Business Modeling and Simulation

Lecture 2
Refresh on Probability, Statisitcs and Linear Algebra

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Outline

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Random variable

- random outcome: e.g., head/tail in coin tossing
- random variable: numerical function on random space: e.g., head $\to X = 1$; tail $\to X = 0$
- discrete random variable: takes on discrete number of values, e.g., $0, 1, ...,$
- continuous random variable: takes on continuum of possible values

"**The Merchant of Venice**" by William Shakespeare:
This contract doesn't give you any blood at all. The words expressly specify "a pound of flesh." So take your penalty of a pound of flesh, but if you shed one drop of Christian blood when you cut it, the state of Venice will confiscate your land and property under Venetian law.–Portia

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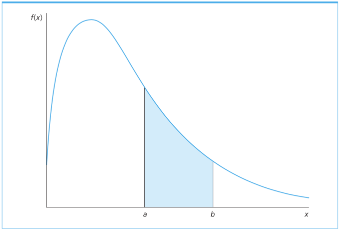

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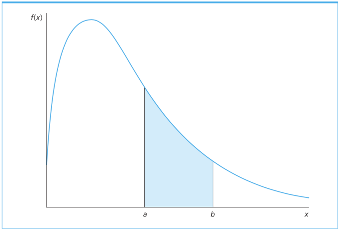

The probability of $P(X = c)$ for continuous variable is ?

Moments of random variables

- Expected value (or mean/expectation) of a random variable X is weighted average of all possible values of X
  - Discrete variable:

$$E(X) = x_1 f(x_1) + x_2 f(x_2) ... + x_n f(x_n) = \sum_{i=1}^{n} x_i f(x_i)$$

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- Standard deviation: $\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$.

Example: normal distribution

normal distribution $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$

- bell shaped
- $E(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$

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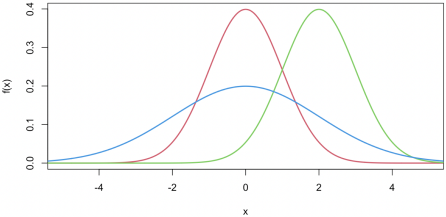

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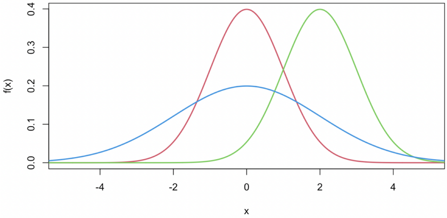

- Guess which is which? $(\mu, \sigma^2) \in \{(0,1), (0, 2^2), (2,1)\}$
  - BLUE: ____________
  - RED: ____________
  - GREEN: ____________

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Property 2	if X and Y are independent, i.e., $F(x, y) = F(x) \times F(y)$, then $\mathrm{Cov}(X, Y) = 0$
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Properties:

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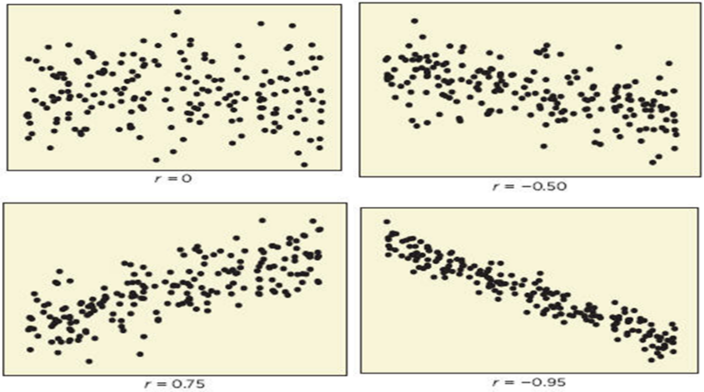
Properties:

$\lvert\mathrm{Corr}(X, Y)\rvert \leq 1$; $\mathrm{Corr}(a_1 X + b_1, a_2 Y + b_2) = \mathrm{Sign(a_1 a_2) Corr(X, Y)}$

Illustration of different correlations



Conditional expectation

- Conditional expectation is the distribution of a random variable conditional on another random variable taking on a specific value; e.g.,
  - $P(\text{Grade in Statistics} = A | \text{Grade in Calculas} = A)$
  - $P(\text{child's height} \leq x | \text{mother's height} \leq y)$
- Bayesian formula

$$P(X \leq x | Y \leq y) = \frac{P(X \leq x, Y \leq y)}{P(Y \leq y)}$$

- Conditional Expectation $E(Y|X = x) = \sum_i y_i P(Y = y_i | X = x)$
- Tower property (iterated law) of conditonal expectation:

$$E(Y) = E(E(Y|X))$$

Independence and Uncorrelatedness

- Two variables X and Y are independent when $P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$ for all x, y.
- If X and Y are independent, $P(X \leq x | Y \leq y) = P(X \leq x)$ and hence $E(X|Y) = E(X)$.
- Uncorrelatedness: $\mathrm{Corr}(X, Y) = 0$.
- Independence implies corelation zero.
- If X and Y are uncorrelated this does not necessarily imply independence!

Normal and related distributions

- standard normal Z, pdf $f(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$
- Chi-Square distribution(χ^2_m): $X = Z_1^2 + Z_2^2 + Z_m^2$, $(Z_i)_{i=1}^m$ are independent standard normal distributed, m is the degree of freedom.
- T distribution (t_m): $T = \frac{Z}{\sqrt{X/m}}$, where Z and X are independent.
- F distribution (F_{m_1,m_2}): $F = \frac{X_1/m_1}{X_2/m_2}$, where X_1 and X_2 are independent Chi-square with df m_1 and m_2 resp.

