MST1252306F1PV1

MST125

Module Examination 2023

Essential mathematics 2

Tuesday 6 June 2023

There are **three sections** in this examination.

In **Section 1** you should **attempt all 18 questions**. Each question is worth 2% of the total mark. *Each question has ONE correct answer from five options.*

An incorrectly answered question will get zero marks.

Submit your answers to Section 1 using the interactive Computer-marked Examination (iCME), following the on-screen instructions. Give yourself time to check you have entered your answers correctly.

In **Section 2** you should **submit answers to all 5 questions**. Each question is worth 8% of the total mark.

In **Section 3** you should **submit answers to 2 out of the 3 questions**. Each question is worth 12% of the total mark.

Do not submit more than the required number of answers for Section 3. If you do, only the first two answers submitted for Section 3 will be marked.

For **Sections 2** and **3**:

Include all your working, as some marks are awarded for this.

Handwritten answers must be in pen, though you may draw diagrams in pencil.

Start your answer to each question on a new page, clearly indicating the number of the question.

Crossed out work will not be marked.

Follow the instructions in the online timed examination for how to submit your work.

Further information about completing and submitting your examination work is in the *Instructions and guidance for your remote examination* document on the module website.

Submit your exam using the iCMA system (iCME81). Make sure that the name of the PDF file containing your answers for Sections 2 and 3 includes your PI and the module code e.g. X1234567MST125.

PLAGIARISM WARNING – the use of assessment help services and websites

The **work that you submit for any assessment/examination on any module should be your own**. Submitting work produced by or with another person, or a web service or an automated system, **as if it is your own** is cheating. It is **strictly forbidden** by the University.

You should not:

- provide any assessment question to a website, online service, social media platform or any individual or organisation, as this is an infringement of copyright.
- request answers or solutions to an assessment question on any website, via an online service or social media platform, or from any individual or organisation.
- use an automated system (other than one prescribed by the module) to obtain answers or solutions to an assessment question and submit the output as your own work.
- discuss examination questions with any other person, including your tutor.

The University actively monitors websites, online services and social media platforms for answers and solutions to assessment questions, and for assessment questions posted by students. Work submitted by students for assessment is also monitored for plagiarism.

A student who is found to have posted a question or answer to a website, online service or social media platform and/or to have used any resulting, or otherwise obtained, output as if it is their own work has committed a disciplinary offence under **Section SD 1.2** of our Code of Practice for Student Discipline. **This means the academic reputation and integrity of the University has been undermined.**

The Open University's Plagiarism policy defines plagiarism in part as:

- using text obtained from assignment writing sites, organisations or private individuals.
- obtaining work from other sources and submitting it as your own.

If it is found that you have used the services of a website, online service or social media platform, or that you have otherwise obtained the work you submit from another person, this is considered serious academic misconduct and you will be referred to the Central Disciplinary Committee for investigation.

Section 1

You should ***submit answers to all questions in this section.***
Each question is worth 2%.

Question 1

If $a \equiv 8 \pmod{19}$ and $b \equiv 6 \pmod{19}$, which of the following is the least residue of $a \times b$ modulo 19?

- 2
- 10
- 12
- 14
- 48

Question 2

Which of the following is congruent to 5^{3p+1} modulo p for every prime number p?

- 5
- 25
- 625
- 5^{3p}
- 5^{3p-3}

Question 3

Which of the following values of x is a solution of the linear congruence

$$ax \equiv 18 \pmod{n}$$

given that a and n are coprime integers and 3 is a multiplicative inverse of a modulo n?

- 2
- 6
- 9
- 54
- 162

Question 4

What type of conic has an equation of the form

$$6x^2 + 7xy + 9y^2 + Dx + Ey + F = 0$$

where D, E and F are constants?

- A circle.
- An ellipse.
- A parabola.
- A hyperbola.
- There is not enough information to identify the type of conic.

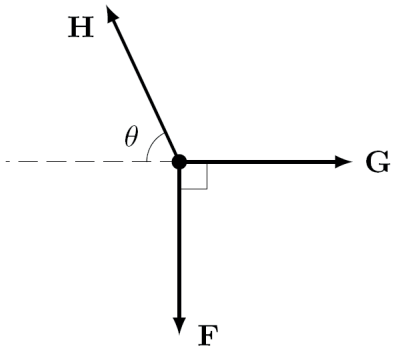
Question 5

A box rests on a flat surface. The normal reaction of the surface on the box is 6 N and the coefficient of static friction between the box and the surface is 0.14. A horizontal force is applied to the box so that it is on the point of slipping. To two significant figures, what is the magnitude of the horizontal force in newtons?

