Section 2

Question 19

a) $112 - 4x^2 = 7y^2$

i. $0 = 4x^2 + 7y^2 - 112$ divide by 112

$= \frac{x^2}{28} + \frac{y^2}{16} - 1$

$\therefore \frac{x^2}{28} + \frac{y^2}{16} = 1$

which is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(where $a > b > 0$)

ii. Directrices of ellipse, $x = \pm \frac{a}{e}$

where $e = \sqrt{1 - \frac{b^2}{a^2}}$

$= \sqrt{1 - \frac{16}{28}}$

$= \sqrt{3/7}$

$= \frac{\sqrt{21}}{7}$

hence

$x = \pm \frac{\sqrt{28}}{\sqrt{21}/7}$

$= \pm \frac{14\sqrt{3}}{3}$

b) $x = 2t^2$, $y = 4t$ $(t > 0)$

i. This is a parabola because the standard parametrisation for a parabola is

$x = at^2$

$y = 2at$

with $a = 2$

ii $x = 2t^2$, $y = 4t$

after transation of 2 units left ie -2 units righ[t]
and 5 units down ie -5 units up

$x = 2t^2 - 2$ $y = 4t - 5$

Question 20

$$\int \frac{1}{(121+x^2)^2}\,dx$$

a) this should be solved using a trigonometric substitution, where $a^2 = 121$ and $a = 11$, and $x = a\tan u$ hence $x = 11\tan u$

b) with $x = 11\tan u$ $\quad dx = 11\sec^2 u\,du$

and

$$121 + x^2 = 121 + (11\tan u)^2$$

$$= 121 + 121\tan^2 u$$

$$= 121(1+\tan^2 u)$$

using symetry identities

$$= 121\sec^2 u$$

$$\int \frac{1}{(121+x^2)^2}\,dx = \int \frac{1}{(121\sec^2 u)^2} \cdot 11\sec^2 u\,du$$

$$= \int \frac{1}{121^2\sec^4 u} \cdot 11\sec^2 u\,du$$

$$= \int \frac{11\sec^2 u}{121^2\sec^4 u}\,du$$

$$= \frac{11}{121^2}\int \frac{\sec^2 u}{\sec^4 u}\,du$$

$$= \frac{11}{121^2}\int \frac{1}{\sec^2 u}\,du$$

using $\sec\theta = \frac{1}{\cos\theta}$

$$= \frac{11}{121^2}\int \cos^2 u\,du$$

using half-angle identities, $\cos^2 u = \frac{1}{2}(1+\cos 2u)$

$$= \frac{1}{2662}\int 1+\cos 2u\,du$$

$$= \frac{1}{2662}\left[\int 1\,du + \int \cos 2u\,du\right]$$

$$= \frac{1}{2662}\left[u + \tfrac{1}{2}\sin 2u\right] + C$$

using double angle identities

$$= \frac{1}{2662}\left[u + \sin u\cos u\right] + C$$

now $\quad x = 11\tan u$

$$\frac{x}{11} = \tan u = \frac{\sin u}{\cos u}$$

Question 21

$$\frac{x^2-6}{x(x^2-x+6)} = \frac{2x-1}{x^2-x+6} + \frac{A}{x}$$

a) $$\frac{\cancel{x(x^2-x+6)}(x^2-6)}{\cancel{x(x^2-x+6)}} = x(2x-1) + A(x^2-x+6)$$

$$= 2x^2 - x + Ax^2 - Ax + 6A$$

$$x^2-6 = x^2(2+A) - x(1+A) + 6A$$

So x^2 — $2+A = 1$
x — $-(1+A) = 0$
$6A = -1$

$\therefore$ $A = -1$

b) $$\int \frac{x^2-6}{x(x^2-x+6)}\,dx = \int \frac{2x-1}{x^2-x+6}\,dx - \int \frac{1}{x}\,dx$$

Let $u = x^2 - x + 6$ $\quad du = 2x - 1$

$$= \int \frac{1}{u}\,du - \ln x + C$$

$$= \ln u - \ln x + C$$

substitute back

$$= \ln(x^2-x+6) - \ln x + C$$

b) $$\frac{dy}{dx} + \frac{(x^2-6)y}{x(x^2-x+6)} = \cos x \qquad (x>0)$$

$$p(x) = \exp \int g(x)\,dx$$

using part a)

$$= \exp\left[\ln(x^2-x+6) - \ln x\right]$$

$$= e^{\ln(x^2-x-6)} \cdot e^{-\ln(x)}$$

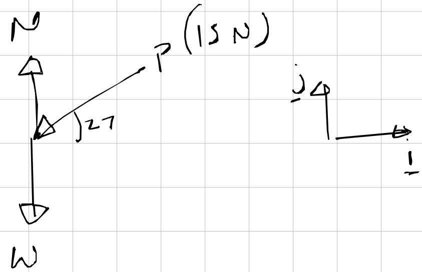
$$= (x^2-x-6) \cdot x^{-1}$$

$$= \frac{x^2-x-6}{x}$$

as required.

Question 22

a)



$\underline{W} = mg$

$\underline{N} = -mg$

$\underline{P} = 15 \cos 27$

b) $\underline{W} = -19g\,\underline{j}$

$\underline{N} = 19g\,\underline{j}$

$\underline{P} = -15 \cos 27\,\underline{i} - 15 \sin 27\,\underline{j}$

c) $\underline{F} = m\underline{a}$

Horizontal force $= 15 \cos 27$

$15 \cos 27 = 19a$

$\dfrac{15 \cos 27}{19} = a$

$= 0.70\,m/s^2$

(to 2 s.f)

Question 23

a)

$$A = \begin{pmatrix} 8 & -2 \\ 15 & -3 \end{pmatrix}$$

$\text{tr } A = 8 + -3$
$= 5$

$\det A = 8(-3) - 15(-2)$
$= 6$

characteristic equation :

$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda - 2)(\lambda - 3)$$

$\lambda = 2$
$\lambda = 3$

for $\lambda = 2$

$$\begin{bmatrix} 6 & -2 \\ 15 & -5 \end{bmatrix}$$

$3, -1$

$3x - y = 0$

$y = 3x$

eigenvector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

For $\lambda = 3$

$$\begin{bmatrix} 5 & -2 \\ 15 & -6 \end{bmatrix}$$

$5x - 2y = 0$

$5x = 2y$

$x = \frac{2}{5} y$

eigen vector $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

c) $A = \begin{pmatrix} 8 & -2 \\ 15 & -3 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ $\quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

inverse $= \frac{1}{5-6}$

$= \frac{1}{-1} \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$

$= \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$

$$PDP^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot 2 + 2\cdot 0 & 1\cdot 0 + 2\cdot 3 \\ 3\cdot 2 + 5\cdot 0 & 3\cdot 0 + 5\cdot 3 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

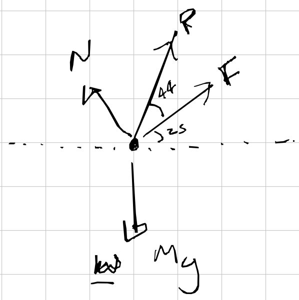
$$= \begin{pmatrix} 2 & 6 \\ 6 & 15 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\cdot -5 + 6\cdot 3 & 2\cdot 2 + 6\cdot -1 \\ 6\cdot -5 + 15\cdot 3 & 6\cdot 2 + 15\cdot -1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & -2 \\ -15 & -3 \end{pmatrix}$$

Section 3

Question 24



$\underline{N}$ = normal reaction

$\underline{w} = mg$

$\underline{F} = \mu N$

$\underline{P}$ = Force applied

$\mu = 0.37$

$\underline{\underline{w}} = 55g\cos 25\,\underline{i} + 55g\sin 25$