Question 1:

a)
If if the following statments are true:

If it is Robin's birthday, then Robin eats cake. Robin is eating cake.

Let P be the statement "It is Robin's birthday" and Q be the statement "Robin eats cake". The first statement can be written as $P \implies Q$ and the second statement can be written as Q.

The statment;

It is Robin's birthday.

P does not imply Q. Robin could be eating cake for any number of reasons. This is an example of the formal fallacy affirming the consequences.

b)

$$\text{For all positive integers } n, \text{ We have } 5^n \leq (6+1)^6$$

This can be proved false with;

Using $n = 9$

$$5^9 = 1953125$$

$$(9+1)^6 = 10^6 = 1000000$$

Thus

$$1953125 \leq 1000000 \text{ is false.}$$

Question 2:

a)

$$y = \frac{4x-1}{3x-4}$$

Given $x \neq \frac{4}{3}$ and $y \neq \frac{4}{3}$
When $x = \frac{4y-1}{3y-4}$

$$y = \frac{4x-1}{3x-4}$$

$$= \frac{4\left(\frac{4y-1}{3y-4}\right)-1}{3\left(\frac{4y-1}{3y-4}\right)-4}$$

Distributing the 4 and 3

$$= \frac{\frac{16y-4}{3y-4}-1}{\frac{12y-3}{3y-4}-4}$$

Using the common denominator of $3y-4$

$$= \frac{\frac{16y-4-(3y-4)}{3y-4}}{\frac{12y-3-4(3y-4)}{3y-4}}$$

Simplifying the numerator and denominator

$$= \frac{\frac{16y-4-3y+4}{3y-4}}{\frac{12y-3-12y+16}{3y-4}}$$

$$= \frac{\frac{13y}{3y-4}}{\frac{13}{3y-4}}$$

Cancelling the common factor of $3y-4$

$$= \frac{13y}{13}$$

$$= y$$

Assume $x = \frac{4y-1}{3y-4}$
Then $y = \frac{4x-1}{3x-4}$

$$x = \frac{4y-1}{3y-4}$$

$$= \frac{4\left(\frac{4x-1}{3x-4}\right) - 1}{3\left(\frac{4x-1}{3x-4}\right) - 4}$$

Distributing the 4 and 3

$$= \frac{\frac{16x-4}{3x-4} - 1}{\frac{12x-3}{3x-4} - 4}$$

Using the common denominator of $3x - 4$

$$= \frac{\frac{16x-4-(3x-4)}{3x-4}}{\frac{12x-3-4(3x-4)}{3x-4}}$$

Simplifying the numerator and denominator

$$= \frac{\frac{16x-4-3x+4}{3x-4}}{\frac{12x-3-12x+16}{3x-4}}$$

$$= \frac{\frac{13x}{3x-4}}{\frac{13}{3x-4}}$$

Cancelling the common factor of $3x - 4$

$$= \frac{13x}{13}$$

$$= x$$

Thus the function is it's own inverse. Hence,

$$y = \frac{4x-1}{3x-4} \text{if and only if} x = \frac{4y-1}{3y-4}$$

for all real numbers such that $y \neq \frac{4}{3}$ and $x \neq \frac{4}{3}$.

b)

To prove

$n + 1$ is even if and only if $2(n + 3)$ is a multiple of 4

Assume $n + 1$ is even, therefore it can be writen as $n + 1 = 2k$ for some integer k.

It follows that we can write;

$$2(n + 3) = 2(n + 1) + 4$$

Substituting $n + 1 = 2k$

$$= 2(2k) + 4$$

$$= 4k + 4$$

$$= 4(k + 1)$$

and thus a multiple of 4

Conversly, assume $2(n + 1)$ is a multiple of 4, therefore it can be written as $2(n + 1) = 4k$ for some integer k.

$$2(n + 3) = 4k$$

Dividing both sides by 2

$$n + 3 = 2k$$

Rearranging gives

$$n = 2k - 3$$

$$n + 1 = 2k - 2$$

$$= 2(k - 1)$$

and thus an even number

Question 3:

a)

$(3n)! \geq (n!)^3$, for all $n \in \mathbb{N}$

Proof by induction.

Base case: $n = 1$

$$(3 \cdot 1)! = 3! = 6$$

$$(1!)^3 = 1^3 = 1$$

Thus

$6 \geq 1$ is true.

Inductive step: Assume $(3n)! \geq (n!)^3$ is true for some $n \in \mathbb{N}$.

We need to show that $(3(n+1))! \geq ((n+1)!)^3$.

