11076CH01

Chapter 1

SETS

❖*In these days of conflict between ancient and modern studies; there must surely be something to be said for a study which did not begin with Pythagoras and will not end with Einstein; but is the oldest and the youngest. — G.H. HARDY* ❖

1.1 Introduction

The concept of set serves as a fundamental part of the present day mathematics. Today this concept is being used in almost every branch of mathematics. Sets are used to define the concepts of relations and functions. The study of geometry, sequences, probability, etc. requires the knowledge of sets.

The theory of sets was developed by German mathematician Georg Cantor (1845-1918). He first encountered sets while working on "problems on trigonometric series". In this Chapter, we discuss some basic definitions and operations involving sets.

Georg Cantor (1845-1918)

1.2 Sets and their Representations

In everyday life, we often speak of collections of objects of a particular kind, such as, a pack of cards, a crowd of people, a cricket team, etc. In mathematics also, we come across collections, for example, of natural numbers, points, prime numbers, etc. More specially, we examine the following collections:

(i) Odd natural numbers less than 10, i.e., 1, 3, 5, 7, 9
(ii) The rivers of India
(iii) The vowels in the English alphabet, namely, *a, e, i, o, u*
(iv) Various kinds of triangles
(v) Prime factors of 210, namely, 2,3,5 and 7
(vi) The solution of the equation: $x^2 - 5x + 6 = 0$, viz, 2 and 3.

We note that each of the above example is a well-defined collection of objects in

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the sense that we can definitely decide whether a given particular object belongs to a given collection or not. For example, we can say that the river Nile does not belong to the collection of rivers of India. On the other hand, the river Ganga does belong to this colleciton.

We give below a few more examples of sets used particularly in mathematics, viz.

N **:** the set of all natural numbers
Z **:** the set of all integers
Q **:** the set of all rational numbers
R **:** the set of real numbers
$\mathbf{Z}^+$ **:** the set of positive integers
$\mathbf{Q}^+$ **:** the set of positive rational numbers, and
$\mathbf{R}^+$ **:** the set of positive real numbers.

The symbols for the special sets given above will be referred to throughout this text.

Again the collection of five most renowned mathematicians of the world is not well-defined, because the criterion for determining a mathematician as most renowned may vary from person to person. Thus, it is not a well-defined collection.

We shall say that ***a set is a well-defined collection of objects.***

The following points may be noted :

(i) Objects, elements and members of a set are synonymous terms.
(ii) Sets are usually denoted by capital letters A, B, C, X, Y, Z, etc.
(iii) The elements of a set are represented by small letters *a, b, c, x, y, z,* etc.

If *a* is an element of a set A, we say that " *a* belongs to A" the Greek symbol $\in$ (epsilon) is used to denote the phrase '*belongs to*'. Thus, we write $a \in$ A. If '*b*' is not an element of a set A, we write $b \notin$ A and read "*b* does not belong to A".

Thus, in the set V of vowels in the English alphabet, $a \in$ V but $b \notin$ V. In the set P of prime factors of 30, $3 \in$ P but $15 \notin$ P.

There are two methods of representing a set :

(i) Roster or tabular form
(ii) Set-builder form.

(i) In roster form, all the elements of a set are listed, the elements are being separated by commas and are enclosed within braces { }. For example, the set of all even positive integers less than 7 is described in roster form as {2, 4, 6}. Some more examples of representing a set in roster form are given below :

(a) The set of all natural numbers which divide 42 is {1, 2, 3, 6, 7, 14, 21, 42}.

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Note In roster form, the order in which the elements are listed is immaterial. Thus, the above set can also be represented as {1, 3, 7, 21, 2, 6, 14, 42}.

(b) The set of all vowels in the English alphabet is {*a, e, i, o, u*}.

(c) The set of odd natural numbers is represented by {1, 3, 5, . . .}. The dots tell us that the list of odd numbers continue indefinitely.

Note It may be noted that while writing the set in roster form an element is not generally repeated, i.e., all the elements are taken as distinct. For example, the set of letters forming the word 'SCHOOL' is { S, C, H, O, L} or {H, O, L, C, S}. Here, the order of listing elements has no relevance.

(ii) In set-builder form, all the elements of a set possess a single common property which is not possessed by any element outside the set. For example, in the set {*a, e, i, o, u*}, all the elements possess a common property, namely, each of them is a vowel in the English alphabet, and no other letter possess this property. Denoting this set by V, we write

V = {x : x is a vowel in English alphabet}

It may be observed that we describe the element of the set by using a symbol x (any other symbol like the letters y, z, etc. could be used) which is followed by a colon " : ". After the sign of colon, we write the characteristic property possessed by the elements of the set and then enclose the whole description within braces. The above description of the set V is read as "the set of all x such that x is a vowel of the English alphabet". In this description the braces stand for "the set of all", the colon stands for "such that". For example, the set

A = {x : x is a natural number and $3 < x < 10$} is read as "the set of all x such that x is a natural number and x lies between 3 and 10." Hence, the numbers 4, 5, 6, 7, 8 and 9 are the elements of the set A.

If we denote the sets described in (*a*), (*b*) and (*c*) above in roster form by A, B, C, respectively, then A, B, C can also be represented in set-builder form as follows:

A= {x : x is a natural number which divides 42}

B= {y : y is a vowel in the English alphabet}

C= {z : z is an odd natural number}

Example 1 Write the solution set of the equation $x^2 + x - 2 = 0$ in roster form.

Solution The given equation can be written as

$$(x - 1)\ (x + 2) = 0, \text{ i. e., } \quad x = 1, -2$$

Therefore, the solution set of the given equation can be written in roster form as {1, – 2}.

Example 2 Write the set {x : x is a positive integer and $x^2 < 40$} in the roster form.

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Solution The required numbers are 1, 2, 3, 4, 5, 6. So, the given set in the roster form is {1, 2, 3, 4, 5, 6}.

Example 3 Write the set A = {1, 4, 9, 16, 25, . . . }in set-builder form.

Solution We may write the set A as

$$A = \{x : x \text{ is the square of a natural number}\}$$

Alternatively, we can write

$$A = \{x : x = n^2, \text{ where } n \in \mathbf{N}\}$$

Example 4 Write the set $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\}$ in the set-builder form.

Solution We see that each member in the given set has the numerator one less than the denominator. Also, the numerator begin from 1 and do not exceed 6. Hence, in the set-builder form the given set is

$$\left\{x : x = \frac{n}{n+1}, \text{ where } n \text{ is a natural number and } 1 \leq n \leq 6\right\}$$

Example 5 Match each of the set on the left described in the roster form with the same set on the right described in the set-builder form :

(i) {P, R, I, N, C, A, L}	(a) { x : x is a positive integer and is a divisor of 18}
(ii) { 0 }	(b) { x : x is an integer and $x^2 – 9 = 0$}
(iii) {1, 2, 3, 6, 9, 18}	(c) {x : x is an integer and $x + 1 = 1$}
(iv) {3, –3}	(d) {x : x is a letter of the word PRINCIPAL}

Solution Since in (d), there are 9 letters in the word PRINCIPAL and two letters P and I are repeated, so (i) matches (d). Similarly, (ii) matches (c) as $x + 1 = 1$ implies $x = 0$. Also, 1, 2 ,3, 6, 9, 18 are all divisors of 18 and so (iii) matches (a). Finally, $x^2 – 9 = 0$ implies $x = 3, –3$ and so (iv) matches (b).

EXERCISE 1.1

1. Which of the following are sets ? Justify your answer.
   (i) The collection of all the months of a year beginning with the letter J.
   (ii) The collection of ten most talented writers of India.
   (iii) A team of eleven best-cricket batsmen of the world.
   (iv) The collection of all boys in your class.
   (v) The collection of all natural numbers less than 100.
   (vi) A collection of novels written by the writer Munshi Prem Chand.
   (vii) The collection of all even integers.

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(viii) The collection of questions in this Chapter.
(ix) A collection of most dangerous animals of the world.

2. Let A = {1, 2, 3, 4, 5, 6}. Insert the appropriate symbol $\in$ or $\notin$ in the blank spaces:

(i) 5. . .A (ii) 8 . . . A (iii) 0. . .A
(iv) 4. . . A (v) 2. . .A (vi) 10. . .A

3. Write the following sets in roster form:

(i) A = $\{x : x$ is an integer and $-3 \le x < 7\}$
(ii) B = $\{x : x$ is a natural number less than $6\}$
(iii) C = $\{x : x$ is a two-digit natural number such that the sum of its digits is $8\}$
(iv) D = $\{x : x$ is a prime number which is divisor of $60\}$
(v) E = The set of all letters in the word TRIGONOMETRY
(vi) F = The set of all letters in the word BETTER

4. Write the following sets in the set-builder form :

(i) (3, 6, 9, 12} (ii) {2,4,8,16,32} (iii) {5, 25, 125, 625}
(iv) {2, 4, 6, . . .} (v) {1,4,9, . . .,100}

5. List all the elements of the following sets :

(i) A = $\{x : x$ is an odd natural number$\}$
(ii) B = $\{x : x$ is an integer, $-\frac{1}{2} < x < \frac{9}{2}\}$
(iii) C = $\{x : x$ is an integer, $x^2 \le 4\}$
(iv) D = $\{x : x$ is a letter in the word "LOYAL"$\}$
(v) E = $\{x : x$ is a month of a year not having 31 days$\}$
(vi) F = $\{x : x$ is a consonant in the English alphabet which precedes $k\}$.

6. Match each of the set on the left in the roster form with the same set on the right described in set-builder form:

(i) {1, 2, 3, 6} (a) $\{x : x$ is a prime number and a divisor of $6\}$
(ii) {2, 3} (b) $\{x : x$ is an odd natural number less than $10\}$
(iii) {M,A,T,H,E,I,C,S} (c) $\{x : x$ is natural number and divisor of $6\}$
(iv) {1, 3, 5, 7, 9} (d) $\{x : x$ is a letter of the word MATHEMATICS$\}$.

1.3 The Empty Set

Consider the set

A = $\{ x : x$ is a student of Class XI presently studying in a school $\}$

We can go to the school and count the number of students presently studying in Class XI in the school. Thus, the set A contains a finite number of elements.

We now write another set B as follows:

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B = { x : x is a student presently studying in both Classes X and XI }

We observe that a student cannot study simultaneously in both Classes X and XI. Thus, the set B contains no element at all.

Definition 1 A set which does not contain any element is called the *empty set* or the *null set* or the *void set*.

According to this definition, B is an empty set while A is not an empty set. The empty set is denoted by the symbol ϕ or { }.

We give below a few examples of empty sets.

(i) Let A = $\{x : 1 < x < 2,\ x$ is a natural number$\}$. Then A is the empty set, because there is no natural number between 1 and 2.

(ii) B = $\{x : x^2 - 2 = 0$ and x is rational number$\}$. Then B is the empty set because the equation $x^2 - 2 = 0$ is not satisfied by any rational value of x.

(iii) C = $\{x : x$ is an even prime number greater than 2$\}$.Then C is the empty set, because 2 is the only even prime number.

(iv) D = $\{ x : x^2 = 4,\ x$ is odd $\}$. Then D is the empty set, because the equation $x^2 = 4$ is not satisfied by any odd value of x.

1.4 Finite and Infinite Sets

Let A = {1, 2, 3, 4, 5}, B = $\{a, b, c, d, e, g\}$

and C = { men living presently in different parts of the world}

We observe that A contains 5 elements and B contains 6 elements. How many elements does C contain? As it is, we do not know the number of elements in C, but it is some natural number which may be quite a big number. By number of elements of a set S, we mean the number of distinct elements of the set and we denote it by n (S). If n (S) is a natural number, then S is *non-empty finite* set.

Consider the set of natural numbers. We see that the number of elements of this set is not finite since there are infinite number of natural numbers. We say that the set of natural numbers is an infinite set. The sets A, B and C given above are finite sets and n(A) = 5, n(B) = 6 and n(C) = some finite number.

Definition 2 A set which is empty or consists of a definite number of elements is called *finite* otherwise, the set is called *infinite*.

Consider some examples :

(i) Let W be the set of the days of the week. Then W is finite.

(ii) Let S be the set of solutions of the equation $x^2 - 16 = 0$. Then S is finite.

(iii) Let G be the set of points on a line. Then G is infinite.

When we represent a set in the roster form, we write all the elements of the set within braces { }. It is not possible to write all the elements of an infinite set within braces { } because the numbers of elements of such a set is not finite. So, we represent

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some infinite set in the roster form by writing a few elements which clearly indicate the structure of the set followed (or preceded) by three dots.

For example, {1, 2, 3 . . .} is the set of natural numbers, {1, 3, 5, 7, . . .} is the set of odd natural numbers, {. . .,–3, –2, –1, 0,1, 2 ,3, . . .} is the set of integers. All these sets are infinite.

Note All infinite sets cannot be described in the roster form. For example, the set of real numbers cannot be described in this form, because the elements of this set do not follow any particular pattern.

Example 6 State which of the following sets are finite or infinite :

(i) $\{x : x \in \text{N and } (x - 1)(x - 2) = 0\}$
(ii) $\{x : x \in \text{N and } x^2 = 4\}$
(iii) $\{x : x \in \text{N and } 2x - 1 = 0\}$
(iv) $\{x : x \in \text{N and } x \text{ is prime}\}$
(v) $\{x : x \in \text{N and } x \text{ is odd}\}$

Solution (i) Given set = {1, 2}. Hence, it is finite.
(ii) Given set = {2}. Hence, it is finite.
(iii) Given set = ϕ. Hence, it is finite.
(iv) The given set is the set of all prime numbers and since set of prime numbers is infinite. Hence the given set is infinite
(v) Since there are infinite number of odd numbers, hence, the given set is infinite.

1.5 Equal Sets

Given two sets A and B, if every element of A is also an element of B and if every element of B is also an element of A, then the sets A and B are said to be equal. Clearly, the two sets have exactly the same elements.

Definition 3 Two sets A and B are said to be *equal* if they have exactly the same elements and we write A = B. Otherwise, the sets are said to be *unequal* and we write A ≠ B.

We consider the following examples :

(i) Let A = {1, 2, 3, 4} and B = {3, 1, 4, 2}. Then A = B.
(ii) Let A be the set of prime numbers less than 6 and P the set of prime factors of 30. Then A and P are equal, since 2, 3 and 5 are the only prime factors of 30 and also these are less than 6.

Note A set does not change if one or more elements of the set are repeated. For example, the sets A = {1, 2, 3} and B = {2, 2, 1, 3, 3} are equal, since each

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element of A is in B and vice-versa. That is why we generally do not repeat any element in describing a set.

Example 7 Find the pairs of equal sets, if any, give reasons:

A = {0}, B = $\{x : x > 15 \text{ and } x < 5\}$,

C = $\{x : x - 5 = 0\}$, D = $\{x: x^2 = 25\}$,

E = $\{x : x$ is an integral positive root of the equation $x^2 - 2x - 15 = 0\}$.

Solution Since 0 ∈ A and 0 does not belong to any of the sets B, C, D and E, it follows that, A ≠ B, A ≠ C, A ≠ D, A ≠ E.

Since B = ϕ but none of the other sets are empty. Therefore B ≠ C, B ≠ D and B ≠ E. Also C = {5} but –5 ∈ D, hence C ≠ D.

Since E = {5}, C = E. Further, D = {–5, 5} and E = {5}, we find that, D ≠ E. Thus, the only pair of equal sets is C and E.

Example 8 Which of the following pairs of sets are equal? Justify your answer.

(i) X, the set of letters in "ALLOY" and B, the set of letters in "LOYAL".

(ii) A = $\{n : n \in \mathrm{Z} \text{ and } n^2 \leq 4\}$ and B = $\{x : x \in \mathrm{R} \text{ and } x^2 - 3x + 2 = 0\}$.

Solution (i) We have, X = {A, L, L, O, Y}, B = {L, O, Y, A, L}. Then X and B are equal sets as repetition of elements in a set do not change a set. Thus,

$$X = \{A, L, O, Y\} = B$$

(ii) A = {–2, –1, 0, 1, 2}, B = {1, 2}. Since 0 ∈ A and 0 ∉ B, A and B are not equal sets.

EXERCISE 1.2

1. Which of the following are examples of the null set
   - (i) Set of odd natural numbers divisible by 2
   - (ii) Set of even prime numbers
   - (iii) $\{ x : x$ is a natural numbers, $x < 5$ and $x > 7 \}$
   - (iv) $\{ y : y$ is a point common to any two parallel lines$\}$
2. Which of the following sets are finite or infinite
   - (i) The set of months of a year
   - (ii) {1, 2, 3, . . .}
   - (iii) {1, 2, 3, . . .99, 100}
   - (iv) The set of positive integers greater than 100
   - (v) The set of prime numbers less than 99
3. State whether each of the following set is finite or infinite:
   - (i) The set of lines which are parallel to the x-axis
   - (ii) The set of letters in the English alphabet
   - (iii) The set of numbers which are multiple of 5

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(iv) The set of animals living on the earth

(v) The set of circles passing through the origin (0,0)

4. In the following, state whether A = B or not:

(i) $A = \{ a, b, c, d \}$ $B = \{ d, c, b, a \}$

(ii) $A = \{ 4, 8, 12, 16 \}$ $B = \{ 8, 4, 16, 18\}$

(iii) $A = \{2, 4, 6, 8, 10\}$ $B = \{ x : x$ is positive even integer and $x \leq 10\}$

(iv) $A = \{ x : x$ is a multiple of $10\}$, $B = \{ 10, 15, 20, 25, 30, \ldots \}$

5. Are the following pair of sets equal ? Give reasons.

(i) $A = \{2, 3\}$, $B = \{x : x$ is solution of $x^2 + 5x + 6 = 0\}$

(ii) $A = \{ x : x$ is a letter in the word FOLLOW$\}$

$B = \{ y : y$ is a letter in the word WOLF$\}$

6. From the sets given below, select equal sets :

$A = \{ 2, 4, 8, 12\}$, $B = \{ 1, 2, 3, 4\}$, $C = \{ 4, 8, 12, 14\}$, $D = \{ 3, 1, 4, 2\}$

$E = \{-1, 1\}$, $F = \{ 0, a\}$, $G = \{1, -1\}$, $H = \{ 0, 1\}$

1.6 Subsets

Consider the sets : X = set of all students in your school, Y = set of all students in your class.

We note that every element of Y is also an element of X; we say that Y is a subset of X. The fact that Y is subset of X is expressed in symbols as $Y \subset X$. The symbol $\subset$ stands for 'is a subset of' or 'is contained in'.

Definition 4 A set A is said to be a subset of a set B if every element of A is also an element of B.

In other words, $A \subset B$ if whenever $a \in A$, then $a \in B$. It is often convenient to use the symbol "$\Rightarrow$" which means *implies*. Using this symbol, we can write the definiton of *subset* as follows:

$$A \subset B \text{ if } a \in A \Rightarrow a \in B$$

We read the above statement as **"***A is a subset of B if a is an element of A implies that a is also an element of B***".** If A is not a subset of B, we write $A \not\subset B$.

We may note that for A to be a subset of B, all that is needed is that every element of A is in B. It is possible that every element of B may or may not be in A. If it so happens that every element of B is also in A, then we shall also have $B \subset A$. In this case, A and B are the same sets so that we have $A \subset B$ and $B \subset A \Leftrightarrow A = B$, where "$\Leftrightarrow$" is a symbol for two way implications, and is usually read as *if and only if* (briefly written as "iff").

It follows from the above definition that every set *A is a subset of itself,* i.e., $A \subset A$. Since the empty set ϕ has no elements, we agree to say that ϕ *is a subset of every set***.** We now consider some examples :

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(i) The set **Q** of rational numbers is a subset of the set **R** of real numbes, and we write **Q** $\subset$ R.

(ii) If A is the set of all divisors of 56 and B the set of all prime divisors of 56, then B is a subset of A and we write B $\subset$ A.

(iii) Let A = {1, 3, 5} and B = {x : x is an odd natural number less than 6}. Then A $\subset$ B and B $\subset$ A and hence A = B.

(iv) Let A = { a, e, i, o, u} and B = { a, b, c, d}. Then A is not a subset of B, also B is not a subset of A.

Let A and B be two sets. If A $\subset$ B and A $\neq$ B , then A is called a *proper subset of* B and B is called *superset* of A. For example,

A = {1, 2, 3} is a proper subset of B = {1, 2, 3, 4}.

If a set A has only one element, we call it a *singleton set*. Thus,{ a } is a singleton set.

Example 9 Consider the sets

$$\phi,\ A = \{ 1, 3 \},\quad B = \{1, 5, 9\},\quad C = \{1, 3, 5, 7, 9\}.$$

Insert the symbol $\subset$ or $\not\subset$ between each of the following pair of sets:

(i) ϕ . . . B (ii) A . . . B (iii) A . . . C (iv) B . . . C

Solution (i) $\phi \subset$ B as ϕ is a subset of every set.

(ii) A $\not\subset$ B as 3 $\in$ A and 3 $\notin$ B

(iii) A $\subset$ C as 1, 3 $\in$ A also belongs to C

(iv) B $\subset$ C as each element of B is also an element of C.

Example 10 Let A = { a, e, i, o, u} and B = { a, b, c, d}. Is A a subset of B ? No. (Why?). Is B a subset of A? No. (Why?)

Example 11 Let A, B and C be three sets. If A $\in$ B and B $\subset$ C, is it true that A $\subset$ C?. If not, give an example.

Solution No. Let A = {1}, B = {{1}, 2} and C = {{1}, 2, 3}. Here A $\in$ B as A = {1} and B $\subset$ C. But A $\not\subset$ C as 1 $\in$ A and 1 $\notin$ C.

Note that an element of a set can never be a subset of itself.

1.6.1 *Subsets of set of real numbers*

As noted in Section 1.6, there are many important subsets of **R**. We give below the names of some of these subsets.

The set of natural numbers **N** = {1, 2, 3, 4, 5, . . .}

The set of integers **Z** = {. . ., –3, –2, –1, 0, 1, 2, 3, . . .}

The set of rational numbers $\mathbf{Q} = \{ x : x = \dfrac{p}{q},\ p, q \in \mathbf{Z} \text{ and } q \neq 0\}$

which is read " **Q** is the set of all numbers x such that x equals the quotient $\frac{p}{q}$, where p and q are integers and q is not zero". Members of **Q** include -5 (which can be expressed as $-\frac{5}{1}$), $\frac{5}{7}$, $3\frac{1}{2}$ (which can be expressed as $\frac{7}{2}$) and $-\frac{11}{3}$.

The set of irrational numbers, denoted by **T**, is composed of all other real numbers. Thus $\mathbf{T} = \{x : x \in \mathbf{R} \text{ and } x \notin \mathbf{Q}\}$, i.e., all real numbers that are not rational. Members of **T** include $\sqrt{2}$, $\sqrt{5}$ and π.

Some of the obvious relations among these subsets are:

$\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q}$, $\mathbf{Q} \subset \mathbf{R}$, $\mathbf{T} \subset \mathbf{R}$, $\mathbf{N} \not\subset \mathbf{T}$.

1.6.2 ***Intervals as subsets of R*** Let $a, b \in \mathbf{R}$ and $a < b$. Then the set of real numbers $\{ y : a < y < b\}$ is called an *open interval* and is denoted by (a, b). All the points between a and b belong to the open interval (a, b) but a, b themselves do not belong to this interval.

The interval which contains the end points also is called *closed interval* and is denoted by $[a, b]$. Thus

$[a, b] = \{x : a \le x \le b\}$

We can also have intervals closed at one end and open at the other, i.e.,

$[a, b) = \{x : a \le x < b\}$ is an *open interval* from a to b, including a but excluding b.

$(a, b] = \{ x : a < x \le b \}$ is an *open interval* from a to b including b but excluding a.

These notations provide an alternative way of designating the subsets of set of real numbers. For example, if $A = (-3, 5)$ and $B = [-7, 9]$, then $A \subset B$. The set $[0, \infty)$ defines the set of non-negative real numbers, while set $(-\infty, 0)$ defines the set of negative real numbers. The set $(-\infty, \infty)$ describes the set of real numbers in relation to a line extending from $-\infty$ to ∞.

On real number line, various types of intervals described above as subsets of **R**, are shown in the Fig 1.1.

Fig 1.1

Here, we note that an interval contains infinitely many points.

For example, the set $\{x : x \in \mathbf{R}, -5 < x \le 7\}$, written in set-builder form, can be written in the form of interval as $(-5, 7]$ and the interval $[-3, 5)$ can be written in set-builder form as $\{x : -3 \le x < 5\}$.

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The number $(b - a)$ is called the *length of any of the intervals* (a, b), $[a, b]$, $[a, b)$ or $(a, b]$.

1.7 Universal Set

Usually, in a particular context, we have to deal with the elements and subsets of a basic set which is relevant to that particular context. For example, while studying the system of numbers, we are interested in the set of natural numbers and its subsets such as the set of all prime numbers, the set of all even numbers, and so forth. This basic set is called the "*Universal Set*". The universal set is usually denoted by U, and all its subsets by the letters A, B, C, etc.

For example, for the set of all integers, the universal set can be the set of rational numbers or, for that matter, the set **R** of real numbers. For another example, in human population studies, the universal set consists of all the people in the world.

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EXERCISE 1.3

1. Make correct statements by filling in the symbols $\subset$ or $\not\subset$ in the blank spaces :

(i) $\{ 2, 3, 4 \} \ldots \{ 1, 2, 3, 4, 5 \}$ (ii) $\{ a, b, c \} \ldots \{ b, c, d \}$

(iii) $\{x : x$ is a student of Class XI of your school$\} \ldots \{x : x$ student of your school$\}$

(iv) $\{x : x$ is a circle in the plane$\} \ldots \{x : x$ is a circle in the same plane with radius 1 unit$\}$

(v) $\{x : x$ is a triangle in a plane$\} \ldots \{x : x$ is a rectangle in the plane$\}$

(vi) $\{x : x$ is an equilateral triangle in a plane$\} \ldots \{x : x$ is a triangle in the same plane$\}$

(vii) $\{x : x$ is an even natural number$\} \ldots \{x : x$ is an integer$\}$

2. Examine whether the following statements are true or false:

(i) $\{ a, b \} \not\subset \{ b, c, a \}$

(ii) $\{ a, e \} \subset \{ x : x$ is a vowel in the English alphabet$\}$

(iii) $\{ 1, 2, 3 \} \subset \{ 1, 3, 5 \}$

(iv) $\{ a \} \subset \{ a, b, c \}$

(v) $\{ a \} \in \{ a, b, c \}$

(vi) $\{ x : x$ is an even natural number less than $6\} \subset \{ x : x$ is a natural number which divides $36\}$

3. Let $A = \{ 1, 2, \{ 3, 4 \}, 5 \}$. Which of the following statements are incorrect and why?

(i) $\{3, 4\} \subset A$ (ii) $\{3, 4\} \in A$ (iii) $\{\{3, 4\}\} \subset A$

(iv) $1 \in A$ (v) $1 \subset A$ (vi) $\{1, 2, 5\} \subset A$

(vii) $\{1, 2, 5\} \in A$ (viii) $\{1, 2, 3\} \subset A$ (ix) $\phi \in A$

(x) $\phi \subset A$ (xi) $\{\phi\} \subset A$

4. Write down all the subsets of the following sets

(i) $\{a\}$ (ii) $\{a, b\}$ (iii) $\{1, 2, 3\}$ (iv) ϕ

5. Write the following as intervals :

(i) $\{x : x \in \mathrm{R}, -4 < x \leq 6\}$ (ii) $\{x : x \in \mathrm{R}, -12 < x < -10\}$

(iii) $\{x : x \in \mathrm{R}, 0 \leq x < 7\}$ (iv) $\{x : x \in \mathrm{R}, 3 \leq x \leq 4\}$

6. Write the following intervals in set-builder form :

(i) $(-3, 0)$ (ii) $[6, 12]$ (iii) $(6, 12]$ (iv) $[-23, 5)$

7. What universal set(s) would you propose for each of the following :

(i) The set of right triangles. (ii) The set of isosceles triangles.

8. Given the sets A = {1, 3, 5}, B = {2, 4, 6} and C = {0, 2, 4, 6, 8}, which of the following may be considered as universal set (s) for all the three sets A, B and C

(i) {0, 1, 2, 3, 4, 5, 6}

(ii) ϕ

(iii) {0,1,2,3,4,5,6,7,8,9,10}

(iv) {1,2,3,4,5,6,7,8}

1.8 Venn Diagrams

Most of the relationships between sets can be represented by means of diagrams which are known as *Venn diagrams*. Venn diagrams are named after the English logician, John Venn (1834-1883). These diagrams consist of rectangles and closed curves usually circles. The universal set is represented usually by a rectangle and its subsets by circles.

In Venn diagrams, the elements of the sets are written in their respective circles (Figs 1.2 and 1.3)

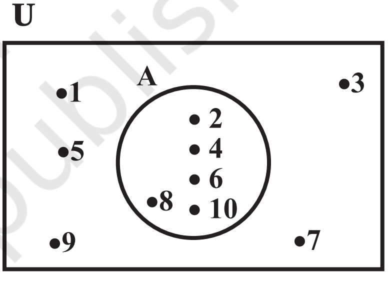

Fig 1.2

Illustration 1 In Fig 1.2, U = {1,2,3, ..., 10} is the universal set of which

A = {2,4,6,8,10} is a subset.

Illustration 2 In Fig 1.3, U = {1,2,3, ..., 10} is the universal set of which

A = {2,4,6,8,10} and B = {4, 6} are subsets, and also B ⊂ A.

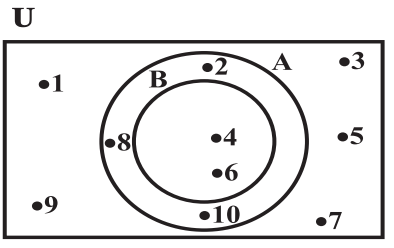

Fig 1.3

The reader will see an extensive use of the Venn diagrams when we discuss the union, intersection and difference of sets.

1.9 Operations on Sets

In earlier classes, we have learnt how to perform the operations of addition, subtraction, multiplication and division on numbers. Each one of these operations was performed on a pair of numbers to get another number. For example, when we perform the operation of addition on the pair of numbers 5 and 13, we get the number 18. Again, performing the operation of multiplication on the pair of numbers 5 and 13, we get 65.

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Similarly, there are some operations which when performed on two sets give rise to another set. We will now define certain operations on sets and examine their properties. Henceforth, we will refer all our sets as subsets of some universal set.

1.9.1 ***Union of sets*** Let A and B be any two sets. The union of A and B is the set which consists of all the elements of A and all the elements of B, the common elements being taken only once. The symbol '$\cup$' is used to denote the *union*. *Symbolically, we write* $A \cup B$ *and usually read as 'A union B'*.

Example 12 Let $A = \{2, 4, 6, 8\}$ and $B = \{6, 8, 10, 12\}$. Find $A \cup B$.

Solution We have $A \cup B = \{2, 4, 6, 8, 10, 12\}$

Note that the common elements 6 and 8 have been taken only once while writing $A \cup B$.

Example 13 Let $A = \{a, e, i, o, u\}$ and $B = \{a, i, u\}$. Show that $A \cup B = A$

Solution We have, $A \cup B = \{a, e, i, o, u\} = A$.

This example illustrates that union of sets A and its subset B is the set A itself, i.e., if $B \subset A$, then $A \cup B = A$.

Example 14 Let X = {Ram, Geeta, Akbar} be the set of students of Class XI, who are in school hockey team. Let Y = {Geeta, David, Ashok} be the set of students from Class XI who are in the school football team. Find $X \cup Y$ and interpret the set.

Solution We have, $X \cup Y$ = {Ram, Geeta, Akbar, David, Ashok}. This is the set of students from Class XI who are in the hockey team or the football team or both.

Thus, we can define the union of two sets as follows:

Definition 5 The union of two sets A and B is the set C which consists of all those elements which are either in A or in B (including those which are in both). In symbols, we write. $A \cup B = \{x : x \in A \text{ or } x \in B\}$

The union of two sets can be represented by a Venn diagram as shown in Fig 1.4.

The shaded portion in Fig 1.4 represents $A \cup B$.

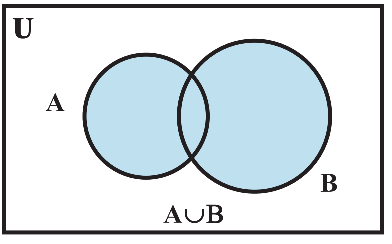

Fig 1.4

Some Properties of the Operation of Union

(i) $A \cup B = B \cup A$ (Commutative law)

(ii) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)

(iii) $A \cup \phi = A$ (Law of identity element, ϕ is the identity of $\cup$)

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(iv) $A \cup A = A$ (Idempotent law)

(v) $U \cup A = U$ (Law of U)

1.9.2 *Intersection of sets* The intersection of sets A and B is the set of all elements which are common to both A and B. The symbol '$\cap$' is used to denote the *intersection*. The intersection of two sets A and B is the set of all those elements which belong to both A and B. Symbolically, we write $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Example 15 Consider the sets A and B of Example 12. Find $A \cap B$.

Solution We see that 6, 8 are the only elements which are common to both A and B. Hence $A \cap B = \{6, 8\}$.

Example 16 Consider the sets X and Y of Example 14. Find $X \cap Y$.

Solution We see that element 'Geeta' is the only element common to both. Hence, $X \cap Y = \{\text{Geeta}\}$.

Example 17 Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $B = \{2, 3, 5, 7\}$. Find $A \cap B$ and hence show that $A \cap B = B$.

Solution We have $A \cap B = \{2, 3, 5, 7\} = B$. We note that $B \subset A$ and that $A \cap B = B$.

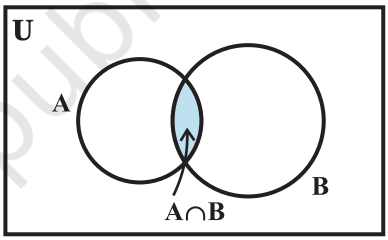


Fig 1.5

Definition 6 The intersection of two sets A and B is the set of all those elements which belong to both A and B. Symbolically, we write

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

The shaded portion in Fig 1.5 indicates the intersection of A and B.

If A and B are two sets such that $A \cap B = \phi$, then A and B are called *disjoint sets*.

For example, let $A = \{2, 4, 6, 8\}$ and $B = \{1, 3, 5, 7\}$. Then A and B are disjoint sets, because there are no elements which are common to A and B. The disjoint sets can be represented by means of Venn diagram as shown in the Fig 1.6

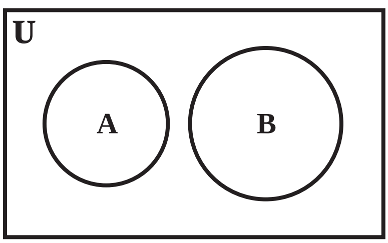


Fig 1.6

In the above diagram, A and B are disjoint sets.

Some Properties of Operation of Intersection

(i) $A \cap B = B \cap A$ (Commutative law).

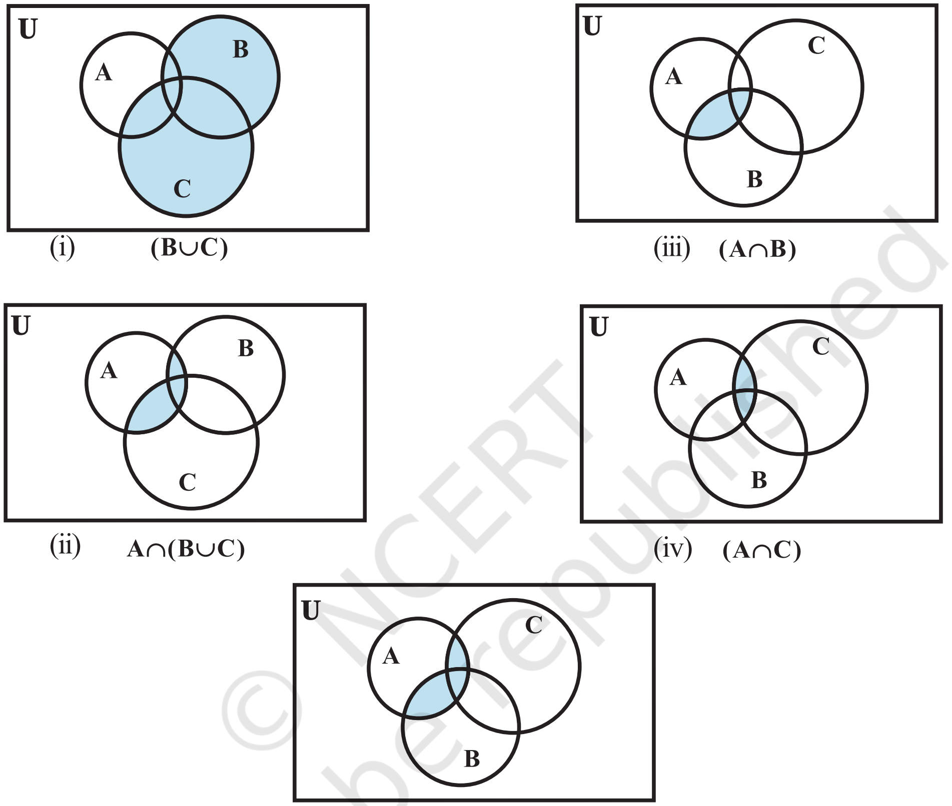
(ii) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law).

(iii) $\phi \cap A = \phi$, $U \cap A = A$ (Law of ϕ and U).

(iv) $A \cap A = A$ (Idempotent law)

(v) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive law) i. e., $\cap$ distributes over $\cup$

This can be seen easily from the following Venn diagrams [Figs 1.7 (i) to (v)].



(i) $(B \cup C)$

(ii) $A \cap (B \cup C)$

(iii) $(A \cap B)$

(iv) $(A \cap C)$

(v) $(A \cap B) \cup (A \cap C)$

Figs 1.7 (i) to (v)

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1.9.3 ***Difference of sets*** The difference of the sets A and B in this order is the set of elements which belong to A but not to B. Symbolically, we write A – B and read as "A minus B".

Example 18 Let A = { 1, 2, 3, 4, 5, 6}, B = { 2, 4, 6, 8 }. Find A – B and B – A.

Solution We have, A – B = { 1, 3, 5 }, since the elements 1, 3, 5 belong to A but not to B and B – A = { 8 }, since the element 8 belongs to B and not to A.
We note that A – B ≠ B – A.

Example 19 Let V = { *a, e, i, o, u* } and B = { *a, i, k, u*}. Find V – B and B – V

Solution We have, V – B = { *e, o* }, since the elements *e, o* belong to V but not to B and B – V = { *k* }, since the element *k* belongs to B but not to V.

We note that V – B ≠ B – V. Using the set-builder notation, we can rewrite the definition of difference as

$$A – B = \{ x : x \in A \text{ and } x \notin B \}$$

The difference of two sets A and B can be represented by Venn diagram as shown in Fig 1.8.

The shaded portion represents the difference of the two sets A and B.

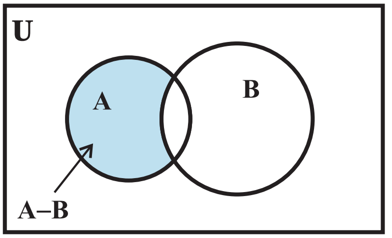

Fig 1.8

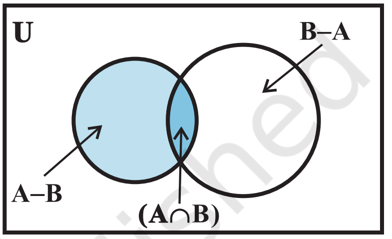

Fig 1.9

Remark The sets A – B, A ∩ B and B – A are mutually disjoint sets, i.e., the intersection of any of these two sets is the null set as shown in Fig 1.9.

EXERCISE 1.4

1. Find the union of each of the following pairs of sets :
   (i) X = {1, 3, 5} Y = {1, 2, 3}
   (ii) A = [*a, e, i, o, u*} B = {*a, b, c*}
   (iii) A = {x : x is a natural number and multiple of 3}
   B = {x : x is a natural number less than 6}
   (iv) A = {x : x is a natural number and $1 < x \le 6$ }
   B = {x : x is a natural number and $6 < x < 10$ }
   (v) A = {1, 2, 3}, B = ϕ
2. Let A = { *a, b* }, B = {*a, b, c*}. Is A ⊂ B ? What is A ∪ B ?
3. If A and B are two sets such that A ⊂ B, then what is A ∪ B ?
4. If A = {1, 2, 3, 4}, B = {3, 4, 5, 6}, C = {5, 6, 7, 8 }and D = { 7, 8, 9, 10 }; find
   (i) A ∪ B (ii) A ∪ C (iii) B ∪ C (iv) B ∪ D
   (v) A ∪ B ∪ C (vi) A ∪ B ∪ D (vii) B ∪ C ∪ D
5. Find the intersection of each pair of sets of question 1 above.
6. If A = { 3, 5, 7, 9, 11 }, B = {7, 9, 11, 13}, C = {11, 13, 15}and D = {15, 17}; find
   (i) A ∩ B (ii) B ∩ C (iii) A ∩ C ∩ D
   (iv) A ∩ C (v) B ∩ D (vi) A ∩ (B ∪ C)
   (vii) A ∩ D (viii) A ∩ (B ∪ D) (ix) (A ∩ B) ∩ (B ∪ C)
   (x) (A ∪ D) ∩ (B ∪ C)

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7. If A = $\{x : x$ is a natural number $\}$, B = $\{x : x$ is an even natural number$\}$
C = $\{x : x$ is an odd natural number$\}$ and D = $\{x : x$ is a prime number $\}$, find
(i) $A \cap B$ (ii) $A \cap C$ (iii) $A \cap D$
(iv) $B \cap C$ (v) $B \cap D$ (vi) $C \cap D$

8. Which of the following pairs of sets are disjoint
(i) $\{1, 2, 3, 4\}$ and $\{x : x$ is a natural number and $4 \leq x \leq 6\}$
(ii) $\{a, e, i, o, u\}$ and $\{c, d, e, f\}$
(iii) $\{x : x$ is an even integer $\}$ and $\{x : x$ is an odd integer$\}$

9. If A = $\{3, 6, 9, 12, 15, 18, 21\}$, B = $\{4, 8, 12, 16, 20\}$,
C = $\{2, 4, 6, 8, 10, 12, 14, 16\}$, D = $\{5, 10, 15, 20\}$; find
(i) $A - B$ (ii) $A - C$ (iii) $A - D$ (iv) $B - A$
(v) $C - A$ (vi) $D - A$ (vii) $B - C$ (viii) $B - D$
(ix) $C - B$ (x) $D - B$ (xi) $C - D$ (xii) $D - C$

10. If X = $\{a, b, c, d\}$ and Y = $\{f, b, d, g\}$, find
(i) $X - Y$ (ii) $Y - X$ (iii) $X \cap Y$

11. If **R** is the set of real numbers and **Q** is the set of rational numbers, then what is **R** – **Q**?

12. State whether each of the following statement is true or false. Justify your answer.
(i) $\{2, 3, 4, 5\}$ and $\{3, 6\}$ are disjoint sets.
(ii) $\{a, e, i, o, u\}$ and $\{a, b, c, d\}$ are disjoint sets.
(iii) $\{2, 6, 10, 14\}$ and $\{3, 7, 11, 15\}$ are disjoint sets.
(iv) $\{2, 6, 10\}$ and $\{3, 7, 11\}$ are disjoint sets.

1.10 Complement of a Set

Let U be the universal set which consists of all prime numbers and A be the subset of U which consists of all those prime numbers that are not divisors of 42. Thus, A = $\{x : x \in$ U and x is not a divisor of 42 $\}$. We see that $2 \in$ U but $2 \notin$ A, because 2 is divisor of 42. Similarly, $3 \in$ U but $3 \notin$ A, and $7 \in$ U but $7 \notin$ A. Now 2, 3 and 7 are the only elements of U which do not belong to A. The set of these three prime numbers, i.e., the set $\{2, 3, 7\}$ is called the *Complement* of A with respect to U, and is denoted by A′. So we have A′ = $\{2, 3, 7\}$. Thus, we see that

A′ = $\{x : x \in$ U and $x \notin$ A $\}$. This leads to the following definition.

Definition 7 Let U be the universal set and A a subset of U. Then the complement of A is the set of all elements of U which are not the elements of A. Symbolically, we write A′ to denote the complement of A with respect to U. Thus,

A′ = $\{x : x \in$ U and $x \notin$ A $\}$. Obviously A′ = U – A

We note that the complement of a set A can be looked upon, alternatively, as the difference between a universal set U and the set A.

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Example 20 Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and A = {1, 3, 5, 7, 9}. Find A′.

Solution We note that 2, 4, 6, 8, 10 are the only elements of U which do not belong to A. Hence A′ = { 2, 4, 6, 8,10 }.

Example 21 Let U be universal set of all the students of Class XI of a coeducational school and A be the set of all girls in Class XI. Find A′.

Solution Since A is the set of all girls, A′ is clearly the set of all boys in the class.

Note If A is a subset of the universal set U, then its complement A′ is also a subset of U.

Again in Example 20 above, we have A′ = { 2, 4, 6, 8, 10 }

Hence $(A')' = \{x : x \in U \text{ and } x \notin A'\}$

$= \{1, 3, 5, 7, 9\} = A$

It is clear from the definition of the complement that for any subset of the universal set U, we have $(A')' = A$

Now, we want to find the results for $(A \cup B)'$ and $A' \cap B'$ in the followng example.

Example 22 Let U = {1, 2, 3, 4, 5, 6}, A = {2, 3} and B = {3, 4, 5}.
Find A′, B′, $A' \cap B'$, $A \cup B$ and hence show that $(A \cup B)' = A' \cap B'$.

Solution Clearly A′ = {1, 4, 5, 6}, B′ = { 1, 2, 6 }. Hence $A' \cap B'$ = { 1, 6 }

Also $A \cup B$ = { 2, 3, 4, 5 }, so that $(A \cup B)'$ = { 1, 6 }

$$(A \cup B)' = \{ 1, 6 \} = A' \cap B'$$

It can be shown that the above result is true in general. If A and B are any two subsets of the universal set U, then

$(A \cup B)' = A' \cap B'$. Similarly, $(A \cap B)' = A' \cup B'$. These two results are stated in words as follows :

The complement of the union of two sets is the intersection of their complements and the complement of the intersection of two sets is the union of their complements. These are called *De Morgan's laws*. These are named after the mathematician De Morgan.

The complement A′ of a set A can be represented by a Venn diagram as shown in Fig 1.10.

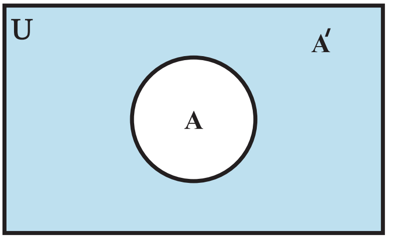

Fig 1.10

The shaded portion represents the complement of the set A.

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Some Properties of Complement Sets

1. Complement laws: (i) $A \cup A' = U$ (ii) $A \cap A' = \phi$

2. De Morgan's law: (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

3. Law of double complementation : $(A')' = A$

4. Laws of empty set and universal set $\phi' = U$ and $U' = \phi$.

These laws can be verified by using Venn diagrams.

EXERCISE 1.5

1. Let U = { 1, 2, 3, 4, 5, 6, 7, 8, 9 }, A = { 1, 2, 3, 4}, B = { 2, 4, 6, 8 } and C = { 3, 4, 5, 6 }. Find (i) A' (ii) B' (iii) $(A \cup C)'$ (iv) $(A \cup B)'$ (v) $(A')'$ (vi) $(B - C)'$

2. If U = { a, b, c, d, e, f, g, h}, find the complements of the following sets :
(i) A = {a, b, c} (ii) B = {d, e, f, g}
(iii) C = {a, c, e, g} (iv) D = { f, g, h, a}

3. Taking the set of natural numbers as the universal set, write down the complements of the following sets:
(i) {x : x is an even natural number} (ii) { x : x is an odd natural number }
(iii) {x : x is a positive multiple of 3} (iv) { x : x is a prime number }
(v) {x : x is a natural number divisible by 3 and 5}
(vi) { x : x is a perfect square } (vii) { x : x is a perfect cube}
(viii) { $x : x + 5 = 8$ } (ix) { $x : 2x + 5 = 9$}
(x) { $x : x \geq 7$ } (xi) { $x : x \in N$ and $2x + 1 > 10$ }

4. If U = {1, 2, 3, 4, 5, 6, 7, 8, 9 }, A = {2, 4, 6, 8} and B = { 2, 3, 5, 7}. Verify that
(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

5. Draw appropriate Venn diagram for each of the following :
(i) $(A \cup B)'$, (ii) $A' \cap B'$, (iii) $(A \cap B)'$, (iv) $A' \cup B'$

6. Let U be the set of all triangles in a plane. If A is the set of all triangles with at least one angle different from 60°, what is A'?

7. Fill in the blanks to make each of the following a true statement :
(i) $A \cup A' = \ldots$ (ii) $\phi' \cap A = \ldots$
(iii) $A \cap A' = \ldots$ (iv) $U' \cap A = \ldots$

Miscellaneous Examples

Example 23 Show that the set of letters needed to spell " CATARACT " and the set of letters needed to spell " TRACT" are equal.

Solution Let X be the set of letters in "CATARACT". Then

$$X = \{ C, A, T, R \}$$

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Let Y be the set of letters in " TRACT". Then

$$Y = \{ T, R, A, C, T \} = \{ T, R, A, C \}$$

Since every element in X is in Y and every element in Y is in X. It follows that X = Y.

Example 24 List all the subsets of the set $\{ -1, 0, 1 \}$.

Solution Let $A = \{ -1, 0, 1 \}$. The subset of A having no element is the empty set ϕ. The subsets of A having one element are $\{ -1 \}, \{ 0 \}, \{ 1 \}$. The subsets of A having two elements are $\{-1, 0\}, \{-1, 1\}, \{0, 1\}$. The subset of A having three elements of A is A itself. So, all the subsets of A are ϕ, $\{-1\}, \{0\}, \{1\}, \{-1, 0\}, \{-1, 1\}$, $\{0, 1\}$ and $\{-1, 0, 1\}$.

Example 25 Show that $A \cup B = A \cap B$ implies $A = B$

Solution Let $a \in A$. Then $a \in A \cup B$. Since $A \cup B = A \cap B$, $a \in A \cap B$. So $a \in B$. Therefore, $A \subset B$. Similarly, if $b \in B$, then $b \in A \cup B$. Since

$A \cup B = A \cap B$, $b \in A \cap B$. So, $b \in A$. Therefore, $B \subset A$. Thus, $A = B$

Miscellaneous Exercise on Chapter 1

1. Decide, among the following sets, which sets are subsets of one and another:
$A = \{ x : x \in \mathbf{R}$ and x satisfy $x^2 - 8x + 12 = 0 \}$,
$B = \{ 2, 4, 6 \}$, $C = \{ 2, 4, 6, 8, \ldots \}$, $D = \{ 6 \}$.

2. In each of the following, determine whether the statement is true or false. If it is true, prove it. If it is false, give an example.

(i) If $x \in A$ and $A \in B$, then $x \in B$

(ii) If $A \subset B$ and $B \in C$, then $A \in C$

(iii) If $A \subset B$ and $B \subset C$, then $A \subset C$

(iv) If $A \not\subset B$ and $B \not\subset C$, then $A \not\subset C$

(v) If $x \in A$ and $A \not\subset B$, then $x \in B$

(vi) If $A \subset B$ and $x \notin B$, then $x \notin A$

3. Let A, B, and C be the sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Show that $B = C$.

4. Show that the following four conditions are equivalent :

(i) $A \subset B$ (ii) $A - B = \phi$ (iii) $A \cup B = B$ (iv) $A \cap B = A$

5. Show that if $A \subset B$, then $C - B \subset C - A$.

6. Show that for any sets A and B,
$A = (A \cap B) \cup (A - B)$ and $A \cup (B - A) = (A \cup B)$

7. Using properties of sets, show that
(i) $A \cup (A \cap B) = A$ (ii) $A \cap (A \cup B) = A$.

8. Show that $A \cap B = A \cap C$ need not imply $B = C$.

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9. Let A and B be sets. If A $\cap$ X = B $\cap$ X = ϕ and A $\cup$ X = B $\cup$ X for some set X, show that A = B.
(**Hints** A = A $\cap$ (A $\cup$ X) , B = B $\cap$ (B $\cup$ X) and use Distributive law)

10. Find sets A, B and C such that A $\cap$ B, B $\cap$ C and A $\cap$ C are non-empty sets and A $\cap$ B $\cap$ C = ϕ.

Summary

This chapter deals with some basic definitions and operations involving sets. These are summarised below:

- A set is a well-defined collection of objects.
- A set which does not contain any element is called *empty set*.
- A set which consists of a definite number of elements is called *finite set*, otherwise, the set is called *infinite set*.
- Two sets A and B are said to be equal if they have exactly the same elements.
- A set A is said to be subset of a set B, if every element of A is also an element of B. Intervals are subsets of **R**.
- The union of two sets A and B is the set of all those elements which are either in A or in B.
- The intersection of two sets A and B is the set of all elements which are common. The difference of two sets A and B in this order is the set of elements which belong to A but not to B.
- The complement of a subset A of universal set U is the set of all elements of U which are not the elements of A.
- For any two sets A and B, (A $\cup$ B)′ = A′ $\cap$ B′ and (A $\cap$ B)′ = A′ $\cup$ B′

Historical Note

The modern theory of sets is considered to have been originated largely by the German mathematician Georg Cantor (1845-1918). His papers on set theory appeared sometimes during 1874 to 1897. His study of set theory came when he was studying trigonometric series of the form $a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + ...$ He published in a paper in 1874 that the set of real numbers could not be put into one-to-one correspondence wih the integers. From 1879 onwards, he publishd several papers showing various properties of abstract sets.

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Cantor's work was well received by another famous mathematician Richard Dedekind (1831-1916). But Kronecker (1810-1893) castigated him for regarding infinite set the same way as finite sets. Another German mathematician Gottlob Frege, at the turn of the century, presented the set theory as principles of logic. Till then the entire set theory was based on the assumption of the existence of the set of all sets. It was the famous Englih Philosopher Bertand Russell (1872-1970) who showed in 1902 that the assumption of existence of a set of all sets leads to a contradiction. This led to the famous Russell's Paradox. Paul R.Halmos writes about it in his book 'Naïve Set Theory' that "nothing contains everything".

The Russell's Paradox was not the only one which arose in set theory. Many paradoxes were produced later by several mathematicians and logicians. As a consequence of all these paradoxes, the first axiomatisation of set theory was published in 1908 by Ernst Zermelo. Another one was proposed by Abraham Fraenkel in 1922. John Von Neumann in 1925 introduced explicitly the axiom of regularity. Later in 1937 Paul Bernays gave a set of more satisfactory axiomatisation. A modification of these axioms was done by Kurt Gödel in his monograph in 1940. This was known as Von Neumann-Bernays (VNB) or Gödel-Bernays (GB) set theory.

Despite all these difficulties, Cantor's set theory is used in present day mathematics. In fact, these days most of the concepts and results in mathematics are expressed in the set theoretic language.

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Chapter 2

RELATIONS AND FUNCTIONS

❖*Mathematics is the indispensable instrument of all physical research. – BERTHELOT* ❖

2.1 Introduction

G. W. Leibnitz
(1646–1716)

Much of mathematics is about finding a pattern – a recognisable link between quantities that change. In our daily life, we come across many patterns that characterise relations such as brother and sister, father and son, teacher and student. In mathematics also, we come across many relations such as number m is less than number n, line l is parallel to line m, set A is a subset of set B. In all these, we notice that a relation involves pairs of objects in certain order. In this Chapter, we will learn how to link pairs of objects from two sets and then introduce relations between the two objects in the pair. Finally, we will learn about special relations which will qualify to be functions. The concept of function is very important in mathematics since it captures the idea of a mathematically precise correspondence between one quantity with the other.

2.2 Cartesian Products of Sets

Suppose A is a set of 2 colours and B is a set of 3 objects, i.e.,

$$A = \{\text{red, blue}\} \text{ and } B = \{b, c, s\},$$

where b, c and s represent a particular bag, coat and shirt, respectively.

How many pairs of coloured objects can be made from these two sets? Proceeding in a very orderly manner, we can see that there will be 6 distinct pairs as given below:

(red, b), (red, c), (red, s), (blue, b), (blue, c), (blue, s).

Thus, we get 6 distinct objects (Fig 2.1).

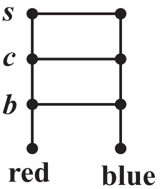

Fig 2.1

Let us recall from our earlier classes that an ordered pair of elements taken from any two sets P and Q is a pair of elements written in small

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brackets and grouped together in a particular order, i.e., (p,q), $p \in$ P and $q \in$ Q . This leads to the following definition:

Definition 1 Given two non-empty sets P and Q. The cartesian product P × Q is the set of all ordered pairs of elements from P and Q, i.e.,

$$\mathrm{P} \times \mathrm{Q} = \{ (p,q) : p \in \mathrm{P}, q \in \mathrm{Q} \}$$

If either P or Q is the null set, then P × Q will also be empty set, i.e., P × Q = ϕ

From the illustration given above we note that

A × B = {(red,b), (red,c), (red,s), (blue,b), (blue,c), (blue,s)}.

Again, consider the two sets:

A = {DL, MP, KA}, where DL, MP, KA represent Delhi, Madhya Pradesh and Karnataka, respectively and B = {01,02, 03}representing codes for the licence plates of vehicles issued by DL, MP and KA .

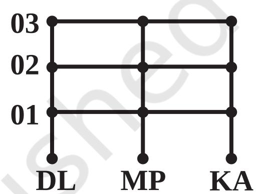

Fig 2.2

If the three states, Delhi, Madhya Pradesh and Karnataka were making codes for the licence plates of vehicles, with the restriction that the code begins with an element from set A, which are the pairs available from these sets and how many such pairs will there be (Fig 2.2)?

The available pairs are:(DL,01), (DL,02), (DL,03), (MP,01), (MP,02), (MP,03), (KA,01), (KA,02), (KA,03) and the product of set A and set B is given by

A × B = {(DL,01), (DL,02), (DL,03), (MP,01), (MP,02), (MP,03), (KA,01), (KA,02), (KA,03)}.

It can easily be seen that there will be 9 such pairs in the Cartesian product, since there are 3 elements in each of the sets A and B. This gives us 9 possible codes. Also note that the order in which these elements are paired is crucial. For example, the code (DL, 01) will not be the same as the code (01, DL).

As a final illustration, consider the two sets A= $\{a_1, a_2\}$ and B = $\{b_1, b_2, b_3, b_4\}$ (Fig 2.3).

A × B = $\{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_1, b_4), (a_2, b_1), (a_2, b_2), (a_2, b_3), (a_2, b_4)\}$.

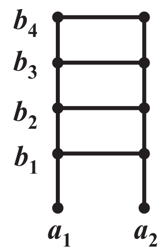

Fig 2.3

The 8 ordered pairs thus formed can represent the position of points in the plane if A and B are subsets of the set of real numbers and it is obvious that the point in the position (a_1, b_2) will be distinct from the point in the position (b_2, a_1).

Remarks

(i) Two ordered pairs are equal, if and only if the corresponding first elements are equal and the second elements are also equal.

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(ii) If there are p elements in A and q elements in B, then there will be pq elements in A × B, i.e., if $n(\text{A}) = p$ and $n(\text{B}) = q$, then $n(\text{A} \times \text{B}) = pq$.

(iii) If A and B are non-empty sets and either A or B is an infinite set, then so is A × B.

(iv) $\text{A} \times \text{A} \times \text{A} = \{(a, b, c) : a, b, c \in \text{A}\}$. Here (a, b, c) is called an *ordered triplet*.

Example 1 If $(x + 1, y - 2) = (3,1)$, find the values of x and y.

Solution Since the ordered pairs are equal, the corresponding elements are equal.

Therefore $x + 1 = 3$ and $y - 2 = 1$.

Solving we get $x = 2$ and $y = 3$.

Example 2 If $\text{P} = \{a, b, c\}$ and $\text{Q} = \{r\}$, form the sets P × Q and Q × P. Are these two products equal?

Solution By the definition of the cartesian product,

$$\text{P} \times \text{Q} = \{(a, r), (b, r), (c, r)\} \text{ and } \text{Q} \times \text{P} = \{(r, a), (r, b), (r, c)\}$$

Since, by the definition of equality of ordered pairs, the pair (a, r) is not equal to the pair (r, a), we conclude that $\text{P} \times \text{Q} \neq \text{Q} \times \text{P}$.

However, the number of elements in each set will be the same.

Example 3 Let $\text{A} = \{1,2,3\}$, $\text{B} = \{3,4\}$ and $\text{C} = \{4,5,6\}$. Find

(i) $\text{A} \times (\text{B} \cap \text{C})$ (ii) $(\text{A} \times \text{B}) \cap (\text{A} \times \text{C})$

(iii) $\text{A} \times (\text{B} \cup \text{C})$ (iv) $(\text{A} \times \text{B}) \cup (\text{A} \times \text{C})$

Solution (i) By the definition of the intersection of two sets, $(\text{B} \cap \text{C}) = \{4\}$.

Therefore, $\text{A} \times (\text{B} \cap \text{C}) = \{(1,4), (2,4), (3,4)\}$.

(ii) Now $(\text{A} \times \text{B}) = \{(1,3), (1,4), (2,3), (2,4), (3,3), (3,4)\}$

and $(\text{A} \times \text{C}) = \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)\}$

Therefore, $(\text{A} \times \text{B}) \cap (\text{A} \times \text{C}) = \{(1, 4), (2, 4), (3, 4)\}$.

(iii) Since, $(\text{B} \cup \text{C}) = \{3, 4, 5, 6\}$, we have

$\text{A} \times (\text{B} \cup \text{C}) = \{(1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6)\}$.

(iv) Using the sets A × B and A × C from part (ii) above, we obtain

$(\text{A} \times \text{B}) \cup (\text{A} \times \text{C}) = \{(1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6)\}$.

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Example 4 If P = {1, 2}, form the set P × P × P.

Solution We have, P × P × P = {(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)}.

Example 5 If **R** is the set of all real numbers, what do the cartesian products **R** × **R** and **R** × **R** × **R** represent?

Solution The Cartesian product **R** × **R** represents the set **R** × **R**={$(x, y) : x, y \in$ **R**} which represents the *coordinates of all the points in two dimensional space* and the cartesian product **R** × **R** × **R** represents the set **R** × **R** × **R** ={$(x, y, z) : x, y, z \in$ **R**} which represents the *coordinates of all the points in three-dimensional space*.

Example 6 If A × B ={$(p, q),(p, r), (m, q), (m, r)$}, find A and B.

Solution A = set of first elements = {p, m}
B = set of second elements = {q, r}.

EXERCISE 2.1

1. If $\left(\frac{x}{3}+1, y-\frac{2}{3}\right)=\left(\frac{5}{3}, \frac{1}{3}\right)$, find the values of x and y.
2. If the set A has 3 elements and the set B = {3, 4, 5}, then find the number of elements in (A×B).
3. If G = {7, 8} and H = {5, 4, 2}, find G × H and H × G.
4. State whether each of the following statements are true or false. If the statement is false, rewrite the given statement correctly.
   (i) If P = {m, n} and Q = {n, m}, then P × Q = {$(m, n),(n, m)$}.
   (ii) If A and B are non-empty sets, then A × B is a non-empty set of ordered pairs (x, y) such that $x \in$ A and $y \in$ B.
   (iii) If A = {1, 2}, B = {3, 4}, then A × (B ∩ φ) = φ.
5. If A = {–1, 1}, find A × A × A.
6. If A × B = {$(a, x),(a, y), (b, x), (b, y)$}. Find A and B.
7. Let A = {1, 2}, B = {1, 2, 3, 4}, C = {5, 6} and D = {5, 6, 7, 8}. Verify that
   (i) A × (B ∩ C) = (A × B) ∩ (A × C). (ii) A × C is a subset of B × D.
8. Let A = {1, 2} and B = {3, 4}. Write A × B. How many subsets will A × B have? List them.
9. Let A and B be two sets such that n(A) = 3 and n(B) = 2. If $(x, 1), (y, 2), (z, 1)$ are in A × B, find A and B, where x, y and z are distinct elements.

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10. The Cartesian product A × A has 9 elements among which are found (–1, 0) and (0,1). Find the set A and the remaining elements of A × A.

2.3 Relations

Consider the two sets P = {*a*, *b*, *c*} and Q = {Ali, Bhanu, Binoy, Chandra, Divya}. The cartesian product of P and Q has 15 ordered pairs which can be listed as P × Q = {(*a*, Ali), (*a*, Bhanu), (*a*, Binoy), ..., (*c*, Divya)}.

We can now obtain a subset of P × Q by introducing a relation R between the first element *x* and the second element *y* of each ordered pair (*x*, *y*) as

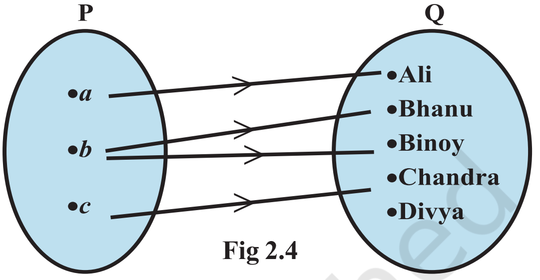

Fig 2.4

R= { (x,y): x is the first letter of the name y, $x \in$ P, $y \in$ Q}.

Then R = {(*a*, Ali), (*b*, Bhanu), (*b*, Binoy), (*c*, Chandra)}

A visual representation of this relation R (called an *arrow diagram*) is shown in Fig 2.4.

Definition 2 A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product A × B. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in A × B. The second element is called the *image* of the first element.

Definition 3 The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the *domain* of the relation R.

Definition 4 The set of all second elements in a relation R from a set A to a set B is called the *range* of the relation R. The whole set B is called the *codomain* of the relation R. Note that range ⊂ codomain.

Remarks (i) A *relation* may be represented algebraically either by the *Roster method* or by the *Set-builder method*.

(ii) An arrow diagram is a visual representation of a relation.

Example 7 Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by

R = $\{(x, y) : y = x + 1\}$

(i) Depict this relation using an arrow diagram.

(ii) Write down the domain, codomain and range of R.

Solution (i) By the definition of the relation,

R = {(1,2), (2,3), (3,4), (4,5), (5,6)}.

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The corresponding arrow diagram is shown in Fig 2.5.

(ii) We can see that the domain $= \{1, 2, 3, 4, 5,\}$

Similarly, the range $= \{2, 3, 4, 5, 6\}$ and the codomain $= \{1, 2, 3, 4, 5, 6\}$.

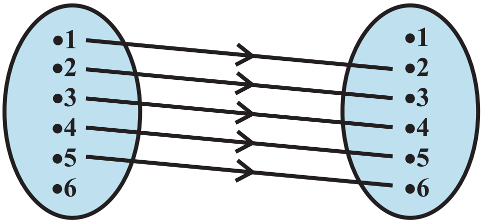

Fig 2.5

Example 8 The Fig 2.6 shows a relation between the sets P and Q. Write this relation (i) in set-builder form, (ii) in roster form. What is its domain and range?

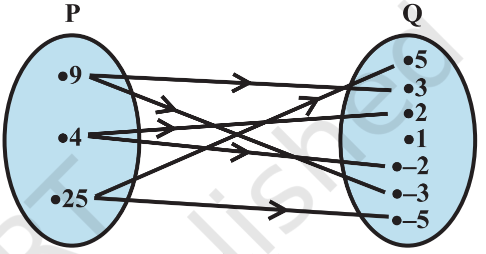

Fig 2.6

Solution It is obvious that the relation R is "x is the square of y".

(i) In set-builder form, $R = \{(x, y): x$ is the square of $y, x \in P, y \in \mathbf{Q}\}$

(ii) In roster form, $R = \{(9, 3), (9, –3), (4, 2), (4, –2), (25, 5), (25, –5)\}$

The domain of this relation is $\{4, 9, 25\}$.

The range of this relation is $\{– 2, 2, –3, 3, –5, 5\}$.

Note that the element 1 is not related to any element in set P.

The set Q is the codomain of this relation.

> **Note** The total number of relations that can be defined from a set A to a set B is the number of possible subsets of $A \times B$. If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$ and the total number of relations is 2^{pq}.

Example 9 Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Find the number of relations from A to B.

Solution We have,

$$A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}.$$

Since $n(A\times B) = 4$, the number of subsets of $A\times B$ is 2^4. Therefore, the number of relations from A into B will be 2^4.

Remark A relation R from A to A is also stated as a relation on A.

EXERCISE 2.2

1. Let $A = \{1, 2, 3,...,14\}$. Define a relation R from A to A by $R = \{(x, y) : 3x – y = 0$, where $x, y \in A\}$. Write down its domain, codomain and range.

2. Define a relation R on the set **N** of natural numbers by R = $\{(x, y) : y = x + 5$, x is a natural number less than 4; $x, y \in \mathbf{N}\}$. Depict this relationship using roster form. Write down the domain and the range.

3. A = {1, 2, 3, 5} and B = {4, 6, 9}. Define a relation R from A to B by R = $\{(x, y)$: the difference between x and y is odd; $x \in$ A, $y \in$ B$\}$. Write R in roster form.

4. The Fig2.7 shows a relationship between the sets P and Q. Write this relation

(i) in set-builder form (ii) roster form. What is its domain and range?

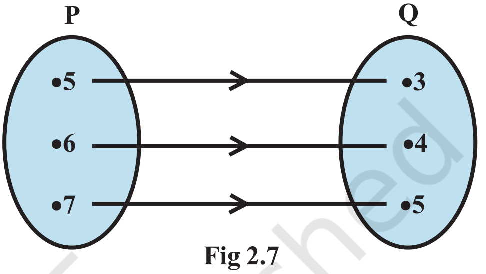

Fig 2.7

5. Let A = {1, 2, 3, 4, 6}. Let R be the relation on A defined by $\{(a, b): a , b \in A, b$ is exactly divisible by $a\}$.

(i) Write R in roster form

(ii) Find the domain of R

(iii) Find the range of R.

6. Determine the domain and range of the relation R defined by R = $\{(x, x + 5) : x \in \{0, 1, 2, 3, 4, 5\}\}$.

7. Write the relation R = $\{(x, x^3) : x$ is a prime number less than 10$\}$ in roster form.

8. Let A = $\{x, y, z\}$ and B = {1, 2}. Find the number of relations from A to B.

9. Let R be the relation on **Z** defined by R = $\{(a,b): a, b \in \mathbf{Z}, a - b$ is an integer$\}$. Find the domain and range of R.

2.4 Functions

In this Section, we study a special type of relation called *function.* It is one of the most important concepts in mathematics. We can, visualise a function as a rule, which produces new elements out of some given elements. There are many terms such as 'map' or 'mapping' used to denote a function.

Definition 5 A relation f from a set A to a set B is said to be a *function* if every element of set A has one and only one image in set B.

In other words, a function f is a relation from a non-empty set A to a non-empty set B such that the domain of f is A and no two distinct ordered pairs in f have the same first element.

If f is a function from A to B and $(a, b) \in f$, then $f(a) = b$, where b is called the *image* of a under f and a is called the *preimage* of b under f.

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The function f from A to B is denoted by $f: A \rightarrow B$.
Looking at the previous examples, we can easily see that the relation in Example 7 is not a function because the element 6 has no image.

Again, the relation in Example 8 is not a function because the elements in the domain are connected to more than one images. Similarly, the relation in Example 9 is also not a function. (*Why*?) In the examples given below, we will see many more relations some of which are functions and others are not.

Example 10 Let **N** be the set of natural numbers and the relation R be defined on N such that $R = \{(x, y) : y = 2x, x, y \in \mathbf{N}\}$.

What is the domain, codomain and range of R? Is this relation a function?

Solution The domain of R is the set of natural numbers **N**. The codomain is also **N**. The range is the set of even natural numbers.

Since every natural number n has one and only one image, this relation is a function.

Example 11 Examine each of the following relations given below and state in each case, giving reasons whether it is a function or not?

(i) $R = \{(2,1),(3,1), (4,2)\}$, (ii) $R = \{(2,2),(2,4),(3,3), (4,4)\}$

(iii) $R = \{(1,2),(2,3),(3,4), (4,5), (5,6), (6,7)\}$

Solution (i) Since 2, 3, 4 are the elements of domain of R having their unique images, this relation R is a function.

(ii) Since the same first element 2 corresponds to two different images 2 and 4, this relation is not a function.

(iii) Since every element has one and only one image, this relation is a function.

Definition 6 A function which has either R or one of its subsets as its range is called a *real valued function*. Further, if its domain is also either R or a subset of R, it is called a *real function*.

Example 12 Let **N** be the set of natural numbers. Define a real valued function $f : \mathbf{N} \rightarrow \mathbf{N}$ by $f(x) = 2x + 1$. Using this definition, complete the table given below.

