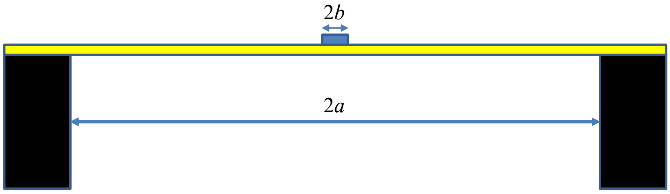

Equation of motion for a stressed membrane:

$$\begin{aligned} \varrho h\ddot{w} - \sigma_0 h \nabla_\perp^2 w &= p \\ \varrho h\ddot{w} - \sigma_0 h \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) &= p, \text{ (rotational symmetry)} \end{aligned}$$

Boundary conditions $w(a) = 0$ and $w(0)$ finite. Eigen-mode equation

$$\omega_n^2 \frac{\varrho}{\sigma_0} w + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \omega_n^2 \frac{\varrho}{\sigma_0} w + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial r^2} = 0$$

with the solution $w(r) = \mathrm{J}_0\left(r\sqrt{\omega_n^2 \frac{\varrho}{\sigma_0}}\right) = \mathrm{J}_0(k_n r)$ with the BC we have $w(a) = \mathrm{J}_0(k_n a) = \mathrm{J}_0\left(a\sqrt{\omega_n^2 \frac{\varrho}{\sigma_0}}\right) = 0.$

The first root of the bessel function is 2.4048, i.e., $a\sqrt{\omega_0^2 \frac{\varrho}{\sigma_0}} = ak_0 = 2.4048$ and the first angular resonant frequency is then

$$\omega_0 = \frac{2.4048}{a}\sqrt{\frac{\sigma_0}{\varrho}}$$

and

$$f_0 = \frac{2.4048}{2\pi a}\sqrt{\frac{\sigma_0}{\varrho}}.$$

Heat transfer: Fourier's law

$$\mathbf{J}_q = -\kappa \nabla T$$

where κ is the thermal conductivity. The heat continuity equation is

$$\begin{aligned} C_V \frac{\partial T}{\partial t} &= -\nabla \cdot \mathbf{J}_q + q \Rightarrow \\ C_V \frac{\partial T}{\partial t} &= \kappa \nabla^2 T + q \end{aligned}$$

where q is the heat dissipated per volume and C_V the heat capacity per unit volume at constant volume. In steady state

$$\kappa \nabla^2 T + q = 0 \Rightarrow \nabla^2 T = -q/\kappa$$

by integration across the thickness of the membrane, and assuming zero temperature gradient in the z-direction, we have

$$\nabla^2 T = -\frac{Q}{\kappa h} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

where Q is the heat dissipated per unit surface area. Assume now that the dissipation is confined to $r < b$ an the temperature at $r = a$ is fixed, then the temperature varies logarithmically for $r > b$, i.e.,

$$\begin{aligned}
\Delta T(r) &= A \ln \frac{r}{a} \\
\frac{\partial T}{\partial r} &= \frac{A}{r} \\
\frac{A}{b} &= -\frac{P}{\kappa h}\frac{1}{2\pi b} \Rightarrow A = -\frac{P}{2\pi\kappa h}
\end{aligned}$$

$$\Delta T(r) = A \ln \frac{r}{a} = -\frac{P}{2\pi\kappa h} \ln \frac{r}{a}, \text{ for } a > r > b$$

where $P = Q\pi b^2$. Inside the heated disc the temperature becomes

$$\Delta T = \frac{1}{4}\frac{Q}{\kappa h}\left(b^2 - r^2\right) - \frac{Q\pi b^2}{2\pi\kappa h}\ln\frac{b}{a} = \frac{1}{4}\frac{P}{\pi\kappa h}\frac{\left(b^2 - r^2\right)}{b^2} - \frac{P}{2\pi\kappa h}\ln\frac{b}{a}$$

Stress-strain and displacement relations (assuming $\sigma_z = 0$ due to the free surface, $v = 0$ due to rotational symmetry)

$$\begin{aligned}
\epsilon_r &= \frac{1}{E}\left(\sigma_r - \nu\sigma_\theta\right) + \alpha\Delta T = \frac{\partial u}{\partial r} \\
\epsilon_\theta &= \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right) + \alpha\Delta T = \frac{u}{r} + \frac{\partial v}{r\partial\theta} = \frac{u}{r} \\
\epsilon_z &= \frac{1}{E}\left(-\nu\sigma_\theta - \nu\sigma_r\right) + \alpha\Delta T = \frac{\partial w}{\partial z} \\
\gamma_{r\theta} &= \frac{\partial u}{r\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = 0
\end{aligned}$$

i.e.,

$$\begin{aligned}
u &= r\left(\frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right) + \alpha\Delta T\right) \\
u &= \int\left(\frac{1}{E}\left(\sigma_r - \nu\sigma_\theta\right) + \alpha\Delta T\right)\mathrm{d}r
\end{aligned}$$

Equation of equilibrium

$$\frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

Timoshenko

$$\begin{aligned}
\sigma_r &= -\alpha E \frac{1}{r^2}\int_0^r T r dr + \frac{E}{1-\nu^2}\left(C_1\left(1+\nu\right) - \frac{C_2}{r^2}\left(1-\nu\right)\right) \\
\sigma_\theta &= \alpha E \frac{1}{r^2}\int_0^r T r dr - \alpha E T + \frac{E}{1-\nu^2}\left(C_1\left(1+\nu\right) + \frac{C_2}{r^2}\left(1-\nu\right)\right) \\
u &= \left(1+\nu\right)\alpha \frac{1}{r}\int_0^r T r dr + C_1 r + \frac{C_2}{r}
\end{aligned}$$

$u(0) = 0 \Rightarrow C_2 = 0$, and $u(a) = 0 \Rightarrow 0 = \left(1+\nu\right)\alpha\frac{1}{a}\int_0^a T r dr + C_1 a \Rightarrow$

$$C_1 = -\left(1+\nu\right)\alpha\frac{1}{a^2}\int_0^a T r dr$$

$$\begin{aligned}
\sigma_r &= -\alpha E \frac{1}{r^2}\int_0^r T r dr - \frac{\left(1+\nu\right)^2 \alpha E}{1-\nu^2}\frac{1}{a^2}\int_0^a T r dr \\
\sigma_r &= -\alpha E \frac{1}{r^2}\int_0^r T r dr - \frac{\left(1+\nu\right) \alpha E}{1-\nu}\frac{1}{a^2}\int_0^a T r dr
\end{aligned}$$

$$\begin{aligned}
\sigma_\theta &= \alpha E \frac{1}{r^2}\int_0^r T r dr - \alpha E T - \frac{\left(1+\nu\right)^2 \alpha E}{1-\nu^2}\frac{1}{a^2}\int_0^a T r dr \\
&= \alpha E \frac{1}{r^2}\int_0^r T r dr - \alpha E T - \frac{\left(1+\nu\right) \alpha E}{1-\nu}\frac{1}{a^2}\int_0^a T r dr
\end{aligned}$$

0.1 Stress profile

Outside the heated area we have $\Delta T(r) = A\ln\frac{r}{a} = -\frac{P}{2\pi\kappa h}\ln\frac{r}{a}$, $A = -\frac{P}{2\pi\kappa h}$ and then

$$\begin{aligned}
\sigma_r &= -\alpha E \frac{1}{r^2}\int_0^r T r dr - \frac{\left(1+\nu\right) \alpha E}{1-\nu}\frac{1}{a^2}\int_0^a T r dr \\
&= -\alpha E \frac{A}{r^2}\int_0^r r\ln\left(\frac{r}{a}\right) dr - \frac{\left(1+\nu\right) \alpha E}{1-\nu}\frac{A}{a^2}\int_0^a r\ln\left(\frac{r}{a}\right) dr
\end{aligned}$$

since $\int r\ln\left(\frac{r}{a}\right) dr = \frac{1}{2}r^2\left(\ln\frac{r}{a} - \frac{1}{2}\right)$ we have $\int_0^r r\ln\left(\frac{r}{a}\right) dr = \frac{1}{2}r^2\left(\ln\frac{r}{a} - \frac{1}{2}\right)$ and $\int_0^a r\ln\left(\frac{r}{a}\right) dr = -\frac{1}{4}a^2$, thus

$$\begin{aligned}
\sigma_r &= -\alpha E \frac{A}{r^2}\frac{1}{2}r^2\left(\ln\frac{r}{a} - \frac{1}{2}\right) - \frac{\left(1+\nu\right) \alpha E}{1-\nu}\frac{A}{a^2}\left(-\frac{1}{4}a^2\right) = \\
&= -\alpha E A\frac{1}{2}\left(\ln\frac{r}{a} - \frac{1}{2}\right) + \frac{\left(1+\nu\right) \alpha E}{1-\nu}\frac{A}{4} = \\
\sigma_r &= \frac{1}{2}A\alpha E\left(\frac{1}{1-\nu} - \ln\frac{r}{a}\right) = -\frac{P}{2\pi\kappa h}\frac{\alpha E}{2}\left(\frac{1}{1-\nu} - \ln\frac{r}{a}\right) \\
\sigma_r &= -\frac{\alpha E P}{4\pi\kappa h}\left(\frac{1}{1-\nu} - \ln\frac{r}{a}\right) = \frac{\alpha E P}{4\pi\kappa h}\left(\ln\frac{r}{a} - \frac{1}{1-\nu}\right)
\end{aligned}$$