Since

$$\begin{aligned}(3(n+1))! &= (3n+3)! \\ &= (3n+3)(3n+2)(3n+1)\,(3n)!\end{aligned}$$

and

$$\begin{aligned}((n+1)!)^3 &= ((n+1)(n!))^3 \\ &= (n+1)^3(n!)^3\end{aligned}$$

Now using our assumption for the inductive step, it suffices to show:

$$(3n+3)(3n+2)(3n+1) \geq (n+1)^3$$

Expanding the LHS:

$$(3n+3)(3n+2) = 9n^2 + 15n + 6$$

$$(9n^2 + 15n + 6)(3n+1) = 27n^3 + 54n^2 + 33n + 6$$

Expanding the RHS:

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$

Subtracting the RHS from the LHS:

$$(27n^3 + 54n^2 + 33n + 6) - (n^3 + 3n^2 + 3n + 1) = 26n^3 + 51n^2 + 30n + 5 \geq 0$$

This is true for all $n \in \mathbb{N}$.

Question 4:

a)
Prove that no such value of x exists such that x is a real positive number.

$$\frac{7x}{x+3} \leq \frac{x-3}{7x}$$

Assume that x is a positive real number.

$$\frac{7x}{x+3} \leq \frac{x-3}{7x}$$

Cross multiplying gives

$$(7x)(7x) \leq (x-3)(x+3)$$

Expanding both sides

$$49x^2 \leq x^2 - 9$$

Rearranging gives

$$48x^2 + 9 \leq 0$$

This is not possible as $48x^2$ is always positive for all real numbers x and 9 is a positive constant.

Thus, we have a contradiction.

We can conclude that no such value of x exists such that x is a positive real number.

b)
Prove that:

If $n^3 + 2n^2$ is not a multiple of 16, then n is odd.

Let us consider the contraposition of this statement;

If n is even, then $n^3 + 2n^2$ is a multiple of 16.

Assume n is even, therefore it can be written as $n = 2k$ for some integer k.

$$\begin{aligned} n^3 + 2n^2 &= (2k)^3 + 2(2k)^2 \\ &= 8k^3 + 2(4k^2) \\ &= 8k^3 + 8k^2 \\ &= 8(k^3 + k^2) \\ &= 8k^2(k+1) \\ &= (8k)(k(k+1)) \end{aligned}$$

As $k(k+1)$ is even, we can write it as $2l$ for some integer l.

$$\begin{aligned} &= (8k)(2l) \\ &= 16kl \end{aligned}$$

Hence a multiple of 16

Thus by proof by contraposition:

If $n^3 + 2n^2$ is not a multiple of 16, then n is odd.

Question 5:

a)

$$x_0 = 0\,\mathrm{m}, \quad x_1 = 300\,\mathrm{m}, \quad v_0 = 0\,\mathrm{m/s}, \quad \mathrm{a} = \mathrm{g} = 9.8\,\mathrm{m/s}$$

Using $x = v_0 t + \frac{1}{2}at^2$ to find t

$$x_1 = v_0 t + \frac{1}{2}at^2$$

$$300 = \frac{1}{2}gt^2$$

$$600 = gt^2$$

$$t^2 = \frac{600}{g}$$

$$t = \sqrt{\frac{600}{g}}$$

$$= \sqrt{\frac{600}{9.8}}$$

$$= 7.824\ldots$$

$$= 7.82\,\mathrm{s}$$

to 2 s.f.