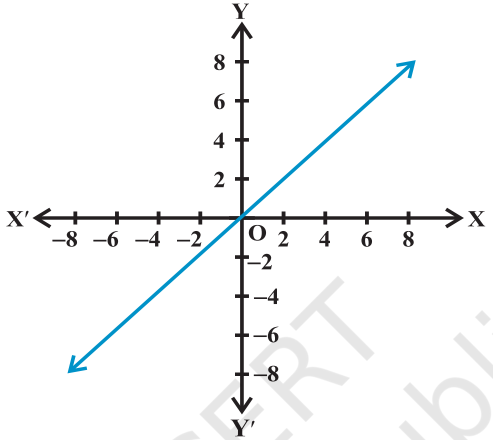
x	1	2	3	4	5	6	7
y	$f(1) = \ldots$	$f(2) = \ldots$	$f(3) = \ldots$	$f(4) = \ldots$	$f(5) = \ldots$	$f(6) = \ldots$	$f(7) = \ldots$

Solution The completed table is given by

x	1	2	3	4	5	6	7
y	$f(1) = 3$	$f(2) = 5$	$f(3) = 7$	$f(4) = 9$	$f(5) = 11$	$f(6) = 13$	$f(7) = 15$

2.4.1 *Some functions and their graphs*

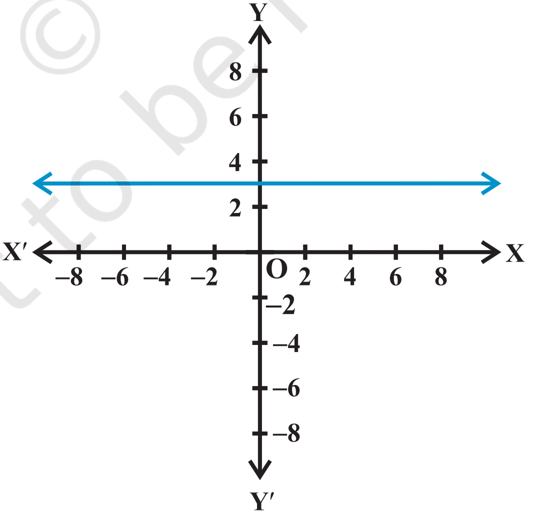
(i) **Identity function** Let **R** be the set of real numbers. Define the real valued function f : **R** $\rightarrow$ **R** by $y = f(x) = x$ for each $x \in$ **R**. Such a function is called the *identity function*. Here the domain and range of f are **R**. The graph is a straight line as shown in Fig 2.8. It passes through the origin.



$f(x) = x$

Fig 2.8

(ii) **Constant function** Define the function f: **R** $\rightarrow$ **R** by $y = f(x) = c$, $x \in$ **R** where c is a constant and each $x \in$ **R**. Here domain of f is **R** and its range is $\{c\}$.



$f(x) = 3$

Fig 2.9

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The graph is a line parallel to x-axis. For example, if $f(x)=3$ for each $x \in \mathbf{R}$, then its graph will be a line as shown in the Fig 2.9.

(iii) **Polynomial function** A function $f : \mathbf{R} \to \mathbf{R}$ is said to be *polynomial function* if for each x in $\mathbf{R}$, $y = f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$, where n is a non-negative integer and $a_0, a_1, a_2, ..., a_n \in \mathbf{R}$.

The functions defined by $f(x) = x^3 - x^2 + 2$, and $g(x) = x^4 + \sqrt{2}\, x$ are some examples of polynomial functions, whereas the function h defined by $h(x) = x^{\frac{2}{3}} + 2x$ is not a polynomial function.(*Why*?)

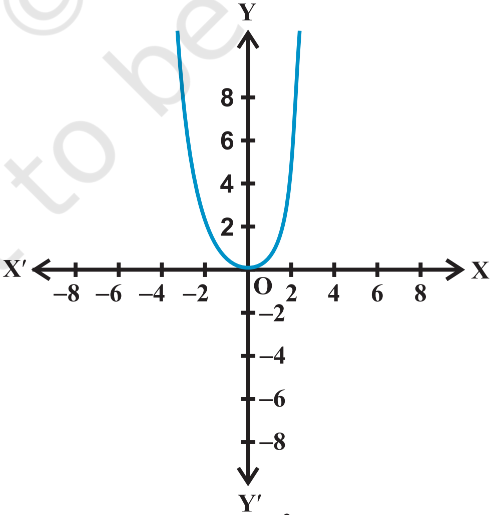
Example 13 Define the function $f: \mathbf{R} \to \mathbf{R}$ by $y = f(x) = x^2$, $x \in \mathbf{R}$. Complete the Table given below by using this definition. What is the domain and range of this function? Draw the graph of f.

x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = x^2$									

Solution The completed Table is given below:

x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = x^2$	16	9	4	1	0	1	4	9	16

Domain of $f = \{x : x \in \mathbf{R}\}$. Range of $f = \{x^2 : x \in \mathbf{R}\}$. The graph of f is given by Fig 2.10



$f(x) = x^2$ **Fig 2.10**

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Example 14 Draw the graph of the function f :**R** → **R** defined by $f(x) = x^3$, $x \in$ **R**.

Solution We have

$f(0) = 0$, $f(1) = 1$, $f(-1) = -1$, $f(2) = 8$, $f(-2) = -8$, $f(3) = 27$; $f(-3) = -27$, etc.

Therefore, $f = \{(x, x^3): x \in \mathbf{R}\}$.

The graph of f is given in Fig 2.11.

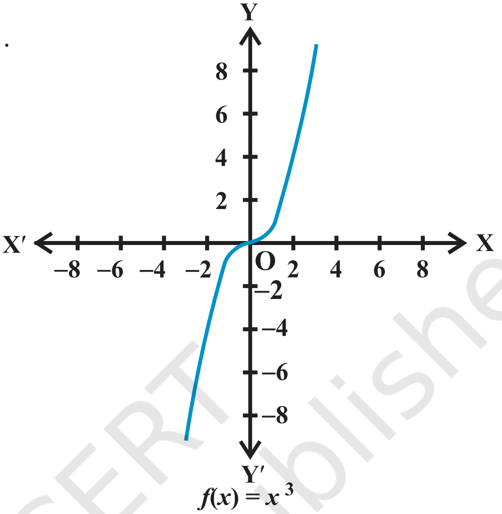


Fig 2.11

(iv) **Rational functions** are functions of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$.

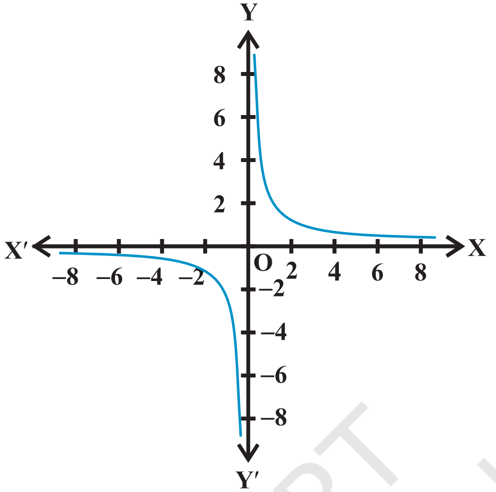
Example 15 Define the real valued function f : **R** – {0} → **R** defined by $f(x) = \frac{1}{x}$, $x \in \mathbf{R} - \{0\}$. Complete the Table given below using this definition. What is the domain and range of this function?

x	–2	–1.5	–1	–0.5	0.25	0.5	1	1.5	2
$y = \frac{1}{x}$	...	...	...	...	...	...	...	...	...

Solution The completed Table is given by

x	–2	–1.5	–1	–0.5	0.25	0.5	1	1.5	2
$y = \frac{1}{x}$	– 0.5	– 0.67	–1	– 2	4	2	1	0.67	0.5

The domain is all real numbers except 0 and its range is also all real numbers except 0. The graph of f is given in Fig 2.12.



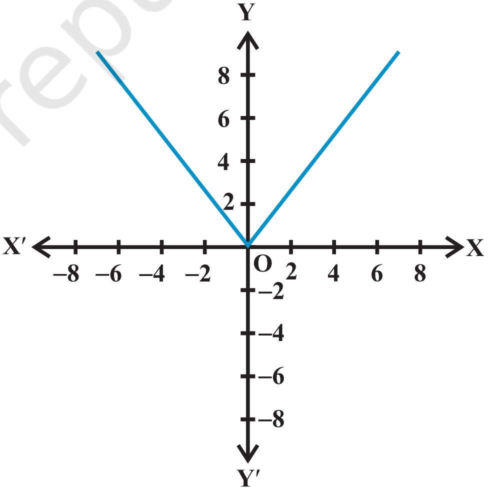
$f(x) = \frac{1}{x}$

Fig 2.12

(v) **The Modulus function** The function f: $\mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = |x|$ for each $x \in \mathbf{R}$ is called *modulus function*. For each non-negative value of x, $f(x)$ is equal to x. But for negative values of x, the value of $f(x)$ is the negative of the value of x, i.e.,

$$f(x) = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

The graph of the modulus function is given in Fig 2.13.



$f(x) = |x|$

Fig 2.13

(vi) **Signum function** The function f:$\mathbf{R} \rightarrow \mathbf{R}$ defined by

$$f(x) = \begin{cases} 1, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ -1, \text{if } x < 0 \end{cases}$$

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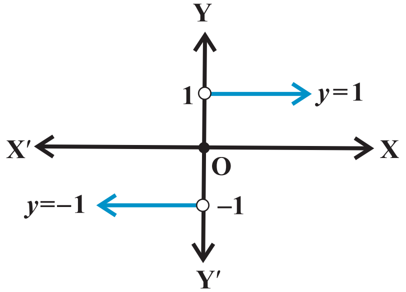
is called the *signum function*. The domain of the signum function is **R** and the range is the set $\{-1, 0, 1\}$. The graph of the signum function is given by the Fig 2.14.



$$f(x) = \frac{|x|}{x},\ x \neq 0 \text{ and } 0 \text{ for } x = 0$$

Fig 2.14

(vii) **Greatest integer function** The function f: **R** → **R** defined by $f(x) = [x]$, $x \in$ **R** assumes the value of the greatest integer, less than or equal to x. Such a function is called the *greatest integer function*.

From the definition of $[x]$, we can see that

$[x] = -1$ for $-1 \leq x < 0$

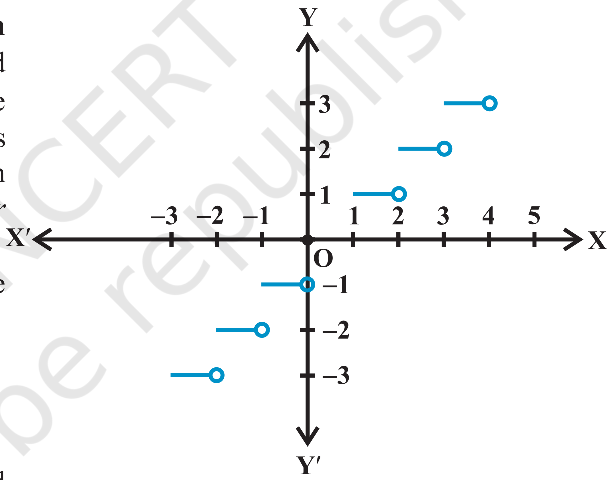
$[x] = 0$ for $0 \leq x < 1$

$[x] = 1$ for $1 \leq x < 2$

$[x] = 2$ for $2 \leq x < 3$ and

so on.

The graph of the function is shown in Fig 2.15.



$f(x) = [x]$

Fig 2.15

2.4.2 ***Algebra of real functions*** In this Section, we shall learn how to add two real functions, subtract a real function from another, multiply a real function by a scalar (here by a scalar we mean a real number), multiply two real functions and divide one real function by another.

(i) **Addition of two real functions** Let $f : X \to$ **R** and $g : X \to$ **R** be any two real functions, where $X \subset$ **R**. Then, we define $(f + g)$: X → **R** by

$(f + g)(x) = f(x) + g(x)$, for all $x \in X$.

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(ii) **Subtraction of a real function from another** Let $f: X \to \mathbf{R}$ and $g: X \to \mathbf{R}$ be any two real functions, where $X \subset \mathbf{R}$. Then, we define $(f - g) : X \to \mathbf{R}$ by $(f-g)(x) = f(x) - g(x)$, for all $x \in X$.

(iii) **Multiplication by a scalar** Let $f: X \to \mathbf{R}$ be a real valued function and α be a scalar. Here by scalar, we mean a real number. Then the product αf is a function from X to $\mathbf{R}$ defined by $(\alpha f)(x) = \alpha f(x)$, $x \in X$.

(iv) **Multiplication of two real functions** The product (or multiplication) of two real functions $f: X \to \mathbf{R}$ and $g: X \to \mathbf{R}$ is a function $fg: X \to \mathbf{R}$ defined by $(fg)(x) = f(x)\, g(x)$, for all $x \in X$.
This is also called *pointwise multiplication.*

(v) **Quotient of two real functions** Let f and g be two real functions defined from $X \to \mathbf{R}$, where $X \subset \mathbf{R}$. The quotient of f by g denoted by $\frac{f}{g}$ is a function defined by ,

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$, $x \in X$

Example 16 Let $f(x) = x^2$ and $g(x) = 2x + 1$ be two real functions.Find

$$(f+g)(x),\ (f-g)(x),\ (fg)(x), \left(\frac{f}{g}\right)(x).$$

Solution We have,

$$(f+g)(x) = x^2 + 2x + 1,\quad (f-g)(x) = x^2 - 2x - 1,$$

$$(fg)(x) = x^2(2x+1) = 2x^3 + x^2,\quad \left(\frac{f}{g}\right)(x) = \frac{x^2}{2x+1},\ x \neq -\frac{1}{2}$$

Example 17 Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$.

Solution We have

$$(f+g)(x) = \sqrt{x} + x,\quad (f-g)(x) = \sqrt{x} - x,$$

$$(fg)\,x = \sqrt{x}(x) = x^{\frac{3}{2}} \text{ and } \left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{x} = x^{-\frac{1}{2}},\ x \neq 0$$

EXERCISE 2.3

1. Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range.

(i) $\{(2,1), (5,1), (8,1), (11,1), (14,1), (17,1)\}$

(ii) $\{(2,1), (4,2), (6,3), (8,4), (10,5), (12,6), (14,7)\}$

(iii) $\{(1,3), (1,5), (2,5)\}$.

2. Find the domain and range of the following real functions:

(i) $f(x) = -|x|$ (ii) $f(x) = \sqrt{9-x^2}$.

3. A function f is defined by $f(x) = 2x - 5$. Write down the values of

(i) $f(0)$, (ii) $f(7)$, (iii) $f(-3)$.

4. The function 't' which maps temperature in degree Celsius into temperature in degree Fahrenheit is defined by $t(C) = \frac{9C}{5} + 32$.

Find (i) $t(0)$ (ii) $t(28)$ (iii) $t(-10)$ (iv) The value of C, when $t(C) = 212$.

5. Find the range of each of the following functions.

(i) $f(x) = 2 - 3x,\ x \in \mathbf{R},\ x > 0$.

(ii) $f(x) = x^2 + 2$, x is a real number.

(iii) $f(x) = x$, x is a real number.

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Miscellaneous Examples

Example 18 Let **R** be the set of real numbers. Define the real function

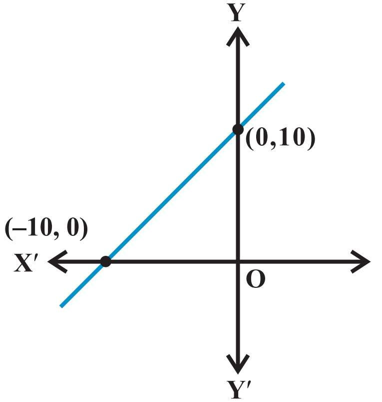
$$f: \mathbf{R} \rightarrow \mathbf{R} \text{ by } f(x) = x + 10$$

and sketch the graph of this function.

Solution Here $f(0) = 10, f(1) = 11, f(2) = 12, ..., f(10) = 20$, etc., and

$f(-1) = 9, f(-2) = 8, ..., f(-10) = 0$ and so on.

Therefore, shape of the graph of the given function assumes the form as shown in Fig 2.16.



$f(x) = x + 10$

Fig 2.16

Remark The function f defined by $f(x) = mx + c$, $x \in \mathbf{R}$, is called *linear function*, where m and c are constants. Above function is an example of a *linear function*.

Example 19 Let R be a relation from **Q** to **Q** defined by R = {(a,b): a,b ∈ **Q** and $a - b$ ∈ **Z**}. Show that

(i) $(a,a) \in$ R for all $a \in$ **Q**

(ii) $(a,b) \in$ R implies that $(b, a) \in$ R

(iii) $(a,b) \in$ R and $(b,c) \in$ R implies that $(a,c) \in$R

Solution (i) Since, $a - a = 0 \in$ **Z**, if follows that $(a, a) \in$ R.

(ii) $(a,b) \in$ R implies that $a - b \in$ **Z**. So, $b - a \in$ **Z**. Therefore, $(b, a) \in$ R

(iii) (a, b) and $(b, c) \in$ R implies that $a - b \in$ **Z**. $b - c \in$ **Z**. So, $a - c = (a - b) + (b - c) \in$ **Z**. Therefore, $(a,c) \in$ R

Example 20 Let f = {(1,1), (2,3), (0,–1), (–1,–3)} be a linear function from **Z** into **Z**. Find $f(x)$.

Solution Since f is a linear function, $f(x) = mx + c$. Also, since (1, 1), (0, – 1) ∈ R, $f(1) = m + c = 1$ and $f(0) = c = -1$. This gives $m = 2$ and $f(x) = 2x - 1$.

Example 21 Find the domain of the function $f(x) = \dfrac{x^2 + 3x + 5}{x^2 - 5x + 4}$

Solution Since $x^2 - 5x + 4 = (x - 4)(x - 1)$, the function f is defined for all real numbers except at $x = 4$ and $x = 1$. Hence the domain of f is **R** – {1, 4}.

Example 22 The function f is defined by

$$f(x) = \begin{cases} 1 - x, & x < 0 \\ 1, & x = 0 \\ x + 1, & x > 0 \end{cases}$$

Draw the graph of $f(x)$.

Solution Here, $f(x) = 1 - x,\ x < 0$, this gives

$f(-4) = 1 - (-4) = 5$;

$f(-3) = 1 - (-3) = 4$,

$f(-2) = 1 - (-2) = 3$

$f(-1) = 1 - (-1) = 2$; etc,

and $f(1) = 2, f(2) = 3, f(3) = 4$

$f(4) = 5$ and so on for $f(x) = x + 1,\ x > 0$.

Thus, the graph of f is as shown in Fig 2.17

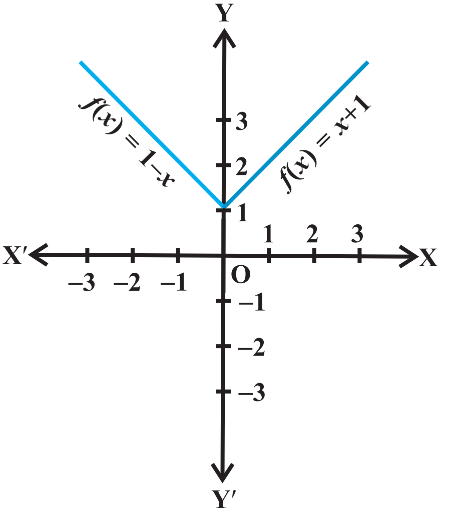


Fig 2.17

Miscellaneous Exercise on Chapter 2

1. The relation f is defined by $f(x)=\begin{cases} x^2, 0\le x\le 3\\ 3x, 3\le x\le 10\end{cases}$

The relation g is defined by $g(x)=\begin{cases} x^2, 0\le x\le 2\\ 3x, 2\le x\le 10\end{cases}$

Show that f is a function and g is not a function.

2. If $f(x) = x^2$, find $\dfrac{f(1.1)-f(1)}{(1.1-1)}$.

3. Find the domain of the function $f(x) = \dfrac{x^2+2x+1}{x^2-8x+12}$.

4. Find the domain and the range of the real function f defined by $f(x) = \sqrt{(x-1)}$.

5. Find the domain and the range of the real function f defined by $f(x) = |x-1|$.

6. Let $f = \left\{\left(x, \dfrac{x^2}{1+x^2}\right) : x \in \mathbf{R}\right\}$ be a function from **R** into **R**. Determine the range of f.

7. Let $f, g : \mathbf{R}\rightarrow\mathbf{R}$ be defined, respectively by $f(x) = x + 1$, $g(x) = 2x - 3$. Find $f + g$, $f - g$ and $\dfrac{f}{g}$.

8. Let $f = \{(1,1), (2,3), (0,-1), (-1, -3)\}$ be a function from **Z** to **Z** defined by $f(x) = ax + b$, for some integers a, b. Determine a, b.

9. Let R be a relation from **N** to **N** defined by $R = \{(a, b) : a, b \in \mathbf{N} \text{ and } a = b^2\}$. Are the following true?

(i) $(a,a) \in R$, for all $a \in \mathbf{N}$ (ii) $(a,b) \in R$, implies $(b,a) \in R$

(iii) $(a,b) \in R$, $(b,c) \in R$ implies $(a,c) \in R$.

Justify your answer in each case.

10. Let $A=\{1,2,3,4\}$, $B = \{1,5,9,11,15,16\}$ and $f = \{(1,5), (2,9), (3,1), (4,5), (2,11)\}$ Are the following true?

(i) f is a relation from A to B (ii) f is a function from A to B.

Justify your answer in each case.

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11. Let f be the subset of $\mathbf{Z} \times \mathbf{Z}$ *defined by* $f = \{(ab, a + b) : a, b \in \mathbf{Z}\}$. Is f a function from $\mathbf{Z}$ to $\mathbf{Z}$? Justify your answer.

12. Let A = {9,10,11,12,13} and let f : A→**N** be defined by $f(n)$ = the highest prime factor of n. Find the range of f.

Summary

In this Chapter, we studied about relations and functions.The main features of this Chapter are as follows:

- ***Ordered pair*** A pair of elements grouped together in a particular order.
- ***Cartesian product*** A × B of two sets A and B is given by

  $A \times B = \{(a, b): a \in A, b \in B\}$

  In particular $\mathbf{R} \times \mathbf{R} = \{(x, y): x, y \in \mathbf{R}\}$

  and $\mathbf{R} \times \mathbf{R} \times \mathbf{R} = \{(x, y, z): x, y, z \in \mathbf{R}\}$
- If $(a, b) = (x, y)$, then $a = x$ and $b = y$.
- If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.
- $A \times \phi = \phi$
- In general, $A \times B \neq B \times A$.
- ***Relation*** A relation R from a set A to a set B is a subset of the cartesian product A × B obtained by describing a relationship between the first element x and the second element y of the ordered pairs in A × B.
- The ***image*** of an element x under a relation R is given by y, where $(x, y) \in R$,
- The ***domain*** of R is the set of all first elements of the ordered pairs in a relation R.
- The ***range*** of the relation R is the set of all second elements of the ordered pairs in a relation R.
- ***Function*** A function f from a set A to a set B is a specific type of relation for which every element x of set A has one and only one image y in set B.

  We write f: A→B, where $f(x) = y$.
- A is the domain and B is the codomain of f.

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- The range of the function is the set of images.
- A real function has the set of real numbers or one of its subsets both as its domain and as its range.
- ***Algebra of functions*** For functions $f: X \rightarrow \mathbf{R}$ and $g: X \rightarrow \mathbf{R}$, we have

$$(f+g)(x) = f(x) + g(x),\ x \in X$$

$$(f-g)(x) = f(x) - g(x),\ x \in X$$

$$(f.g)(x) = f(x) .g(x),\ x \in X$$

$$(kf)(x) = k(f(x)),\ x \in X, \text{ where } k \text{ is a real number.}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ x \in X,\ g(x) \neq 0$$

Historical Note

The word FUNCTION first appears in a Latin manuscript "Methodus tangentium inversa, seu de fuctionibus" written by Gottfried Wilhelm Leibnitz (1646-1716) in 1673; Leibnitz used the word in the non-analytical sense. He considered a function in terms of "mathematical job" – the "employee" being just a curve.

On July 5, 1698, Johan Bernoulli, in a letter to Leibnitz, for the first time deliberately assigned a specialised use of the term *function* in the analytical sense. At the end of that month, Leibnitz replied showing his approval.

Function is found in English in 1779 in Chambers' Cyclopaedia: "The term function is used in algebra, for an analytical expression any way compounded of a variable quantity, and of numbers, or constant quantities".

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Chapter 3

TRIGONOMETRIC FUNCTIONS

❖ A mathematician knows how to solve a problem, he can not solve it. – MILNE ❖

3.1 Introduction

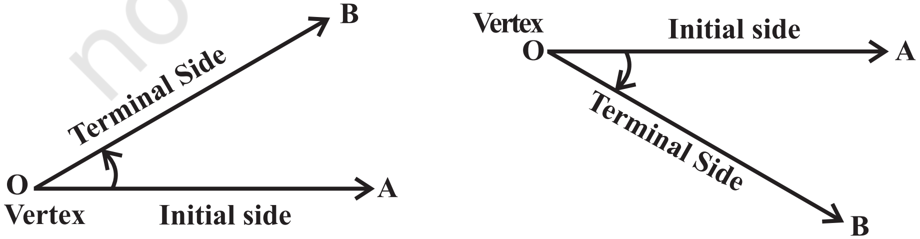

Arya Bhatt
(476-550)

The word 'trigonometry' is derived from the Greek words '*trigon*' and '*metron*' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analysing a musical tone and in many other areas.

In earlier classes, we have studied the trigonometric ratios of acute angles as the ratio of the sides of a right angled triangle. We have also studied the trigonometric identities and application of trigonometric ratios in solving the problems related to heights and distances. In this Chapter, we will generalise the concept of trigonometric ratios to trigonometric functions and study their properties.

3.2 Angles

Angle is a measure of rotation of a given ray about its initial point. The original ray is



(i)Positive angle Fig 3.1 (ii) Negative angle

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called the *initial side* and the final position of the ray after rotation is called the *terminal side* of the angle. The point of rotation is called the *vertex*. If the direction of rotation is anticlockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is *negative* (Fig 3.1).

The measure of an angle is the amount of rotation performed to get the terminal side from the initial side. There are several units for measuring angles. The definition of an angle suggests a unit, viz. *one complete revolution* from the position of the initial side as indicated in Fig 3.2.

Fig 3.2

This is often convenient for large angles. For example, we can say that a rapidly spinning wheel is making an angle of say 15 revolution per second. We shall describe two other units of measurement of an angle which are most commonly used, viz. degree measure and radian measure.

3.2.1 ***Degree measure*** If a rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{th}$ of a revolution, the angle is said to have a measure of one *degree*, written as 1°. A degree is divided into 60 minutes, and a minute is divided into 60 seconds . One sixtieth of a degree is called a *minute*, written as 1′, and one sixtieth of a minute is called a *second*, written as 1″.

Thus, $1° = 60'$, $1' = 60''$

Some of the angles whose measures are 360°,180°, 270°, 420°, – 30°, – 420° are shown in Fig 3.3.

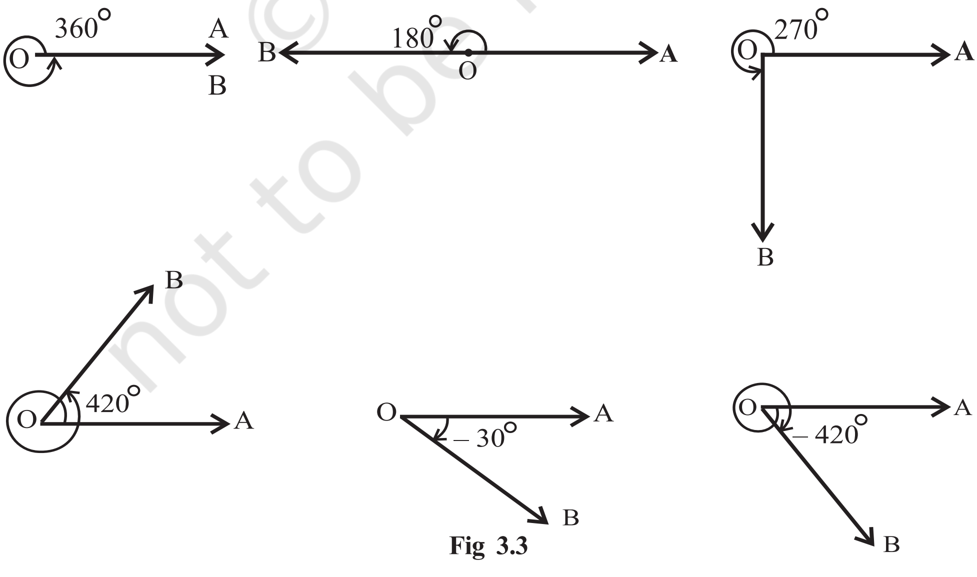

Fig 3.3

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3.2.2 *Radian measure* There is another unit for measurement of an angle, called the *radian* measure. Angle subtended at the centre by an arc of length 1 unit in a unit circle (circle of radius 1 unit) is said to have a measure of 1 radian. In the Fig 3.4(i) to (iv), OA is the initial side and OB is the terminal side. The figures show the angles whose measures are 1 radian, –1 radian, $1\frac{1}{2}$ radian and $-1\frac{1}{2}$ radian.

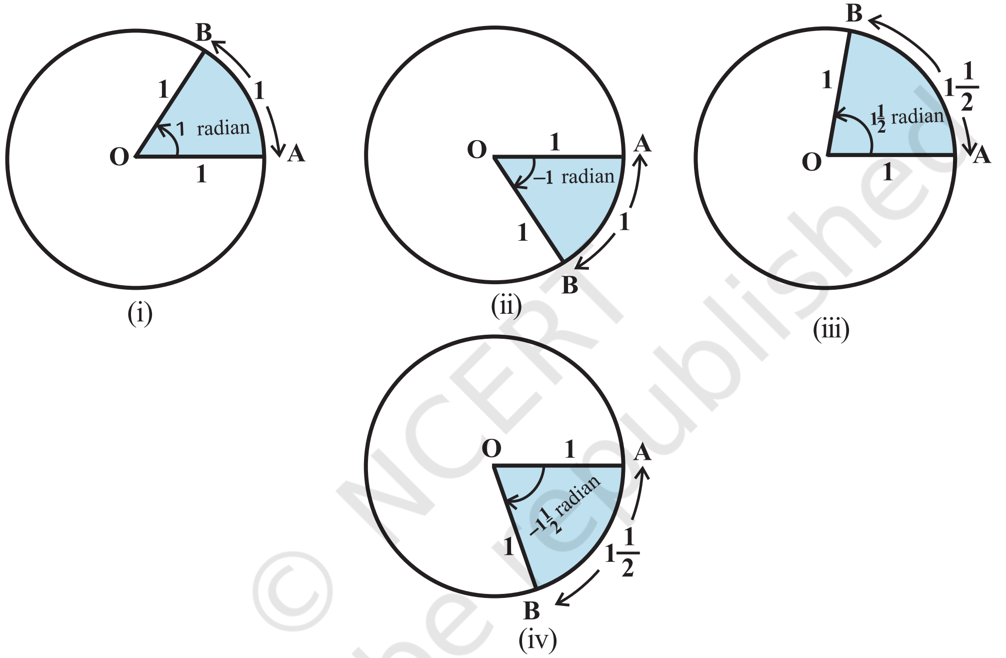

Fig 3.4 (i) to (iv)

We know that the circumference of a circle of radius 1 unit is 2π. Thus, one complete revolution of the initial side subtends an angle of 2π radian.

More generally, in a circle of radius r, an arc of length r will subtend an angle of 1 radian. It is well-known that equal arcs of a circle subtend equal angle at the centre. Since in a circle of radius r, an arc of length r subtends an angle whose measure is 1 radian, an arc of length l will subtend an angle whose measure is $\frac{l}{r}$ radian. Thus, if in a circle of radius r, an arc of length l subtends an angle θ radian at the centre, we have

$$\theta = \frac{l}{r} \text{ or } l = r\,\theta.$$

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3.2.3 *Relation between radian and real numbers*

Consider the unit circle with centre O. Let A be any point on the circle. Consider OA as initial side of an angle. Then the length of an arc of the circle will give the radian measure of the angle which the arc will subtend at the centre of the circle. Consider the line PAQ which is tangent to the circle at A. Let the point A represent the real number zero, AP represents positive real number and AQ represents negative real numbers (Fig 3.5). If we rope the line AP in the anticlockwise direction along the circle, and AQ in the clockwise direction, then every real number will correspond to a radian measure and conversely. Thus, radian measures and real numbers can be considered as one and the same.

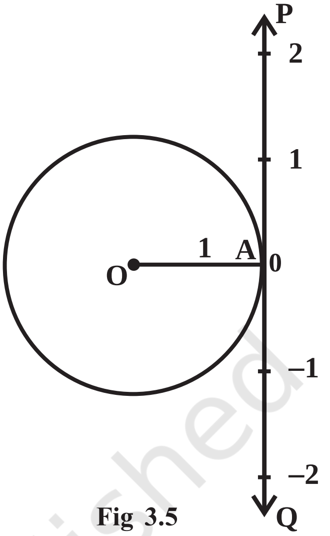

Fig 3.5

3.2.4 *Relation between degree and radian*

Since a circle subtends at the centre an angle whose radian measure is 2π and its degree measure is 360°, it follows that

$$2\pi \text{ radian} = 360° \quad \text{or} \quad \pi \text{ radian} = 180°$$

The above relation enables us to express a radian measure in terms of degree measure and a degree measure in terms of radian measure. Using approximate value of π as $\frac{22}{7}$, we have

$$1 \text{ radian} = \frac{180°}{\pi} = 57°\ 16' \text{ approximately.}$$

Also $$1° = \frac{\pi}{180} \text{ radian} = 0.01746 \text{ radian approximately.}$$

The relation between degree measures and radian measure of some common angles are given in the following table:

Degree	30°	45°	60°	90°	180°	270°	360°
Radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

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Notational Convention

Since angles are measured either in degrees or in radians, we adopt the convention that whenever we write angle $\theta°$, we mean the angle whose degree measure is θ and whenever we write angle β, we mean the angle whose radian measure is β.

Note that when an angle is expressed in radians, the word 'radian' is frequently omitted. Thus, $\pi = 180°$ and $\frac{\pi}{4} = 45°$ are written with the understanding that π and $\frac{\pi}{4}$ are radian measures. Thus, we can say that

$$\text{Radian measure} = \frac{\pi}{180} \times \text{Degree measure}$$

$$\text{Degree measure} = \frac{180}{\pi} \times \text{Radian measure}$$

Example 1 Convert $40°\ 20'$ into radian measure.

Solution We know that $180° = \pi$ radian.

Hence $\quad 40°\ 20' = 40\frac{1}{3}$ degree $= \frac{\pi}{180} \times \frac{121}{3}$ radian $= \frac{121\pi}{540}$ radian.

Therefore $\quad 40°\ 20' = \frac{121\pi}{540}$ radian.

Example 2 Convert 6 radians into degree measure.

Solution We know that π radian = 180°.

Hence $\quad 6 \text{ radians} = \frac{180}{\pi} \times 6 \text{ degree} = \frac{1080 \times 7}{22} \text{degree}$

$$= 343\frac{7}{11} \text{degree} \quad = 343° + \frac{7 \times 60}{11} \text{ minute} \quad [\text{as } 1° = 60']$$

$$= 343° + 38' + \frac{2}{11} \text{ minute} \qquad [\text{as } 1' = 60'']$$

$$= 343° + 38' + 10.9'' \quad = 343° 38'\ 11'' \text{ approximately.}$$

Hence $\quad$ 6 radians = $343°\ 38'\ 11''$ approximately.

Example 3 Find the radius of the circle in which a central angle of 60° intercepts an arc of length 37.4 cm (use $\pi = \frac{22}{7}$).

Solution Here $l = 37.4$ cm and $\theta = 60° = \frac{60\pi}{180}$ radian $= \frac{\pi}{3}$

Hence, by $r = \frac{l}{\theta}$, we have

$$r = \frac{37.4 \times 3}{\pi} = \frac{37.4 \times 3 \times 7}{22} = 35.7 \text{ cm}$$

Example 4 The minute hand of a watch is 1.5 cm long. How far does its tip move in 40 minutes? (Use $\pi = 3.14$).

Solution In 60 minutes, the minute hand of a watch completes one revolution. Therefore, in 40 minutes, the minute hand turns through $\frac{2}{3}$ of a revolution. Therefore, $\theta = \frac{2}{3} \times 360°$ or $\frac{4\pi}{3}$ radian. Hence, the required distance travelled is given by

$$l = r\,\theta = 1.5 \times \frac{4\pi}{3} \text{ cm} = 2\pi \text{ cm} = 2 \times 3.14 \text{ cm} = 6.28 \text{ cm}.$$

Example 5 If the arcs of the same lengths in two circles subtend angles 65°and 110° at the centre, find the ratio of their radii.

Solution Let r_1 and r_2 be the radii of the two circles. Given that

$$\theta_1 = 65° = \frac{\pi}{180} \times 65 = \frac{13\pi}{36} \text{ radian}$$

and $$\theta_2 = 110° = \frac{\pi}{180} \times 110 = \frac{22\pi}{36} \text{ radian}$$

Let l be the length of each of the arc. Then $l = r_1\theta_1 = r_2\theta_2$, which gives

$$\frac{13\pi}{36} \times r_1 = \frac{22\pi}{36} \times r_2, \text{ i.e., } \frac{r_1}{r_2} = \frac{22}{13}$$

Hence $r_1 : r_2 = 22 : 13$.

EXERCISE 3.1

1. Find the radian measures corresponding to the following degree measures:

(i) 25° (ii) – 47°30′ (iii) 240° (iv) 520°

2. Find the degree measures corresponding to the following radian measures (Use $\pi = \frac{22}{7}$).

(i) $\frac{11}{16}$ (ii) -4 (iii) $\frac{5\pi}{3}$ (iv) $\frac{7\pi}{6}$

3. A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?

4. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm (Use $\pi = \frac{22}{7}$).

5. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord.

6. If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii.

7. Find the angle in radian through which a pendulum swings if its length is 75 cm and th e tip describes an arc of length

(i) 10 cm (ii) 15 cm (iii) 21 cm

3.3 Trigonometric Functions

In earlier classes, we have studied trigonometric ratios for acute angles as the ratio of sides of a right angled triangle. We will now extend the definition of trigonometric ratios to any angle in terms of radian measure and study them as trigonometric functions.

Consider a unit circle with centre at origin of the coordinate axes. Let P (a, b) be any point on the circle with angle AOP = x radian, i.e., length of arc AP = x (Fig 3.6).

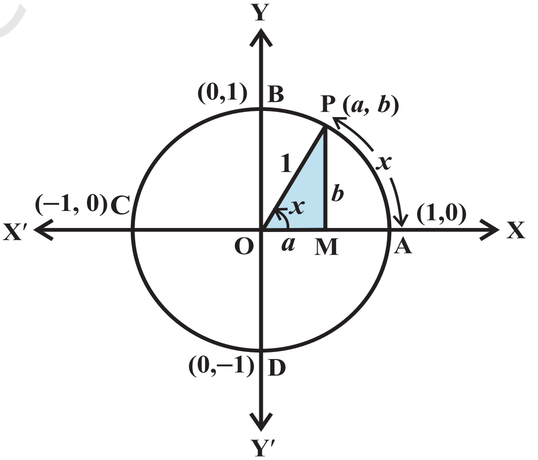

Fig 3.6

We define $\cos x = a$ and $\sin x = b$

Since ΔOMP is a right triangle, we have

$OM^2 + MP^2 = OP^2$ or $a^2 + b^2 = 1$

Thus, for every point on the unit circle, we have

$a^2 + b^2 = 1$ or $\cos^2 x + \sin^2 x = 1$

Since one complete revolution subtends an angle of 2π radian at the centre of the circle, $\angle AOB = \frac{\pi}{2}$,

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$\angle AOC = \pi$ and $\angle AOD = \frac{3\pi}{2}$. All angles which are integral multiples of $\frac{\pi}{2}$ are called *quadrantal angles*. The coordinates of the points A, B, C and D are, respectively, (1, 0), (0, 1), (–1, 0) and (0, –1). Therefore, for quadrantal angles, we have

$$\cos 0^\circ = 1 \qquad \sin 0^\circ = 0,$$

$$\cos \frac{\pi}{2} = 0 \qquad \sin \frac{\pi}{2} = 1$$

$$\cos\pi = -1 \qquad \sin\pi = 0$$

$$\cos \frac{3\pi}{2} = 0 \qquad \sin \frac{3\pi}{2} = -1$$

$$\cos 2\pi = 1 \qquad \sin 2\pi = 0$$

Now, if we take one complete revolution from the point P, we again come back to same point P. Thus, we also observe that if x increases (or decreases) by any integral multiple of 2π, the values of sine and cosine functions do not change. Thus,

$$\sin (2n\pi + x) = \sin x,\ n \in \mathbf{Z},\ \cos (2n\pi + x) = \cos x,\ n \in \mathbf{Z}$$

Further, $\sin x = 0$, if $x = 0, \pm \pi, \pm 2\pi, \pm 3\pi, ...$, i.e., when x is an integral multiple of π and $\cos x = 0$, if $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, ...$ i.e., $\cos x$ vanishes when x is an odd multiple of $\frac{\pi}{2}$. Thus

sin x = 0 implies $x = n\pi$, where n is any integer

cos x = 0 implies $x = (2n + 1)\frac{\pi}{2}$, where n is any integer.

We now define other trigonometric functions in terms of sine and cosine functions:

$$\text{cosec } x = \frac{1}{\sin x},\ x \neq n\pi, \text{ where } n \text{ is any integer.}$$

$$\sec x = \frac{1}{\cos x},\ x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

$$\tan x = \frac{\sin x}{\cos x},\ x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

$$\cot x = \frac{\cos x}{\sin x},\ x \neq n\pi, \text{ where } n \text{ is any integer.}$$

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We have shown that for all real x, $\sin^2 x + \cos^2 x = 1$

It follows that

$$1 + \tan^2 x = \sec^2 x \qquad \text{(why?)}$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x \qquad \text{(why?)}$$

In earlier classes, we have discussed the values of trigonometric ratios for 0°, 30°, 45°, 60° and 90°. The values of trigonometric functions for these angles are same as that of trigonometric ratios studied in earlier classes. Thus, we have the following table:

	0°	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	– 1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	– 1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0

The values of cosec x, sec x and cot x are the reciprocal of the values of sin x, cos x and tan x, respectively.

3.3.1 *Sign of trigonometric functions*

Let P (a, b) be a point on the unit circle with centre at the origin such that $\angle AOP = x$. If $\angle AOQ = -x$, then the coordinates of the point Q will be $(a, -b)$ (Fig 3.7). Therefore

$$\cos(-x) = \cos x$$

and $\quad \sin(-x) = -\sin x$

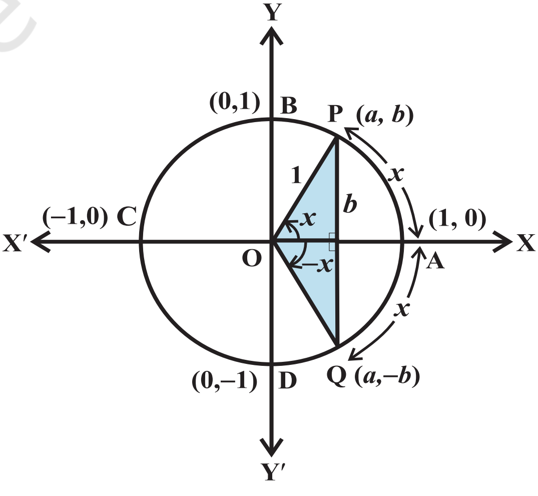

Fig 3.7

Since for every point P (a, b) on the unit circle, $-1 \le a \le 1$ and

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$-1 \le b \le 1$, we have $-1 \le \cos x \le 1$ and $-1 \le \sin x \le 1$ for all x. We have learnt in previous classes that in the first quadrant ($0 < x < \frac{\pi}{2}$) a and b are both positive, in the second quadrant ($\frac{\pi}{2} < x < \pi$) a is negative and b is positive, in the third quadrant ($\pi < x < \frac{3\pi}{2}$) a and b are both negative and in the fourth quadrant ($\frac{3\pi}{2} < x < 2\pi$) a is positive and b is negative. Therefore, $\sin x$ is positive for $0 < x < \pi$, and negative for $\pi < x < 2\pi$. Similarly, $\cos x$ is positive for $0 < x < \frac{\pi}{2}$, negative for $\frac{\pi}{2} < x < \frac{3\pi}{2}$ and also positive for $\frac{3\pi}{2} < x < 2\pi$. Likewise, we can find the signs of other trigonometric functions in different quadrants. In fact, we have the following table.

	I	II	III	IV
$\sin x$	+	+	–	–
$\cos x$	+	–	–	+
$\tan x$	+	–	+	–
$\operatorname{cosec} x$	+	+	–	–
$\sec x$	+	–	–	+
$\cot x$	+	–	+	–

3.3.2 ***Domain and range of trigonometric functions*** From the definition of sine and cosine functions, we observe that they are defined for all real numbers. Further, we observe that for each real number x,

$$-1 \le \sin x \le 1 \text{ and } -1 \le \cos x \le 1$$

Thus, domain of $y = \sin x$ and $y = \cos x$ is the set of all real numbers and range is the interval $[-1, 1]$, i.e., $-1 \le y \le 1$.

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Since cosec $x = \dfrac{1}{\sin x}$, the domain of y = cosec x is the set $\{ x : x \in \mathbf{R}$ and $x \neq n\pi, n \in \mathbf{Z}\}$ and range is the set $\{y : y \in \mathbf{R}, y \geq 1 \text{ or } y \leq -1\}$. Similarly, the domain of $y = \sec x$ is the set $\{x : x \in \mathbf{R}$ and $x \neq (2n + 1)\dfrac{\pi}{2}, n \in \mathbf{Z}\}$ and range is the set $\{y : y \in \mathbf{R}, y \leq -1 \text{ or } y \geq 1\}$. The domain of $y = \tan x$ is the set $\{x : x \in \mathbf{R}$ and $x \neq (2n + 1)\dfrac{\pi}{2}, n \in \mathbf{Z}\}$ and range is the set of all real numbers. The domain of $y = \cot x$ is the set $\{x : x \in \mathbf{R}$ and $x \neq n\pi, n \in \mathbf{Z}\}$ and the range is the set of all real numbers.

We further observe that in the first quadrant, as x increases from 0 to $\dfrac{\pi}{2}$, $\sin x$ increases from 0 to 1, as x increases from $\dfrac{\pi}{2}$ to π, $\sin x$ decreases from 1 to 0. In the third quadrant, as x increases from π to $\dfrac{3\pi}{2}$, $\sin x$ decreases from 0 to –1and finally, in the fourth quadrant, $\sin x$ increases from –1 to 0 as x increases from $\dfrac{3\pi}{2}$ to 2π. Similarly, we can discuss the behaviour of other trigonometric functions. In fact, we have the following table:

	I quadrant	II quadrant	III quadrant	IV quadrant
sin	increases from 0 to 1	decreases from 1 to 0	decreases from 0 to –1	increases from –1 to 0
cos	decreases from 1 to 0	decreases from 0 to – 1	increases from –1 to 0	increases from 0 to 1
tan	increases from 0 to ∞	increases from –∞to 0	increases from 0 to ∞	increases from –∞to 0
cot	decreases from ∞ to 0	decreases from 0 to–∞	decreases from ∞ to 0	decreases from 0to –∞
sec	increases from 1 to ∞	increases from –∞to–1	decreases from –1to–∞	decreases from ∞ to 1
cosec	decreases from ∞ to 1	increases from 1 to ∞	increases from –∞to–1	decreases from–1to–∞

Remark In the above table, the statement $\tan x$ increases from 0 to ∞ (infinity) for $0 < x < \dfrac{\pi}{2}$ simply means that $\tan x$ increases as x increases for $0 < x < \dfrac{\pi}{2}$ and

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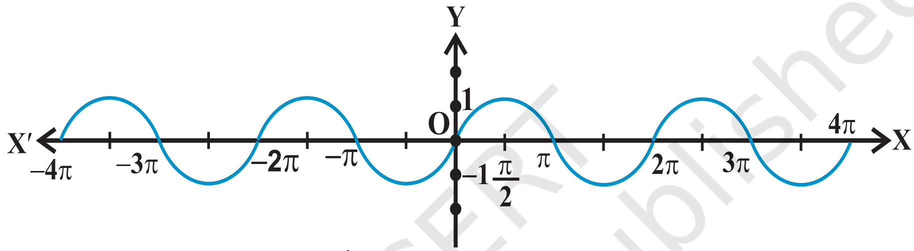
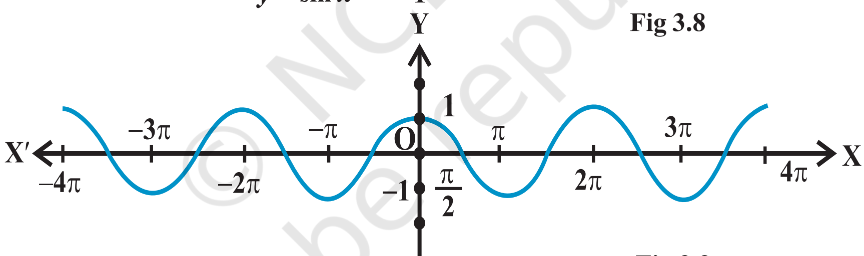
assumes arbitraily large positive values as x approaches to $\frac{\pi}{2}$. Similarly, to say that cosec x decreases from –1 to $-\infty$ (minus infinity) in the fourth quadrant means that cosec x decreases for $x \in (\frac{3\pi}{2}, 2\pi)$ and assumes arbitrarily large negative values as x approaches to 2π. The symbols ∞ and $-\infty$ simply specify certain types of behaviour of functions and variables.

We have already seen that values of sin x and cos x repeats after an interval of 2π. Hence, values of cosec x and sec x will also repeat after an interval of 2π. We



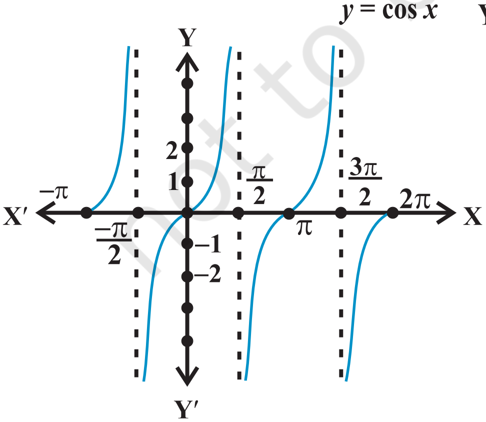
y* = sin *x

Fig 3.8

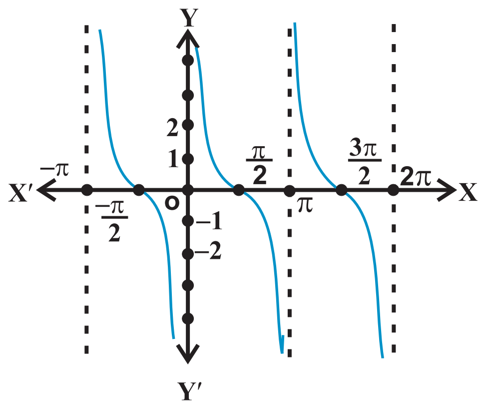


y* = cos *x

Fig 3.9



y* = tan *x **Fig 3.10**



y* = cot *x **Fig 3.11**

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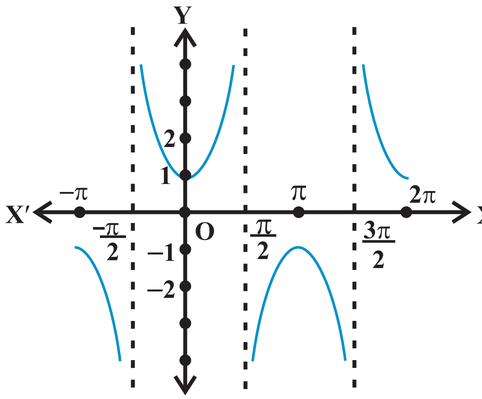
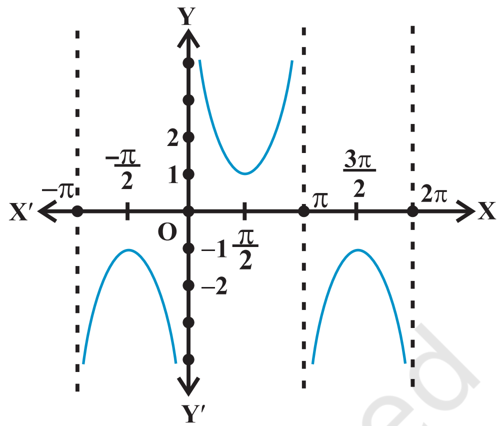


$y = \sec x$

Fig 3.12



$y = \text{cosec}\, x$

Fig 3.13

shall see in the next section that $\tan(\pi + x) = \tan x$. Hence, values of $\tan x$ will repeat after an interval of π. Since $\cot x$ is reciprocal of $\tan x$, its values will also repeat after an interval of π. Using this knowledge and behaviour of trigonometic functions, we can sketch the graph of these functions. The graph of these functions are given above:

Example 6 If $\cos x = -\frac{3}{5}$, x lies in the third quadrant, find the values of other five trigonometric functions.

Solution Since $\cos x = -\frac{3}{5}$, we have $\sec x = -\frac{5}{3}$

Now $\quad \sin^2 x + \cos^2 x = 1$, i.e., $\sin^2 x = 1 - \cos^2 x$

or $\quad \sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Hence $\quad \sin x = \pm \frac{4}{5}$

Since x lies in third quadrant, $\sin x$ is negative. Therefore

$$\sin x = -\frac{4}{5}$$

which also gives

$$\text{cosec}\, x = -\frac{5}{4}$$

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Further, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{4}{3} \quad \text{and} \quad \cot x = \frac{\cos x}{\sin x} = \frac{3}{4}.$$

Example 7 If $\cot x = -\frac{5}{12}$, x lies in second quadrant, find the values of other five trigonometric functions.

Solution Since $\cot x = -\frac{5}{12}$, we have $\tan x = -\frac{12}{5}$

Now $\sec^2 x = 1 + \tan^2 x = 1 + \frac{144}{25} = \frac{169}{25}$

Hence $\sec x = \pm \frac{13}{5}$

Since x lies in second quadrant, $\sec x$ will be negative. Therefore

$$\sec x = -\frac{13}{5},$$

which also gives

$$\cos x = -\frac{5}{13}$$

Further, we have

$$\sin x = \tan x \cos x = \left(-\frac{12}{5}\right) \times \left(-\frac{5}{13}\right) = \frac{12}{13}$$

and $$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{13}{12}.$$

Example 8 Find the value of $\sin \frac{31\pi}{3}$.

Solution We know that values of $\sin x$ repeats after an interval of 2π. Therefore

$$\sin \frac{31\pi}{3} = \sin \left(10\pi + \frac{\pi}{3}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

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Example 9 Find the value of cos (–1710°).

Solution We know that values of cos x repeats after an interval of 2π or 360°. Therefore, $\cos(-1710°) = \cos(-1710° + 5 \times 360°)$

$$= \cos(-1710° + 1800°) = \cos 90° = 0.$$

EXERCISE 3.2

Find the values of other five trigonometric functions in Exercises 1 to 5.

1. $\cos x = -\frac{1}{2}$, x lies in third quadrant.
2. $\sin x = \frac{3}{5}$, x lies in second quadrant.
3. $\cot x = \frac{3}{4}$, x lies in third quadrant.
4. $\sec x = \frac{13}{5}$, x lies in fourth quadrant.
5. $\tan x = -\frac{5}{12}$, x lies in second quadrant.

Find the values of the trigonometric functions in Exercises 6 to 10.

6. sin 765°
7. cosec (– 1410°)
8. $\tan \frac{19\pi}{3}$
9. $\sin (-\frac{11\pi}{3})$
10. $\cot (-\frac{15\pi}{4})$

3.4 Trigonometric Functions of Sum and Difference of Two Angles

In this Section, we shall derive expressions for trigonometric functions of the sum and difference of two numbers (angles) and related expressions. The basic results in this connection are called *trigonometric identities*. We have seen that

1. **$\sin(-x) = -\sin x$**
2. **$\cos(-x) = \cos x$**

We shall now prove some more results:

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3. $\cos (x + y) = \cos x \cos y - \sin x \sin y$

Consider the unit circle with centre at the origin. Let x be the angle P_4OP_1 and y be the angle P_1OP_2. Then $(x + y)$ is the angle P_4OP_2. Also let $(-y)$ be the angle P_4OP_3. Therefore, P_1, P_2, P_3 and P_4 will have the coordinates $P_1(\cos x, \sin x)$, $P_2 [\cos (x + y), \sin (x + y)]$, $P_3 [\cos (-y), \sin (-y)]$ and $P_4 (1, 0)$ (Fig 3.14).

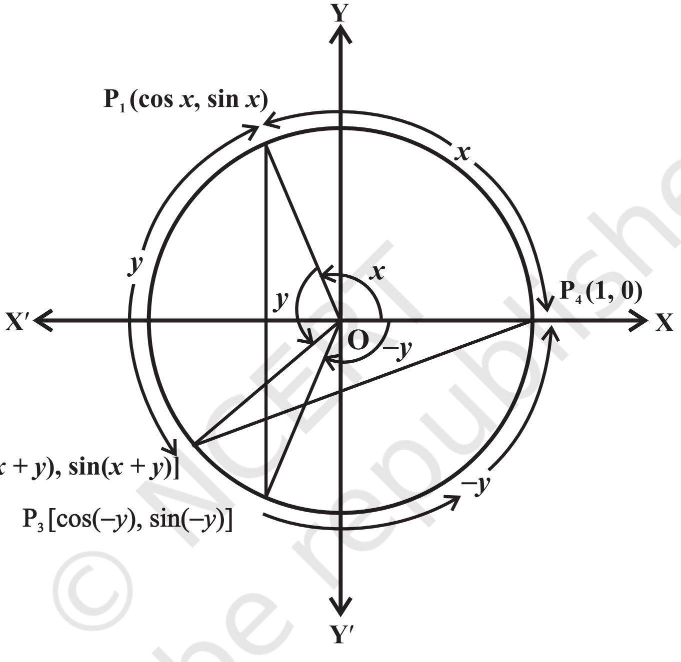

Fig 3.14

Consider the triangles P_1OP_3 and P_2OP_4. They are congruent (Why?). Therefore, P_1P_3 and P_2P_4 are equal. By using distance formula, we get

$$P_1P_3^{\,2} = [\cos x - \cos (-y)]^2 + [\sin x - \sin(-y]^2$$

$$= (\cos x - \cos y)^2 + (\sin x + \sin y)^2$$

$$= \cos^2 x + \cos^2 y - 2 \cos x \cos y + \sin^2 x + \sin^2 y + 2\sin x \sin y$$

$$= 2 - 2 (\cos x \cos y - \sin x \sin y) \qquad \text{(Why?)}$$

Also, $$P_2P_4^{\,2} = [1 - \cos (x + y)]^2 + [0 - \sin (x + y)]^2$$

$$= 1 - 2\cos (x + y) + \cos^2 (x + y) + \sin^2 (x + y)$$

$$= 2 - 2 \cos (x + y)$$

Since $P_1P_3 = P_2P_4$, we have $P_1P_3^2 = P_2P_4^2$.
Therefore, $2 - 2 (\cos x \cos y - \sin x \sin y) = 2 - 2 \cos (x + y)$.

Hence $\cos (x + y) = \cos x \cos y - \sin x \sin y$

4. $\mathbf{\cos (x - y) = \cos x \cos y + \sin x \sin y}$

Replacing y by $-y$ in identity 3, we get

$$\cos (x + (-y)) = \cos x \cos (-y) - \sin x \sin (-y)$$

or $\quad \cos (x - y) = \cos x \cos y + \sin x \sin y$

5. $\mathbf{\cos \left(\frac{\pi}{2} - x\right) = \sin x}$

If we replace x by $\frac{\pi}{2}$ and y by x in Identity (4), we get

$$\cos \left(\frac{\pi}{2} - x\right) = \cos \frac{\pi}{2} \cos x + \sin \frac{\pi}{2} \sin x = \sin x.$$

6. $\mathbf{\sin \left(\frac{\pi}{2} - x\right) = \cos x}$

Using the Identity 5, we have

$$\sin \left(\frac{\pi}{2} - x\right) = \cos \left[\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right] = \cos x.$$

7. $\mathbf{\sin (x + y) = \sin x \cos y + \cos x \sin y}$

We know that

$$\sin (x + y) = \cos \left(\frac{\pi}{2} - (x + y)\right) = \cos \left(\left(\frac{\pi}{2} - x\right) - y\right)$$

$$= \cos \left(\frac{\pi}{2} - x\right) \cos y + \sin \left(\frac{\pi}{2} - x\right) \sin y$$

$$= \sin x \cos y + \cos x \sin y$$

8. $\mathbf{\sin (x - y) = \sin x \cos y - \cos x \sin y}$

If we replace y by $-y$, in the Identity 7, we get the result.

9. By taking suitable values of x and y in the identities 3, 4, 7 and 8, we get the following results:

$$\cos \left(\frac{\pi}{2} + x\right) = -\sin x \qquad \sin \left(\frac{\pi}{2} + x\right) = \cos x$$

$$\cos (\pi - x) = -\cos x \qquad \sin (\pi - x) = \sin x$$

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$$\cos(\pi + x) = -\cos x \qquad \sin(\pi + x) = -\sin x$$
$$\cos(2\pi - x) = \cos x \qquad \sin(2\pi - x) = -\sin x$$

Similar results for tan x, cot x, sec x and cosec x can be obtained from the results of sin x and cos x.

10. If none of the angles x, y and $(x + y)$ is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Since none of the x, y and $(x + y)$ is an odd multiple of $\frac{\pi}{2}$, it follows that cos x, cos y and cos $(x + y)$ are non-zero. Now

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

Dividing numerator and denominator by cos x cos y, we have

$$\tan(x+y) = \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

11. $$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

If we replace y by $-y$ in Identity 10, we get

$$\tan(x-y) = \tan[x + (-y)]$$

$$= \frac{\tan x + \tan(-y)}{1 - \tan x \tan(-y)} = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

12. If none of the angles x, y and $(x + y)$ is a multiple of π, then

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

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Since, none of the x, y and $(x + y)$ is multiple of π, we find that $\sin x \;\; \sin y$ and $\sin (x + y)$ are non-zero. Now,

$$\cot (x + y) = \frac{\cos (x + y)}{\sin (x + y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}$$

Dividing numerator and denominator by $\sin x \sin y$, we have

$$\cot (x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

13. $\boldsymbol{\cot (x - y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}}$ if none of angles x, y and $x–y$ is a multiple of π

If we replace y by $–y$ in identity 12, we get the result

14. $\boldsymbol{\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}}$

We know that

$$\cos (x + y) = \cos x \;\; \cos y - \sin x \sin y$$

Replacing y by x, we get

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= \cos^2 x - (1 - \cos^2 x) = 2 \cos^2 x - 1$$

Again, $\cos 2x = \cos^2 x - \sin^2 x$

$$= 1 - \sin^2 x - \sin^2 x = 1 - 2 \sin^2 x.$$

We have $\cos 2x = \cos^2 x - \sin^2 x = \dfrac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}$

Dividing numerator and denominator by $\cos^2 x$, we get

$$\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}, \quad x \neq n\pi + \frac{\pi}{2}, \text{ where } n \text{ is an integer}$$

15. $\boldsymbol{\sin 2x = 2 \sin x \cos x = \dfrac{2 \tan x}{1 + \tan^2 x}}$ $\;x \neq n\pi + \dfrac{\pi}{2}$, where n is an integer

We have

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$

Replacing y by x, we get $\sin 2x = 2 \sin x \cos x$.

Again $\sin 2x = \dfrac{2 \sin x \cos x}{\cos^2 x + \sin^2 x}$

Dividing each term by $\cos^2 x$, we get

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

16. $\mathbf{\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}}$ if $2x \neq n\pi + \dfrac{\pi}{2}$, where n is an integer

We know that

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Replacing y by x, we get $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

17. $\mathbf{\sin 3x = 3\sin x - 4\sin^3 x}$

We have,

$$\begin{aligned}
\sin 3x &= \sin(2x + x)\\
&= \sin 2x \cos x + \cos 2x \sin x\\
&= 2\sin x \cos x \cos x + (1 - 2\sin^2 x)\sin x\\
&= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x\\
&= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x\\
&= 3\sin x - 4\sin^3 x
\end{aligned}$$

18. $\mathbf{\cos 3x = 4\cos^3 x - 3\cos x}$

We have,

$$\begin{aligned}
\cos 3x &= \cos(2x + x)\\
&= \cos 2x \cos x - \sin 2x \sin x\\
&= (2\cos^2 x - 1)\cos x - 2\sin x \cos x \sin x\\
&= (2\cos^2 x - 1)\cos x - 2\cos x(1 - \cos^2 x)\\
&= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x\\
&= 4\cos^3 x - 3\cos x.
\end{aligned}$$

19. $\mathbf{\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}}$ if $3x \neq n\pi + \dfrac{\pi}{2}$, where n is an integer

We have $\tan 3x = \tan(2x + x)$

$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} = \frac{\dfrac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \dfrac{2\tan x \,.\, \tan x}{1-\tan^2 x}}$$

$$= \frac{2\tan x + \tan x - \tan^3 x}{1 - \tan^2 x - 2\tan^2 x} = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

20. **(i)** $\mathbf{\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}}$

(ii) $\mathbf{\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}}$

(iii) $\mathbf{\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}}$

(iv) $\mathbf{\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}}$

We know that

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \quad \text{... (1)}$$

and $$\cos(x - y) = \cos x \cos y + \sin x \sin y \quad \text{... (2)}$$

Adding and subtracting (1) and (2), we get

$$\cos(x + y) + \cos(x - y) = 2\cos x \cos y \quad \text{... (3)}$$

and $$\cos(x + y) - \cos(x - y) = -2\sin x \sin y \quad \text{... (4)}$$

Further $$\sin(x + y) = \sin x \cos y + \cos x \sin y \quad \text{... (5)}$$

and $$\sin(x - y) = \sin x \cos y - \cos x \sin y \quad \text{... (6)}$$

Adding and subtracting (5) and (6), we get

$$\sin(x + y) + \sin(x - y) = 2\sin x \cos y \quad \text{... (7)}$$

$$\sin(x + y) - \sin(x - y) = 2\cos x \sin y \quad \text{... (8)}$$

Let $x + y = \theta$ and $x - y = \phi$. Therefore

$$x = \left(\frac{\theta+\phi}{2}\right) \text{ and } y = \left(\frac{\theta-\phi}{2}\right)$$

Substituting the values of x and y in (3), (4), (7) and (8), we get

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

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$$\sin \theta - \sin \phi = 2 \cos \left(\frac{\theta+\phi}{2}\right) \sin \left(\frac{\theta-\phi}{2}\right)$$

Since θ and ϕ can take any real values, we can replace θ by x and ϕ by y.
Thus, we get

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}\ ;\ \cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2},$$

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}\ ;\ \sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}.$$

Remark As a part of identities given in 20, we can prove the following results:

21. **(i)** $\mathbf{2 \cos x \cos y = \cos (x + y) + \cos (x - y)}$
(ii) $\mathbf{-2 \sin x \sin y = \cos (x + y) - \cos (x - y)}$
(iii) $\mathbf{2 \sin x \cos y = \sin (x + y) + \sin (x - y)}$
(iv) $\mathbf{2 \cos x \sin y = \sin (x + y) - \sin (x - y)}$.

Example 10 Prove that

$$3\sin\frac{\pi}{6}\sec\frac{\pi}{3}-4\sin\frac{5\pi}{6}\cot\frac{\pi}{4}=1$$

Solution We have

$$\text{L.H.S.} = 3\sin\frac{\pi}{6}\sec\frac{\pi}{3}-4\sin\frac{5\pi}{6}\cot\frac{\pi}{4}$$

$$= 3 \times \frac{1}{2} \times 2 - 4 \sin \left(\pi - \frac{\pi}{6}\right) \times 1 = 3 - 4 \sin \frac{\pi}{6}$$

$$= 3 - 4 \times \frac{1}{2} = 1 = \text{R.H.S.}$$

Example 11 Find the value of sin 15°.

Solution We have

$$\sin 15^\circ = \sin (45^\circ - 30^\circ)$$
$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$
$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Example 12 Find the value of $\tan \frac{13\pi}{12}$.

Solution We have

$$\tan\frac{13\pi}{12} = \tan\left(\pi + \frac{\pi}{12}\right) = \tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}$$

Example 13 Prove that

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}.$$

Solution We have

$$\text{L.H.S.} \quad = \frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x\cos y + \cos x\sin y}{\sin x\cos y - \cos x\sin y}$$

Dividing the numerator and denominator by $\cos x \cos y$, we get

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}.$$

Example 14 Show that

$$\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$$

Solution We know that $3x = 2x + x$

Therefore, $\tan 3x = \tan(2x + x)$

or $\quad \tan 3x = \dfrac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$

or $\quad \tan 3x - \tan 3x \tan 2x \tan x = \tan 2x + \tan x$

or $\quad \tan 3x - \tan 2x - \tan x = \tan 3x \tan 2x \tan x$

or $\quad \tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x.$

Example 15 Prove that

$$\cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} - x\right) = \sqrt{2}\cos x$$

Solution Using the Identity 20(i), we have

$$\text{L.H.S.} \quad = \cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)$$

$$= 2\cos\left(\frac{\frac{\pi}{4}+x+\frac{\pi}{4}-x}{2}\right)\cos\left(\frac{\frac{\pi}{4}+x-(\frac{\pi}{4}-x)}{2}\right)$$

$$= 2\cos\frac{\pi}{4}\cos x = 2\times\frac{1}{\sqrt{2}}\cos x = \sqrt{2}\cos x = \text{R.H.S.}$$

Example 16 Prove that $\dfrac{\cos 7x+\cos 5x}{\sin 7x-\sin 5x} = \cot x$

Solution Using the Identities 20 (i) and 20 (iv), we get

$$\text{L.H.S.} \quad = \frac{2\cos\frac{7x+5x}{2}\cos\frac{7x-5x}{2}}{2\cos\frac{7x+5x}{2}\sin\frac{7x-5x}{2}} = \frac{\cos x}{\sin x} = \cot x = \text{R.H.S.}$$

Example 17 Prove that $= \dfrac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x} = \tan x$

Solution We have

$$\text{L.H.S.} \quad = \frac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x} = \frac{\sin 5x+\sin x-2\sin 3x}{\cos 5x-\cos x}$$

$$= \frac{2\sin 3x\cos 2x-2\sin 3x}{-2\sin 3x\sin 2x} = -\frac{\sin 3x\,(\cos 2x-1)}{\sin 3x\sin 2x}$$

$$= \frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x\cos x} = \tan x = \text{R.H.S.}$$

EXERCISE 3.3

Prove that:

1. $\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{4} = -\frac{1}{2}$

2. $2\sin^2 \frac{\pi}{6} + \text{cosec}^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3} = \frac{3}{2}$

3. $\cot^2 \frac{\pi}{6} + \text{cosec}\frac{5\pi}{6} + 3\tan^2 \frac{\pi}{6} = 6$

4. $2\sin^2 \frac{3\pi}{4} + 2\cos^2 \frac{\pi}{4} + 2\sec^2 \frac{\pi}{3} = 10$

5. Find the value of:

(i) sin 75° (ii) tan 15°

Prove the following:

6. $\cos\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - y\right) - \sin\left(\frac{\pi}{4} - x\right)\sin\left(\frac{\pi}{4} - y\right) = \sin(x+y)$

7. $\dfrac{\tan\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} - x\right)} = \left(\dfrac{1+\tan x}{1-\tan x}\right)^2$

8. $\dfrac{\cos(\pi + x)\cos(-x)}{\sin(\pi - x)\cos\left(\frac{\pi}{2} + x\right)} = \cot^2 x$

9. $\cos\left(\frac{3\pi}{2} + x\right)\cos(2\pi + x)\left[\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x)\right] = 1$

10. $\sin(n+1)x \sin(n+2)x + \cos(n+1)x \cos(n+2)x = \cos x$

11. $\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -\sqrt{2}\sin x$

12. $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$

13. $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$

14. $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$

15. $\cot 4x (\sin 5x + \sin 3x) = \cot x (\sin 5x - \sin 3x)$

16. $\dfrac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\dfrac{\sin 2x}{\cos 10x}$

17. $\dfrac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$

18. $\dfrac{\sin x - \sin y}{\cos x + \cos y} = \tan\dfrac{x-y}{2}$

19. $\dfrac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$

20. $\dfrac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x$

21. $\dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$

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22. $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$

23. $\tan 4x = \dfrac{4\tan x\,(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x}$ **24.** $\cos 4x = 1 - 8\sin^2 x \cos^2 x$

25. $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$

Miscellaneous Examples

Example 18 If $\sin x = \dfrac{3}{5}$, $\cos y = -\dfrac{12}{13}$, where x and y both lie in second quadrant, find the value of $\sin(x+y)$.

Solution We know that

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad \ldots (1)$$

Now $\cos^2 x = 1 - \sin^2 x = 1 - \dfrac{9}{25} = \dfrac{16}{25}$

Therefore $\cos x = \pm\dfrac{4}{5}$.

Since x lies in second quadrant, $\cos x$ is negative.

Hence $\cos x = -\dfrac{4}{5}$

Now $\sin^2 y = 1 - \cos^2 y = 1 - \dfrac{144}{169} = \dfrac{25}{169}$

i.e. $\sin y = \pm\dfrac{5}{13}$.

