**	0.051	0.061
	0.05	0.086	0.111	**0.116**	0.09	0.099
	0.10	0.114	0.149	**0.154**	0.118	0.127
0.85	0.15	0.136	0.181	**0.188**	0.141	0.148
	0.20	0.153	0.21	**0.22**	0.162	0.166
	0.25	0.175	0.242	**0.252**	0.187	0.184
	0.30	0.197	0.272	**0.284**	0.212	0.204
	0.01	0.035	0.038	0.04	0.034	**0.041**
	0.05	0.057	0.067	**0.076**	0.061	0.074
	0.10	0.081	0.094	**0.107**	0.089	0.096
0.95	0.15	0.099	0.117	**0.131**	0.108	0.11
	0.20	0.116	0.134	**0.151**	0.124	0.124
	0.25	0.129	0.154	**0.173**	0.138	0.135
	0.30	0.141	0.169	**0.195**	0.156	0.146

Table A34: Emprical power with $u = 1.5$ and $V = 2$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART
0.65	0.01	0.219	0.292	**0.296**	0.237	0.253
	0.05	0.332	0.456	**0.46**	0.362	0.381
	0.10	0.405	0.562	**0.567**	0.443	0.466
	0.15	0.455	0.639	**0.644**	0.503	0.528
	0.20	0.499	0.7	**0.705**	0.554	0.581
	0.25	0.537	0.75	**0.756**	0.599	0.626
	0.30	0.573	0.793	**0.799**	0.641	0.669
0.75	0.01	0.203	0.258	**0.262**	0.214	0.232
	0.05	0.307	0.401	**0.407**	0.326	0.346
	0.10	0.376	0.496	**0.501**	0.4	0.422
	0.15	0.424	0.566	**0.574**	0.456	0.478
	0.20	0.467	0.622	**0.632**	0.5	0.522
	0.25	0.501	0.671	**0.68**	0.541	0.564
	0.30	0.534	0.714	**0.725**	0.578	0.601
0.85	0.01	0.185	0.224	**0.232**	0.19	0.211
	0.05	0.28	0.342	**0.352**	0.29	0.309
	0.10	0.337	0.42	**0.43**	0.352	0.369
	0.15	0.381	0.478	**0.488**	0.395	0.413
	0.20	0.417	0.525	**0.538**	0.436	0.448
	0.25	0.45	0.571	**0.585**	0.472	0.481
	0.30	0.481	0.616	**0.628**	0.503	0.512
0.95	0.01	0.149	0.162	0.17	0.152	**0.172**
	0.05	0.218	0.24	**0.254**	0.221	0.238
	0.10	0.261	0.293	**0.31**	0.268	0.279
	0.15	0.295	0.335	**0.352**	0.301	0.309
	0.20	0.324	0.372	**0.395**	0.332	0.334
	0.25	0.353	0.408	**0.436**	0.365	0.36
	0.30	0.376	0.443	**0.471**	0.392	0.382

Table A35: Posterior mean MSE with $u = 2$

V	ω	SNP	DNP	Grouplinear	XGB.M	XGB.SB	GMLEB	EBayesThresh	NEST	ash
1.0	0.55	0.433	**0.425**	0.537	0.489	0.53	0.426	0.516	0.473	0.517
	0.75	0.307	**0.296**	0.38	0.35	0.373	0.297	0.339	0.372	0.338
	0.95	**0.085**	**0.085**	0.124	0.095	0.116	0.092	0.092	0.199	0.089
1.5	0.55	0.517	**0.515**	0.685	0.614	0.677	0.52	0.675	0.551	0.673
	0.75	0.385	**0.379**	0.522	0.478	0.513	0.382	0.462	0.434	0.46
	0.95	0.122	**0.119**	0.17	0.146	0.163	0.124	0.131	0.221	0.128
2.0	0.55	**0.607**	**0.607**	0.83	0.737	0.822	0.613	0.82	0.628	0.81
	0.65	**0.546**	0.548	0.778	0.692	0.77	0.551	0.715	0.58	0.706
	0.75	0.463	**0.46**	0.677	0.613	0.668	0.464	0.582	0.495	0.576
	0.85	0.337	**0.332**	0.512	0.467	0.502	0.336	0.408	0.382	0.405
	0.95	0.159	**0.154**	0.234	0.213	0.226	0.159	0.174	0.241	0.172
2.5	0.55	**0.675**	0.678	0.928	0.843	0.933	0.683	0.935	0.687	0.909
	0.65	**0.611**	0.614	0.898	0.807	0.896	0.619	0.825	0.632	0.805
	0.75	0.522	**0.521**	0.824	0.736	0.81	0.526	0.681	0.542	0.668
	0.85	0.386	**0.382**	0.654	0.588	0.641	0.388	0.486	0.414	0.479
	0.95	0.191	**0.183**	0.311	0.29	0.302	0.191	0.215	0.257	0.213
3.0	0.55	**0.711**	0.715	0.986	0.927	1.007	0.72	1.016	0.717	0.97
	0.65	**0.645**	0.651	0.968	0.899	0.984	0.654	0.902	0.656	0.868
	0.75	**0.553**	**0.553**	0.928	0.838	0.92	0.559	0.751	0.564	0.726
	0.85	0.415	**0.41**	0.781	0.698	0.766	0.418	0.543	0.428	0.529
	0.95	0.211	**0.203**	0.395	0.371	0.387	0.212	0.248	0.266	0.245