0.2 Equation of motion

When the stress is dependent on radial position the equation of motion must be modified

$$
\begin{aligned}
\varrho h\ddot{w} - \frac{1}{r}\frac{\partial}{\partial r}\left(\sigma h r\frac{\partial w}{\partial r}\right) &= p \\
\varrho \ddot{w} - \frac{1}{r}\frac{\partial}{\partial r}\left(\sigma r\frac{\partial w}{\partial r}\right) &= \frac{p}{h} \\
\varrho \ddot{w} - \sigma\left[w''_{rr} + \frac{1}{r}w'_r\right] - \sigma'_r w'_r &= \frac{p}{h}
\end{aligned}
$$

where $\sigma = \sigma_0 + \sigma_r(r)$

This equation can probably not be solved with the logarithmic dependent stress. But with Rayleighs quotiont we may get some progress anyway. Assuming harmonic solutions to the homogeneous equation we get

$$
\omega^2\varrho w + \sigma\left[w''_{rr} + \frac{1}{r}w'_r\right] + \sigma'_r w'_r = 0
$$

and then

$$
\begin{aligned}
\omega^2 &= -\frac{\int\left(\sigma\left[w''_{rr} + \frac{1}{r}w'_r\right] + \sigma'_r w'_r\right) wrdr}{\int(\varrho w)\, wrdr} = -\frac{\int_0^a\left(\sigma\left[y''_{rr} + \frac{1}{r}y'_r\right] + \sigma'_r y'_r\right) yrdr}{\int_0^a(\varrho y)\, yrdr} \\
&\simeq -\frac{\int_0^a\left(\sigma_0\left[y''_{rr} + \frac{1}{r}y'_r\right]\right) yrdr + \int_0^a\left(\sigma_r\left[y''_{rr} + \frac{1}{r}y'_r\right] + \sigma'_r y'_r\right) yrdr}{\int_0^a(\varrho y)\, yrdr} \\
&= \omega_0^2\left[1 + \frac{\int_0^a\left(\sigma_r\left[y''_{rr} + \frac{1}{r}y'_r\right] + \sigma'_r y'_r\right) yrdr}{\int_0^a\left(\sigma_0\left[y''_{rr} + \frac{1}{r}y'_r\right]\right) yrdr}\right]
\end{aligned}
$$

where $y(r) = \mathrm{J}_0(k_0 r)$ is the eigenmode for $\sigma = \sigma_0$. Here $y''_{rr} + \frac{1}{r}y'_r = -k_0^2\,\mathrm{BesselJ}_0(rk_0) = -k_0^2 y$

Thus

$$
\begin{aligned}
\omega^2 &= \omega_0^2\left[1 + \frac{\int_0^a\left(\sigma_r\left[y''_{rr} + \frac{1}{r}y'_r\right] + \sigma'_r y'_r\right) yrdr}{\int_0^a\left(\sigma_0\left[y''_{rr} + \frac{1}{r}y'_r\right]\right) yrdr}\right] = \omega_0^2\left[1 + \frac{\int_0^a\left(-k_0^2 y\sigma_r + \sigma'_r y'_r\right) yrdr}{\int_0^a\left(-k_0^2 y\sigma_0\right) yrdr}\right] \\
&= \omega_0^2\left[1 + \frac{\int_0^a\left(-k_0^2 y^2\sigma_r + \sigma'_r y'_r y\right) rdr}{\int_0^a\left(-k_0^2 y^2\sigma_0\right) rdr}\right] = \omega_0^2\left[1 + \frac{\int_0^a\left(\sigma_r y^2 - \frac{\sigma'_r}{k_0^2}y'_r y\right) rdr}{\sigma_0\int_0^a y^2 rdr}\right]
\end{aligned}
$$

The integrals

$\int_0^a y^2 rdr = \int_0^a \mathrm{J}_0^2(k_0 r)\, rdr = \left[\frac{1}{2}r^2\left(\mathrm{J}_0^2(rk_0) + \mathrm{J}_1^2(rk_0)\right)\right]_0^a = \frac{1}{2}a^2\mathrm{J}_1^2(ak_0)$