Since y lies in second quadrant, hence $\sin y$ is positive. Therefore, $\sin y = \dfrac{5}{13}$. Substituting the values of $\sin x$, $\sin y$, $\cos x$ and $\cos y$ in (1), we get

$$\sin(x+y) = \frac{3}{5}\times\left(-\frac{12}{13}\right)+\left(-\frac{4}{5}\right)\times\frac{5}{13} = -\frac{36}{65}-\frac{20}{65} = -\frac{56}{65}.$$

Example 19 Prove that

$$\cos 2x \cos\frac{x}{2} - \cos 3x \cos\frac{9x}{2} = \sin 5x \sin\frac{5x}{2}.$$

Solution We have

$$\text{L.H.S.} \quad = \frac{1}{2}\left[2\cos 2x \cos\frac{x}{2} - 2\cos\frac{9x}{2}\cos 3x\right]$$

$$= \frac{1}{2}\left[\cos\left(2x+\frac{x}{2}\right) + \cos\left(2x-\frac{x}{2}\right) - \cos\left(\frac{9x}{2}+3x\right) - \cos\left(\frac{9x}{2}-3x\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{5x}{2} + \cos\frac{3x}{2} - \cos\frac{15x}{2} - \cos\frac{3x}{2}\right] = \frac{1}{2}\left[\cos\frac{5x}{2} - \cos\frac{15x}{2}\right]$$

$$= \frac{1}{2}\left[-2\sin\left\{\frac{\frac{5x}{2}+\frac{15x}{2}}{2}\right\}\sin\left\{\frac{\frac{5x}{2}-\frac{15x}{2}}{2}\right\}\right]$$

$$= -\sin 5x \sin\left(-\frac{5x}{2}\right) = \sin 5x \sin\frac{5x}{2} = \text{R.H.S.}$$

Example 20 Find the value of $\tan\frac{\pi}{8}$.

Solution Let $x = \frac{\pi}{8}$. Then $2x = \frac{\pi}{4}$.

Now $\qquad \tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

or $\qquad \tan\frac{\pi}{4} = \dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

Let $y = \tan\frac{\pi}{8}$. Then $1 = \dfrac{2y}{1-y^2}$

or $\qquad y^2 + 2y - 1 = 0$

Therefore $\qquad y = \dfrac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

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Since $\frac{\pi}{8}$ lies in the first quadrant, $y = \tan\frac{\pi}{8}$ is positve. Hence

$$\tan\frac{\pi}{8} = \sqrt{2} - 1.$$

Example 21 If $\tan x = \frac{3}{4}, \pi < x < \frac{3\pi}{2}$, find the value of $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$.

Solution Since $\pi < x < \frac{3\pi}{2}$, $\cos x$ is negative.

Also $\qquad \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$.

Therefore, $\sin\frac{x}{2}$ is positive and $\cos\frac{x}{2}$ is negative.

Now $\qquad \sec^2 x = 1 + \tan^2 x = 1 + \frac{9}{16} = \frac{25}{16}$

Therefore $\qquad \cos^2 x = \frac{16}{25}$ or $\cos x = -\frac{4}{5}$ (Why?)

Now $\qquad 2\sin^2\frac{x}{2} = 1 - \cos x = 1 + \frac{4}{5} = \frac{9}{5}$.

Therefore $\qquad \sin^2\frac{x}{2} = \frac{9}{10}$

or $\qquad \sin\frac{x}{2} = \frac{3}{\sqrt{10}}$ (Why?)

Again $\qquad 2\cos^2\frac{x}{2} = 1 + \cos x = 1 - \frac{4}{5} = \frac{1}{5}$

Therefore $\qquad \cos^2\frac{x}{2} = \frac{1}{10}$

or $\qquad \cos\frac{x}{2} = -\frac{1}{\sqrt{10}}$ (Why?)

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Hence $\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{3}{\sqrt{10}} \times \left(\frac{-\sqrt{10}}{1}\right) = -3.$

Example 22 Prove that $\cos^2 x + \cos^2\left(x+\frac{\pi}{3}\right)+\cos^2\left(x-\frac{\pi}{3}\right) = \frac{3}{2}$.

Solution We have

$$\text{L.H.S.} = \frac{1+\cos 2x}{2} + \frac{1+\cos\left(2x+\frac{2\pi}{3}\right)}{2} + \frac{1+\cos\left(2x-\frac{2\pi}{3}\right)}{2}.$$

$$= \frac{1}{2}\left[3+\cos 2x+\cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(2x-\frac{2\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\frac{2\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\left(\pi-\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x-2\cos 2x\cos\frac{\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x-\cos 2x\right] = \frac{3}{2} = \text{R.H.S.}$$

Miscellaneous Exercise on Chapter 3

Prove that:

1. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$

2. $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = 0$

3. $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4\cos^2\frac{x+y}{2}$

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4. $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4 \sin^2 \dfrac{x-y}{2}$

5. $\sin x + \sin 3x + \sin 5x + \sin 7x = 4 \cos x \cos 2x \sin 4x$

6. $\dfrac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$

7. $\sin 3x + \sin 2x - \sin x = 4\sin x \cos \dfrac{x}{2} \cos \dfrac{3x}{2}$

Find $\sin \dfrac{x}{2}$, $\cos \dfrac{x}{2}$ and $\tan \dfrac{x}{2}$ in each of the following :

8. $\tan x = -\dfrac{4}{3}$, x in quadrant II

9. $\cos x = -\dfrac{1}{3}$, x in quadrant III

10. $\sin x = \dfrac{1}{4}$, x in quadrant II

Summary

- If in a circle of radius r, an arc of length l subtends an angle of θ radians, then $l = r\,\theta$
- Radian measure $= \dfrac{\pi}{180} \times$ Degree measure
- Degree measure $= \dfrac{180}{\pi} \times$ Radian measure
- $\cos^2 x + \sin^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \operatorname{cosec}^2 x$
- $\cos (2n\pi + x) = \cos x$
- $\sin (2n\pi + x) = \sin x$
- $\sin (-x) = -\sin x$
- $\cos (-x) = \cos x$
- $\cos (x + y) = \cos x \cos y - \sin x \sin y$
- $\cos (x - y) = \cos x \cos y + \sin x \sin y$
- $\cos \left(\dfrac{\pi}{2} - x\right) = \sin x$

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- $\sin\left(\frac{\pi}{2}-x\right) = \cos x$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\sin(x - y) = \sin x \cos y - \cos x \sin y$
- $\cos\left(\frac{\pi}{2}+x\right) = -\sin x$ $\qquad$ $\sin\left(\frac{\pi}{2}+x\right) = \cos x$

  $\cos(\pi - x) = -\cos x$ $\qquad$ $\sin(\pi - x) = \sin x$

  $\cos(\pi + x) = -\cos x$ $\qquad$ $\sin(\pi + x) = -\sin x$

  $\cos(2\pi - x) = \cos x$ $\qquad$ $\sin(2\pi - x) = -\sin x$

- If none of the angles x, y and $(x \pm y)$ is an odd multiple of $\frac{\pi}{2}$, then

  $$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

- $$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$
- If none of the angles x, y and $(x \pm y)$ is a multiple of π, then

  $$\cot(x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

- $$\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$$
- $$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$
- $$\sin 2x = 2 \sin x \cos x = \frac{2\tan x}{1 + \tan^2 x}$$
- $$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$
- $\sin 3x = 3\sin x - 4\sin^3 x$
- $\cos 3x = 4\cos^3 x - 3\cos x$

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- $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

- (i) $\cos x + \cos y = 2\cos \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

  (ii) $\cos x - \cos y = -2\sin \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$

  (iii) $\sin x + \sin y = 2\sin \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

  (iv) $\sin x - \sin y = 2\cos \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$

- (i) $2\cos x \cos y = \cos (x + y) + \cos (x - y)$

  (ii) $-2\sin x \sin y = \cos (x + y) - \cos (x - y)$

  (iii) $2\sin x \cos y = \sin (x + y) + \sin (x - y)$

  (iv) $2\cos x \sin y = \sin (x + y) - \sin (x - y)$.

Historical Note

The study of trigonometry was first started in India. The ancient Indian Mathematicians, Aryabhatta (476), Brahmagupta (598), Bhaskara I (600) and Bhaskara II (1114) got important results. All this knowledge first went from India to middle-east and from there to Europe. The Greeks had also started the study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughout the world.

In India, the predecessor of the modern trigonometric functions, known as the sine of an angle, and the introduction of the sine function represents the main contribution of the *siddhantas* (Sanskrit astronomical works) to the history of mathematics.

Bhaskara I (about 600) gave formulae to find the values of sine functions for angles more than 90°. A sixteenth century Malayalam work *Yuktibhasa* (period) contains a proof for the expansion of sin (A + B). Exact expression for sines or cosines of 18°, 36°, 54°, 72°, etc., are given by Bhaskara II.

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The symbols $\sin^{-1} x$, $\cos^{-1} x$, etc., for arc sin x, arc cos x, etc., were suggested by the astronomer Sir John F.W. Hersehel (1813) The names of Thales (about 600 B.C.) is invariably associated with height and distance problems. He is credited with the determination of the height of a great pyramid in Egypt by measuring shadows of the pyramid and an auxiliary staff (or gnomon) of known height, and comparing the ratios:

$$\frac{\mathrm{H}}{\mathrm{S}} = \frac{h}{s} = \tan\text{ (sun's altitude)}$$

Thales is also said to have calculated the distance of a ship at sea through the proportionality of sides of similar triangles. Problems on height and distance using the similarity property are also found in ancient Indian works.

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Chapter 4

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

❖ Mathematics is the Queen of Sciences and Arithmetic is the Queen of Mathematics. – GAUSS ❖

4.1 Introduction

In earlier classes, we have studied linear equations in one and two variables and quadratic equations in one variable. We have seen that the equation $x^2 + 1 = 0$ has no real solution as $x^2 + 1 = 0$ gives $x^2 = -1$ and square of every real number is non-negative. So, we need to extend the real number system to a larger system so that we can find the solution of the equation $x^2 = -1$. In fact, the main objective is to solve the equation $ax^2 + bx + c = 0$, where $D = b^2 - 4ac < 0$, which is not possible in the system of real numbers.

W. R. Hamilton (1805-1865)

4.2 Complex Numbers

Let us denote $\sqrt{-1}$ by the symbol i. Then, we have $i^2 = -1$. This means that i is a solution of the equation $x^2 + 1 = 0$.

A number of the form $a + ib$, where a and b are real numbers, is defined to be a complex number. For example, $2 + i3$, $(-1) + i\sqrt{3}$, $4 + i\left(\frac{-1}{11}\right)$ are complex numbers.

For the complex number $z = a + ib$, a is called the *real part*, denoted by Re z and b is called the *imaginary part* denoted by Im z *of the complex number* z. For example, if $z = 2 + i5$, then Re $z = 2$ and Im $z = 5$.

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$.

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Example 1 If $4x + i(3x - y) = 3 + i(-6)$, where x and y are real numbers, then find the values of x and y.

Solution We have

$$4x + i(3x - y) = 3 + i(-6) \qquad \text{... (1)}$$

Equating the real and the imaginary parts of (1), we get

$$4x = 3,\ 3x - y = -6,$$

which, on solving simultaneously, give $x = \frac{3}{4}$ and $y = \frac{33}{4}$.

4.3 Algebra of Complex Numbers

In this Section, we shall develop the algebra of complex numbers.

4.3.1 ***Addition of two complex numbers*** Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the sum $z_1 + z_2$ is defined as follows:

$z_1 + z_2 = (a + c) + i(b + d)$, which is again a complex number.

For example, $(2 + i3) + (-6 + i5) = (2 - 6) + i(3 + 5) = -4 + i\,8$

The addition of complex numbers satisfy the following properties:

(i) *The closure law* The sum of two complex numbers is a complex number, i.e., $z_1 + z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) *The commutative law* For any two complex numbers z_1 and z_2, $z_1 + z_2 = z_2 + z_1$

(iii) *The associative law* For any three complex numbers z_1, z_2, z_3, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

(iv) *The existence of additive identity* There exists the complex number $0 + i\,0$ (denoted as 0), called the *additive identity* or the *zero complex number,* such that, for every complex number z, $z + 0 = z$.

(v) *The existence of additive inverse* To every complex number $z = a + ib$, we have the complex number $-a + i(-b)$ (denoted as $-z$), called the *additive inverse* or *negative of* z. We observe that $z + (-z) = 0$ (the additive identity).

4.3.2 ***Difference of two complex numbers*** Given any two complex numbers z_1 and z_2, the difference $z_1 - z_2$ is defined as follows:

$$z_1 - z_2 = z_1 + (-z_2).$$

For example, $(6 + 3i) - (2 - i) = (6 + 3i) + (-2 + i) = 4 + 4i$

and $(2 - i) - (6 + 3i) = (2 - i) + (-6 - 3i) = -4 - 4i$

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4.3.3 ***Multiplication of two complex numbers*** Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the product $z_1 z_2$ is defined as follows:

$$z_1 z_2 = (ac - bd) + i(ad + bc)$$

For example, $(3 + i5)(2 + i6) = (3 \times 2 - 5 \times 6) + i(3 \times 6 + 5 \times 2) = -24 + i28$

The multiplication of complex numbers possesses the following properties, which we state without proofs.

(i) **The closure law** The product of two complex numbers is a complex number, the product $z_1 z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) **The commutative law** For any two complex numbers z_1 and z_2,

$$z_1 z_2 = z_2 z_1.$$

(iii) **The associative law** For any three complex numbers z_1, z_2, z_3,

$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

(iv) **The existence of multiplicative identity** There exists the complex number $1 + i\,0$ (denoted as 1), called the *multiplicative identity* such that $z.1 = z$, for every complex number z.

(v) **The existence of multiplicative inverse** For every non-zero complex number $z = a + ib$ or $a + bi (a \neq 0, b \neq 0)$, we have the complex number

$\dfrac{a}{a^2+b^2} + i\dfrac{-b}{a^2+b^2}$ (denoted by $\dfrac{1}{z}$ or z^{-1}), called the *multiplicative inverse* of z such that

$z.\dfrac{1}{z} = 1$ (the multiplicative identity).

(vi) **The distributive law** For any three complex numbers z_1, z_2, z_3,

(a) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

(b) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

4.3.4 ***Division of two complex numbers*** Given any two complex numbers z_1 and z_2, where $z_2 \neq 0$, the quotient $\dfrac{z_1}{z_2}$ is defined by

$$\frac{z_1}{z_2} = z_1 \frac{1}{z_2}$$

For example, let $z_1 = 6 + 3i$ and $z_2 = 2 - i$

Then $$\frac{z_1}{z_2} = \left((6+3i) \times \frac{1}{2-i}\right) = (6+3i)\left(\frac{2}{2^2+(-1)^2} + i\frac{-(-1)}{2^2+(-1)^2}\right)$$

$$= (6+3i)\left(\frac{2+i}{5}\right) = \frac{1}{5}\left[12-3+i(6+6)\right] = \frac{1}{5}(9+12i)$$

4.3.5 ***Power of i*** we know that

$$i^3 = i^2 i = (-1)\, i = -i\,, \qquad i^4 = \left(i^2\right)^2 = (-1)^2 = 1$$

$$i^5 = \left(i^2\right)^2 i = (-1)^2\, i = i\,, \qquad i^6 = \left(i^2\right)^3 = (-1)^3 = -1\,, \text{ etc.}$$

Also, we have $\quad i^{-1} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i, \quad i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{1} = i, \quad i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

In general, for any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

4.3.6 ***The square roots of a negative real number***

Note that $i^2 = -1$ and $(-i)^2 = i^2 = -1$

Therefore, the square roots of -1 are $i, -i$. However, by the symbol $\sqrt{-1}$, we would mean i only.

Now, we can see that i and $-i$ both are the solutions of the equation $x^2 + 1 = 0$ or $x^2 = -1$.

Similarly $\qquad \left(\sqrt{3}\, i\right)^2 = \left(\sqrt{3}\right)^2 i^2 = 3\,(-1) = -3$

$$\left(-\sqrt{3}\, i\right)^2 = \left(-\sqrt{3}\right)^2 i^2 = -3$$

Therefore, the square roots of -3 are $\sqrt{3}\, i$ and $-\sqrt{3}\, i$.

Again, the symbol $\sqrt{-3}$ is meant to represent $\sqrt{3}\, i$ only, i.e., $\sqrt{-3} = \sqrt{3}\, i$.

Generally, if a is a positive real number, $\sqrt{-a} = \sqrt{a}\,\sqrt{-1} = \sqrt{a}\, i$,

We already know that $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all positive real number a and b. This result also holds true when either $a > 0,\ b < 0$ or $a < 0,\ b > 0$. What if $a < 0,\ b < 0$? Let us examine.

Note that

$i^2 = \sqrt{-1}\,\sqrt{-1} = \sqrt{(-1)(-1)}$ (by assuming $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all real numbers)

$= \sqrt{1} = 1$, which is a contradiction to the fact that $i^2 = -1$.

Therefore, $\sqrt{a} \times \sqrt{b} \neq \sqrt{ab}$ if both a and b are negative real numbers.

Further, if any of a and b is zero, then, clearly, $\sqrt{a} \times \sqrt{b} = \sqrt{ab} = 0$.

4.3.7 *Identities* We prove the following identity

$(z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1 z_2$, for all complex numbers z_1 and z_2.

Proof We have, $(z_1 + z_2)^2 = (z_1 + z_2)(z_1 + z_2)$,

$$= (z_1 + z_2)\, z_1 + (z_1 + z_2)\, z_2 \quad \text{(Distributive law)}$$

$$= z_1^2 + z_2 z_1 + z_1 z_2 + z_2^2 \quad \text{(Distributive law)}$$

$$= z_1^2 + z_1 z_2 + z_1 z_2 + z_2^2 \quad \text{(Commutative law of multiplication)}$$

$$= z_1^2 + 2z_1 z_2 + z_2^2$$

Similarly, we can prove the following identities:

(i) $(z_1 - z_2)^2 = z_1^2 - 2z_1 z_2 + z_2^2$

(ii) $(z_1 + z_2)^3 = z_1^3 + 3z_1^2 z_2 + 3z_1 z_2^2 + z_2^3$

(iii) $(z_1 - z_2)^3 = z_1^3 - 3z_1^2 z_2 + 3z_1 z_2^2 - z_2^3$

(iv) $z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)$

In fact, many other identities which are true for all real numbers, can be proved to be true for all complex numbers.

Example 2 Express the following in the form of $a + bi$:

(i) $(-5i)\left(\frac{1}{8}i\right)$ (ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3$

Solution (i) $(-5i)\left(\frac{1}{8}i\right) = \frac{-5}{8}i^2 = \frac{-5}{8}(-1) = \frac{5}{8} = \frac{5}{8} + i0$

(ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3 = 2 \times \frac{1}{8 \times 8 \times 8} \times i^5 = \frac{1}{256}\left(i^2\right)^2 i = \frac{1}{256}i$.

Example 3 Express $(5 - 3i)^3$ in the form $a + ib$.

Solution We have, $(5 - 3i)^3 = 5^3 - 3 \times 5^2 \times (3i) + 3 \times 5\,(3i)^2 - (3i)^3$

$$= 125 - 225i - 135 + 27i = -10 - 198i.$$

Example 4 Express $\left(-\sqrt{3}+\sqrt{-2}\right)\left(2\sqrt{3}-i\right)$ in the form of $a + ib$

Solution We have, $\left(-\sqrt{3}+\sqrt{-2}\right)\left(2\sqrt{3}-i\right) = \left(-\sqrt{3}+\sqrt{2}\,i\right)\left(2\sqrt{3}-i\right)$

$$= -6+\sqrt{3}i+2\sqrt{6}i-\sqrt{2}\,i^2 = \left(-6+\sqrt{2}\right)+\sqrt{3}\left(1+2\sqrt{2}\right)i$$

4.4 The Modulus and the Conjugate of a Complex Number

Let $z = a + ib$ be a complex number. Then, the modulus of z, denoted by $|z|$, is defined to be the non-negative real number $\sqrt{a^2+b^2}$, i.e., $|z| = \sqrt{a^2+b^2}$ and the conjugate of z, denoted as $\bar{z}$, is the complex number $a - ib$, i.e., $\bar{z} = a - ib$.

For example, $\quad |3+i| = \sqrt{3^2+1^2} = \sqrt{10}$, $|2-5i| = \sqrt{2^2+(-5)^2} = \sqrt{29}$,

and $\quad \overline{3+i} = 3-i$, $\overline{2-5i} = 2+5i$, $\overline{-3i-5} = 3i-5$

Observe that the multiplicative inverse of the non-zero complex number z is given by

$$z^{-1} = \frac{1}{a+ib} = \frac{a}{a^2+b^2}+i\frac{-b}{a^2+b^2} = \frac{a-ib}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$$

or $\quad z\,\bar{z} = |z|^2$

Furthermore, the following results can easily be derived.

For any two compex numbers z_1 and z_2, we have

(i) $|z_1 z_2| = |z_1||z_2|$ (ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ provided $|z_2| \neq 0$

(iii) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$ (iv) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$ (v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ provided $z_2 \neq 0$.

Example 5 Find the multiplicative inverse of $2 - 3i$.

Solution Let $z = 2 - 3i$

Then $\bar{z} = 2 + 3i$ and $|z|^2 = 2^2 + (-3)^2 = 13$

Therefore, the multiplicative inverse of $2 - 3i$ is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

The above working can be reproduced in the following manner also,

$$z^{-1} = \frac{1}{2-3i} = \frac{2+3i}{(2-3i)(2+3i)}$$

$$= \frac{2+3i}{2^2-(3i)^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

Example 6 Express the following in the form $a + ib$

(i) $\dfrac{5+\sqrt{2}i}{1-\sqrt{2}i}$ (ii) i^{-35}

Solution (i) We have, $\dfrac{5+\sqrt{2}i}{1-\sqrt{2}i} = \dfrac{5+\sqrt{2}i}{1-\sqrt{2}i} \times \dfrac{1+\sqrt{2}i}{1+\sqrt{2}i} = \dfrac{5+5\sqrt{2}i+\sqrt{2}i-2}{1-\left(\sqrt{2}i\right)^2}$

$$= \frac{3+6\sqrt{2}i}{1+2} = \frac{3(1+2\sqrt{2}i)}{3} = 1+2\sqrt{2}i.$$

(ii) $i^{-35} = \dfrac{1}{i^{35}} = \dfrac{1}{\left(i^2\right)^{17} i} = \dfrac{1}{-i} \times \dfrac{i}{i} = \dfrac{i}{-i^2} = i$

EXERCISE 4.1

Express each of the complex number given in the Exercises 1 to 10 in the form $a + ib$.

1. $(5i)\left(-\dfrac{3}{5}i\right)$ **2.** $i^9 + i^{19}$ **3.** i^{-39}

4. $3(7 + i7) + i\,(7 + i7)$ **5.** $(1 - i) - (-1 + i6)$

6. $\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$ **7.** $\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$

8. $(1 - i)^4$ **9.** $\left(\frac{1}{3}+3i\right)^3$ **10.** $\left(-2-\frac{1}{3}i\right)^3$

Find the multiplicative inverse of each of the complex numbers given in the Exercises 11 to 13.

11. $4 - 3i$ **12.** $\sqrt{5}+3i$ **13.** $-i$

14. Express the following expression in the form of $a + ib$:

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}\,i\right)-\left(\sqrt{3}-i\sqrt{2}\right)}$$

4.5 Argand Plane and Polar Representation

We already know that corresponding to each ordered pair of real numbers (x, y), we get a unique point in the XY-plane and vice-versa with reference to a set of mutually perpendicular lines known as the x-axis and the y-axis. The complex number $x + iy$ which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point $P(x, y)$ in the XY-plane and vice-versa.

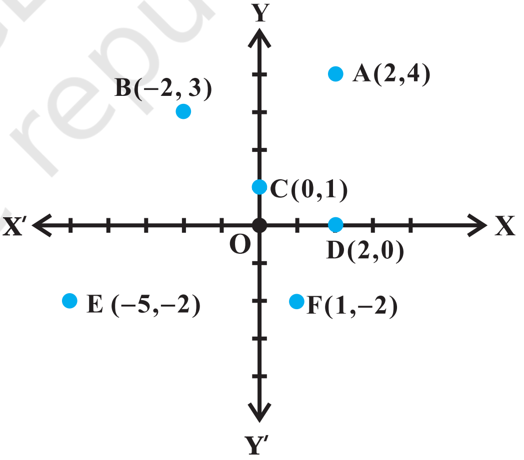

Fig 4.1

Some complex numbers such as $2 + 4i$, $-2 + 3i$, $0 + 1i$, $2 + 0i$, $-5 - 2i$ and $1 - 2i$ which correspond to the ordered pairs (2, 4), (– 2, 3), (0, 1), (2, 0), (–5, –2), and (1, – 2), respectively, have been represented geometrically by the points A, B, C, D, E, and F, respectively in the Fig 4.1.

The plane having a complex number assigned to each of its point is called the *complex plane* or the *Argand plane*.

Obviously, in the Argand plane, the modulus of the complex number $x + iy = \sqrt{x^2 + y^2}$ is the distance between the point P(x, y) and the origin O (0, 0) (Fig 4.2). The points on the x-axis corresponds to the complex numbers of the form $a + i\,0$ and the points on the y-axis corresponds to the complex numbers of the form

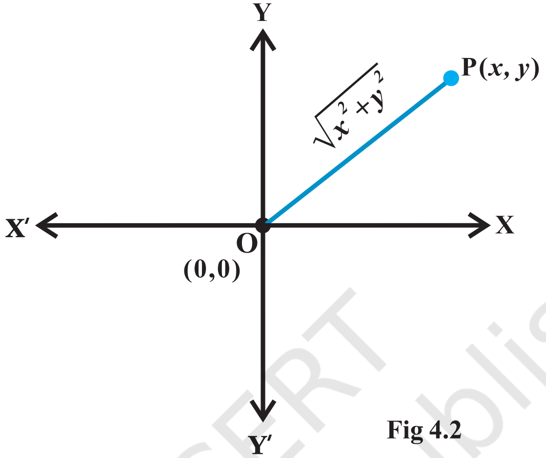

Fig 4.2

$0 + i\,b$. The x-axis and y-axis in the Argand plane are called, respectively, the *real axis* and the *imaginary axis*.

The representation of a complex number $z = x + iy$ and its conjugate $\bar{z} = x - iy$ in the Argand plane are, respectively, the points P (x, y) and Q (x, $-y$).

Geometrically, the point (x, $-y$) is the mirror image of the point (x, y) on the real axis (Fig 4.3).

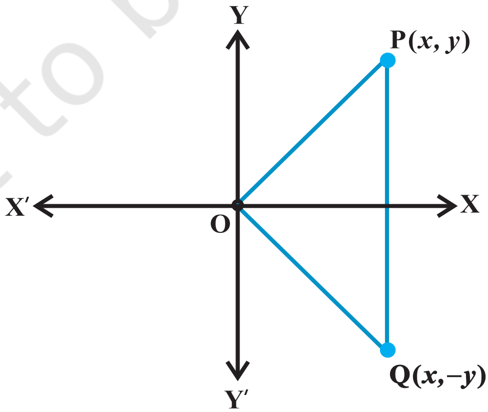

Fig 4.3

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Miscellaneous Examples

Example 7 Find the conjugate of $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$.

Solution We have , $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$

$$= \frac{6+9i-4i+6}{2-i+4i+2} = \frac{12+5i}{4+3i}\times\frac{4-3i}{4-3i}$$

$$= \frac{48-36i+20i+15}{16+9} = \frac{63-16i}{25} = \frac{63}{25}-\frac{16}{25}i$$

Therefore, conjugate of $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ is $\dfrac{63}{25}+\dfrac{16}{25}i$.

Example 8 If $x + iy = \dfrac{a+ib}{a-ib}$, prove that $x^2 + y^2 = 1$.

Solution We have,

$$x + iy = \frac{(a+ib)(a+ib)}{(a-ib)(a+ib)} = \frac{a^2-b^2+2abi}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2}+\frac{2ab}{a^2+b^2}i$$

So that, $x - iy = \dfrac{a^2-b^2}{a^2+b^2}-\dfrac{2ab}{a^2+b^2}i$

Therefore,

$$x^2 + y^2 = (x + iy)(x - iy) = \frac{(a^2-b^2)^2}{(a^2+b^2)^2}+\frac{4a^2b^2}{(a^2+b^2)^2} = \frac{(a^2+b^2)^2}{(a^2+b^2)^2} = 1$$

Miscellaneous Exercise on Chapter 4

1. Evaluate: $\left[i^{18}+\left(\dfrac{1}{i}\right)^{25}\right]^3$.

2. For any two complex numbers z_1 and z_2, prove that
$\text{Re}\,(z_1 z_2) = \text{Re}\, z_1 \,\text{Re}\, z_2 - \text{Im}z_1\, \text{Im}z_2$

3. Reduce $\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form .

4. If $x-iy=\sqrt{\frac{a-ib}{c-id}}$ prove that $\left(x^2+y^2\right)^2=\frac{a^2+b^2}{c^2+d^2}$.

5. If $z_1 = 2 - i$, $z_2 = 1 + i$, find $\left|\frac{z_1+z_2+1}{z_1-z_2+1}\right|$.

6. If $a + ib = \frac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \frac{(x^2+1)^2}{\left(2x^2+1\right)^2}$.

7. Let $z_1 = 2 - i$, $z_2 = -2 + i$. Find

(i) $\text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$, (ii) $\text{Im}\left(\frac{1}{z_1\bar{z}_1}\right)$.

8. Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

9. Find the modulus of $\frac{1+i}{1-i}-\frac{1-i}{1+i}$.

10. If $(x + iy)^3 = u + iv$, then show that $\frac{u}{x}+\frac{v}{y}=4(x^2-y^2)$.

11. If α and β are different complex numbers with $|\beta|=1$, then find $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$.

12. Find the number of non-zero integral solutions of the equation $|1-i|^x=2^x$.

13. If $(a + ib)(c + id)(e + if)(g + ih) = A + iB$, then show that

$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$

14. If $\left(\frac{1+i}{1-i}\right)^m=1$, then find the least positive integral value of m.

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Summary

- A number of the form $a + ib$, where a and b are real numbers, is called a *complex number*, a is called the *real part* and b is called the *imaginary* part of the complex number.
- Let $z_1 = a + ib$ and $z_2 = c + id$. Then
  (i) $z_1 + z_2 = (a + c) + i\,(b + d)$
  (ii) $z_1\, z_2 = (ac - bd) + i\,(ad + bc)$
- For any non-zero complex number $z = a + ib$ ($a \neq 0$, $b \neq 0$), there exists the complex number $\dfrac{a}{a^2+b^2} + i\dfrac{-b}{a^2+b^2}$, denoted by $\dfrac{1}{z}$ or z^{-1}, called the *multiplicative inverse* of z such that $(a + ib)\left(\dfrac{a}{a^2+b^2} + i\dfrac{-b}{a^2+b^2}\right) = 1 + i0$ $=1$
- For any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$
- The conjugate of the complex number $z = a + ib$, denoted by $\bar{z}$, is given by $\bar{z} = a - ib$.

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Historical Note

The fact that square root of a negative number does not exist in the real number system was recognised by the Greeks. But the credit goes to the Indian mathematician *Mahavira* (850) who first stated this difficulty clearly. "He mentions in his work '*Ganitasara Sangraha*' as in the nature of things a negative (quantity) is not a square (quantity)', it has, therefore, no square root". *Bhaskara*, another Indian mathematician, also writes in his work *Bijaganita*, written in 1150. "There is no square root of a negative quantity, for it is not a square." *Cardan* (1545) considered the problem of solving

$$x + y = 10,\ xy = 40.$$

He obtained $x = 5 + \sqrt{-15}$ and $y = 5 - \sqrt{-15}$ as the solution of it, which was discarded by him by saying that these numbers are 'useless'. *Albert Girard* (about 1625) accepted square root of negative numbers and said that this will enable us to get as many roots as the degree of the polynomial equation. *Euler* was the first to introduce the symbol i for $\sqrt{-1}$ and *W.R. Hamilton* (about 1830) regarded the complex number $a + ib$ as an ordered pair of real numbers (a, b) thus giving it a purely mathematical definition and avoiding use of the so called '*imaginary numbers*'.

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Chapter 5

LINEAR INEQUALITIES

❖ *Mathematics is the art of saying many things in many different ways.* – *MAXWELL* ❖

5.1 Introduction

In earlier classes, we have studied equations in one variable and two variables and also solved some statement problems by translating them in the form of equations. Now a natural question arises: 'Is it always possible to translate a statement problem in the form of an equation? For example, the height of all the students in your class is less than 160 cm. Your classroom can occupy atmost 60 tables or chairs or both. Here we get certain statements involving a sign '$<$' (less than), '$>$' (greater than), '$\leq$' (less than or equal) and $\geq$ (greater than or equal) which are known as *inequalities*.

In this Chapter, we will study linear inequalities in one and two variables. The study of inequalities is very useful in solving problems in the field of science, mathematics, statistics, economics, psychology, etc.

5.2 Inequalities

Let us consider the following situations:

(i) Ravi goes to market with ₹200 to buy rice, which is available in packets of 1kg. The price of one packet of rice is ₹30. If x denotes the number of packets of rice, which he buys, then the total amount spent by him is ₹ $30x$. Since, he has to buy rice in packets only, he may not be able to spend the entire amount of ₹200. (Why?) Hence

$$30x < 200 \quad \ldots (1)$$

Clearly the statement (i) is not an equation as it does not involve the sign of equality.

(ii) Reshma has ₹ 120 and wants to buy some registers and pens. The cost of one register is ₹40 and that of a pen is ₹20. In this case, if x denotes the number of registers and y, the number of pens which Reshma buys, then the total amount spent by her is ₹ $(40x + 20y)$ and we have

$$40x + 20y \leq 120 \quad \ldots (2)$$

Since in this case the total amount spent may be upto ₹ 120. Note that the statement (2) consists of two statements

$$40x + 20y < 120 \quad \text{... (3)}$$

and
$$40x + 20y = 120 \quad \text{... (4)}$$

Statement (3) is not an equation, i.e., it is an inequality while statement (4) is an equation.

Definition 1 Two real numbers or two algebraic expressions related by the symbol '<', '>', '≤' or '≥' form an *inequality*.

Statements such as (1), (2) and (3) above are inequalities.

$3 < 5$; $7 > 5$ are the examples of *numerical inequalities* while

$x < 5$; $y > 2$; $x \geq 3$, $y \leq 4$ are some examples of *literal inequalities*.

$3 < 5 < 7$ (read as 5 is greater than 3 and less than 7), $3 \leq x < 5$ (read as x is greater than or equal to 3 and less than 5) and $2 < y \leq 4$ are the examples of *double inequalities*.

Some more examples of inequalities are:

$$ax + b < 0 \quad \text{... (5)}$$

$$ax + b > 0 \quad \text{... (6)}$$

$$ax + b \leq 0 \quad \text{... (7)}$$

$$ax + b \geq 0 \quad \text{... (8)}$$

$$ax + by < c \quad \text{... (9)}$$

$$ax + by > c \quad \text{... (10)}$$

$$ax + by \leq c \quad \text{... (11)}$$

$$ax + by \geq c \quad \text{... (12)}$$

$$ax^2 + bx + c \leq 0 \quad \text{... (13)}$$

$$ax^2 + bx + c > 0 \quad \text{... (14)}$$

Inequalities (5), (6), (9), (10) and (14) are *strict inequalities* while inequalities (7), (8), (11), (12), and (13) are *slack inequalities*. Inequalities from (5) to (8) are *linear inequalities* in one variable x when $a \neq 0$, while inequalities from (9) to (12) *are linear inequalities in two variables x and y* when $a \neq 0$, $b \neq 0$.

Inequalities (13) and (14) are not linear *(in fact, these are quadratic inequalities in one variable x when* $a \neq 0$).

In this Chapter, we shall confine ourselves to the study of linear inequalities in one and two variables only.

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5.3 Algebraic Solutions of Linear Inequalities in One Variable and their Graphical Representation

Let us consider the inequality (1) of Section 6.2, viz, $30x < 200$
Note that here x denotes the number of packets of rice.

Obviously, x cannot be a negative integer or a fraction. Left hand side (L.H.S.) of this inequality is $30x$ and right hand side (RHS) is 200. Therefore, we have

For $x = 0$, L.H.S. = 30 (0) = 0 < 200 (R.H.S.), which is true.

For $x = 1$, L.H.S. = 30 (1) = 30 < 200 (R.H.S.), which is true.

For $x = 2$, L.H.S. = 30 (2) = 60 < 200, which is true.

For $x = 3$, L.H.S. = 30 (3) = 90 < 200, which is true.

For $x = 4$, L.H.S. = 30 (4) = 120 < 200, which is true.

For $x = 5$, L.H.S. = 30 (5) = 150 < 200, which is true.

For $x = 6$, L.H.S. = 30 (6) = 180 < 200, which is true.

For $x = 7$, L.H.S. = 30 (7) = 210 < 200, which is false.

In the above situation, we find that the values of x, which makes the above inequality a true statement, are 0,1,2,3,4,5,6. These values of x, which make above inequality a true statement, are called *solutions* of inequality and the set {0,1,2,3,4,5,6} is called its *solution set.*

Thus, any solution of an inequality in one variable is a value of the variable which makes it a true statement.

We have found the solutions of the above inequality by *trial and error* method which is not very efficient. Obviously, this method is time consuming and sometimes not feasible. We must have some better or systematic techniques for solving inequalities. Before that we should go through some more properties of numerical inequalities and follow them as rules while solving the inequalities.

You will recall that while solving linear equations, we followed the following rules:

Rule 1 Equal numbers may be added to (or subtracted from) both sides of an equation.

Rule 2 Both sides of an equation may be multiplied (or divided) by the same non-zero number.

In the case of solving inequalities, we again follow the same rules except with a difference that in Rule 2, the sign of inequality is reversed (i.e., '<' becomes '>', ≤' becomes '≥' and so on) whenever we multiply (or divide) both sides of an inequality by a negative number. It is evident from the facts that

$$3 > 2 \text{ while } -3 < -2,$$
$$-8 < -7 \text{ while } (-8)(-2) > (-7)(-2)\text{ , i.e., } 16 > 14.$$

Thus, we state the following rules for solving an inequality:

Rule 1 Equal numbers may be added to (or subtracted from) both sides of an inequality without affecting the sign of inequality.

Rule 2 Both sides of an inequality can be multiplied (or divided) by the same positive number. But when both sides are multiplied or divided by a negative number, then the sign of inequality is *reversed.*

Now, let us consider some examples.

Example 1 Solve $30\,x < 200$ when

(i) x is a natural number, (ii) x is an integer.

Solution We are given $30\,x < 200$

or $\frac{30x}{30} < \frac{200}{30}$ (Rule 2), i.e., $x < 20 / 3$.

(i) When x is a natural number, in this case the following values of x make the statement true.

1, 2, 3, 4, 5, 6.

The solution set of the inequality is $\{1,2,3,4,5,6\}$.

(ii) When x is an integer, the solutions of the given inequality are

$..., -3, -2, -1, 0, 1, 2, 3, 4, 5, 6$

The solution set of the inequality is $\{...,-3, -2,-1, 0, 1, 2, 3, 4, 5, 6\}$

Example 2 Solve $5x - 3 < 3x + 1$ when

(i) x is an integer, (ii) x is a real number.

Solution We have, $5x - 3 < 3x + 1$

or $5x - 3 + 3 < 3x + 1 + 3$ (Rule 1)

or $5x < 3x + 4$

or $5x - 3x < 3x + 4 - 3x$ (Rule 1)

or $2x < 4$

or $x < 2$ (Rule 2)

(i) When x is an integer, the solutions of the given inequality are

$..., -4, -3, -2, -1, 0, 1$

(ii) When x is a real number, the solutions of the inequality are given by $x < 2$, i.e., all real numbers x which are less than 2. Therefore, the solution set of the inequality is $x \in (-\infty, 2)$.

We have considered solutions of inequalities in the set of natural numbers, set of integers and in the set of real numbers. Henceforth, unless stated otherwise, we shall solve the inequalities in this Chapter in the set of real numbers.

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Example 3 Solve $4x + 3 < 6x + 7$.

Solution We have, $4x + 3 < 6x + 7$

or $4x - 6x < 6x + 4 - 6x$

or $-2x < 4$ or $x > -2$

i.e., all the real numbers which are greater than –2, are the solutions of the given inequality. Hence, the solution set is $(-2, \infty)$.

Example 4 Solve $\frac{5-2x}{3} \le \frac{x}{6} - 5$.

Solution We have

$$\frac{5-2x}{3} \le \frac{x}{6} - 5$$

or $2(5 - 2x) \le x - 30.$

or $10 - 4x \le x - 30$

or $-5x \le -40$, i.e., $x \ge 8$

Thus, all real numbers x which are greater than or equal to 8 are the solutions of the given inequality, i.e., $x \in [8, \infty)$.

Example 5 Solve $7x + 3 < 5x + 9$. Show the graph of the solutions on number line.

Solution We have $7x + 3 < 5x + 9$ or

$2x < 6$ or $x < 3$

The graphical representation of the solutions are given in Fig 5.1.

Fig 5.1

Example 6 Solve $\frac{3x-4}{2} \ge \frac{x+1}{4} - 1$. Show the graph of the solutions on number line.

Solution We have

$$\frac{3x-4}{2} \ge \frac{x+1}{4} - 1$$

or $$\frac{3x-4}{2} \ge \frac{x-3}{4}$$

or $2(3x - 4) \ge (x - 3)$

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or $\quad 6x - 8 \geq x - 3$

or $\quad 5x \geq 5$ or $x \geq 1$

The graphical representation of solutions is given in Fig 5.2.

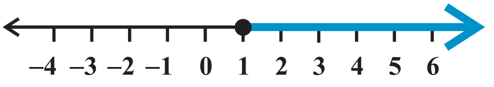

Fig 5.2

Example 7 The marks obtained by a student of Class XI in first and second terminal examination are 62 and 48, respectively. Find the minimum marks he should get in the annual examination to have an average of at least 60 marks.

Solution Let x be the marks obtained by student in the annual examination. Then

$$\frac{62+48+x}{3} \geq 60$$

or $\quad 110 + x \geq 180$

or $\quad x \geq 70$

Thus, the student must obtain a minimum of 70 marks to get an average of at least 60 marks.

Example 8 Find all pairs of consecutive odd natural numbers, both of which are larger than 10, such that their sum is less than 40.

Solution Let x be the smaller of the two consecutive odd natural number, so that the other one is x +2. Then, we should have

$x > 10 \quad$ ... (1)

and $\quad x + (x + 2) < 40 \quad$ ... (2)

Solving (2), we get

$2x + 2 < 40$

i.e., $\quad x < 19 \quad$ ... (3)

From (1) and (3), we get

$10 < x < 19$

Since x is an odd number, x can take the values 11, 13, 15, and 17. So, the required possible pairs will be

(11, 13), (13, 15), (15, 17), (17, 19)

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EXERCISE 5.1

1. Solve $24x < 100$, when
(i) x is a natural number. (ii) x is an integer.

2. Solve $-12x > 30$, when
(i) x is a natural number. (ii) x is an integer.

3. Solve $5x - 3 < 7$, when
(i) x is an integer. (ii) x is a real number.

4. Solve $3x + 8 > 2$, when
(i) x is an integer. (ii) x is a real number.

Solve the inequalities in Exercises 5 to 16 for real x.

5. $4x + 3 < 5x + 7$

6. $3x - 7 > 5x - 1$

7. $3(x - 1) \le 2(x - 3)$

8. $3(2 - x) \ge 2(1 - x)$

9. $x + \frac{x}{2} + \frac{x}{3} < 11$

10. $\frac{x}{3} > \frac{x}{2} + 1$

11. $\frac{3(x-2)}{5} \le \frac{5(2-x)}{3}$

12. $\frac{1}{2}\left(\frac{3x}{5} + 4\right) \ge \frac{1}{3}(x - 6)$

13. $2(2x + 3) - 10 < 6(x - 2)$

14. $37 - (3x + 5) \ge 9x - 8(x - 3)$

15. $\frac{x}{4} < \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$

16. $\frac{(2x-1)}{3} \ge \frac{(3x-2)}{4} - \frac{(2-x)}{5}$

Solve the inequalities in Exercises 17 to 20 and show the graph of the solution in each case on number line

17. $3x - 2 < 2x + 1$

18. $5x - 3 \ge 3x - 5$

19. $3(1 - x) < 2(x + 4)$

20. $\frac{x}{2} \ge \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$

21. Ravi obtained 70 and 75 marks in first two unit test. Find the minimum marks he should get in the third test to have an average of at least 60 marks.

22. To receive Grade 'A' in a course, one must obtain an average of 90 marks or more in five examinations (each of 100 marks). If Sunita's marks in first four examinations are 87, 92, 94 and 95, find minimum marks that Sunita must obtain in fifth examination to get grade 'A' in the course.

23. Find all pairs of consecutive odd positive integers both of which are smaller than 10 such that their sum is more than 11.

24. Find all pairs of consecutive even positive integers, both of which are larger than 5 such that their sum is less than 23.

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25. The longest side of a triangle is 3 times the shortest side and the third side is 2 cm shorter than the longest side. If the perimeter of the triangle is at least 61 cm, find the minimum length of the shortest side.

26. A man wants to cut three lengths from a single piece of board of length 91cm. The second length is to be 3cm longer than the shortest and the third length is to be twice as long as the shortest. What are the possible lengths of the shortest board if the third piece is to be at least 5cm longer than the second?
[***Hint***: If x is the length of the shortest board, then x , $(x + 3)$ and $2x$ are the lengths of the second and third piece, respectively. Thus, $x + (x + 3) + 2x \leq 91$ and $2x \geq (x + 3) + 5$].

Miscellaneous Examples

Example 9 Solve $-8 \leq 5x - 3 < 7$.

Solution In this case, we have two inequalities, $-8 \leq 5x - 3$ and $5x - 3 < 7$, which we will solve simultaneously. We have $-8 \leq 5x - 3 < 7$

or $\quad -5 \leq 5x < 10 \qquad$ or $\quad -1 \leq x < 2$

Example 10 Solve $-5 \leq \dfrac{5-3x}{2} \leq 8$.

Solution We have $\qquad -5 \leq \dfrac{5-3x}{2} \leq 8$

or $\quad -10 \leq 5 - 3x \leq 16 \qquad$ or $\quad -15 \leq -3x \leq 11$

or $\quad 5 \geq x \geq -\dfrac{11}{3}$

which can be written as $\dfrac{-11}{3} \leq x \leq 5$

Example 11 Solve the system of inequalities:

$$3x - 7 < 5 + x \qquad \text{... (1)}$$

$$11 - 5x \leq 1 \qquad \text{... (2)}$$

and represent the solutions on the number line.

Solution From inequality (1), we have

$3x - 7 < 5 + x$

or $\quad x < 6 \qquad$ ... (3)

Also, from inequality (2), we have

$11 - 5x \leq 1$

or $\quad -5x \leq -10 \qquad$ i.e., $x \geq 2 \qquad$ ... (4)

If we draw the graph of inequalities (3) and (4) on the number line, we see that the values of x, which are common to both, are shown by bold line in Fig 5.3.

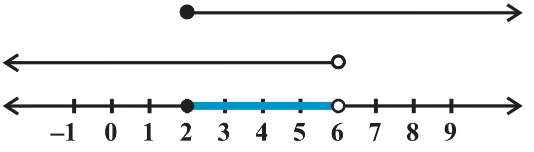

Fig 5.3

Thus, solution of the system are real numbers x lying between 2 and 6 including 2, i.e.,

$$2 \leq x < 6$$

Example 12 In an experiment, a solution of hydrochloric acid is to be kept between 30° and 35° Celsius. What is the range of temperature in degree Fahrenheit if conversion formula is given by $C = \frac{5}{9}(F - 32)$, where C and F represent temperature in degree Celsius and degree Fahrenheit, respectively.

Solution It is given that $30 < C < 35$.

Putting $\quad C = \frac{5}{9}(F - 32)$, we get

$$30 < \frac{5}{9}(F - 32) < 35,$$

or $\quad \frac{9}{5} \times (30) < (F - 32) < \frac{9}{5} \times (35)$

or $\quad 54 < (F - 32) < 63$

or $\quad 86 < F < 95.$

Thus, the required range of temperature is between 86° F and 95° F.

Example 13 A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18%?

Solution Let x litres of 30% acid solution is required to be added. Then

Total mixture = $(x + 600)$ litres

Therefore $\quad 30\%\ x + 12\%$ of $600 > 15\%$ of $(x + 600)$

and $\quad 30\%\ x + 12\%$ of $600 < 18\%$ of $(x + 600)$

or $\quad \frac{30x}{100} + \frac{12}{100}(600) > \frac{15}{100}(x + 600)$

and $\frac{30x}{100} + \frac{12}{100}(600) < \frac{18}{100}(x + 600)$

or $30x + 7200 > 15x + 9000$

and $30x + 7200 < 18x + 10800$

or $15x > 1800$ and $12x < 3600$

or $x > 120$ and $x < 300$,

i.e. $120 < x < 300$

Thus, the number of litres of the 30% solution of acid will have to be more than 120 litres but less than 300 litres.

Miscellaneous Exercise on Chapter 5

Solve the inequalities in Exercises 1 to 6.

1. $2 \leq 3x - 4 \leq 5$

2. $6 \leq -3(2x - 4) < 12$

3. $-3 \leq 4 - \frac{7x}{2} \leq 18$

4. $-15 < \frac{3(x-2)}{5} \leq 0$

5. $-12 < 4 - \frac{3x}{-5} \leq 2$

6. $7 \leq \frac{(3x+11)}{2} \leq 11$.

Solve the inequalities in Exercises 7 to 10 and represent the solution graphically on number line.

7. $5x + 1 > -24$, $5x - 1 < 24$

8. $2(x - 1) < x + 5$, $3(x + 2) > 2 - x$

9. $3x - 7 > 2(x - 6)$, $6 - x > 11 - 2x$

10. $5(2x - 7) - 3(2x + 3) \leq 0$, $2x + 19 \leq 6x + 47$.

11. A solution is to be kept between 68° F and 77° F. What is the range in temperature in degree Celsius (C) if the Celsius / Fahrenheit (F) conversion formula is given by

$$F = \frac{9}{5}C + 32\ ?$$

12. A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be more than 4% but less than 6% boric acid. If we have 640 litres of the 8% solution, how many litres of the 2% solution will have to be added?

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13. How many litres of water will have to be added to 1125 litres of the 45% solution of acid so that the resulting mixture will contain more than 25% but less than 30% acid content?

14. IQ of a person is given by the formula

$$\text{IQ} = \frac{\text{MA}}{\text{CA}} \times 100,$$

where MA is mental age and CA is chronological age. If $80 \leq \text{IQ} \leq 140$ for a group of 12 years old children, find the range of their mental age.

Summary

- Two real numbers or two algebraic expressions related by the symbols $<, >, \leq$ or $\geq$ form an inequality.
- Equal numbers may be added to (or subtracted from) both sides of an inequality.
- Both sides of an inequality can be multiplied (or divided) by the same positive number. But when both sides are multiplied (or divided) by a negative number, then the inequality is reversed.
- The values of x, which make an inequality a true statement, are called *solutions of the inequality*.
- To represent $x < a$ (or $x > a$) on a number line, put a circle on the number a and dark line to the left (or right) of the number a.
- To represent $x \leq a$ (or $x \geq a$) on a number line, put a dark circle on the number a and dark the line to the left (or right) of the number x.

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Chapter 6

PERMUTATIONS AND COMBINATIONS

❖ *Every body of discovery is mathematical in form because there is no other guidance we can have – DARWIN* ❖

6.1 Introduction

Suppose you have a suitcase with a number lock. The number lock has 4 wheels each labelled with 10 digits from 0 to 9. The lock can be opened if 4 specific digits are arranged in a particular sequence with no repetition. Some how, you have forgotten this specific sequence of digits. You remember only the first digit which is 7. In order to open the lock, how many sequences of 3-digits you may have to check with? To answer this question, you may, immediately, start listing all possible arrangements of 9 remaining digits taken 3 at a time. But, this method will be tedious, because the number of possible sequences may be large. Here, in this Chapter, we shall learn some basic counting techniques which will enable us to answer this question without actually listing 3-digit arrangements. In fact, these techniques will be useful in determining the number of different ways of arranging and selecting objects without actually listing them. As a first step, we shall examine a principle which is most fundamental to the learning of these techniques.

Jacob Bernoulli (1654-1705)

6.2 Fundamental Principle of Counting

Let us consider the following problem. Mohan has 3 pants and 2 shirts. How many different pairs of a pant and a shirt, can he dress up with? There are 3 ways in which a pant can be chosen, because there are 3 pants available. Similarly, a shirt can be chosen in 2 ways. For every choice of a pant, there are 2 choices of a shirt. Therefore, there are $3 \times 2 = 6$ pairs of a pant and a shirt.

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Let us name the three pants as P_1, P_2, P_3 and the two shirts as S_1, S_2. Then, these six possibilities can be illustrated in the Fig. 6.1.

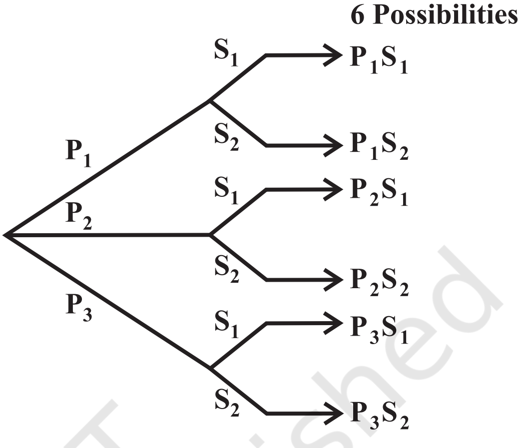

Fig 6.1

Let us consider another problem of the same type.

Sabnam has 2 school bags, 3 tiffin boxes and 2 water bottles. In how many ways can she carry these items (choosing one each).

A school bag can be chosen in 2 different ways. After a school bag is chosen, a tiffin box can be chosen in 3 different ways. Hence, there are $2 \times 3 = 6$ pairs of school bag and a tiffin box. For each of these pairs a water bottle can be chosen in 2 different ways. Hence, there are $6 \times 2 = 12$ different ways in which, Sabnam can carry these items to school. If we name the 2 school bags as B_1, B_2, the three tiffin boxes as T_1, T_2, T_3 and the two water bottles as W_1, W_2, these possibilities can be illustrated in the Fig. 6.2.

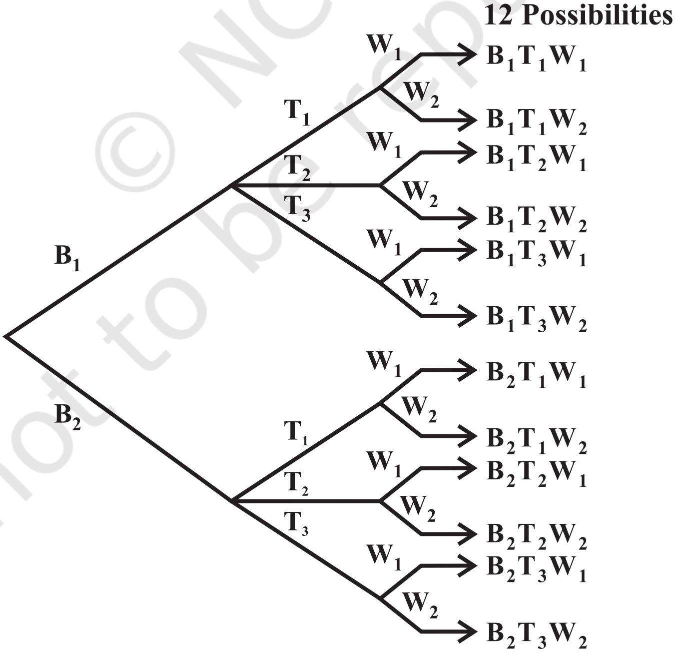

Fig 6.2

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In fact, the problems of the above types are solved by applying the following principle known as the *fundamental principle of counting*, or, simply, the *multiplication principle*, which states that

"If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is m×n."

The above principle can be generalised for any finite number of events. For example, for 3 events, the principle is as follows:

'If an event can occur in m different ways, following which another event can occur in n different ways, following which a third event can occur in p different ways, then the total number of occurrence to 'the events in the given order is $m \times n \times p$."

In the first problem, the required number of ways of wearing a pant and a shirt was the number of different ways of the occurence of the following events in succession:

(i) the event of choosing a pant
(ii) the event of choosing a shirt.

In the second problem, the required number of ways was the number of different ways of the occurence of the following events in succession:

(i) the event of choosing a school bag
(ii) the event of choosing a tiffin box
(iii) the event of choosing a water bottle.

Here, in both the cases, the events in each problem could occur in various possible orders. But, we have to choose any one of the possible orders and count the number of different ways of the occurence of the events in this chosen order.

Example 1 Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, where the repetition of the letters is not allowed.

Solution There are as many words as there are ways of filling in 4 vacant places $\square\,\square\,\square\,\square$ by the 4 letters, keeping in mind that the repetition is not allowed. The first place can be filled in 4 different ways by anyone of the 4 letters R,O,S,E. Following which, the second place can be filled in by anyone of the remaining 3 letters in 3 different ways, following which the third place can be filled in 2 different ways; following which, the fourth place can be filled in 1 way. Thus, the number of ways in which the 4 places can be filled, by the multiplication principle, is $4 \times 3 \times 2 \times 1 = 24$. Hence, the required number of words is 24.

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Note If the repetition of the letters was allowed, how many words can be formed? One can easily understand that each of the 4 vacant places can be filled in succession in 4 different ways. Hence, the required number of words = $4 \times 4 \times 4 \times 4 = 256$.

Example 2 Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of 2 flags one below the other?

Solution There will be as many signals as there are ways of filling in 2 vacant places in succession by the 4 flags of different colours. The upper vacant place can be filled in 4 different ways by anyone of the 4 flags; following which, the lower vacant place can be filled in 3 different ways by anyone of the remaining 3 different flags. Hence, by the multiplication principle, the required number of signals = $4 \times 3 = 12$.

Example 3 How many 2 digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

Solution There will be as many ways as there are ways of filling 2 vacant places in succession by the five given digits. Here, in this case, we start filling in unit's place, because the options for this place are 2 and 4 only and this can be done in 2 ways; following which the ten's place can be filled by any of the 5 digits in 5 different ways as the digits can be repeated. Therefore, by the multiplication principle, the required number of two digits even numbers is 2×5, i.e., 10.

Example 4 Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.

Solution A signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags. Now, let us count the possible number of signals consisting of 2 flags, 3 flags, 4 flags and 5 flags separately and then add the respective numbers.

There will be as many 2 flag signals as there are ways of filling in 2 vacant places in succession by the 5 flags available. By Multiplication rule, the number of ways is $5 \times 4 = 20$.

Similarly, there will be as many 3 flag signals as there are ways of filling in 3 vacant places in succession by the 5 flags.

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The number of ways is $5 \times 4 \times 3 = 60$.

Continuing the same way, we find that

The number of 4 flag signals $= 5 \times 4 \times 3 \times 2 = 120$

and the number of 5 flag signals $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

Therefore, the required no of signals $= 20 + 60 + 120 + 120 = 320$.

EXERCISE 6.1

1. How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming that

(i) repetition of the digits is allowed?

(ii) repetition of the digits is not allowed?

2. How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?

3. How many 4-letter code can be formed using the first 10 letters of the English alphabet, if no letter can be repeated?

4. How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 and no digit appears more than once?

5. A coin is tossed 3 times and the outcomes are recorded. How many possible outcomes are there?

6. Given 5 flags of different colours, how many different signals can be generated if each signal requires the use of 2 flags, one below the other?

6.3 Permutations

In Example 1 of the previous Section, we are actually counting the different possible arrangements of the letters such as ROSE, REOS, ..., etc. Here, in this list, each arrangement is different from other. In other words, the order of writing the letters is important. Each arrangement is called a *permutation of* 4 *different letters taken all at a time*. Now, if we have to determine the number of 3-letter words, with or without meaning, which can be formed out of the letters of the word NUMBER, where the repetition of the letters is not allowed, we need to count the arrangements NUM, NMU, MUN, NUB, ..., etc. Here, we are counting the permutations of 6 different letters taken 3 at a time. The required number of words $= 6 \times 5 \times 4 = 120$ (by using multiplication principle).

If the repetition of the letters was allowed, the required number of words would be $6 \times 6 \times 6 = 216$.

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Definition 1 A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.

In the following sub-section, we shall obtain the formula needed to answer these questions immediately.

6.3.1 *Permutations when all the objects are distinct*

Theorem 1 The number of permutations of n different objects taken r at a time, where $0 < r \le n$ and the objects do not repeat is $n(n-1)(n-2)\ldots(n-r+1)$, which is denoted by ^nP_r.

Proof There will be as many permutations as there are ways of filling in r vacant places $\square\ \square\ \square \ldots \square$ by

$\leftarrow r$ vacant places $\rightarrow$

the n objects. The first place can be filled in n ways; following which, the second place can be filled in $(n-1)$ ways, following which the third place can be filled in $(n-2)$ ways,..., the rth place can be filled in $(n-(r-1))$ ways. Therefore, the number of ways of filling in r vacant places in succession is $n(n-1)(n-2)\ldots(n-(r-1))$ or $n(n-1)(n-2)\ldots(n-r+1)$

This expression for ^nP_r is cumbersome and we need a notation which will help to reduce the size of this expression. The symbol $n!$ (read as factorial n or n factorial) comes to our rescue. In the following text we will learn what actually $n!$ means.

6.3.2 ***Factorial notation*** The notation $n!$ represents the product of first n natural numbers, i.e., the product $1 \times 2 \times 3 \times \ldots \times (n-1) \times n$ is denoted as $n!$. We read this symbol as 'n factorial'. Thus, $1 \times 2 \times 3 \times 4 \ldots \times (n-1) \times n = n!$

$$1 = 1!$$
$$1 \times 2 = 2!$$
$$1 \times 2 \times 3 = 3!$$
$$1 \times 2 \times 3 \times 4 = 4! \text{ and so on.}$$

We define $0! = 1$

We can write $5! = 5 \times 4! = 5 \times 4 \times 3! = 5 \times 4 \times 3 \times 2!$
$= 5 \times 4 \times 3 \times 2 \times 1!$

Clearly, for a natural number n

$$
\begin{aligned}
n! &= n(n-1)! \\
&= n(n-1)(n-2)! && \text{[provided } (n \ge 2)] \\
&= n(n-1)(n-2)(n-3)! && \text{[provided } (n \ge 3)]
\end{aligned}
$$

and so on.

Example 5 Evaluate (i) 5 ! (ii) 7 ! (iii) 7 ! – 5!

Solution (i) $5\,! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

(ii) $7\,! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$

and (iii) $7\,! - 5! = 5040 - 120 = 4920$.

Example 6 Compute (i) $\frac{7!}{5!}$ (ii) $\frac{12!}{(10!)\,(2!)}$

Solution (i) We have $\frac{7!}{5!} = \frac{7 \times 6 \times 5!}{5!} = 7 \times 6 = 42$

and (ii) $\frac{12!}{(10!)\,(2!)} = \frac{12 \times 11 \times (10!)}{(10!) \times (2)} = 6 \times 11 = 66$.

Example 7 Evaluate $\frac{n!}{r!(n-r)!}$, when $n = 5$, $r = 2$.

Solution We have to evaluate $\frac{5!}{2!(5-2)!}$ (since $n = 5$, $r = 2$)

We have $\frac{5!}{2\,!(5-2)!} = \frac{5!}{2! \times 3!} = \frac{5 \times 4}{2} = 10$.

Example 8 If $\frac{1}{8!} + \frac{1}{9!} = \frac{x}{10!}$, find x.

Solution We have $\frac{1}{8!} + \frac{1}{9 \times 8!} = \frac{x}{10 \times 9 \times 8!}$

Therefore $1 + \frac{1}{9} = \frac{x}{10 \times 9}$ or $\frac{10}{9} = \frac{x}{10 \times 9}$

So $x = 100$.

EXERCISE 6.2

1. Evaluate

(i) 8 ! (ii) 4 ! – 3 !

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2. Is 3 ! + 4 ! = 7 ! ?   **3.** Compute $\dfrac{8!}{6!\times 2!}$ **4.** If $\dfrac{1}{6!}+\dfrac{1}{7!}=\dfrac{x}{8!}$, find x

5. Evaluate $\dfrac{n!}{(n-r)!}$, when

(i) $n = 6, r = 2$ (ii) $n = 9, r = 5$.

6.3.3 *Derivation of the formula for ^nP_r*

$$^n\mathrm{P}_r=\frac{n!}{(n-r)!},\ 0\le r\le n$$

Let us now go back to the stage where we had determined the following formula:

$$^n\mathrm{P}_r = n\ (n-1)\ (n-2)\ .\ .\ .\ \ (n-r+1)$$

Multiplying numerator and denomirator by $(n-r)\ (n-r-1)\ .\ .\ .\ 3\times 2\times 1$, we get

$$^n\mathrm{P}_r=\frac{n(n-1)\,(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots3\times2\times1}{(n-r)(n-r-1)\ldots3\times2\times1}=\frac{n!}{(n-r)!},$$

Thus $\qquad ^n\mathrm{P}_r=\dfrac{n!}{(n-r)!}$, where $0 < r \le n$

This is a much more convenient expression for $^n\mathrm{P}_r$ than the previous one.

In particular, when $r = n$, $^n\mathrm{P}_n=\dfrac{n!}{0!}=n!$

Counting permutations is merely counting the number of ways in which some or all objects at a time are rearranged. Arranging no object at all is the same as leaving behind all the objects and we know that there is only one way of doing so. Thus, we can have

$$^n\mathrm{P}_0 = 1 = \frac{n!}{n!}=\frac{n!}{(n-0)!} \qquad \ldots (1)$$

Therefore, the formula (1) is applicable for $r = 0$ also.

Thus $\qquad ^n\mathrm{P}_r=\dfrac{n!}{(n-r)!},\ 0\le r\le n$.

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Theorem 2 The number of permutations of n different objects taken r at a time, where repetition is allowed, is n^r.

Proof is very similar to that of Theorem 1 and is left for the reader to arrive at.

Here, we are solving some of the problems of the pervious Section using the formula for ${}^nP_r$ to illustrate its usefulness.

In Example 1, the required number of words = ${}^4P_4 = 4! = 24$. Here repetition is not allowed. If repetition is allowed, the required number of words would be $4^4 = 256$.

The number of 3-letter words which can be formed by the letters of the word NUMBER = ${}^6P_3 = \dfrac{6!}{3!} = 4 \times 5 \times 6 = 120$. Here, in this case also, the repetition is not allowed. If the repetition is allowed,the required number of words would be $6^3 = 216$.

The number of ways in which a Chairman and a Vice-Chairman can be chosen from amongst a group of 12 persons assuming that one person can not hold more than one position, clearly ${}^{12}P_2 = \dfrac{12!}{10!} = 11 \times 12 = 132$.

6.3.4 ***Permutations when all the objects are not distinct objects*** Suppose we have to find the number of ways of rearranging the letters of the word ROOT. In this case, the letters of the word are not all different. There are 2 Os, which are of the same kind. Let us treat, temporarily, the 2 Os as different, say, O_1 and O_2. The number of permutations of 4-different letters, in this case, taken all at a time is 4!. Consider one of these permutations say, RO_1O_2T. Corresponding to this permutation,we have 2 ! permutations RO_1O_2T and RO_2O_1T which will be exactly the same permutation if O_1 and O_2 are not treated as different, i.e., if O_1 and O_2 are the same O at both places.

Therefore, the required number of permutations = $\dfrac{4!}{2!} = 3 \times 4 = 12$.

Permutations when O_1, O_2 are different.		Permutations when O_1, O_2 are the same O.
RO_1O_2T RO_2O_1T	$\longrightarrow$	R O O T
TO_1O_2R TO_2O_1R	$\longrightarrow$	T O O R

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$$\left.\begin{array}{l} R\,O_1T\,O_2 \\ R\,O_2T\,O_1 \end{array}\right] \longrightarrow \text{R O T O}$$

$$\left.\begin{array}{l} T\,O_1R\,O_2 \\ T\,O_2R\,O_1 \end{array}\right] \longrightarrow \text{T O R O}$$

$$\left.\begin{array}{l} R\,T\,O_1\,O_2 \\ R\,T\,O_2\,O_1 \end{array}\right] \longrightarrow \text{R T O O}$$

$$\left.\begin{array}{l} T\,R\,O_1\,O_2 \\ T\,R\,O_2\,O_1 \end{array}\right] \longrightarrow \text{T R O O}$$

$$\left.\begin{array}{l} O_1\;O_2\;R\;T \\ O_2\;O_1\;T\;R \end{array}\right] \longrightarrow \text{O O R T}$$

$$\left.\begin{array}{l} O_1\,R\,O_2\,T \\ O_2\,R\,O_1\,T \end{array}\right] \longrightarrow \text{O R O T}$$

$$\left.\begin{array}{l} O_1\;T\,O_2\;R \\ O_2\;T\,O_1\;R \end{array}\right] \longrightarrow \text{O T O R}$$

$$\left.\begin{array}{l} O_1\;R\,T\,O_2 \\ O_2\;R\,T\,O_1 \end{array}\right] \longrightarrow \text{O R T O}$$

$$\left.\begin{array}{l} O_1\;T\,R\,O_2 \\ O_2\;T\,R\,O_1 \end{array}\right] \longrightarrow \text{O T R O}$$

$$\left.\begin{array}{l} O_1\;O_2T\,R \\ O_2\;O_1\,T\,R \end{array}\right] \longrightarrow \text{O O T R}$$

Let us now find the number of ways of rearranging the letters of the word INSTITUTE. In this case there are 9 letters, in which I appears 2 times and T appears 3 times.

Temporarily, let us treat these letters different and name them as I_1, I_2, T_1, T_2, T_3. The number of permutations of 9 different letters, in this case, taken all at a time is 9 !. Consider one such permutation, say, $I_1\,NT_1\,SI_2\,T_2\,U\,E\,T_3$. Here if I_1, I_2 are not same

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and T_1, T_2, T_3 are not same, then I_1, I_2 can be arranged in 2! ways and T_1, T_2, T_3 can be arranged in 3! ways. Therefore, 2! × 3! permutations will be just the same permutation corresponding to this chosen permutation $I_1NT_1SI_2T_2UET_3$. Hence, total number of different permutations will be $\frac{9!}{2!\,3!}$

We can state (without proof) the following theorems:

Theorem 3 The number of permutations of n objects, where p objects are of the same kind and rest are all different $= \frac{n!}{p!}$.

In fact, we have a more general theorem.

Theorem 4 The number of permutations of n objects, where p_1 objects are of one kind, p_2 are of second kind, ..., p_k are of k^{th} kind and the rest, if any, are of different kind is $\frac{n!}{p_1!\, p_2! \ldots p_k!}$.

Example 9 Find the number of permutations of the letters of the word ALLAHABAD.

Solution Here, there are 9 objects (letters) of which there are 4A's, 2 L's and rest are all different.

Therefore, the required number of arrangements $= \frac{9!}{4!\,2!} = \frac{5 \times 6 \times 7 \times 8 \times 9}{2} = 7560$

Example 10 How many 4-digit numbers can be formed by using the digits 1 to 9 if repetition of digits is not allowed?

Solution Here order matters for example 1234 and 1324 are two different numbers. Therefore, there will be as many 4 digit numbers as there are permutations of 9 different digits taken 4 at a time.

Therefore, the required 4 digit numbers $= {}^9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = 9 \times 8 \times 7 \times 6 = 3024.$

Example 11 How many numbers lying between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5, if the repetition of the digits is not allowed?

Solution Every number between 100 and 1000 is a 3-digit number. We, first, have to count the permutations of 6 digits taken 3 at a time. This number would be $^6P_3$. But, these permutations will include those also where 0 is at the 100's place. For example, 092, 042, . . ., etc are such numbers which are actually 2-digit numbers and hence the number of such numbers has to be subtracted from $^6P_3$ to get the required number. To get the number of such numbers, we fix 0 at the 100's place and rearrange the remaining 5 digits taking 2 at a time. This number is $^5P_2$. So

The required number $= {}^6P_3 - {}^5P_2 = \dfrac{6!}{3!} - \dfrac{5!}{3!}$

$= 4 \times 5 \times 6 - 4 \times 5 = 100$

Example 12 Find the value of n such that

(i) ${}^nP_5 = 42\ {}^nP_3,\ n > 4$ (ii) $\dfrac{{}^nP_4}{{}^{n-1}P_4} = \dfrac{5}{3},\ n > 4$

Solution (i) Given that

${}^nP_5 = 42\ {}^nP_3$

or $n\,(n-1)\,(n-2)\,(n-3)\,(n-4) = 42\,n(n-1)\,(n-2)$

Since $n > 4$ so $n(n-1)\,(n-2) \neq 0$

Therefore, by dividing both sides by $n(n-1)\,(n-2)$, we get

$(n - 3\,(n-4) = 42$

or $n^2 - 7n - 30 = 0$

or $n^2 - 10n + 3n - 30$

or $(n-10)\,(n+3) = 0$

or $n - 10 = 0$ or $n + 3 = 0$

or $n = 10$ or $n = -3$

As n cannot be negative, so $n = 10$.

(ii) Given that $\dfrac{{}^nP_4}{{}^{n-1}P_4} = \dfrac{5}{3}$

Therefore $3n\,(n-1)\,(n-2)\,(n-3) = 5(n-1)\,(n-2)\,(n-3)\,(n-4)$

or $3n = 5\,(n-4)$ [as $(n-1)\,(n-2)\,(n-3) \neq 0,\ n > 4$]

or $n = 10$.

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Example 13 Find r, if $5\ {}^4P_r = 6\ {}^5P_{r-1}$.

Solution We have $5\ {}^4P_r = 6\ {}^5P_{r-1}$

or $$5 \times \frac{4!}{(4-r)!} = 6 \times \frac{5!}{(5-r+1)!}$$

or $$\frac{5!}{(4-r)!} = \frac{6 \times 5!}{(5-r+1)(5-r)(5-r-1)!}$$

or $(6 - r)(5 - r) = 6$

or $r^2 - 11r + 24 = 0$

or $r^2 - 8r - 3r + 24 = 0$

or $(r - 8)(r - 3) = 0$

or $r = 8$ or $r = 3$.

Hence $r = 8, 3$.

Example 14 Find the number of different 8-letter arrangements that can be made from the letters of the word DAUGHTER so that

(i) all vowels occur together (ii) all vowels do not occur together.

Solution (i) There are 8 different letters in the word DAUGHTER, in which there are 3 vowels, namely, A, U and E. Since the vowels have to occur together, we can for the time being, assume them as a single object (AUE). This single object together with 5 remaining letters (objects) will be counted as 6 objects. Then we count permutations of these 6 objects taken all at a time. This number would be ${}^6P_6 = 6!$. Corresponding to each of these permutations, we shall have 3! permutations of the three vowels A, U, E taken all at a time . Hence, by the multiplication principle the required number of permutations $= 6\,! \times 3\,! = 4320$.

(ii) If we have to count those permutations in which all vowels are never together, we first have to find all possible arrangments of 8 letters taken all at a time, which can be done in 8! ways. Then, we have to subtract from this number, the number of permutations in which the vowels are always together.

Therefore, the required number $8\,! - 6\,! \times 3\,! = 6\,!\,(7{\times}8 - 6)$

$= 2 \times 6\,!\,(28 - 3)$

$= 50 \times 6\,! = 50 \times 720 = 36000$

Example 15 In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same colour are indistinguishable ?

Solution Total number of discs are $4 + 3 + 2 = 9$. Out of 9 discs, 4 are of the first kind

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(red), 3 are of the second kind (yellow) and 2 are of the third kind (green).

Therefore, the number of arrangements $\frac{9!}{4!\,3!\,2!} = 1260$.

Example 16 Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements,

(i) do the words start with P
(ii) do all the vowels always occur together
(iii) do the vowels never occur together
(iv) do the words begin with I and end in P?

Solution There are 12 letters, of which N appears 3 times, E appears 4 times and D appears 2 times and the rest are all different. Therefore

The required number of arrangements $= \frac{12!}{3!\,4!\,2!} = 1663200$

(i) Let us fix P at the extreme left position, we, then, count the arrangements of the remaining 11 letters. Therefore, the required number of words starting with P

$$= \frac{11!}{3!\,2!\,4!} = 138600.$$

(ii) There are 5 vowels in the given word, which are 4 Es and 1 I. Since, they have to always occur together, we treat them as a single object $\boxed{\text{EEEEI}}$ for the time being. This single object together with 7 remaining objects will account for 8 objects. These 8 objects, in which there are 3Ns and 2 Ds, can be rearranged in $\frac{8!}{3!\,2!}$ ways. Corresponding to each of these arrangements, the 5 vowels E, E, E, E and I can be rearranged in $\frac{5!}{4!}$ ways. Therefore, by multiplication principle, the required number of arrangements

$$= \frac{8!}{3!\,2!} \times \frac{5!}{4!} = 16800$$

(iii) The required number of arrangements
= the total number of arrangements (without any restriction) – the number of arrangements where all the vowels occur together.

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$$= 1663200 - 16800 = 1646400$$

(iv) Let us fix I and P at the extreme ends (I at the left end and P at the right end). We are left with 10 letters.

Hence, the required number of arrangements

$$= \frac{10!}{3!\,2!\,4!} = 12600$$

EXERCISE 6.3

1. How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated?
2. How many 4-digit numbers are there with no digit repeated?
3. How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 6, 7, if no digit is repeated?
4. Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 if no digit is repeated. How many of these will be even?
5. From a committee of 8 persons, in how many ways can we choose a chairman and a vice chairman assuming one person can not hold more than one position?
6. Find n if ${}^{n-1}P_3 : {}^{n}P_4 = 1 : 9$.
7. Find r if (i) ${}^5P_r = 2\,{}^6P_{r-1}$ (ii) ${}^5P_r = {}^6P_{r-1}$.
8. How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once?
9. How many words, with or without meaning can be made from the letters of the word MONDAY, assuming that no letter is repeated, if.
   (i) 4 letters are used at a time, (ii) all letters are used at a time,
   (iii) all letters are used but first letter is a vowel?
10. In how many of the distinct permutations of the letters in MISSISSIPPI do the four I's not come together?
11. In how many ways can the letters of the word PERMUTATIONS be arranged if the
   (i) words start with P and end with S, (ii) vowels are all together,
   (iii) there are always 4 letters between P and S?

6.4 Combinations

Let us now assume that there is a group of 3 lawn tennis players X, Y, Z. A team consisting of 2 players is to be formed. In how many ways can we do so? Is the team of X and Y different from the team of Y and X ? Here, order is not important. In fact, there are only 3 possible ways in which the team could be constructed.

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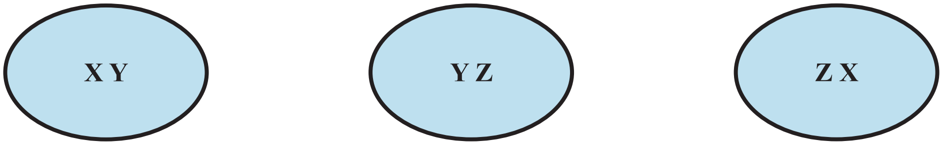

Fig. 6.3

These are XY, YZ and ZX (Fig 6.3).

Here, each selection is called a *combination of 3 different objects taken 2 at a time*. In a combination, the order is not important.

Now consider some more illustrations.

Twelve persons meet in a room and each shakes hand with all the others. How do we determine the number of hand shakes. X shaking hands with Y and Y with X will not be two different hand shakes. Here, order is not important. There will be as many hand shakes as there are combinations of 12 different things taken 2 at a time.

Seven points lie on a circle. How many chords can be drawn by joining these points pairwise? There will be as many chords as there are combinations of 7 different things taken 2 at a time.

Now, we obtain the formula for finding the number of combinations of n different objects taken r at a time, denoted by ${}^nC_r$..

Suppose we have 4 different objects A, B, C and D. Taking 2 at a time, if we have to make combinations, these will be AB, AC, AD, BC, BD, CD. Here, AB and BA are the same combination as order does not alter the combination. This is why we have not included BA, CA, DA, CB, DB and DC in this list. There are as many as 6 combinations of 4 different objects taken 2 at a time, i.e., ${}^4C_2 = 6$.

Corresponding to each combination in the list, we can arrive at 2! permutations as 2 objects in each combination can be rearranged in 2! ways. Hence, the number of permutations = ${}^4C_2 \times 2!$.

On the other hand, the number of permutations of 4 different things taken 2 at a time = ${}^4P_2$.

Therefore $\quad \mathbf{{}^4P_2 = {}^4C_2 \times 2!}$ or $\dfrac{4!}{(4-2)!\,2!} = {}^4C_2$

Now, let us suppose that we have 5 different objects A, B, C, D, E. Taking 3 at a time, if we have to make combinations, these will be ABC, ABD, ABE, BCD, BCE, CDE, ACE, ACD, ADE, BDE. Corresponding to each of these ${}^5C_3$ combinations, there are 3! permutations, because, the three objects in each combination can be

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rearranged in 3 ! ways. Therefore, the total of permutations = ${}^5C_3 \times 3!$

Therefore $\quad {}^5\mathbf{P}_3 = {}^5\mathbf{C}_3 \times \mathbf{3!} \quad$ or $\quad \dfrac{5!}{(5-3)!\ 3!} = {}^5C_3$

These examples suggest the following theorem showing relationship between permutaion and combination:

Theorem 5 ${}^nP_r = {}^nC_r\ r!$, $0 < r \le n$.

Proof Corresponding to each combination of ${}^nC_r$, we have r ! permutations, because r objects in every combination can be rearranged in r ! ways.

Hence, the total number of permutations of n different things taken r at a time is ${}^nC_r \times r!$. On the other hand, it is ${}^nP_r$. Thus

$$ {}^nP_r = {}^nC_r \times r!,\ 0 < r \le n . $$

Remarks **1.** From above $\dfrac{n!}{(n-r)!} = {}^nC_r \times r!$, i.e., $\quad {}^nC_r = \dfrac{n!}{r!(n-r)!}$.

In particular, if $r = n$, ${}^nC_n = \dfrac{n!}{n!\ 0!} = 1$.

2. We define ${}^nC_0 = 1$, i.e., the number of combinations of n different things taken nothing at all is considered to be 1. Counting combinations is merely counting the number of ways in which some or all objects at a time are selected. Selecting nothing at all is the same as leaving behind all the objects and we know that there is only one way of doing so. This way we define ${}^nC_0 = 1$.

3. As $\dfrac{n!}{0!(n-0)!} = 1 = {}^nC_0$, the formula ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ is applicable for $r = 0$ also.

Hence

$$ {}^nC_r = \frac{n!}{r!\ (n-r)!},\ 0 \le r \le n. $$

4. $\quad {}^nC_{n-r} = \dfrac{n!}{(n-r)!(n-(n-r))!} = \dfrac{n!}{(n-r)!r!} = {}^nC_r$,

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i.e., selecting r objects out of n objects is same as rejecting $(n - r)$ objects.

5. ${}^nC_a = {}^nC_b \Rightarrow a = b$ or $a = n - b$, i.e., $n = a + b$

Theorem 6 ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Proof We have $${}^nC_r + {}^nC_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!}{r \times (r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!} \times \frac{n-r+1+r}{r(n-r+1)} = \frac{(n+1)!}{r!(n+1-r)!} = {}^{n+1}C_r$$

Example 17 If ${}^nC_9 = {}^nC_8$, find ${}^nC_{17}$.

Solution We have ${}^nC_9 = {}^nC_8$

i.e., $$\frac{n!}{9!(n-9)!} = \frac{n!}{(n-8)!\,8!}$$

or $$\frac{1}{9} = \frac{1}{n-8} \quad \text{or} \quad n - 8 = 9 \quad \text{or} \quad n = 17$$

Therefore ${}^nC_{17} = {}^{17}C_{17} = 1$.

Example 18 A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?

Solution Here, order does not matter. Therefore, we need to count combinations. There will be as many committees as there are combinations of 5 different persons taken 3 at a time. Hence, the required number of ways $= {}^5C_3 = \frac{5!}{3!\,2!} = \frac{4 \times 5}{2} = 10$.

Now, 1 man can be selected from 2 men in ${}^2C_1$ ways and 2 women can be selected from 3 women in ${}^3C_2$ ways. Therefore, the required number of committees

$$= {}^2C_1 \times {}^3C_2 = \frac{2!}{1!\ 1!} \times \frac{3!}{2!\ 1!} = 6.$$

Example 19 What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these

(i) four cards are of the same suit,

(ii) four cards belong to four different suits,

(iii) are face cards,

(iv) two are red cards and two are black cards,

(v) cards are of the same colour?

Solution There will be as many ways of choosing 4 cards from 52 cards as there are combinations of 52 different things, taken 4 at a time. Therefore

The required number of ways = $^{52}C_4 = \frac{52!}{4!\ 48!} = \frac{49 \times 50 \times 51 \times 52}{2 \times 3 \times 4}$

$$= 270725$$

(i) There are four suits: diamond, club, spade, heart and there are 13 cards of each suit. Therefore, there are $^{13}C_4$ ways of choosing 4 diamonds. Similarly, there are $^{13}C_4$ ways of choosing 4 clubs, $^{13}C_4$ ways of choosing 4 spades and $^{13}C_4$ ways of choosing 4 hearts. Therefore

The required number of ways $= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$.

$$= 4 \times \frac{13!}{4!\ 9!} = 2860$$

(ii) There are13 cards in each suit.

Therefore, there are $^{13}C_1$ ways of choosing 1 card from 13 cards of diamond, $^{13}C_1$ ways of choosing 1 card from 13 cards of hearts, $^{13}C_1$ ways of choosing 1 card from 13 cards of clubs, $^{13}C_1$ ways of choosing 1 card from 13 cards of spades. Hence, by multiplication principle, the required number of ways

$$= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4$$

(iii) There are 12 face cards and 4 are to be selected out of these 12 cards. This can be done in $^{12}C_4$ ways. Therefore, the required number of ways = $\frac{12!}{4!\ 8!} = 495$.

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(iv) There are 26 red cards and 26 black cards. Therefore, the required number of ways = ${}^{26}C_2 \times {}^{26}C_2$

$$= \left(\frac{26!}{2!\ 24!}\right)^2 = (325)^2 = 105625$$

(v) 4 red cards can be selected out of 26 red cards in ${}^{26}C_4$ ways. 4 black cards can be selected out of 26 black cards in ${}^{26}C_4$ways.

Therefore, the required number of ways = ${}^{26}C_4 + {}^{26}C_4$

$$= 2 \times \frac{26!}{4!\ 22!} = 29900.$$

EXERCISE 6.4

1. If ${}^{n}C_8 = {}^{n}C_2$, find ${}^{n}C_2$.

2. Determine n if

(i) ${}^{2n}C_3 : {}^{n}C_3 = 12 : 1$ (ii) ${}^{2n}C_3 : {}^{n}C_3 = 11 : 1$

3. How many chords can be drawn through 21 points on a circle?

4. In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls?

5. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.

6. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in each combination.

7. In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers?

8. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected.

9. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student?

Miscellaneous Examples

Example 20 How many words, with or without meaning, each of 3 vowels and 2 consonants can be formed from the letters of the word INVOLUTE ?

Solution In the word INVOLUTE, there are 4 vowels, namely, I,O,E,Uand 4 consonants, namely, N, V, L and T.

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The number of ways of selecting 3 vowels out of 4 = ${}^4C_3 = 4$.
The number of ways of selecting 2 consonants out of 4 = ${}^4C_2 = 6$.

Therefore, the number of combinations of 3 vowels and 2 consonants is $4 \times 6 = 24$.

Now, each of these 24 combinations has 5 letters which can be arranged among themselves in 5 ! ways. Therefore, the required number of different words is $24 \times 5\,! = 2880$.

Example 21 A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has (i) no girl ? (ii) at least one boy and one girl ? (iii) at least 3 girls ?

Solution (i) Since, the team will not include any girl, therefore, only boys are to be selected. 5 boys out of 7 boys can be selected in ${}^7C_5$ ways. Therefore, the required

number of ways $= {}^7C_5 = \dfrac{7!}{5!\ 2!} = \dfrac{6 \times 7}{2} = 21$

(ii) Since, at least one boy and one girl are to be there in every team. Therefore, the team can consist of

(a) 1 boy and 4 girls (b) 2 boys and 3 girls

(c) 3 boys and 2 girls (d) 4 boys and 1 girl.

1 boy and 4 girls can be selected in ${}^7C_1 \times {}^4C_4$ ways.

2 boys and 3 girls can be selected in ${}^7C_2 \times {}^4C_3$ ways.

3 boys and 2 girls can be selected in ${}^7C_3 \times {}^4C_2$ ways.

4 boys and 1 girl can be selected in ${}^7C_4 \times {}^4C_1$ ways.

Therefore, the required number of ways

$= {}^7C_1 \times {}^4C_4 + {}^7C_2 \times {}^4C_3 + {}^7C_3 \times {}^4C_2 + {}^7C_4 \times {}^4C_1$

$= 7 + 84 + 210 + 140 = 441$

(iii) Since, the team has to consist of at least 3 girls, the team can consist of
(a) 3 girls and 2 boys, or (b) 4 girls and 1 boy.

Note that the team cannot have all 5 girls, because, the group has only 4 girls.

3 girls and 2 boys can be selected in ${}^4C_3 \times {}^7C_2$ ways.
4 girls and 1 boy can be selected in ${}^4C_4 \times {}^7C_1$ ways.

Therefore, the required number of ways

$= {}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1 = 84 + 7 = 91$

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Example 22 Find the number of words with or without meaning which can be made using all the letters of the word AGAIN. If these words are written as in a dictionary, what will be the 50th word?

Solution There are 5 letters in the word AGAIN, in which A appears 2 times. Therefore, the required number of words = $\frac{5!}{2!} = 60$.

To get the number of words starting with A, we fix the letter A at the extreme left position, we then rearrange the remaining 4 letters taken all at a time. There will be as many arrangements of these 4 letters taken 4 at a time as there are permutations of 4 different things taken 4 at a time. Hence, the number of words starting with A = $4! = 24$. Then, starting with G, the number of words $= \frac{4!}{2!} = 12$ as after placing G at the extreme left position, we are left with the letters A, A, I and N. Similarly, there are 12 words starting with the next letter I. Total number of words so far obtained = 24 + 12 + 12 =48.

The 49th word is NAAGI. The 50th word is NAAIG.

Example 23 How many numbers greater than 1000000 can be formed by using the digits 1, 2, 0, 2, 4, 2, 4?

Solution Since, 1000000 is a 7-digit number and the number of digits to be used is also 7. Therefore, the numbers to be counted will be 7-digit only. Also, the numbers have to be greater than 1000000, so they can begin either with 1, 2 or 4.

The number of numbers beginning with 1 = $\frac{6!}{3!\,2!} = \frac{4 \times 5 \times 6}{2}$ = 60, as when 1 is fixed at the extreme left position, the remaining digits to be rearranged will be 0, 2, 2, 2, 4, 4, in which there are 3, 2*s* and 2, 4*s*.

Total numbers begining with 2

$$= \frac{6!}{2!\;2!} = \frac{3 \times 4 \times 5 \times 6}{2} = 180$$

and total numbers begining with 4 $= \frac{6!}{3!} = 4 \times 5 \times 6 = 120$

Therefore, the required number of numbers $= 60 + 180 + 120 = 360$.

Alternative Method

The number of 7-digit arrangements, clearly, $\frac{7!}{3!\ 2!} = 420$. But, this will include those numbers also, which have 0 at the extreme left position. The number of such arrangements $\frac{6!}{3!\ 2!}$ (by fixing 0 at the extreme left position) = 60.

Therefore, the required number of numbers = 420 – 60 = 360.

Note If one or more than one digits given in the list is repeated, it will be understood that in any number, the digits can be used as many times as is given in the list, e.g., in the above example 1 and 0 can be used only once whereas 2 and 4 can be used 3 times and 2 times, respectively.

Example 24 In how many ways can 5 girls and 3 boys be seated in a row so that no two boys are together?

Solution Let us first seat the 5 girls. This can be done in 5! ways. For each such arrangement, the three boys can be seated only at the cross marked places.

$$\times\ G \times G \times G \times G \times G \times.$$

There are 6 cross marked places and the three boys can be seated in $^6P_3$ ways. Hence, by multiplication principle, the total number of ways

$$= 5! \times {}^6P_3 = 5! \times \frac{6!}{3!}$$

$$= 4 \times 5 \times 2 \times 3 \times 4 \times 5 \times 6 = 14400.$$

Miscellaneous Exercise on Chapter 6

1. How many words, with or without meaning, each of 2 vowels and 3 consonants can be formed from the letters of the word DAUGHTER ?
2. How many words, with or without meaning, can be formed using all the letters of the word EQUATION at a time so that the vowels and consonants occur together?
3. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of:

   (i) exactly 3 girls ? (ii) atleast 3 girls ? (iii) atmost 3 girls ?
4. If the different permutations of all the letter of the word EXAMINATION are

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listed as in a dictionary, how many words are there in this list before the first word starting with E ?

5. How many 6-digit numbers can be formed from the digits 0, 1, 3, 5, 7 and 9 which are divisible by 10 and no digit is repeated ?
6. The English alphabet has 5 vowels and 21 consonants. How many words with two different vowels and 2 different consonants can be formed from the alphabet ?
7. In an examination, a question paper consists of 12 questions divided into two parts i.e., Part I and Part II, containing 5 and 7 questions, respectively. A student is required to attempt 8 questions in all, selecting at least 3 from each part. In how many ways can a student select the questions ?
8. Determine the number of 5-card combinations out of a deck of 52 cards if each selection of 5 cards has exactly one king.
9. It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible ?
10. From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can the excursion party be chosen ?
11. In how many ways can the letters of the word ASSASSINATION be arranged so that all the S's are together ?

Summary

- *Fundamental principle of counting* If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is $m \times n$.
- The number of permutations of n different things taken r at a time, where repetition is not allowed, is denoted by ${}^nP_r$ and is given by ${}^nP_r = \dfrac{n!}{(n-r)!}$, where $0 \le r \le n$.
- $n! = 1 \times 2 \times 3 \times ... \times n$
- $n! = n \times (n - 1)\,!$
- The number of permutations of n different things, taken r at a time, where repeatition is allowed, is n^r.
- The number of permutations of n objects taken all at a time, where p_1 objects

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are of first kind, p_2 objects are of the second kind, ..., p_k objects are of the k^{th} kind and rest, if any, are all different is $\dfrac{n!}{p_1!\, p_2! \ldots p_k!}$.

- The number of combinations of n different things taken r at a time, denoted by ${}^nC_r$, is given by ${}^nC_r = = \dfrac{n!}{r!(n-r)!}$, $0 \le r \le n$.

Historical Note

The concepts of permutations and combinations can be traced back to the advent of Jainism in India and perhaps even earlier. The credit, however, goes to the Jains who treated its subject matter as a self-contained topic in mathematics, under the name *Vikalpa*.

Among the Jains, *Mahavira*, (around 850) is perhaps the world's first mathematician credited with providing the general formulae for permutations and combinations.

In the 6th century B.C., *Sushruta,* in his medicinal work, *Sushruta Samhita*, asserts that 63 combinations can be made out of 6 different tastes, taken one at a time, two at a time, etc. *Pingala*, a Sanskrit scholar around third century B.C., gives the method of determining the number of combinations of a given number of letters, taken one at a time, two at a time, etc. in his work *Chhanda Sutra*. *Bhaskaracharya* (born 1114) treated the subject matter of permutations and combinations under the name *Anka Pasha* in his famous work *Lilavati.* In addition to the general formulae for ${}^nC_r$ and ${}^nP_r$ already provided by *Mahavira, Bhaskaracharya* gives several important theorems and results concerning the subject.

Outside India, the subject matter of permutations and combinations had its humble beginnings in China in the famous book I–King (Book of changes). It is difficult to give the approximate time of this work, since in 213 B.C., the emperor had ordered all books and manuscripts in the country to be burnt which fortunately was not completely carried out. Greeks and later Latin writers also did some scattered work on the theory of permutations and combinations.

Some Arabic and Hebrew writers used the concepts of permutations and combinations in studying astronomy. *Rabbi ben Ezra,* for instance, determined the number of combinations of known planets taken two at a time, three at a time and so on. This was around 1140. It appears that *Rabbi ben Ezra* did not know

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the formula for ${}^nC_r$. However, he was aware that ${}^nC_r = {}^nC_{n-r}$ for specific values n and r. In 1321, *Levi Ben Gerson*, another Hebrew writer came up with the formulae for ${}^nP_r$, ${}^nP_n$ *and the general formula for* ${}^nC_r$.

The first book which gives a complete treatment of the subject matter of permutations and combinations is Ars Conjectandi written by a Swiss, *Jacob Bernoulli* (1654 – 1705), posthumously published in 1713. This book contains essentially the theory of permutations and combinations as is known today.

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Chapter 7

BINOMIAL THEOREM

❖*Mathematics is a most exact science and its conclusions are capable of absolute proofs.* – *C.P. STEINMETZ*❖

7.1 Introduction

In earlier classes, we have learnt how to find the squares and cubes of binomials like $a + b$ and $a - b$. Using them, we could evaluate the numerical values of numbers like $(98)^2 = (100 - 2)^2$, $(999)^3 = (1000 - 1)^3$, etc. However, for higher powers like $(98)^5$, $(101)^6$, etc., the calculations become difficult by using repeated multiplication. This difficulty was overcome by a theorem known as binomial theorem. It gives an easier way to expand $(a + b)^n$, where n is an integer or a rational number. In this Chapter, we study binomial theorem for positive integral indices only.

Blaise Pascal
(1623-1662)

7.2 Binomial Theorem for Positive Integral Indices

Let us have a look at the following identities done earlier:

$$(a+b)^0 = 1 \qquad a + b \neq 0$$
$$(a+b)^1 = a + b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = (a+b)^3\,(a+b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

In these expansions, we observe that

(i) The total number of terms in the expansion is one more than the index. For example, in the expansion of $(a + b)^2$, number of terms is 3 whereas the index of $(a + b)^2$ is 2.

(ii) Powers of the first quantity 'a' go on decreasing by 1 whereas the powers of the second quantity 'b' increase by 1, in the successive terms.

(iii) In each term of the expansion, the sum of the indices of a and b is the same and is equal to the index of $a + b$.

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We now arrange the coefficients in these expansions as follows (Fig 7.1):

Index	**Coefficients**
0	**1**
1	**1 1**
2	**1 2 1**
3	**1 3 3 1**
4	**1 4 6 4 1**

Fig 7.1

Do we observe any pattern in this table that will help us to write the next row? Yes we do. It can be seen that the addition of 1's in the row for index 1 gives rise to 2 in the row for index 2. The addition of 1, 2 and 2, 1 in the row for index 2, gives rise to 3 and 3 in the row for index 3 and so on. Also, 1 is present at the beginning and at the end of each row. This can be continued till any index of our interest.

We can extend the pattern given in Fig 7.2 by writing a few more rows.

Index	**Coefficients**
0	**1**
1	**1 1**
2	**1 2 1**
3	**1 3 3 1**
4	**1 4 6 4 1**

Fig 7.2

Pascal's Triangle

The structure given in Fig 7.2 looks like a triangle with 1 at the top vertex and running down the two slanting sides. This array of numbers is known as *Pascal's triangle*, after the name of French mathematician Blaise Pascal. It is also known as *Meru Prastara* by Pingla.

Expansions for the higher powers of a binomial are also possible by using Pascal's triangle. Let us expand $(2x + 3y)^5$ by using Pascal's triangle. The row for index 5 is

1 5 10 10 5 1

Using this row and our observations (i), (ii) and (iii), we get

$$(2x + 3y)^5 = (2x)^5 + 5(2x)^4 (3y) + 10(2x)^3 (3y)^2 + 10 (2x)^2 (3y)^3 + 5(2x)(3y)^4 + (3y)^5$$
$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5.$$

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Now, if we want to find the expansion of $(2x + 3y)^{12}$, we are first required to get the row for index 12. This can be done by writing all the rows of the Pascal's triangle till index 12. This is a slightly lengthy process. The process, as you observe, will become more difficult, if we need the expansions involving still larger powers.

We thus try to find a rule that will help us to find the expansion of the binomial for any power without writing all the rows of the Pascal's triangle, that come before the row of the desired index.

For this, we make use of the concept of combinations studied earlier to rewrite the numbers in the Pascal's triangle. We know that ${}^nC_r = \dfrac{n!}{r!(n-r)!}$, $0 \le r \le n$ and n is a non-negative integer. Also, ${}^nC_0 = 1 = {}^nC_n$

The Pascal's triangle can now be rewritten as (Fig 7.3)

Index	**Coefficients**
0	${}^0C_0$ (=1)
1	${}^1C_0$ (=1) ${}^1C_1$ (=1)
2	${}^2C_0$ (=1) ${}^2C_1$ (=2) ${}^2C_2$ (=1)
3	${}^3C_0$ (=1) ${}^3C_1$ (=3) ${}^3C_2$ (=3) ${}^3C_3$ (=1)
4	${}^4C_0$ (=1) ${}^4C_1$ (=4) ${}^4C_2$ (=6) ${}^4C_3$ (=4) ${}^4C_4$ (=1)
5	${}^5C_0$ (=1) ${}^5C_1$ (=5) ${}^5C_2$ (=10) ${}^5C_3$ (=10) ${}^5C_4$ (=5) ${}^5C_5$ (=1)

Fig 7.3 Pascal's triangle

Observing this pattern, we can now write the row of the Pascal's triangle for any index without writing the earlier rows. For example, for the index 7 the row would be

$$ {}^7C_0 \quad {}^7C_1 \quad {}^7C_2 \quad {}^7C_3 \quad {}^7C_4 \quad {}^7C_5 \quad {}^7C_6 \quad {}^7C_7. $$

Thus, using this row and the observations (i), (ii) and (iii), we have

$$(a + b)^7 = {}^7C_0\, a^7 + 7C_1a^6b + {}^7C_2a^5b^2 + {}^7C_3a^4b^3 + 7C_4a^3b^4 + {}^7C_5a^2b^5 + {}^7C_6ab^6 + {}^7C_7b^7$$

An expansion of a binomial to any positive integral index say n can now be visualised using these observations. We are now in a position to write the expansion of a binomial to any positive integral index.

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7.2.1 *Binomial theorem for any positive integer n,*

$$(a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + ...+ {}^nC_{n-1}a.b^{n-1} + {}^nC_nb^n$$

Proof The proof is obtained by applying principle of mathematical induction.

Let the given statement be

$$P(n) : (a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + ...+ {}^nC_{n-1}a.b^{n-1} + {}^nC_nb^n$$

For $n = 1$, we have

$$P(1) : (a + b)^1 = {}^1C_0a^1 + {}^1C_1b^1 = a + b$$

Thus, P (1) is true.

Suppose P (k) is true for some positive integer k, i.e.

$$(a + b)^k = {}^kC_0a^k + {}^kC_1a^{k-1}b + {}^kC_2a^{k-2}b^2 + ...+ {}^kC_kb^k \quad ... (1)$$

We shall prove that P(k + 1) is also true, i.e.,

$$(a + b)^{k+1} = {}^{k+1}C_0\, a^{k+1} + {}^{k+1}C_1\, a^kb + {}^{k+1}C_2\, a^{k-1}b^2 + ...+ {}^{k+1}C_{k+1}\, b^{k+1}$$

Now, $(a + b)^{k+1} = (a + b)(a + b)^k$

$$= (a + b)({}^kC_0\, a^k + {}^kC_1a^{k-1}\, b + {}^kC_2\, a^{k-2}\, b^2 +...+ {}^kC_{k-1}\, ab^{k-1} + {}^kC_k\, b^k) \quad \text{[from (1)]}$$

$$= {}^kC_0\, a^{k+1} + {}^kC_1\, a^kb + {}^kC_2a^{k-1}b^2 +...+ {}^kC_{k-1}\, a^2b^{k-1} + {}^kC_k\, ab^k + {}^kC_0\, a^kb$$
$$+ {}^kC_1a^{k-1}b^2 + {}^kC_2a^{k-2}b^3+...+ {}^kC_{k-1}ab^k + {}^kC_kb^{k+1} \quad \text{[by actual multiplication]}$$

$$= {}^kC_0a^{k+1} + ({}^kC_1 + {}^kC_0)\, a^kb + ({}^kC_2 + {}^kC_1)a^{k-1}b^2 + ...$$
$$+ ({}^kC_k + {}^kC_{k-1})\, ab^k + {}^kC_kb^{k+1} \quad \text{[grouping like terms]}$$

$$= {}^{k+1}C_0a^{k+1} + {}^{k+1}C_1a^kb + {}^{k+1}C_2\, a^{k-1}b^2 +...+ {}^{k+1}C_kab^k + {}^{k+1}C_{k+1}\, b^{k+1}$$

(by using ${}^{k+1}C_0=1$, ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$ and ${}^kC_k = 1 = {}^{k+1}C_{k+1}$)

Thus, it has been proved that P (k + 1) is true whenever P(k) is true. Therefore, by principle of mathematical induction, P(n) is true for every positive integer n.

We illustrate this theorem by expanding $(x + 2)^6$:

$$(x + 2)^6 = {}^6C_0x^6 + {}^6C_1x^5.2 + {}^6C_2x^42^2 + {}^6C_3x^3.2^3 + {}^6C_4x^2.2^4 + {}^6C_5x.2^5 + {}^6C_6.2^6.$$
$$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$$

Thus $(x + 2)^6 = x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64.$

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Observations

1. The notation $\sum_{k=0}^{n} {}^nC_k\, a^{n-k} b^k$ stands for

$${}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + ...+ {}^nC_r a^{n-r} b^r + ...+ {}^nC_n a^{n-n} b^n, \text{ where } b^0 = 1 = a^{n-n}.$$

Hence the theorem can also be stated as

$$(a+b)^n = \sum_{k=0}^{n} {}^nC_k\, a^{n-k} b^k .$$

2. The coefficients ${}^nC_r$ occuring in the binomial theorem are known as binomial coefficients.

3. There are $(n+1)$ terms in the expansion of $(a+b)^n$, i.e., one more than the index.

4. In the successive terms of the expansion the index of a goes on decreasing by unity. It is n in the first term, $(n–1)$ in the second term, and so on ending with zero in the last term. At the same time the index of b increases by unity, starting with zero in the first term, 1 in the second and so on ending with n in the last term.

5. In the expansion of $(a+b)^n$, the sum of the indices of a and b is $n + 0 = n$ in the first term, $(n – 1) + 1 = n$ in the second term and so on $0 + n = n$ in the last term. Thus, it can be seen that the sum of the indices of a and b is n in every term of the expansion.

7.2.2 ***Some special cases*** In the expansion of $(a + b)^n$,

(i) Taking $a = x$ and $b = – y$, we obtain

$$\begin{aligned}(x - y)^n &= [x + (-y)]^n \\ &= {}^nC_0 x^n + {}^nC_1 x^{n-1}(-y) + {}^nC_2 x^{n-2}(-y)^2 + {}^nC_3 x^{n-3}(-y)^3 + ... + {}^nC_n (-y)^n \\ &= {}^nC_0 x^n - {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 - {}^nC_3 x^{n-3} y^3 + ... + (-1)^n\, {}^nC_n\, y^n\end{aligned}$$

Thus $(x–y)^n = {}^nC_0 x^n - {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + ... + (-1)^n\, {}^nC_n\, y^n$

Using this, we have

$$\begin{aligned}(x-2y)^5 &= {}^5C_0 x^5 - {}^5C_1 x^4 (2y) + {}^5C_2 x^3 (2y)^2 - {}^5C_3 x^2 (2y)^3 + {}^5C_4 x(2y)^4 - {}^5C_5 (2y)^5 \\ &= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5.\end{aligned}$$

(ii) Taking $a = 1$, $b = x$, we obtain

$$\begin{aligned}(1 + x)^n &= {}^nC_0 (1)^n + {}^nC_1 (1)^{n-1} x + {}^nC_2 (1)^{n-2} x^2 + ... + {}^nC_n x^n \\ &= {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + ... + {}^nC_n x^n\end{aligned}$$

Thus $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + ... + {}^nC_n x^n$

In particular, for $x = 1$, we have

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n.$$

(iii) Taking $a = 1$, $b = -x$, we obtain

$$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - \ldots + (-1)^n\, {}^nC_n x^n$$

In particular, for $x = 1$, we get

$$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots + (-1)^n\, {}^nC_n$$

Example 1 Expand $\left(x^2 + \frac{3}{x}\right)^4,\ x \neq 0$

Solution By using binomial theorem, we have

$$\left(x^2 + \frac{3}{x}\right)^4 = {}^4C_0(x^2)^4 + {}^4C_1(x^2)^3\left(\frac{3}{x}\right) + {}^4C_2(x^2)^2\left(\frac{3}{x}\right)^2 + {}^4C_3(x^2)\left(\frac{3}{x}\right)^3 + {}^4C_4\left(\frac{3}{x}\right)^4$$

$$= x^8 + 4.x^6 . \frac{3}{x} + 6.x^4 . \frac{9}{x^2} + 4.x^2.\frac{27}{x^3} + \frac{81}{x^4}$$

$$= x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}.$$

Example 2 Compute $(98)^5$.

Solution We express 98 as the sum or difference of two numbers whose powers are easier to calculate, and then use Binomial Theorem.

Write $98 = 100 - 2$

Therefore, $(98)^5 = (100 - 2)^5$

$$= {}^5C_0\,(100)^5 - {}^5C_1\,(100)^4.2 + {}^5C_2\,(100)^3 2^2$$
$$- {}^5C_3\,(100)^2\,(2)^3 + {}^5C_4\,(100)\,(2)^4 - {}^5C_5\,(2)^5$$
$$= 10000000000 - 5 \times 100000000 \times 2 + 10 \times 1000000 \times 4 - 10 \times 10000 \times 8 + 5 \times 100 \times 16 - 32$$
$$= 10040008000 - 1000800032 = 9039207968.$$

Example 3 Which is larger $(1.01)^{1000000}$ or 10,000?

Solution Splitting 1.01 and using binomial theorem to write the first few terms we have

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$$\begin{aligned}(1.01)^{1000000} &= (1 + 0.01)^{1000000}\\ &= {}^{1000000}C_0 + {}^{1000000}C_1(0.01) + \text{other positive terms}\\ &= 1 + 1000000 \times 0.01 + \text{other positive terms}\\ &= 1 + 10000 + \text{other positive terms}\\ &> 10000\end{aligned}$$

Hence $(1.01)^{1000000} > 10000$

Example 4 Using binomial theorem, prove that $6^n–5n$ always leaves remainder 1 when divided by 25.

Solution For two numbers a and b if we can find numbers q and r such that $a = bq + r$, then we say that b divides a with q as quotient and r as remainder. Thus, in order to show that $6^n – 5n$ leaves remainder 1 when divided by 25, we prove that $6^n – 5n = 25k + 1$, where k is some natural number.

We have

$$(1 + a)^n = {}^nC_0 + {}^nC_1a + {}^nC_2a^2 + ... + {}^nC_na^n$$

For $a = 5$, we get

$$(1 + 5)^n = {}^nC_0 + {}^nC_15 + {}^nC_25^2 + ... + {}^nC_n5^n$$

i.e. $(6)^n = 1 + 5n + 5^2.{}^nC_2 + 5^3.{}^nC_3 + ... + 5^n$

i.e. $6^n – 5n = 1+5^2 ({}^nC_2 + {}^nC_35 + ... + 5^{n-2})$

or $6^n – 5n = 1+ 25 ({}^nC_2 + 5 .{}^nC_3 + ... + 5^{n-2})$

or $6^n – 5n = 25k+1$ where $k = {}^nC_2 + 5 .{}^nC_3 + ... + 5^{n–2}$.

This shows that when divided by 25, $6^n – 5n$ leaves remainder 1.

EXERCISE 7.1

Expand each of the expressions in Exercises 1 to 5.

1. $(1–2x)^5$ **2.** $\left(\frac{2}{x}-\frac{x}{2}\right)^5$ **3.** $(2x – 3)^6$

4. $\left(\frac{x}{3}+\frac{1}{x}\right)^5$ **5.** $\left(x+\frac{1}{x}\right)^6$

Using binomial theorem, evaluate each of the following:

6. $(96)^3$ **7.** $(102)^5$ **8.** $(101)^4$

9. $(99)^5$

10. Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

11. Find $(a + b)^4 – (a – b)^4$. Hence, evaluate $(\sqrt{3}+\sqrt{2})^4 - (\sqrt{3}-\sqrt{2})^4$.

12. Find $(x + 1)^6 + (x – 1)^6$. Hence or otherwise evaluate $(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$.

13. Show that $9^{n+1} – 8n – 9$ is divisible by 64, whenever n is a positive integer.

14. Prove that $\sum_{r=0}^{n} 3^r\ {}^nC_r = 4^n$.

Miscellaneous Exercise on Chapter 7

1. If a and b are distinct integers, prove that $a – b$ is a factor of $a^n – b^n$, whenever n is a positive integer.
[**Hint** write $a^n = (a – b + b)^n$ and expand]

2. Evaluate $\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6$.

3. Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4+\left(a^2-\sqrt{a^2-1}\right)^4$.

4. Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

5. Expand using Binomial Theorem $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4, x \neq 0$.

6. Find the expansion of $(3x^2 – 2ax + 3a^2)^3$ using binomial theorem.

Summary

- The expansion of a binomial for any positive integral n is given by Binomial Theorem, which is $(a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + ...+ {}^nC_{n-1}a.b^{n-1} + {}^nC_nb^n$.
- The coefficients of the expansions are arranged in an array. This array is called *Pascal's triangle*.

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Historical Note

The ancient Indian mathematicians knew about the coefficients in the expansions of $(x + y)^n$, $0 \le n \le 7$. The arrangement of these coefficients was in the form of a diagram called *Meru-Prastara*, provided by Pingla in his book *Chhanda shastra* (200B.C.). This triangular arrangement is also found in the work of Chinese mathematician Chu-shi-kie in 1303. The term binomial coefficients was first introduced by the German mathematician, Michael Stipel (1486-1567) in approximately 1544. Bombelli (1572) also gave the coefficients in the expansion of $(a + b)^n$, for $n = 1,2 ...,7$ and Oughtred (1631) gave them for $n = 1, 2,..., 10$. The arithmetic triangle, popularly known as *Pascal's triangle* and similar to the *Meru-Prastara* of Pingla was constructed by the French mathematician Blaise Pascal (1623-1662) in 1665.

The present form of the binomial theorem for integral values of n appeared in *Trate du triange arithmetic*, written by Pascal and published posthumously in 1665.

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Chapter 8

SEQUENCES AND SERIES

❖ *Natural numbers are the product of human spirit. – DEDEKIND* ❖

8.1 Introduction

Fibonacci
(1175-1250)

In mathematics, the word, "*sequence*" is used in much the same way as it is in ordinary English. When we say that a collection of objects is listed in a sequence, we usually mean that the collection is ordered in such a way that it has an identified first member, second member, third member and so on. For example, population of human beings or bacteria at different times form a sequence. The amount of money deposited in a bank, over a number of years form a sequence. Depreciated values of certain commodity occur in a sequence. Sequences have important applications in several spheres of human activities.

Sequences, following specific patterns are called *progressions*. In previous class, we have studied about *arithmetic progression* (A.P). In this Chapter, besides discussing more about A.P.; *arithmetic mean, geometric mean, relationship between A.M. and G.M., special series in forms of sum to n terms of consecutive natural numbers, sum to n terms of squares of natural numbers and sum to n terms of cubes of natural numbers* will also be studied.

8.2 Sequences

Let us consider the following examples:

Assume that there is a generation gap of 30 years, we are asked to find the number of ancestors, i.e., parents, grandparents, great grandparents, etc. that a person might have over 300 years.

Here, the total number of generations $= \dfrac{300}{30} = 10$

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The number of person's ancestors for the first, second, third, …, tenth generations are 2, 4, 8, 16, 32, …, 1024. These numbers form what we call a *sequence*.

Consider the successive quotients that we obtain in the division of 10 by 3 at different steps of division. In this process we get 3,3.3,3.33,3.333, ... and so on. These quotients also form a sequence. The various numbers occurring in a sequence are called its *terms*. We denote the terms of a sequence by $a_1, a_2, a_3, ..., a_n$, …, etc., the subscripts denote the position of the term. The n^{th} term is the number at the n^{th} position of the sequence and is denoted by a_n. The n^{th} term is also called the *general term* of the sequence.

Thus, the terms of the sequence of person's ancestors mentioned above are:

$$a_1 = 2, a_2 = 4, a_3 = 8, ..., a_{10} = 1024.$$

Similarly, in the example of successive quotients

$$a_1 = 3, a_2 = 3.3, a_3 = 3.33, …, a_6 = 3.33333, \text{ etc.}$$

A sequence containing finite number of terms is called a *finite sequence*. For example, sequence of ancestors is a finite sequence since it contains 10 terms (a fixed number).

A sequence is called *infinite*, if it is not a finite sequence. For example, the sequence of successive quotients mentioned above is an *infinite sequence*, infinite in the sense that it never ends.

Often, it is possible to express the rule, which yields the various terms of a sequence in terms of algebraic formula. Consider for instance, the sequence of even natural numbers 2, 4, 6, …

Here $a_1 = 2 = 2 \times 1$ $\quad a_2 = 4 = 2 \times 2$

$a_3 = 6 = 2 \times 3$ $\quad a_4 = 8 = 2 \times 4$

....

....

$a_{23} = 46 = 2 \times 23$, $a_{24} = 48 = 2 \times 24$, and so on.

In fact, we see that the n^{th} term of this sequence can be written as $a_n = 2n$, where n is a natural number. Similarly, in the sequence of odd natural numbers 1,3,5, …, the n^{th} term is given by the formula, $a_n = 2n - 1$, where n is a natural number.

In some cases, an arrangement of numbers such as 1, 1, 2, 3, 5, 8,.. has no visible pattern, but the sequence is generated by the recurrence relation given by

$$a_1 = a_2 = 1$$
$$a_3 = a_1 + a_2$$
$$a_n = a_{n-2} + a_{n-1}, n > 2$$

This sequence is called *Fibonacci sequence*.

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In the sequence of primes 2,3,5,7,…, we find that there is no formula for the n^{th} prime. Such sequence can only be described by verbal description.

In every sequence, we should not expect that its terms will necessarily be given by a specific formula. However, we expect a theoretical scheme or a rule for generating the terms $a_1, a_2, a_3,\ldots,a_n,\ldots$ in succession.

In view of the above, *a sequence can be regarded as a function whose domain is the set of natural numbers or some subset of it. Sometimes, we use the functional notation a(n) for* a_n.

8.3 Series

Let $a_1, a_2, a_3,\ldots,a_n,$ be a given sequence. Then, the expression

$$a_1 + a_2 + a_3 +,\ldots+ a_n + \ldots$$

is called the *series associated with the given sequence* .The series is finite or infinite according as the given sequence is finite or infinite. Series are often represented in compact form, called *sigma notation*, using the Greek letter $\sum$ (sigma) as means of indicating the summation involved. Thus, the series $a_1 + a_2 + a_3 + \ldots + a_n$ is abbreviated as $\sum_{k=1}^{n} a_k$.

Remark When the series is used, it refers to the indicated sum not to the sum itself. For example, 1 + 3 + 5 + 7 is a finite series with four terms. When we use the phrase "*sum of a series*," we will mean the number that results from adding the terms, the sum of the series is 16.

We now consider some examples.

Example 1 Write the first three terms in each of the following sequences defined by the following:

(i) $a_n = 2n + 5$, (ii) $a_n = \dfrac{n-3}{4}$.

Solution (i) Here $a_n = 2n + 5$

Substituting $n = 1, 2, 3$, we get

$$a_1 = 2(1) + 5 = 7,\ a_2 = 9,\ a_3 = 11$$

Therefore, the required terms are 7, 9 and 11.

(ii) Here $a_n = \dfrac{n-3}{4}$. Thus, $a_1 = \dfrac{1-3}{4} = -\dfrac{1}{2}$, $a_2 = -\dfrac{1}{4}$, $a_3 = 0$

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Hence, the first three terms are $-\frac{1}{2}, -\frac{1}{4}$ and 0.

Example 2 What is the 20th term of the sequence defined by

$$a_n = (n - 1)(2 - n)(3 + n)\ ?$$

Solution Putting $n = 20$, we obtain

$$\begin{aligned} a_{20} &= (20 - 1)(2 - 20)(3 + 20) \\ &= 19 \times (-18) \times (23) = -7866. \end{aligned}$$

Example 3 Let the sequence a_n be defined as follows:

$$a_1 = 1,\ \ a_n = a_{n-1} + 2 \text{ for } n \geq 2.$$

Find first five terms and write corresponding series.

Solution We have

$$a_1 = 1,\ a_2 = a_1 + 2 = 1 + 2 = 3,\ a_3 = a_2 + 2 = 3 + 2 = 5,$$

$$a_4 = a_3 + 2 = 5 + 2 = 7,\ a_5 = a_4 + 2 = 7 + 2 = 9.$$

Hence, the first five terms of the sequence are 1,3,5,7 and 9. The corresponding series is $1 + 3 + 5 + 7 + 9 + ...$

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EXERCISE 8.1

Write the first five terms of each of the sequences in Exercises 1 to 6 whose n^{th} terms are:

1. $a_n = n(n + 2)$ **2.** $a_n = \frac{n}{n+1}$ **3.** $a_n = 2^n$

4. $a_n = \frac{2n-3}{6}$ **5.** $a_n = (-1)^{n-1}\, 5^{n+1}$ **6.** $a_n = n\frac{n^2+5}{4}$.

Find the indicated terms in each of the sequences in Exercises 7 to 10 whose n^{th} terms are:

7. $a_n = 4n - 3;\ a_{17},\ a_{24}$ **8.** $a_n = \frac{n^2}{2^n};\ a_7$

9. $a_n = (-1)^{n-1}n^3;\ a_9$ **10.** $a_n = \frac{n(n-2)}{n+3};\ a_{20}$.

Write the first five terms of each of the sequences in Exercises 11 to 13 and obtain the corresponding series:

11. $a_1 = 3, a_n = 3a_{n-1} + 2$ for all $n > 1$ **12.** $a_1 = -1, a_n = \dfrac{a_{n-1}}{n}, n \geq 2$

13. $a_1 = a_2 = 2, a_n = a_{n-1} - 1, n > 2$

14. The Fibonacci sequence is defined by

$$1 = a_1 = a_2 \text{ and } a_n = a_{n-1} + a_{n-2}, n > 2.$$

Find $\dfrac{a_{n+1}}{a_n}$, for $n = 1, 2, 3, 4, 5$

8.4 Geometric Progression (G. P.)

Let us consider the following sequences:

$$\text{(i) } 2,4,8,16,..., \text{ (ii) } \frac{1}{9}, \frac{-1}{27}, \frac{1}{81}, \frac{-1}{243} ... \text{ (iii) } .01,.0001,.000001,...$$

In each of these sequences, how their terms progress? We note that each term, except the first progresses in a definite order.

In (i), we have $a_1 = 2, \dfrac{a_2}{a_1} = 2, \dfrac{a_3}{a_2} = 2, \dfrac{a_4}{a_3} = 2$ and so on.

In (ii), we observe, $a_1 = \dfrac{1}{9}, \dfrac{a_2}{a_1} = -\dfrac{1}{3}, \dfrac{a_3}{a_2} = -\dfrac{1}{3}, \dfrac{a_4}{a_3} = -\dfrac{1}{3}$ and so on.

Similarly, state how do the terms in (iii) progress? It is observed that in each case, every term except the first term bears a constant ratio to the term immediately preceding it. In (i), this constant ratio is 2; in (ii), it is $-\dfrac{1}{3}$ and in (iii), the constant ratio is 0.01. Such sequences are called *geometric sequence* or *geometric progression* abbreviated as G.P.

A sequence $a_1, a_2, a_3, ..., a_n, ...$ is called *geometric progression*, if each term is non-zero and $\dfrac{a_{k+1}}{a_k} = r$ (constant), for $k \geq 1$.

By letting $a_1 = a$, we obtain a geometric progression, $a, ar, ar^2, ar^3, ...$, where a is called the *first term* and r is called the *common ratio* of the G.P. Common ratio in geometric progression (i), (ii) and (iii) above are 2, $-\dfrac{1}{3}$ and 0.01, respectively.

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As in case of arithmetic progression, the problem of finding the n^{th} term or sum of n terms of a geometric progression containing a large number of terms would be difficult without the use of the formulae which we shall develop in the next Section. We shall use the following notations with these formulae:

a = the first term, r = the common ratio, l = the last term,

n = the numbers of terms,

S_n = the sum of first n terms.

8.4.1 General term of a G.P. Let us consider a G.P. with first non-zero term 'a' and common ratio 'r'. Write a few terms of it. The second term is obtained by multiplying a by r, thus $a_2 = ar$. Similarly, third term is obtained by multiplying a_2 by r. Thus, $a_3 = a_2 r = ar^2$, and so on.

We write below these and few more terms.

1^{st} term = $a_1 = a = ar^{1–1}$, 2^{nd} term = $a_2 = ar = ar^{2–1}$, 3^{rd} term = $a_3 = ar^2 = ar^{3–1}$
4^{th} term = $a_4 = ar^3 = ar^{4–1}$, 5^{th} term = $a_5 = ar^4 = ar^{5–1}$
Do you see a pattern? What will be 16^{th} term?

$$a_{16} = ar^{16–1} = ar^{15}$$

Therefore, the pattern suggests that the n^{th} term of a G.P. is given by $a_n = ar^{n–1}$.

Thus, a, G.P. can be written as $a, ar, ar^2, ar^3, \ldots ar^{n-1}$; $a, ar, ar^2, \ldots, ar^{n-1} \ldots$;according as G.P. is *finite* or *infinite*, respectively.

The series $a + ar + ar^2 + \ldots + ar^{n–1}$ or $a + ar + ar^2 + \ldots + ar^{n–1} + \ldots$are called *finite* or *infinite geometric series*, respectively.

8.4.2. Sum to n terms of a G.P. Let the first term of a G.P. be a and the common ratio be r. Let us denote by S_n the sum to first n terms of G.P. Then

$$S_n = a + ar + ar^2 + \ldots + ar^{n–1} \quad \ldots (1)$$

Case 1 If $r = 1$, we have $S_n = a + a + a + \ldots + a$ (n terms) $= na$

Case 2 If $r \neq 1$, multiplying (1) by r, we have

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n \quad \ldots (2)$$

Subtracting (2) from (1), we get $(1 – r)\, S_n = a – ar^n = a(1 – r^n)$

This gives or $S_n = \dfrac{a(r^n - 1)}{r - 1}$

Example 4 Find the 10^{th} and n^{th} terms of the G.P. 5, 25,125,… .

Solution Here $a = 5$ and $r = 5$. Thus, $a_{10} = 5(5)^{10–1} = 5(5)^9 = 5^{10}$
and $a_n = ar^{n–1} = 5(5)^{n–1} = 5^n$.

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Example 5 Which term of the G.P., 2,8,32, ... up to n terms is 131072?

Solution Let 131072 be the n^{th} term of the given G.P. Here $a = 2$ and $r = 4$.
Therefore $131072 = a_n = 2(4)^{n-1}$ or $65536 = 4^{n-1}$

This gives $4^8 = 4^{n-1}$.

So that $n - 1 = 8$, i.e., $n = 9$. Hence, 131072 is the 9^{th} term of the G.P.

Example 6 In a G.P., the 3^{rd} term is 24 and the 6^{th} term is 192.Find the 10^{th} term.

Solution Here, $a_3 = ar^2 = 24$ ... (1)

and $a_6 = ar^5 = 192$ ... (2)

Dividing (2) by (1), we get $r = 2$. Substituting $r = 2$ in (1), we get $a = 6$.
Hence $a_{10} = 6\,(2)^9 = 3072$.

Example 7 Find the sum of first n terms and the sum of first 5 terms of the geometric series $1+\frac{2}{3}+\frac{4}{9}+...$

Solution Here $a = 1$ and $r = \frac{2}{3}$. Therefore

$$S_n = \frac{a\,(1-r^n)}{1-r} = \frac{\left[1-\left(\frac{2}{3}\right)^n\right]}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right]$$

In particular, $$S_5 = 3\left[1-\left(\frac{2}{3}\right)^5\right] = 3\times\frac{211}{243} = \frac{211}{81}.$$

Example 8 How many terms of the G.P. $3, \frac{3}{2}, \frac{3}{4}, ...$ are needed to give the sum $\frac{3069}{512}$?

Solution Let n be the number of terms needed. Given that $a = 3$, $r = \frac{1}{2}$ and $S_n = \frac{3069}{512}$

Since $$S_n = \frac{a\,(1-r^n)}{1-r}$$

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Therefore $$\frac{3069}{512}=\frac{3(1-\frac{1}{2^n})}{1-\frac{1}{2}}=6\left(1-\frac{1}{2^n}\right)$$

or $$\frac{3069}{3072} = 1-\frac{1}{2^n}$$

or $$\frac{1}{2^n} = 1-\frac{3069}{3072}=\frac{3}{3072}=\frac{1}{1024}$$

or $2^n = 1024 = 2^{10}$, which gives $n = 10$.

Example 9 The sum of first three terms of a G.P. is $\frac{13}{12}$ and their product is – 1. Find the common ratio and the terms.

Solution Let $\frac{a}{r}$, a, ar be the first three terms of the G.P. Then

$$\frac{a}{r}+ar+a=\frac{13}{12} \qquad \text{... (1)}$$

and $$\left(\frac{a}{r}\right)(a)\,(ar)=-1 \qquad \text{... (2)}$$

From (2), we get $a^3 = – 1$, i.e., $a = – 1$ (considering only real roots)

Substituting $a=-1$ in (1), we have

$$-\frac{1}{r}-1-r=\frac{13}{12} \text{ or } 12r^2 + 25r + 12 = 0.$$

This is a quadratic in r, solving, we get $r=-\frac{3}{4}$ or $-\frac{4}{3}$.

Thus, the three terms of G.P. are : $\frac{4}{3},-1,\frac{3}{4}$ for $r = \frac{-3}{4}$ and $\frac{3}{4},-1, \frac{4}{3}$ for $r = \frac{-4}{3}$,

Example10 Find the sum of the sequence 7, 77, 777, 7777, ... to n terms.

Solution This is not a G.P., however, we can relate it to a G.P. by writing the terms as

$$S_n = 7 + 77 + 777 + 7777 + ... \text{ to } n \text{ terms}$$

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$$= \frac{7}{9}[9+99+999+9999+ \text{...to } n \text{ term}]$$

$$= \frac{7}{9}[(10-1)+(10^2-1)+(10^3-1)+(10^4-1)+ \text{...} n \text{ terms}]$$

$$= \frac{7}{9}[(10+10^2+10^3+\text{...} n \text{ terms})-(1+1+1+\text{...} n \text{ terms})]$$

$$= \frac{7}{9}\left[\frac{10(10^n-1)}{10-1}-n\right] = \frac{7}{9}\left[\frac{10\,(10^n-1)}{9}-n\right].$$

Example 11 A person has 2 parents, 4 grandparents, 8 great grandparents, and so on. Find the number of his ancestors during the ten generations preceding his own.

Solution Here $a = 2$, $r = 2$ and $n = 10$

Using the sum formula $\quad S_n = \dfrac{a\,(r^n-1)}{r-1}$

We have $\quad S_{10} = 2(2^{10} - 1) = 2046$

Hence, the number of ancestors preceding the person is 2046.

8.4.3 *Geometric Mean (G.M.)* The geometric mean of two positive numbers a and b is the number $\sqrt{ab}$. Therefore, the geometric mean of 2 and 8 is 4. We observe that the three numbers 2,4,8 are consecutive terms of a G.P. This leads to a generalisation of the concept of geometric means of two numbers.

Given any two positive numbers a and b, we can insert as many numbers as we like between them to make the resulting sequence in a G.P.

Let $G_1, G_2, \ldots, G_n$ be n numbers between positive numbers a and b such that $a, G_1, G_2, G_3, \ldots, G_n, b$ is a G.P. Thus, b being the $(n + 2)^{\text{th}}$ term, we have

$$b = ar^{n+1}, \quad \text{or} \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

Hence $\quad G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, \; G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \; G_3 = ar^3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}},$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Example12 Insert three numbers between 1 and 256 so that the resulting sequence is a G.P.

Solution Let G_1, G_2,G_3 be three numbers between 1 and 256 such that 1, G_1,G_2,G_3 ,256 is a G.P.

Therefore $256 = r^4$ giving $r = \pm 4$ (Taking real roots only)

For $r = 4$, we have $G_1 = ar = 4$, $G_2 = ar^2 = 16$, $G_3 = ar^3 = 64$

Similarly, for $r = -4$, numbers are -4,16 and -64.

Hence, we can insert 4, 16, 64 between 1 and 256 so that the resulting sequences are in G.P.

8.5 Relationship Between A.M. and G.M.

Let A and G be A.M. and G.M. of two given positive real numbers a and b, respectively. Then

$$A=\frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

Thus, we have

$$A - G = \frac{a+b}{2}-\sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} \geq 0 \quad \text{... (1)}$$

From (1), we obtain the relationship $A \geq G$.

Example 13 If A.M. and G.M. of two positive numbers a and b are 10 and 8, respectively, find the numbers.

Solution Given that $$\text{A.M.}=\frac{a+b}{2}=10 \quad \text{... (1)}$$

and $$\text{G.M.}=\sqrt{ab}=8 \quad \text{... (2)}$$

From (1) and (2), we get

$$a + b = 20 \quad \text{... (3)}$$
$$ab = 64 \quad \text{... (4)}$$

Putting the value of a and b from (3), (4) in the identity $(a - b)^2 = (a + b)^2 - 4ab$, we get

$$(a - b)^2 = 400 - 256 = 144$$

or $$a - b = \pm 12 \quad \text{... (5)}$$

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Solving (3) and (5), we obtain

$$a = 4,\ b = 16 \text{ or } a = 16,\ b = 4$$

Thus, the numbers a and b are 4, 16 or 16, 4 respectively.

EXERCISE 8.2

1. Find the 20th and n^{th} terms of the G.P. $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots$
2. Find the 12th term of a G.P. whose 8th term is 192 and the common ratio is 2.
3. The 5th, 8th and 11th terms of a G.P. are p, q and s, respectively. Show that $q^2 = ps$.
4. The 4th term of a G.P. is square of its second term, and the first term is – 3. Determine its 7th term.
5. Which term of the following sequences:

   (a) $2, 2\sqrt{2}, 4, \ldots$ is 128 ? (b) $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729 ?

   (c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ is $\frac{1}{19683}$?

6. For what values of x, the numbers $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P.?

Find the sum to indicated number of terms in each of the geometric progressions in Exercises 7 to 10:

7. 0.15, 0.015, 0.0015, ... 20 terms.
8. $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$ n terms.
9. $1, -a, a^2, -a^3, \ldots$ n terms (if $a \neq -1$).
10. $x^3, x^5, x^7, \ldots$ n terms (if $x \neq \pm 1$).
11. Evaluate $\sum_{k=1}^{11} (2 + 3^k)$.
12. The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.
13. How many terms of G.P. $3, 3^2, 3^3, \ldots$ are needed to give the sum 120?
14. The sum of first three terms of a G.P. is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the G.P.
15. Given a G.P. with $a = 729$ and 7th term 64, determine S_7.

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16. Find a G.P. for which sum of the first two terms is – 4 and the fifth term is 4 times the third term.

17. If the 4^{th}, 10^{th} and 16^{th} terms of a G.P. are x, y and z, respectively. Prove that x, y, z are in G.P.

18. Find the sum to n terms of the sequence, 8, 88, 888, 8888… .

19. Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, $\frac{1}{2}$.

20. Show that the products of the corresponding terms of the sequences a, ar, ar^2, $\ldots ar^{n-1}$ and A, AR, AR^2, … AR^{n-1} form a G.P, and find the common ratio.

21. Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the 4^{th} by 18.

22. If the p^{th}, q^{th} and r^{th} terms of a G.P. are a, b and c, respectively. Prove that

$$a^{q-r}\, b^{r-p} c^{P-q} = 1.$$

23. If the first and the n^{th} term of a G.P. are a and b, respectively, and if P is the product of n terms, prove that $P^2 = (ab)^n$.

24. Show that the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.

25. If a, b, c and d are in G.P. show that

$(a^2 + b^2 + c^2)\,(b^2 + c^2 + d^2) = (ab + bc + cd)^2$.

26. Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

27. Find the value of n so that $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b.

28. The sum of two numbers is 6 times their geometric mean, show that numbers are in the ratio $\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right)$.

29. If A and G be A.M. and G.M., respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$.

30. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2^{nd} hour, 4^{th} hour and n^{th} hour ?

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31. What will Rs 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

32. If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.

Miscellaneous Examples

Example 14 If a, b, c, d and p are different real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)\,p + (b^2 + c^2 + d^2) \le 0$, then show that a, b, c and d are in G.P.

Solution Given that

$$(a^2 + b^2 + c^2)\,p^2 - 2\,(ab + bc + cd)\,p + (b^2 + c^2 + d^2) \le 0 \quad \ldots (1)$$

But L.H.S.

$$= (a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) + (c^2p^2 - 2cdp + d^2),$$

which gives $(ap - b)^2 + (bp - c)^2 + (cp - d)^2 \ge 0$ ... (2)

Since the sum of squares of real numbers is non negative, therefore, from (1) and (2), we have, $(ap - b)^2 + (bp - c)^2 + (cp - d)^2 = 0$

or $ap - b = 0,\ bp - c = 0,\ cp - d = 0$

This implies that $\dfrac{b}{a} = \dfrac{c}{b} = \dfrac{d}{c} = p$

Hence a, b, c and d are in G.P.

Miscellaneous Exercise On Chapter 8

1. If f is a function satisfying $f(x + y) = f(x)\,f(y)$ for all $x, y \in \mathbf{N}$ such that $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$, find the value of n.

2. The sum of some terms of G.P. is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.

3. The first term of a G.P. is 1. The sum of the third term and fifth term is 90. Find the common ratio of G.P.

4. The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

5. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

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6. If $\dfrac{a+bx}{a-bx}=\dfrac{b+cx}{b-cx}=\dfrac{c+dx}{c-dx}(x\neq 0)$, then show that a, b, c and d are in G.P.

7. Let S be the sum, P the product and R the sum of reciprocals of n terms in a G.P. Prove that $P^2R^n = S^n$.

8. If a, b, c, d are in G.P, prove that $(a^n + b^n), (b^n + c^n), (c^n + d^n)$ are in G.P.

9. If a and b are the roots of $x^2 - 3x + p = 0$ and c, d are roots of $x^2 - 12x + q = 0$, where a, b, c, d form a G.P. Prove that $(q + p) : (q - p) = 17:15$.

10. The ratio of the A.M. and G.M. of two positive numbers a and b, *is* $m : n$. Show that $a:b=\left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$.

11. Find the sum of the following series up to n terms:
(i) 5 + 55 +555 + … (ii) .6 +. 66 +. 666+…

12. Find the 20th term of the series $2 \times 4 + 4 \times 6 + 6 \times 8 + ... + n$ terms.

13. A farmer buys a used tractor for Rs 12000. He pays Rs 6000 cash and agrees to pay the balance in annual instalments of Rs 500 plus 12% interest on the unpaid amount. How much will the tractor cost him?

14. Shamshad Ali buys a scooter for Rs 22000. He pays Rs 4000 cash and agrees to pay the balance in annual instalment of Rs 1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

15. A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8th set of letter is mailed.

16. A man deposited Rs 10000 in a bank at the rate of 5% simple interest annually. Find the amount in 15th year since he deposited the amount and also calculate the total amount after 20 years.

17. A manufacturer reckons that the value of a machine, which costs him Rs. 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

18. 150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

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Summary

- By a *sequence*, we mean an arrangement of number in definite order according to some rule. Also, we define a sequence as a function whose domain is the set of natural numbers or some subsets of the type $\{1, 2, 3, ...k\}$. A sequence containing a finite number of terms is called a *finite sequence*. A sequence is called *infinite* if it is not a finite sequence.
- Let $a_1, a_2, a_3, ...$ be the sequence, then the sum expressed as $a_1 + a_2 + a_3 + ...$ is called *series*. A series is called *finite series* if it has got finite number of terms.
- A sequence is said to be a *geometric progression* or *G.P.*, if the ratio of any term to its preceding term is same throughout. This constant factor is called the *common ratio*. Usually, we denote the first term of a G.P. by a and its common ratio by r. The general or the n^{th} term of G.P. is given by $a_n = ar^{n-1}$. The sum S_n of the first n terms of G.P. is given by

$$S_n = \frac{a\left(r^n - 1\right)}{r-1} \; or \; \frac{a\left(1-r^n\right)}{1-r}, \; if \; r \neq 1$$

- The geometric mean (G.M.) of any two positive numbers a and b is given by $\sqrt{ab}$ i.e., the sequence a, G, b is G.P.

Historical Note

Evidence is found that Babylonians, some 4000 years ago, knew of arithmetic and geometric sequences. According to Boethius (510), arithmetic and geometric sequences were known to early Greek writers. Among the Indian mathematician, Aryabhatta (476) was the first to give the formula for the sum of squares and cubes of natural numbers in his famous work Aryabhatiyam, written around 499. He also gave the formula for finding the sum to n terms of an arithmetic sequence starting with p^{th} term. Noted Indian mathematicians Brahmgupta

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(598), Mahavira (850) and Bhaskara (1114-1185) also considered the sum of squares and cubes. Another specific type of sequence having important applications in mathematics, called *Fibonacci sequence*, was discovered by Italian mathematician Leonardo Fibonacci (1170-1250). Seventeenth century witnessed the classification of series into specific forms. In 1671 James Gregory used the term infinite series in connection with infinite sequence. It was only through the rigorous development of algebraic and set theoretic tools that the concepts related to sequence and series could be formulated suitably.

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Chapter 9

STRAIGHT LINES

❖ *Geometry, as a logical system, is a means and even the most powerful means to make children feel the strength of the human spirit that is of their own spirit. – H. FREUDENTHAL* ❖

9.1 Introduction


René Descartes
(1596 -1650)

We are familiar with two-dimensional *coordinate geometry* from earlier classes. Mainly, it is a combination of *algebra* and *geometry*. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician René Descartes, in his book 'La Géométry, published in 1637. This book introduced the notion of the equation of a curve and related analytical methods into the study of geometry. The resulting combination of analysis and geometry is referred now as *analytical geometry.* In the earlier classes, we initiated the study of coordinate geometry, where we studied about coordinate axes, coordinate plane, plotting of points in a plane, distance between two points, section formulae, etc. All these concepts are the basics of coordinate geometry.

Let us have a brief recall of coordinate geometry done in earlier classes. To recapitulate, the location of the points (6, – 4) and (3, 0) in the XY-plane is shown in Fig 9.1.

We may note that the point (6, – 4) is at 6 units distance from the y-axis measured along the positive x-axis and at 4 units distance from the x-axis measured along the negative y-axis. Similarly, the point (3, 0) is at 3 units distance from the y-axis measured along the positive x-axis and has zero distance from the x-axis.

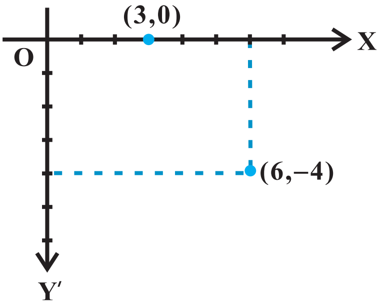

Fig 9.1

We also studied there following important formulae:

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I. Distance between the points P (x_1, y_1) and Q (x_2, y_2) is

$$PQ=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$$

For example, distance between the points (6, – 4) and (3, 0) is

$$\sqrt{(3-6)^2+(0+4)^2}=\sqrt{9+16}=5 \text{ units.}$$

II. The coordinates of a point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) internally, in the ratio $m: n$ are $\left(\dfrac{m\,x_2+n\,x_1}{m+n}, \dfrac{m\,y_2+n\,y_1}{m+n}\right)$.

For example, the coordinates of the point which divides the line segment joining A (1, –3) and B (–3, 9) internally, in the ratio 1: 3 are given by $x=\dfrac{1.(-3)+3.1}{1+3}=0$

and $y=\dfrac{1.9+3.(-3)}{1+3}=0.$

III. In particular, if $m = n$, the coordinates of the mid-point of the line segment joining the points (x_1, y_1) and (x_2, y_2) are $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$.

IV. Area of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\left|\, x_1\left(y_2-y_3\right)+x_2\left(y_3-y_1\right)+x_3\left(y_1-y_2\right)\right| .$$

For example, the area of the triangle, whose vertices are (4, 4), (3, – 2) and (– 3, 16) is

$$\frac{1}{2}\left|4(-2-16)+3(16-4)+(-3)(4+2)\right|=\frac{|-54|}{2}=27.$$

Remark If the area of the triangle ABC is zero, then three points A, B and C lie on a line, i.e., they are collinear.

In the this Chapter, we shall continue the study of coordinate geometry to study properties of the simplest geometric figure – *straight line.* Despite its simplicity, the line is a vital concept of geometry and enters into our daily experiences in numerous interesting and useful ways. Main focus is on representing the line algebraically, for which *slope* is most essential.

9.2 Slope of a Line

A line in a coordinate plane forms two angles with the x-axis, which are supplementary.

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The angle (say) θ made by the line l with positive direction of x-axis and measured anti clockwise is called the *inclination of the line*. Obviously $0° \le \theta \le 180°$ (Fig 9.2).

We observe that lines parallel to x-axis, or coinciding with x-axis, have inclination of 0°. The inclination of a vertical line (parallel to or coinciding with y-axis) is 90°.

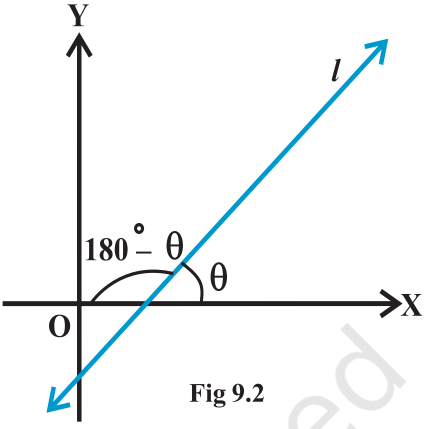

Fig 9.2

Definition 1 If θ is the inclination of a line l, then tan θ is called the *slope* or *gradient* of the line l.

The slope of a line whose inclination is 90° is not defined.

The slope of a line is denoted by m.

Thus, $m = \tan\theta,\ \theta \ne 90°$

It may be observed that the slope of x-axis is zero and slope of y-axis is not defined.

9.2.1 *Slope of a line when coordinates of any two points on the line are given*

We know that a line is completely determined when we are given two points on it. Hence, we proceed to find the slope of a line in terms of the coordinates of two points on the line.

Let P(x_1, y_1) and Q(x_2, y_2) be two points on non-vertical line l whose inclination is θ. Obviously, $x_1 \ne x_2$, otherwise the line will become perpendicular to x-axis and its slope will not be defined. The inclination of the line l may be acute or obtuse. Let us take these two cases.

Draw perpendicular QR to x-axis and PM perpendicular to RQ as shown in Figs. 9.3 (i) and (ii).

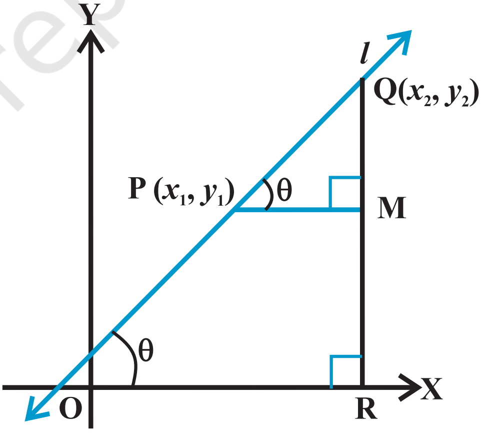

Fig 9. 3 (i)

Case 1 When angle θ is acute:

In Fig 9.3 (i), ∠MPQ = θ. ... (1)

Therefore, slope of line $l = m = \tan\theta$.

But in ΔMPQ, we have $\tan\theta = \frac{MQ}{MP} = \frac{y_2 - y_1}{x_2 - x_1}$. ... (2)

From equations (1) and (2), we have

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

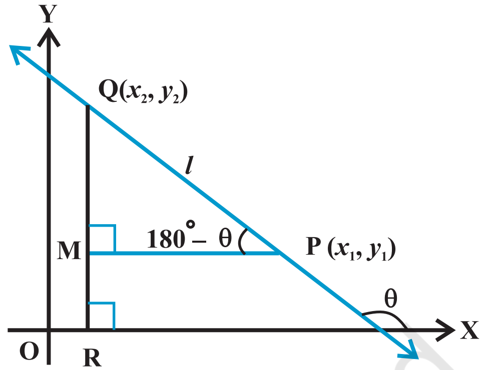

Fig 9. 3 (ii)

Case II When angle θ is obtuse:
In Fig 9.3 (ii), we have
$\angle MPQ = 180° - \theta$.

Therefore, $\theta = 180° - \angle MPQ$.

Now, slope of the line l

$$m = \tan\theta$$
$$= \tan(180° - \angle MPQ) = -\tan \angle MPQ$$
$$= -\frac{MQ}{MP} = -\frac{y_2 - y_1}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Consequently, we see that in both the cases the slope m of the line through the points (x_1, y_1) and (x_2, y_2) is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

9.2.2 ***Conditions for parallelism and perpendicularity of lines in terms of their slopes*** In a coordinate plane, suppose that non-vertical *lines* l_1 and l_2 have slopes m_1 *and* m_2, respectively. Let their inclinations be α and β, respectively.

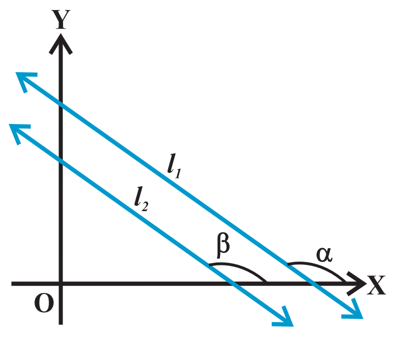

Fig 9. 4

If the line l_1 is parallel to l_2 (Fig 9.4), then their inclinations are equal, i.e.,

$$\alpha = \beta, \text{ and hence, } \tan\alpha = \tan\beta$$

Therefore $m_1 = m_2$, i.e., their slopes are equal.

Conversely, if the slope of two lines l_1 and l_2 is same, i.e.,

$$m_1 = m_2.$$

Then $\tan\alpha = \tan\beta$.

By the property of tangent function (between 0° and 180°), α = β.

Therefore, the lines are parallel.

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Hence, two non vertical lines l_1 and l_2 are parallel if and only if their slopes are equal.

If the lines l_1 and l_2 are perpendicular (Fig 9.5), then $\beta = \alpha + 90°$.

Therefore, $\tan \beta = \tan (\alpha + 90°)$

$$= -\cot \alpha = -\frac{1}{\tan \alpha}$$

i.e., $m_2 = -\frac{1}{m_1}$ or $m_1\ m_2 = -1$

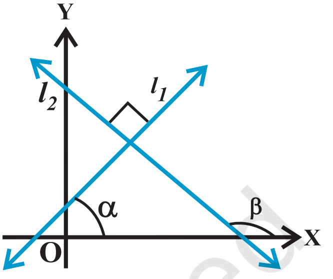

Fig 9. 5

Conversely, if $m_1\ m_2 = -1$, i.e., $\tan \alpha \tan \beta = -1$.

Then $\tan \alpha = -\cot \beta = \tan (\beta + 90°)$ or $\tan (\beta - 90°)$

Therefore, α and β differ by 90°.

Thus, lines l_1 and l_2 are perpendicular to each other.

Hence, *two non-vertical lines are perpendicular to each other if and only if their slopes are negative reciprocals of each other,*

i.e., $m_2 = -\frac{1}{m_1}$ or, $m_1\ m_2 = -1$.

Let us consider the following example.

Example 1 Find the slope of the lines:

(a) Passing through the points (3, – 2) and (–1, 4),

(b) Passing through the points (3, – 2) and (7, – 2),

(c) Passing through the points (3, – 2) and (3, 4),

(d) Making inclination of 60° with the positive direction of x-axis.

Solution (a) The slope of the line through (3, – 2) and (– 1, 4) is

$$m = \frac{4-(-2)}{-1-3} = \frac{6}{-4} = -\frac{3}{2}.$$

(b) The slope of the line through the points (3, – 2) and (7, – 2) is

$$m = \frac{-2-(-2)}{7-3} = \frac{0}{4} = 0.$$

(c) The slope of the line through the points (3, – 2) and (3, 4) is

$$m = \frac{4-(-2)}{3-3} = \frac{6}{0}, \text{ which is not defined.}$$

(d) Here inclination of the line $\alpha = 60°$. Therefore, slope of the line is

$m = \tan 60° = \sqrt{3}$.

9.2.3 ***Angle between two lines*** When we think about more than one line in a plane, then we find that these lines are either intersecting or parallel. Here we will discuss the angle between two lines in terms of their slopes.

Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2, respectively. If α_1 and α_2 are the inclinations of lines L_1 and L_2, respectively. Then

$$m_1 = \tan\alpha_1 \text{ and } m_2 = \tan\alpha_2 .$$

We know that when two lines intersect each other, they make two pairs of vertically opposite angles such that sum of any two adjacent angles is 180°. Let θ and ϕ be the adjacent angles between the lines L_1 and L_2 (Fig 9.6). Then

$$\theta = \alpha_2 - \alpha_1 \text{ and } \alpha_1, \alpha_2 \neq 90°.$$

Therefore $\tan\theta = \tan(\alpha_2 - \alpha_1) = \dfrac{\tan\alpha_2 - \tan\alpha_1}{1+\tan\alpha_1 \tan\alpha_2} = \dfrac{m_2 - m_1}{1+m_1 m_2}$ (as $1 + m_1 m_2 \neq 0$)

and $\phi = 180° - \theta$ so that

$$\tan\phi = \tan(180° - \theta) = -\tan\theta = -\frac{m_2 - m_1}{1+m_1 m_2}, \text{ as } 1 + m_1 m_2 \neq 0$$

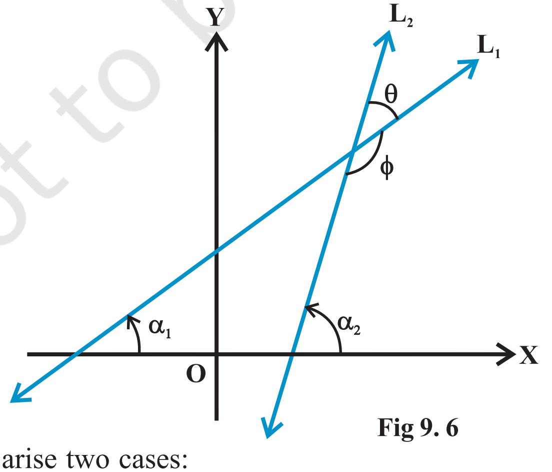

Fig 9. 6

Now, there arise two cases:

Case I If $\dfrac{m_2 - m_1}{1 + m_1 m_2}$ is positive, then tan θ will be positive and tan ϕ will be negative, which means θ will be acute and ϕ will be obtuse.

Case II If $\dfrac{m_2 - m_1}{1 + m_1 m_2}$ is negative, then tan θ will be negative and tan ϕ will be positive, which means that θ will be obtuse and ϕ will be acute.

Thus, the acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2, respectively, is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|, \text{ as } \quad 1 + m_1 m_2 \neq 0 \qquad \text{... (1)}$$

The obtuse angle (say ϕ) can be found by using $\phi = 180^0 - \theta$.

Example 2 If the angle between two lines is $\dfrac{\pi}{4}$ and slope of one of the lines is $\dfrac{1}{2}$, find the slope of the other line.

Solution We know that the acute angle θ between two lines with slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \qquad \text{... (1)}$$

Let $m_1 = \dfrac{1}{2}$, $m_2 = m$ and $\theta = \dfrac{\pi}{4}$.

Now, putting these values in (1), we get

$$\tan\frac{\pi}{4} = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| \quad \text{or} \quad 1 = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right|,$$

which gives $$\frac{m - \frac{1}{2}}{1 + \frac{1}{2}\,m} = 1 \quad \text{or} \quad \frac{m - \frac{1}{2}}{1 + \frac{1}{2}\,m} = -1.$$

Therefore $m = 3$ or $m = -\frac{1}{3}$.

Hence, slope of the other line is 3 or $-\frac{1}{3}$. Fig 9.7 explains the reason of two answers.

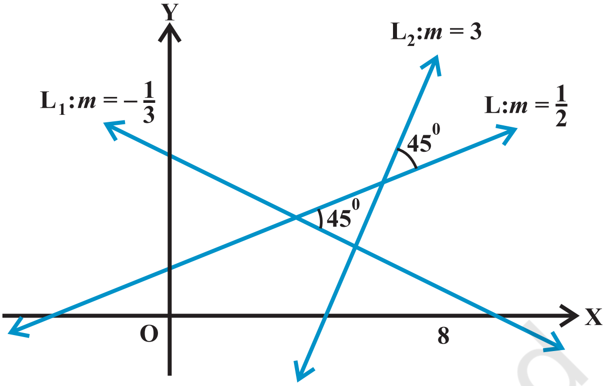

Fig 9.7

Example 3 Line through the points (–2, 6) and (4, 8) is perpendicular to the line through the points (8, 12) and (x, 24). Find the value of x.

Solution Slope of the line through the points (– 2, 6) and (4, 8) is

$$m_1 = \frac{8-6}{4-(-2)} = \frac{2}{6} = \frac{1}{3}$$

Slope of the line through the points (8, 12) and (x, 24) is

$$m_2 = \frac{24-12}{x-8} = \frac{12}{x-8}$$

Since two lines are perpendicular,

$m_1\, m_2 = -1$, which gives

$$\frac{1}{3} \times \frac{12}{x-8} = -1 \text{ or } x = 4.$$

EXERCISE 9.1

1. Draw a quadrilateral in the Cartesian plane, whose vertices are (– 4, 5), (0, 7), (5, – 5) and (– 4, –2). Also, find its area.
2. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.

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3. Find the distance between P (x_1, y_1) and Q (x_2, y_2) when : (i) PQ is parallel to the y-axis, (ii) PQ is parallel to the x-axis.

4. Find a point on the x-axis, which is equidistant from the points (7, 6) and (3, 4).

5. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points P (0, – 4) and B (8, 0).

6. Without using the Pythagoras theorem, show that the points (4, 4), (3, 5) and (–1, –1) are the vertices of a right angled triangle.

7. Find the slope of the line, which makes an angle of 30° with the positive direction of y-axis measured anticlockwise.

8. Without using distance formula, show that points (– 2, – 1), (4, 0), (3, 3) and (–3, 2) are the vertices of a parallelogram.

9. Find the angle between the x-axis and the line joining the points (3,–1) and (4,–2).

10. The slope of a line is double of the slope of another line. If tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines.

11. A line passes through (x_1, y_1) and (h, k). If slope of the line is m, show that

$$k - y_1 = m\,(h - x_1).$$

9.3 Various Forms of the Equation of a Line

We know that every line in a plane contains infinitely many points on it. This relationship between line and points leads us to find the solution of the following problem:

How can we say that a given point lies on the given line? Its answer may be that for a given line we should have a definite condition on the points lying on the line. Suppose P (x, y) is an arbitrary point in the XY-plane and L is the given line. For the equation of L, we wish to construct a *statement* or *condition* for the point P that is true, when P is on L, otherwise false. Of course the statement is merely an algebraic equation involving the variables x and y. Now, we will discuss the equation of a line under different conditions.

9.3.1 *Horizontal and vertical lines* If a horizontal line L is at a distance a from the x-axis then ordinate of every point lying on the line is either a or $-a$ [Fig 9.8 (a)]. Therefore, equation of the line L is either $y = a$ or $y = -a$. Choice of sign will depend upon the position of the line according as the line is above or below the y-axis. Similarly, the equation of a vertical line at a distance b from the y-axis is either $x = b$ or $x = -b$ [Fig 9.8(b)].

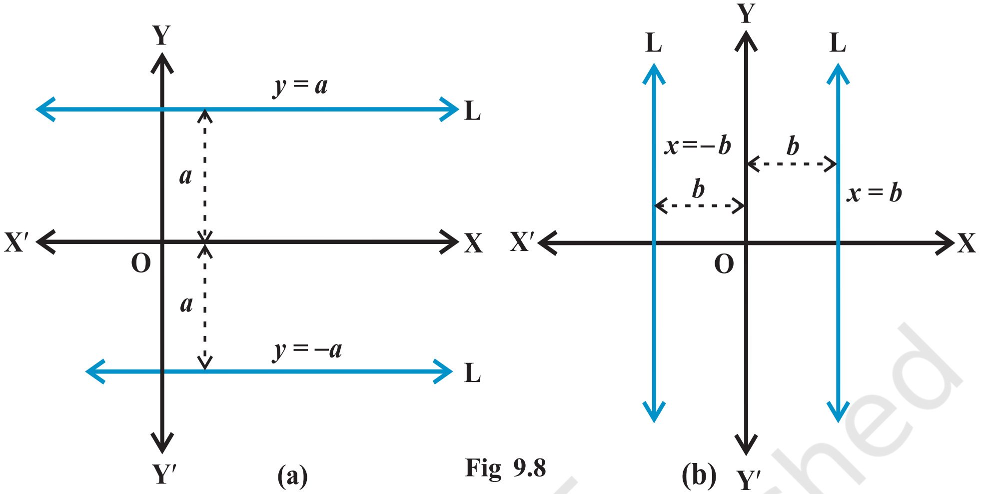

Fig 9.8

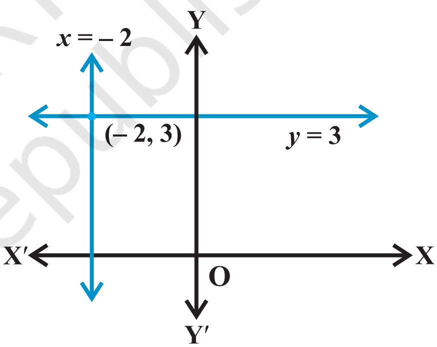

Fig 9.9

Example 4 Find the equations of the lines parallel to axes and passing through (– 2, 3).

Solution Position of the lines is shown in the Fig 9.9. The y-coordinate of every point on the line parallel to x-axis is 3, therefore, equation of the line parallel to x-axis and passing through (– 2, 3) is $y = 3$. Similarly, equation of the line parallel to y-axis and passing through (– 2, 3) is $x = -2$.

9.3.2 *Point-slope form*

Suppose that $P_0 (x_0, y_0)$ is a fixed point on a non-vertical line L, whose slope is m. Let P (x, y) be an arbitrary point on L (Fig 9.10).

Then, by the definition, the slope of L is given by

$$m = \frac{y - y_0}{x - x_0}, \text{ i.e., } y - y_0 = m(x - x_0) \qquad \ldots(1)$$

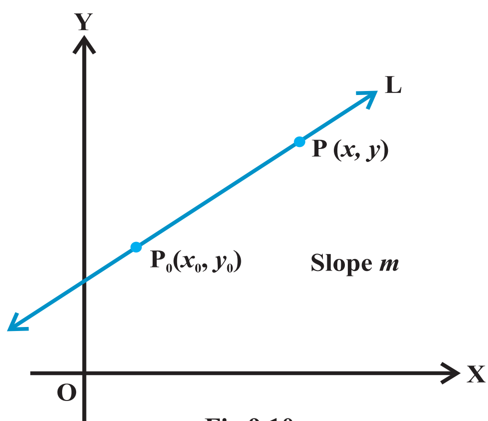

Fig 9.10

Since the point $P_0 (x_0, y_0)$ along with all points (x, y) on L satisfies (1) and no other point in the plane satisfies (1). Equation (1) is indeed the equation for the given line L.

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Thus, the point (x, y) lies on the line with slope m through the fixed point (x_0, y_0), if and only if, its coordinates satisfy the equation

$$y - y_0 = m\,(x - x_0)$$

Example 5 Find the equation of the line through $(-2, 3)$ with slope -4.

Solution Here $m = -4$ and given point (x_0, y_0) is $(-2, 3)$.

By slope-intercept form formula (1) above, equation of the given line is

$y - 3 = -4\,(x + 2)$ or $4x + y + 5 = 0$, which is the required equation.

9.3.3 *Two-point form* Let the line L passes through two given points $P_1\,(x_1, y_1)$ and $P_2\,(x_2, y_2)$. Let $P\,(x, y)$ be a general point on L (Fig 9.11).

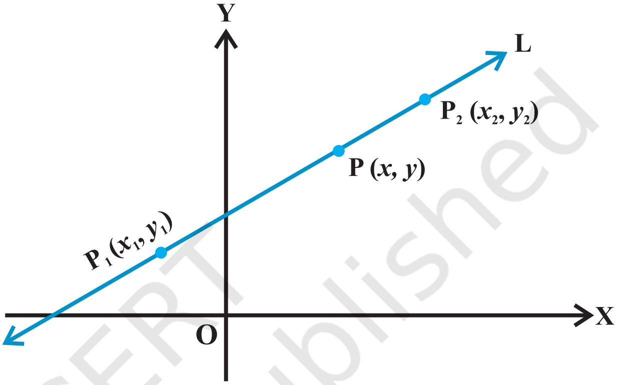

Fig 9.11

The three points P_1, P_2 and P are collinear, therefore, we have

slope of P_1P = slope of P_1P_2

i.e., $\dfrac{y-y_1}{x-x_1} = \dfrac{y_2-y_1}{x_2-x_1}$, or $y - y_1 = \dfrac{y_2-y_1}{x_2-x_1}(x - x_1)$.

Thus, equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1) \qquad \text{... (2)}$$

Example 6 Write the equation of the line through the points $(1, -1)$ and $(3, 5)$.

Solution Here $x_1 = 1$, $y_1 = -1$, $x_2 = 3$ and $y_2 = 5$. Using two-point form (2) above for the equation of the line, we have

$$y - (-1) = \frac{5-(-1)}{3-1}(x - 1)$$

or $\qquad -3x + y + 4 = 0$, which is the required equation.

9.3.4 *Slope-intercept form* Sometimes a line is known to us with its slope and an intercept on one of the axes. We will now find equations of such lines.

Case I Suppose a line L with slope m cuts the y-axis at a distance c from the origin (Fig 9.12). The distance c is called the y-*intercept* of the line L. Obviously, coordinates of the point where the line meet the y-axis are $(0, c)$. Thus, L has slope m and passes through a fixed point $(0, c)$. Therefore, by point-slope form, the equation of L is

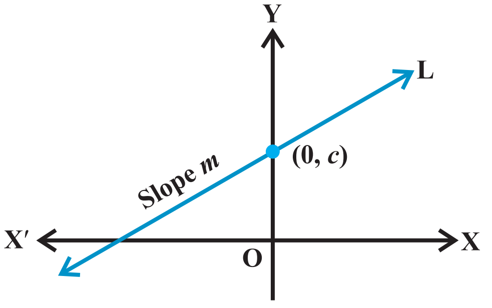

Fig 9.12

$$y - c = m(x - 0) \quad \text{or} \quad y = mx + c$$

Thus, the point (x, y) on the line with slope m and y-intercept c lies on the line if and only if

$$y = mx + c \qquad \ldots(3)$$

Note that the value of c will be positive or negative according as the intercept is made on the positive or negative side of the y-axis, respectively.

Case II Suppose line L with slope m makes x-intercept d. Then equation of L is

$$y = m(x - d) \qquad \ldots (4)$$

Students may derive this equation themselves by the same method as in Case I.

Example 7 Write the equation of the lines for which $\tan\theta = \frac{1}{2}$, where θ is the inclination of the line and (i) y-intercept is $-\frac{3}{2}$ (ii) x-intercept is 4.

Solution (i) Here, slope of the line is $m = \tan\theta = \frac{1}{2}$ and y - intercept $c = -\frac{3}{2}$.

Therefore, by slope-intercept form (3) above, the equation of the line is

$$y = \frac{1}{2}x - \frac{3}{2} \quad \text{or} \quad 2y - x + 3 = 0,$$

which is the required equation.

(ii) Here, we have $m = \tan\theta = \frac{1}{2}$ and $d = 4$.

Therefore, by slope-intercept form (4) above, the equation of the line is

$$y = \frac{1}{2}(x - 4) \text{ or } 2y - x + 4 = 0,$$

which is the required equation.

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9.3.5 *Intercept - form* Suppose a line L makes x-intercept a and y-intercept b on the axes. Obviously L meets x-axis at the point $(a, 0)$ and y-axis at the point $(0, b)$ (Fig .9.13). By two-point form of the equation of the line, we have

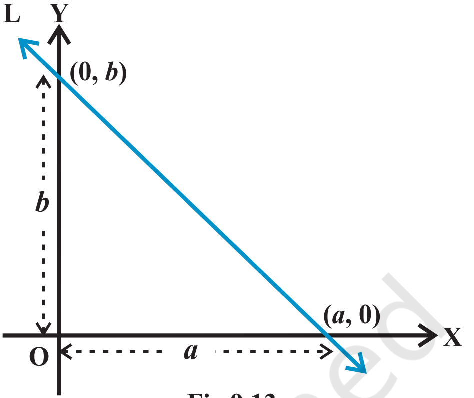

Fig 9.13

$$y-0=\frac{b-0}{0-a}(x-a) \quad \text{or} \quad ay=-bx+ab\,,$$

i.e., $$\frac{x}{a}+\frac{y}{b}=1.$$

Thus, equation of the line making intercepts a and b on x-and y-axis, respectively, is

$$\frac{x}{a}+\frac{y}{b}=1 \qquad \dots (5)$$

Example 8 Find the equation of the line, which makes intercepts –3 and 2 on the x- and y-axes respectively.

Solution Here $a = –3$ and $b = 2$. By intercept form (5) above, equation of the line is

$$\frac{x}{-3}+\frac{y}{2}=1 \quad \text{or} \quad 2x-3y+6=0.$$

Any equation of the form $Ax + By + C = 0$, where A and B are not zero simultaneously is called *general linear equation* or *general equation of a line*.

EXERCISE 9.2

In Exercises 1 to 8, find the equation of the line which satisfy the given conditions:

1. Write the equations for the x-and y-axes.
2. Passing through the point (– 4, 3) with slope $\frac{1}{2}$.
3. Passing through (0, 0) with slope m.
4. Passing through $\left(2, 2\sqrt{3}\right)$ and inclined with the x-axis at an angle of 75°.
5. Intersecting the x-axis at a distance of 3 units to the left of origin with slope –2.
6. Intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with positive direction of the x-axis.

7. Passing through the points (–1, 1) and (2, – 4).

8. The vertices of Δ PQR are P (2, 1), Q (–2, 3) and R (4, 5). Find equation of the median through the vertex R.

9. Find the equation of the line passing through (–3, 5) and perpendicular to the line through the points (2, 5) and (–3, 6).

10. A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio 1: n. Find the equation of the line.

11. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point (2, 3).

12. Find equation of the line passing through the point (2, 2) and cutting off intercepts on the axes whose sum is 9.

13. Find equation of the line through the point (0, 2) making an angle $\frac{2\pi}{3}$ with the positive x-axis. Also, find the equation of line parallel to it and crossing the y-axis at a distance of 2 units below the origin.

14. The perpendicular from the origin to a line meets it at the point (–2, 9), find the equation of the line.

15. The length L (in centimetre) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if L = 124.942 when C = 20 and L= 125.134 when C = 110, express L in terms of C.

16. The owner of a milk store finds that, he can sell 980 litres of milk each week at Rs 14/litre and 1220 litres of milk each week at Rs 16/litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at Rs 17/litre?

17. P (a, b) is the mid-point of a line segment between axes. Show that equation of the line is $\frac{x}{a}+\frac{y}{b}=2$.

18. Point R (h, k) divides a line segment between the axes in the ratio 1: 2. Find equation of the line.

19. By using the concept of equation of a line, prove that the three points (3, 0), (– 2, – 2) and (8, 2) are collinear.

9.4 Distance of a Point From a Line

The distance of a point from a line is the length of the perpendicular drawn from the point to the line. Let L : Ax + By + C = 0 be a line, whose distance from the point P (x_1, y_1) is d. Draw a perpendicular PM from the point P to the line L (Fig 9.14). If the

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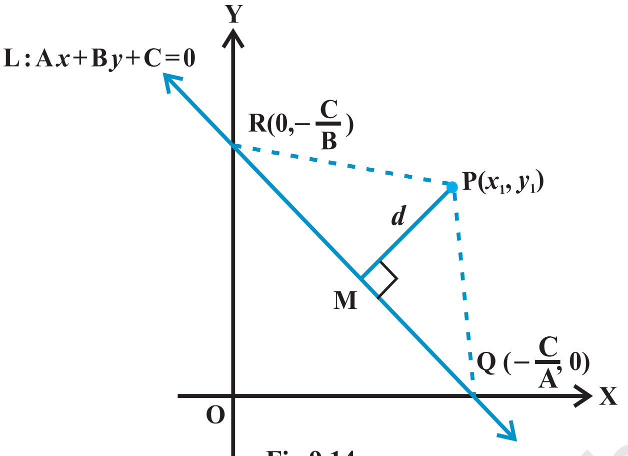

Fig 9.14

line meets the x-and y-axes at the points Q and R, respectively. Then, coordinates of the points are $Q\left(-\frac{C}{A},\ 0\right)$ and $R\left(0,\ -\frac{C}{B}\right)$. Thus, the area of the triangle PQR is given by

$$\text{area }(\Delta PQR) = \frac{1}{2}PM.QR\text{, which gives } PM = \frac{2\ \text{ area }(\Delta PQR)}{QR} \qquad ...(1)$$

Also, area $(\Delta PQR) = \frac{1}{2}\left| x_1\left(0+\frac{C}{B}\right) + \left(-\frac{C}{A}\right)\left(-\frac{C}{B} - y_1\right) + 0(y_1 - 0)\right|$

$$= \frac{1}{2}\left| x_1\frac{C}{B} + y_1\frac{C}{A} + \frac{C^2}{AB}\right|$$

or $\quad 2 \text{ area }(\Delta PQR) = \left|\frac{C}{AB}\right|.\ \left|Ax_1 + By_1 + C\right|$, and

$$QR = \sqrt{\left(0+\frac{C}{A}\right)^2 + \left(\frac{C}{B} - 0\right)^2} = \left|\frac{C}{AB}\right|\ \sqrt{A^2+B^2}$$

Substituting the values of area (ΔPQR) and QR in (1), we get

$$PM = \frac{\left| Ax_1 + By_1 + C\right|}{\sqrt{A^2+B^2}}$$

or $$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

Thus, the perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is given by

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

9.4.1 *Distance between two parallel lines*

We know that slopes of two parallel lines are equal. Therefore, two parallel lines can be taken in the form

$$y = mx + c_1 \qquad \text{... (1)}$$

and $$y = mx + c_2 \qquad \text{... (2)}$$

Line (1) will intersect x-axis at the point $A\left(-\frac{c_1}{m},\ 0\right)$ as shown in Fig 9.15.

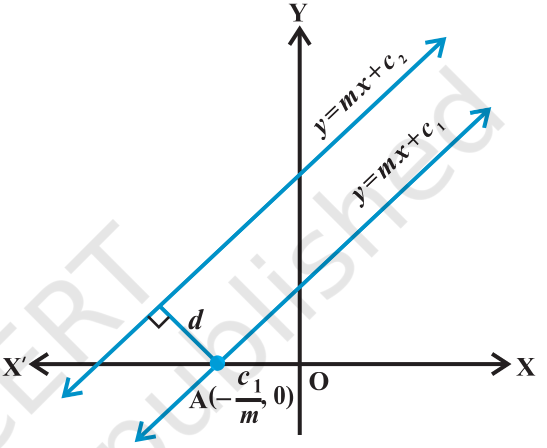

Fig 9.15

Distance between two lines is equal to the length of the perpendicular from point A to line (2). Therefore, distance between the lines (1) and (2) is

$$\frac{\left|(-m)\left(-\frac{c_1}{m}\right) + (-c_2)\right|}{\sqrt{1+m^2}} \quad \text{or} \quad d = \frac{|c_1 - c_2|}{\sqrt{1+m^2}}.$$

Thus, the distance d between two parallel lines $y = mx + c_1$ and $y = mx + c_2$ is given by

$$d = \frac{|c_1 - c_2|}{\sqrt{1+m^2}}.$$

If lines are given in general form, i.e., $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$,

then above formula will take the form $d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$

Students can derive it themselves.

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Example 9 Find the distance of the point (3, – 5) from the line $3x - 4y - 26 = 0$.

Solution Given line is $3x - 4y - 26 = 0$ ... (1)

Comparing (1) with general equation of line $Ax + By + C = 0$, we get

$$A = 3,\ B = -4 \text{ and } C = -26.$$

Given point is $(x_1, y_1) = (3, -5)$. The distance of the given point from given line is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|3.3 + (-4)(-5) - 26|}{\sqrt{3^2 + (-4)^2}} = \frac{3}{5}.$$

Example 10 Find the distance between the parallel lines $3x - 4y + 7 = 0$ and

$$3x - 4y + 5 = 0$$

Solution Here $A = 3$, $B = -4$, $C_1 = 7$ and $C_2 = 5$. Therefore, the required distance is

$$d = \frac{|7 - 5|}{\sqrt{3^2 + (-4)^2}} = \frac{2}{5}.$$

EXERCISE 9.3

1. Reduce the following equations into slope - intercept form and find their slopes and the y - intercepts.
   (i) $x + 7y = 0$, (ii) $6x + 3y - 5 = 0$, (iii) $y = 0$.
2. Reduce the following equations into intercept form and find their intercepts on the axes.
   (i) $3x + 2y - 12 = 0$, (ii) $4x - 3y = 6$, (iii) $3y + 2 = 0$.
3. Find the distance of the point (–1, 1) from the line $12(x + 6) = 5(y - 2)$.
4. Find the points on the x-axis, whose distances from the line $\frac{x}{3} + \frac{y}{4} = 1$ are 4 units.
5. Find the distance between parallel lines
   (i) $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$ (ii) $l(x + y) + p = 0$ and $l(x + y) - r = 0$.
6. Find equation of the line parallel to the line $3x - 4y + 2 = 0$ and passing through the point (–2, 3).
7. Find equation of the line perpendicular to the line $x - 7y + 5 = 0$ and having x intercept 3.
8. Find angles between the lines $\sqrt{3}x + y = 1$ and $x + \sqrt{3}y = 1$.
9. The line through the points $(h, 3)$ and $(4, 1)$ intersects the line $7x - 9y - 19 = 0$. at right angle. Find the value of h.

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10. Prove that the line through the point (x_1, y_1) and parallel to the line Ax + By + C = 0 is

$$A(x - x_1) + B(y - y_1) = 0.$$

11. Two lines passing through the point (2, 3) intersects each other at an angle of 60°. If slope of one line is 2, find equation of the other line.

12. Find the equation of the right bisector of the line segment joining the points (3, 4) and (–1, 2).

13. Find the coordinates of the foot of perpendicular from the point (–1, 3) to the line $3x - 4y - 16 = 0$.

14. The perpendicular from the origin to the line $y = mx + c$ meets it at the point (–1, 2). Find the values of m and c.

15. If p and q are the lengths of perpendiculars from the origin to the lines $x\cos\theta - y\sin\theta = k\cos 2\theta$ and $x \sec\theta + y \operatorname{cosec}\theta = k$, respectively, prove that $p^2 + 4q^2 = k^2$.

16. In the triangle ABC with vertices A (2, 3), B (4, –1) and C (1, 2), find the equation and length of altitude from the vertex A.

17. If p is the length of perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

Miscellaneous Examples

Example 11 If the lines $2x + y - 3 = 0$, $5x + ky - 3 = 0$ and $3x - y - 2 = 0$ are concurrent, find the value of k.

Solution Three lines are said to be concurrent, if they pass through a common point, i.e., point of intersection of any two lines lies on the third line. Here given lines are

$$2x + y - 3 = 0 \quad \ldots (1)$$

$$5x + ky - 3 = 0 \quad \ldots (2)$$

$$3x - y - 2 = 0 \quad \ldots (3)$$

Solving (1) and (3) by cross-multiplication method, we get

$$\frac{x}{-2-3} = \frac{y}{-9+4} = \frac{1}{-2-3} \quad \text{or} \quad x = 1,\ y = 1.$$

Therefore, the point of intersection of two lines is (1, 1). Since above three lines are concurrent, the point (1, 1) will satisfy equation (2) so that

$$5.1 + k.1 - 3 = 0 \text{ or } k = -2.$$

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Example 12 Find the distance of the line $4x - y = 0$ from the point P (4, 1) measured along the line making an angle of 135° with the positive x-axis.

Solution Given line is $4x - y = 0$ ... (1)

In order to find the distance of the line (1) from the point P (4, 1) along another line, we have to find the point of intersection of both the lines. For this purpose, we will first find the equation of the second line (Fig 9.16). Slope of second line is $\tan 135° = -1$. Equation of the line with slope -1 through the point P (4, 1) is

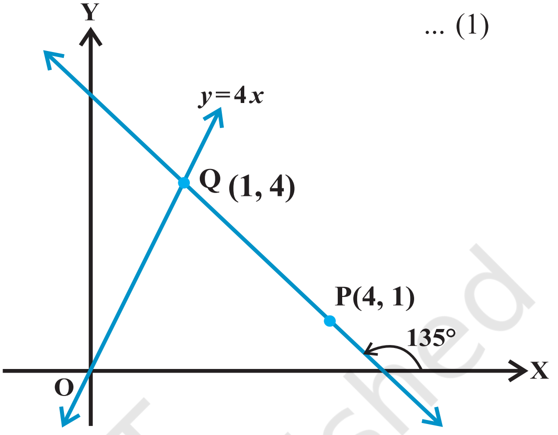

Fig 9.16

$$y - 1 = -1\ (x - 4) \text{ or } x + y - 5 = 0 \qquad \text{... (2)}$$

Solving (1) and (2), we get $x = 1$ and $y = 4$ so that point of intersection of the two lines is Q (1, 4). Now, distance of line (1) from the point P (4, 1) along the line (2)

= the distance between the points P (4, 1) and Q (1, 4).

$$= \sqrt{(1-4)^2 + (4-1)^2} = 3\sqrt{2} \text{ units.}$$

Example 13 Assuming that straight lines work as the plane mirror for a point, find the image of the point (1, 2) in the line $x - 3y + 4 = 0$.

Solution Let Q (h, k) is the image of the point P (1, 2) in the line

$$x - 3y + 4 = 0 \qquad \text{... (1)}$$

Therefore, the line (1) is the perpendicular bisector of line segment PQ (Fig 9.17).

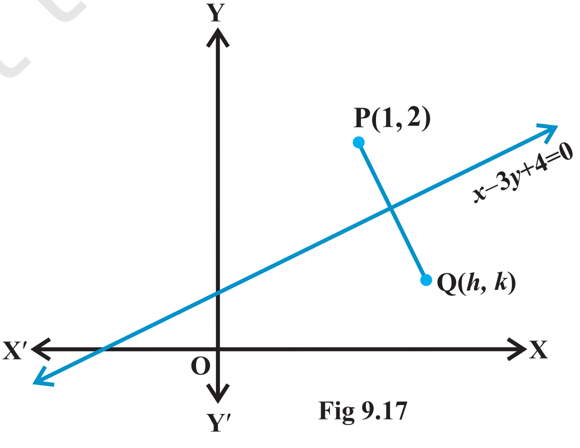

Fig 9.17

Hence Slope of line PQ = $\dfrac{-1}{\text{Slope of line } \quad x-3y+4=0}$,

so that $\dfrac{k-2}{h-1}=\dfrac{-1}{\frac{1}{3}}$ or $3h+k=5$ ... (2)

and the mid-point of PQ, i.e., point $\left(\dfrac{h+1}{2},\dfrac{k+2}{2}\right)$ will satisfy the equation (1) so that

$$\frac{h+1}{2}-3\left(\frac{k+2}{2}\right)+4=0 \text{ or } h-3k=-3 \qquad \text{... (3)}$$

Solving (2) and (3), we get $h=\dfrac{6}{5}$ and $k=\dfrac{7}{5}$.

Hence, the image of the point (1, 2) in the line (1) is $\left(\dfrac{6}{5},\dfrac{7}{5}\right)$.

Example 14 Show that the area of the triangle formed by the lines

$y=m_1x+c_1$, $y=m_2x+c_2$ and $x=0$ is $\dfrac{(c_1-c_2)^2}{2|m_1-m_2|}$.

Solution Given lines are

$y=m_1x+c_1$ ... (1)

$y=m_2x+c_2$ ... (2)

$x=0$ ... (3)

We know that line $y = mx + c$ meets the line $x = 0$ (y-axis) at the point $(0, c)$. Therefore, two vertices of the triangle formed by lines (1) to (3) are P $(0, c_1)$ and Q $(0, c_2)$ (Fig 9.18).

Third vertex can be obtained by solving equations (1) and (2). Solving (1) and (2), we get

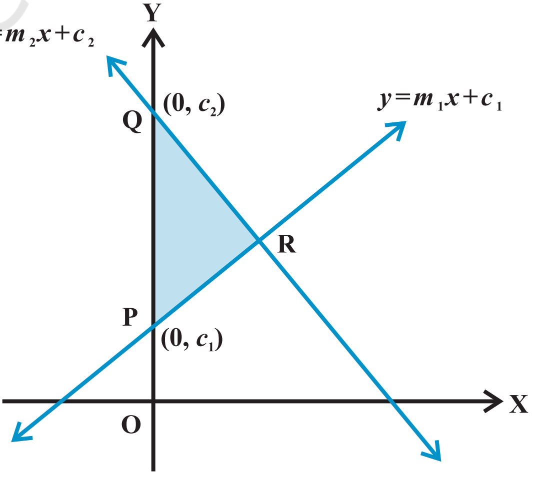

Fig 9.18

$$x=\frac{(c_2-c_1)}{(m_1-m_2)} \text{ and } y=\frac{(m_1c_2-m_2c_1)}{(m_1-m_2)}$$

Therefore, third vertex of the triangle is R $\left(\frac{(c_2-c_1)}{(m_1-m_2)}, \frac{(m_1c_2-m_2c_1)}{(m_1-m_2)}\right)$.

Now, the area of the triangle is

$$=\frac{1}{2}\left| 0 \left(\frac{m_1c_2-m_2c_1}{m_1-m_2}-c_2\right)+\frac{c_2-c_1}{m_1-m_2}(c_2-c_1)+0\left(c_1-\frac{m_1c_2-m_2c_1}{m_1-m_2}\right)\right| = \frac{(c_2-c_1)^2}{2|m_1-m_2|}$$

Example 15 A line is such that its segment between the lines $5x - y + 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point (1, 5). Obtain its equation.

Solution Given lines are

$5x - y + 4 = 0$ ... (1)

$3x + 4y - 4 = 0$ ... (2)

Let the required line intersects the lines (1) and (2) at the points, (α_1, β_1) and (α_2, β_2), respectively (Fig 9.19). Therefore

$$5\alpha_1 - \beta_1 + 4 = 0 \text{ and}$$
$$3\,\alpha_2 + 4\,\beta_2 - 4 = 0$$

or $\quad \beta_1 = 5\alpha_1 + 4$ and $\beta_2 = \dfrac{4-3\alpha_2}{4}$.

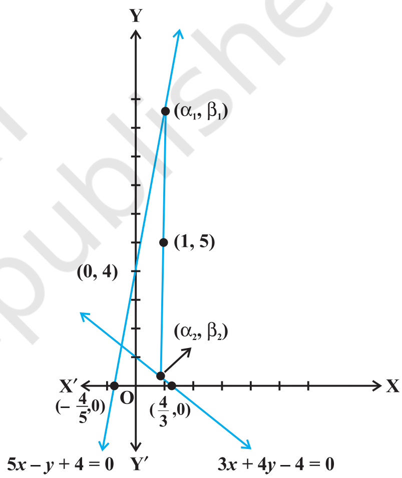

Fig 9.19

We are given that the mid point of the segment of the required line between (α_1, β_1) and (α_2, β_2) is (1, 5). Therefore

$$\frac{\alpha_1+\alpha_2}{2}=1 \text{ and } \frac{\beta_1+\beta_2}{2}=5,$$

or
$$\alpha_1+\alpha_2=2 \text{ and } \frac{5\alpha_1+4+\dfrac{4-3\alpha_2}{4}}{2}=5,$$

$$\text{or} \quad \alpha_1+\alpha_2=2 \text{ and } 20\alpha_1-3\alpha_2=20 \qquad \text{... (3)}$$

Solving equations in (3) for α_1 and α_2, we get

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$$\alpha_1 = \frac{26}{23} \text{ and } \alpha_2 = \frac{20}{23} \text{ and hence, } \beta_1 = 5.\frac{26}{23} + 4 = \frac{222}{23}.$$

Equation of the required line passing through (1, 5) and (α_1, β_1) is

$$y - 5 = \frac{\beta_1 - 5}{\alpha_1 - 1}(x - 1) \text{ or } y - 5 = \frac{\frac{222}{23} - 5}{\frac{26}{23} - 1}(x - 1)$$

or $\qquad 107x - 3y - 92 = 0,$

which is the equation of required line.

Example 16 Show that the path of a moving point such that its distances from two lines $3x - 2y = 5$ and $3x + 2y = 5$ are equal is a straight line.

Solution Given lines are

$$3x - 2y = 5 \qquad \ldots (1)$$

and
$$3x + 2y = 5 \qquad \ldots (2)$$

Let (h, k) is any point, whose distances from the lines (1) and (2) are equal. Therefore

$$\frac{|3h - 2k - 5|}{\sqrt{9+4}} = \frac{|3h + 2k - 5|}{\sqrt{9+4}} \text{ or } |3h - 2k - 5| = |3h + 2k - 5|,$$

which gives $3h - 2k - 5 = 3h + 2k - 5$ or $-(3h - 2k - 5) = 3h + 2k - 5$.

Solving these two relations we get $k = 0$ or $h = \frac{5}{3}$. Thus, the point (h, k) satisfies the equations $y = 0$ or $x = \frac{5}{3}$, which represent straight lines. Hence, path of the point equidistant from the lines (1) and (2) is a straight line.

Miscellaneous Exercise on Chapter 9

1. Find the values of k for which the line $(k-3)\,x - (4 - k^2)\,y + k^2 - 7k + 6 = 0$ is
   (a) Parallel to the x-axis,
   (b) Parallel to the y-axis,
   (c) Passing through the origin.
2. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and – 6, respectively.

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3. What are the points on the y-axis whose distance from the line $\frac{x}{3}+\frac{y}{4}=1$ is 4 units.
4. Find perpendicular distance from the origin to the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$.
5. Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines $x - 7y + 5 = 0$ and $3x + y = 0$.
6. Find the equation of a line drawn perpendicular to the line $\frac{x}{4}+\frac{y}{6}=1$ through the point, where it meets the y-axis.
7. Find the area of the triangle formed by the lines $y - x = 0$, $x + y = 0$ and $x - k = 0$.
8. Find the value of p so that the three lines $3x + y - 2 = 0$, $px + 2y - 3 = 0$ and $2x - y - 3 = 0$ may intersect at one point.
9. If three lines whose equations are $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent, then show that $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$.
10. Find the equation of the lines through the point (3, 2) which make an angle of 45° with the line $x - 2y = 3$.
11. Find the equation of the line passing through the point of intersection of the lines $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$ that has equal intercepts on the axes.
12. Show that the equation of the line passing through the origin and making an angle θ with the line $y = mx + c$ is $\frac{y}{x} = \frac{m \pm \tan ,}{1 \mp m\tan ,}$.
13. In what ratio, the line joining (–1, 1) and (5, 7) is divided by the line $x + y = 4$?
14. Find the distance of the line $4x + 7y + 5 = 0$ from the point (1, 2) along the line $2x - y = 0$.
15. Find the direction in which a straight line must be drawn through the point (–1, 2) so that its point of intersection with the line $x + y = 4$ may be at a distance of 3 units from this point.
16. The hypotenuse of a right angled triangle has its ends at the points (1, 3) and (– 4, 1). Find an equation of the legs (perpendicular sides) of the triangle which are parallel to the axes.
17. Find the image of the point (3, 8) with respect to the line $x + 3y = 7$ assuming the line to be a plane mirror.
18. If the lines $y = 3x + 1$ and $2y = x + 3$ are equally inclined to the line $y = mx + 4$, find the value of m.
19. If sum of the perpendicular distances of a variable point P (x, y) from the lines $x + y - 5 = 0$ and $3x - 2y + 7 = 0$ is always 10. Show that P must move on a line.

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20. Find equation of the line which is equidistant from parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.

21. A ray of light passing through the point (1, 2) reflects on the x-axis at point A and the reflected ray passes through the point (5, 3). Find the coordinates of A.

22. Prove that the product of the lengths of the perpendiculars drawn from the points $\left(\sqrt{a^2-b^2}, 0\right)$ and $\left(-\sqrt{a^2-b^2}, 0\right)$ to the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is b^2.

23. A person standing at the junction (crossing) of two straight paths represented by the equations $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ wants to reach the path whose equation is $6x - 7y + 8 = 0$ in the least time. Find equation of the path that he should follow.

Summary

- *Slope* (m) of a non-vertical line passing through the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$, $x_1 \neq x_2$.
- If a line makes an angle á with the positive direction of x-axis, then the slope of the line is given by $m = \tan\alpha$, $\alpha \neq 90°$.
- Slope of horizontal line is zero and slope of vertical line is undefined.
- An acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2 is given by $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|, 1 + m_1 m_2 \neq 0$.
- Two lines are *parallel* if and only if their slopes are equal.
- Two lines are *perpendicular* if and only if product of their slopes is –1.
- Three points A, B and C are collinear, if and only if slope of AB = slope of BC.
- Equation of the horizontal line having distance a from the x-axis is either $y = a$ or $y = -a$.
- Equation of the vertical line having distance b from the y-axis is either $x = b$ or $x = -b$.
- The point (x, y) lies on the line with slope m and through the fixed point (x_o, y_o), if and only if its coordinates satisfy the equation $y - y_o = m(x - x_o)$.
- Equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

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- The point (x, y) on the line with slope m and y-intercept c lies on the line if and only if $y = mx + c$.
- If a line with slope m makes x-intercept d. Then equation of the line is $y = m (x - d)$.
- Equation of a line making intercepts a and b on the x-and y-axis, respectively, is $\frac{x}{a} + \frac{y}{b} = 1$.
- Any equation of the form Ax + By + C = 0, with A and B are not zero, simultaneously, is called the *general linear equation* or *general equation of a line*.
- The perpendicular distance (d) of a line Ax + By+ C = 0 from a point (x_1, y_1) is given by $d = \frac{|\mathrm{A}x_1 + \mathrm{B}y_1 + \mathrm{C}|}{\sqrt{\mathrm{A}^2 + \mathrm{B}^2}}$.
- Distance between the parallel lines Ax + By + C_1 = 0 and Ax + By + C_2 = 0, is given by $d = \frac{|\mathrm{C}_1 - \mathrm{C}_2|}{\sqrt{\mathrm{A}^2 + \mathrm{B}^2}}$.

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Chapter 10

CONIC SECTIONS

❖ *Let the relation of knowledge to real life be very visible to your pupils and let them understand how by knowledge the world could be transformed.* – *BERTRAND RUSSELL* ❖

10.1 Introduction

Apollonius
(262 B.C. -190 B.C.)

In the preceding Chapter 10, we have studied various forms of the equations of a line. In this Chapter, we shall study about some other curves, viz., circles, ellipses, parabolas and hyperbolas. The names parabola and hyperbola are given by Apollonius. These curves are in fact, known as *conic sections* or more commonly *conics* because they can be obtained as intersections of a plane with a double napped right circular cone. These curves have a very wide range of applications in fields such as planetary motion, design of telescopes and antennas, reflectors in flashlights and automobile headlights, etc. Now, in the subsequent sections we will see how the intersection of a plane with a double napped right circular cone results in different types of curves.

10.2 Sections of a Cone

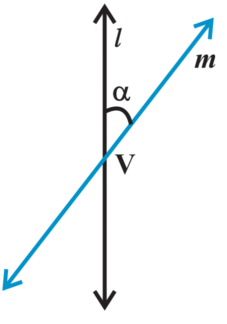

Fig 10. 1

Let l be a fixed vertical line and m be another line intersecting it at a fixed point V and inclined to it at an angle α (Fig10.1).

Suppose we rotate the line m around the line l in such a way that the angle α remains constant. Then the surface generated is a double-napped right circular hollow cone herein after referred as

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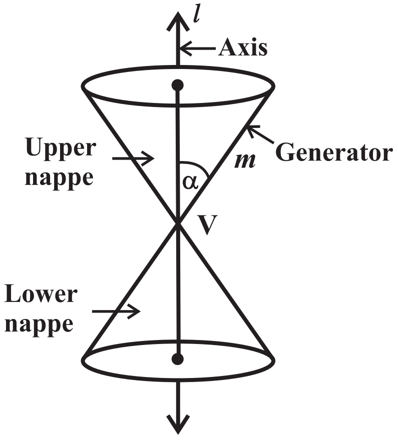

Fig 10. 2

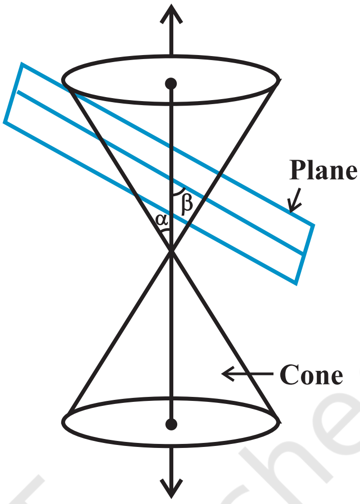

Fig 10. 3

cone and extending indefinitely far in both directions (Fig10.2).

The point V is called the *vertex*; the line l is the *axis* of the cone. The rotating line m is called a *generator* of the cone. The *vertex* separates the cone into two parts called *nappes*.

If we take the intersection of a plane with a cone, the section so obtained is called a *conic section*. Thus, conic sections are the curves obtained by intersecting a right circular cone by a plane.

We obtain different kinds of conic sections depending on the position of the intersecting plane with respect to the cone and by the angle made by it with the vertical axis of the cone. Let β be the angle made by the intersecting plane with the vertical axis of the cone (Fig10.3).

The intersection of the plane with the cone can take place either at the vertex of the cone or at any other part of the nappe either below or above the vertex.

10.2.1 *Circle, ellipse, parabola and hyperbola* When the plane cuts the nappe (other than the vertex) of the cone, we have the following situations:

(a) When $\beta = 90^\circ$, the section is a *circle* (Fig10.4).

(b) When $\alpha < \beta < 90^\circ$, the section is an *ellipse* (Fig10.5).

(c) When $\beta = \alpha$; the section is a *parabola* (Fig10.6).

(In each of the above three situations, the plane cuts entirely across one nappe of the cone).

(d) When $0 \leq \beta < \alpha$; the plane cuts through both the nappes and the curves of intersection is a *hyperbola* (Fig10.7).

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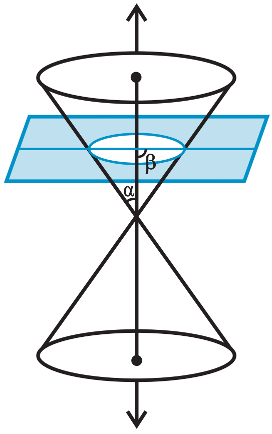

Fig 10. 4

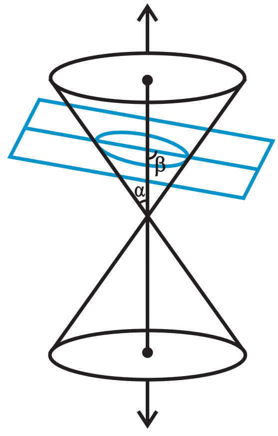

Fig 10. 5

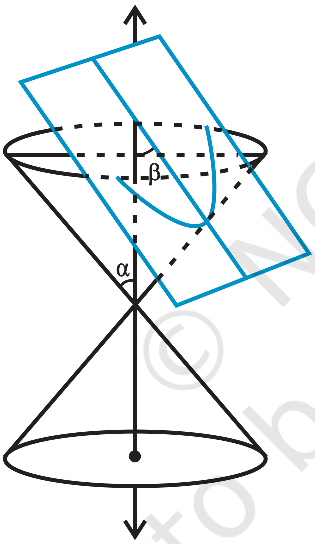

Fig 10. 6

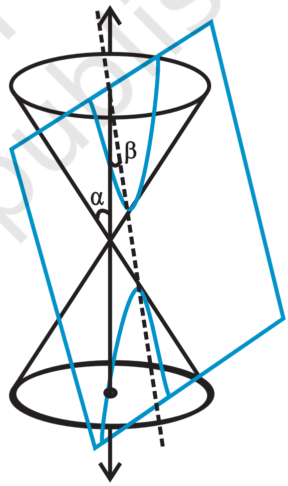

Fig 10. 7

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10.2.2 *Degenerated conic sections*

When the plane cuts at the vertex of the cone, we have the following different cases:

(a) When $\alpha < \beta \le 90^o$, then the section is a point (Fig10.8).

(b) When $\beta = \alpha$, the plane contains a generator of the cone and the section is a straight line (Fig10.9).

It is the degenerated case of a parabola.

(c) When $0 \le \beta < \alpha$, the section is a pair of intersecting straight lines (Fig10.10). It is the degenerated case of a *hyperbola*.

In the following sections, we shall obtain the equations of each of these conic sections in standard form by defining them based on geometric properties.

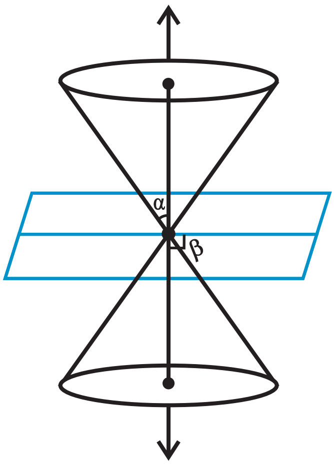


Fig 10. 8

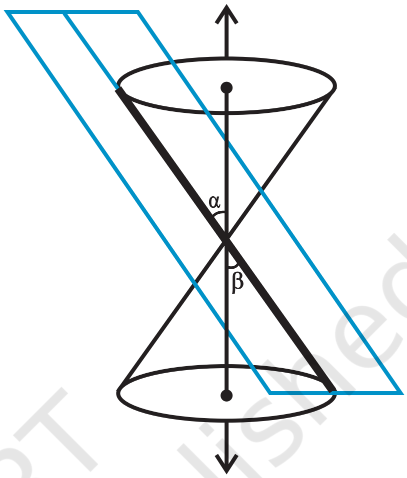
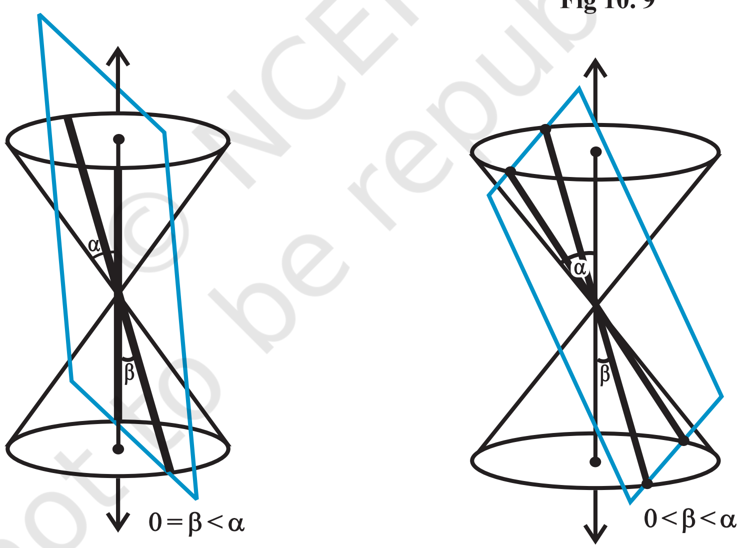


Fig 10. 9



(a) Fig 10. 10 (b)

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10.3 Circle

Definition 1 A circle is the set of all points in a plane that are equidistant from a fixed point in the plane.

The fixed point is called the *centre of the circle* and the distance from the centre to a point on the circle is called the *radius* of the circle (Fig 10.11).

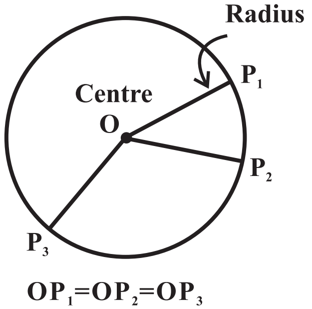

Fig 10. 11

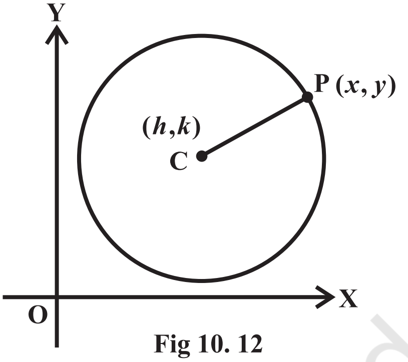

Fig 10. 12

The equation of the circle is simplest if the centre of the circle is at the origin. However, we derive below the equation of the circle with a given centre and radius (Fig 10.12).

Given C (h, k) be the centre and r the radius of circle. Let P(x, y) be any point on the circle (Fig10.12). Then, by the definition, | CP | = r . By the distance formula, we have

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

i.e. $$(x-h)^2+(y-k)^2=r^2$$

This is the required equation of the circle with centre at (h,k) and radius r .

Example 1 Find an equation of the circle with centre at (0,0) and radius r.

Solution Here $h = k = 0$. Therefore, the equation of the circle is $x^2 + y^2 = r^2$.

Example 2 Find the equation of the circle with centre (–3, 2) and radius 4.

Solution Here h = –3, k = 2 and r = 4. Therefore, the equation of the required circle is

$$(x+3)^2+(y-2)^2=16$$

Example 3 Find the centre and the radius of the circle $x^2 + y^2 + 8x + 10y - 8 = 0$

Solution The given equation is

$$(x^2+8x)+(y^2+10y)=8$$

Now, completing the squares within the parenthesis, we get

$$(x^2+8x+16)+(y^2+10y+25)=8+16+25$$

i.e. $$(x+4)^2+(y+5)^2=49$$

i.e. $$\{x-(-4)\}^2+\{y-(-5)\}^2=7^2$$

Therefore, the given circle has centre at (– 4, –5) and radius 7.

Example 4 Find the equation of the circle which passes through the points (2, – 2), and (3,4) and whose centre lies on the line $x + y = 2$.

Solution Let the equation of the circle be $(x - h)^2 + (y - k)^2 = r^2$.

Since the circle passes through (2, – 2) and (3,4), we have

$$(2 - h)^2 + (-2 - k)^2 = r^2 \quad \text{... (1)}$$

and $$(3 - h)^2 + (4 - k)^2 = r^2 \quad \text{... (2)}$$

Also since the centre lies on the line $x + y = 2$, we have

$$h + k = 2 \quad \text{... (3)}$$

Solving the equations (1), (2) and (3), we get

$$h = 0.7, \quad k = 1.3 \text{ and } r^2 = 12.58$$

Hence, the equation of the required circle is

$$(x - 0.7)^2 + (y - 1.3)^2 = 12.58.$$

EXERCISE 10.1

In each of the following Exercises 1 to 5, find the equation of the circle with

1. centre (0,2) and radius 2

2. centre (–2,3) and radius 4

3. centre $\left(\frac{1}{2}, \frac{1}{4}\right)$ and radius $\frac{1}{12}$

4. centre (1,1) and radius $\sqrt{2}$

5. centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$.

In each of the following Exercises 6 to 9, find the centre and radius of the circles.

6. $(x + 5)^2 + (y - 3)^2 = 36$

7. $x^2 + y^2 - 4x - 8y - 45 = 0$

8. $x^2 + y^2 - 8x + 10y - 12 = 0$

9. $2x^2 + 2y^2 - x = 0$

10. Find the equation of the circle passing through the points (4,1) and (6,5) and whose centre is on the line $4x + y = 16$.

11. Find the equation of the circle passing through the points (2,3) and (–1,1) and whose centre is on the line $x - 3y - 11 = 0$.

12. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point (2,3).

13. Find the equation of the circle passing through (0,0) and making intercepts a and b on the coordinate axes.

14. Find the equation of a circle with centre (2,2) and passes through the point (4,5).

15. Does the point (–2.5, 3.5) lie inside, outside or on the circle $x^2 + y^2 = 25$?

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10.4 Parabola

Definition 2 A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point (not on the line) in the plane.

The fixed line is called the *directrix* of the parabola and the fixed point F is called the *focus* (Fig 10.13). ('Para' means 'for' and 'bola' means 'throwing', i.e., the shape described when you throw a ball in the air).

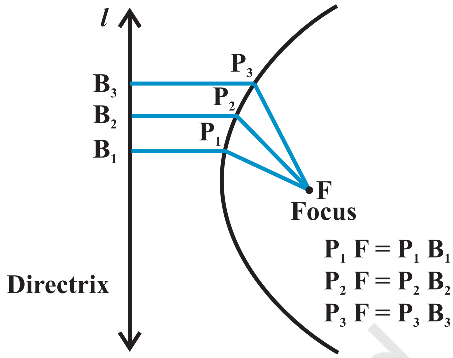


Fig 10. 13

> **Note** If the fixed point lies on the fixed line, then the set of points in the plane, which are equidistant from the fixed point and the fixed line is the straight line through the fixed point and perpendicular to the fixed line. We call this straight line as *degenerate case* of the parabola.

A line through the focus and perpendicular to the *directrix* is called the *axis* of the parabola. The point of intersection of parabola with the axis is called the vertex of the parabola (Fig10.14).

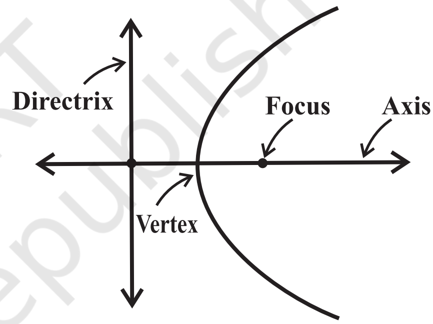
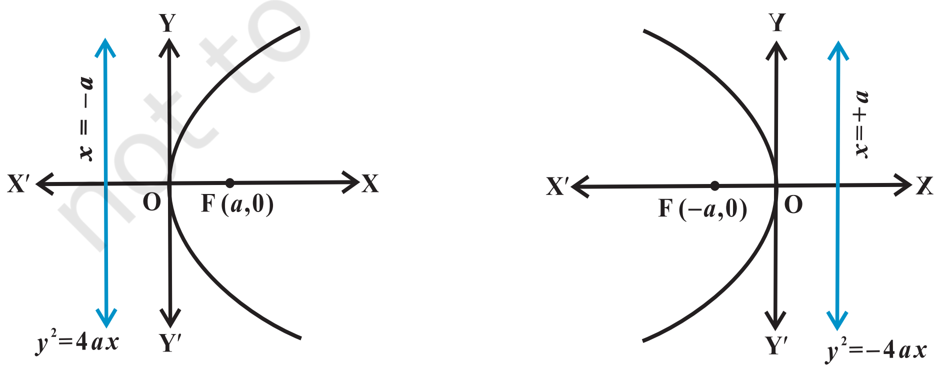


Fig 10.14

10.4.1 *Standard equations of parabola*

The equation of a *parabola* is simplest if the vertex is at the origin and the axis of symmetry is along the x-axis or y-axis. The four possible such orientations of parabola are shown below in Fig10.15 (a) to (d).



(a) (b)

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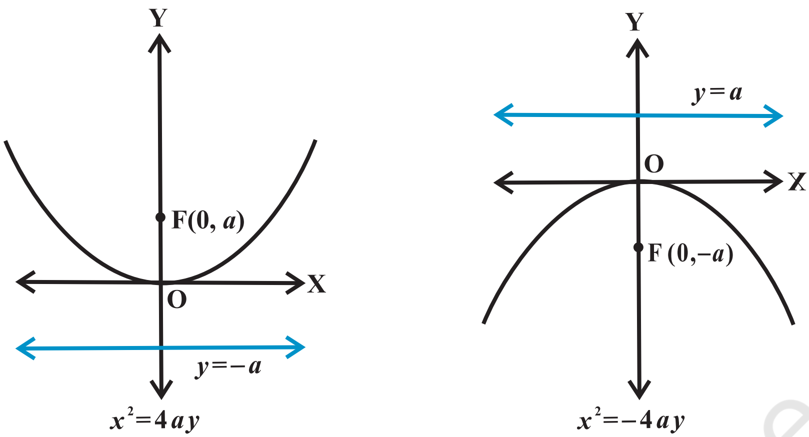

(c) Fig 10.15 (a) to (d) (d)

We will derive the equation for the parabola shown above in Fig 10.15 (a) with *focus* at $(a, 0)$ $a > 0$; and directricx $x = -a$ as below:

Let F be the *focus* and l the *directrix*. Let FM be perpendicular to the *directrix* and bisect FM at the point O. Produce MO to X. By the definition of parabola, the mid-point O is on the parabola and is called the *vertex* of the parabola. Take O as origin, OX the x-axis and OY perpendicular to it as the y-axis. Let the distance from the directrix to the focus be $2a$. Then, the coordinates of the *focus* are $(a, 0)$, and the equation of the *directrix* is $x + a = 0$ as in Fig10.16. Let $P(x, y)$ be any point on the parabola such that

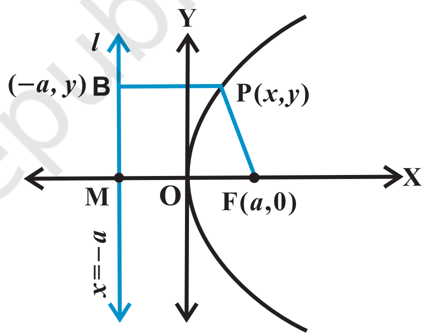

Fig 10.16

$$PF = PB, \quad \ldots (1)$$

where PB is perpendicular to l. The coordinates of B are $(-a, y)$. By the distance formula, we have

$$PF = \sqrt{(x-a)^2 + y^2} \text{ and } PB = \sqrt{(x+a)^2}$$

Since PF = PB, we have

$$\sqrt{(x-a)^2 + y^2} = \sqrt{(x+a)^2}$$

i.e. $(x - a)^2 + y^2 = (x + a)^2$

or $x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$

or $y^2 = 4ax$ ($a > 0$).

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Hence, any point on the parabola satisfies

$$y^2 = 4ax. \qquad \text{... (2)}$$

Conversely, let P(x, y) satisfy the equation (2)

$$\text{PF} = \sqrt{(x-a)^2 + y^2} = \sqrt{(x-a)^2 + 4ax}$$

$$= \sqrt{(x+a)^2} = \text{PB} \qquad \text{... (3)}$$

and so P(x,y) lies on the parabola.

Thus, from (2) and (3) we have proved that the equation to the parabola with vertex at the origin, focus at (a,0) and directrix $x = -a$ is $y^2 = 4ax$.

Discussion In equation (2), since $a > 0$, x can assume any positive value or zero but no negative value and the curve extends indefinitely far into the first and the fourth quadrants. The axis of the parabola is the positive x-axis.

Similarly, we can derive the equations of the parabolas in:

Fig 11.15 (b) as $y^2 = -4ax$,
Fig 11.15 (c) as $x^2 = 4ay$,
Fig 11.15 (d) as $x^2 = -4ay$,

These four equations are known as *standard equations* of parabolas.

Note The standard equations of parabolas have focus on one of the coordinate axis; vertex at the *origin* and thereby the directrix is parallel to the other coordinate axis. However, the study of the equations of parabolas with focus at any point and any line as directrix is beyond the scope here.

From the standard equations of the parabolas, Fig10.15, we have the following observations:

1. Parabola is symmetric with respect to the axis of the parabola.If the equation has a y^2 term, then the axis of symmetry is along the x-axis and if the equation has an x^2 term, then the axis of symmetry is along the y-axis.

2. When the axis of symmetry is along the x-axis the parabola opens to the
(a) right if the coefficient of x is positive,
(b) left if the coefficient of x is negative.

3. When the axis of symmetry is along the y-axis the parabola opens
(c) upwards if the coefficient of y is positive.
(d) downwards if the coefficient of y is negative.

10.4.2 *Latus rectum*

Definition 3 Latus rectum of a parabola is a line segment perpendicular to the axis of the parabola, through the focus and whose end points lie on the parabola (Fig10.17).

To find the Length of the latus rectum of the parabola $y^2 = 4ax$ (Fig 10.18).

By the definition of the parabola, AF = AC.

But AC = FM = 2a

Hence AF = 2a.

And since the parabola is symmetric with respect to x-axis AF = FB and so

AB = Length of the latus rectum = 4a.

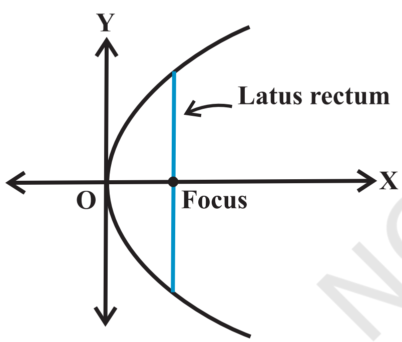

Fig 10.17

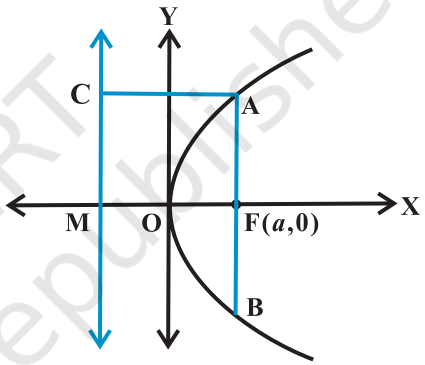

Fig 10.18

Example 5 Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$.

Solution The given equation involves y^2, so the axis of symmetry is along the x-axis.
The coefficient of x is positive so the parabola opens to the right. Comparing with the given equation $y^2 = 4ax$, we find that $a = 2$.

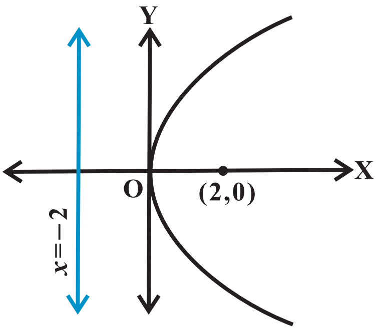

Fig 10.19

Thus, the focus of the parabola is (2, 0) and the equation of the directrix of the parabola is $x = -2$ (Fig 10.19).

Length of the latus rectum is $4a = 4 \times 2 = 8$.

Example 6 Find the equation of the parabola with focus (2,0) and directrix $x = -2$.

Solution Since the focus (2,0) lies on the x-axis, the x-axis itself is the axis of the parabola. Hence the equation of the parabola is of the form either $y^2 = 4ax$ or $y^2 = -4ax$. Since the directrix is $x = -2$ and the focus is (2,0), the parabola is to be of the form $y^2 = 4ax$ with $a = 2$. Hence the required equation is

$$y^2 = 4(2)x = 8x$$

Example 7 Find the equation of the parabola with vertex at (0, 0) and focus at (0, 2).

Solution Since the vertex is at (0,0) and the focus is at (0,2) which lies on y-axis, the y-axis is the axis of the parabola. Therefore, equation of the parabola is of the form $x^2 = 4ay$. thus, we have

$$x^2 = 4(2)y, \text{ i.e., } x^2 = 8y.$$

Example 8 Find the equation of the parabola which is symmetric about the y-axis, and passes through the point (2,–3).

Solution Since the parabola is symmetric about y-axis and has its vertex at the origin, the equation is of the form $x^2 = 4ay$ or $x^2 = -4ay$, where the sign depends on whether the parabola opens upwards or downwards. But the parabola passes through (2,–3) which lies in the fourth quadrant, it must open downwards. Thus the equation is of the form $x^2 = -4ay$.

Since the parabola passes through (2,–3), we have

$$2^2 = -4a(-3), \text{ i.e., } a = \frac{1}{3}$$

Therefore, the equation of the parabola is

$$x^2 = -4\left(\frac{1}{3}\right)y, \text{ i.e., } 3x^2 = -4y.$$

EXERCISE 10.2

In each of the following Exercises 1 to 6, find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum.

1. $y^2 = 12x$ **2.** $x^2 = 6y$ **3.** $y^2 = -8x$

4. $x^2 = -16y$ **5.** $y^2 = 10x$ **6.** $x^2 = -9y$

In each of the Exercises 7 to 12, find the equation of the parabola that satisfies the given conditions:

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7. Focus (6,0); directrix $x = -6$

8. Focus (0,–3); directrix $y = 3$

9. Vertex (0,0); focus (3,0)

10. Vertex (0,0); focus (–2,0)

11. Vertex (0,0) passing through (2,3) and axis is along x-axis.

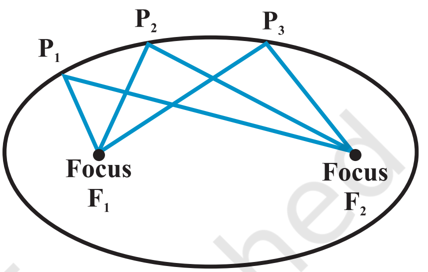
12. Vertex (0,0), passing through (5,2) and symmetric with respect to y-axis.

10.5 Ellipse

Definition 4 An *ellipse* is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant.

The two fixed points are called the *foci* (plural of '*focus*') of the ellipse (Fig10.20).

> **Note** The constant which is the sum of the distances of a point on the ellipse from the two fixed points is always greater than the distance between the two fixed points.



$$P_1F_1 + P_1F_2 = P_2F_1 + P_2F_2 = P_3F_1 + P_3F_2$$

Fig 10.20

The mid point of the line segment joining the foci is called the *centre* of the ellipse. The line segment through the foci of the ellipse is called the *major axis* and the line segment through the centre and perpendicular to the major axis is called the *minor axis*. The end points of the major axis are called the *vertices* of the ellipse(Fig 10.21).

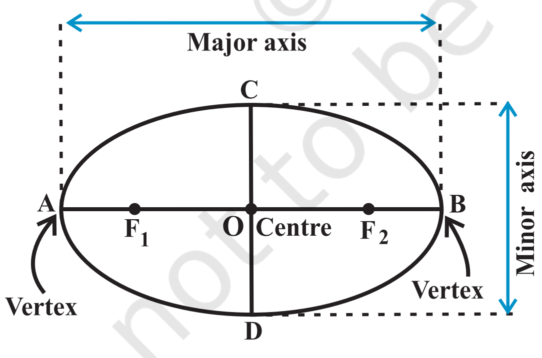


Fig 10.21

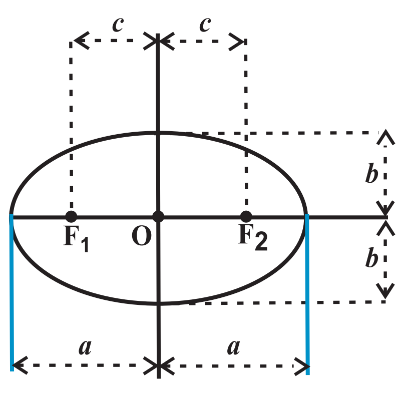


Fig 10.22

We denote the length of the major axis by $2a$, the length of the minor axis by $2b$ and the distance between the foci by $2c$. Thus, the length of the semi major axis is a and semi-minor axis is b (Fig10.22).

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10.5.1 *Relationship between semi-major axis, semi-minor axis and the distance of the focus from the centre of the ellipse (Fig 10.23).*

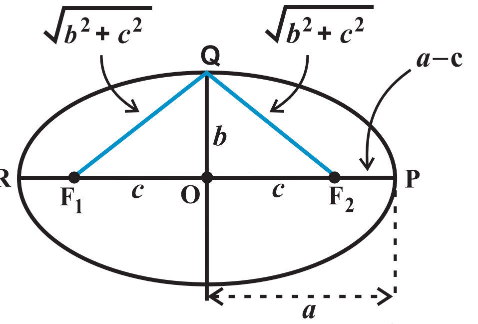

Fig 10.23

Take a point P at one end of the major axis.

Sum of the distances of the point P to the foci is $F_1P + F_2P = F_1O + OP + F_2P$

(Since, $F_1P = F_1O + OP$)

$= c + a + a - c = 2a$

Take a point Q at one end of the minor axis.

Sum of the distances from the point Q to the foci is

$$F_1Q + F_2Q = \sqrt{b^2 + c^2} + \sqrt{b^2 + c^2} = 2\sqrt{b^2 + c^2}$$

Since both P and Q lies on the ellipse.

By the definition of ellipse, we have

$$2\sqrt{b^2 + c^2} = 2a, \text{ i.e.,} \qquad a = \sqrt{b^2 + c^2}$$

or $$a^2 = b^2 + c^2, \text{ i.e., } c = \sqrt{a^2 - b^2}.$$

10.5.2 *Eccentricity*

Definition 5 The eccentricity of an ellipse is the ratio of the distances from the centre of the ellipse to one of the foci and to one of the vertices of the ellipse (eccentricity is denoted by *e*) i.e., $e = \dfrac{c}{a}$.

Then since the focus is at a distance of *c* from the centre, in terms of the eccentricity the focus is at a distance of *ae* from the centre.

10.5.3 *Standard equations of an ellipse* The equation of an ellipse is simplest if the centre of the ellipse is at the origin and the foci are on the *x*-axis or *y*-axis. The two such possible orientations are shown in Fig 10.24.

We will derive the equation for the ellipse shown above in Fig 10.24 (a) with foci on the *x*-axis.

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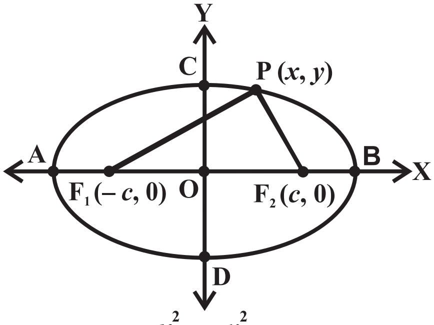
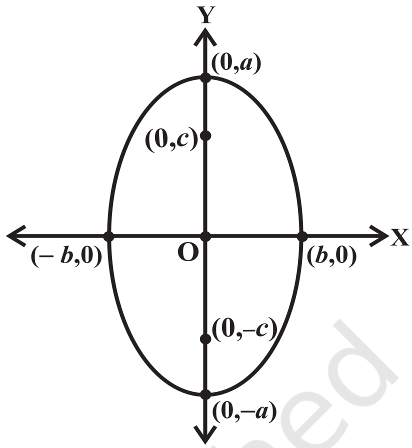


$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

(a)



$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

(b)

Fig 10.24

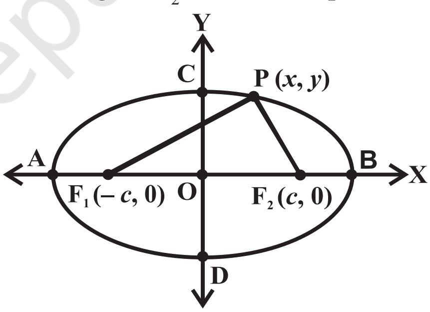
Let F_1 and F_2 be the foci and O be the mid-point of the line segment F_1F_2. Let O be the origin and the line from O through F_2 be the positive x-axis and that through F_1 as the negative x-axis. Let, the line through O perpendicular to the x-axis be the y-axis. Let the coordinates of F_1 be $(-c, 0)$ and F_2 be $(c, 0)$ (Fig 10.25).

Let $P(x, y)$ be any point on the ellipse such that the sum of the distances from P to the two foci be $2a$ so given

$$PF_1 + PF_2 = 2a. \qquad \text{... (1)}$$

Using the distance formula, we have

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Fig 10.25

i.e., $\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$

Squaring both sides, we get

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

which on simplification gives

$$\sqrt{(x-c)^2+y^2} = a - \frac{c}{a}x$$

Squaring again and simplifying, we get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

i.e., $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{(Since } c^2 = a^2 - b^2\text{)}$$

Hence any point on the ellipse satisfies

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \text{... (2)}$$

Conversely, let P (x, y) satisfy the equation (2) with $0 < c < a$. Then

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

Therefore, $$\mathrm{PF}_1 = \sqrt{(x+c)^2 + y^2}$$

$$= \sqrt{(x+c)^2 + b^2\left(\frac{a^2 - x^2}{a^2}\right)}$$

$$= \sqrt{(x+c)^2 + (a^2 - c^2)\left(\frac{a^2 - x^2}{a^2}\right)} \quad \text{(since } b^2 = a^2 - c^2\text{)}$$

$$= \sqrt{\left(a + \frac{cx}{a}\right)^2} = a + \frac{c}{a}x$$

Similarly $$\mathrm{PF}_2 = a - \frac{c}{a}x$$

Hence $$\mathrm{PF}_1 + \mathrm{PF}_2 = a + \frac{c}{a}x + a - \frac{c}{a}x = 2a \qquad \text{... (3)}$$

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So, any point that satisfies $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, satisfies the geometric condition and so P(x, y) lies on the ellipse.

Hence from (2) and (3), we proved that the equation of an ellipse with centre of the origin and major axis along the x-axis is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Discussion From the equation of the ellipse obtained above, it follows that for every point P (x, y) on the ellipse, we have

$$\frac{x^2}{a^2}=1-\frac{y^2}{b^2}\le 1, \text{ i.e., } x^2 \le a^2, \text{ so } -a \le x \le a.$$

Therefore, the ellipse lies between the lines $x = -a$ and $x = a$ and touches these lines.

Similarly, the ellipse lies between the lines $y = -b$ and $y = b$ and touches these lines.

Similarly, we can derive the equation of the ellipse in Fig 10.24 (b) as $\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$.

These two equations are known as *standard equations* of the ellipses.

Note The standard equations of ellipses have centre at the origin and the major and minor axis are coordinate axes. However, the study of the ellipses with centre at any other point, and any line through the centre as major and the minor axes passing through the centre and perpendicular to major axis are beyond the scope here.

From the standard equations of the ellipses (Fig10.24), we have the following observations:

1. Ellipse is symmetric with respect to both the coordinate axes since if (x, y) is a point on the ellipse, then $(-x, y)$, $(x, -y)$ and $(-x, -y)$ are also points on the ellipse.

2. The foci always lie on the major axis. The major axis can be determined by finding the intercepts on the axes of symmetry. That is, major axis is along the x-axis if the coefficient of x^2 has the larger denominator and it is along the y-axis if the coefficient of y^2 has the larger denominator.

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10.5.4 *Latus rectum*

Definition 6 Latus rectum of an ellipse is a line segment perpendicular to the major axis through any of the foci and whose end points lie on the ellipse (Fig 10.26).

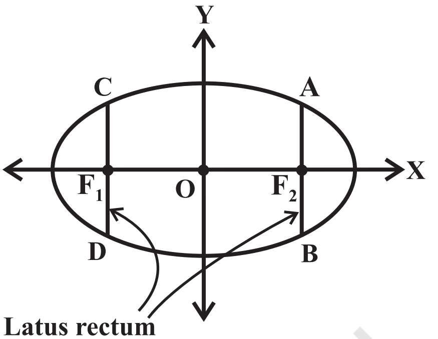

Fig 10. 26

To find the length of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let the length of AF_2 be l.

Then the coordinates of A are (c, l), i.e., (ae, l)

Since A lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have

$$\frac{(ae)^2}{a^2}+\frac{l^2}{b^2}=1$$

$$\Rightarrow l^2 = b^2 (1 - e^2)$$

But $$e^2=\frac{c^2}{a^2}=\frac{a^2-b^2}{a^2}=1-\frac{b^2}{a^2}$$

Therefore $$l^2=\frac{b^4}{a^2}, \text{ i.e., } l=\frac{b^2}{a}$$

Since the ellipse is symmetric with respect to y-axis (of course, it is symmetric w.r.t. both the coordinate axes), $AF_2 = F_2B$ and so length of the latus rectum is $\frac{2b^2}{a}$.

Example 9 Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the latus rectum of the ellipse

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

Solution Since denominator of $\frac{x^2}{25}$ is larger than the denominator of $\frac{y^2}{9}$, the major axis is along the x-axis. Comparing the given equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$a = 5$ and $b = 3$. Also

$$c=\sqrt{a^2-b^2}=\sqrt{25-9}=4$$

Therefore, the coordinates of the foci are (– 4,0) and (4,0), vertices are (– 5, 0) and (5, 0). Length of the major axis is 10 units length of the minor axis $2b$ is 6 units and the eccentricity is $\frac{4}{5}$ and latus rectum is $\frac{2b^2}{a}=\frac{18}{5}$.

Example 10 Find the coordinates of the foci, the vertices, the lengths of major and minor axes and the eccentricity of the ellipse $9x^2 + 4y^2 = 36$.

Solution The given equation of the ellipse can be written in standard form as

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

Since the denominator of $\frac{y^2}{9}$ is larger than the denominator of $\frac{x^2}{4}$, the major axis is along the y-axis. Comparing the given equation with the standard equation

$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, we have $b = 2$ and $a = 3$.

Also $\qquad c=\sqrt{a^2-b^2}=\sqrt{9-4}=\sqrt{5}$

and $\qquad e=\frac{c}{a}=\frac{\sqrt{5}}{3}$

Hence the foci are $(0, \sqrt{5})$ and $(0, -\sqrt{5})$, vertices are (0,3) and (0, –3), length of the major axis is 6 units, the length of the minor axis is 4 units and the eccentricity of the ellipse is $\frac{\sqrt{5}}{3}$.

Example 11 Find the equation of the ellipse whose vertices are (± 13, 0) and foci are (± 5, 0).

Solution Since the vertices are on x-axis, the equation will be of the form

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a is the semi-major axis.

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Given that $a = 13, c = \pm 5$.

Therefore, from the relation $c^2 = a^2 - b^2$, we get

$$25 = 169 - b^2 \text{, i.e., } b = 12$$

Hence the equation of the ellipse is $\frac{x^2}{169} + \frac{y^2}{144} = 1$.

Example 12 Find the equation of the ellipse, whose length of the major axis is 20 and foci are $(0, \pm 5)$.

Solution Since the foci are on y-axis, the major axis is along the y-axis. So, equation of the ellipse is of the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

Given that

$$a = \text{semi-major axis} = \frac{20}{2} = 10$$

and the relation $\quad c^2 = a^2 - b^2$ gives

$$5^2 = 10^2 - b^2 \text{ i.e., } b^2 = 75$$

Therefore, the equation of the ellipse is

$$\frac{x^2}{75} + \frac{y^2}{100} = 1$$

Example 13 Find the equation of the ellipse, with major axis along the x-axis and passing through the points (4, 3) and (– 1,4).

Solution The standard form of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Since the points (4, 3) and (–1, 4) lie on the ellipse, we have

$$\frac{16}{a^2} + \frac{9}{b^2} = 1 \qquad \text{... (1)}$$

and

$$\frac{1}{a^2} + \frac{16}{b^2} = 1 \qquad \text{....(2)}$$

Solving equations (1) and (2), we find that $a^2 = \frac{247}{7}$ and $b^2 = \frac{247}{15}$.

Hence the required equation is

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$$\frac{x^2}{\left(\frac{247}{7}\right)} + \frac{y^2}{\frac{247}{15}} = 1, \text{ i.e., } 7x^2 + 15y^2 = 247.$$

EXERCISE 10.3

In each of the Exercises 1 to 9, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.

1. $\frac{x^2}{36}+\frac{y^2}{16}=1$ **2.** $\frac{x^2}{4}+\frac{y^2}{25}=1$ **3.** $\frac{x^2}{16}+\frac{y^2}{9}=1$

4. $\frac{x^2}{25}+\frac{y^2}{100}=1$ **5.** $\frac{x^2}{49}+\frac{y^2}{36}=1$ **6.** $\frac{x^2}{100}+\frac{y^2}{400}=1$

7. $36x^2 + 4y^2 = 144$ **8.** $16x^2 + y^2 = 16$ **9.** $4x^2 + 9y^2 = 36$

In each of the following Exercises 10 to 20, find the equation for the ellipse that satisfies the given conditions:

10. Vertices (± 5, 0), foci (± 4, 0)

11. Vertices (0, ± 13), foci (0, ± 5)

12. Vertices (± 6, 0), foci (± 4, 0)

13. Ends of major axis (± 3, 0), ends of minor axis (0, ± 2)

14. Ends of major axis $(0, \pm\sqrt{5})$, ends of minor axis (± 1, 0)

15. Length of major axis 26, foci (± 5, 0)

16. Length of minor axis 16, foci (0, ± 6).

17. Foci (± 3, 0), $a = 4$

18. $b = 3$, $c = 4$, centre at the origin; foci on the x axis.

19. Centre at (0,0), major axis on the y-axis and passes through the points (3, 2) and (1,6).

20. Major axis on the x-axis and passes through the points (4,3) and (6,2).

10.6 Hyperbola

Definition 7 A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.

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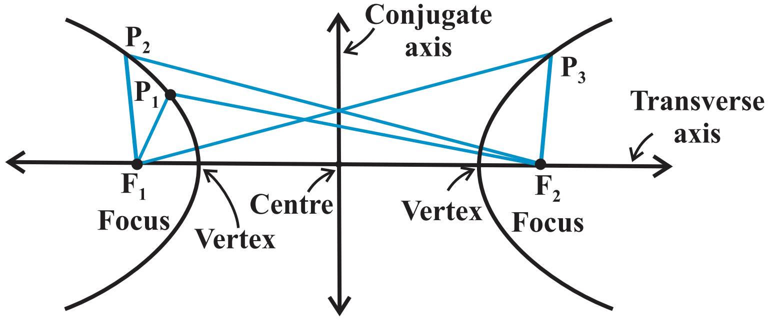

$$P_1F_2 - P_1F_1 = P_2F_2 - P_2F_1 = P_3F_1 - P_3F_2$$

Fig 10.27

The term "*difference*" that is used in the definition means the distance to the farther point minus the distance to the closer point. The two fixed points are called the foci of the hyperbola. The mid-point of the line segment joining the foci is called the *centre of the hyperbola*. The line through the foci is called the *transverse axis* and the line through the centre and perpendicular to the transverse axis is called the *conjugate axis*. The points at which the hyperbola intersects the transverse axis are called the *vertices of the hyperbola* (Fig 10.27).

We denote the distance between the two foci by $2c$, the distance between two vertices (the length of the transverse axis) by $2a$ and we define the quantity b as

$b = \sqrt{c^2 - a^2}$

Also $2b$ is the length of the conjugate axis (Fig 10.28).

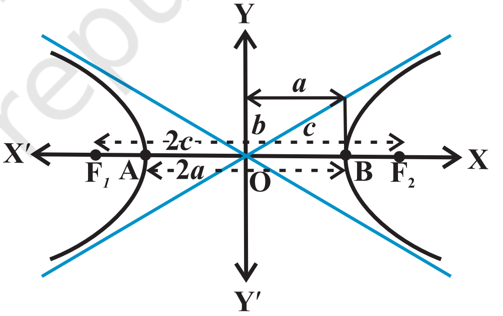

Fig 10.28

To find the constant $P_1F_2 - P_1F_1$:

By taking the point P at A and B in the Fig 10.28, we have

$BF_1 - BF_2 = AF_2 - AF_1$ (by the definition of the hyperbola)

$BA + AF_1 - BF_2 = AB + BF_2 - AF_1$

i.e., $AF_1 = BF_2$

So that, $BF_1 - BF_2 = BA + AF_1 - BF_2 = BA = 2a$

10.6.1 *Eccentricity*

Definition 8 Just like an ellipse, the ratio $e = \dfrac{c}{a}$ is called the *eccentricity of the hyperbola*. Since $c \geq a$, the eccentricity is never less than one. In terms of the eccentricity, the foci are at a distance of ae from the centre.

10.6.2 ***Standard equation of Hyperbola*** The equation of a hyperbola is simplest if the centre of the hyperbola is at the origin and the foci are on the x-axis or y-axis. The two such possible orientations are shown in Fig10.29.

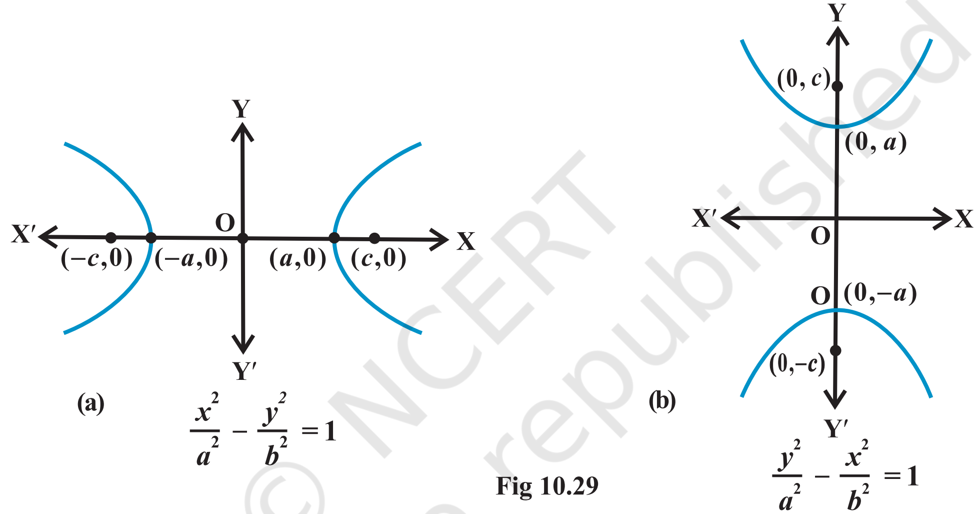

Fig 10.29

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We will derive the equation for the hyperbola shown in Fig 10.29(a) with *foci* on the x-axis.

Let F_1 and F_2 be the foci and O be the mid-point of the line segment F_1F_2. Let O be the origin and the line through O through F_2 be the positive x-axis and that through F_1 as the negative x-axis. The line through O perpendicular to the x-axis be the y-axis. Let the coordinates of F_1 be $(-c,0)$ and F_2 be $(c,0)$ (Fig 10.30).

Let $P(x, y)$ be any point on the hyperbola such that the difference of the distances from P to the farther point minus the closer point be $2a$. So given, $PF_1 - PF_2 = 2a$

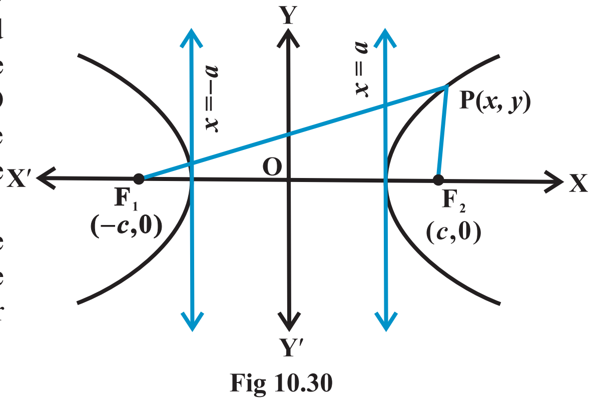

Fig 10.30

Using the distance formula, we have

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=2a$$

i.e.,
$$\sqrt{(x+c)^2+y^2}=2a+\sqrt{(x-c)^2+y^2}$$

Squaring both side, we get

$$(x+c)^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

and on simplifying, we get

$$\frac{cx}{a}-a=\sqrt{(x-c)^2+y^2}$$

On squaring again and further simplifying, we get

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1$$

i.e.,
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{(Since } c^2-a^2=b^2\text{)}$$

Hence any point on the hyperbola satisfies $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Conversely, let P(x, y) satisfy the above equation with $0 < a < c$. Then

$$y^2 = b^2\left(\frac{x^2-a^2}{a^2}\right)$$

Therefore,
$$\text{PF}_1 = +\sqrt{(x+c)^2+y^2}$$

$$= +\sqrt{(x+c)^2+b^2\left(\frac{x^2-a^2}{a^2}\right)} = a+\frac{c}{a}x$$

Similarly,
$$\text{PF}_2 = a-\frac{a}{c}x$$

In hyperbola $c > a$; and since P is to the right of the line $x = a$, $x > a$, $\frac{c}{a}x > a$. Therefore,

$a-\frac{c}{a}x$ becomes negative. Thus, $\text{PF}_2 = \frac{c}{a}x - a$.

Therefore $\qquad \text{PF}_1 - \text{PF}_2 = a + \frac{c}{a}x - \frac{cx}{a} + a = 2a$

Also, note that if P is to the left of the line $x = -a$, then

$$\text{PF}_1 = -\left(a + \frac{c}{a}x\right), \ \text{PF}_2 = a - \frac{c}{a}x.$$

In that case P $\text{F}_2 - \text{PF}_1 = 2a$. So, any point that satisfies $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, lies on the hyperbola.

Thus, we proved that the equation of hyperbola with origin (0,0) and transverse axis along x-axis is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

> **Note** A hyperbola in which $a = b$ is called an *equilateral hyperbola*.

Discussion From the equation of the hyperbola we have obtained, it follows that, we have for every point (x, y) on the hyperbola, $\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1$.

i.e, $\left|\frac{x}{a}\right| \geq 1$, i.e., $x \leq -a$ or $x \geq a$. Therefore, no portion of the curve lies between the lines $x = +a$ and $x = -a$, (i.e. no real intercept on the conjugate axis).

Similarly, we can derive the equation of the hyperbola in Fig 11.31 (b) as $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

These two equations are known as the *standard equations* of *hyperbolas*.

> **Note** The standard equations of hyperbolas have transverse and conjugate axes as the coordinate axes and the centre at the origin. However, there are hyperbolas with any two perpendicular lines as transverse and conjugate axes, but the study of such cases will be dealt in higher classes.

From the standard equations of hyperbolas (Fig10.27), we have the following observations:

1. Hyperbola is symmetric with respect to both the axes, since if (x, y) is a point on the hyperbola, then $(-x, y)$, $(x, -y)$ and $(-x, -y)$ are also points on the hyperbola.

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2. The foci are always on the transverse axis. It is the positive term whose denominator gives the transverse axis. For example, $\frac{x^2}{9}-\frac{y^2}{16}=1$ has transverse axis along x-axis of length 6, while $\frac{y^2}{25}-\frac{x^2}{16}=1$ has transverse axis along y-axis of length 10.

10.6.3 *Latus rectum*

Definition 9 Latus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola.

As in ellipse, it is easy to show that the length of the latus rectum in hyperbola is $\frac{2b^2}{a}$.

Example 14 Find the coordinates of the foci and the vertices, the eccentricity,the length of the latus rectum of the hyperbolas:

(i) $\frac{x^2}{9}-\frac{y^2}{16}=1$, (ii) $y^2 - 16x^2 = 16$

Solution **(i) Comparing the equation** $\frac{x^2}{9}-\frac{y^2}{16}=1$ **with the standard equation**

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Here, $a = 3$, $b = 4$ and $c = \sqrt{a^2+b^2}=\sqrt{9+16}=5$

Therefore, the coordinates of the foci are (± 5, 0) and that of vertices are (± 3, 0).Also,

The eccentricity $e = \frac{c}{a}=\frac{5}{3}$. The latus rectum $=\frac{2b^2}{a}=\frac{32}{3}$

(ii) Dividing the equation by 16 on both sides, we have $\frac{y^2}{16}-\frac{x^2}{1}=1$

Comparing the equation with the standard equation $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, we find that

$a = 4$, $b = 1$ and $c=\sqrt{a^2+b^2}=\sqrt{16+1}=\sqrt{17}$.

Therefore, the coordinates of the foci are $(0, \pm\sqrt{17})$ and that of the vertices are $(0, \pm 4)$. Also,

The eccentricity $e = \frac{c}{a} = \frac{\sqrt{17}}{4}$. The latus rectum $= \frac{2b^2}{a} = \frac{1}{2}$.

Example 15 Find the equation of the hyperbola with foci $(0, \pm 3)$ and vertices $(0, \pm \frac{\sqrt{11}}{2})$.

Solution Since the foci is on y-axis, the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Since vertices are $(0, \pm \frac{\sqrt{11}}{2})$, $a = \frac{\sqrt{11}}{2}$

Also, since foci are $(0, \pm 3)$; $c = 3$ and $b^2 = c^2 - a^2 = \frac{25}{4}$.

Therefore, the equation of the hyperbola is

$$\frac{y^2}{\left(\frac{11}{4}\right)} - \frac{x^2}{\left(\frac{25}{4}\right)} = 1, \text{ i.e., } 100\,y^2 - 44\,x^2 = 275.$$

Example 16 Find the equation of the hyperbola where foci are $(0, \pm 12)$ and the length of the latus rectum is 36.

Solution Since foci are $(0, \pm 12)$, it follows that $c = 12$.

Length of the latus rectum $= \frac{2b^2}{a} = 36$ or $b^2 = 18a$

Therefore $c^2 = a^2 + b^2$; gives

$144 = a^2 + 18a$

i.e., $a^2 + 18a - 144 = 0$,

So $a = -24, 6$.

Since a cannot be negative, we take $a = 6$ and so $b^2 = 108$.

Therefore, the equation of the required hyperbola is $\frac{y^2}{36} - \frac{x^2}{108} = 1$, i.e., $3y^2 - x^2 = 108$

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EXERCISE 10.4

In each of the Exercises 1 to 6, find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbolas.

1. $\frac{x^2}{16}-\frac{y^2}{9}=1$ **2.** $\frac{y^2}{9}-\frac{x^2}{27}=1$ **3.** $9y^2 - 4x^2 = 36$

4. $16x^2 - 9y^2 = 576$ **5.** $5y^2 - 9x^2 = 36$ **6.** $49y^2 - 16x^2 = 784$.

In each of the Exercises 7 to 15, find the equations of the hyperbola satisfying the given conditions.

7. Vertices $(\pm 2, 0)$, foci $(\pm 3, 0)$
8. Vertices $(0, \pm 5)$, foci $(0, \pm 8)$
9. Vertices $(0, \pm 3)$, foci $(0, \pm 5)$
10. Foci $(\pm 5, 0)$, the transverse axis is of length 8.
11. Foci $(0, \pm 13)$, the conjugate axis is of length 24.
12. Foci $(\pm 3\sqrt{5}, 0)$, the latus rectum is of length 8.
13. Foci $(\pm 4, 0)$, the latus rectum is of length 12
14. vertices $(\pm 7,0)$, $e = \frac{4}{3}$.
15. Foci $(0, \pm \sqrt{10})$, passing through (2,3)

Miscellaneous Examples

Example 17 The focus of a parabolic mirror as shown in Fig 10.31 is at a distance of 5 cm from its vertex. If the mirror is 45 cm deep, find the distance AB (Fig 10.31).

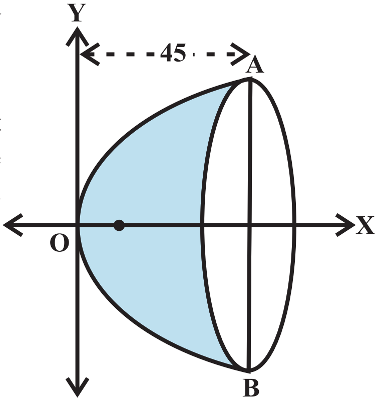

Fig 10.31

Solution Since the distance from the focus to the vertex is 5 cm. We have, $a = 5$. If the origin is taken at the vertex and the axis of the mirror lies along the positive x-axis, the equation of the parabolic section is

$$y^2 = 4\,(5)\,x = 20\,x$$

Note that $x = 45$. Thus

$$y^2 = 900$$

Therefore $y = \pm 30$

Hence AB = $2y = 2 \times 30 = 60$ cm.

Example 18 A beam is supported at its ends by supports which are 12 metres apart. Since the load is concentrated at its centre, there

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is a deflection of 3 cm at the centre and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm?

Solution Let the vertex be at the lowest point and the axis vertical. Let the coordinate axis be chosen as shown in Fig 10.32.

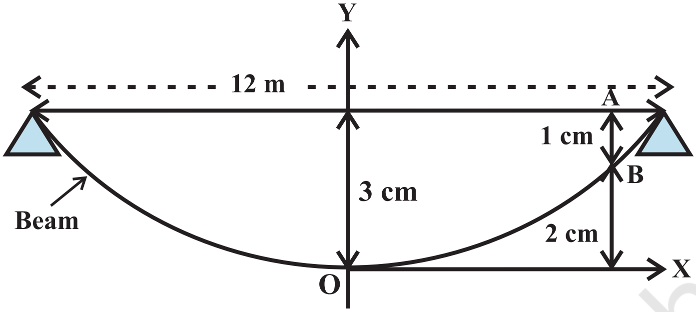

Fig 10.32

The equation of the parabola takes the form $x^2 = 4ay$. Since it passes through

$\left(6, \frac{3}{100}\right)$, we have $(6)^2 = 4a\left(\frac{3}{100}\right)$, i.e., $a = \frac{36 \times 100}{12} = 300$ m

Let AB be the deflection of the beam which is $\frac{1}{100}$ m. Coordinates of B are $(x, \frac{2}{100})$.

Therefore $\qquad x^2 = 4 \times 300 \times \frac{2}{100} = 24$

i.e. $\qquad x = \sqrt{24} = 2\sqrt{6}$ metres

Example 19 A rod AB of length 15 cm rests in between two coordinate axes in such a way that the end point A lies on x-axis and end point B lies on y-axis. A point P(x, y) is taken on the rod in such a way that AP = 6 cm. Show that the locus of P is an ellipse.

Solution Let AB be the rod making an angle θ with OX as shown in Fig 10.33 and P (x, y) the point on it such that AP = 6 cm.

Since $\qquad$ AB = 15 cm, we have

$\qquad$ PB = 9 cm.

From P draw PQ and PR perpendiculars on y-axis and x-axis, respectively.

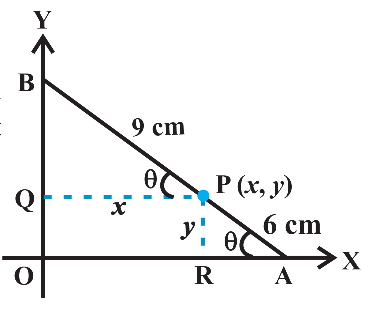

Fig 10.33

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From $\qquad \Delta$ PBQ, $\cos\theta = \dfrac{x}{9}$

From $\qquad \Delta$ PRA, $\sin\theta = \dfrac{y}{6}$

Since $\quad \cos^2\theta + \sin^2\theta = 1$

$$\left(\frac{x}{9}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

or
$$\frac{x^2}{81} + \frac{y^2}{36} = 1$$

Thus the locus of P is an ellipse.

Miscellaneous Exercise on Chapter 10

1. If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus.

2. An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola?

3. The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.

4. An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.

5. A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the x-axis.

6. Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.

7. A man running a racecourse notes that the sum of the distances from the two flag posts from him is always 10 m and the distance between the flag posts is 8 m. Find the equation of the posts traced by the man.

8. An equilateral triangle is inscribed in the parabola $y^2 = 4\,ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

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Summary

In this Chapter the following concepts and generalisations are studied.

- A circle is the set of all points in a plane that are equidistant from a fixed point in the plane.
- The equation of a circle with centre (h, k) and the radius r is

$$(x - h)^2 + (y - k)^2 = r^2.$$

- A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point in the plane.
- The equation of the parabola with focus at $(a, 0)$ $a > 0$ and directrix $x = - a$ is

$$y^2 = 4ax.$$

- Latus rectum of a parabola is a line segment perpendicular to the axis of the parabola, through the focus and whose end points lie on the parabola.
- Length of the latus rectum of the parabola $y^2 = 4ax$ is $4a$.
- An *ellipse* is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant.
- The equation of an ellipse with foci on the x-axis is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.
- Latus rectum of an ellipse is a line segment perpendicular to the major axis through any of the foci and whose end points lie on the ellipse.
- Length of the latus rectum of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $\dfrac{2b^2}{a}$.
- The eccentricity of an ellipse is the ratio between the distances from the centre of the ellipse to one of the foci and to one of the vertices of the ellipse.
- A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.
- The equation of a hyperbola with foci on the x-axis is : $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

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- Latus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola.

- Length of the latus rectum of the hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is : $\frac{2b^2}{a}$.

- The eccentricity of a hyperbola is the ratio of the distances from the centre of the hyperbola to one of the foci and to one of the vertices of the hyperbola.

Historical Note

Geometry is one of the most ancient branches of mathematics. The Greek geometers investigated the properties of many curves that have theoretical and practical importance. Euclid wrote his treatise on geometry around 300 B.C. He was the first who organised the geometric figures based on certain axioms suggested by physical considerations. Geometry as initially studied by the ancient Indians and Greeks, who made essentially no use of the process of algebra. The synthetic approach to the subject of geometry as given by Euclid and in *Sulbasutras*, etc., was continued for some 1300 years. In the 200 B.C., Apollonius wrote a book called '*The Conic*' which was all about conic sections with many important discoveries that have remained unsurpassed for eighteen centuries.

Modern analytic geometry is called '*Cartesian*' after the name of Rene Descartes (1596-1650) whose relevant 'La Geometrie' was published in 1637. But the fundamental principle and method of analytical geometry were already discovered by Pierre de Fermat (1601-1665). Unfortunately, Fermats treatise on the subject, entitled *Ad Locus Planos et So LIDOS Isagoge* (Introduction to Plane and Solid Loci) was published only posthumously in 1679. So, Descartes came to be regarded as the unique inventor of the analytical geometry.

Isaac Barrow avoided using cartesian method. Newton used method of undetermined coefficients to find equations of curves. He used several types of coordinates including polar and bipolar. Leibnitz used the terms '*abscissa*', 'ordinate' and 'coordinate'. L'Hospital (about 1700) wrote an important textbook on analytical geometry.

Clairaut (1729) was the first to give the distance formula although in clumsy form. He also gave the intercept form of the linear equation. Cramer (1750) made formal use of the two axes and gave the equation of a circle as

$$(y - a)^2 + (b - x)^2 = r$$

He gave the best exposition of the analytical geometry of his time. Monge (1781) gave the modern 'point-slope' form of equation of a line as

$$y - y' = a\,(x - x')$$

and the condition of perpendicularity of two lines as $aa' + 1 = 0$.

S.F. Lacroix (1765–1843) was a prolific textbook writer, but his contributions to analytical geometry are found scattered. He gave the 'two-point' form of equation of a line as

$$y - \beta = \frac{\beta' - \beta}{\alpha' - \alpha}\,(x - \alpha)$$

and the length of the perpendicular from (α, β) on $y = ax + b$ as $\dfrac{(\beta - a - b)}{\sqrt{1 + a^2}}$.

His formula for finding angle between two lines was $\tan\theta = \left(\dfrac{a' - a}{1 + aa'}\right)$. It is, of course, surprising that one has to wait for more than 150 years after the invention of analytical geometry before finding such essential basic formula. In 1818, C. Lame, a civil engineer, gave $mE + m'E' = 0$ as the curve passing through the points of intersection of two loci $E = 0$ and $E' = 0$.

Many important discoveries, both in Mathematics and Science, have been linked to the conic sections. The Greeks particularly Archimedes (287–212 B.C.) and Apollonius (200 B.C.) studied conic sections for their own beauty. These curves are important tools for present day exploration of outer space and also for research into behaviour of atomic particles.

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Chapter 11

INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

❖ *Mathematics is both the queen and the hand-maiden of all sciences – E.T. BELL* ❖

11.1 Introduction

You may recall that to locate the position of a point in a plane, we need two intersecting mutually perpendicular lines in the plane. These lines are called the *coordinate axes* and the two numbers are called the *coordinates of the point with respect to the axes*. In actual life, we do not have to deal with points lying in a plane only. For example, consider the position of a ball thrown in space at different points of time or the position of an aeroplane as it flies from one place to another at different times during its flight.

Leonhard Euler (1707-1783)

Similarly, if we were to locate the position of the lowest tip of an electric bulb hanging from the ceiling of a room or the position of the central tip of the ceiling fan in a room, we will not only require the perpendicular distances of the point to be located from two perpendicular walls of the room but also the height of the point from the floor of the room. Therefore, we need not only two but three numbers representing the perpendicular distances of the point from three mutually perpendicular planes, namely the floor of the room and two adjacent walls of the room. The three numbers representing the three distances are called the *coordinates of the point with reference to the three coordinate planes*. So, a point in space has three coordinates. In this Chapter, we shall study the basic concepts of geometry in three dimensional space.*

* For various activities in three dimensional geometry one may refer to the Book, "***A Hand Book for designing Mathematics Laboratory in Schools***", NCERT, 2005.

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11.2 Coordinate Axes and Coordinate Planes in Three Dimensional Space

Consider three planes intersecting at a point O such that these three planes are mutually perpendicular to each other (Fig 11.1). These three planes intersect along the lines X′OX, Y′OY and Z′OZ, called the *x*, *y* and *z-axes*, respectively. We may note that these lines are mutually perpendicular to each other. These lines constitute the *rectangular coordinate system*. The planes XOY, YOZ and ZOX, called, respectively the XY-*plane*, YZ-*plane* and the ZX-*plane*, are known as the three coordinate planes. We take the XOY plane as the plane of the paper and the line Z′OZ as perpendicular to the plane XOY. If the plane of the paper is considered as horizontal, then the line Z′OZ will be vertical. The distances measured from XY-plane upwards in the direction of OZ are taken as positive and those measured downwards in the direction of OZ′ are taken as negative. Similarly, the distance measured to the right of ZX-plane along OY are taken as positive, to the left of ZX-plane and along OY′ as negative, in front of the YZ-plane along OX as positive and to the back of it along OX′ as negative. The point O is called the *origin* of the coordinate system. The three coordinate planes divide the space into eight parts known as *octants*. These octants could be named as XOYZ, X′OYZ, X′OY′Z, XOY′Z, XOYZ′, X′OYZ′, X′OY′Z′ and XOY′Z′. and denoted by I, II, III, ..., VIII , respectively.

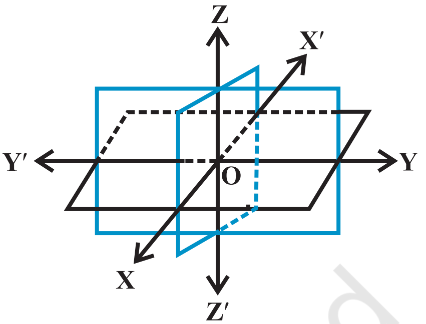

Fig 11.1

11.3 Coordinates of a Point in Space

Having chosen a fixed coordinate system in the space, consisting of coordinate axes, coordinate planes and the origin, we now explain, as to how, given a point in the space, we associate with it three coordinates (*x,y,z*) and conversely, given a triplet of three numbers (*x, y, z*), how, we locate a point in the space.

Given a point P in space, we drop a perpendicular PM on the XY-plane with M as the foot of this perpendicular (Fig 11.2). Then, from the point M, we draw a perpendicular ML to the *x*-axis, meeting it at L. Let OL be *x*, LM be *y* and MP be *z*. Then *x,y* and *z* are called the *x*, *y* and *z coordinates*, respectively, of the point P in the space. In Fig 11.2, we may note that the point P (*x*, *y*, *z*) lies in the octant XOYZ and so all *x*, *y*, *z* are positive. If P was in any other octant, the signs of *x*, *y* and *z* would change

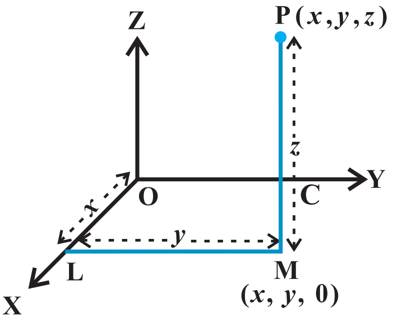

Fig 11.2

accordingly. Thus, to each point P in the space there corresponds an ordered triplet (x, y, z) of real numbers.

Conversely, given any triplet (x, y, z), we would first fix the point L on the x-axis corresponding to x, then locate the point M in the XY-plane such that (x, y) are the coordinates of the point M in the XY-plane. Note that LM is perpendicular to the x-axis or is parallel to the y-axis. Having reached the point M, we draw a perpendicular MP to the XY-plane and locate on it the point P corresponding to z. The point P so obtained has then the coordinates (x, y, z). Thus, there is a one to one correspondence between the points in space and ordered triplet (x, y, z) of real numbers.

Alternatively, through the point P in the space, we draw three planes parallel to the coordinate planes, meeting the x-axis, y-axis and z-axis in the points A, B and C, respectively (Fig 11.3). Let OA = x, OB = y and OC = z. Then, the point P will have the coordinates x, y and z and we write P (x, y, z). Conversely, given x, y and z, we locate the three points A, B and C on the three coordinate axes. Through the points A, B and C we draw planes parallel to the YZ-plane, ZX-plane and XY-plane, respectively. The point of interesection of these three planes, namely, ADPF, BDPE and CEPF is obviously the point P, corresponding to the ordered triplet (x, y, z). We observe that if P (x, y, z) is any point in the space, then x, y and z are perpendicular distances from YZ, ZX and XY planes, respectively.

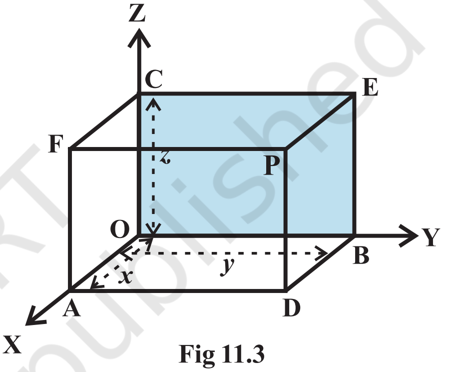

Fig 11.3

Note The coordinates of the origin O are (0,0,0). The coordinates of any point on the x-axis will be as $(x,0,0)$ and the coordinates of any point in the YZ-plane will be as $(0, y, z)$.

Remark The sign of the coordinates of a point determine the octant in which the point lies. The following table shows the signs of the coordinates in eight octants.

Table 11.1

Octants / Coordinates	I	II	III	IV	V	VI	VII	VIII
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

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Example 1 In Fig 11.3, if P is (2,4,5), find the coordinates of F.

Solution For the point F, the distance measured along OY is zero. Therefore, the coordinates of F are (2,0,5).

Example 2 Find the octant in which the points (–3,1,2) and (–3,1,– 2) lie.

Solution From the Table 11.1, the point (–3,1, 2) lies in second octant and the point (–3, 1, – 2) lies in octant VI.

EXERCISE 11.1

1. A point is on the x-axis. What are its y-coordinate and z-coordinates?
2. A point is in the XZ-plane. What can you say about its y-coordinate?
3. Name the octants in which the following points lie:
   (1, 2, 3), (4, –2, 3), (4, –2, –5), (4, 2, –5), (– 4, 2, –5), (– 4, 2, 5), (–3, –1, 6) (– 2, – 4, –7).
4. Fill in the blanks:
   (i) The x-axis and y-axis taken together determine a plane known as_______.
   (ii) The coordinates of points in the XY-plane are of the form _______.
   (iii) Coordinate planes divide the space into ______ octants.

11.4 Distance between Two Points

We have studied about the distance between two points in two-dimensional coordinate system. Let us now extend this study to three-dimensional system.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points referred to a system of rectangular axes OX, OY and OZ. Through the points P and Q draw planes parallel to the coordinate planes so as to form a rectangular parallelopiped with one diagonal PQ (Fig 11.4).

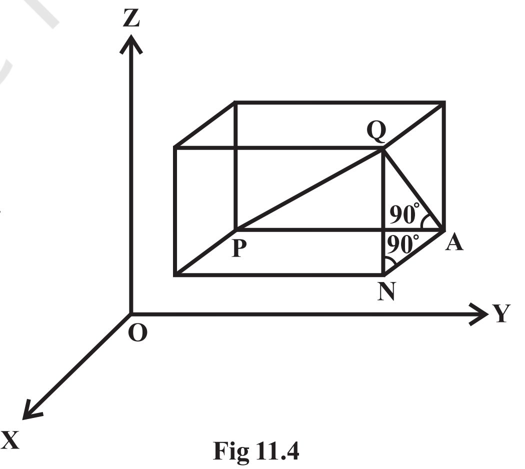

Fig 11.4

Now, since $\angle PAQ$ is a right angle, it follows that, in triangle PAQ,

$$PQ^2 = PA^2 + AQ^2 \quad \ldots (1)$$

Also, triangle ANQ is right angle triangle with $\angle ANQ$ a right angle.

Therefore $AQ^2 = AN^2 + NQ^2$ ... (2)

From (1) and (2), we have

$$PQ^2 = PA^2 + AN^2 + NQ^2$$

Now $PA = y_2 - y_1$, $AN = x_2 - x_1$ and $NQ = z_2 - z_1$

Hence $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

Therefore $PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

This gives us the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

In particular, if $x_1 = y_1 = z_1 = 0$, i.e., point P is origin O, then $OQ = \sqrt{x_2^2+y_2^2+z_2^2}$, which gives the distance between the origin O and any point Q (x_2, y_2, z_2).

Example 3 Find the distance between the points P(1, –3, 4) and Q (– 4, 1, 2).

Solution The distance PQ between the points P (1,–3, 4) and Q (– 4, 1, 2) is

$$PQ = \sqrt{(-4-1)^2+(1+3)^2+(2-4)^2}$$

$$= \sqrt{25+16+4}$$

$$= \sqrt{45} = 3\sqrt{5} \text{ units}$$

Example 4 Show that the points P (–2, 3, 5), Q (1, 2, 3) and R (7, 0, –1) are collinear.

Solution We know that points are said to be collinear if they lie on a line.

Now, $PQ = \sqrt{(1+2)^2+(2-3)^2+(3-5)^2} = \sqrt{9+1+4} = \sqrt{14}$

$QR = \sqrt{(7-1)^2+(0-2)^2+(-1-3)^2} = \sqrt{36+4+16} = \sqrt{56} = 2\sqrt{14}$

and $PR = \sqrt{(7+2)^2+(0-3)^2+(-1-5)^2} = \sqrt{81+9+36} = \sqrt{126} = 3\sqrt{14}$

Thus, PQ + QR = PR. Hence, P, Q and R are collinear.

Example 5 Are the points A (3, 6, 9), B (10, 20, 30) and C (25, – 41, 5), the vertices of a right angled triangle?

Solution By the distance formula, we have

$$AB^2 = (10-3)^2 + (20-6)^2 + (30-9)^2$$

$$= 49 + 196 + 441 = 686$$

$$BC^2 = (25-10)^2 + (-41-20)^2 + (5-30)^2$$

$$= 225 + 3721 + 625 = 4571$$

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$$\begin{aligned} CA^2 &= (3-25)^2 + (6+41)^2 + (9-5)^2 \\ &= 484 + 2209 + 16 = 2709 \end{aligned}$$

We find that $CA^2 + AB^2 \neq BC^2$.

Hence, the triangle ABC is not a right angled triangle.

Example 6 Find the equation of set of points P such that $PA^2 + PB^2 = 2k^2$, where A and B are the points (3, 4, 5) and (–1, 3, –7), respectively.

Solution Let the coordinates of point P be (x, y, z).

Here $PA^2 = (x-3)^2 + (y-4)^2 + (z-5)^2$

$PB^2 = (x+1)^2 + (y-3)^2 + (z+7)^2$

By the given condition $PA^2 + PB^2 = 2k^2$, we have

$$(x-3)^2 + (y-4)^2 + (z-5)^2 + (x+1)^2 + (y-3)^2 + (z+7)^2 = 2k^2$$

i.e., $2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z = 2k^2 - 109$.

EXERCISE 11.2

1. Find the distance between the following pairs of points:
   (i) (2, 3, 5) and (4, 3, 1) (ii) (–3, 7, 2) and (2, 4, –1)
   (iii) (–1, 3, – 4) and (1, –3, 4) (iv) (2, –1, 3) and (–2, 1, 3).
2. Show that the points (–2, 3, 5), (1, 2, 3) and (7, 0, –1) are collinear.
3. Verify the following:
   (i) (0, 7, –10), (1, 6, – 6) and (4, 9, – 6) are the vertices of an isosceles triangle.
   (ii) (0, 7, 10), (–1, 6, 6) and (– 4, 9, 6) are the vertices of a right angled triangle.
   (iii) (–1, 2, 1), (1, –2, 5), (4, –7, 8) and (2, –3, 4) are the vertices of a parallelogram.
4. Find the equation of the set of points which are equidistant from the points (1, 2, 3) and (3, 2, –1).
5. Find the equation of the set of points P, the sum of whose distances from A (4, 0, 0) and B (– 4, 0, 0) is equal to 10.

Miscellaneous Examples

Example 7 Show that the points A (1, 2, 3), B (–1, –2, –1), C (2, 3, 2) and D (4, 7, 6) are the vertices of a parallelogram ABCD, but it is not a rectangle.

Solution To show ABCD is a parallelogram we need to show opposite side are equal Note that.

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$$AB = \sqrt{(-1-1)^2+(-2-2)^2+(-1-3)^2} = \sqrt{4+16+16} = 6$$

$$BC = \sqrt{(2+1)^2+(3+2)^2+(2+1)^2} = \sqrt{9+25+9} = \sqrt{43}$$

$$CD = \sqrt{(4-2)^2+(7-3)^2+(6-2)^2} = \sqrt{4+16+16} = 6$$

$$DA = \sqrt{(1-4)^2+(2-7)^2+(3-6)^2} = \sqrt{9+25+9}=\sqrt{43}$$

Since AB = CD and BC = AD, ABCD is a parallelogram.

Now, it is required to prove that ABCD is not a rectangle. For this, we show that diagonals AC and BD are unequal. We have

$$AC = \sqrt{(2-1)^2+(3-2)^2+(2-3)^2} = \sqrt{1+1+1}=\sqrt{3}$$

$$BD = \sqrt{(4+1)^2+(7+2)^2+(6+1)^2} = \sqrt{25+81+49}=\sqrt{155}.$$

Since AC ≠ BD, ABCD is not a rectangle.

Note We can also show that ABCD is a parallelogram, using the property that diagonals AC and BD bisect each other.

Example 8 Find the equation of the set of the points P such that its distances from the points A (3, 4, –5) and B (– 2, 1, 4) are equal.

Solution If P (x, y, z) be any point such that PA = PB.

Now $$\sqrt{(x-3)^2+(y-4)^2+(z+5)^2} = \sqrt{(x+2)^2+(y-1)^2+(z-4)^2}$$

or $$(x-3)^2+(y-4)^2+(z+5)^2 = (x+2)^2+(y-1)^2+(z-4)^2$$

or $$10x + 6y - 18z - 29 = 0.$$

Example 9 The centroid of a triangle ABC is at the point (1, 1, 1). If the coordinates of A and B are (3, –5, 7) and (–1, 7, – 6), respectively, find the coordinates of the point C.

Solution Let the coordinates of C be (x, y, z) and the coordinates of the centroid G be (1, 1, 1). Then

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$$\frac{x+3-1}{3}=1, \text{ i.e., } x=1; \quad \frac{y-5+7}{3}=1, \text{ i.e., } y=1; \quad \frac{z+7-6}{3}=1, \text{ i.e., } z=2.$$

Hence, coordinates of C are (1, 1, 2).

Miscellaneous Exercise on Chapter 11

1. Three vertices of a parallelogram ABCD are A(3, – 1, 2), B (1, 2, – 4) and C (– 1, 1, 2). Find the coordinates of the fourth vertex.

2. Find the lengths of the medians of the triangle with vertices A (0, 0, 6), B (0,4, 0) and (6, 0, 0).

3. If the origin is the centroid of the triangle PQR with vertices P ($2a$, 2, 6), Q (– 4, $3b$, –10) and R(8, 14, $2c$), then find the values of a, b and c.

4. If A and B be the points (3, 4, 5) and (–1, 3, –7), respectively, find the equation of the set of points P such that $PA^2 + PB^2 = k^2$, where k is a constant.

Summary

- In three dimensions, the coordinate axes of a rectangular Cartesian coordinate system are three mutually perpendicular lines. The axes are called the x, y and z-axes.
- The three planes determined by the pair of axes are the coordinate planes, called XY, YZ and ZX-*planes*.
- The three coordinate planes divide the space into eight parts known as *octants*.
- The coordinates of a point P in three dimensional geometry is always written in the form of triplet like (x, y, z). Here x, y and z are the distances from the YZ, ZX and XY-planes.
- (i) Any point on x-axis is of the form (x, 0, 0)

  (ii) Any point on y-axis is of the form (0, y, 0)

  (iii) Any point on z-axis is of the form (0, 0, z).
- Distance between two points P(x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by

$$PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

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Historical Note

Rene' Descartes (1596–1650), the father of analytical geometry, essentially dealt with plane geometry only in 1637. The same is true of his co-inventor Pierre Fermat (1601-1665) and La Hire (1640-1718). Although suggestions for the three dimensional coordinate geometry can be found in their works but no details. Descartes had the idea of coordinates in three dimensions but did not develop it.
J.Bernoulli (1667-1748) in a letter of 1715 to Leibnitz introduced the three coordinate planes which we use today. It was Antoinne Parent (1666-1716), who gave a systematic development of analytical solid geometry for the first time in a paper presented to the French Academy in 1700.
L.Euler (1707-1783) took up systematically the three dimensional coordinate geometry, in Chapter 5 of the appendix to the second volume of his "Introduction to Geometry" in 1748.
It was not until the middle of the nineteenth century that geometry was extended to more than three dimensions, the well-known application of which is in the Space-Time Continuum of Einstein's Theory of Relativity.

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Chapter 12

LIMITS AND DERIVATIVES

❖ *With the Calculus as a key, Mathematics can be successfully applied to the explanation of the course of Nature – WHITEHEAD* ❖

12.1 Introduction

This chapter is an introduction to Calculus. Calculus is that branch of mathematics which mainly deals with the study of change in the value of a function as the points in the domain change. First, we give an intuitive idea of derivative (without actually defining it). Then we give a naive definition of limit and study some algebra of limits. Then we come back to a definition of derivative and study some algebra of derivatives. We also obtain derivatives of certain standard functions.

Sir Issac Newton (1642-1727)

12.2 Intuitive Idea of Derivatives

Physical experiments have confirmed that the body dropped from a tall cliff covers a distance of $4.9t^2$ metres in t seconds, i.e., distance s in metres covered by the body as a function of time t in seconds is given by $s = 4.9t^2$.

The adjoining Table 13.1 gives the distance travelled in metres at various intervals of time in seconds of a body dropped from a tall cliff.

The objective is to find the veloctiy of the body at time $t = 2$ seconds from this data. One way to approach this problem is to find the average velocity for various intervals of time ending at $t = 2$ seconds and hope that these throw some light on the velocity at $t = 2$ seconds.

Average velocity between $t = t_1$ and $t = t_2$ equals distance travelled between $t = t_1$ and $t = t_2$ seconds divided by $(t_2 - t_1)$. Hence the average velocity in the first two seconds

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$$= \frac{\text{Distance travelled between } t_2 = 2 \textit{ and } t_1 = 0}{\text{Time interval } (t_2 - t_1)}$$

$$= \frac{(19.6 - 0)m}{(2 - 0)s} = 9.8 m/s .$$

Similarly, the average velocity between $t = 1$ and $t = 2$ is

$$\frac{(19.6 - 4.9)m}{(2 - 1)s} = 14.7 \; m/s$$

Likewise we compute the average velocitiy between $t = t_1$ and $t = 2$ for various t_1. The following Table 13.2 gives the average velocity (v), $t = t_1$ seconds and $t = 2$ seconds.

Table 12.1

t	s
0	0
1	4.9
1.5	11.025
1.8	15.876
1.9	17.689
1.95	18.63225
2	19.6
2.05	20.59225
2.1	21.609
2.2	23.716
2.5	30.625
3	44.1
4	78.4

Table 12.2

t_1	0	1	1.5	1.8	1.9	1.95	1.99
v	9.8	14.7	17.15	18.62	19.11	19.355	19.551

From Table 12.2, we observe that the average velocity is gradually increasing. As we make the time intervals ending at $t = 2$ smaller, we see that we get a better idea of the velocity at $t = 2$. Hoping that nothing really dramatic happens between 1.99 seconds and 2 seconds, we conclude that the average velocity at $t = 2$ seconds is just above $19.551 m/s$.

This conclusion is somewhat strengthened by the following set of computation. Compute the average velocities for various time intervals starting at $t = 2$ seconds. As before the average velocity v between $t = 2$ seconds and $t = t_2$ seconds is

$$= \frac{\text{Distance travelled between 2 seconds and } t_2 \text{ seconds}}{t_2 - 2}$$

$$= \frac{\text{Distance travelled in } t_2 \text{ seconds } - \text{ Distance travelled in 2 seconds}}{t_2 - 2}$$

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$$= \frac{\text{Distance travelled in } t_2 \text{ seconds } - 19.6}{t_2 - 2}$$

The following Table 12.3 gives the average velocity v in metres per second between $t = 2$ seconds and t_2 seconds.

Table 12.3

t_2	4	3	2.5	2.2	2.1	2.05	2.01
v	29.4	24.5	22.05	20.58	20.09	19.845	19.649

Here again we note that if we take smaller time intervals starting at $t = 2$, we get better idea of the velocity at $t = 2$.

In the first set of computations, what we have done is to find average velocities in increasing time intervals ending at $t = 2$ and then hope that nothing dramatic happens just before $t = 2$. In the second set of computations, we have found the average velocities decreasing in time intervals ending at $t = 2$ and then hope that nothing dramatic happens just after $t = 2$. Purely on the physical grounds, both these sequences of average velocities must approach a common limit. We can safely conclude that the velocity of the body at $t = 2$ is between 19.551*m/s* and 19.649 *m/s*. Technically, we say that the instantaneous velocity at $t = 2$ is between 19.551 *m/s* and 19.649 *m/s*. As is well-known, *velocity is the rate of change of displacement*. Hence what we have accomplished is the following. From the given data of distance covered at various time instants we have estimated the rate of change of the distance at a given instant of time. We say that the *derivative* of the distance function $s = 4.9t^2$ at $t = 2$ is between 19.551 and 19.649.

An alternate way of viewing this limiting process is shown in Fig 12.1. This is a plot of distance s of the body from the top of the cliff versus the time t elapsed. In the limit as the sequence of time intervals $h_1, h_2, ...,$ approaches zero, the sequence of average velocities approaches the same limit as does the sequence of ratios

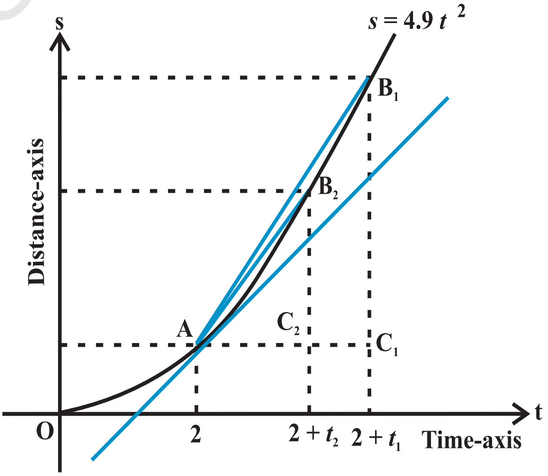

Fig 12.1

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$$\frac{C_1B_1}{AC_1}, \frac{C_2B_2}{AC_2}, \frac{C_3B_3}{AC_3}, \ldots$$

where $C_1B_1 = s_1 - s_0$ is the distance travelled by the body in the time interval $h_1 = AC_1$, etc. From the Fig 12.1 it is safe to conclude that this latter sequence approaches the slope of the tangent to the curve at point A. In other words, the instantaneous velocity $v(t)$ of a body at time $t = 2$ is equal to the slope of the tangent of the curve $s = 4.9t^2$ at $t = 2$.

12.3 Limits

The above discussion clearly points towards the fact that we need to understand limiting process in greater clarity. We study a few illustrative examples to gain some familiarity with the concept of limits.

Consider the function $f(x) = x^2$. Observe that as x takes values very close to 0, the value of $f(x)$ also moves towards 0 (See Fig 2.10 Chapter 2). We say

$$\lim_{x \to 0} f(x) = 0$$

(to be read as limit of $f(x)$ as x tends to zero equals zero). The limit of $f(x)$ as x tends to zero is to be thought of as the value $f(x)$ should assume at $x = 0$.

In general as $x \to a$, $f(x) \to l$, then l is called *limit of the function* $f(x)$ which is symbolically written as $\lim_{x \to a} f(x) = l$.

Consider the following function $g(x) = |x|, x \neq 0$. Observe that $g(0)$ is not defined. Computing the value of $g(x)$ for values of x very near to 0, we see that the value of $g(x)$ moves towards 0. So, $\lim_{x \to 0} g(x) = 0$. This is intuitively clear from the graph of $y = |x|$ for $x \neq 0$. (See Fig 2.13, Chapter 2).

Consider the following function.

$$h(x) = \frac{x^2 - 4}{x - 2}, x \neq 2.$$

Compute the value of $h(x)$ for values of x very near to 2 (but not at 2). Convince yourself that all these values are near to 4. This is somewhat strengthened by considering the graph of the function $y = h(x)$ given here (Fig 12.2).

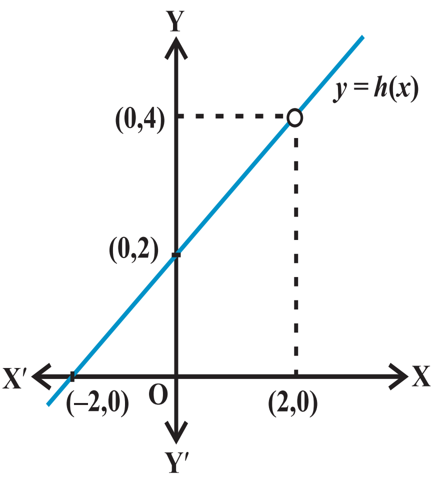

Fig 12.2

In all these illustrations the value which the function should assume at a given point $x = a$ did not really depend on how is x tending to a. Note that there are essentially two ways x could approach a number a either from left or from right, i.e., all the values of x near a could be less than a or could be greater than a. This naturally leads to two limits – the *right hand limit* and the *left hand limit*. *Right hand limit* of a function $f(x)$ is that value of $f(x)$ which is dictated by the values of $f(x)$ when x tends to a from the right. Similarly, the *left hand limit*. To illustrate this, consider the function

$$f(x)=\begin{cases}1, & x \le 0\\ 2, & x > 0\end{cases}$$

Graph of this function is shown in the Fig 12.3. It is clear that the value of f at 0 dictated by values of $f(x)$ with $x \le 0$ equals 1, i.e., the left hand limit of $f(x)$ at 0 is

$$\lim_{x\to 0^-} f(x)=1.$$

Similarly, the value of f at 0 dictated by values of $f(x)$ with $x > 0$ equals 2, i.e., the right hand limit of $f(x)$ at 0 is

$$\lim_{x\to 0^+} f(x)=2.$$

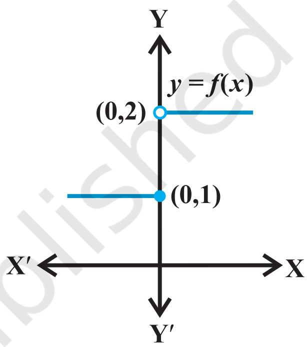

Fig 12.3

In this case the right and left hand limits are different, and hence we say that the limit of $f(x)$ as x tends to zero does not exist (even though the function is defined at 0).

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Summary

We say $\lim_{x\to a^-} f(x)$ is the expected value of f at $x = a$ given the values of f near x to the left of a. This value is called the *left hand limit* of f at a.

We say $\lim_{x\to a^+} f(x)$ is the expected value of f at $x = a$ given the values of f near x to the right of a. This value is called the *right hand limit* of $f(x)$ at a.

If the right and left hand limits coincide, we call that common value as the limit of $f(x)$ at $x = a$ and denote it by $\lim_{x\to a} f(x)$.

Illustration 1 Consider the function $f(x) = x + 10$. We want to find the limit of this function at $x = 5$. Let us compute the value of the function $f(x)$ for x very near to 5. Some of the points near and to the left of 5 are 4.9, 4.95, 4.99, 4.995. . ., etc. Values of the function at these points are tabulated below. Similarly, the real number 5.001, 5.01, 5.1 are also points near and to the right of 5. Values of the function at these points are also given in the Table 12.4.

Table 12.4

x	4.9	4.95	4.99	4.995	5.001	5.01	5.1
$f(x)$	14.9	14.95	14.99	14.995	15.001	15.01	15.1

From the Table 12.4, we deduce that value of $f(x)$ at $x = 5$ should be greater than 14.995 and less than 15.001 assuming nothing dramatic happens between $x = 4.995$ and 5.001. It is reasonable to assume that the value of the $f(x)$ at $x = 5$ as dictated by the numbers to the left of 5 is 15, i.e.,

$$\lim_{x \to 5^-} f(x) = 15.$$

Similarly, when x approaches 5 from the right, $f(x)$ should be taking value 15, i.e.,

$$\lim_{x \to 5^+} f(x) = 15.$$

Hence, it is likely that the left hand limit of $f(x)$ and the right hand limit of $f(x)$ are both equal to 15. Thus,

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = \lim_{x \to 5} f(x) = 15.$$

This conclusion about the limit being equal to 15 is somewhat strengthened by seeing the graph of this function which is given in Fig 2.16, Chapter 2. In this figure, we note that as x approaches 5 from either right or left, the graph of the function $f(x) = x + 10$ approaches the point (5, 15).

We observe that the value of the function at $x = 5$ also happens to be equal to 15.

Illustration 2 Consider the function $f(x) = x^3$. Let us try to find the limit of this function at $x = 1$. Proceeding as in the previous case, we tabulate the value of $f(x)$ at x near 1. This is given in the Table 12.5.

Table 12.5

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.729	0.970299	0.997002999	1.003003001	1.030301	1.331

From this table, we deduce that value of $f(x)$ at $x = 1$ should be greater than 0.997002999 and less than 1.003003001 assuming nothing dramatic happens between

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$x = 0.999$ and 1.001. It is reasonable to assume that the value of the $f(x)$ at $x = 1$ as dictated by the numbers to the left of 1 is 1, i.e.,

$$\lim_{x \to 1^-} f(x) = 1.$$

Similarly, when x approaches 1 from the right, $f(x)$ should be taking value 1, i.e.,

$$\lim_{x \to 1^+} f(x) = 1.$$

Hence, it is likely that the left hand limit of $f(x)$ and the right hand limit of $f(x)$ are both equal to 1. Thus,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 1.$$

This conclusion about the limit being equal to 1 is somewhat strengthened by seeing the graph of this function which is given in Fig 2.11, Chapter 2. In this figure, we note that as x approaches 1 from either right or left, the graph of the function $f(x) = x^3$ approaches the point (1, 1).

We observe, again, that the value of the function at $x = 1$ also happens to be equal to 1.

Illustration 3 Consider the function $f(x) = 3x$. Let us try to find the limit of this function at $x = 2$. The following Table 12.6 is now self-explanatory.

Table 12.6

x	1.9	1.95	1.99	1.999	2.001	2.01	2.1
$f(x)$	5.7	5.85	5.97	5.997	6.003	6.03	6.3

As before we observe that as x approaches 2 from either left or right, the value of $f(x)$ seem to approach 6. We record this as

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} f(x) = 6$$

Its graph shown in Fig 12.4 strengthens this fact.

Here again we note that the value of the function at $x = 2$ coincides with the limit at $x = 2$.

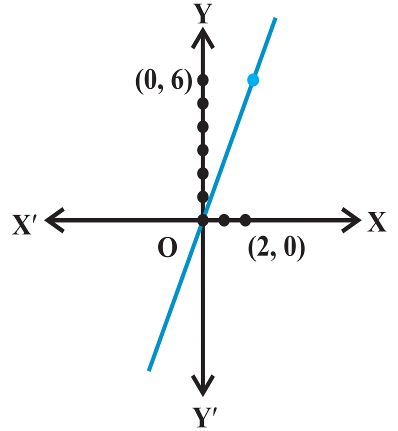

Fig 12.4

Illustration 4 Consider the constant function $f(x) = 3$. Let us try to find its limit at $x = 2$. This function being the constant function takes the same value (3, in this case) everywhere, i.e., its value at points close to 2 is 3. Hence

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} f(x) = 3$$

Graph of $f(x) = 3$ is anyway the line parallel to x-axis passing through (0, 3) and is shown in Fig 2.9, Chapter 2. From this also it is clear that the required limit is 3. In fact, it is easily observed that $\lim_{x \to a} f(x) = 3$ for any real number a.

Illustration 5 Consider the function $f(x) = x^2 + x$. We want to find $\lim_{x \to 1} f(x)$. We tabulate the values of $f(x)$ near $x = 1$ in Table 12.7.