Using $v = v_0 + at$

$$v_1 = v_0 + at$$

$$= 0 + g\sqrt{\frac{600}{g}}$$

$$= g\sqrt{\frac{600}{g}}$$

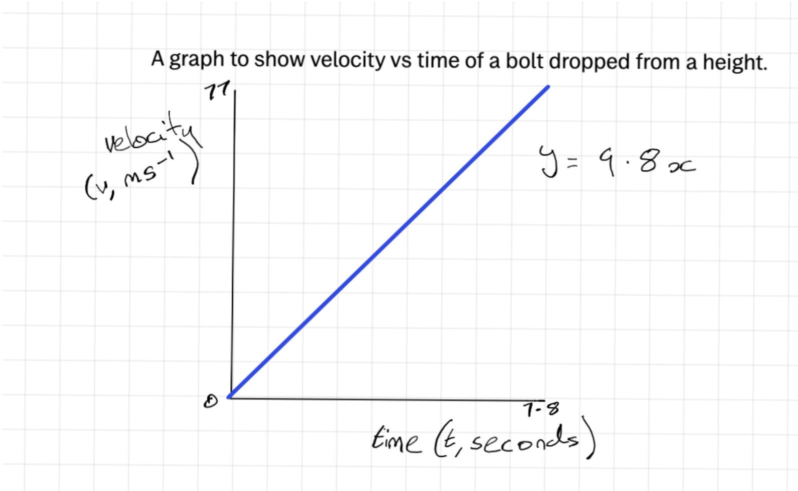
$$= \sqrt{600g}$$

$$= \sqrt{600 \times 9.8}$$

$$= 77.46\ldots$$

$$= 77\,\mathrm{m\,s^{-1}}$$ *to 2 s.f*

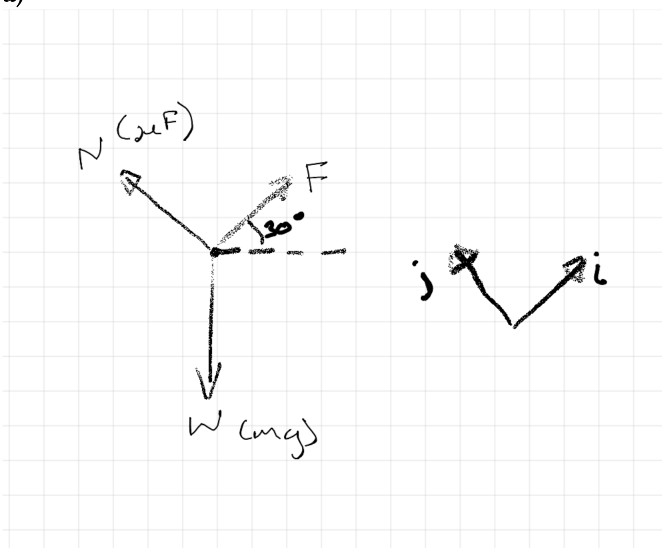
b)


A graph to show velocity vs time of a bolt dropped from a height.
77
velocity
(v, ms^{-1})
$y = 9.8x$
0
7.8
time (t, seconds)

Question 6:

$$v_0 = 0\,\text{meter/s}, \quad x_0 = 0\,\text{m}, \quad v_1 = 9\,\text{m}\,\text{s}^{-1}, \quad x_1 = 30\,\text{m}$$

a)



Using

$$v_1 = v_0 + at$$

$$9 = at$$

and

$$x_1 = v_0 t + \frac{1}{2}at^2$$

$$30 = \frac{1}{2}at^2$$

$$60 = at^2$$

Substituting $at = 9$

$$
\begin{aligned}
60 &= 9t \\
t &= \frac{60}{9} \\
&= \frac{20}{3} \\
&= 6.67\,\mathrm{s}
\end{aligned}
$$

and

$$
\begin{aligned}
a &= \frac{9}{t} \\
a &= \frac{9}{\frac{20}{3}} \\
a &= \frac{27}{20} \\
&= 1.35\,\mathrm{m\,s^{-2}}
\end{aligned}
$$

b)

$$
\begin{aligned}
\mathbf{F} &= \mu|N| \\
\mathbf{N} &= |N| \\
\mathbf{W} &= -\sin(30)mg - \cos(30)mg \\
F_i &= \sin(30)mg - \mathbf{F} \\
&= \sin(30)mg - \mu|N| \\
N_j &= \cos(30)mg
\end{aligned}
$$

thus

$$
F = \sin 30mg - \mu\cos(30)mg
$$

And using $F = ma$

$$ma = \sin(30)mg - \mu\cos(30)mg$$

Divinding through by m

$$a = \sin(30)g - \mu\cos(30)g$$

$$1.35 = \sin(30)g - \mu\cos(30)g$$

Rearranging

$$\mu\cos(30)g = \sin(30)g - 1.35$$

$$\mu = \frac{\sin(30)g - 1.35}{\cos(30)g}$$

$$= \frac{\sin(30)9.8 - 1.35}{\cos(30)9.8}$$

$$= \frac{4.9 - 1.35}{8.487}$$

$$= \frac{3.55}{8.487}$$

$$= 0.418\ldots$$

$$= 0.42$$

to 2 s.f.