- 0.023
- 0.84
- 1.4
- 8.2
- 43

Question 6

Three forces $\mathbf{F}, \mathbf{G}$ and $\mathbf{H}$ act on a particle that is in equilibrium. The force $\mathbf{F}$ acts vertically down, the force $\mathbf{G}$ acts horizontally to the right and the force $\mathbf{H}$ acts up and to the left at an angle of θ to the horizontal as indicated in the diagram below.



If $|\mathbf{F}| = 3$ and $|\mathbf{G}| = 6$ what is the angle θ in degrees to two significant figures?

- $27°$
- $30°$
- $60°$
- $63°$
- $72°$

Question 7

Which of the statements below is true for the linear transformation

$f(\mathbf{x}) = \begin{pmatrix} 2 & -2 \\ 1 & 6 \end{pmatrix} \mathbf{x}$?

- f is not one-to-one
- f preserves orientation and scales areas by the factor 10
- f preserves orientation and scales areas by the factor 14
- f reverses orientation and scales areas by the factor 10
- f reverses orientation and scales areas by the factor 14

Question 8

If (x, y) is a fixed point of the affine transformation

$$f(\mathbf{x}) = \begin{pmatrix} 6 & -6 \\ 3 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -5 \end{pmatrix},$$

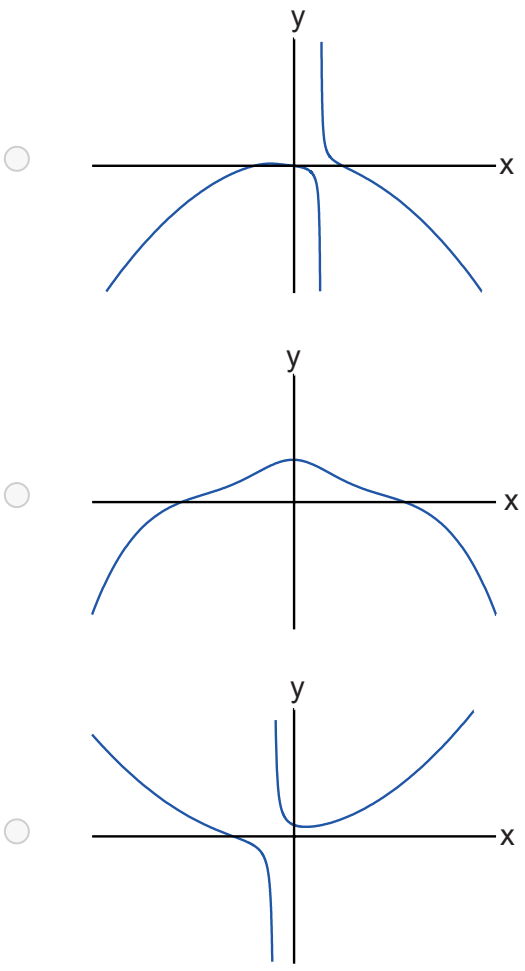
then x and y satisfy which of the following pairs of simultaneous equations?

- $5x - 6y = -1, \quad 3x + y = 5.$
- $6x - 6y = 1, \quad 3x + 2y = -5.$
- $7x - 6y = 1, \quad 3x + 3y = -5.$
- $5x + 3y = -1, \quad -6x + y = 5.$
- $6x + 3y = 1, \quad -6x + 2y = -5.$

Question 9

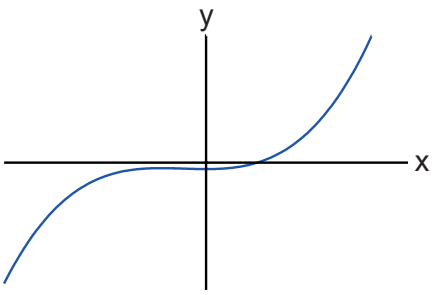
Which of the sketches below shows the graph $y = f(x)$ of a function f with all of the following properties?

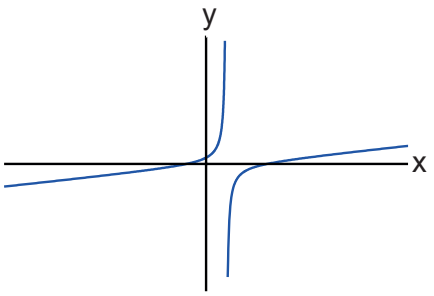
- f has a single vertical asymptote.
- $f(x) \to -\infty$ as $x \to -\infty$.
- $f(x) \to -\infty$ as $x \to \infty$.



Question 9 continues overleaf

Question 9 Continued





Question 10

Using the substitution $x = \cosh(8u)$ transforms the integral

$$\int \frac{\sqrt{x^2 - 1}}{x}\,\mathrm{d}x \qquad (x \geq 1)$$

into which of the following integrals?