Table 12.7

x	0.9	0.99	0.999	1.01	1.1	1.2
$f(x)$	1.71	1.9701	1.997001	2.0301	2.31	2.64

From this it is reasonable to deduce that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 2.$$

From the graph of $f(x) = x^2 + x$ shown in the Fig 12.5, it is clear that as x approaches 1, the graph approaches (1, 2).

Here, again we observe that the

$$\lim_{x \to 1} f(x) = f(1)$$

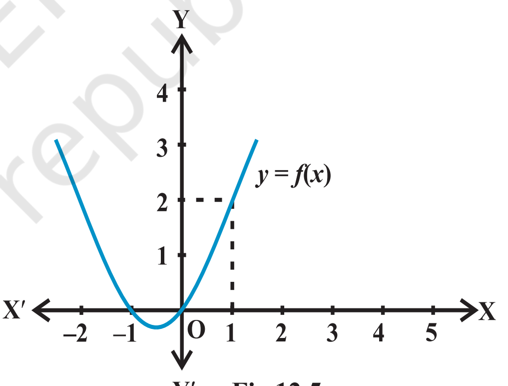

Fig 12.5

Now, convince yourself of the following three facts:

$$\lim_{x \to 1} x^2 = 1, \quad \lim_{x \to 1} x = 1 \text{ and } \lim_{x \to 1} x + 1 = 2$$

Then $$\lim_{x \to 1} x^2 + \lim_{x \to 1} x = 1 + 1 = 2 = \lim_{x \to 1}\left[x^2 + x\right].$$

Also $$\lim_{x \to 1} x . \lim_{x \to 1} (x+1) = 1.2 = 2 = \lim_{x \to 1}\left[x(x+1)\right] = \lim_{x \to 1}\left[x^2 + x\right].$$

Illustration 6 Consider the function $f(x) = \sin x$. We are interested in $\lim\limits_{x\to\frac{\pi}{2}} \sin x$, where the angle is measured in radians.

Here, we tabulate the (approximate) value of $f(x)$ near $\frac{\pi}{2}$ (Table 12.8). From this, we may deduce that

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}} f(x) = 1 .$$

Further, this is supported by the graph of $f(x) = \sin x$ which is given in the Fig 3.8 (Chapter 3). In this case too, we observe that $\lim\limits_{x\to\frac{\pi}{2}} \sin x = 1$.

Table 12.8

x	$\frac{\pi}{2} - 0.1$	$\frac{\pi}{2} - 0.01$	$\frac{\pi}{2} + 0.01$	$\frac{\pi}{2} + 0.1$
$f(x)$	0.9950	0.9999	0.9999	0.9950

Illustration 7 Consider the function $f(x) = x + \cos x$. We want to find the $\lim\limits_{x\to 0} f(x)$.

Here we tabulate the (approximate) value of $f(x)$ near 0 (Table 12.9).

Table 12.9

x	– 0.1	– 0.01	– 0.001	0.001	0.01	0.1
$f(x)$	0.9850	0.98995	0.9989995	1.0009995	1.00995	1.0950

From the Table 13.9, we may deduce that

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = 1$$

In this case too, we observe that $\lim\limits_{x\to 0} f(x) = f(0) = 1$.

Now, can you convince yourself that

$$\lim_{x\to 0}[x + \cos x] = \lim_{x\to 0} x + \lim_{x\to 0} \cos x$$ is indeed true?

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Illustration 8 Consider the function $f(x)=\frac{1}{x^2}$ for $x>0$. We want to know $\lim\limits_{x\to 0} f(x)$.

Here, observe that the domain of the function is given to be all positive real numbers. Hence, when we tabulate the values of $f(x)$, it does not make sense to talk of x approaching 0 from the left. Below we tabulate the values of the function for positive x close to 0 (in this table n denotes any positive integer).

From the Table 12.10 given below, we see that as x tends to 0, $f(x)$ becomes larger and larger. What we mean here is that the value of $f(x)$ may be made larger than any given number.

Table 12.10

x	1	0.1	0.01	10^{-n}
$f(x)$	1	100	10000	10^{2n}

Mathematically, we say

$$\lim_{x\to 0} f(x)=+\infty$$

We also remark that we will not come across such limits in this course.

Illustration 9 We want to find $\lim\limits_{x\to 0} f(x)$, where

$$f(x)=\begin{cases} x-2, & x<0 \\ 0\ \ , & x=0 \\ x+2, & x>0 \end{cases}$$

As usual we make a table of x near 0 with $f(x)$. Observe that for negative values of x we need to evaluate $x-2$ and for positive values, we need to evaluate $x+2$.

Table 12.11

x	– 0.1	– 0.01	– 0.001	0.001	0.01	0.1
$f(x)$	– 2.1	– 2.01	– 2.001	2.001	2.01	2.1

From the first three entries of the Table 12.11, we deduce that the value of the function is decreasing to –2 and hence.

$$\lim_{x\to 0^-} f(x)=-2$$

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From the last three entires of the table we deduce that the value of the function is increasing from 2 and hence

$$\lim_{x \to 0^+} f(x) = 2$$

Since the left and right hand limits at 0 do not coincide, we say that the limit of the function at 0 does not exist.

Graph of this function is given in the Fig12.6. Here, we remark that the value of the function at $x = 0$ is well defined and is, indeed, equal to 0, but the limit of the function at $x = 0$ is not even defined.

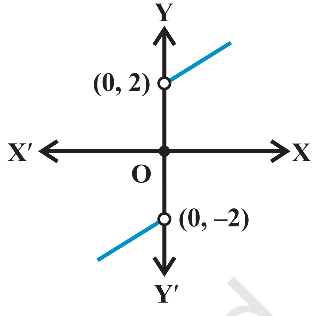

Fig 12.6

Illustration 10 As a final illustration, we find $\lim_{x \to 1} f(x)$, where

$$f(x) = \begin{cases} x + 2 & x \neq 1 \\ 0 & x = 1 \end{cases}$$

Table 12.12

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	2.9	2.99	2.999	3.001	3.01	3.1

As usual we tabulate the values of $f(x)$ for x near 1. From the values of $f(x)$ for x less than 1, it seems that the function should take value 3 at $x = 1$., i.e.,

$$\lim_{x \to 1^-} f(x) = 3.$$

Similarly, the value of $f(x)$ should be 3 as dictated by values of $f(x)$ at x greater than 1. i.e.

$$\lim_{x \to 1^+} f(x) = 3.$$

But then the left and right hand limits coincide and hence

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 3.$$

Graph of function given in Fig 12.7 strengthens our deduction about the limit. Here, we

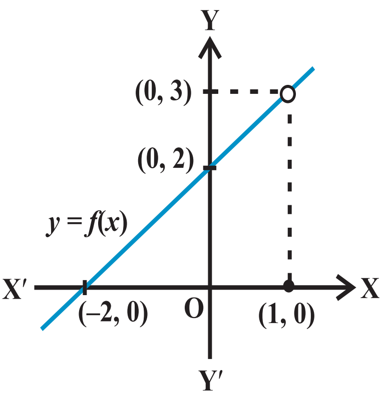

Fig 12.7

note that in general, at a given point the value of the function and its limit may be different (even when both are defined).

12.3.1 ***Algebra of limits*** In the above illustrations, we have observed that the limiting process respects addition, subtraction, multiplication and division as long as the limits and functions under consideration are well defined. This is not a coincidence. In fact, below we formalise these as a theorem without proof.

Theorem 1 Let f and g be two functions such that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

(i) Limit of sum of two functions is sum of the limits of the functions, i.e.,

$$\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

(ii) Limit of difference of two functions is difference of the limits of the functions, i.e.,

$$\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x).$$

(iii) Limit of product of two functions is product of the limits of the functions, i.e.,

$$\lim_{x\to a} [f(x) \,.\, g(x)] = \lim_{x\to a} f(x).\ \lim_{x\to a} g(x).$$

(iv) Limit of quotient of two functions is quotient of the limits of the functions (whenever the denominator is non zero), i.e.,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

Note In particular as a special case of (iii), when g is the constant function such that $g(x) = \lambda$, for some real number λ, we have

$$\lim_{x\to a}\left[(\lambda.f)(x)\right] = \lambda.\lim_{x\to a} f(x).$$

In the next two subsections, we illustrate how to exploit this theorem to evaluate limits of special types of functions.

12.3.2 ***Limits of polynomials and rational functions*** A function f is said to be a polynomial function of degree n $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where a_is are real numbers such that $a_n \neq 0$ for some natural number n.

We know that $\lim_{x\to a} x = a$. Hence

$$\lim_{x \to a} x^2 = \lim_{x \to a}(x.x) = \lim_{x \to a} x . \lim_{x \to a} x = a.\, a = a^2$$

An easy exercise in induction on n tells us that

$$\lim_{x \to a} x^n = a^n$$

Now, let $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ be a polynomial function. Thinking of each of $a_0, a_1\, x, a_2 x^2, ..., a_n\, x^n$ as a function, we have

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a}\left[a_0 + a_1 x + a_2 x^2 + ... + a_n x^n\right] \\
&= \lim_{x \to a} a_0 + \lim_{x \to a} a_1 x + \lim_{x \to a} a_2 x^2 + ... + \lim_{x \to a} a_n x^n \\
&= a_0 + a_1 \lim_{x \to a} x + a_2 \lim_{x \to a} x^2 + ... + a_n \lim_{x \to a} x^n \\
&= a_0 + a_1 a + a_2 a^2 + ... + a_n a^n \\
&= f(a)
\end{aligned}$$

(Make sure that you understand the justification for each step in the above!)

A function f is said to be a rational function, if $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials such that $h(x) \neq 0$. Then

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{g(x)}{h(x)} = \frac{\lim_{x \to a} g(x)}{\lim_{x \to a} h(x)} = \frac{g(a)}{h(a)}$$

However, if $h(a) = 0$, there are two scenarios – (i) when $g(a) \neq 0$ and (ii) when $g(a) = 0$. In the former case we say that the limit does not exist. In the latter case we can write $g(x) = (x - a)^k g_1(x)$, where k is the maximum of powers of $(x - a)$ in $g(x)$ Similarly, $h(x) = (x - a)^l h_1(x)$ as $h(a) = 0$. Now, if $k > l$, we have

$$\lim_{x \to a} f(x) = \frac{\lim_{x \to a} g(x)}{\lim_{x \to a} h(x)} = \frac{\lim_{x \to a} (x - a)^k g_1(x)}{\lim_{x \to a} (x - a)^l h_1(x)}$$

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$$= \frac{\lim_{x\to a}(x-a)^{(k-l)}\, g_1(x)}{\lim_{x\to a} h_1(x)} = \frac{0.g_1(a)}{h_1(a)} = 0$$

If $k < l$, the limit is not defined.

Example 1 Find the limits: (i) $\lim_{x\to 1}\left[x^3 - x^2 + 1\right]$ (ii) $\lim_{x\to 3}\left[x(x+1)\right]$

(iii) $\lim_{x\to -1}\left[1 + x + x^2 + \ldots + x^{10}\right]$.

Solution The required limits are all limits of some polynomial functions. Hence the limits are the values of the function at the prescribed points. We have

(i) $\lim_{x\to 1}\ [x^3 - x^2 + 1] = 1^3 - 1^2 + 1 = 1$

(ii) $\lim_{x\to 3}\left[x(x+1)\right] = 3(3+1) = 3(4) = 12$

(iii) $\lim_{x\to -1}\left[1 + x + x^2 + \ldots + x^{10}\right] = 1 + (-1) + (-1)^2 + \ldots + (-1)^{10}$

$= 1 - 1 + 1 \ldots + 1 = 1$.

Example 2 Find the limits:

(i) $\lim_{x\to 1}\left[\frac{x^2+1}{x+100}\right]$ (ii) $\lim_{x\to 2}\left[\frac{x^3 - 4x^2 + 4x}{x^2 - 4}\right]$

(iii) $\lim_{x\to 2}\left[\frac{x^2-4}{x^3 - 4x^2 + 4x}\right]$ (iv) $\lim_{x\to 2}\left[\frac{x^3 - 2x^2}{x^2 - 5x + 6}\right]$

(v) $\lim_{x\to 1}\left[\frac{x-2}{x^2 - x} - \frac{1}{x^3 - 3x^2 + 2x}\right]$.

Solution All the functions under consideration are rational functions. Hence, we first evaluate these functions at the prescribed points. If this is of the form $\frac{0}{0}$, we try to rewrite the function cancelling the factors which are causing the limit to be of the form $\frac{0}{0}$.

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(i) We have $\lim_{x \to 1} \frac{x^2 + 1}{x + 100} = \frac{1^2 + 1}{1 + 100} = \frac{2}{101}$

(ii) Evaluating the function at 2, it is of the form $\frac{0}{0}$.

Hence $\lim_{x \to 2} \frac{x^3 - 4x^2 + 4x}{x^2 - 4} = \lim_{x \to 2} \frac{x(x-2)^2}{(x+2)(x-2)}$

$= \lim_{x \to 2} \frac{x(x-2)}{(x+2)}$ as $x \neq 2$

$= \frac{2(2-2)}{2+2} = \frac{0}{4} = 0$.

(iii) Evaluating the function at 2, we get it of the form $\frac{0}{0}$.

Hence $\lim_{x \to 2} \frac{x^2 - 4}{x^3 - 4x^2 + 4x} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x(x-2)^2}$

$= \lim_{x \to 2} \frac{(x+2)}{x(x-2)} = \frac{2+2}{2(2-2)} = \frac{4}{0}$

which is not defined.

(iv) Evaluating the function at 2, we get it of the form $\frac{0}{0}$.

Hence $\lim_{x \to 2} \frac{x^3 - 2x^2}{x^2 - 5x + 6} = \lim_{x \to 2} \frac{x^2(x-2)}{(x-2)(x-3)}$

$= \lim_{x \to 2} \frac{x^2}{(x-3)} = \frac{(2)^2}{2-3} = \frac{4}{-1} = -4$.

(v) First, we rewrite the function as a rational function.

$$\left[\frac{x-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]=\left[\frac{x-2}{x(x-1)}-\frac{1}{x\left(x^2-3x+2\right)}\right]$$

$$=\left[\frac{x-2}{x(x-1)}-\frac{1}{x(x-1)(x-2)}\right]$$

$$=\left[\frac{x^2-4x+4-1}{x(x-1)(x-2)}\right]$$

$$=\frac{x^2-4x+3}{x(x-1)(x-2)}$$

Evaluating the function at 1, we get it of the form $\frac{0}{0}$.

Hence $$\lim_{x\to 1}\left[\frac{x^2-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]=\lim_{x\to 1}\frac{x^2-4x+3}{x(x-1)(x-2)}$$

$$=\lim_{x\to 1}\frac{(x-3)(x-1)}{x(x-1)(x-2)}$$

$$=\lim_{x\to 1}\frac{x-3}{x(x-2)}=\frac{1-3}{1(1-2)}=2.$$

We remark that we could cancel the term (x – 1) in the above evaluation because $x \neq 1$.

Evaluation of an important limit which will be used in the sequel is given as a theorem below.

Theorem 2 For any positive integer *n*,

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}.$$

Remark The expression in the above theorem for the limit is true even if *n* is any rational number and *a* is positive.

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Proof Dividing $(x^n - a^n)$ by $(x - a)$, we see that

$$x^n - a^n = (x-a)\,(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + ... + x\,a^{n-2} + a^{n-1})$$

Thus, $\lim\limits_{x\to a}\dfrac{x^n - a^n}{x-a} = \lim\limits_{x\to a}(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + ... + x\,a^{n-2} + a^{n-1})$

$$= a^{n-1} + a\,a^{n-2} + . . . + a^{n-2}(a) + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + ... + a^{n-1} + a^{n-1} \; (n \text{ terms})$$

$$= na^{n-1}$$

Example 3 Evaluate:

(i) $\lim\limits_{x\to 1}\dfrac{x^{15}-1}{x^{10}-1}$ (ii) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}$

Solution (i) We have

$$\lim_{x\to 1}\frac{x^{15}-1}{x^{10}-1} = \lim_{x\to 1}\left[\frac{x^{15}-1}{x-1} \div \frac{x^{10}-1}{x-1}\right]$$

$$= \lim_{x\to 1}\left[\frac{x^{15}-1}{x-1}\right] \div \lim_{x\to 1}\left[\frac{x^{10}-1}{x-1}\right]$$

$$= 15\,(1)^{14} \div 10(1)^9 \quad \text{(by the theorem above)}$$

$$= 15 \div 10 = \frac{3}{2}$$

(ii) Put $y = 1 + x$, so that $y \to 1$ as $x \to 0$.

Then $$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{y\to 1}\frac{\sqrt{y}-1}{y-1}$$

$$= \lim_{y\to 1}\frac{y^{\frac{1}{2}} - 1^{\frac{1}{2}}}{y-1}$$

$$= \frac{1}{2}(1)^{\frac{1}{2}-1} \quad \text{(by the remark above)} \quad = \frac{1}{2}$$

12.4 Limits of Trigonometric Functions

The following facts (stated as theorems) about functions in general come in handy in calculating limits of some trigonometric functions.

Theorem 3 Let f and g be two real valued functions with the same domain such that $f(x) \leq g(x)$ for all x in the domain of definition, For some a, if both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$. This is illustrated in Fig 12.8.

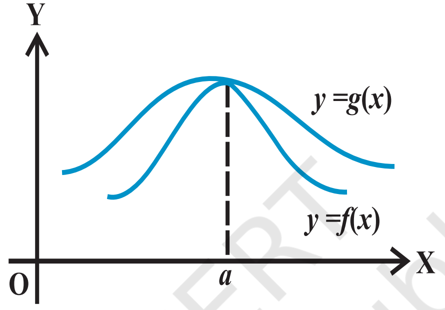

Fig 12.8

Theorem 4 (Sandwich Theorem) Let f, g and h be real functions such that $f(x) \leq g(x) \leq h(x)$ for all x in the common domain of definition. For some real number a, if $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$, then $\lim_{x \to a} g(x) = l$. This is illustrated in Fig 12.9.

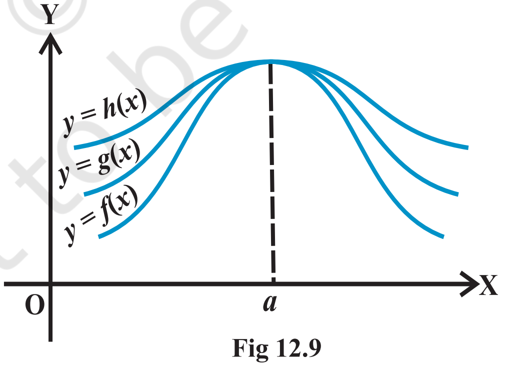

Fig 12.9

Given below is a beautiful geometric proof of the following important inequality relating trigonometric functions.

$$\cos x < \frac{\sin x}{x} < 1 \quad \text{for } 0 < |x| < \frac{\pi}{2} \qquad (*)$$

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Proof We know that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$. Hence, it is sufficient to prove the inequality for $0 < x < \frac{\pi}{2}$.

In the Fig 12.10, O is the centre of the unit circle such that the angle AOC is x radians and $0 < x < \frac{\pi}{2}$. Line segments B A and CD are perpendiculars to OA. Further, join AC. Then

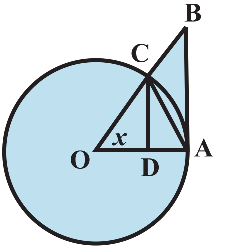

Fig 12.10

Area of $\Delta OAC <$ Area of sector $OAC <$ Area of Δ OAB.

i.e., $\frac{1}{2}OA.CD < \frac{x}{2\pi}.\pi.(OA)^2 < \frac{1}{2}OA.AB$.

i.e., CD < x . OA < AB.

From Δ OCD,

$\sin x = \frac{CD}{OA}$ (since OC = OA) and hence CD = OA $\sin x$. Also $\tan x = \frac{AB}{OA}$ and hence AB = OA. $\tan x$. Thus

OA $\sin x$ < OA. x < OA. $\tan x$.

Since length OA is positive, we have

$\sin x < x < \tan x$.

Since $0 < x < \frac{\pi}{2}$, $\sin x$ is positive and thus by dividing throughout by $\sin x$, we have

$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$. Taking reciprocals throughout, we have

$\cos x < \frac{\sin x}{x} < 1$

which complete the proof.

Theorem 5 The following are two important limits.

(i) $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$. (ii) $\lim\limits_{x \to 0} \frac{1-\cos x}{x} = 0$.

Proof (i) The inequality in (*) says that the function $\frac{\sin x}{x}$ is sandwiched between the function $\cos x$ and the constant function which takes value 1.

Further, since $\lim\limits_{x\to 0}$ cos x = 1, we see that the proof of (i) of the theorem is complete by sandwich theorem.

To prove (ii), we recall the trigonometric identity 1 – cos x = 2 $\sin^2\left(\frac{x}{2}\right)$.

Then

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x} = \lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.\sin\left(\frac{x}{2}\right)$$

$$= \lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.\lim_{x\to 0}\sin\left(\frac{x}{2}\right) = 1.0 = 0$$

Observe that we have implicitly used the fact that $x \to 0$ is equivalent to $\frac{x}{2} \to 0$. This may be justified by putting $y = \frac{x}{2}$.

Example 4 Evaluate: (i) $\lim\limits_{x\to 0}\frac{\sin 4x}{\sin 2x}$ (ii) $\lim\limits_{x\to 0}\frac{\tan x}{x}$

Solution (i) $\lim\limits_{x\to 0}\frac{\sin 4x}{\sin 2x} = \lim\limits_{x\to 0}\left[\frac{\sin 4x}{4x}.\frac{2x}{\sin 2x}.2\right]$

$$= 2.\lim_{x\to 0}\left[\frac{\sin 4x}{4x}\right] \div \left[\frac{\sin 2x}{2x}\right]$$

$$= 2.\lim_{4x\to 0}\left[\frac{\sin 4x}{4x}\right] \div \lim_{2x\to 0}\left[\frac{\sin 2x}{2x}\right]$$

$= 2.1.1 = 2$ (as $x \to 0$, $4x \to 0$ and $2x \to 0$)

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(ii) We have $\lim\limits_{x\to 0}\dfrac{\tan x}{x} = \lim\limits_{x\to 0}\dfrac{\sin x}{x\cos x} = \lim\limits_{x\to 0}\dfrac{\sin x}{x} \cdot \lim\limits_{x\to 0}\dfrac{1}{\cos x} = 1.1 = 1$

A general rule that needs to be kept in mind while evaluating limits is the following.

Say, given that the limit $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ exists and we want to evaluate this. First we check the value of $f(a)$ and $g(a)$. If both are 0, then we see if we can get the factor which is causing the terms to vanish, i.e., see if we can write $f(x) = f_1(x) f_2(x)$ so that $f_1(a) = 0$ and $f_2(a) \neq 0$. Similarly, we write $g(x) = g_1(x) g_2(x)$, where $g_1(a) = 0$ and $g_2(a) \neq 0$. Cancel out the common factors from $f(x)$ and $g(x)$ (if possible) and write

$$\frac{f(x)}{g(x)} = \frac{p(x)}{q(x)}, \text{ where } q(x) \neq 0.$$

Then $$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{p(a)}{q(a)}.$$

EXERCISE 12.1

Evaluate the following limits in Exercises 1 to 22.

1. $\lim\limits_{x\to 3} x+3$

2. $\lim\limits_{x\to \pi}\left(x-\dfrac{22}{7}\right)$

3. $\lim\limits_{r\to 1} \pi r^2$

4. $\lim\limits_{x\to 4}\dfrac{4x+3}{x-2}$

5. $\lim\limits_{x\to -1}\dfrac{x^{10}+x^5+1}{x-1}$

6. $\lim\limits_{x\to 0}\dfrac{(x+1)^5-1}{x}$

7. $\lim\limits_{x\to 2}\dfrac{3x^2-x-10}{x^2-4}$

8. $\lim\limits_{x\to 3}\dfrac{x^4-81}{2x^2-5x-3}$

9. $\lim\limits_{x\to 0}\dfrac{ax+b}{cx+1}$

10. $\lim\limits_{z\to 1}\dfrac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

11. $\lim\limits_{x\to 1}\dfrac{ax^2+bx+c}{cx^2+bx+a}, a+b+c\neq 0$

12. $\lim\limits_{x\to -2}\dfrac{\frac{1}{x}+\frac{1}{2}}{x+2}$

13. $\lim\limits_{x\to 0}\dfrac{\sin ax}{bx}$

14. $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}, a,b\neq 0$

15. $\lim\limits_{x \to \pi} \dfrac{\sin(\pi - x)}{\pi(\pi - x)}$ **16.** $\lim\limits_{x \to 0} \dfrac{\cos x}{\pi - x}$ **17.** $\lim\limits_{x \to 0} \dfrac{\cos 2x - 1}{\cos x - 1}$

18. $\lim\limits_{x \to 0} \dfrac{ax + x\cos x}{b\sin x}$ **19.** $\lim\limits_{x \to 0} x\sec x$

20. $\lim\limits_{x \to 0} \dfrac{\sin ax + bx}{ax + \sin bx}\ a, b, a + b \neq 0,$ **21.** $\lim\limits_{x \to 0} (\operatorname{cosec} x - \cot x)$

22. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\tan 2x}{x - \dfrac{\pi}{2}}$

23. Find $\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 1} f(x)$, where $f(x) = \begin{cases} 2x + 3, & x \leq 0 \\ 3(x + 1), & x > 0 \end{cases}$

24. Find $\lim\limits_{x \to 1} f(x)$, where $f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x^2 - 1, & x > 1 \end{cases}$

25. Evaluate $\lim\limits_{x \to 0} f(x)$, where $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

26. Find $\lim\limits_{x \to 0} f(x)$, where $f(x) = \begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

27. Find $\lim\limits_{x \to 5} f(x)$, where $f(x) = |x| - 5$

28. Suppose $f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$

and if $\lim\limits_{x \to 1} f(x) = f(1)$ what are possible values of a and b?

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29. Let $a_1, a_2, \ldots, a_n$ be fixed real numbers and define a function

$$f(x) = (x - a_1)(x - a_2)\ldots(x - a_n).$$

What is $\lim_{x \to a_1} f(x)$? For some $a \neq a_1, a_2, \ldots, a_n$, compute $\lim_{x \to a} f(x)$.

30. If $f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$.

For what value (s) of a does $\lim_{x \to a} f(x)$ exists?

31. If the function $f(x)$ satisfies $\lim_{x \to 1} \dfrac{f(x) - 2}{x^2 - 1} = \pi$, evaluate $\lim_{x \to 1} f(x)$.

32. If $f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \leq x \leq 1 \\ nx^3 + m, & x > 1 \end{cases}$. For what integers m and n does both $\lim_{x \to 0} f(x)$

and $\lim_{x \to 1} f(x)$ exist?

12.5 Derivatives

We have seen in the Section 13.2, that by knowing the position of a body at various time intervals it is possible to find the rate at which the position of the body is changing. It is of very general interest to know a certain parameter at various instants of time and try to finding the rate at which it is changing. There are several real life situations where such a process needs to be carried out. For instance, people maintaining a reservoir need to know when will a reservoir overflow knowing the depth of the water at several instances of time, Rocket Scientists need to compute the precise velocity with which the satellite needs to be shot out from the rocket knowing the height of the rocket at various times. Financial institutions need to predict the changes in the value of a particular stock knowing its present value. In these, and many such cases it is desirable to know how a particular parameter is changing with respect to some other parameter. The heart of the matter is derivative of a function at a given point in its domain of definition.

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Definition 1 *Suppose f is a real valued function and a is a point in its domain of definition. The derivative of f at a is defined by*

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

provided this limit exists. Derivative of f (x) at a is denoted by f′(a).

Observe that $f'(a)$ quantifies the change in $f(x)$ at a with respect to x.

Example 5 Find the derivative at $x = 2$ of the function $f(x) = 3x$.

Solution We have

$$f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{3(2+h)-3(2)}{h}$$

$$= \lim_{h\to 0}\frac{6+3h-6}{h} = \lim_{h\to 0}\frac{3h}{h} = \lim_{h\to 0}3 = 3.$$

The derivative of the function $3x$ at $x = 2$ is 3.

Example 6 Find the derivative of the function $f(x) = 2x^2 + 3x - 5$ at $x = -1$. Also prove that $f'(0) + 3f'(-1) = 0$.

Solution We first find the derivatives of $f(x)$ at $x = -1$ and at $x = 0$. We have

$$f'(-1) = \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}$$

$$= \lim_{h\to 0}\frac{\left[2(-1+h)^2+3(-1+h)-5\right]-\left[2(-1)^2+3(-1)-5\right]}{h}$$

$$= \lim_{h\to 0}\frac{2h^2-h}{h} = \lim_{h\to 0}(2h-1) = 2(0)-1 = -1$$

and $$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{\left[2(0+h)^2+3(0+h)-5\right]-\left[2(0)^2+3(0)-5\right]}{h}$$

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$$= \lim_{h\to 0}\frac{2h^2+3h}{h} = \lim_{h\to 0}(2h+3) = 2(0)+3 = 3$$

Clearly $\qquad f'(0)+3f'(-1)=0$

Remark At this stage note that evaluating derivative at a point involves effective use of various rules, limits are subjected to. The following illustrates this.

Example 7 Find the derivative of sin x at $x = 0$.

Solution Let $f(x)$ = sin x. Then

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(0+h)-\sin(0)}{h} = \lim_{h\to 0}\frac{\sin h}{h} = 1$$

Example 8 Find the derivative of $f(x) = 3$ at $x = 0$ and at $x = 3$.

Solution Since the derivative measures the change in function, intuitively it is clear that the derivative of the constant function must be zero at every point. This is indeed, supported by the following computation.

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{3-3}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$$

Similarly $\qquad f'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{3-3}{h} = 0.$

We now present a geometric interpretation of derivative of a function at a point. Let $y = f(x)$ be a function and let P $= (a, f(a))$ and Q $= (a + h, f(a + h)$ be two points close to each other on the graph of this function. The Fig 12.11 is now self explanatory.

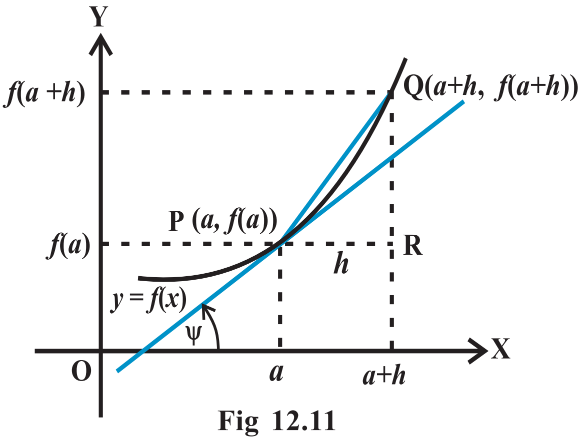

Fig 12.11

We know that $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

From the triangle PQR, it is clear that the ratio whose limit we are taking is precisely equal to tan(QPR) which is the slope of the chord PQ. In the limiting process, as h tends to 0, the point Q tends to P and we have

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{Q \to P} \frac{QR}{PR}$$

This is equivalent to the fact that the chord PQ tends to the tangent at P of the curve $y = f(x)$. Thus the limit turns out to be equal to the slope of the tangent. Hence

$$f'(a) = \tan \psi .$$

For a given function f we can find the derivative at every point. If the derivative exists at every point, it defines a new function called the derivative of f. Formally, we define derivative of a function as follows.

Definition 2 *Suppose f is a real valued function, the function defined by*

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

wherever the limit exists is defined to be the derivative of f at x and is denoted by $f'(x)$. This definition of derivative is also called the first principle of derivative.

Thus $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Clearly the domain of definition of $f'(x)$ is wherever the above limit exists. There are different notations for derivative of a function. Sometimes $f'(x)$ is denoted by $\frac{d}{dx}(f(x))$ or if $y = f(x)$, it is denoted by $\frac{dy}{dx}$. This is referred to as derivative of $f(x)$ or y with respect to x. It is also denoted by D $(f(x))$. Further, derivative of f at $x = a$ is also denoted by $\frac{d}{dx} f(x) \Big|_a$ or $\frac{df}{dx}\Big|_a$ or even $\left(\frac{df}{dx}\right)_{x=a}$.

Example 9 Find the derivative of $f(x) = 10x$.

Solution Since $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{10(x+h) - 10(x)}{h}$$

$$= \lim_{h \to 0} \frac{10h}{h} = \lim_{h \to 0} (10) = 10.$$

Example 10 Find the derivative of $f(x) = x^2$.

Solution We have, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x+h)^2 - (x)^2}{h} = \lim_{h \to 0} (h + 2x) = 2x$$

Example 11 Find the derivative of the constant function $f(x) = a$ for a fixed real number a.

Solution We have, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{a - a}{h} = \lim_{h \to 0} \frac{0}{h} = 0 \text{ as } h \neq 0$$

Example 12 Find the derivative of $f(x) = \frac{1}{x}$

Solution We have $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{x - (x+h)}{x(x+h)} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{-h}{x(x+h)} \right] = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

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12.5.1 ***Algebra of derivative of functions*** Since the very definition of derivatives involve limits in a rather direct fashion, we expect the rules for derivatives to follow closely that of limits. We collect these in the following theorem.

Theorem 5 Let f and g be two functions such that their derivatives are defined in a common domain. Then

(i) Derivative of sum of two functions is sum of the derivatives of the functions.

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x).$$

(ii) Derivative of difference of two functions is difference of the derivatives of the functions.

$$\frac{d}{dx}\left[f(x)-g(x)\right]=\frac{d}{dx}f(x)-\frac{d}{dx}g(x).$$

(iii) Derivative of product of two functions is given by the following *product rule.*

$$\frac{d}{dx}\left[f(x)\ .\ g(x)\right]=\frac{d}{dx}f(x).g(x)+f(x).\frac{d}{dx}g(x)$$

(iv) Derivative of quotient of two functions is given by the following *quotient rule* (whenever the denominator is non–zero).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{d}{dx}f(x).g(x)-f(x)\ \frac{d}{dx}g(x)}{\left(g(x)\right)^2}$$

The proofs of these follow essentially from the analogous theorem for limits. We will not prove these here. As in the case of limits this theorem tells us how to compute derivatives of special types of functions. The last two statements in the theorem may be restated in the following fashion which aids in recalling them easily:

Let $u=f(x)$ and $v = g(x)$. Then

$$(uv)' = u'v+uv'$$

This is referred to a Leibnitz rule for differentiating product of functions or the product rule. Similarly, the quotient rule is

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Now, let us tackle derivatives of some standard functions.

It is easy to see that the derivative of the function $f(x) = x$ is the constant function 1. This is because $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{x+h-x}{h}$

$$= \lim_{h\to 0} 1 = 1 .$$

We use this and the above theorem to compute the derivative of $f(x) = 10x = x + ... + x$ (ten terms). By (*i*) of the above theorem

$$\frac{df(x)}{dx} = \frac{d}{dx}\left(x + ... + x\right) \text{ (ten terms)}$$

$$= \frac{d}{dx}x + \ldots + \frac{d}{dx}x \text{ (ten terms)}$$

$$= 1 + ... + 1 \text{ (ten terms)} = 10.$$

We note that this limit may be evaluated using product rule too. Write $f(x) = 10x = uv$, where u is the constant function taking value 10 everywhere and $v(x) = x$. Here, $f(x) = 10x = uv$ we know that the derivative of u equals 0. Also derivative of $v(x) = x$ equals 1. Thus by the product rule we have

$$f'(x) = (10x)' = (uv)' = u'v + uv' = 0.x + 10.1 = 10$$

On similar lines the derivative of $f(x) = x^2$ may be evaluated. We have $f(x) = x^2 = x .x$ and hence

$$\frac{df}{dx} = \frac{d}{dx}(x.x) = \frac{d}{dx}(x).x + x.\frac{d}{dx}(x)$$

$$= 1.x + x.1 = 2x .$$

More generally, we have the following theorem.

Theorem 6 Derivative of $f(x) = x^n$ is nx^{n-1} for any positive integer n.

Proof By definition of the derivative function, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} .$$

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Binomial theorem tells that $(x + h)^n = \left({}^nC_0\right)x^n + \left({}^nC_1\right)x^{n-1}h + ... + \left({}^nC_n\right)h^n$ and hence $(x + h)^n - x^n = h(nx^{n-1} + ... + h^{n-1})$. Thus

$$\frac{df(x)}{dx} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{h\left(nx^{n-1} + ... + h^{n-1}\right)}{h}$$

$$= \lim_{h\to 0}\left(nx^{n-1} + ... + h^{n-1}\right) = nx^{n-1}.$$

Alternatively, we may also prove this by induction on n and the product rule as follows. The result is true for $n = 1$, which has been proved earlier. We have

$$\frac{d}{dx}\left(x^n\right) = \frac{d}{dx}\left(x.x^{n-1}\right)$$

$$= \frac{d}{dx}(x).\left(x^{n-1}\right) + x.\frac{d}{dx}\left(x^{n-1}\right) \text{ (by product rule)}$$

$$= 1.x^{n-1} + x.\left((n-1)x^{n-2}\right) \text{ (by induction hypothesis)}$$

$$= x^{n-1} + (n-1)x^{n-1} = nx^{n-1}.$$

Remark The above theorem is true for all powers of x, i.e., n can be any real number (but we will not prove it here).

12.5.2 *Derivative of polynomials and trigonometric functions* We start with the following theorem which tells us the derivative of a polynomial function.

Theorem 7 Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ be a polynomial function, where a_is are all real numbers and $a_n \neq 0$. Then, the derivative function is given by

$$\frac{df(x)}{dx} = na_n x^{n-1} + (n-1)a_{n-1}x^{x-2} + ... + 2a_2 x + a_1.$$

Proof of this theorem is just putting together part (i) of Theorem 5 and Theorem 6.

Example 13 Compute the derivative of $6x^{100} - x^{55} + x$.

Solution A direct application of the above theorem tells that the derivative of the above function is $600x^{99} - 55x^{54} + 1$.

Example 14 Find the derivative of $f(x) = 1 + x + x^2 + x^3 + ... + x^{50}$ at $x = 1$.

Solution A direct application of the above Theorem 6 tells that the derivative of the above function is $1 + 2x + 3x^2 + . . . + 50x^{49}$. At $x = 1$ the value of this function equals

$$1 + 2(1) + 3(1)^2 + .. . + 50(1)^{49} = 1 + 2 + 3 + . . . + 50 = \frac{(50)(51)}{2} = 1275.$$

Example 15 Find the derivative of $f(x) = \frac{x+1}{x}$

Solution Clearly this function is defined everywhere except at $x = 0$. We use the quotient rule with $u = x + 1$ and $v = x$. Hence $u' = 1$ and $v' = 1$. Therefore

$$\frac{df(x)}{dx} = \frac{d}{dx}\left(\frac{x+1}{x}\right) = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2} = \frac{1(x)-(x+1)1}{x^2} = -\frac{1}{x^2}$$

Example 16 Compute the derivative of $\sin x$.

Solution Let $f(x) = \sin x$. Then

$$\frac{df(x)}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} \text{ (using formula for } \sin A - \sin B)$$

$$= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right).\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}} = \cos x.1 = \cos x.$$

Example 17 Compute the derivative of $\tan x$.

Solution Let $f(x) = \tan x$. Then

$$\frac{df(x)}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$$

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$$= \lim_{h \to 0} \left[\frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x} \right]$$

$$= \lim_{h \to 0} \frac{\sin(x+h-x)}{h\cos(x+h)\cos x} \text{ (using formula for sin (A + B))}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} . \lim_{h \to 0} \frac{1}{\cos(x+h)\cos x}$$

$$= 1 . \frac{1}{\cos^2 x} = \sec^2 x .$$

Example 18 Compute the derivative of $f(x) = \sin^2 x$.

Solution We use the Leibnitz product rule to evaluate this.

$$\frac{df(x)}{dx} = \frac{d}{dx}(\sin x \sin x)$$

$$= (\sin x)' \sin x + \sin x (\sin x)'$$

$$= (\cos x)\sin x + \sin x(\cos x)$$

$$= 2\sin x \cos x = \sin 2x \cdot$$

EXERCISE 12.2

1. Find the derivative of $x^2 - 2$ at $x = 10$.
2. Find the derivative of x at $x = 1$.
3. Find the derivative of $99x$ at $x = 100$.
4. Find the derivative of the following functions from first principle.

   (i) $x^3 - 27$ (ii) $(x-1)(x-2)$

   (iii) $\frac{1}{x^2}$ (iv) $\frac{x+1}{x-1}$

5. For the function

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + . . . + \frac{x^2}{2} + x + 1 .$$

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Prove that $f'(1) = 100 f'(0)$.

6. Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^{n-1}x + a^n$ for some fixed real number a.

7. For some constants a and b, find the derivative of

(i) $(x-a)(x-b)$ (ii) $(ax^2+b)^2$ (iii) $\dfrac{x-a}{x-b}$

8. Find the derivative of $\dfrac{x^n - a^n}{x-a}$ for some constant a.

9. Find the derivative of

(i) $2x - \dfrac{3}{4}$ (ii) $(5x^3+3x-1)(x-1)$

(iii) $x^{-3}(5+3x)$ (iv) $x^5(3-6x^{-9})$

(v) $x^{-4}(3-4x^{-5})$ (vi) $\dfrac{2}{x+1} - \dfrac{x^2}{3x-1}$

10. Find the derivative of $\cos x$ from first principle.

11. Find the derivative of the following functions:

(i) $\sin x \cos x$ (ii) $\sec x$ (iii) $5\sec x + 4\cos x$

(iv) $\operatorname{cosec} x$ (v) $3\cot x + 5\operatorname{cosec} x$

(vi) $5\sin x - 6\cos x + 7$ (vii) $2\tan x - 7\sec x$

Miscellaneous Examples

Example 19 Find the derivative of *f* from the first principle, where *f* is given by

(i) $f(x) = \dfrac{2x+3}{x-2}$ (ii) $f(x) = x + \dfrac{1}{x}$

Solution (i) Note that function is not defined at $x = 2$. But, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{2(x+h)+3}{x+h-2} - \dfrac{2x+3}{x-2}}{h}$$

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$$= \lim_{h\to 0} \frac{(2x+2h+3)(x-2)-(2x+3)(x+h-2)}{h(x-2)(x+h-2)}$$

$$= \lim_{h\to 0} \frac{(2x+3)(x-2)+2h(x-2)-(2x+3)(x-2)-h(2x+3)}{h(x-2)(x+h-2)}$$

$$= \lim_{h\to 0} \frac{-7}{(x-2)\,(x+h-2)} = -\frac{7}{(x-2)^2}$$

Again, note that the function f' is also not defined at $x = 2$.

(ii) The function is not defined at $x = 0$. But, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left(x+h+\frac{1}{x+h}\right)-\left(x+\frac{1}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[h+\frac{1}{x+h}-\frac{1}{x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[h+\frac{x-x-h}{x(x+h)}\right] = \lim_{h\to 0} \frac{1}{h}\left[h\left(1-\frac{1}{x(x+h)}\right)\right]$$

$$= \lim_{h\to 0} \left[1-\frac{1}{x(x+h)}\right] = 1-\frac{1}{x^2}$$

Again, note that the function f' is not defined at $x = 0$.

Example 20 Find the derivative of *f(x)* from the first principle, where *f(x)* is

(i) $\sin x + \cos x$ (ii) $x \sin x$

Solution (i) we have $f'(x) = \dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{\sin(x+h)+\cos(x+h)-\sin x-\cos x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x\,\cos h+\cos x\,\sin h+\cos x\cos h-\sin x\,\sin h-\sin x-\cos x}{h}$$

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$$= \lim_{h\to 0} \frac{\sin h(\cos x - \sin x) + \sin x(\cos h - 1) + \cos x(\cos h - 1)}{h}$$

$$= \lim_{h\to 0} \frac{\sin h}{h}(\cos x - \sin x) + \lim_{h\to 0} \sin x \frac{(\cos h - 1)}{h} \quad + \lim_{h\to 0} \cos x \frac{(\cos h - 1)}{h}$$

$$= \cos x - \sin x$$

(ii) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)\sin(x+h) - x\sin x}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)(\sin x \cos h + \sin h \cos x) - x\sin x}{h}$$

$$= \lim_{h\to 0} \frac{x\sin x(\cos h - 1) + x\cos x \sin h + h(\sin x \cos h + \sin h \cos x)}{h}$$

$$= \lim_{h\to 0} \frac{x\sin x(\cos h - 1)}{h} + \lim_{h\to 0} x\cos x \frac{\sin h}{h} + \lim_{h\to 0}(\sin x \cos h + \sin h \cos x)$$

$$= x\cos x + \sin x$$

Example 21 Compute derivative of

(i) $f(x) = \sin 2x$ (ii) $g(x) = \cot x$

Solution (i) Recall the trigonometric formula $\sin 2x = 2 \sin x \cos x$. Thus

$$\frac{df(x)}{dx} = \frac{d}{dx}(2\sin x \cos x) = 2\frac{d}{dx}(\sin x \cos x)$$

$$= 2\left[(\sin x)' \cos x + \sin x (\cos x)'\right]$$

$$= 2\left[(\cos x)\cos x + \sin x(-\sin x)\right]$$

$$= 2\left(\cos^2 x - \sin^2 x\right)$$

(ii) By definition, $g(x) = \cot x = \dfrac{\cos x}{\sin x}$. We use the quotient rule on this function wherever it is defined. $\dfrac{dg}{dx} = \dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right)$

$$= \frac{(\cos x)'(\sin x) - (\cos x)(\sin x)'}{(\sin x)^2}$$

$$= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{(\sin x)^2}$$

$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\operatorname{cosec}^2 x$$

Alternatively, this may be computed by noting that $\cot x = \frac{1}{\tan x}$. Here, we use the fact that the derivative of *tan x* is *sec*^{2} *x* which we saw in Example 17 and also that the derivative of the constant function is 0.

$$\frac{dg}{dx} = \frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right)$$

$$= \frac{(1)'(\tan x) - (1)(\tan x)'}{(\tan x)^2}$$

$$= \frac{(0)(\tan x) - (\sec x)^2}{(\tan x)^2}$$

$$= \frac{-\sec^2 x}{\tan^2 x} = -\operatorname{cosec}^2 x$$

Example 22 Find the derivative of

(i) $\frac{x^5 - \cos x}{\sin x}$ (ii) $\frac{x + \cos x}{\tan x}$

Solution (i) Let $h(x) = \frac{x^5 - \cos x}{\sin x}$. We use the quotient rule on this function wherever it is defined.

$$h'(x) = \frac{(x^5 - \cos x)'\sin x - (x^5 - \cos x)(\sin x)'}{(\sin x)^2}$$

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$$= \frac{(5x^4 + \sin x)\sin x - (x^5 - \cos x)\cos x}{\sin^2 x}$$

$$= \frac{-x^5 \cos x + 5x^4 \sin x + 1}{(\sin x)^2}$$

(ii) We use quotient rule on the function $\frac{x + \cos x}{\tan x}$ wherever it is defined.

$$h'(x) = \frac{(x + \cos x)' \tan x - (x + \cos x)(\tan x)'}{(\tan x)^2}$$

$$= \frac{(1 - \sin x)\tan x - (x + \cos x)\sec^2 x}{(\tan x)^2}$$

Miscellaneous Exercise on Chapter 12

1. Find the derivative of the following functions from first principle:

(i) $-x$ (ii) $(-x)^{-1}$ (iii) $\sin(x + 1)$ (iv) $\cos\left(x - \frac{\pi}{8}\right)$

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):

2. $(x + a)$

3. $(px + q)\left(\frac{r}{x} + s\right)$

4. $(ax + b)(cx + d)^2$

5. $\frac{ax + b}{cx + d}$

6. $\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$

7. $\frac{1}{ax^2 + bx + c}$

8. $\frac{ax + b}{px^2 + qx + r}$

9. $\frac{px^2 + qx + r}{ax + b}$

10. $\frac{a}{x^4} - \frac{b}{x^2} + \cos x$

11. $4\sqrt{x} - 2$

12. $(ax + b)^n$

13. $(ax + b)^n (cx + d)^m$

14. $\sin(x + a)$

15. $\operatorname{cosec} x \cot x$

16. $\frac{\cos x}{1 + \sin x}$

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17. $\dfrac{\sin x+\cos x}{\sin x-\cos x}$ **18.** $\dfrac{\sec x-1}{\sec x+1}$ **19.** $\sin^n x$

20. $\dfrac{a+b\sin x}{c+d\cos x}$ **21.** $\dfrac{\sin(x+a)}{\cos x}$ **22.** $x^4(5\sin x-3\cos x)$

23. $\left(x^2+1\right)\cos x$ **24.** $\left(ax^2+\sin x\right)\left(p+q\cos x\right)$

25. $(x+\cos x)\,(x-\tan x)$ **26.** $\dfrac{4x+5\sin x}{3x+7\cos x}$ **27.** $\dfrac{x^2\cos\left(\dfrac{\pi}{4}\right)}{\sin x}$

28. $\dfrac{x}{1+\tan x}$ **29.** $(x+\sec x)\,(x-\tan x)$ **30.** $\dfrac{x}{\sin^n x}$

Summary

- The expected value of the function as dictated by the points to the left of a point defines the left hand limit of the function at that point. Similarly the right hand limit.
- Limit of a function at a point is the common value of the left and right hand limits, if they coincide.
- For a function f and a real number a, $\lim\limits_{x\to a} f(x)$ and $f(a)$ may not be same (In fact, one may be defined and not the other one).
- For functions f and g the following holds:

$$\lim_{x\to a}[f(x)\pm g(x)]=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$$

$$\lim_{x\to a}[f(x).g(x)]=\lim_{x\to a} f(x).\lim_{x\to a} g(x)$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$$

- Following are some of the standard limits

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$$

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$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

- The derivative of a function f at a is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- Derivative of a function f at any point x is defined by

$$f'(x) = \frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- For functions u and v the following holds:

$$(u \pm v)' = u' \pm v'$$

$$(uv)' = u'v + uv'$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$ provided all are defined.

- Following are some of the standard derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

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Historical Note

In the history of mathematics two names are prominent to share the credit for inventing calculus, Issac Newton (1642 – 1727) and G.W. Leibnitz (1646 – 1717). Both of them independently invented calculus around the seventeenth century. After the advent of calculus many mathematicians contributed for further development of calculus. The rigorous concept is mainly attributed to the great mathematicians, A.L. Cauchy, J.L.Lagrange and Karl Weierstrass. Cauchy gave the foundation of calculus as we have now generally accepted in our textbooks. Cauchy used D' Alembert's limit concept to define the derivative of a function. Starting with definition of a limit, Cauchy gave examples such as the limit of $\frac{\sin\alpha}{\alpha}$ for $\alpha = 0$. He wrote $\frac{\Delta y}{\Delta x} = \frac{f(x+i)-f(x)}{i}$, and called the limit for $i \to 0$, the "function derive'e, y' for $f'(x)$".

Before 1900, it was thought that calculus is quite difficult to teach. So calculus became beyond the reach of youngsters. But just in 1900, John Perry and others in England started propagating the view that essential ideas and methods of calculus were simple and could be taught even in schools. F.L. Griffin, pioneered the teaching of calculus to first year students. This was regarded as one of the most daring act in those days.

Today not only the mathematics but many other subjects such as Physics, Chemistry, Economics and Biological Sciences are enjoying the fruits of calculus.

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Chapter 13

STATISTICS

❖ *"Statistics may be rightly called the science of averages and their estimates." – A.L.BOWLEY & A.L. BODDINGTON* ❖

13.1 Introduction

Karl Pearson
(1857-1936)

We know that statistics deals with data collected for specific purposes. We can make decisions about the data by analysing and interpreting it. In earlier classes, we have studied methods of representing data graphically and in tabular form. This representation reveals certain salient features or characteristics of the data. We have also studied the methods of finding a representative value for the given data. This value is called the measure of central tendency. Recall mean (arithmetic mean), median and mode are three measures of central tendency. A *measure of central tendency* gives us a rough idea where data points are centred. But, in order to make better interpretation from the data, we should also have an idea how the data are scattered or how much they are bunched around a measure of central tendency.

Consider now the runs scored by two batsmen in their last ten matches as follows:

Batsman A : 30, 91, 0, 64, 42, 80, 30, 5, 117, 71
Batsman B : 53, 46, 48, 50, 53, 53, 58, 60, 57, 52

Clearly, the mean and median of the data are

	Batsman A	Batsman B
Mean	53	53
Median	53	53

Recall that, we calculate the mean of a data (denoted by $\bar{x}$) by dividing the sum of the observations by the number of observations, i.e.,

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$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Also, the median is obtained by first arranging the data in ascending or descending order and applying the following rule.

If the number of observations is odd, then the median is $\left(\frac{n+1}{2}\right)^{\text{th}}$ observation.

If the number of observations is even, then median is the mean of $\left(\frac{n}{2}\right)^{\text{th}}$ and $\left(\frac{n}{2}+1\right)^{\text{th}}$ observations.

We find that the mean and median of the runs scored by both the batsmen A and B are same i.e., 53. Can we say that the performance of two players is same? Clearly No, because the variability in the scores of batsman A is from 0 (minimum) to 117 (maximum). Whereas, the range of the runs scored by batsman B is from 46 to 60.

Let us now plot the above scores as dots on a number line. We find the following diagrams:

For batsman A

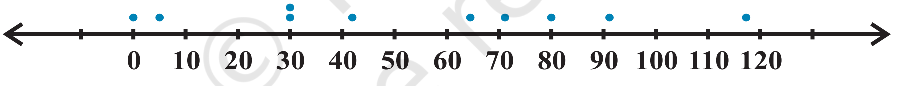

Fig 13.1

For batsman B

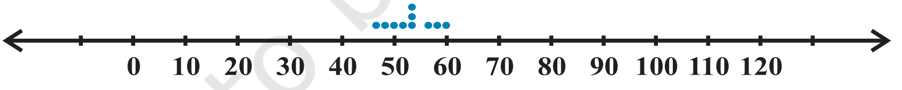

Fig 13.2

We can see that the dots corresponding to batsman B are close to each other and are clustering around the measure of central tendency (mean and median), while those corresponding to batsman A are scattered or more spread out.

Thus, the measures of central tendency are not sufficient to give complete information about a given data. Variability is another factor which is required to be studied under statistics. Like '*measures of central tendency*' we want to have a single number to describe variability. This single number is called a '*measure of dispersion*'. In this Chapter, we shall learn some of the important measures of dispersion and their methods of calculation for ungrouped and grouped data.

13.2 Measures of Dispersion

The dispersion or scatter in a data is measured on the basis of the observations and the types of the measure of central tendency, used there. There are following measures of dispersion:

(i) Range, (ii) Quartile deviation, (iii) Mean deviation, (iv) Standard deviation.

In this Chapter, we shall study all of these measures of dispersion except the quartile deviation.

13.3 Range

Recall that, in the example of runs scored by two batsmen A and B, we had some idea of variability in the scores on the basis of minimum and maximum runs in each series. To obtain a single number for this, we find the difference of maximum and minimum values of each series. This difference is called the 'Range' of the data.

In case of batsman A, Range = 117 – 0 = 117 and for batsman B, Range = 60 – 46 = 14. Clearly, Range of A > Range of B. Therefore, the scores are scattered or dispersed in case of A while for B these are close to each other.

Thus, Range of a series = Maximum value – Minimum value.

The range of data gives us a rough idea of variability or scatter but does not tell about the dispersion of the data from a measure of central tendency. For this purpose, we need some other measure of variability. Clearly, such measure must depend upon the difference (or deviation) of the values from the central tendency.

The important measures of dispersion, which depend upon the deviations of the observations from a central tendency are mean deviation and standard deviation. Let us discuss them in detail.

13.4 Mean Deviation

Recall that the deviation of an observation x from a fixed value 'a' is the difference $x - a$. In order to find the dispersion of values of x from a central value 'a', we find the deviations about a. An absolute measure of dispersion is the mean of these deviations. To find the mean, we must obtain the sum of the deviations. But, we know that a measure of central tendency lies between the maximum and the minimum values of the set of observations. Therefore, some of the deviations will be negative and some positive. Thus, the sum of deviations may vanish. Moreover, the sum of the deviations from mean ($\bar{x}$) is zero.

Also $$\text{Mean of deviations} = \frac{\text{Sum of deviations}}{\text{Number of observations}} = \frac{0}{n} = 0$$

Thus, finding the mean of deviations about mean is not of any use for us, as far as the measure of dispersion is concerned.

Remember that, in finding a suitable measure of dispersion, we require the distance of each value from a central tendency or a fixed number 'a'. Recall, that the absolute value of the difference of two numbers gives the distance between the numbers when represented on a number line. Thus, to find the measure of dispersion from a fixed number 'a' we may take the mean of the absolute values of the deviations from the central value. This mean is called the '*mean deviation*'. Thus mean deviation about a central value 'a' is the mean of the absolute values of the deviations of the observations from 'a'. The mean deviation from 'a' is denoted as M.D. (a). Therefore,

$$\text{M.D.}(a) = \frac{\text{Sum of absolute values of deviations from } 'a'}{\text{Number of observations}}.$$

Remark Mean deviation may be obtained from any measure of central tendency. However, mean deviation from mean and median are commonly used in statistical studies.

Let us now learn how to calculate mean deviation about mean and mean deviation about median for various types of data

13.4.1 ***Mean deviation for ungrouped data*** Let n observations be $x_1, x_2, x_3, ..., x_n$. The following steps are involved in the calculation of mean deviation about mean or median:

Step 1 Calculate the measure of central tendency about which we are to find the mean deviation. Let it be 'a'.

Step 2 Find the deviation of each x_i from a, i.e., $x_1 - a, x_2 - a, x_3 - a, . . . , x_n - a$

Step 3 Find the absolute values of the deviations, i.e., drop the minus sign (–), if it is there, i.e., $|x_1 - a|, |x_2 - a|, |x_3 - a|, ..., |x_n - a|$

Step 4 Find the mean of the absolute values of the deviations. This mean is the mean deviation about a, i.e.,

$$\text{M.D.}(a) = \frac{\sum_{i=1}^{n} |x_i - a|}{n}$$

Thus $$\text{M.D.}(\bar{x}) = \frac{1}{n} \sum_{i=1}^{n} |x_i - \bar{x}|, \text{ where } \bar{x} = \text{Mean}$$

and $$\text{M.D.}(\text{M}) = \frac{1}{n} \sum_{i=1}^{n} |x_i - \text{M}|, \text{ where M = Median}$$

Note In this Chapter, we shall use the symbol M to denote median unless stated otherwise.Let us now illustrate the steps of the above method in following examples.

Example 1 Find the mean deviation about the mean for the following data:

6, 7, 10, 12, 13, 4, 8, 12

Solution We proceed step-wise and get the following:

Step 1 Mean of the given data is

$$\bar{x} = \frac{6+7+10+12+13+4+8+12}{8} = \frac{72}{8} = 9$$

Step 2 The deviations of the respective observations from the mean $\bar{x}$, i.e., $x_i - \bar{x}$ are

6 – 9, 7 – 9, 10 – 9, 12 – 9, 13 – 9, 4 – 9, 8 – 9, 12 – 9,

or –3, –2, 1, 3, 4, –5, –1, 3

Step 3 The absolute values of the deviations, i.e., $|x_i - \bar{x}|$ are

3, 2, 1, 3, 4, 5, 1, 3

Step 4 The required mean deviation about the mean is

$$\text{M.D.}\ (\bar{x}) = \frac{\sum_{i=1}^{8}|x_i - \bar{x}|}{8}$$

$$= \frac{3+2+1+3+4+5+1+3}{8} = \frac{22}{8} = 2.75$$

Note Instead of carrying out the steps every time, we can carry on calculation, step-wise without referring to steps.

Example 2 Find the mean deviation about the mean for the following data :

12, 3, 18, 17, 4, 9, 17, 19, 20, 15, 8, 17, 2, 3, 16, 11, 3, 1, 0, 5

Solution We have to first find the mean ($\bar{x}$) of the given data

$$\bar{x} = \frac{1}{20}\sum_{i=1}^{20} x_i = \frac{200}{20} = 10$$

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The respective absolute values of the deviations from mean, i.e., $|x_i - \bar{x}|$ are

2, 7, 8, 7, 6, 1, 7, 9, 10, 5, 2, 7, 8, 7, 6, 1, 7, 9, 10, 5

Therefore $$\sum_{i=1}^{20} |x_i - \bar{x}| = 124$$

and $$\text{M.D. }(\bar{x}) = \frac{124}{20} = 6.2$$

Example 3 Find the mean deviation about the median for the following data:

3, 9, 5, 3, 12, 10, 18, 4, 7, 19, 21.

Solution Here the number of observations is 11 which is odd. Arranging the data into ascending order, we have 3, 3, 4, 5, 7, 9, 10, 12, 18, 19, 21

Now $$\text{Median} = \left(\frac{11 + 1}{2}\right)^{\text{th}} \text{ or } 6^{\text{th}} \text{ observation} = 9$$

The absolute values of the respective deviations from the median, i.e., $|x_i - \text{M}|$ are

6, 6, 5, 4, 2, 0, 1, 3, 9, 10, 12

Therefore $$\sum_{i=1}^{11} |x_i - \text{M}| = 58$$

and $$\text{M.D.}(\text{M}) = \frac{1}{11}\sum_{i=1}^{11} |x_i - \text{M}| = \frac{1}{11} \times 58 = 5.27$$

13.4.2 ***Mean deviation for grouped data*** We know that data can be grouped into two ways :

(a) Discrete frequency distribution,

(b) Continuous frequency distribution.

Let us discuss the method of finding mean deviation for both types of the data.

(a) Discrete frequency distribution Let the given data consist of n distinct values $x_1, x_2, ..., x_n$ occurring with frequencies $f_1, f_2, ..., f_n$ respectively. This data can be represented in the tabular form as given below, and is called *discrete frequency distribution*:

$$\begin{array}{llll} x : x_1 & x_2 & x_3 \ldots & x_n \\ f : f_1 & f_2 & f_3 \ldots & f_n \end{array}$$

(i) Mean deviation about mean

First of all we find the mean $\bar{x}$ of the given data by using the formula

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i} = \frac{1}{N} \sum_{i=1}^{n} x_i f_i,$$

where $\sum_{i=1}^{n} x_i f_i$ denotes the sum of the products of observations x_i with their respective frequencies f_i and $N = \sum_{i=1}^{n} f_i$ is the sum of the frequencies.

Then, we find the deviations of observations x_i from the mean $\bar{x}$ and take their absolute values, i.e., $|x_i - \bar{x}|$ for all $i = 1, 2, ..., n$.

After this, find the mean of the absolute values of the deviations, which is the required mean deviation about the mean. Thus

$$\text{M.D.}(\bar{x}) = \frac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{\sum_{i=1}^{n} f_i} = \frac{1}{N} \sum_{i=1}^{n} f_i |x_i - \bar{x}|$$

(ii) Mean deviation about median To find mean deviation about median, we find the median of the given discrete frequency distribution. For this the observations are arranged in ascending order. After this the cumulative frequencies are obtained. Then, we identify the observation whose cumulative frequency is equal to or just greater than $\frac{N}{2}$, where N is the sum of frequencies. This value of the observation lies in the middle of the data, therefore, it is the required median. After finding median, we obtain the mean of the absolute values of the deviations from median.Thus,

$$\text{M.D.(M)} = \frac{1}{N} \sum_{i=1}^{n} f_i |x_i - M|$$

Example 4 Find mean deviation about the mean for the following data :

x_i	2	5	6	8	10	12
f_i	2	8	10	7	8	5

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Solution Let us make a Table 13.1 of the given data and append other columns after calculations.

Table 13.1

x_i	f_i	$f_i x_i$	$\lvert x_i - \bar{x}\rvert$	$f_i \lvert x_i - \bar{x}\rvert$
2	2	4	5.5	11
5	8	40	2.5	20
6	10	60	1.5	15
8	7	56	0.5	3.5
10	8	80	2.5	20
12	5	60	4.5	22.5
	40	300		92

$$\mathrm{N} = \sum_{i=1}^{6} f_i = 40, \quad \sum_{i=1}^{6} f_i x_i = 300, \quad \sum_{i=1}^{6} f_i \lvert x_i - \bar{x}\rvert = 92$$

Therefore $$\bar{x} = \frac{1}{\mathrm{N}} \sum_{i=1}^{6} f_i x_i = \frac{1}{40} \times 300 = 7.5$$

and $$\mathrm{M.D.}\ (\bar{x}) = \frac{1}{\mathrm{N}} \sum_{i=1}^{6} f_i \ \lvert x_i - \bar{x}\rvert = \frac{1}{40} \times 92 = 2.3$$

Example 5 Find the mean deviation about the median for the following data:

x_i	3	6	9	12	13	15	21	22
f_i	3	4	5	2	4	5	4	3

Solution The given observations are already in ascending order. Adding a row corresponding to cumulative frequencies to the given data, we get (Table 13.2).

Table 13.2

x_i	3	6	9	12	13	15	21	22
f_i	3	4	5	2	4	5	4	3
$c.f.$	3	7	12	14	18	23	27	30

Now, N=30 which is even.

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Median is the mean of the 15th and 16th observations. Both of these observations lie in the cumulative frequency 18, for which the corresponding observation is 13.

$$\text{Therefore, Median M} = \frac{15^{\text{th}} \text{ observation} + 16^{\text{th}} \text{ observation}}{2} = \frac{13+13}{2} = 13$$

Now, absolute values of the deviations from median, i.e., $|x_i - \text{M}|$ are shown in Table 13.3.

Table 13.3

$\lvert x_i - \text{M}\rvert$	10	7	4	1	0	2	8	9
f_i	3	4	5	2	4	5	4	3
$f_i\lvert x_i - \text{M}\rvert$	30	28	20	2	0	10	32	27

We have $$\sum_{i=1}^{8} f_i = 30 \quad \text{and} \quad \sum_{i=1}^{8} f_i\,|x_i - \text{M}| = 149$$

Therefore $$\text{M.D.(M)} = \frac{1}{\text{N}} \sum_{i=1}^{8} f_i |x_i - \text{M}|$$

$$= \frac{1}{30} \times 149 = 4.97.$$

(b) Continuous frequency distribution A continuous frequency distribution is a series in which the data are classified into different class-intervals without gaps alongwith their respective frequencies.

For example, marks obtained by 100 students are presented in a continuous frequency distribution as follows :

Marks obtained	0-10	10-20	20-30	30-40	40-50	50-60
Number of Students	12	18	27	20	17	6

(i) Mean deviation about mean While calculating the mean of a continuous frequency distribution, we had made the assumption that the frequency in each class is centred at its mid-point. Here also, we write the mid-point of each given class and proceed further as for a discrete frequency distribution to find the mean deviation.

Let us take the following example.

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Example 6 Find the mean deviation about the mean for the following data.

Marks obtained	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Number of students	2	3	8	14	8	3	2

Solution We make the following Table 13.4 from the given data :

Table 13.4

Marks obtained	Number of students f_i	Mid-points x_i	$f_i x_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
10-20	2	15	30	30	60
20-30	3	25	75	20	60
30-40	8	35	280	10	80
40-50	14	45	630	0	0
50-60	8	55	440	10	80
60-70	3	65	195	20	60
70-80	2	75	150	30	60
	40		1800		400

Here $$N = \sum_{i=1}^{7} f_i = 40, \sum_{i=1}^{7} f_i x_i = 1800, \sum_{i=1}^{7} f_i \lvert x_i - \bar{x} \rvert = 400$$

Therefore $$\bar{x} = \frac{1}{N}\sum_{i=1}^{7} f_i x_i = \frac{1800}{40} = 45$$

and $$\text{M.D.}(\bar{x}) = \frac{1}{N}\sum_{i=1}^{7} f_i \lvert x_i - \bar{x} \rvert = \frac{1}{40} \times 400 = 10$$

Shortcut method for calculating mean deviation about mean We can avoid the tedious calculations of computing $\bar{x}$ by following step-deviation method. Recall that in this method, we take an assumed mean which is in the middle or just close to it in the data. Then deviations of the observations (or mid-points of classes) are taken from the

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assumed mean. This is nothing but the shifting of origin from zero to the assumed mean on the number line, as shown in Fig 13.3

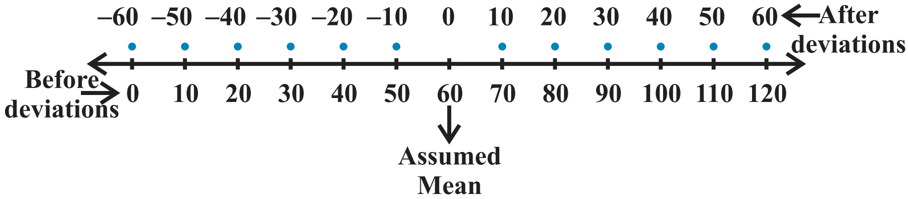

Fig 13.3

If there is a common factor of all the deviations, we divide them by this common factor to further simplify the deviations. These are known as step-deviations. The process of taking step-deviations is the change of scale on the number line as shown in Fig 13.4

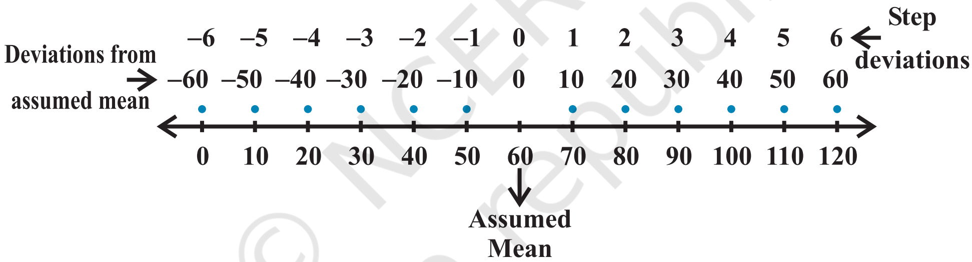

Fig 13.4

The deviations and step-deviations reduce the size of the observations, so that the computations viz. multiplication, etc., become simpler. Let, the new variable be denoted by $d_i = \frac{x_i - a}{h}$, where 'a' is the assumed mean and h is the common factor. Then, the mean $\bar{x}$ by step-deviation method is given by

$$\bar{x} = a + \frac{\sum_{i=1}^{n} f_i \, d_i}{\mathrm{N}} \times h$$

Let us take the data of Example 6 and find the mean deviation by using step-deviation method.

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Take the assumed mean $a = 45$ and $h = 10$, and form the following Table 13.5.

Table 13.5

Marks obtained	Number of students	Mid-points	$d_i = \frac{x_i - 45}{10}$	$f_i d_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
	f_i	x_i				
10-20	2	15	– 3	– 6	30	60
20-30	3	25	– 2	– 6	20	60
30-40	8	35	– 1	– 8	10	80
40-50	14	45	0	0	0	0
50-60	8	55	1	8	10	80
60-70	3	65	2	6	20	60
70-80	2	75	3	6	30	60
	40			0		400

Therefore
$$\bar{x} = a + \frac{\sum_{i=1}^{7} f_i d_i}{N} \times h$$

$$= 45 + \frac{0}{40} \times 10 = 45$$

and
$$\text{M.D. } (\bar{x}) = \frac{1}{N} \sum_{i=1}^{7} f_i \lvert x_i - \bar{x} \rvert = \frac{400}{40} = 10$$

Note The step deviation method is applied to compute $\bar{x}$. Rest of the procedure is same.

(ii) Mean deviation about median The process of finding the mean deviation about median for a continuous frequency distribution is similar as we did for mean deviation about the mean. The only difference lies in the replacement of the mean by median while taking deviations.

Let us recall the process of finding median for a continuous frequency distribution.

The data is first arranged in ascending order. Then, the median of continuous frequency distribution is obtained by first identifying the class in which median lies (median class) and then applying the formula

$$\text{Median} = l + \frac{\frac{N}{2} - C}{f} \times h$$

where median class is the class interval whose cumulative frequency is just greater than or equal to $\frac{N}{2}$, N is the sum of frequencies, l, f, h and C are, respectively the lower limit , the frequency, the width of the median class and C the cumulative frequency of the class just preceding the median class. After finding the median, the absolute values of the deviations of mid-point x_i of each class from the median i.e., $|x_i - M|$ are obtained.

Then $$\text{M.D. (M)} = \frac{1}{N} \sum_{i=1}^{n} f_i |x_i - M|$$

The process is illustrated in the following example:

Example 7 Calculate the mean deviation about median for the following data :

Class	0-10	10-20	20-30	30-40	40-50	50-60
Frequency	6	7	15	16	4	2

Solution Form the following Table 13.6 from the given data :

Table 13.6

Class	Frequency f_i	Cumulative frequency $(c.f.)$	Mid-points x_i	$\lvert x_i - \text{Med.}\rvert$	$f_i \lvert x_i - \text{Med.}\rvert$
0-10	6	6	5	23	138
10-20	7	13	15	13	91
20-30	15	28	25	3	45
30-40	16	44	35	7	112
40-50	4	48	45	17	68
50-60	2	50	55	27	54
	50				508

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The class interval containing $\left(\frac{N}{2}\right)^{th}$ or 25th item is 20-30. Therefore, 20–30 is the median class. We know that

$$\text{Median} = l + \frac{\frac{N}{2} - C}{f} \times h$$

Here $l = 20$, C = 13, f = 15, h = 10 and N = 50

Therefore, $$\text{Median} = 20 + \frac{25 - 13}{15} \times 10 = 20 + 8 = 28$$

Thus, Mean deviation about median is given by

$$\text{M.D. (M)} = \frac{1}{N}\sum_{i=1}^{6} f_i |x_i - M| = \frac{1}{50} \times 508 = 10.16$$

EXERCISE 13.1

Find the mean deviation about the mean for the data in Exercises 1 and 2.