Question 7:

a)
The vector expression for the acceleration is

$$\mathbf{a} = -g\mathbf{j}$$

b)
The initial velocity vector is

$$\mathbf{v}_0 = 12\cos(50^\circ)\,\mathbf{i} + 12\sin(50^\circ)\,\mathbf{j}$$

Integrating the acceleration vector to find the velocity vector:

$$\begin{aligned}\mathbf{v}(t) &= \int \mathbf{a}\ \mathrm{d}t \\ &= \int -g\mathbf{j}\ \mathrm{d}t \\ &= -gt\,\mathbf{j} + \mathbf{C}_1\end{aligned}$$

Using the initial velocity to find the constant of integration:

$$\mathbf{v}(0) = \mathbf{v}_0 \Rightarrow \mathbf{C}_1 = \mathbf{v}_0$$

Thus, the velocity vector is:

$$\mathbf{v}(t) = \mathbf{v}_0 - gt\,\mathbf{j}$$

Integrating the velocity vector to find the position vector:

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)\ \mathrm{d}t \\ &= \int (\mathbf{v}_0 - gt\,\mathbf{j})\,\mathrm{d}t \\ &= \mathbf{v}_0 t - \frac{1}{2}gt^2\,\mathbf{j} + \mathbf{C}_2\end{aligned}$$

Taking the initial position as the origin:

$$\mathbf{r}(0) = \mathbf{0} \Rightarrow \mathbf{C}_2 = \mathbf{0}$$

Therefore

$$\mathbf{r}(t) = \mathbf{v}_0 t - \frac{1}{2}gt^2\,\mathbf{j}$$

Substituting the expression for $\mathbf{v}_0$:

$$\mathbf{r} = (12t\cos(50°))\,\mathbf{i} + \left(12t\sin(50°) - \frac{1}{2}gt^2\right)\mathbf{j}$$ *(as required)*

c)

i.

Given the position vector $\mathbf{r}$ is from the orign, we want to find the time t for which the $\mathbf{j}$-component is -1.5, that is,

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^{2-4ac}}}{2a}$$

$$12t\sin(50°) - \frac{1}{2}gt^2 = -1.5$$

Rewriting

$$-\frac{1}{2}gt^2 + 12t\sin(50°) + 1.5 = 0$$

This is a quadratic equation in t:

$$t = \frac{-12\sin(50) \pm \sqrt{(12\sin(50)^2 - 4(-\frac{1}{2}g)(1.5))}}{2(-\frac{1}{2}g)}$$

$$= \frac{-12\sin(50) \pm \sqrt{(12\sin(50))^2 + \frac{147}{5}}}{-g}$$

$$= -0.151\ldots \text{ and } 2.027\ldots$$

since we can reject the negaive value for time, we have

$$= 2.027\ldots$$

$$= 2.0\,\mathrm{s}$$

to 2 s.f.

ii.
The horizonal distance traveled by the ball is given by the $\mathbf{i}$-component of the position vector $\mathbf{r}$ at time t:

$$\mathbf{r_i} = 12t\cos(50)\mathbf{i}$$

Substituting $t = 2.027\ldots$

$$= 12(2.027\ldots)\cos(50)\mathbf{i}$$

$$= 15.635\ldots$$

$$= 16\,\mathrm{m}$$

to 2 s.f.

Question 8:

$$\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 18 & 2 \end{pmatrix}$$

a)
The determinant of matrix $\mathbf{A}$ is given by:

$$\det(\mathbf{A}) = 5 \cdot 2 - 6 \cdot 18$$

$$= 10 - 108$$

$$= -98$$

The trace of matrix $\mathbf{A}$ is given by the sum of the diagonal elements:

$$\text{tr}(\mathbf{A}) = 5 + 2$$

$$= 7$$

The characteristic equation of a 2×2 matrix $\mathbf{A}$ is given by:

$$\lambda^2 - (tr\mathbf{A})\lambda + \det \mathbf{A} = 0$$

where λ is the eigenvalue.

Hence, the characteristic equation of matrix $\mathbf{A}$ is:

$$\lambda^2 - 7\lambda - 98 = 0$$

$$(\lambda - 14)(\lambda + 7) = 0$$

Hence the eigenvalues are:

$$\lambda_1 = 14$$

and

$$\lambda_2 = -7$$

The corresponding eigenvectors can be found by solving the equation:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

$$\begin{pmatrix} 5-\lambda & 6 \\ 18 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For

$$\lambda_1 = 14 :$$

$$\begin{pmatrix} 5-14 & 6 \\ 18 & 2-14 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -9 & 6 \\ 18 & -12 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives the system of equations:

$$-9x + 6y = 0$$

$$18x - 12y = 0$$

Hence

$$-9x + 6y = 18x - 12y$$

Rearranging gives

$$-27x + 18y = 0$$

$$18y = 27x$$

$$2y = 3x$$

This gives us the eigenvector:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