- $\displaystyle\int \frac{1}{8\,\cosh(8\,u)}\,\mathrm{d}u$
- $\displaystyle\int \frac{\sinh(8\,u)}{\cosh(8\,u)}\,\mathrm{d}u$
- $\displaystyle\int 8\,\sinh^2(8\,u)\,\mathrm{d}u$
- $\displaystyle\int \frac{\sinh^2(8\,u)}{\cosh(8\,u)}\,\mathrm{d}u$
- $\displaystyle\int \frac{8\,\sinh^2(8\,u)}{\cosh(8\,u)}\,\mathrm{d}u$

Question 11

Suppose that $p(x)$ is a quadratic expression in x. If

$$\frac{x-2}{(x-6)p(x)} = \frac{A}{x-6} + \frac{Bx+C}{p(x)}$$

where A, B and C are constants and $p(6) = 3$, what is the value of A?

- $\frac{1}{6}$
- $\frac{3}{4}$
- $\frac{4}{3}$
- 6
- There is not enough information to find the value of A

Question 12

Which of the following describes the first order differential equation

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3\,y^2 + \sin(8\,x)\,y + e^{2\,x} + 13$?

- Directly integrable
- Separable
- Linear homogeneous
- Linear non-homogeneous
- None of the above

Question 13

If the relationship between the quantities x and y is described by the initial value problem

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^3$, where $y = 7$ when $x = 0$,

what is the value of y when $x = 4$, to two significant figures?

- 12
- 15
- 68
- 71
- 92

Question 14

Separating the variables in the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin(4\,x)}{y^7},$$

leads to which of the following equations?

- $\displaystyle\int y^7\,\mathrm{d}y = \int \sin(4x)\,\mathrm{d}x$
- $\displaystyle\int y^7\,\mathrm{d}y = \int \frac{1}{\sin(4x)}\,\mathrm{d}x$
- $\displaystyle\int \mathrm{d}y = \int \frac{\sin(4x)}{y^7}\,\mathrm{d}x$
- $\displaystyle\int \frac{1}{y^7}\,\mathrm{d}y = \int \sin(4x)\,\mathrm{d}x$
- $\displaystyle\int \frac{1}{y^7}\,\mathrm{d}y = \int \frac{1}{\sin(4x)}\,\mathrm{d}x$

Question 15

Which of the following is the negation of the statement

'There exists $x > 0$ such that $\cosh(x) = 19$.' ?

- $\cosh(x) \neq 19$ for all $x \leq 0$.
- $\cosh(x) \neq 19$ for some $x > 0$.
- $\cosh(x) \neq 19$ for all $x > 0$.
- $\cosh(x) = 19$ for all $x > 0$.
- $\cosh(x) = 19$ for all $x \leq 0$.

Question 16

Suppose that P, Q and R are three statements and that we know:

1. the statement $P \Rightarrow Q$ is true,
2. the statement $Q \Leftrightarrow R$ is true,
3. the statement R is true.

Which of the following conclusions can be deduced from this information?

- The statement P is false.
- The statement P is true.
- The statement P AND Q is false.
- The statement P AND Q is true.
- The statement Q AND R is true.

Question 17

The acceleration of a particle at time t is given by

$$\mathbf{a} = \cosh(9t)\,\mathbf{i} + 18t\,\mathbf{k}.$$

Given that initially it has velocity $4\,\mathbf{i} + 3\,\mathbf{j}$, what is its velocity at time t?

- $\left(-\frac{1}{9}\sinh(9\,t) + 4\right)\,\mathbf{i} + 3\,\mathbf{j} + 9\,t^2\,\mathbf{k}$
- $\left(\frac{1}{9}\sinh(9\,t) + 4\right)\,\mathbf{i} + 3\,\mathbf{j} + 9\,t^2\,\mathbf{k}$
- $\frac{1}{9}\sinh(9\,t)\,\mathbf{i} + 9\,t^2\,\mathbf{k}$
- $9\,\sinh(9\,t)\,\mathbf{i} + 18\,\mathbf{k}$
- $(9\,\sinh(9\,t) + 4)\,\mathbf{i} + 3\,\mathbf{j} + 18\,\mathbf{k}$

Question 18

A particle of mass 5 kg moves along a straight line. At time t its distance in metres from its starting point is given by

$$9t^2 + 8t.$$

What is the magnitude in newtons of the force acting on the particle at time t?

- 18
- 45
- 90
- $5\,(18t + 8)$
- $5\left(\frac{3t^4}{4} + \frac{4t^3}{3}\right)$

SECTION 2

*You should **submit answers to all questions in this section**, write in **pen** and start your answer to each question on a new page.*

***Include all your working**, as most marks are awarded for this. Answers without appropriate supporting working as directed by the question **will not be given credit**.*

Each question is worth 8%.

Question 19

A hyperbola in standard position has foci $(\pm 2\sqrt{3}, 0)$ and directrices $x = \pm\sqrt{3}$.