1. 4, 7, 8, 9, 10, 12, 13, 17

2. 38, 70, 48, 40, 42, 55, 63, 46, 54, 44

Find the mean deviation about the median for the data in Exercises 3 and 4.

3. 13, 17, 16, 14, 11, 13, 10, 16, 11, 18, 12, 17

4. 36, 72, 46, 42, 60, 45, 53, 46, 51, 49

Find the mean deviation about the mean for the data in Exercises 5 and 6.

5.

x_i	5	10	15	20	25
f_i	7	4	6	3	5

6.

x_i	10	30	50	70	90
f_i	4	24	28	16	8

Find the mean deviation about the median for the data in Exercises 7 and 8.

7.

x_i	5	7	9	10	12	15
f_i	8	6	2	2	2	6

8.

x_i	15	21	27	30	35
f_i	3	5	6	7	8

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Find the mean deviation about the mean for the data in Exercises 9 and 10.

9.

Income per day in ₹	0-100	100-200	200-300	300-400	400-500	500-600	600-700	700-800
Number of persons	4	8	9	10	7	5	4	3

10.

Height in cms	95-105	105-115	115-125	125-135	135-145	145-155
Number of boys	9	13	26	30	12	10

11. Find the mean deviation about median for the following data :

Marks	0-10	10-20	20-30	30-40	40-50	50-60
Number of Girls	6	8	14	16	4	2

12. Calculate the mean deviation about median age for the age distribution of 100 persons given below:

Age (in years)	16-20	21-25	26-30	31-35	36-40	41-45	46-50	51-55
Number	5	6	12	14	26	12	16	9

[**Hint** Convert the given data into continuous frequency distribution by subtracting 0.5 from the lower limit and adding 0.5 to the upper limit of each class interval]

13.4.3 ***Limitations of mean deviation*** In a series, where the degree of variability is very high, the median is not a representative central tendency. Thus, the mean deviation about median calculated for such series can not be fully relied.

The sum of the deviations from the mean (minus signs ignored) is more than the sum of the deviations from median. Therefore, the mean deviation about the mean is not very scientific.Thus, in many cases, mean deviation may give unsatisfactory results. Also mean deviation is calculated on the basis of absolute values of the deviations and therefore, cannot be subjected to further algebraic treatment. This implies that we must have some other measure of dispersion. Standard deviation is such a measure of dispersion.

13.5 Variance and Standard Deviation

Recall that while calculating mean deviation about mean or median, the absolute values of the deviations were taken. The absolute values were taken to give meaning to the mean deviation, otherwise the deviations may cancel among themselves.

Another way to overcome this difficulty which arose due to the signs of deviations, is to take squares of all the deviations. Obviously all these squares of deviations are

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non-negative. Let $x_1, x_2, x_3, ..., x_n$ be n observations and $\bar{x}$ be their mean. Then

$$(x_1-\bar{x})^2 + (x_2 - \bar{x})^2 + ...+ (x_n-\bar{x})^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2 .$$

If this sum is zero, then each $(x_i - \bar{x})$ has to be zero. This implies that there is no dispersion at all as all observations are equal to the mean $\bar{x}$.

If $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is small , this indicates that the observations $x_1, x_2, x_3,...,x_n$ are close to the mean $\bar{x}$ and therefore, there is a lower degree of dispersion. On the contrary, if this sum is large, there is a higher degree of dispersion of the observations from the mean $\bar{x}$. Can we thus say that the sum $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is a reasonable indicator of the degree of dispersion or scatter?

Let us take the set A of six observations 5, 15, 25, 35, 45, 55. The mean of the observations is $\bar{x}$ = 30. The sum of squares of deviations from $\bar{x}$ for this set is

$$\sum_{i=1}^{6}(x_i - \bar{x})^2 = (5–30)^2 + (15–30)^2 + (25–30)^2 + (35–30)^2 + (45–30)^2 +(55–30)^2$$

$$= 625 + 225 + 25 + 25 + 225 + 625 = 1750$$

Let us now take another set B of 31 observations 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45. The mean of these observations is $\bar{y}$ = 30

Note that both the sets A and B of observations have a mean of 30.

Now, the sum of squares of deviations of observations for set B from the mean $\bar{y}$ is given by

$$\sum_{i=1}^{31}(y_i - \bar{y})^2 = (15–30)^2 +(16–30)^2 + (17–30)^2 + ...+ (44–30)^2 +(45–30)^2$$

$$= (–15)^2 +(–14)^2 + ...+ (–1)^2 + 0^2 + 1^2 + 2^2 + 3^2 + ...+ 14^2 + 15^2$$

$$= 2\,[15^2 + 14^2 + ... + 1^2]$$

$$= 2\times\frac{15\times(15+1)\,(30+1)}{6} = 5 \times 16 \times 31 = 2480$$

(Because sum of squares of first n natural numbers = $\frac{n\,(n+1)\,(2n+1)}{6}$. Here n = 15)

If $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is simply our measure of dispersion or scatter about mean, we will tend to say that the set A of six observations has a lesser dispersion about the mean than the set B of 31 observations, even though the observations in set A are more scattered from the mean (the range of deviations being from –25 to 25) than in the set B (where the range of deviations is from –15 to 15).

This is also clear from the following diagrams.

For the set A, we have

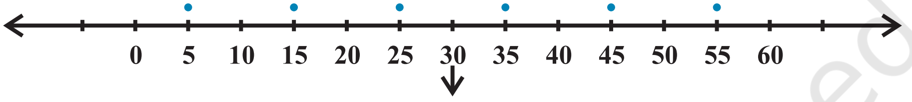

Fig 13.5

For the set B, we have

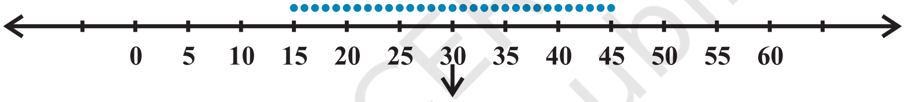

Fig 13.6

Thus, we can say that the sum of squares of deviations from the mean is not a proper measure of dispersion. To overcome this difficulty we take the mean of the squares of the deviations, i.e., we take $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$. In case of the set A, we have

Mean $= \frac{1}{6} \times 1750 = 291.67$ and in case of the set B, it is $\frac{1}{31} \times 2480 = 80$.

This indicates that the scatter or dispersion is more in set A than the scatter or dispersion in set B, which confirms with the geometrical representation of the two sets.

Thus, we can take $\frac{1}{n}\sum(x_i - \bar{x})^2$ as a quantity which leads to a proper measure of dispersion. This number, i.e., mean of the squares of the deviations from mean is called the ***variance*** and is denoted by σ^2 (read as sigma square). Therefore, the variance of n observations $x_1, x_2,..., x_n$ is given by

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

13.5.1 ***Standard Deviation*** In the calculation of variance, we find that the units of individual observations x_i and the unit of their mean $\bar{x}$ are different from that of variance, since variance involves the sum of squares of $(x_i - \bar{x})$. For this reason, the proper measure of dispersion about the mean of a set of observations is expressed as positive square-root of the variance and is called *standard deviation*. Therefore, the standard deviation, usually denoted by σ, is given by

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} \qquad \dots (1)$$

Let us take the following example to illustrate the calculation of variance and hence, standard deviation of ungrouped data.

Example 8 Find the variance of the following data:

6, 8, 10, 12, 14, 16, 18, 20, 22, 24

Solution From the given data we can form the following Table 13.7. The mean is calculated by step-deviation method taking 14 as assumed mean. The number of observations is $n = 10$

Table 13.7

x_i	$d_i = \frac{x_i - 14}{2}$	Deviations from mean $(x_i - \bar{x})$	$(x_i - \bar{x})$
6	–4	–9	81
8	–3	–7	49
10	–2	–5	25
12	–1	–3	9
14	0	–1	1
16	1	1	1
18	2	3	9
20	3	5	25
22	4	7	49
24	5	9	81
	5		330

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Therefore Mean $\bar{x}$ = assumed mean + $\dfrac{\sum_{i=1}^{n} d_i}{n} \times h$ = $14 + \dfrac{5}{10} \times 2 = 15$

and Variance $(\sigma^2) = \dfrac{1}{n}\sum_{i=1}^{10}(x_i - \bar{x})^2 = \dfrac{1}{10} \times 330 = 33$

Thus Standard deviation $(\sigma) = \sqrt{33} = 5.74$

13.5.2 *Standard deviation of a discrete frequency distribution* Let the given discrete frequency distribution be

$$x: \quad x_1, \quad x_2, \quad x_3, \ldots, x_n$$
$$f: \quad f_1, \quad f_2, \quad f_3, \ldots, f_n$$

In this case standard deviation $(\sigma) = \sqrt{\dfrac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}$ ... (2)

where $N = \sum_{i=1}^{n} f_i$.

Let us take up following example.

Example 9 Find the variance and standard deviation for the following data:

x_i	4	8	11	17	20	24	32
f_i	3	5	9	5	4	3	1

Solution Presenting the data in tabular form (Table 13.8), we get

Table 13.8

x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
4	3	12	–10	100	300
8	5	40	–6	36	180
11	9	99	–3	9	81
17	5	85	3	9	45
20	4	80	6	36	144
24	3	72	10	100	300
32	1	32	18	324	324
	30	420			1374

$$\mathrm{N} = 30, \ \sum_{i=1}^{7} f_i x_i = 420, \ \sum_{i=1}^{7} f_i \left(x_i - \bar{x}\right)^2 = 1374$$

Therefore
$$\bar{x} = \frac{\sum_{i=1}^{7} f_i x_i}{\mathrm{N}} = \frac{1}{30} \times 420 = 14$$

Hence
$$\text{variance } (\sigma^2) = \frac{1}{\mathrm{N}} \sum_{i=1}^{7} f_i (x_i - \bar{x})^2$$
$$= \frac{1}{30} \times 1374 = 45.8$$

and
$$\text{Standard deviation } (\sigma) = \sqrt{45.8} = 6.77$$

13.5.3 *Standard deviation of a continuous frequency distribution* The given continuous frequency distribution can be represented as a discrete frequency distribution by replacing each class by its mid-point. Then, the standard deviation is calculated by the technique adopted in the case of a discrete frequency distribution.

If there is a frequency distribution of n classes each class defined by its mid-point x_i with frequency f_i, the standard deviation will be obtained by the formula

$$\sigma = \sqrt{\frac{1}{\mathrm{N}} \sum_{i=1}^{n} f_i (x_i - \bar{x})^2},$$

where $\bar{x}$ is the mean of the distribution and $\mathrm{N} = \sum_{i=1}^{n} f_i$.

Another formula for standard deviation We know that

$$\text{Variance } (\sigma^2) = \frac{1}{\mathrm{N}} \sum_{i=1}^{n} f_i (x_i - \bar{x})^2 = \frac{1}{\mathrm{N}} \sum_{i=1}^{n} f_i (x_i^2 + \bar{x}^2 - 2\bar{x}\, x_i)$$

$$= \frac{1}{\mathrm{N}} \left[\sum_{i=1}^{n} f_i x_i^2 + \sum_{i=1}^{n} \bar{x}^2 f_i - \sum_{i=1}^{n} 2\bar{x}\, f_i x_i \right]$$

$$= \frac{1}{\mathrm{N}} \left[\sum_{i=1}^{n} f_i x_i^2 + \bar{x}^2 \sum_{i=1}^{n} f_i - 2\bar{x} \sum_{i=1}^{n} x_i f_i \right]$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 + \bar{x}^2\,N - 2\bar{x}\,.\,N\,\bar{x} \quad \left[\text{Here } \frac{1}{N}\sum_{i=1}^{n} x_i f_i = \bar{x} \text{ or } \sum_{i=1}^{n} x_i f_i = N\bar{x}\right]$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 + \bar{x}^2 - 2\bar{x}^2 = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \bar{x}^2$$

or $$\sigma^2 = \frac{1}{N}\sum_{i-1}^{n} f_i x_i^2 - \left(\frac{\sum_{i=1}^{n} f_i x_i}{N}\right)^2 = \frac{1}{N^2}\left[N\sum_{i=1}^{n} f_i x_i^2 - \left(\sum_{i=1}^{n} f_i x_i\right)^2\right]$$

Thus, standard deviation $(\sigma) = \frac{1}{N}\sqrt{N\sum_{i=1}^{n} f_i x_i^2 - \left(\sum_{i=1}^{n} f_i x_i\right)^2}$ ... (3)

Example 10 Calculate the mean, variance and standard deviation for the following distribution :

Class	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

Solution From the given data, we construct the following Table 13.9.

Table 13.9

Class	Frequency (f_i)	Mid-point (x_i)	$f_i x_i$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
30-40	3	35	105	729	2187
40-50	7	45	315	289	2023
50-60	12	55	660	49	588
60-70	15	65	975	9	135
70-80	8	75	600	169	1352
80-90	3	85	255	529	1587
90-100	2	95	190	1089	2178
	50		3100		10050

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Thus $\qquad \text{Mean } \bar{x} = \frac{1}{\text{N}} \sum_{i=1}^{7} f_i x_i = \frac{3100}{50} = 62$

$$\text{Variance } \left(\sigma^2\right) = \frac{1}{\text{N}} \sum_{i=1}^{7} f_i (x_i - \bar{x})^2$$

$$= \frac{1}{50} \times 10050 = 201$$

and $\qquad \text{Standard deviation } (\sigma) = \sqrt{201} = 14.18$

Example 11 Find the standard deviation for the following data :

x_i	3	8	13	18	23
f_i	7	10	15	10	6

Solution Let us form the following Table 13.10:

Table 13.10

x_i	f_i	$f_i x_i$	x_i^2	$f_i x_i^2$
3	7	21	9	63
8	10	80	64	640
13	15	195	169	2535
18	10	180	324	3240
23	6	138	529	3174
	48	614		9652

Now, by formula (3), we have

$$\sigma = \frac{1}{\text{N}} \sqrt{\text{N} \sum f_i x_i^2 - \left(\sum f_i x_i\right)^2}$$

$$= \frac{1}{48} \sqrt{48 \times 9652 - (614)^2}$$

$$= \frac{1}{48} \sqrt{463296 - 376996}$$

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$$= \frac{1}{48} \times 293.77 = 6.12$$

Therefore, Standard deviation $(\sigma) = 6.12$

13.5.4. ***Shortcut method to find variance and standard deviation*** Sometimes the values of x_i in a discrete distribution or the mid points x_i of different classes in a continuous distribution are large and so the calculation of mean and variance becomes tedious and time consuming. By using step-deviation method, it is possible to simplify the procedure.

Let the assumed mean be 'A' and the scale be reduced to $\frac{1}{h}$ times (h being the width of class-intervals). Let the step-deviations or the new values be y_i.

i.e. $$y_i = \frac{x_i - \text{A}}{h} \text{ or } x_i = \text{A} + hy_i \quad \text{... (1)}$$

We know that $$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\text{N}} \quad \text{... (2)}$$

Replacing x_i from (1) in (2), we get

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i(\text{A} + hy_i)}{\text{N}}$$

$$= \frac{1}{\text{N}}\left(\sum_{i=1}^{n} f_i\text{A} + \sum_{i=1}^{n} h\, f_i y_i\right) = \frac{1}{\text{N}}\left(\text{A}\sum_{i=1}^{n} f_i + h\sum_{i=1}^{n} f_i y_i\right)$$

$$= \text{A}\,.\frac{\text{N}}{\text{N}} + h\frac{\sum_{i=1}^{n} f_i y_i}{\text{N}} \qquad \left(\text{because} \sum_{i=1}^{n} f_i = \text{N}\right)$$

Thus $$\bar{x} = \text{A} + h\,\bar{y} \quad \text{... (3)}$$

Now Variance of the variable x, $$\sigma_x^{\,2} = \frac{1}{\text{N}}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2$$

$$= \frac{1}{\text{N}}\sum_{i=1}^{n} f_i\,(\text{A} + hy_i - \text{A} - h\,\bar{y})^2 \qquad \text{(Using (1) and (3))}$$

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$$= \frac{1}{N}\sum_{i=1}^{n} f_i\, h^2\, (y_i - \bar{y})^2$$

$$= \frac{h^2}{N}\sum_{i=1}^{n} f_i\, (y_i - \bar{y})^2 = h^2 \times \text{ variance of the variable } y_i$$

i.e. $\sigma_x^{\ 2} = h^2 \sigma_y^{\ 2}$

or $\sigma_x = h\sigma_y$ ... (4)

From (3) and (4), we have

$$\sigma_x = \frac{h}{N}\sqrt{N\sum_{i=1}^{n} f_i y_i^{\ 2} - \left(\sum_{i=1}^{n} f_i y_i\right)^2} \quad \text{... (5)}$$

Let us solve Example 11 by the short-cut method and using formula (5)

Examples 12 Calculate mean, variance and standard deviation for the following distribution.

Classes	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

Solution Let the assumed mean A = 65. Here h = 10

We obtain the following Table 13.11 from the given data :

Table 13.11

Class	Frequency f_i	Mid-point x_i	$y_i = \frac{x_i - 65}{10}$	$y_i^{\ 2}$	$f_i\, y_i$	$f_i y_i^{\ 2}$
30-40	3	35	– 3	9	– 9	27
40-50	7	45	– 2	4	– 14	28
50-60	12	55	– 1	1	– 12	12
60-70	15	65	0	0	0	0
70-80	8	75	1	1	8	8
80-90	3	85	2	4	6	12
9 0-100	2	95	3	9	6	18
	N=50				– 15	105

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Therefore $\bar{x} = A + \frac{\sum f_i y_i}{50} \times h = 65 - \frac{15}{50} \times 10 = 62$

Variance $\sigma^2 = \frac{h^2}{N^2}\left[N\Sigma f_i y_i^2 - \left(\Sigma f_i y_i\right)^2\right]$

$$= \frac{(10)^2}{(50)^2}\left[50 \times 105 - (-15)^2\right]$$

$$= \frac{1}{25}[5250 - 225] = 201$$

and standard deviation $(\sigma) = \sqrt{201} = 14.18$

EXERCISE 13.2

Find the mean and variance for each of the data in Exercies 1 to 5.

1. 6, 7, 10, 12, 13, 4, 8, 12

2. First n natural numbers

3. First 10 multiples of 3

4.

x_i	6	10	14	18	24	28	30
f_i	2	4	7	12	8	4	3

5.

x_i	92	93	97	98	102	104	109
f_i	3	2	3	2	6	3	3

6. Find the mean and standard deviation using short-cut method.

x_i	60	61	62	63	64	65	66	67	68
f_i	2	1	12	29	25	12	10	4	5

Find the mean and variance for the following frequency distributions in Exercises 7 and 8.

7.

Classes	0-30	30-60	60-90	90-120	120-150	150-180	180-210
Frequencies	2	3	5	10	3	5	2

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8.

Classes	0-10	10-20	20-30	30-40	40-50
Frequencies	5	8	15	16	6

9. Find the mean, variance and standard deviation using short-cut method

Height in cms	70-75	75-80	80-85	85-90	90-95	95-100	100-105	105-110	110-115
No. of children	3	4	7	7	15	9	6	6	3

10. The diameters of circles (in mm) drawn in a design are given below:

Diameters	33-36	37-40	41-44	45-48	49-52
No. of circles	15	17	21	22	25

Calculate the standard deviation and mean diameter of the circles.

[**Hint** First make the data continuous by making the classes as 32.5-36.5, 36.5-40.5, 40.5-44.5, 44.5 - 48.5, 48.5 - 52.5 and then proceed.]

Miscellaneous Examples

Example 13 The variance of 20 observations is 5. If each observation is multiplied by 2, find the new variance of the resulting observations.

Solution Let the observations be $x_1, x_2, ..., x_{20}$ and $\overline{x}$ be their mean. Given that variance = 5 and n = 20. We know that

$$\text{Variance}\left(\sigma^2\right) = \frac{1}{n}\sum_{i=1}^{20}(x_i - \overline{x})^2 \text{, i.e., } 5 = \frac{1}{20}\sum_{i=1}^{20}(x_i - \overline{x})^2$$

or $$\sum_{i=1}^{20}(x_i - \overline{x})^2 = 100$$

If each observation is multiplied by 2, and the new resulting observations are y_i, then

$$y_i = 2x_i \text{ i.e., } x_i = \frac{1}{2}y_i \quad \text{... (1)}$$

Therefore $\quad \bar{y} = \frac{1}{n}\sum_{i=1}^{20} y_i = \frac{1}{20}\sum_{i=1}^{20} 2x_i = 2 \cdot \frac{1}{20}\sum_{i=1}^{20} x_i$

i.e. $\quad \bar{y} = 2\bar{x}$ or $\bar{x} = \frac{1}{2}\bar{y}$

Substituting the values of x_i and $\bar{x}$ in (1), we get

$$\sum_{i=1}^{20}\left(\frac{1}{2}y_i - \frac{1}{2}\bar{y}\right)^2 = 100 \text{, i.e., } \sum_{i=1}^{20}(y_i - \bar{y})^2 = 400$$

Thus the variance of new observations $= \frac{1}{20} \times 400 = 20 = 2^2 \times 5$

Note The reader may note that if each observation is multiplied by a constant k, the variance of the resulting observations becomes k^2 times the original variance.

Example14 The mean of 5 observations is 4.4 and their variance is 8.24. If three of the observations are 1, 2 and 6, find the other two observations.

Solution Let the other two observations be x and y.
Therefore, the series is 1, 2, 6, x, y.

Now $\quad$ Mean $\bar{x} = 4.4 = \frac{1+2+6+x+y}{5}$

or $\quad 22 = 9 + x + y$

Therefore $\quad x + y = 13 \quad$ ... (1)

Also $\quad$ variance $= 8.24 = \frac{1}{n}\sum_{i=1}^{5}(x_i - \bar{x})^2$

i.e. $\quad 8.24 = \frac{1}{5}\left[(3.4)^2 + (2.4)^2 + (1.6)^2 + x^2 + y^2 - 2\times 4.4\,(x+y) + 2\times(4.4)^2\right]$

or $41.20 = 11.56 + 5.76 + 2.56 + x^2 + y^2 - 8.8 \times 13 + 38.72$

Therefore $\quad x^2 + y^2 = 97 \quad$ ... (2)

But from (1), we have

$x^2 + y^2 + 2xy = 169 \quad$ ... (3)

From (2) and (3), we have

$2xy = 72 \quad$ ... (4)

Subtracting (4) from (2), we get

$$x^2 + y^2 - 2xy = 97 - 72 \text{ i.e. } (x - y)^2 = 25$$

or $$x - y = \pm 5 \quad \text{... (5)}$$

So, from (1) and (5), we get

$x = 9, y = 4$ when $x - y = 5$

or $x = 4, y = 9$ when $x - y = -5$

Thus, the remaining observations are 4 and 9.

Example 15 If each of the observation $x_1, x_2, ..., x_n$ is increased by 'a', where a is a negative or positive number, show that the variance remains unchanged.

Solution Let $\bar{x}$ be the mean of $x_1, x_2, ..., x_n$. Then the variance is given by

$$\sigma_1^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

If 'a is added to each observation, the new observations will be

$$y_i = x_i + a \quad \text{... (1)}$$

Let the mean of the new observations be $\bar{y}$. Then

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\sum_{i=1}^{n}(x_i + a)$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} a\right] = \frac{1}{n}\sum_{i=1}^{n} x_i + \frac{na}{n} = \bar{x} + a$$

i.e. $$\bar{y} = \bar{x} + a \quad \text{... (2)}$$

Thus, the variance of the new observations

$$\sigma_2^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i + a - \bar{x} - a)^2 \quad \text{[Using (1) and (2)]}$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sigma_1^2$$

Thus, the variance of the new observations is same as that of the original observations.

Note We may note that adding (or subtracting) a positive number to (or from) each observation of a group does not affect the variance.

Example 16 The mean and standard deviation of 100 observations were calculated as 40 and 5.1, respectively by a student who took by mistake 50 instead of 40 for one observation. What are the correct mean and standard deviation?

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Solution Given that number of observations (n) = 100

Incorrect mean ($\bar{x}$) = 40,

Incorrect standard deviation (σ) = 5.1

We know that $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

i.e. $40 = \frac{1}{100}\sum_{i=1}^{100} x_i$ or $\sum_{i=1}^{100} x_i = 4000$

i.e. Incorrect sum of observations = 4000

Thus the correct sum of observations = Incorrect sum – 50 + 40

$= 4000 - 50 + 40 = 3990$

Hence Correct mean $= \frac{\text{correct sum}}{100} = \frac{3990}{100} = 39.9$

Also Standard deviation $\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{n} x_i\right)^2}$

$$= \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\bar{x})^2}$$

i.e. $5.1 = \sqrt{\frac{1}{100} \times \text{Incorrect} \sum_{i=1}^{n} x_i^2 - (40)^2}$

or $26.01 = \frac{1}{100} \times \text{Incorrect} \sum_{i=1}^{n} x_i^2 - 1600$

Therefore Incorrect $\sum_{i=1}^{n} x_i^2 = 100\,(26.01 + 1600) = 162601$

Now Correct $\sum_{i=1}^{n} x_i^2 = \text{Incorrect} \sum_{i=1}^{n} x_i^2 - (50)^2 + (40)^2$

$= 162601 - 2500 + 1600 = 161701$

Therefore Correct standard deviation

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$$= \sqrt{\frac{\text{Correct } \sum x_i^2}{n} - (\text{Correct mean})^2}$$

$$= \sqrt{\frac{161701}{100} - (39.9)^2}$$

$$= \sqrt{1617.01 - 1592.01} = \sqrt{25} = 5$$

Miscellaneous Exercise On Chapter 13

1. The mean and variance of eight observations are 9 and 9.25, respectively. If six of the observations are 6, 7, 10, 12, 12 and 13, find the remaining two observations.

2. The mean and variance of 7 observations are 8 and 16, respectively. If five of the observations are 2, 4, 10, 12, 14. Find the remaining two observations.

3. The mean and standard deviation of six observations are 8 and 4, respectively. If each observation is multiplied by 3, find the new mean and new standard deviation of the resulting observations.

4. Given that $\bar{x}$ is the mean and σ^2 is the variance of n observations $x_1, x_2, ..., x_n$. Prove that the mean and variance of the observations $ax_1, ax_2, ax_3, ..., ax_n$ are $a\bar{x}$ and $a^2 \sigma^2$, respectively, $(a \neq 0)$.

5. The mean and standard deviation of 20 observations are found to be 10 and 2, respectively. On rechecking, it was found that an observation 8 was incorrect. Calculate the correct mean and standard deviation in each of the following cases:
(i) If wrong item is omitted. (ii) If it is replaced by 12.

6. The mean and standard deviation of a group of 100 observations were found to be 20 and 3, respectively. Later on it was found that three observations were incorrect, which were recorded as 21, 21 and 18. Find the mean and standard deviation if the incorrect observations are omitted.

Summary

- **Measures of dispersion** Range, Quartile deviation, mean deviation, variance, standard deviation are measures of dispersion.
Range = Maximum Value – Minimum Value
- **Mean deviation for ungrouped data**

$$\text{M.D.}(\bar{x}) = \frac{\sum |x_i - \bar{x}|}{n}, \quad \text{M.D. (M)} = \frac{\sum |x_i - \text{M}|}{n}$$

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◆ **Mean deviation for grouped data**

$$\text{M.D.}(\bar{x}) = \frac{\sum f_i |x_i \quad \bar{x}|}{\text{N}}, \qquad \text{M.D.}(\text{M}) = \frac{\sum f_i |x_i \quad \text{M}|}{\text{N}}, \text{ where N} = \sum f_i$$

◆ **Variance and standard deviation for ungrouped data**

$$\sigma^2 = \frac{1}{n} \sum (x_i - \bar{x})^2, \qquad \sigma = \sqrt{\frac{1}{n} \sum (x_i - \bar{x})^2}$$

◆ **Variance and standard deviation of a discrete frequency distribution**

$$\sigma^2 = \frac{1}{\text{N}} \sum f_i (x_i - \bar{x})^2, \qquad \sigma = \sqrt{\frac{1}{\text{N}} \sum f_i (x_i - \bar{x})^2}$$

◆ **Variance and standard deviation of a continuous frequency distribution**

$$\sigma^2 = \frac{1}{\text{N}} \sum f_i (x_i - \bar{x})^2, \qquad \sigma = \frac{1}{\text{N}} \sqrt{\text{N} \sum f_i x_i^2 - \left(\sum f_i x_i\right)^2}$$

◆ **Shortcut method to find variance and standard deviation.**

$$\sigma^2 = \frac{h^2}{\text{N}^2} \left[\text{N} \sum f_i y_i^2 - \left(\sum f_i y_i\right)^2\right], \sigma = \frac{h}{\text{N}} \sqrt{\text{N} \sum f_i y_i^2 - \left(\sum f_i y_i\right)^2},$$

where $y_i = \dfrac{x_i - \text{A}}{h}$

Historical Note

'Statistics' is derived from the Latin word 'status' which means a political state. This suggests that statistics is as old as human civilisation. In the year 3050 B.C., perhaps the first census was held in Egypt. In India also, about 2000 years ago, we had an efficient system of collecting administrative statistics, particularly, during the regime of Chandra Gupta Maurya (324-300 B.C.). The system of collecting data related to births and deaths is mentioned in Kautilya's *Arthshastra* (around 300 B.C.) A detailed account of administrative surveys conducted during Akbar's regime is given in *Ain-I-Akbari* written by Abul Fazl.

Captain John Graunt of London (1620-1674) is known as father of vital statistics due to his studies on statistics of births and deaths. Jacob Bernoulli (1654-1705) stated the Law of Large numbers in his book "Ars Conjectandi', published in 1713.

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The theoretical development of statistics came during the mid seventeenth century and continued after that with the introduction of theory of games and chance (i.e., probability). Francis Galton (1822-1921), an Englishman, pioneered the use of statistical methods, in the field of Biometry. Karl Pearson (1857-1936) contributed a lot to the development of statistical studies with his discovery of *Chi square test* and foundation of *statistical laboratory* in England (1911). Sir Ronald A. Fisher (1890-1962), known as the Father of modern statistics, applied it to various diversified fields such as Genetics, Biometry, Education, Agriculture, etc.

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Chapter 14

PROBABILITY

❖ *Where a mathematical reasoning can be had, it is as great a folly to make use of any other, as to grope for a thing in the dark, when you have a candle in your hand. – JOHN ARBUTHNOT* ❖

14.1 Event

We have studied about random experiment and sample space associated with an experiment. The sample space serves as an universal set for all questions concerned with the experiment.

Consider the experiment of tossing a coin two times. An associated sample space is S = {HH, HT, TH, TT}.

Now suppose that we are interested in those outcomes which correspond to the occurrence of exactly one head. We find that HT and TH are the only elements of S corresponding to the occurrence of this happening (event). These two elements form the set E = { HT, TH}

We know that the set E is a subset of the sample space S . Similarly, we find the following correspondence between events and subsets of S.

Description of events	Corresponding subset of 'S'
Number of tails is exactly 2	A = {TT}
Number of tails is atleast one	B = {HT, TH, TT}
Number of heads is atmost one	C = {HT, TH, TT}
Second toss is not head	D = { HT, TT}
Number of tails is atmost two	S = {HH, HT, TH, TT}
Number of tails is more than two	ϕ

The above discussion suggests that a subset of sample space is associated with an event and an event is associated with a subset of sample space. In the light of this we define an event as follows.

Definition Any subset E of a sample space S is called *an event*.

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14.1.1 *Occurrence of an event* Consider the experiment of throwing a die. Let E denotes the event " a number less than 4 appears". If actually '1' had appeared on the die then we say that event E has occurred. As a matter of fact if outcomes are 2 or 3, we say that event E has occurred

Thus, the event E of a sample space S is said to have occurred if the outcome ω of the experiment is such that $\omega \in$ E. If the outcome ω is such that $\omega \notin$ E, we say that the event E has not occurred.

14.1.2 *Types of events* Events can be classified into various types on the basis of the elements they have.

1. Impossible and Sure Events The empty set ϕ and the sample space S describe events. In fact ϕ is called an *impossible event* and S, i.e., the whole sample space is called the *sure event*.

To understand these let us consider the experiment of rolling a die. The associated sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let E be the event " the number appears on the die is a multiple of 7". Can you write the subset associated with the event E?

Clearly no outcome satisfies the condition given in the event, i.e., no element of the sample space ensures the occurrence of the event E. Thus, we say that the empty set only correspond to the event E. In other words we can say that it is impossible to have a multiple of 7 on the upper face of the die. Thus, the event E = ϕ is an impossible event.

Now let us take up another event F "the number turns up is odd or even". Clearly F = {1, 2, 3, 4, 5, 6,} = S, i.e., all outcomes of the experiment ensure the occurrence of the event F. Thus, the event F = S is a sure event.

2. Simple Event If an event E has only one sample point of a sample space, it is called a *simple* (or *elementary*) *event.*

In a sample space containing *n* distinct elements, there are exactly *n* simple events.

For example in the experiment of tossing two coins, a sample space is

$$S=\{HH, HT, TH, TT\}$$

There are four simple events corresponding to this sample space. These are

$$E_1= \{HH\},\ E_2=\{HT\},\ E_3= \{ TH\} \text{ and } E_4=\{TT\}.$$

3. Compound Event If an event has more than one sample point, it is called a *Compound event*.

For example, in the experiment of "tossing a coin thrice" the events

E: 'Exactly one head appeared'

F: 'Atleast one head appeared'

G: 'Atmost one head appeared' etc.

are all compound events. The subsets of S associated with these events are

E={HTT,THT,TTH}

F={HTT,THT, TTH, HHT, HTH, THH, HHH}

G= {TTT, THT, HTT, TTH}

Each of the above subsets contain more than one sample point, hence they are all compound events.

14.1.3 *Algebra of events* In the Chapter on Sets, we have studied about different ways of combining two or more sets, viz, union, intersection, difference, complement of a set etc. Like-wise we can combine two or more events by using the analogous set notations.

Let A, B, C be events associated with an experiment whose sample space is S.

1. Complementary Event For every event A, there corresponds another event A' called the complementary event to A. It is also called the *event 'not* A'.

For example, take the experiment 'of tossing three coins'. An associated sample space is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Let A={HTH, HHT, THH} be the event 'only one tail appears'

Clearly for the outcome HTT, the event A has not occurred. But we may say that the event 'not A' has occurred. Thus, with every outcome which is not in A, we say that 'not A' occurs.

Thus the complementary event 'not A' to the event A is

$$A' = \{HHH, HTT, THT, TTH, TTT\}$$

or $$A' = \{\omega : \omega \in S \text{ and } \omega \notin A\} = S - A.$$

2. The Event 'A or B' Recall that union of two sets A and B denoted by $A \cup B$ contains all those elements which are either in A or in B or in both.

When the sets A and B are two events associated with a sample space, then '$A \cup B$' is the event 'either A or B or both'. This event '$A \cup B$' is also called 'A or B'.

Therefore $$\text{Event 'A or B'} = A \cup B$$
$$= \{\omega : \omega \in A \text{ or } \omega \in B\}$$

3. The Event 'A and B' We know that intersection of two sets $A \cap B$ is the set of those elements which are common to both A and B. i.e., which belong to both 'A and B'.

If A and B are two events, then the set $A \cap B$ denotes the event 'A and B'.

Thus, $A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$

For example, in the experiment of 'throwing a die twice' Let A be the event 'score on the first throw is six' and B is the event 'sum of two scores is atleast 11' then

$$A = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}, \text{ and } B = \{(5,6), (6,5), (6,6)\}$$

so $A \cap B = \{(6,5), (6,6)\}$

Note that the set $A \cap B = \{(6,5), (6,6)\}$ may represent the event 'the score on the first throw is six and the sum of the scores is atleast 11'.

4. The Event 'A but not B' We know that A–B is the set of all those elements which are in A but not in B. Therefore, the set A–B may denote the event 'A but not B'.We know that

$$A - B = A \cap B'$$

Example 1 Consider the experiment of rolling a die. Let A be the event 'getting a prime number', B be the event 'getting an odd number'. Write the sets representing the events (i) Aor B (ii) A and B (iii) A but not B (iv) 'not A'.

Solution Here $S = \{1, 2, 3, 4, 5, 6\}, A = \{2, 3, 5\}$ and $B = \{1, 3, 5\}$

Obviously

(i) 'A or B' $= A \cup B = \{1, 2, 3, 5\}$

(ii) 'A and B' $= A \cap B = \{3,5\}$

(iii) 'A but not B' $= A - B = \{2\}$

(iv) 'not A' $= A' = \{1,4,6\}$

14.1.4 ***Mutually exclusive events*** In the experiment of rolling a die, a sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Consider events, A 'an odd number appears' and B 'an even number appears'

Clearly the event A excludes the event B and vice versa. In other words, there is no outcome which ensures the occurrence of events A and B simultaneously. Here

$A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$

Clearly $A \cap B = \phi$, i.e., A and B are disjoint sets.

In general, two events A and B are called *mutually exclusive* events if the occurrence of any one of them excludes the occurrence of the other event, i.e., if they can not occur simultaneously. In this case the sets A and B are disjoint.

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Again in the experiment of rolling a die, consider the events A 'an odd number appears' and event B 'a number less than 4 appears'

Obviously A = {1, 3, 5} and B = {1, 2, 3}

Now $3 \in A$ as well as $3 \in B$

Therefore, A and B are not mutually exclusive events.

Remark Simple events of a sample space are always mutually exclusive.

14.1.5 *Exhaustive events* Consider the experiment of throwing a die. We have S = {1, 2, 3, 4, 5, 6}. Let us define the following events

A: 'a number less than 4 appears',

B: 'a number greater than 2 but less than 5 appears'

and C: 'a number greater than 4 appears'.

Then A = {1, 2, 3}, B = {3,4} and C = {5, 6}. We observe that

$$A \cup B \cup C = \{1, 2, 3\} \cup \{3, 4\} \cup \{5, 6\} = S.$$

Such events A, B and C are called exhaustive events. In general, if $E_1, E_2, ..., E_n$ are n events of a sample space S and if

$$E_1 \cup E_2 \cup E_3 \cup ... \cup E_n = \bigcup_{i=1}^{n} E_i = S$$

then $E_1, E_2, ..., E_n$ are called *exhaustive events*.In other words, events $E_1, E_2, ..., E_n$ are said to be exhaustive if atleast one of them necessarily occurs whenever the experiment is performed.

Further, if $E_i \cap E_j = \phi$ for $i \neq j$ i.e., events E_i and E_j are pairwise disjoint and $\bigcup_{i=1}^{n} E_i = S$, then events $E_1, E_2, ..., E_n$ are called *mutually exclusive and exhaustive events*.

We now consider some examples.

Example 2 Two dice are thrown and the sum of the numbers which come up on the dice is noted. Let us consider the following events associated with this experiment

A: 'the sum is even'.

B: 'the sum is a multiple of 3'.

C: 'the sum is less than 4'.

D: 'the sum is greater than 11'.

Which pairs of these events are mutually exclusive?

Solution There are 36 elements in the sample space S = $\{(x, y): x, y = 1, 2, 3, 4, 5, 6\}$. Then

A = {(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)}

B = {(1, 2), (2, 1), (1, 5), (5, 1), (3, 3), (2, 4), (4, 2), (3, 6), (6, 3), (4, 5), (5, 4), (6, 6)}

C = {(1, 1), (2, 1), (1, 2)} and D = {(6, 6)}

We find that

$$A \cap B = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 6)\} \neq \phi$$

Therefore, A and B are not mutually exclusive events.

Similarly $A \cap C \neq \phi$, $A \cap D \neq \phi$, $B \cap C \neq \phi$ and $B \cap D \neq \phi$.

Thus, the pairs of events, (A, C), (A, D), (B, C), (B, D) are not mutually exclusive events.

Also $C \cap D = \phi$ and so C and D are mutually exclusive events.

Example 3 A coin is tossed three times, consider the following events.

A: 'No head appears', B: 'Exactly one head appears' and C: 'Atleast two heads appear'.

Do they form a set of mutually exclusive and exhaustive events?

Solution The sample space of the experiment is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

and A = {TTT}, B = {HTT, THT, TTH}, C = {HHT, HTH, THH, HHH}

Now

$A \cup B \cup C$ = {TTT, HTT, THT, TTH, HHT, HTH, THH, HHH} = S

Therefore, A, B and C are exhaustive events.

Also, $A \cap B = \phi$, $A \cap C = \phi$ and $B \cap C = \phi$

Therefore, the events are pair-wise disjoint, i.e., they are mutually exclusive.

Hence, A, B and C form a set of mutually exclusive and exhaustive events.

EXERCISE 14.1

1. A die is rolled. Let E be the event "die shows 4" and F be the event "die shows even number". Are E and F mutually exclusive?
2. A die is thrown. Describe the following events:

   (i) A: a number less than 7 (ii) B: a number greater than 7
   (iii) C: a multiple of 3 (iv) D: a number less than 4
   (v) E: an even number greater than 4 (vi) F: a number not less than 3

   Also find $A \cup B$, $A \cap B$, $B \cup C$, $E \cap F$, $D \cap E$, $A - C$, $D - E$, $E \cap F'$, F'

3. An experiment involves rolling a pair of dice and recording the numbers that come up. Describe the following events:
A: the sum is greater than 8, B: 2 occurs on either die
C: the sum is at least 7 and a multiple of 3.
Which pairs of these events are mutually exclusive?

4. Three coins are tossed once. Let A denote the event 'three heads show", B denote the event "two heads and one tail show", C denote the event" three tails show and D denote the event 'a head shows on the first coin". Which events are
(i) mutually exclusive? (ii) simple? (iii) Compound?

5. Three coins are tossed. Describe
(i) Two events which are mutually exclusive.
(ii) Three events which are mutually exclusive and exhaustive.
(iii) Two events, which are not mutually exclusive.
(iv) Two events which are mutually exclusive but not exhaustive.
(v) Three events which are mutually exclusive but not exhaustive.

6. Two dice are thrown. The events A, B and C are as follows:
A: getting an even number on the first die.
B: getting an odd number on the first die.
C: getting the sum of the numbers on the dice ≤ 5.
Describe the events
(i) A' (ii) not B (iii) A or B
(iv) A and B (v) A but not C (vi) B or C
(vii) B and C (viii) $A \cap B' \cap C'$

7. Refer to question 6 above, state true or false: (give reason for your answer)
(i) A and B are mutually exclusive
(ii) A and B are mutually exclusive and exhaustive
(iii) $A = B'$
(iv) A and C are mutually exclusive
(v) A and B' are mutually exclusive.
(vi) A', B', C are mutually exclusive and exhaustive.

14.2 Axiomatic Approach to Probability

In earlier sections, we have considered random experiments, sample space and events associated with these experiments. In our day to day life we use many words about the chances of occurrence of events. Probability theory attempts to quantify these chances of occurrence or non occurrence of events.

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In earlier classes, we have studied some methods of assigning probability to an event associated with an experiment having known the number of total outcomes.

Axiomatic approach is another way of describing probability of an event. In this approach some axioms or rules are depicted to assign probabilities.

Let S be the sample space of a random experiment. The probability P is a real valued function whose domain is the power set of S and range is the interval [0,1] satisfying the following axioms

(i) For any event E, $P(E) \geq 0$ (ii) $P(S) = 1$

(iii) If E and F are mutually exclusive events, then $P(E \cup F) = P(E) + P(F)$.

It follows from (iii) that $P(\phi) = 0$. To prove this, we take $F = \phi$ and note that E and ϕ are disjoint events. Therefore, from axiom (iii), we get

$P(E \cup \phi) = P(E) + P(\phi)$ or $P(E) = P(E) + P(\phi)$ i.e. $P(\phi) = 0$.

Let S be a sample space containing outcomes $\omega_1, \omega_2, ..., \omega_n$, i.e.,

$$S = \{\omega_1, \omega_2, ..., \omega_n\}$$

It follows from the axiomatic definition of probability that

(i) $0 \leq P(\omega_i) \leq 1$ for each $\omega_i \in S$

(ii) $P(\omega_1) + P(\omega_2) + ... + P(\omega_n) = 1$

(iii) For any event A, $P(A) = \sum P(\omega_i)$, $\omega_i \in A$.

Note It may be noted that the singleton $\{\omega_i\}$ is called elementary event and for notational convenience, we write $P(\omega_i)$ for $P(\{\omega_i\})$.

For example, in 'a coin tossing' experiment we can assign the number $\frac{1}{2}$ to each of the outcomes H and T.

i.e. $$P(H) = \frac{1}{2} \text{ and } P(T) = \frac{1}{2}$$

(1)

Clearly this assignment satisfies both the conditions i.e., each number is neither less than zero nor greater than 1 and

$$P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1$$

Therefore, in this case we can say that probability of H = $\frac{1}{2}$, and probability of T = $\frac{1}{2}$

If we take $P(H) = \frac{1}{4}$ and $P(T) = \frac{3}{4}$ ... (2)

Does this assignment satisfy the conditions of axiomatic approach?

Yes, in this case, probability of H = $\frac{1}{4}$ and probability of T = $\frac{3}{4}$.

We find that both the assignments (1) and (2) are valid for probability of H and T.

In fact, we can assign the numbers p and $(1 - p)$ to both the outcomes such that $0 \le p \le 1$ and $P(H) + P(T) = p + (1 - p) = 1$

This assignment, too, satisfies both conditions of the axiomatic approach of probability. Hence, we can say that there are many ways (rather infinite) to assign probabilities to outcomes of an experiment. We now consider some examples.

Example 4 Let a sample space be S = $\{\omega_1, \omega_2, ..., \omega_6\}$.Which of the following assignments of probabilities to each outcome are valid?

Outcomes	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
(a)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
(b)	1	0	0	0	0	0
(c)	$\frac{1}{8}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{3}$
(d)	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{3}{2}$
(e)	0.1	0.2	0.3	0.4	0.5	0.6

Solution (a) Condition (i): Each of the number p(ω_i) is positive and less than one.

Condition (ii): Sum of probabilities

$$= \frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=1$$

Therefore, the assignment is valid

(b) Condition (i): Each of the number $p(\omega_i)$ is either 0 or 1.
Condition (ii) Sum of the probabilities = 1 + 0 + 0 + 0 + 0 + 0 = 1
Therefore, the assignment is valid

(c) Condition (i) Two of the probabilities $p(\omega_5)$ and $p(\omega_6)$ are negative, the assignment is not valid

(d) Since $p(\omega_6) = \frac{3}{2} > 1$, the assignment is not valid

(e) Since, sum of probabilities = 0.1 + 0.2 + 0.3 + 0.4 + 0.5 + 0.6 = 2.1, the assignment is not valid.

14.2.1 ***Probability of an event*** Let S be a sample space associated with the experiment 'examining three consecutive pens produced by a machine and classified as Good (non-defective) and bad (defective)'. We may get 0, 1, 2 or 3 defective pens as result of this examination.

A sample space associated with this experiment is

$$S = \{BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG\},$$

where B stands for a defective or bad pen and G for a non – defective or good pen.

Let the probabilities assigned to the outcomes be as follows

Sample point:	BBB	BBG	BGB	GBB	BGG	GBG	GGB	GGG
Probability:	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Let event A: there is exactly one defective pen and event B: there are atleast two defective pens.

Hence A = {BGG, GBG, GGB} and B = {BBG, BGB, GBB, BBB}

Now $\quad P(A) = \sum P(\omega_i), \forall \omega_i \in A$

$$= P(BGG) + P(GBG) + P(GGB) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

and $\quad P(B) = \sum P(\omega_i), \forall \omega_i \in B$

$$= P(BBG) + P(BGB) + P(GBB) + P(BBB) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$$

Let us consider another experiment of 'tossing a coin "twice"

The sample space of this experiment is S = {HH, HT, TH, TT}

Let the following probabilities be assigned to the outcomes

$$P(HH) = \frac{1}{4},\ P(HT) = \frac{1}{7},\ P(TH) = \frac{2}{7},\ P(TT) = \frac{9}{28}$$

Clearly this assignment satisfies the conditions of axiomatic approach. Now, let us find the probability of the event E: 'Both the tosses yield the same result'.

Here $\quad E = \{HH, TT\}$

Now $\quad P(E) = \Sigma\, P(w_i)$, for all $w_i \in E$

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$$= P(HH) + P(TT) = \frac{1}{4} + \frac{9}{28} = \frac{4}{7}$$

For the event F: 'exactly two heads', we have F = {HH}

and $\qquad P(F) = P(HH) = \frac{1}{4}$

14.2.2 ***Probabilities of equally likely outcomes*** Let a sample space of an experiment be

$$S = \{\omega_1, \omega_2, ..., \omega_n\}.$$

Let all the outcomes are equally likely to occur, i.e., the chance of occurrence of each simple event must be same.

i.e. $\qquad P(\omega_i) = p$, for all $\omega_i \in S$ where $0 \le p \le 1$

Since $\qquad \sum_{i=1}^{n} P(\omega_i) = 1$ i.e., $p + p + ... + p$ (n times) = 1

or $\qquad np = 1$ i.e., $p = \frac{1}{n}$

Let S be a sample space and E be an event, such that $n(S) = n$ and $n(E) = m$. If each out come is equally likely, then it follows that

$$P(E) = \frac{m}{n} = \frac{\text{Number of outcomes favourable to E}}{\text{Total possible outcomes}}$$

14.2.3 ***Probability of the event 'A or B'*** Let us now find the probability of event 'A or B', i.e., P $(A \cup B)$

Let A = {HHT, HTH, THH} and B = {HTH, THH, HHH} be two events associated with 'tossing of a coin thrice'

Clearly $A \cup B$ = {HHT, HTH, THH, HHH}

Now $\qquad P(A \cup B) = P(HHT) + P(HTH) + P(THH) + P(HHH)$

If all the outcomes are equally likely, then

$$P(A \cup B) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Also $\qquad P(A) = P(HHT) + P(HTH) + P(THH) = \frac{3}{8}$

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and $\qquad P(B) = P(HTH) + P(THH) + P(HHH) = \dfrac{3}{8}$

Therefore $\qquad P(A) + P(B) = \dfrac{3}{8}+\dfrac{3}{8}=\dfrac{6}{8}$

It is clear that $\quad P(A\cup B) \neq P(A) + P(B)$

The points HTH and THH are common to both A and B. In the computation of P(A) + P(B) the probabilities of points HTH and THH, i.e., the elements of $A \cap B$ are included twice. Thus to get the probability $P(A \cup B)$ we have to subtract the probabilities of the sample points in $A \cap B$ from P(A) + P(B)

i.e.
$$P(A \cup B) = P(A) + P(B) - \sum P(\omega_i),\ \forall \omega_i \in A \cap B$$
$$= P(A) + P(B) - P(A \cap B)$$

Thus we observe that, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

In general, if A and B are any two events associated with a random experiment, then by the definition of probability of an event, we have

$$P(A \cup B) = \sum p(\omega_i),\ \forall \omega_i \in A \cup B.$$

Since $\qquad A \cup B = (A–B) \cup (A \cap B) \cup (B–A)$,

we have

$$P(A \cup B) = \left[\sum P(\omega_i)\forall \omega_i \in (A–B)\right] + \left[\sum P(\omega_i)\forall \omega_i \in A \cap B\right] + \left[\sum P(\omega_i)\forall \omega_i \in B - A\right]$$

(because A–B, $A \cap B$ and B – A are mutually exclusive) $\qquad$ ... (1)

Also $P(A) + P(B) = \left[\sum p(\omega_i)\ \forall \omega_i \in A\right] + \left[\sum p(\omega_i)\ \forall \omega_i \in B\right]$

$$= \left[\sum P(\omega_i)\forall \omega_i \in (A–B) \cup (A \cap B)\right] + \left[\sum P(\omega_i)\forall \omega_i \in (B-A) \cup (A \cap B)\right]$$

$$= \left[\sum P(\omega_i)\forall \omega_i \in (A-B)\right] + \left[\sum P(\omega_i)\forall \omega_i \in (A \cap B)\right] + \left[\sum P(\omega_i)\forall \omega_i \in (B–A)\right] +$$
$$\left[\sum P(\omega_i)\forall \omega_i \in (A \cap B)\right]$$

$$= P(A \cup B) + \left[\sum P(\omega_i)\forall \omega_i \in A \cap B\right] \quad \text{[using (1)]}$$

$$= P(A \cup B) + P(A \cap B).$$

Hence $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Alternatively, it can also be proved as follows:

$A \cup B = A \cup (B - A)$, where A and B – A are mutually exclusive,

and $\quad B = (A \cap B) \cup (B - A)$, where $A \cap B$ and B – A are mutually exclusive.

Using Axiom (iii) of probability, we get

$$P(A \cup B) = P(A) + P(B - A) \quad \text{... (2)}$$

and $$P(B) = P(A \cap B) + P(B - A) \quad \text{... (3)}$$

Subtracting (3) from (2) gives

$$P(A \cup B) - P(B) = P(A) - P(A \cap B)$$

or $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

The above result can further be verified by observing the Venn Diagram (Fig 14.1)

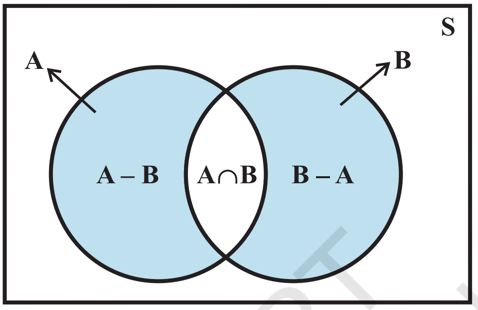

Fig 14.1

If A and B are disjoint sets, i.e., they are mutually exclusive events, then $A \cap B = \phi$

Therefore $P(A \cap B) = P(\phi) = 0$

Thus, for mutually exclusive events A and B, we have

$$P(A \cup B) = P(A) + P(B),$$

which is Axiom (iii) of probability.

14.2.4 ***Probability of event 'not A'*** Consider the event A = {2, 4, 6, 8} associated with the experiment of drawing a card from a deck of ten cards numbered from 1 to 10. Clearly the sample space is S = {1, 2, 3, ...,10}

If all the outcomes 1, 2, ...,10 are considered to be equally likely, then the probability of each outcome is $\frac{1}{10}$

Now $$P(A) = P(2) + P(4) + P(6) + P(8)$$

$$= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$$

Also event 'not A' = $A' = \{1, 3, 5, 7, 9, 10\}$

Now $$P(A') = P(1) + P(3) + P(5) + P(7) + P(9) + P(10)$$

$$= \frac{6}{10} = \frac{3}{5}$$

Thus, $\quad P(A') = \frac{3}{5} = 1 - \frac{2}{5} = 1 - P(A)$

Also, we know that A′ and A are mutually exclusive and exhaustive events i.e.,

$A \cap A' = \phi$ and $A \cup A' = S$

or $\quad P(A \cup A') = P(S)$

Now $\quad P(A) + P(A') = 1$, $\quad$ by using axioms (ii) and (iii).

or $\quad P(A') = P(\text{not } A) = 1 - P(A)$

We now consider some examples and exercises having equally likely outcomes unless stated otherwise.

Example 5 One card is drawn from a well shuffled deck of 52 cards. If each outcome is equally likely, calculate the probability that the card will be

(i) a diamond
(ii) not an ace
(iii) a black card (i.e., a club or, a spade)
(iv) not a diamond
(v) not a black card.

Solution When a card is drawn from a well shuffled deck of 52 cards, the number of possible outcomes is 52.

(i) Let A be the event 'the card drawn is a diamond'
Clearly the number of elements in set A is 13.

Therefore, $\quad P(A) = \frac{13}{52} = \frac{1}{4}$

i.e. probability of a diamond card $= \frac{1}{4}$

(ii) We assume that the event ‘Card drawn is an ace’ is B

Therefore ‘Card drawn is not an ace’ should be B′.

We know that $P(B') = 1 - P(B) = 1 - \frac{4}{52} = 1 - \frac{1}{13} = \frac{12}{13}$

(iii) Let C denote the event ‘card drawn is black card’

Therefore, number of elements in the set C = 26

i.e. $\quad P(C) = \frac{26}{52} = \frac{1}{2}$

Thus, probability of a black card = $\frac{1}{2}$.

(iv) We assumed in (i) above that A is the event 'card drawn is a diamond', so the event 'card drawn is not a diamond' may be denoted as A' or 'not A'

Now P(not A) = 1 – P(A) = $1-\frac{1}{4}=\frac{3}{4}$

(v) The event 'card drawn is not a black card' may be denoted as C′ or 'not C'.

We know that P(not C) = 1 – P(C) = $1-\frac{1}{2}=\frac{1}{2}$

Therefore, probability of not a black card = $\frac{1}{2}$

Example 6 A bag contains 9 discs of which 4 are red, 3 are blue and 2 are yellow. The discs are similar in shape and size. A disc is drawn at random from the bag. Calculate the probability that it will be (i) red, (ii) yellow, (iii) blue, (iv) not blue, (v) either red or blue.

Solution There are 9 discs in all so the total number of possible outcomes is 9. Let the events A, B, C be defined as

A: 'the disc drawn is red'

B: 'the disc drawn is yellow'

C: 'the disc drawn is blue'.

(i) The number of red discs = 4, i.e., n (A) = 4

Hence P(A) = $\frac{4}{9}$

(ii) The number of yellow discs = 2, i.e., n (B) = 2

Therefore, P(B) = $\frac{2}{9}$

(iii) The number of blue discs = 3, i.e., n(C) = 3

Therefore, P(C) = $\frac{3}{9}=\frac{1}{3}$

(iv) Clearly the event 'not blue' is 'not C'. We know that P(not C) = 1 – P(C)

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Therefore $\quad P(\text{not } C) = 1-\frac{1}{3}=\frac{2}{3}$

(v) The event 'either red or blue' may be described by the set 'A or C'

Since, A and C are mutually exclusive events, we have

$$P(A \text{ or } C) = P(A \cup C) = P(A) + P(C) = \frac{4}{9}+\frac{1}{3}=\frac{7}{9}$$

Example 7 Two students Anil and Ashima appeared in an examination. The probability that Anil will qualify the examination is 0.05 and that Ashima will qualify the examination is 0.10. The probability that both will qualify the examination is 0.02. Find the probability that

(a) Both Anil and Ashima will not qualify the examination.

(b) Atleast one of them will not qualify the examination and

(c) Only one of them will qualify the examination.

Solution Let E and F denote the events that Anil and Ashima will qualify the examination, respectively. Given that

$$P(E) = 0.05,\ P(F) = 0.10 \text{ and } P(E \cap F) = 0.02.$$

Then

(a) The event 'both Anil and Ashima will not qualify the examination' may be expressed as $E' \cap F'$.

Since, E' is 'not E', i.e., Anil will not qualify the examination and F' is 'not F', i.e., Ashima will not qualify the examination.

Also $\quad E' \cap F' = (E \cup F)'$ (by Demorgan's Law)

Now $\quad P(E \cup F) = P(E) + P(F) - P(E \cap F)$

or $\quad P(E \cup F) = 0.05 + 0.10 - 0.02 = 0.13$

Therefore $P(E' \cap F') = P(E \cup F)' = 1 - P(E \cup F) = 1 - 0.13 = 0.87$

(b) P (atleast one of them will not qualify)

= 1 – P(both of them will qualify)

= 1 – 0.02 = 0.98

(c) The event only one of them will qualify the examination is same as the event either (Anil will qualify, and Ashima will not qualify) or (Anil will not qualify and Ashima will qualify) i.e., $E \cap F'$ or $E' \cap F$, where $E \cap F'$ and $E' \cap F$ are mutually exclusive.

Therefore, P(only one of them will qualify) $= P(E \cap F' \text{ or } E' \cap F)$

$$= P(E \cap F') + P(E' \cap F) = P(E) - P(E \cap F) + P(F) - P(E \cap F)$$

$$= 0.05 - 0.02 + 0.10 - 0.02 = 0.11$$

Example 8 A committee of two persons is selected from two men and two women. What is the probability that the committee will have (a) no man? (b) one man? (c) two men?

Solution The total number of persons = 2 + 2 = 4. Out of these four person, two can be selected in ${}^4C_2$ ways.

(a) No men in the committee of two means there will be two women in the committee.

Out of two women, two can be selected in ${}^2C_2 = 1$ way.

Therefore $$P(\text{no man}) = \frac{{}^2C_2}{{}^4C_2} = \frac{1 \times 2 \times 1}{4 \times 3} = \frac{1}{6}$$

(b) One man in the committee means that there is one woman. One man out of 2 can be selected in ${}^2C_1$ ways and one woman out of 2 can be selected in ${}^2C_1$ ways. Together they can be selected in ${}^2C_1 \times {}^2C_1$ ways.

Therefore $$P(\text{One man}) = \frac{{}^2C_1 \times {}^2C_1}{{}^4C_2} = \frac{2 \times 2}{2 \times 3} = \frac{2}{3}$$

(c) Two men can be selected in ${}^2C_2$ way.

Hence $$P(\text{Two men}) = \frac{{}^2C_2}{{}^4C_2} = \frac{1}{{}^4C_2} = \frac{1}{6}$$

EXERCISE 14.2

1. Which of the following can not be valid assignment of probabilities for outcomes of sample Space S $= \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7\}$

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Assignment	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
(a)	0.1	0.01	0.05	0.03	0.01	0.2	0.6
(b)	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
(c)	0.1	0.2	0.3	0.4	0.5	0.6	0.7
(d)	– 0.1	0.2	0.3	0.4	– 0.2	0.1	0.3
(e)	$\frac{1}{14}$	$\frac{2}{14}$	$\frac{3}{14}$	$\frac{4}{14}$	$\frac{5}{14}$	$\frac{6}{14}$	$\frac{15}{14}$

2. A coin is tossed twice, what is the probability that atleast one tail occurs?

3. A die is thrown, find the probability of following events:

(i) A prime number will appear,
(ii) A number greater than or equal to 3 will appear,
(iii) A number less than or equal to one will appear,
(iv) A number more than 6 will appear,
(v) A number less than 6 will appear.

4. A card is selected from a pack of 52 cards.

(a) How many points are there in the sample space?
(b) Calculate the probability that the card is an ace of spades.
(c) Calculate the probability that the card is (i) an ace (ii) black card.

5. A fair coin with 1 marked on one face and 6 on the other and a fair die are both tossed. find the probability that the sum of numbers that turn up is (i) 3 (ii) 12

6. There are four men and six women on the city council. If one council member is selected for a committee at random, how likely is it that it is a woman?

7. A fair coin is tossed four times, and a person win Re 1 for each head and lose Rs 1.50 for each tail that turns up.
From the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts.

8. Three coins are tossed once. Find the probability of getting

(i) 3 heads	(ii) 2 heads	(iii) atleast 2 heads
(iv) atmost 2 heads	(v) no head	(vi) 3 tails
(vii) exactly two tails	(viii) no tail	(ix) atmost two tails

9. If $\frac{2}{11}$ is the probability of an event, what is the probability of the event 'not A'.

10. A letter is chosen at random from the word 'ASSASSINATION'. Find the probability that letter is (i) a vowel (ii) a consonant

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11. In a lottery, a person choses six different natural numbers at random from 1 to 20, and if these six numbers match with the six numbers already fixed by the lottery committee, he wins the prize. What is the probability of winning the prize in the game? [**Hint** order of the numbers is not important.]

12. Check whether the following probabilities P(A) and P(B) are consistently defined
(i) $P(A) = 0.5,\ P(B) = 0.7,\ P(A \cap B) = 0.6$
(ii) $P(A) = 0.5, P(B) = 0.4, P(A \cup B) = 0.8$

13. Fill in the blanks in following table:

	P(A)	**P(B)**	**P(A ∩ B)**	**P(A ∪ B)**
(i)	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{15}$	. . .
(ii)	0.35	. . .	0.25	0.6
(iii)	0.5	0.35	. . .	0.7

14. Given $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$. Find P(A or B), if A and B are mutually exclusive events.

15. If E and F are events such that $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{2}$ and $P(E \text{ and } F) = \frac{1}{8}$, find (i) P(E or F), (ii) P(not E and not F).

16. Events E and F are such that P(not E or not F) = 0.25, State whether E and F are mutually exclusive.

17. A and B are events such that P(A) = 0.42, P(B) = 0.48 and P(A and B) = 0.16. Determine (i) P(not A), (ii) P(not B) and (iii) P(A or B)

18. In Class XI of a school 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class, find the probability that he will be studying Mathematics or Biology.

19. In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing atleast one of them is 0.95. What is the probability of passing both?

20. The probability that a student will pass the final examination in both English and Hindi is 0.5 and the probability of passing neither is 0.1. If the probability of passing the English examination is 0.75, what is the probability of passing the Hindi examination?

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21. In a class of 60 students, 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random, find the probability that

(i) The student opted for NCC or NSS.

(ii) The student has opted neither NCC nor NSS.

(iii) The student has opted NSS but not NCC.

Miscellaneous Examples

Example 9 On her vacations Veena visits four cities (A, B, C and D) in a random order. What is the probability that she visits

(i) A before B? (ii) A before B and B before C?

(iii) A first and B last? (iv) A either first or second?

(v) A just before B?

Solution The number of arrangements (orders) in which Veena can visit four cities A, B, C, or D is 4! i.e., 24.Therefore, n (S) = 24.

Since the number of elements in the sample space of the experiment is 24 all of these outcomes are considered to be equally likely. A sample space for the experiment is

S = {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB
BACD, BADC, BDAC, BDCA, BCAD, BCDA
CABD, CADB, CBDA, CBAD, CDAB, CDBA
DABC, DACB, DBCA, DBAC, DCAB, DCBA}

(i) Let the event 'she visits A before B' be denoted by E

Therefore, E = {ABCD, CABD, DABC, ABDC, CADB, DACB
ACBD, ACDB, ADBC, CDAB, DCAB, ADCB}

Thus $$P(E) = \frac{n(E)}{n(S)} = \frac{12}{24} = \frac{1}{2}$$

(ii) Let the event 'Veena visits A before B and B before C' be denoted by F.
Here F = {ABCD, DABC, ABDC, ADBC}

Therefore, $$P(F) = \frac{n(F)}{n(S)} = \frac{4}{24} = \frac{1}{6}$$

Students are advised to find the probability in case of (iii), (iv) and (v).

Example 10 Find the probability that when a hand of 7 cards is drawn from a well shuffled deck of 52 cards, it contains (i) all Kings (ii) 3 Kings (iii) atleast 3 Kings.

Solution Total number of possible hands = $^{52}C_7$

(i) Number of hands with 4 Kings = $^{4}C_4 \times {}^{48}C_3$ (other 3 cards must be chosen from the rest 48 cards)

Hence P (a hand will have 4 Kings) = $\dfrac{^{4}C_4 \times {}^{48}C_3}{^{52}C_7} = \dfrac{1}{7735}$

(ii) Number of hands with 3 Kings and 4 non-King cards = $^{4}C_3 \times {}^{48}C_4$

Therefore P (3 Kings) = $\dfrac{^{4}C_3 \times {}^{48}C_4}{^{52}C_7} = \dfrac{9}{1547}$

(iii) P(atleast 3 King) = P(3 Kings or 4 Kings)
= P(3 Kings) + P(4 Kings)

$$= \frac{9}{1547} + \frac{1}{7735} = \frac{46}{7735}$$

Example 11 If A, B, C are three events associated with a random experiment, prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Solution Consider E = B ∪ C so that

$$P(A \cup B \cup C) = P(A \cup E)$$

$$= P(A) + P(E) - P(A \cap E) \quad \ldots (1)$$

Now

$$P(E) = P(B \cup C)$$

$$= P(B) + P(C) - P(B \cap C) \quad \ldots (2)$$

Also $A \cap E = A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ [using distribution property of intersection of sets over the union]. Thus

$$P(A \cap E) = P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$

$$= P(A \cap B) + P(A \cap C) - P[A \cap B \cap C] \qquad \text{... (3)}$$

Using (2) and (3) in (1), we get

$$P[A \cup B \cup C] = P(A) + P(B) + P(C) - P(B \cap C)$$

$$- P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

Example 12 In a relay race there are five teams A, B, C, D and E.

(a) What is the probability that A, B and C finish first, second and third, respectively.

(b) What is the probability that A, B and C are first three to finish (in any order) (Assume that all finishing orders are equally likely)

Solution If we consider the sample space consisting of all finishing orders in the first three places, we will have ${}^5P_3$, i.e., $\frac{5!}{(5-3)!} = 5 \times 4 \times 3 = 60$ sample points, each with a probability of $\frac{1}{60}$.

(a) A, B and C finish first, second and third, respectively. There is only one finishing order for this, i.e., ABC.

Thus P(A, B and C finish first, second and third respectively) = $\frac{1}{60}$.

(b) A, B and C are the first three finishers. There will be 3! arrangements for A, B and C. Therefore, the sample points corresponding to this event will be 3! in number.

So P (A, B and C are first three to finish) $= \frac{3!}{60} = \frac{6}{60} = \frac{1}{10}$

Miscellaneous Exercise on Chapter 14

1. A box contains 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn from the box, what is the probability that
(i) all will be blue? (ii) atleast one will be green?

2. 4 cards are drawn from a well – shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade?

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3. A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If die is rolled once, determine

(i) P(2) (ii) P(1 or 3) (iii) P(not 3)

4. In a certain lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (a) one ticket (b) two tickets (c) 10 tickets.

5. Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, what is the probability that

(a) you both enter the same section?
(b) you both enter the different sections?

6. Three letters are dictated to three persons and an envelope is addressed to each of them, the letters are inserted into the envelopes at random so that each envelope contains exactly one letter. Find the probability that at least one letter is in its proper envelope.

7. A and B are two events such that P(A) = 0.54, P(B) = 0.69 and $P(A \cap B) = 0.35$. Find (i) $P(A \cup B)$ (ii) $P(A' \cap B')$ (iii) $P(A \cap B')$ (iv) $P(B \cap A')$

8. From the employees of a company, 5 persons are selected to represent them in the managing committee of the company. Particulars of five persons are as follows:

S. No.	Name	Sex	Age in years
1.	Harish	M	30
2.	Rohan	M	33
3.	Sheetal	F	46
4.	Alis	F	28
5.	Salim	M	41

A person is selected at random from this group to act as a spokesperson. What is the probability that the spokesperson will be either male or over 35 years?

9. If 4-digit numbers greater than 5,000 are randomly formed from the digits 0, 1, 3, 5, and 7, what is the probability of forming a number divisible by 5 when, (i) the digits are repeated? (ii) the repetition of digits is not allowed?

10. The number lock of a suitcase has 4 wheels, each labelled with ten digits i.e., from 0 to 9. The lock opens with a sequence of four digits with no repeats. What is the probability of a person getting the right sequence to open the suitcase?

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Summary

In this Chapter, we studied about the axiomatic approach of probability. The main features of this Chapter are as follows:

- ***Event***: A subset of the sample space
- ***Impossible event*** : The empty set
- ***Sure event***: The whole sample space
- ***Complementary event or 'not event'*** : The set A′ or S – A
- ***Event* A *or* B**: The set $A \cup B$
- ***Event* A *and* B**: The set $A \cap B$
- ***Event* A *and* not B**: The set A – B
- ***Mutually exclusive event***: A and B are mutually exclusive if $A \cap B = \phi$
- ***Exhaustive and mutually exclusive events***: Events $E_1, E_2, ..., E_n$ are mutually exclusive and exhaustive if $E_1 \cup E_2 \cup ... \cup E_n = S$ and $E_i \cap E_j = \phi \ \forall \ i \neq j$
- ***Probability***: Number $P(\omega_i)$ associated with sample point ω_i such that

  (i) $0 \leq P(\omega_i) \leq 1$ (ii) $\sum P(\omega_i)$ for all $\omega_i \in S = 1$

  (iii) $P(A) = \sum P(\omega_i)$ for all $\omega_i \in A$. The number $P(\omega_i)$ is called *probability of the outcome* ω_i.
- ***Equally likely outcomes***: All outcomes with equal probability
- ***Probability of an event***: For a finite sample space with equally likely outcomes

  Probability of an event $P(A) = \dfrac{n(A)}{n(S)}$, where $n(A)$ = number of elements in the set A, $n(S)$ = number of elements in the set S.
- If A and B are any two events, then

  P(A or B) = P(A) + P(B) – P(A and B)

  equivalently, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If A and B are mutually exclusive, then P(A or B) = P(A) + P(B)
- If A is any event, then

  P(not A) = 1 – P(A)

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Historical Note

Probability theory like many other branches of mathematics, evolved out of practical consideration. It had its origin in the 16th century when an Italian physician and mathematician Jerome Cardan (1501–1576) wrote the first book on the subject "Book on Games of Chance" (Biber de Ludo Aleae). It was published in 1663 after his death.

In 1654, a gambler Chevalier de Metre approached the well known French Philosopher and Mathematician Blaise Pascal (1623–1662) for certain dice problem. Pascal became interested in these problems and discussed with famous French Mathematician Pierre de Fermat (1601–1665). Both Pascal and Fermat solved the problem independently. Besides, Pascal and Fermat, outstanding contributions to probability theory were also made by Christian Huygenes (1629–1665), a Dutchman, J. Bernoulli (1654–1705), De Moivre (1667–1754), a Frenchman Pierre Laplace (1749–1827), the Russian P.L Chebyshev (1821–1897), A. A Markov (1856–1922) and A. N Kolmogorove (1903–1987). Kolmogorov is credited with the axiomatic theory of probability. His book 'Foundations of Probability' published in 1933, introduces probability as a set function and is considered a classic.

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