For

$$\lambda_2 = -7 :$$

$$\begin{pmatrix} 5-(-7) & 6 \\ 18 & 2-(-7) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 & 6 \\ 18 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives the system of equations:

$$12x + 6y = 0$$

$$18x + 9y = 0$$

Hence

$$12x + 6y = 18x + 9y$$

Rearranging gives

$$-6x - 3y = 0$$

$$-3y = 6x$$

$$-y = 2x$$

This gives us the eigenvector:

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

b)

We can express

$$\mathbf{A} = \mathbf{PDP}^{-1}$$

Where $\mathbf{P}$ is $\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$

and $\mathbf{D}$ is $\begin{pmatrix} 14 & 0 \\ 0 & -7 \end{pmatrix}$

and $\mathbf{P}^{-1}$ is the inverse of $\mathbf{P}$; $\begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{pmatrix}$

The inverse of a 2×2 matrix $\mathbf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by:

$$\mathbf{P}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where $ad - bc$ is the determinant of $\mathbf{P}$.

Hence, we can write:

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 14 & 0 \\ 0 & -7 \end{pmatrix} \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{pmatrix}$$

c)

$$\mathbf{A}^5 = \mathbf{P}\mathbf{D}^5\mathbf{P}^{-1}$$

$$\begin{aligned}
\mathbf{A}^5 &= \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 14^5 & 0 \\ 0 & (-7)^5 \end{pmatrix} \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 537824 & 0 \\ 0 & -16807 \end{pmatrix} \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{pmatrix} \\
&= \begin{pmatrix} 2 \cdot 537824 + -1 \cdot 0 & 2 \cdot 0 + -1 \cdot -16807 \\ 3 \cdot 537824 + 2 \cdot 0 & 3 \cdot 0 + 2 \cdot -16807 \end{pmatrix} \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{pmatrix} \\
&= \begin{pmatrix} 1075648 & 16807 \\ 1613472 & -33614 \end{pmatrix} \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1075648 \cdot 2}{7} + \frac{16807 \cdot -3}{7} & \frac{1075648 \cdot 1}{7} + \frac{16807 \cdot 2}{7} \\ \frac{1613472 \cdot 2}{7} + \frac{-33614 \cdot -3}{7} & \frac{1613472 \cdot 1}{7} + \frac{-33614 \cdot 2}{7} \end{pmatrix} \\
&= \begin{pmatrix} \frac{2151296 - 50421}{7} & \frac{1075648 + 33614}{7} \\ \frac{3226944 + 100842}{7} & \frac{1613472 - 67228}{7} \end{pmatrix} \\
&= \begin{pmatrix} \frac{2100875}{7} & \frac{1109262}{7} \\ \frac{3327786}{7} & \frac{1546244}{7} \end{pmatrix} \\
&= \begin{pmatrix} 300125 & 158466 \\ 475398 & 220892 \end{pmatrix}
\end{aligned}$$

d)

Question 8 d)

Define the matrix

```
(%i1) A:matrix([5,6],[18,2]);
```

A $\begin{bmatrix} 5 & 6 \\ 18 & 2 \end{bmatrix}$

Find the eignevales and associated eigenvectors

```
(%i5) eigenvalues(A);
(%o5) [[-7,14],[1,1]]
```

The [1,1] being the multiplicities

```
(%i6) eigenvectors(A);
```

(%o6) $\left[[[-7,14],[1,1]],\left[[[1,-2]],\left[\left[1,\frac{3}{2}\right]\right]\right]\right]$

These are consistant with my results, Maxima uses the normalisation of having 1 as the first value.

Check the values of A^5

```
(%i4) A^^5;
```

(%o4) $\begin{bmatrix} 300125 & 158466 \\ 475398 & 220892 \end{bmatrix}$

e)

$$\dot{x} = 5x + 6y$$

$$\dot{y} = 18x + 2y$$

The general solution of a system of linear differential equations is given by:

$$\mathbf{x} = e^{\lambda_1 t}\mathbf{v}_1 + e^{\lambda_2 t}\mathbf{v}_2$$

where λ_1 and λ_2 are the eigenvalues, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are the corresponding eigenvectors.

$$\mathbf{x} = Ce^{14t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + De^{-7t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

hence we can write:

$$x = 2Ce^{14t} - De^{-7t}$$

$$y = 3Ce^{14t} + 2De^{-7t}$$

where C and D are constants determined by initial conditions.

Question 9: