Question 1:

a)
If if the following statments are true:

If it is Robin's birthday, then Robin eats cake. Robin is eating cake.

The statment;

It is Robin's birthday.

Cannot be deduced as true.

b)

For all positive integers n, We have $5^n \leq (6+1)^6$

This can be proved false with;

$$5^7 = 78125 > 7^6 = 117649$$

Question 2:

a)

$$y = \frac{4x-1}{3x-4}$$

Given $x \neq \frac{4}{3}$ and $y \neq \frac{4}{3}$
When $x = \frac{4y-1}{3y-4}$

$$y = \frac{4x-1}{3x-4}$$

$$= \frac{4\left(\frac{4y-1}{3y-4}\right)-1}{3\left(\frac{4y-1}{3y-4}\right)-4}$$

Distributing the 4 and 3

$$= \frac{\frac{16y-4}{3y-4}-1}{\frac{12y-3}{3y-4}-4}$$

Using the common denominator of $3y-4$

$$= \frac{\frac{16y-4-(3y-4)}{3y-4}}{\frac{12y-3-4(3y-4)}{3y-4}}$$

Simplifying the numerator and denominator

$$= \frac{\frac{16y-4-3y+4}{3y-4}}{\frac{12y-3-12y+16}{3y-4}}$$

$$= \frac{\frac{13y}{3y-4}}{\frac{13}{3y-4}}$$

Cancelling the common factor of $3y-4$

$$= \frac{13y}{13}$$

$$= y$$

Assume $x = \frac{4y-1}{3y-4}$

Then $y = \frac{4x-1}{3x-4}$

$$x = \frac{4y-1}{3y-4}$$

$$= \frac{4\left(\frac{4x-1}{3x-4}\right) - 1}{3\left(\frac{4x-1}{3x-4}\right) - 4}$$

Distributing the 4 and 3

$$= \frac{\frac{16x-4}{3x-4} - 1}{\frac{12x-3}{3x-4} - 4}$$

Using the common denominator of $3x - 4$

$$= \frac{\frac{16x-4-(3x-4)}{3x-4}}{\frac{12x-3-4(3x-4)}{3x-4}}$$

Simplifying the numerator and denominator

$$= \frac{\frac{16x-4-3x+4}{3x-4}}{\frac{12x-3-12x+16}{3x-4}}$$

$$= \frac{\frac{13x}{3x-4}}{\frac{13}{3x-4}}$$

Cancelling the common factor of $3x - 4$

$$= \frac{13x}{13}$$

$$= x$$

Thus the function is it's own inverse. Hence,

$$y = \frac{4x-1}{3x-4} \text{if and only if} x = \frac{4y-1}{3y-4}$$

for all real numbers such that $y \neq \frac{4}{3}$ and $x \neq \frac{4}{3}$.

b)

To prove

$n + 1$ is even if and only if $2(n + 1)$ is a multiple of 4

Assume $n + 1$ is even, therefore it can be writen as $n + 1 = 2k$ for some integer k.

It follows that we can write;

$$2(n+3) = 2(n+1) + 4$$

Substituting $n+1 = 2k$

$$= 2(2k) + 4$$

$$= 4k + 4$$

$$= 4(k+1)$$

and thus a multiple of 4

Conversly, assume $2(n+1)$ is a multiple of 4, therefore it can be written as $2(n+1) = 4k$ for some integer k.

$$2(n+3) = 4k$$

Dividing both sides by 2

$$n+3 = 2k$$

Rearranging gives

$$n = 2k - 3$$

$$n+1 = 2k - 2$$

$$= 2(k-1)$$

and thus an even number

Question 3:

a)

$$(3n)! \geq (n!)^3, \text{ for all } n \in \mathbb{N}$$

Prooof by induction.

Base case: $n = 1$

$$(3 \cdot 1)! = 3! = 6$$

$$(1!)^3 = 1^3 = 1$$

Thus

$$6 \geq 1 \text{ is true.}$$

Inductive step: Assume $(3n)! \geq (n!)^3$ is true for some $n \in \mathbb{N}$.

We need to show that $(3(n+1))! \geq ((n+1)!)^3$.

Since

$$3(k+1)! = 3(k+1)(k!)$$

$$= (3k+3)(3k+2)(3k+1)(3k!)$$

and

$$((k+1)!)^3 = ((k+1)(k!))^3$$

$$= (k+1)^3(k!)^3$$

Now using our assumption for the inductive step, we have to show;

$$(3k+3)(3k+2)(3k+1) \geq (k+1)^3$$

Expanding the LHS:

$$(9k^2 + 9k + 6k + 6)(3k+1) \geq (k+1)^3$$

$$27k^3 + 54k^2 + 33k + 6 \geq (k+1)^3$$

Expanding the RHS:

$$27k^3 + 54k^2 + 33k + 6 \geq (K+1)(k+1)(k+1)$$

$$\geq (k^2 + 2k + 2)(k+1)$$

$$\geq k^3 + 3k^2 + 3k + 1$$

Reagrranging

$$26k^3 + 51k^2 + 30k + 5 \geq 0$$

This is true for all $k \in \mathbb{N}$

Question 4:

a)
Prove that no such value of x exists such that x is a real posive number.

$$\frac{7x}{x+3} \leq \frac{x-3}{7x}$$

Assume that x is a positive real number.

$$frac7xx+3 \leq \frac{x-3}{7x}$$

Cross multiplying gives

$$(7x)(7x) \leq (x-3)(x+3)$$

Expanding both sides

$$49x^2 \leq x^2 - 9$$

Rearranging gives

$$48x^2 + 9 \leq 0$$

This is not possible as $48x^2$ is always positive for all real numbers x and 9 is a positive constant.

Thus, we have a contradiction.

We can conclude that no such value of x exists such that x is a positive real number.

b)
Prove that:

If $n^3 + 2n^2$ is not a multiple of 16, then n is odd.

Let us consider the contraposition of this stament;

If n is even, then $n^3 + 2n^2$ is a multiple of 16.

Assume n is even, therefore it can be written as $n = 2k$ for some integer k.

$$\begin{aligned} n^3 + 2n^2 &= (2k)^3 + 2(2k)^2 \\ &= 8k^3 + 2(4k^2) \\ &= 8k^3 + 8k^2 \\ &= 8(k^3 + k^2) \\ &= 8k^2(k+1) \\ &= (8k)(k(k+1)) \end{aligned}$$

As $k(k+1)$ is even, we can write it as $2l$ for some integer l.

$$\begin{aligned} &= (8k)(2l) \\ &= 16kl \end{aligned}$$

Hence a multiple of 16

Thus by proof by contraposition:

If $n^3 + 2n^2$ is not a multiple of 16, then n is odd.

Question 5:

a)

$$x_0 = 0\,\mathrm{m}, \quad x_1 = 300\,\mathrm{m}, \quad v_0 = 0\,\mathrm{m/s}, \quad \mathrm{a} = \mathrm{g} = 9.8\,\mathrm{m/s}$$

b)

inset graph here

Question 6:

$$v_0 = 0\,\text{meter/s}, \quad x_0 = 0\,\text{m}, \quad v_1 = 9\,\text{m}\,\text{s}^{-1}, \quad x_1 = 30\,\text{m}$$

a)

b)

$$\begin{aligned}
\mathbf{F} &= \mu|N| \\
\mathbf{N} &= |N| \\
\mathbf{W} &= -\sin(30)mg - \cos(30)mg \\
F_i &= \sin(30)mg - \mathbf{F} \\
&= \sin(30)mg - \mu|N| \\
N_j &= \cos(30)mg
\end{aligned}$$

thus

$$F = \sin 30mg - \mu\cos(30)mg$$

And using $F = ma$

$$ma = \sin(30)mg - \mu\cos(30)mg$$

Divinding through by $ma = \sin(30)g - \mu\cos(30)g$

$$1.35 = \sin(30)g - \mu\cos(30)g$$

Rearranging $\mu\cos(30)g = \sin(30)g - 1.35$

$$\begin{aligned}\mu &= \frac{\sin(30)g - 1.35}{\cos(30)g}\\ &= \frac{\sin(30)9.8 - 1.35}{\cos(30)9.8}\\ &= \frac{4.9 - 1.35}{8.487}\\ &= \frac{3.55}{8.487}\\ &= 0.418\ldots\\ &= 0.42\end{aligned}$$

to 2 s.f.