Table A36: Posterior mean MSE Relative ratio with $u = 2$

V	ω	SNP	DNP	Grouplinear	XGB.M	XGB.SB	GMLEB	EBayesThresh	NEST	ash
1.0	0.55	1	**0.982**	1.24	1.13	1.224	0.985	1.191	1.092	1.195
	0.75	1	**0.965**	1.239	1.141	1.214	0.969	1.104	1.211	1.103
	0.95	**1**	1.004	1.463	1.123	1.369	1.087	1.087	2.34	1.052
1.5	0.55	1	**0.996**	1.324	1.187	1.31	1.005	1.306	1.065	1.302
	0.75	1	**0.984**	1.355	1.241	1.332	0.992	1.201	1.127	1.196
	0.95	1	**0.977**	1.397	1.202	1.336	1.015	1.075	1.813	1.053
2.0	0.55	**1**	**1**	1.368	1.215	1.355	1.01	1.352	1.035	1.336
	0.65	**1**	1.002	1.424	1.267	1.409	1.009	1.308	1.062	1.293
	0.75	1	**0.994**	1.463	1.324	1.443	1.003	1.259	1.071	1.246
	0.85	1	**0.987**	1.521	1.389	1.491	0.999	1.212	1.134	1.203
	0.95	1	**0.967**	1.47	1.339	1.422	0.997	1.094	1.515	1.079
2.5	0.55	**1**	1.005	1.376	1.25	1.383	1.012	1.385	1.018	1.348
	0.65	**1**	1.005	1.469	1.321	1.466	1.013	1.35	1.034	1.317
	0.75	1	**0.999**	1.58	1.41	1.553	1.009	1.305	1.038	1.28
	0.85	1	**0.989**	1.693	1.522	1.661	1.005	1.259	1.071	1.24
	0.95	1	**0.959**	1.626	1.514	1.582	0.997	1.125	1.346	1.112
3.0	0.55	**1**	1.006	1.388	1.305	1.417	1.013	1.43	1.009	1.366
	0.65	**1**	1.009	1.499	1.393	1.525	1.014	1.398	1.017	1.345
	0.75	**1**	**1**	1.677	1.514	1.663	1.01	1.357	1.02	1.312
	0.85	1	**0.989**	1.883	1.683	1.847	1.007	1.309	1.033	1.277
	0.95	1	**0.965**	1.877	1.763	1.836	1.007	1.177	1.265	1.162

Table A37: Posterior mode MSE with $u = 2$

V	ω	SNP	DNP	SSLASSO	SLOPE	GMLEB	EBayesThresh
1.0	0.55	**0.455**	0.508	0.512	0.524	0.538	0.472
	0.75	**0.268**	0.284	0.355	0.291	0.405	0.286
	0.95	0.057	**0.056**	0.222	0.059	0.214	0.075
1.5	0.55	**0.548**	0.657	0.633	0.699	0.588	0.566
	0.75	**0.329**	0.369	0.392	0.389	0.454	0.341
	0.95	**0.073**	0.075	0.243	0.078	0.223	0.086
2.0	0.55	0.634	0.813	0.788	0.904	**0.62**	0.646
	0.65	**0.518**	0.638	0.608	0.705	0.572	0.523
	0.75	**0.387**	0.46	0.442	0.503	0.491	0.389
	0.85	**0.245**	0.278	0.313	0.302	0.38	0.249
	0.95	**0.089**	0.094	0.242	0.1	0.233	0.097
2.5	0.55	0.686	0.951	0.968	1.123	**0.634**	0.699
	0.65	0.569	0.75	0.728	0.875	0.584	**0.562**
	0.75	0.429	0.544	0.502	0.625	0.5	**0.419**
	0.85	0.274	0.329	0.324	0.375	0.405	**0.272**
	0.95	**0.102**	0.112	0.231	0.125	0.241	0.107
3.0	0.55	0.697	1.056	1.167	1.346	**0.625**	0.725
	0.65	0.589	0.837	0.865	1.049	**0.568**	0.574
	0.75	0.448	0.611	0.573	0.749	0.498	**0.428**
	0.85	0.291	0.372	0.338	0.45	0.403	**0.282**
	0.95	**0.109**	0.128	0.217	0.15	0.248	0.113

Table A38: Posterior mode MSE Relative ratio with $u = 2$

V	ω	SNP	DNP	SSLASSO	SLOPE	GMLEB	EBayesThresh
1.0	0.55	**1**	1.115	1.125	1.151	1.181	1.036
	0.75	**1**	1.061	1.325	1.088	1.515	1.068
	0.95	1	**0.986**	3.882	1.033	3.742	1.309
1.5	0.55	**1**	1.198	1.155	1.275	1.072	1.033
	0.75	**1**	1.121	1.193	1.182	1.382	1.036
	0.95	**1**	1.022	3.317	1.061	3.038	1.173
2.0	0.55	1	1.282	1.242	1.426	**0.978**	1.018
	0.65	**1**	1.23	1.172	1.36	1.104	1.009
	0.75	**1**	1.188	1.142	1.3	1.268	1.004
	0.85	**1**	1.137	1.279	1.235	1.553	1.017
	0.95	**1**	1.062	2.728	1.133	2.625	1.097
2.5	0.55	1	1.386	1.409	1.636	**0.924**	1.019
	0.65	1	1.317	1.279	1.538	1.025	**0.988**
	0.75	1	1.27	1.172	1.458	1.168	**0.977**
	0.85	1	1.2	1.181	1.368	1.477	**0.991**
	0.95	**1**	1.098	2.265	1.221	2.362	1.047
3.0	0.55	1	1.515	1.673	1.93	**0.897**	1.039
	0.65	1	1.422	1.468	1.782	**0.965**	0.975
	0.75	1	1.365	1.281	1.674	1.113	**0.957**
	0.85	1	1.279	1.161	1.547	1.384	**0.971**
	0.95	**1**	1.169	1.985	1.369	2.266	1.036