Simple random sampling

- If $Y_1, ... Y_n$ are independent random variables with the common probability density $f(y; \theta)$, then $\{Y_1, ..., Y_n\}$ is said to ba a random sample from $f(y; \theta)$
- When $\{Y_1, ..., Y_n\}$ is a random sample from $f(y; \theta)$, we also say they are independent, identically distributed (i.i.d.) random variables. e.g.,
    - repeated experiment of coin tossing.

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Given a set of data $(x_i)_{i=1}^n$

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  - Given two data sets $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$

$$\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{x}) = \sum_{i=1}^n x_i y_i - n\overline{x}\,\overline{y}$$

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- Bias of an Estimator: $\text{Bias}(W) = E(W) - \theta$

Efficiency of unbiased estimator and MSE

- an unbiased estimator W_1 is efficient relative to an unbiased estimator W_2 if $\mathrm{Var}(W_1) \leq \mathrm{Var}(W_2)$ for all θ with strict inequality for at least one θ.
- e.g., for i.i.d. variables with $\mathrm{Var}(Y) = \sigma^2$:

$$\mathrm{Var}(\overline{Y}) = \frac{\sigma^2}{n}$$

  $\overline{Y}$ is efficient relative to Y_1, because $\mathrm{Var}(\overline{Y}) < \mathrm{Var}(Y_1) = \sigma^2$.
- Mean squared error (MSE): If we do not restrict our attention to unbiased estimators, then comparing variances is meaningless. Instead, we can look at mean squared error (MSE)

$$E((W - \theta)^2) = \mathrm{Var}(W) + (\mathrm{bias}(W))^2$$

Consistency

- W_n depends on a sample $Y_1, ..., Y_n$. W_n is consistent estimator of θ if

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write $\mathrm{plim(W_n)} = \theta$ (convergence in probability).
- Law of large numbers (LNN): for i.i.d. sample $Y_1, ..., Y_n$ with expectation μ
$$\mathrm{plim}(\overline{\mathrm{Y}}) = \mu$$
- Slustky' Theorem

If $\mathrm{plim(T_n) = T}$*, and* $\mathrm{plim}(\overline{\mathrm{U}}_\mathrm{n}) = \beta$*, where T is a random variable and* β *is a constant, then* $\mathrm{plim(T_n + U_n) = T} + \beta$*;*
$\mathrm{plim(T_n \cdot U_n) = T}\beta$*;*
$\mathrm{plim(T_n/U_n) = T}/\beta$ *provided that* $\beta \neq 0$

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The 95% confidence interval of μ is $[\overline{Y} - 1.96/\sqrt{n}, \overline{Y} + 1.96/\sqrt{n}]$. It means the **random interval** contains μ with probability 0.95. In other words, **before the random sample** is drawn, there is a 95% chance that the CI contains μ.

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- Let θ denote the true proportion of population voting. Hypothesis test: $H_0 : \theta \leq 0.42$ v.s. $H_1 : \theta > 0.42$
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- How "strong" is strong? Is 100 sample sufficient?

Hypothesis test, cont'd

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- Type II error: failing to reject H_0 when it is actually false
- Want to get small Type I error
- Significance level: α is probability of a Type I error.

$$\alpha = P(\text{Reject } H_0 | H_0)$$

- Commonly used levels $\alpha = 0.01, 0.05, 0.1$

Test statistic and p-values

- Construct test statistic:

$$T = \frac{\overline{Y} - \theta_0}{s/\sqrt{n}}$$

For the **election example**:

- $\theta_0 = 42\%$, $n = 100$, $\overline{Y} = 53\%$,
- s is estimated standard deviation of Y_i where Y_i is binary variable (indicator of voting for candidate A), $s = \sqrt{\overline{Y}(1 - \overline{Y})}$.
- observed value of T is $t = \frac{0.53 - 0.42}{\sqrt{100 \times 0.42 \times 0.58}} \approx 2.229$

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- p-value$= P(T > 2.229|H_0) = 1 - \Phi(2.229) = 0.013$, where $\Phi(\cdot)$ is cdf of standard normal.
- Given significance level $\alpha = 0.05$,
- p-value=0.013<α;

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- Given significance level $\alpha = 0.05$,
- p-value=0.013<α; reject H_0.

Outline

Matrix Operations

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- Inverse: $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where $\mathbf{I} = \mathrm{diag}(1, ..., 1)$.

Quadratic forms and Positive definite matrices

- quadratic form

$$\mathbf{x}^t\mathbf{A}\mathbf{x} = \sum_{ij} x_i x_j a_{ij} = \sum_i x_i^2 a_{ii}^2 + 2\sum_{i<j} a_{ij} x_i x_j,$$

  where $\mathbf{x}$ is $n \times 1$ vector, and $\mathbf{A}$ is $n \times n$ symetric matrix, $a_{ij} = a_{ji}$.

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  symetric matrix $\mathbf{A}$ is called positive definite (or semi-positive definite) when

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  property of idempotent matrix: $\mathrm{tr}(\mathbf{A}) = \mathrm{rank}(\mathbf{A})$.

Outline

Summary

Reference: 

- GC Chapter 3; LP Chapter 4–7

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Come next: 

- Refresh on Python
- Random number generator

Survey and Q&A time