$\int_0^a\left(\ln\left(\frac{r}{a}\right)\right) y^2 rdr = \int_0^a\left(\ln\left(\frac{r}{a}\right)\right)\mathrm{J}_0^2(k_0 r)\, rdr = \Big[\frac{r^2}{4}\big[\left(2\ln\frac{r}{a} - 1\right)\mathrm{J}_0^2(k_0 r) + 2\left(\ln\frac{r}{a} - 1\right)\mathrm{J}_1^2(k_0 r) + \mathrm{J}_0(k_0 r)\,\mathrm{J}_2($

$\int_0^a\left(\ln\left(\frac{r}{a}\right)\right) y^2 rdr = \frac{a^2}{4}\left[-2\mathrm{J}_1^2(k_0 a)\right] = -\frac{a^2}{2}\mathrm{J}_1^2(k_0 a)$

$\int_0^a \frac{1}{r} y_r' y r dr = -k_0 \int_0^a \mathrm{J}_1(k_0 r)\, \mathrm{J}_0(k_0 r)\, dr = \left[\frac{1}{2}\mathrm{J}_0^2(k_0 r)\right]_0^a = -\frac{1}{2}$
Since we have

$$\begin{aligned}
\sigma_r &= -\frac{\alpha EP}{4\pi\kappa h}\left(\frac{1}{1-\nu} - \ln\frac{r}{a}\right) = \frac{\alpha EP}{4\pi\kappa h}\left(\ln\frac{r}{a} - \frac{1}{1-\nu}\right) = B\left(\ln\frac{r}{a} - \frac{1}{1-\nu}\right) \\
\sigma_r' &= \frac{B}{r}
\end{aligned}$$

we get

$$\begin{aligned}
\omega^2 &= \omega_0^2\left[1 + \frac{\int_0^a \left(\sigma_r y^2 - \frac{\sigma_r'}{k_0^2} y_r' y\right) r dr}{\sigma_0 \int_0^a y^2 r dr}\right] = \omega_0^2\left[1 + \frac{-\frac{a^2}{2}\mathrm{J}_1^2(k_0 a)\, B - \frac{1}{1-\nu} B \frac{1}{2} a^2 \mathrm{J}_1^2(ak_0) - \frac{B}{k_0^2}\left(-\frac{1}{2}\right)}{\sigma_0 \frac{1}{2} a^2 \mathrm{J}_1^2(ak_0)}\right] \\
&= \omega_0^2\left[1 + \frac{B}{\sigma_0}\left(-1 - \frac{1}{1-\nu} + \frac{1}{k_0^2 a^2 \mathrm{J}_1^2(ak_0)}\right)\right] = \omega_0^2\left[1 + \frac{B}{\sigma_0}\left(-\frac{2-\nu}{1-\nu} + \frac{1}{k_0^2 a^2 \mathrm{J}_1^2(ak_0)}\right)\right] = \\
&= \omega_0^2\left[1 + \frac{B}{\sigma_0}\left(-\frac{2-\nu}{1-\nu} + \frac{1}{1.5587}\right)\right] = \omega_0^2\left[1 + \frac{\alpha EP}{4\pi\kappa h}\frac{1}{\sigma_0}\left(-\frac{2-\nu}{1-\nu} + \frac{1}{1.5587}\right)\right] \\
\omega^2 &= \omega_0^2\left[1 - \frac{\alpha EP}{4\pi\kappa h}\frac{1}{\sigma_0}\left(\frac{2-\nu}{1-\nu} - \frac{1}{1.5587}\right)\right] = \omega_0^2\left[1 - \frac{\alpha EP}{4\pi\kappa h}\frac{1}{\sigma_0}\left(\frac{1.3584 - 0.35844\nu}{1-\nu}\right)\right]
\end{aligned}$$

where $ak_0 = 2.4048$
The final result is that

$$\begin{aligned}
\frac{\omega^2}{\omega_0^2} &= 1 - \frac{\alpha EP}{4\pi\kappa h}\frac{1}{\sigma_0}\left(\frac{1.3584 - 0.35844\nu}{1-\nu}\right) \\
\frac{\omega}{\omega_0} &= \sqrt{1 - \frac{\alpha EP}{4\pi\kappa h}\frac{1}{\sigma_0}\left(\frac{1.3584 - 0.35844\nu}{1-\nu}\right)} \\
&\simeq 1 - \frac{\alpha EP}{8\pi\kappa h}\frac{1}{\sigma_0}\left(\frac{1.3584 - 0.35844\nu}{1-\nu}\right)
\end{aligned}$$

or

$$\frac{\Delta f}{f_0} \simeq -\frac{\alpha EP}{8\pi\kappa h}\frac{1}{\sigma_0}\left(\frac{1.3584 - 0.35844\nu}{1-\nu}\right)$$