Table A39: Bias of sparsity with $u = 2$

V	ω	SNP	DNP	SSLASSO	GMLEB	EBayesThresh	HART	ash
1.0	0.55	0.263	0.141	0.323	-0.329	-0.448	0.382	**-0.004**
	0.75	0.172	0.066	0.105	-0.487	-0.456	0.218	**-0.063**
	0.95	0.042	**-0.027**	-0.058	-0.507	-0.529	0.048	-0.115
1.5	0.55	0.15	**0.056**	0.345	-0.297	-0.485	0.342	-0.09
	0.75	0.119	**0.027**	0.125	-0.499	-0.412	0.196	-0.096
	0.95	**0.035**	-0.036	-0.103	-0.512	-0.413	0.046	-0.116
2.0	0.55	0.075	**-0.002**	0.368	-0.272	-0.511	0.294	-0.165
	0.65	0.08	**-0.005**	0.258	-0.432	-0.457	0.232	-0.143
	0.75	0.074	**-0.01**	0.146	-0.472	-0.39	0.169	-0.126
	0.85	0.056	**-0.026**	0.027	-0.515	-0.313	0.107	-0.11
	0.95	**0.027**	-0.045	-0.107	-0.488	-0.289	0.043	-0.116
2.5	0.55	**0.039**	-0.044	0.393	-0.284	-0.519	0.245	-0.204
	0.65	0.041	**-0.034**	0.281	-0.397	-0.458	0.194	-0.179
	0.75	0.044	**-0.026**	0.169	-0.456	-0.371	0.142	-0.146
	0.85	**0.037**	-0.041	0.049	-0.503	-0.277	0.09	-0.112
	0.95	**0.018**	-0.05	-0.098	-0.484	-0.212	0.038	-0.113
3.0	0.55	**0.023**	-0.058	0.418	-0.254	-0.51	0.197	-0.228
	0.65	**0.023**	-0.052	0.303	-0.343	-0.441	0.156	-0.198
	0.75	**0.028**	-0.033	0.19	-0.411	-0.347	0.116	-0.153
	0.85	**0.027**	-0.045	0.07	-0.479	-0.249	0.075	-0.112
	0.95	**0.012**	-0.051	-0.084	-0.469	-0.16	0.033	-0.105

Table A40: Coverage of credible interval with $u = 2$

V	ω	SNP	DNP	EBayesThresh	ash
1.0	0.55	0.92	0.91	0.97	0.94
	0.75	0.93	0.93	0.98	0.94
	0.95	0.97	0.97	0.99	0.98
1.5	0.55	0.94	0.94	0.96	0.94
	0.75	0.95	0.95	0.97	0.95
	0.95	0.97	0.98	0.99	0.97
2.0	0.55	0.96	0.96	0.94	0.95
	0.65	0.96	0.96	0.95	0.95
	0.75	0.96	0.96	0.96	0.95
	0.85	0.96	0.97	0.97	0.96
	0.95	0.97	0.98	0.99	0.97
2.5	0.55	0.96	0.96	0.89	0.96
	0.65	0.96	0.97	0.91	0.96
	0.75	0.97	0.97	0.93	0.96
	0.85	0.97	0.97	0.96	0.96
	0.95	0.98	0.98	0.98	0.98
3.0	0.55	0.97	0.97	0.82	0.96
	0.65	0.97	0.97	0.86	0.96
	0.75	0.97	0.97	0.90	0.97
	0.85	0.98	0.98	0.94	0.97
	0.95	0.98	0.98	0.98	0.98

Table A41: Length of credible interval with $u = 2$

V	ω	SNP	DNP	EBayesThresh	ash
	0.55	2.132	**1.999**	3.664	2.696
1.0	0.75	**1.646**	1.655	3.453	2.199
	0.95	0.911	**0.851**	3.324	1.334
	0.55	2.403	**2.253**	3.533	3.059
1.5	0.75	1.964	**1.866**	3.235	2.48
	0.95	0.985	**0.984**	2.897	1.406
	0.55	2.652	**2.516**	3.364	3.375
	0.65	2.467	**2.33**	3.228	3.095
2.0	0.75	2.186	**2.07**	3.037	2.731
	0.85	1.748	**1.721**	2.74	2.231
	0.95	**1.049**	1.06	2.422	1.489
	0.55	2.844	**2.764**	3.174	3.607
	0.65	2.645	**2.528**	3.045	3.32
2.5	0.75	2.33	**2.212**	2.83	2.913
	0.85	1.863	**1.83**	2.508	2.346
	0.95	1.129	**1.119**	2.046	1.553
	0.55	2.963	**2.918**	2.977	3.765
	0.65	2.744	**2.678**	2.836	3.456
3.0	0.75	2.393	**2.285**	2.61	3.004
	0.85	1.893	**1.848**	2.277	2.373
	0.95	1.18	**1.105**	1.716	1.546