Question 7:

a)
The vector expression for the acceleration is

$$\mathbf{a} = -g\mathbf{j}$$

b)
The initial velocity vector is

$$\mathbf{v}_0 = 12\cos(50^\circ)\,\mathbf{i} + 12\sin(50^\circ)\,\mathbf{j}$$

Integrating the acceleration vector to find the velocity vector:

$$\begin{aligned}
\mathbf{v}(t) &= \int \mathbf{a}\ \mathrm{d}t \\
&= \int -g\mathbf{j}\ \mathrm{d}t \\
&= -gt\,\mathbf{j} + \mathbf{C}_1
\end{aligned}$$

Using the initial velocity to find the constant of integration:

$$\mathbf{v}(0) = \mathbf{v}_0 \Rightarrow \mathbf{C}_1 = \mathbf{v}_0$$

Thus, the velocity vector is:

$$\mathbf{v}(t) = \mathbf{v}_0 - gt\,\mathbf{j}$$

Integrating the velocity vector to find the position vector:

$$\begin{aligned}
\mathbf{r}(t) &= \int \mathbf{v}(t)\ \mathrm{d}t \\
&= \int (\mathbf{v}_0 - gt\,\mathbf{j})\,\mathrm{d}t \\
&= \mathbf{v}_0 t - \frac{1}{2}gt^2\,\mathbf{j} + \mathbf{C}_2
\end{aligned}$$

Taking the initial position as the origin:

$$\mathbf{r}(0) = \mathbf{0} \Rightarrow \mathbf{C}_2 = \mathbf{0}$$

Therefore

$$\mathbf{r}(t) = \mathbf{v}_0 t - \frac{1}{2} g t^2 \,\mathbf{j}$$

Substituting the expression for $\mathbf{v}_0$:

$$\mathbf{r} = (12t\cos(50°))\,\mathbf{i} + \left(12t\sin(50°) - \frac{1}{2}gt^2\right)\mathbf{j}$$ *(as required)*

c)
i.
Given the position vector $\mathbf{r}$ is from the orign, we want to find the time t for which the $\mathbf{j}$-component is -1.5, that is,

The quadractic formula

$$x = \frac{-b \pm \sqrt{b^{2-4ac}}}{2a}$$

$$12t\sin(50°) - \frac{1}{2}gt^2 = -1.5$$

Rewriting

$$-\frac{1}{2}gt^2 + 12t\sin(50°) + 1.5 = 0$$

This is a quadratic equation in t:

$$\begin{aligned} t &= \frac{-12\sin(50) \pm \sqrt{(12\sin(50)^2 - 4(-\frac{1}{2}g)(1.5))}}{2(-\frac{1}{2}g)} \\ &= \frac{-12\sin(50) \pm \sqrt{(12\sin(50))^2 + \frac{147}{5}}}{-g} \\ &= -0.151\ldots \text{ and } 2.027\ldots \end{aligned}$$

since we can reject the negaive value for time, we have

$$= 2.027\ldots$$

$$= 2.0\,\mathrm{s}$$

to 2 s.f.

ii.
The horizonal distance traveled by the ball is given by the **i**-component of the position vector **r** at time t:

$$\mathbf{r_i} = 12t\cos(50)\mathbf{i}$$

Substituting $t = 2.027\ldots$

$$= 12(2.027\ldots)\cos(50)\mathbf{i}$$

$$= 15.635\ldots$$

$$= 16\,\mathrm{m}$$

to 2 s.f.

Question 8: