MST1252209I3PV1

MST125

Module Examination 2022

Essential mathematics 2

Monday 12 September 2022

There are **three sections** in this examination.

In **Section 1** you should **attempt all 18 questions**. Each question is worth 2% of the total mark. *Each question has ONE correct answer from five options.*

An incorrectly answered question will get zero marks.

Submit your answers to Section 1 using the interactive Computer-marked Examination (iCME), following the on-screen instructions. Give yourself time to check you have entered your answers correctly.

In **Section 2** you should **submit answers to all 5 questions**. Each question is worth 8% of the total mark.

In **Section 3** you should **submit answers to 2 out of the 3 questions**. Each question is worth 12% of the total mark.

Do not submit more than the required number of answers for Section 3. If you do, only the first two answers submitted for Section 3 will be marked.

For **Sections 2** and **3**:

Include all your working, as some marks are awarded for this.

Handwritten answers must be in pen, though you may draw diagrams in pencil.

Start your answer to each question on a new page, clearly indicating the number of the question.

Crossed out work will not be marked.

Follow the instructions in the online timed examination for how to submit your work.

Further information about completing and submitting your examination work is in the *Instructions and guidance for your remote examination* document on the module website.

Submit your exam using the iCMA system (iCME81). Make sure that the name of the PDF file containing your answers for Sections 2 and 3 includes your PI and the module code e.g. X1234567MST125.

PLAGIARISM WARNING – the use of assessment help services and websites

The **work that you submit for any assessment/examination on any module should be your own**. Submitting work produced by or with another person, or a web service or an automated system, **as if it is your own** is cheating. It is **strictly forbidden** by the University.

You should not:

- provide any assessment question to a website, online service, social media platform or any individual or organisation, as this is an infringement of copyright.
- request answers or solutions to an assessment question on any website, via an online service or social media platform, or from any individual or organisation.
- use an automated system (other than one prescribed by the module) to obtain answers or solutions to an assessment question and submit the output as your own work.
- discuss examination questions with any other person, including your tutor.

The University actively monitors websites, online services and social media platforms for answers and solutions to assessment questions, and for assessment questions posted by students. Work submitted by students for assessment is also monitored for plagiarism.

A student who is found to have posted a question or answer to a website, online service or social media platform and/or to have used any resulting, or otherwise obtained, output as if it is their own work has committed a disciplinary offence under **Section SD 1.2** of our Code of Practice for Student Discipline. **This means the academic reputation and integrity of the University has been undermined.**

The Open University's Plagiarism policy defines plagiarism in part as:

- using text obtained from assignment writing sites, organisations or private individuals.
- obtaining work from other sources and submitting it as your own.

If it is found that you have used the services of a website, online service or social media platform, or that you have otherwise obtained the work you submit from another person, this is considered serious academic misconduct and you will be referred to the Central Disciplinary Committee for investigation.

Section 1

You should ***submit answers to all questions in this section.***
Each question is worth 2%.

Question 1

Which of the following is a value for a for which the linear congruence

$$ax \equiv 31 \pmod{36}$$

has solutions?

- 2
- 4
- 12
- 13
- 18

Question 2

What is the multiplicative inverse of 13 modulo 56?

- 0.077 to 2 s.f.
- 3
- 13
- 43
- There is no multiplicative inverse.

Question 3

Suppose that $p > 8$ is a prime number and that a is an integer such that $a^4 \equiv 2 \pmod{p}$.

What is the least residue of $a^{(p-1)+12}$ modulo p?

- 1
- 2
- 4
- 8
- $4a^3$

Question 4

What is the eccentricity of the conic with equation

$$16cx^2 + 25y^2 = 400c$$

where $c > 0$?

- 1
- $\sqrt{1 - \frac{16c}{25}}$
- $\sqrt{\frac{16c}{25} + 1}$
- $\sqrt{1 - \frac{25}{16c}}$
- It is not possible to determine the eccentricity from the given information.

Question 5

Which of the following is a parameterisation of the straight line passing through the points $(7, 0)$ and $(2, -3a)$ where a is a real number?

- $x = 7 - 5t, \quad y = -3at.$
- $x = 2 - 7t, \quad y = 5t - 3a.$
- $x = 7 - 7t, \quad y = (-3a - 2)\, t.$
- $x = 5t, \quad y = 3at + 2.$
- $x = (-3a - 2)\, t + 2, \quad y = -7t - 3a.$

Question 6

A rock rests on a horizontal plane. The plane is slowly tilted so that the angle between the surface and the horizontal increases to 20 degrees, at which point the rock is about to slip.

What is the coefficient of static friction between the rock and the surface, to two significant figures?

- 0.36
- 2.2
- 3.4
- 9.4
- 13

Question 7

The magnitude F (in newtons) of a force $\mathbf{F}$ and the magnitude G (in newtons) of a force $\mathbf{G}$ satisfy the vector equation

$$2F\,\mathbf{i} - 3F\,\mathbf{j} + 2G\,\mathbf{i} + 2G\,\mathbf{j} - 3\,\mathbf{i} = \mathbf{0}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the Cartesian unit vectors. What is the value of F?

- $-\frac{9}{10}$
- $-\frac{3}{5}$
- $\frac{3}{5}$
- $\frac{9}{10}$
- 6

Question 8

Let f and g be the linear transformations represented by the matrices $\begin{pmatrix} 0 & -4 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 3 \\ -5 & -4 \end{pmatrix}$, respectively. What is the image of the point $(-1, 1)$ under the composite transformation $g \circ f$?

- $(-4, 1)$
- $(-14, 25)$
- $(-12, 6)$
- $(-4, 0)$
- $(-15, 24)$

Question 9

Using the substitution $x = \dfrac{10}{3}\sinh(6u)$ transforms the integral

$$\int \frac{12}{\sqrt{100+9x^2}}\,\mathrm{d}x$$

into which of the following integrals?

- $\displaystyle\int \frac{2}{3}\,\mathrm{d}u$
- $\displaystyle\int 4\,\mathrm{d}u$
- $\displaystyle\int 24\,\mathrm{d}u$
- $\displaystyle\int \frac{6}{5\,\cosh(6\,u)}\,\mathrm{d}u$
- $\displaystyle\int \frac{36}{5\,\cosh(6\,u)}\,\mathrm{d}u$

Question 10

Suppose that $a > 0$. Which of the following functions has all of the following features?

- One negative x-intercept
- One positive y-intercept
- One vertical asymptote
- One horizontal asymptote

- $f(x) = \dfrac{x+7}{a-x}$
- $f(x) = \dfrac{1-7x}{x^2+a}$
- $f(x) = \dfrac{x+7}{x^2+a}$
- $f(x) = \dfrac{x^2+a}{x-7}$
- $f(x) = \dfrac{a-x}{x+7}$

Question 11

Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x,$$

where $y = 1$ when $x = 0$.

What is the value of y when $x = 2$?

- $\dfrac{1}{2}$
- 1
- $\dfrac{9}{2}$
- 7
- e^6

Question 12

For which of the following differential equations could we find a general solution using the integrating factor method with integrating factor $\exp\left(\frac{\sinh(6x)}{3}\right)$?

- $y\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\cosh(6x)}{18}y = \tanh(6x)$
- $\frac{\mathrm{d}y}{\mathrm{d}x} + 2\cosh(6x)y = \sinh(6y)$
- $\frac{\mathrm{d}y}{\mathrm{d}x} + 2\cosh(6x)y = 6x^2$
- $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\sinh(6x)}{3}y = \cosh(6x)$
- $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\cosh(6x)}{3}y = \sinh(6x)$

Question 13

Which of the options below is the negation of the following statement?

There exists $n \in \mathbb{Z}$ such that $n^2 \geq 36$.

- For every $n^2 < 36$, $n \in \mathbb{Z}$.
- There exists $n \in \mathbb{Z}$ such that $n^2 < 36$.
- For every $n \in \mathbb{Z}$, $n^2 < 36$.
- There exists $n^2 \geq 36$ such that $n \in \mathbb{Z}$.
- For every $n \in \mathbb{Z}$, $n^2 \geq 36$.

Question 14

Suppose that

- $P(x)$ is the statement '$x \geq 11$',
- $Q(x)$ is the statement '$x \leq -2$',
- $R(x)$ is the statement '$(x-3)(x+2) \geq 0$'.

Which of the following statements is true?

- $P(x) \Rightarrow Q(x)$
- $R(x) \Rightarrow Q(x)$
- $R(x) \Leftrightarrow P(x)$
- $R(x) \Leftrightarrow Q(x)$
- $P(x) \Rightarrow R(x)$

Question 15

The position of a particle is given in terms of the time t by

$$\mathbf{r} = 3\cos(t)\,\mathbf{i} + \left(2t^5 + 5t^2\right)\mathbf{j} + 5e^{2t}\,\mathbf{k},$$

where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are the Cartesian unit vectors. What is the velocity of the particle at time t?