Table A42: Empirical FDR with $u = 2$ and $V = 1$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART	ash
	0.01	0.01	0.02	0.02	0.02	0.01	0.02
	0.05	0.03	0.04	0.05	0.03	0.03	0.03
	0.10	0.04	0.05	0.07	0.05	0.04	0.05
0.35	0.15	0.06	0.07	0.10	0.07	0.05	0.06
	0.20	0.07	0.09	0.12	0.09	0.06	0.08
	0.25	0.09	0.11	0.15	0.10	0.06	0.09
	0.30	0.10	0.13	0.18	0.12	0.07	0.10
	0.01	0.02	0.02	0.04	0.02	0.02	0.02
	0.05	0.05	0.05	0.08	0.05	0.05	0.06
	0.10	0.08	0.07	0.11	0.08	0.07	0.08
0.55	0.15	0.10	0.09	0.14	0.10	0.08	0.10
	0.20	0.12	0.11	0.16	0.12	0.09	0.12
	0.25	0.13	0.13	0.19	0.14	0.10	0.14
	0.30	0.15	0.15	0.22	0.17	0.12	0.16
	0.01	0.04	0.03	0.07	0.05	0.04	0.04
	0.05	0.10	0.08	0.14	0.10	0.10	0.10
	0.10	0.13	0.11	0.17	0.14	0.12	0.14
0.75	0.15	0.16	0.13	0.21	0.17	0.14	0.17
	0.20	0.19	0.15	0.24	0.20	0.16	0.20
	0.25	0.22	0.17	0.27	0.23	0.18	0.22
	0.30	0.24	0.18	0.30	0.25	0.20	0.24
	0.01	0.26	0.19	0.21	0.18	0.11	0.13
	0.05	0.30	0.23	0.31	0.22	0.15	0.28
	0.10	0.31	0.25	0.42	0.29	0.17	0.37
0.95	0.15	0.36	0.26	0.47	0.35	0.21	0.42
	0.20	0.40	0.28	0.53	0.42	0.24	0.44
	0.25	0.47	0.27	0.58	0.47	0.27	0.49
	0.30	0.48	0.30	0.62	0.50	0.32	0.53

Table A43: Empirical FDR with $u = 2$ and $V = 2$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART	ash
	0.01	0.01	0.01	0.02	0.01	0.01	0.01
	0.05	0.02	0.03	0.05	0.03	0.02	0.02
	0.10	0.04	0.07	0.09	0.06	0.03	0.04
0.35	0.15	0.05	0.11	0.15	0.08	0.04	0.05
	0.20	0.06	0.16	0.21	0.10	0.05	0.07
	0.25	0.08	0.21	0.27	0.13	0.07	0.09
	0.30	0.09	0.26	0.31	0.16	0.09	0.12
	0.01	0.01	0.01	0.02	0.01	0.01	0.01
	0.05	0.03	0.04	0.04	0.04	0.02	0.02
	0.10	0.04	0.06	0.08	0.06	0.03	0.04
0.45	0.15	0.06	0.10	0.13	0.09	0.04	0.05
	0.20	0.08	0.14	0.19	0.11	0.06	0.06
	0.25	0.10	0.19	0.25	0.14	0.07	0.08
	0.30	0.11	0.24	0.30	0.17	0.10	0.10
	0.01	0.01	0.02	0.02	0.01	0.01	0.01
	0.05	0.04	0.04	0.05	0.04	0.03	0.03
	0.10	0.06	0.07	0.09	0.07	0.04	0.05
0.55	0.15	0.08	0.10	0.13	0.10	0.05	0.06
	0.20	0.10	0.14	0.17	0.13	0.07	0.08
	0.25	0.12	0.18	0.23	0.16	0.09	0.09
	0.30	0.14	0.23	0.28	0.19	0.11	0.11
	0.01	0.02	0.02	0.03	0.02	0.02	0.02
	0.05	0.05	0.05	0.06	0.06	0.04	0.04
	0.10	0.08	0.07	0.09	0.09	0.06	0.06
0.65	0.15	0.10	0.10	0.13	0.12	0.07	0.07
	0.20	0.12	0.13	0.17	0.14	0.09	0.09
	0.25	0.15	0.16	0.22	0.17	0.11	0.10
	0.30	0.17	0.21	0.27	0.20	0.13	0.12
	0.01	0.02	0.02	0.04	0.03	0.02	0.02
	0.05	0.06	0.05	0.08	0.07	0.05	0.05
	0.10	0.10	0.08	0.11	0.10	0.07	0.07
0.75	0.15	0.13	0.10	0.15	0.13	0.10	0.09
	0.20	0.15	0.13	0.18	0.17	0.12	0.11
	0.25	0.18	0.16	0.22	0.20	0.14	0.12
	0.30	0.21	0.20	0.27	0.23	0.16	0.14
	0.01	0.03	0.02	0.05	0.03	0.03	0.03
	0.05	0.09	0.07	0.10	0.09	0.08	0.07
	0.10	0.12	0.10	0.15	0.13	0.10	0.10
0.85	0.15	0.16	0.12	0.19	0.17	0.13	0.12
	0.20	0.20	0.14	0.23	0.20	0.16	0.15
	0.25	0.22	0.18	0.27	0.24	0.18	0.17
	0.30	0.26	0.21	0.31	0.27	0.21	0.19
	0.01	0.07	0.04	0.07	0.07	0.06	0.09
	0.05	0.12	0.09	0.21	0.12	0.10	0.15
	0.10	0.19	0.14	0.30	0.20	0.14	0.20
0.95	0.15	0.26	0.18	0.35	0.27	0.18	0.26
	0.20	0.30	0.20	0.39	0.32	0.22	0.30
	0.25	0.35	0.23	0.43	0.37	0.25	0.33
	0.30	0.38	0.26	0.47	0.39	0.28	0.37

Table A44: Emprical power with $u = 2$ and $V = 1$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART
	0.01	0.036	0.056	**0.057**	0.038	0.043
	0.05	0.065	0.1	**0.103**	0.068	0.073
	0.10	0.086	0.138	**0.14**	0.09	0.094
0.65	0.15	0.104	0.171	**0.175**	0.109	0.111
	0.20	0.121	0.207	**0.212**	0.128	0.128
	0.25	0.138	0.243	**0.25**	0.146	0.143
	0.30	0.152	0.282	**0.292**	0.162	0.161
	0.01	0.032	0.045	**0.047**	0.033	0.037
	0.05	0.055	0.081	**0.083**	0.055	0.06
	0.10	0.072	0.111	**0.114**	0.074	0.077
0.75	0.15	0.086	0.138	**0.141**	0.089	0.091
	0.20	0.101	0.163	**0.169**	0.105	0.104
	0.25	0.116	0.191	**0.197**	0.121	0.117
	0.30	0.131	0.22	**0.228**	0.136	0.131
	0.01	0.029	0.039	**0.041**	0.029	0.031
	0.05	0.051	0.068	**0.071**	0.052	0.052
	0.10	0.065	0.09	**0.096**	0.067	0.064
0.85	0.15	0.077	0.109	**0.116**	0.08	0.075
	0.20	0.092	0.126	**0.136**	0.093	0.085
	0.25	0.102	0.144	**0.157**	0.105	0.095
	0.30	0.111	0.163	**0.181**	0.116	0.104
	0.01	0.031	0.032	**0.034**	0.029	0.022
	0.05	0.044	0.048	**0.055**	0.044	0.034
	0.10	0.051	0.062	**0.074**	0.052	0.04
0.95	0.15	0.06	0.075	**0.09**	0.063	0.044
	0.20	0.073	0.088	**0.105**	0.081	0.048
	0.25	0.084	0.099	**0.12**	0.091	0.052
	0.30	0.097	0.108	**0.135**	0.103	0.056

Table A45: Emprical power with $u = 2$ and $V = 2$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART
0.65	0.01	0.137	0.192	**0.194**	0.143	0.158
	0.05	0.213	0.316	**0.318**	0.226	0.242
	0.10	0.271	0.414	**0.418**	0.289	0.304
	0.15	0.314	0.497	**0.501**	0.337	0.355
	0.20	0.351	0.571	**0.576**	0.378	0.402
	0.25	0.385	0.639	**0.644**	0.417	0.448
	0.30	0.417	0.7	**0.706**	0.456	0.494
0.75	0.01	0.125	0.166	**0.168**	0.129	0.143
	0.05	0.193	0.266	**0.269**	0.2	0.216
	0.10	0.243	0.346	**0.351**	0.252	0.268
	0.15	0.281	0.41	**0.417**	0.293	0.309
	0.20	0.316	0.472	**0.48**	0.33	0.349
	0.25	0.346	0.532	**0.541**	0.364	0.386
	0.30	0.378	0.587	**0.596**	0.398	0.422
0.85	0.01	0.116	0.143	**0.148**	0.117	0.128
	0.05	0.175	0.222	**0.23**	0.178	0.188
	0.10	0.215	0.281	**0.291**	0.219	0.23
	0.15	0.249	0.331	**0.342**	0.251	0.261
	0.20	0.276	0.374	**0.387**	0.282	0.289
	0.25	0.304	0.418	**0.435**	0.31	0.316
	0.30	0.327	0.462	**0.48**	0.335	0.34
0.95	0.01	0.094	0.101	**0.108**	0.094	0.088
	0.05	0.137	0.148	**0.159**	0.139	0.128
	0.10	0.167	0.181	**0.197**	0.168	0.151
	0.15	0.188	0.21	**0.227**	0.192	0.168
	0.20	0.206	0.237	**0.255**	0.214	0.181
	0.25	0.226	0.26	**0.284**	0.229	0.195
	0.30	0.241	0.285	**0.312**	0.245	0.208

Table A46: Posterior mean MSE with $u = 2.5$

V	ω	SNP	DNP	Grouplinear	XGB.M	XGB.SB	GMLEB	EBayesThresh	NEST	ash
	0.55	0.487	**0.466**	0.58	0.574	0.572	0.47	0.575	0.525	0.581
1.0	0.75	0.347	**0.321**	0.4	0.399	0.391	0.323	0.367	0.419	0.369
	0.95	0.09	**0.088**	0.134	0.113	0.128	0.105	0.093	0.235	0.091
	0.55	0.588	**0.581**	0.768	0.741	0.76	0.59	0.78	0.616	0.784
1.5	0.75	0.447	**0.428**	0.564	0.558	0.555	0.43	0.521	0.492	0.523
	0.95	0.133	**0.126**	0.178	0.159	0.169	0.13	0.137	0.26	0.135
	0.55	0.704	**0.701**	0.986	0.919	0.977	0.714	0.978	0.711	0.976
2.0	0.75	0.548	**0.537**	0.763	0.738	0.752	0.542	0.684	0.567	0.683
	0.95	0.18	**0.171**	0.245	0.24	0.235	0.173	0.189	0.285	0.188
	0.55	0.805	0.805	1.197	1.079	1.187	0.818	1.143	**0.789**	1.128
2.5	0.75	0.636	0.63	0.973	0.914	0.959	0.638	0.831	**0.629**	0.823
	0.95	0.227	**0.213**	0.325	0.326	0.314	0.218	0.244	0.306	0.244
	0.55	0.87	0.872	1.371	1.207	1.355	0.884	1.267	**0.839**	1.227
3.0	0.75	0.697	0.693	1.169	1.068	1.153	0.701	0.948	**0.668**	0.929
	0.95	0.266	**0.25**	0.42	0.425	0.408	0.258	0.297	0.321	0.296

Table A47: Posterior mean MSE Relative ratio with $u = 2.5$

V	ω	SNP	DNP	Grouplinear	XGB.M	XGB.SB	GMLEB	EBayesThresh	NEST	ash
	0.55	1	**0.958**	1.192	1.18	1.176	0.966	1.181	1.078	1.193
1.0	0.75	1	**0.924**	1.15	1.148	1.125	0.929	1.056	1.207	1.063
	0.95	1	**0.974**	1.485	1.246	1.417	1.16	1.027	2.599	1.001
	0.55	1	**0.988**	1.307	1.261	1.292	1.003	1.328	1.048	1.333
1.5	0.75	1	**0.957**	1.262	1.248	1.242	0.963	1.166	1.101	1.17
	0.95	1	**0.949**	1.345	1.198	1.275	0.981	1.035	1.964	1.016
	0.55	1	**0.995**	1.4	1.305	1.387	1.014	1.389	1.01	1.386
2.0	0.75	1	**0.98**	1.392	1.346	1.372	0.99	1.247	1.035	1.246
	0.95	1	**0.951**	1.363	1.336	1.306	0.965	1.053	1.587	1.046
	0.55	1	1	1.488	1.341	1.475	1.016	1.42	**0.981**	1.402
2.5	0.75	1	0.99	1.529	1.437	1.507	1.002	1.305	**0.989**	1.294
	0.95	1	**0.937**	1.429	1.434	1.382	0.96	1.072	1.347	1.073
	0.55	1	1.003	1.577	1.388	1.559	1.017	1.457	**0.964**	1.41
3.0	0.75	1	0.994	1.677	1.532	1.654	1.005	1.359	**0.958**	1.332
	0.95	1	**0.94**	1.582	1.599	1.537	0.972	1.119	1.208	1.114

Table A48: Posterior mode MSE with $u = 2.5$

V	ω	SNP	DNP	SSLASSO	SLOPE	GMLEB	EBayesThresh
	0.55	**0.474**	0.511	0.532	0.524	0.543	0.481
1.0	0.75	**0.275**	0.286	0.368	0.291	0.393	0.284
	0.95	**0.057**	**0.057**	0.115	0.063	0.12	0.068
	0.55	**0.586**	0.668	0.657	0.699	0.618	0.592
1.5	0.75	**0.345**	0.374	0.411	0.389	0.456	0.351
	0.95	**0.074**	0.075	0.149	0.079	0.233	0.085
	0.55	0.697	0.835	0.814	0.904	**0.658**	0.692
2.0	0.75	0.418	0.47	0.465	0.503	0.519	**0.415**
	0.95	**0.092**	0.096	0.201	0.103	0.211	0.098
	0.55	0.788	0.994	0.995	1.123	**0.689**	0.765
2.5	0.75	0.48	0.563	0.528	0.625	0.544	**0.463**
	0.95	**0.108**	0.115	0.23	0.125	0.225	0.111
	0.55	0.835	1.128	1.192	1.346	**0.681**	0.807
3.0	0.75	0.522	0.646	0.6	0.749	0.546	**0.491**
	0.95	**0.121**	0.134	0.227	0.15	0.237	0.122

Table A49: Posterior mode MSE Relative ratio with $u = 2.5$

V	ω	SNP	DNP	SSLASSO	SLOPE	GMLEB	EBayesThresh
	0.55	**1**	1.08	1.124	1.106	1.148	1.016
1.0	0.75	**1**	1.041	1.34	1.061	1.429	1.034
	0.95	1	**0.999**	2.003	1.103	2.094	1.193
	0.55	**1**	1.139	1.121	1.192	1.053	1.009
1.5	0.75	**1**	1.083	1.191	1.125	1.32	1.015
	0.95	**1**	1.018	2.003	1.068	3.144	1.147
	0.55	1	1.198	1.168	1.297	**0.945**	0.993
2.0	0.75	1	1.125	1.112	1.203	1.24	**0.992**
	0.95	**1**	1.041	2.185	1.12	2.287	1.064
	0.55	1	1.261	1.262	1.425	**0.874**	0.97
2.5	0.75	1	1.173	1.1	1.302	1.134	**0.965**
	0.95	**1**	1.067	2.126	1.16	2.078	1.028
	0.55	1	1.351	1.427	1.611	**0.815**	0.966
3.0	0.75	1	1.237	1.149	1.435	1.047	**0.94**
	0.95	**1**	1.103	1.874	1.234	1.956	1.009

Table A50: Bias of sparsity with $u = 2.5$

V	ω	SNP	DNP	SSLASSO	GMLEB	EBayesThresh	HART	ash
1.0	0.55	0.291	0.16	0.336	-0.261	-0.408	0.413	**0.05**
	0.75	0.194	0.087	0.123	-0.435	-0.399	0.241	**-0.01**
	0.95	0.045	**-0.007**	0.164	-0.133	-0.573	0.05	-0.07
1.5	0.55	0.171	0.078	0.354	-0.256	-0.459	0.379	**-0.038**
	0.75	0.143	**0.045**	0.136	-0.442	-0.369	0.224	-0.05
	0.95	0.041	**-0.014**	0.108	-0.543	-0.483	0.05	-0.073
2.0	0.55	0.082	**0.018**	0.375	-0.25	-0.498	0.338	-0.126
	0.75	0.093	**0.016**	0.154	-0.457	-0.357	0.201	-0.086
	0.95	0.033	-0.02	**0.005**	-0.438	-0.3	0.049	-0.076
2.5	0.55	0.04	**-0.027**	0.397	-0.277	-0.513	0.293	-0.18
	0.75	0.058	**-0.008**	0.174	-0.412	-0.346	0.177	-0.113
	0.95	**0.027**	-0.031	-0.06	-0.405	-0.191	0.048	-0.074
3.0	0.55	**0.021**	-0.047	0.421	-0.238	-0.511	0.25	-0.203
	0.75	0.04	**-0.016**	0.193	-0.375	-0.329	0.153	-0.126
	0.95	**0.021**	-0.032	-0.068	-0.397	-0.131	0.046	-0.069

Table A51: Coverage of credible interval with $u = 2.5$

V	ω	SNP	DNP	EBayesThresh	ash
1.0	0.55	0.91	0.90	0.98	0.93
	0.75	0.91	0.92	0.98	0.93
	0.95	0.96	0.97	0.99	0.97
1.5	0.55	0.94	0.93	0.97	0.94
	0.75	0.94	0.94	0.98	0.94
	0.95	0.96	0.97	0.99	0.97
2.0	0.55	0.96	0.95	0.94	0.95
	0.75	0.96	0.96	0.96	0.94
	0.95	0.97	0.98	0.99	0.97
2.5	0.55	0.96	0.96	0.89	0.96
	0.75	0.96	0.97	0.93	0.95
	0.95	0.97	0.98	0.98	0.97
3.0	0.55	0.97	0.97	0.80	0.96
	0.75	0.97	0.97	0.88	0.96
	0.95	0.98	0.98	0.97	0.98

Table A52: Length of credible interval with $u = 2.5$

V	ω	SNP	DNP	EBayesThresh	ash
1.0	0.55	2.143	**1.976**	4.081	2.788
	0.75	**1.489**	1.629	3.794	2.186
	0.95	0.717	**0.645**	4.04	1.061
1.5	0.55	2.462	**2.269**	3.924	3.235
	0.75	1.97	**1.877**	3.543	2.528
	0.95	**0.762**	0.768	3.637	1.167
2.0	0.55	2.778	**2.592**	3.705	3.65
	0.75	2.279	**2.127**	3.304	2.87
	0.95	**0.878**	0.892	2.844	1.265
2.5	0.55	3.042	**2.888**	3.45	3.988
	0.75	2.506	**2.357**	3.049	3.148
	0.95	**0.977**	1.019	2.099	1.339
3.0	0.55	3.245	**3.126**	3.191	4.241
	0.75	2.649	**2.495**	2.779	3.333
	0.95	1.038	**1.016**	1.653	1.36

Table A53: Empirical FDR with $u = 2.5$ and $V = 1$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART	ash
0.35	0.01	0.01	0.02	0.02	0.02	0.01	0.02
	0.05	0.02	0.04	0.05	0.02	0.02	0.03
	0.10	0.04	0.05	0.07	0.04	0.03	0.04
	0.15	0.05	0.06	0.09	0.05	0.04	0.06
	0.20	0.06	0.08	0.11	0.06	0.04	0.07
	0.25	0.07	0.09	0.13	0.08	0.05	0.08
	0.30	0.08	0.11	0.16	0.09	0.06	0.09
0.55	0.01	0.01	0.01	0.04	0.01	0.01	0.02
	0.05	0.04	0.05	0.07	0.04	0.04	0.05
	0.10	0.06	0.07	0.10	0.06	0.05	0.07
	0.15	0.08	0.08	0.13	0.08	0.06	0.09
	0.20	0.10	0.10	0.15	0.10	0.07	0.10
	0.25	0.11	0.11	0.17	0.11	0.08	0.12
	0.30	0.13	0.13	0.20	0.13	0.09	0.14
0.75	0.01	0.04	0.03	0.06	0.04	0.03	0.05
	0.05	0.08	0.07	0.13	0.08	0.06	0.09
	0.10	0.12	0.10	0.16	0.12	0.07	0.12
	0.15	0.14	0.11	0.20	0.14	0.09	0.16
	0.20	0.16	0.13	0.23	0.16	0.10	0.18
	0.25	0.19	0.14	0.25	0.19	0.12	0.20
	0.30	0.21	0.16	0.28	0.21	0.14	0.23
0.95	0.01	0.28	0.27	0.32	0.17	0.00	0.16
	0.05	0.32	0.28	0.35	0.26	0.00	0.24
	0.10	0.34	0.31	0.45	0.28	0.00	0.32
	0.15	0.36	0.31	0.48	0.32	0.00	0.38
	0.20	0.38	0.32	0.51	0.37	0.25	0.43
	0.25	0.40	0.34	0.55	0.41	0.25	0.46
	0.30	0.45	0.36	0.59	0.46	0.25	0.48

Table A54: Empirical FDR with $u = 2.5$ and $V = 2$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART	ash
	0.01	0.01	0.01	0.02	0.01	0.01	0.01
	0.05	0.02	0.03	0.04	0.02	0.01	0.02
	0.10	0.03	0.06	0.08	0.04	0.02	0.03
0.35	0.15	0.04	0.10	0.13	0.06	0.03	0.04
	0.20	0.05	0.15	0.20	0.07	0.03	0.06
	0.25	0.06	0.20	0.26	0.09	0.04	0.07
	0.30	0.07	0.26	0.31	0.11	0.05	0.10
	0.01	0.01	0.02	0.02	0.01	0.01	0.01
	0.05	0.03	0.04	0.05	0.03	0.03	0.03
	0.10	0.05	0.06	0.07	0.06	0.03	0.04
0.55	0.15	0.06	0.08	0.10	0.07	0.04	0.06
	0.20	0.08	0.11	0.14	0.09	0.05	0.07
	0.25	0.10	0.15	0.19	0.11	0.06	0.08
	0.30	0.11	0.19	0.24	0.13	0.07	0.10
	0.01	0.02	0.02	0.04	0.02	0.02	0.02
	0.05	0.06	0.05	0.07	0.06	0.04	0.04
	0.10	0.08	0.07	0.10	0.08	0.06	0.06
0.75	0.15	0.10	0.09	0.13	0.11	0.07	0.08
	0.20	0.12	0.11	0.16	0.13	0.09	0.10
	0.25	0.15	0.13	0.19	0.15	0.10	0.12
	0.30	0.16	0.16	0.22	0.17	0.12	0.13
	0.01	0.08	0.05	0.06	0.06	0.02	0.08
	0.05	0.12	0.08	0.17	0.12	0.03	0.13
	0.10	0.17	0.13	0.26	0.16	0.04	0.20
0.95	0.15	0.22	0.16	0.31	0.23	0.06	0.22
	0.20	0.27	0.19	0.36	0.28	0.06	0.26
	0.25	0.30	0.21	0.39	0.31	0.07	0.29
	0.30	0.33	0.24	0.42	0.34	0.09	0.33

Table A55: Emprical power with $u = 2.5$ and $V = 1$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART
	0.01	0.025	0.04	**0.041**	0.026	0.026
	0.05	0.044	0.073	**0.074**	0.045	0.044
	0.10	0.058	0.099	**0.102**	0.06	0.057
0.65	0.15	0.071	0.123	**0.126**	0.072	0.067
	0.20	0.08	0.146	**0.152**	0.081	0.076
	0.25	0.09	0.172	**0.179**	0.092	0.085
	0.30	0.101	0.202	**0.211**	0.103	0.094
	0.01	0.022	0.032	**0.033**	0.022	0.02
	0.05	0.038	0.057	**0.06**	0.039	0.033
	0.10	0.049	0.077	**0.082**	0.049	0.042
0.75	0.15	0.058	0.095	**0.101**	0.059	0.049
	0.20	0.067	0.112	**0.12**	0.068	0.055
	0.25	0.075	0.131	**0.141**	0.076	0.061
	0.30	0.083	0.15	**0.164**	0.085	0.068
	0.01	0.025	0.032	**0.033**	0.025	0.017
	0.05	0.042	0.054	**0.058**	0.042	0.026
	0.10	0.052	0.072	**0.078**	0.054	0.032
0.85	0.15	0.06	0.086	**0.092**	0.061	0.037
	0.20	0.069	0.1	**0.108**	0.07	0.041
	0.25	0.075	0.112	**0.121**	0.077	0.045
	0.30	0.083	0.125	**0.137**	0.085	0.05
	0.01	**0.03**	**0.03**	**0.03**	**0.03**	0.02
	0.05	**0.04**	**0.04**	**0.04**	**0.04**	0.02
	0.10	0.05	0.07	**0.09**	0.05	0.02
0.95	0.15	0.06	0.08	**0.11**	0.06	0.02
	0.20	0.08	0.09	**0.12**	0.08	0.02
	0.25	0.08	0.11	**0.15**	0.09	0.02
	0.30	0.09	0.11	**0.15**	0.11	0.02

Table A56: Emprical power with $u = 2.5$ and $V = 2$

ω	FDR level	BH	SNP NEB	DNP NEB	Storey	HART
0.65	0.01	0.098	0.143	**0.145**	0.1	0.111
	0.05	0.149	0.233	**0.235**	0.152	0.166
	0.10	0.187	0.311	**0.314**	0.192	0.207
	0.15	0.219	0.382	**0.386**	0.227	0.242
	0.20	0.249	0.452	**0.458**	0.258	0.274
	0.25	0.275	0.522	**0.529**	0.287	0.307
	0.30	0.3	0.594	**0.601**	0.314	0.342
0.75	0.01	0.089	0.122	**0.124**	0.09	0.097
	0.05	0.135	0.194	**0.196**	0.136	0.145
	0.10	0.167	0.251	**0.255**	0.169	0.178
	0.15	0.193	0.304	**0.308**	0.196	0.206
	0.20	0.219	0.356	**0.361**	0.223	0.231
	0.25	0.242	0.407	**0.415**	0.246	0.257
	0.30	0.262	0.463	**0.471**	0.267	0.282
0.85	0.01	0.081	0.105	**0.108**	0.081	0.083
	0.05	0.122	0.16	**0.166**	0.123	0.119
	0.10	0.15	0.201	**0.211**	0.151	0.144
	0.15	0.172	0.237	**0.249**	0.173	0.162
	0.20	0.189	0.272	**0.288**	0.19	0.18
	0.25	0.206	0.307	**0.326**	0.208	0.198
	0.30	0.224	0.343	**0.366**	0.227	0.216
0.95	0.01	0.088	0.096	**0.105**	0.089	0.061
	0.05	0.129	0.143	**0.152**	0.13	0.08
	0.10	0.151	0.168	**0.177**	0.153	0.094
	0.15	0.167	0.187	**0.207**	0.169	0.104
	0.20	0.181	0.208	**0.226**	0.181	0.114
	0.25	0.192	0.224	**0.246**	0.195	0.123
	0.30	0.207	0.244	**0.27**	0.209	0.134

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