- $-3\sin(t)\,\mathbf{i} + \left(10t^4 + 10t\right)\mathbf{j} + 10e^{2t}\,\mathbf{k}$
- $-3\sin(t)\,\mathbf{i} + \left(\dfrac{5t^6}{3} + \dfrac{10t^3}{3}\right)\mathbf{j} + 5e^{2t}\,\mathbf{k}$
- $3\sin(t)\,\mathbf{i} + \left(\dfrac{t^6}{3} + \dfrac{5t^3}{3}\right)\mathbf{j} + \dfrac{5e^{2t}}{2}\,\mathbf{k}$
- $3\sin(t)\,\mathbf{i} + \left(10t^4 + 10t\right)\mathbf{j} + 10e^{2t}\,\mathbf{k}$
- $-3\sin(t)\,\mathbf{i} + \left(10t^4 + 10t\right)\mathbf{j} + 5e^{2t}\,\mathbf{k}$

Question 16

A 2×2 matrix has trace -13. One of its eigenvalues is -9. What is its other eigenvalue?

- -9
- -4
- $\frac{13}{9}$
- $\sqrt{13}$
- 4

Question 17

The 2×2 upper triangular matrix $\mathbf{A}$ has eigenvalues a and -5 (with $a \neq -5$). An eigenvector corresponding to the eigenvalue a is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Which of the following could be the matrix $\mathbf{A}$?

- $\begin{pmatrix} a & 0 \\ 2 & -5 \end{pmatrix}$
- $\begin{pmatrix} a & -5 \\ 0 & 2 \end{pmatrix}$
- $\begin{pmatrix} a & 2 \\ 0 & -5 \end{pmatrix}$
- $\begin{pmatrix} -5 & 2 \\ 0 & a \end{pmatrix}$
- $\begin{pmatrix} -5 & a \\ 2 & 0 \end{pmatrix}$

Question 18

One of the eigenvalues of the matrix

$$\begin{pmatrix} 1 & -a \\ c & a \end{pmatrix}$$

is $2(c+1)$, and a corresponding eigenvector is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$. What is the value of a?

- $-1-2c$
- $2\ (2c+1)$
- $\frac{-4}{3}(c+1)$
- $-2\ (2+c)$
- 0

SECTION 2

*You should **attempt all questions**, write in **pen** and start your answer to each question on a new page.*

***Include all your working**, as some marks are awarded for this.*
Each question is worth 8%.

Question 19

Consider the hyperbola in standard position with foci $(\pm 17, 0)$ and x-intercepts ± 15.

(a) Find the eccentricity e of the hyperbola. [*1*]

(b) Find the equations of the directrices of the hyperbola. [*1*]

(c) Hence show that the equation of the hyperbola is

$$64x^2 - 225y^2 = 14\,400.$$ [*3*]

(d) The hyperbola is translated four units to the left and one unit down.

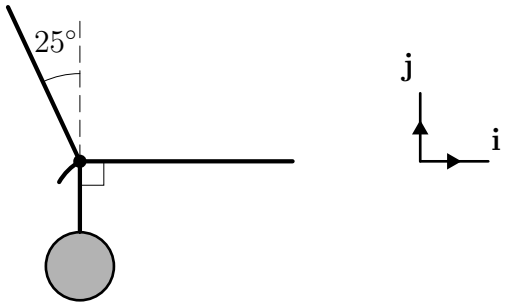
(i) Write down the equations of the directrices of the translated hyperbola. [*1*]

(ii) Find a parametrisation of the translated hyperbola. [*2*]

Question 20

A decoration hangs at rest from a string. A second, horizontal string is knotted to the first, pulling it to the right, as shown below. The top section of the first string is at an angle of 25° to the vertical.
The tension in the horizontal string is 0.8 N.

Take the magnitude of the acceleration due to gravity to be $g = 9.8\,\mathrm{m\,s^{-2}}$.



(a) State the three forces acting on the knot, and draw a force diagram to represent them, labelling the forces appropriately and indicating their directions by marking the sizes of relevant angles. [*3*]

(b) Find expressions for the component forms of the three forces, in terms of unknown magnitudes where appropriate, taking the Cartesian unit vectors **i** and **j** to point horizontally and vertically, respectively, in the directions shown above. [*2*]

(c) Find the mass of the decoration, to two significant figures. [*3*]

Question 21

(a) Show that

$$\sin^4 x = 1 - 2\cos^2 x + \cos^4 x. \qquad [2]$$

(b) Hence find the following integral

$$\int \sin^5 x \cos^n x \, \mathrm{d}x,$$

where $n > 1$ is an integer. [6]

Question 22

This question concerns the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{e^{2y}\cosh t + e^{2y}\sinh t}{\cosh t + e^{y}\cosh t}.$$

(a) Factorise the numerator and denominator of the right-hand side of the differential equation, and hence write it in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)g(y). \qquad [2]$$

(b) Hence find the general solution in implicit form of the differential equation. [4]

(c) Find the particular solution in implicit form of the differential equation that satisfies the initial condition $y = 0$ when $t = 0$. [2]

Question 23

In each of the following statements, n represents a positive integer. One of the statements is true and the other two are false.

(A) If $n!$ is even, then n is not prime.

(B) If n is a square number, then n is not prime.

(C) If n is a prime number and $n \geq 10$, then $n + 2$ is prime or $n + 4$ is prime.

(a) Write down which two statements are false and in each case give a counter-example to show that it is false. [4]

(b) Prove the true statement. [4]

SECTION 3

*You should attempt **two questions**. If you attempt more, the scores from your best two questions will count. Write in **pen** and start your answer to each question on a new page.*

***Include all your working**, as some marks are awarded for this.*
Each question is worth 12%.

Question 24

(a) (i) Write down the matrix of the linear transformation f that maps $(1, 0)$ to $(3, 1)$ and $(0, 1)$ to $(-1, 1)$. [*1*]

(ii) Show that f is invertible and find the matrix of f^{-1}. [*2*]

(iii) Find the equation of the image of the unit circle $x^2 + y^2 = 1$ under the linear transformation f. Write your answer in the form $ax^2 + bxy + cy^2 = d$, where a, b, c and d are integers whose values you should find. [*3*]

(b) (i) Find the matrix that represents the reflection g in the line through the origin with angle of inclination 165°. [*2*]

(ii) The matrix that represents the rotation h through 90° about the origin is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Show that the composite transformation formed from the rotation h followed by the reflection g is the reflection in the line $y = -\sqrt{3}\,x$. [*4*]

Question 25

(a) A rational function g has *derivative*

$$g'(x) = \frac{-2(x+3)}{(x-1)^3}.$$

(i) Find the x-coordinate(s) of the stationary point(s) of g. [*1*]

(ii) Construct a table of signs for g'. [*3*]

(iii) State the interval(s) on which g is increasing and the interval(s) on which g is decreasing. [*1*]

(iv) Determine the nature of the stationary point(s) of g. [*1*]

(b) Determine whether the function

$$h(x) = \frac{x^2+3}{(x-1)^2}$$

is even, odd or neither. [*1*]

(c) The information below about a rational function f was obtained by following the steps of the graph-sketching strategy.

- The domain of f is $(-\infty, 2) \cup (2, \infty)$.
- The graph of f has only one x-intercept, namely 0, and only one y-intercept, namely 0.
- f is increasing on the intervals $(-\infty, -1)$ and $(2, \infty)$, and decreasing on the interval $(-1, 2)$.
- f has a stationary point, which is a local maximum, at $(-1, 3)$.
- The graph of f has two asymptotes: $x = 2$ and $y = 0$.
- f is neither even nor odd.

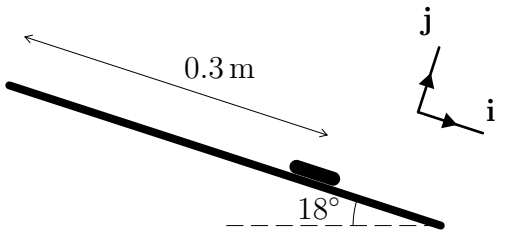
Sketch the graph of f. [*5*]

Question 26

A plastic disc is flicked up a sloping board. It has an initial speed of $1.6\,\mathrm{ms}^{-1}$, but gradually slows down.

The board is inclined at 18° to the horizontal, and when the disc is flicked its centre is 0.3 m from the top of the board, as shown below. The coefficient of sliding friction between the disc and the board is 0.05.

Model the disc as a particle, and take the magnitude of the acceleration due to gravity to be $g = 9.8\,\mathrm{m\,s}^{-2}$.



In your response to this question, underline vectors to distinguish them from scalar quantities. If the magnitude of a vector is unknown, use the vector letter to represent the magnitude. For example, write the magnitude of a vector $\underline{\mathrm{A}}$ as A.

(a) State the three forces acting on the disc while it is sliding up the board, and draw a force diagram to represent them, labelling each force appropriately and indicating the directions of the forces by marking the sizes of suitable angles. [*4*]

(b) Find expressions for the component forms of the three forces, in terms of unknown magnitudes where appropriate, taking the Cartesian unit vectors **i** and **j** to point parallel and perpendicular, respectively, to the board, in the directions shown above. [*2*]

(c) Write down the vector equation obtained by applying Newton's second law of motion to the disc. Hence, or otherwise, find the acceleration of the disc, to two significant figures. [*4*]

(d) Determine whether the disc falls off the top of the board. [*2*]

[END OF QUESTION PAPER]