(a) Find its eccentricity e and its positive x-intercept a. Give exact answers, simplifying where possible. [3]

(b) Hence show that its equation is

$$x^2 - y^2 = 6.$$ [3]

(c) Find its asymptotes. [1]

(d) Is it a rectangular hyperbola? Justify your answer. [1]

Question 20

The linear transformation f is formed from the composite of the $(2, 3)$-scaling followed by the vertical shear with shear factor -2.

(a) Find the matrix $\mathbf{A}$ of f. [2]

(b) Show that f is invertible and find the matrix of f^{-1}. [2]

(c) Find the image of the point $(-1, 4)$ under f, and find the point that has image $(-2, 2)$ under f. [2]

(d) A circle $\mathcal{C}$ has area 12π. Find the area of $f^{-1}(\mathcal{C})$. [2]

Question 21

By using first the identity

$$2\sinh x \cosh x = \sinh(2x)$$

and then the identity

$$\sinh^2 x = \frac{1}{2}(\cosh(2x) - 1),$$

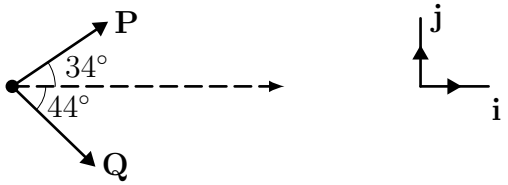
find the integral

$$\int \sinh^2(5t)\cosh^2(5t)\,\mathrm{d}t.$$ [8]

Question 22

A particle of mass 15 kg is acted on by two forces **P** and **Q**, and no other forces, causing it to accelerate in a straight line.

The diagram below shows the angles that **P** and **Q** make with the path of the particle, which is shown as a dashed line. The magnitude of **Q** is 21 N. Let the directions of the Cartesian unit vectors **i** and **j** be as shown, with **i** parallel to the path of the particle. Give numerical answers to two significant figures.



(a) Find expressions for the component forms of **P** and **Q**, denoting the magnitude of **P** by P. [2]

(b) Write down the vector equation obtained by applying Newton's second law of motion to the particle. Hence, or otherwise, find the acceleration of the particle. [4]

(c) The particle starts from rest. Find its speed after it has moved 2 m. [2]

Question 23

(a) Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$.

(i) Without using the characteristic equation of $\mathbf{A}$, show that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$, and find the corresponding eigenvalue. [*1*]

(ii) The characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 6\lambda + 9 = 0.$$

Explain how you can deduce from the characteristic equation that the eigenvalue you found in part (i) is a repeated eigenvalue. [*1*]

(b) A matrix $\mathbf{B}$ has eigenvectors $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, with corresponding eigenvalues 4 and -1 respectively.

(i) Write down a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{P}$ such that $\mathbf{B} = \mathbf{PDP^{-1}}$, and calculate $\mathbf{P^{-1}}$. [*2*]

(ii) Hence calculate $\mathbf{B}^4$. [*4*]

SECTION 3

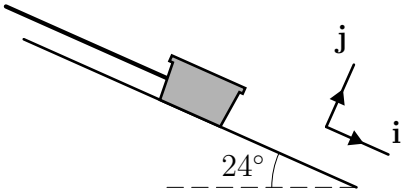
*You should **submit answers to two questions in this section**. If you submit more, only the first two answers in your submission will be marked. Write in **pen** and start your answer to each question on a new page.*

***Include all your working**, as most marks are awarded for this. Answers without appropriate supporting working as directed by the question **will not be given credit**.*

Each question is worth 12%.

Question 24

A crate rests on a rough plane inclined at 24° to the horizontal, as shown below. It is on the point of slipping *down* the plane, but is held in place by a rope parallel to the plane. The tension in the rope is 52 N. The coefficient of static friction between the crate and the plane is 0.27.



Model the crate as a particle and the rope as a model string, and take the magnitude of the acceleration due to gravity to be $g = 9.8\,\mathrm{m\,s^{-2}}$.

In your response to this question, underline vectors to distinguish them from scalar quantities. If the magnitude of a vector is unknown, use the vector letter to represent the magnitude. For example, write the magnitude of a vector $\underline{\mathrm{A}}$ as A.

(a) State the four forces that act on the crate. Draw a force diagram to represent them, labelling the forces appropriately and indicating their directions by marking the sizes of suitable angles. [*3*]

(b) Find expressions for the component forms of the four forces, in terms of unknown magnitudes where appropriate, taking the Cartesian unit vectors **i** and **j** to point parallel and perpendicular, respectively, to the plane, in the directions shown above. [*3*]

(c) Find the mass of the crate, in kilograms to two significant figures. [*4*]

(d) The tension in the rope is provided by running it over a pulley and attaching a metal weight to the other end. Determine the mass of the metal weight, in kilograms to two significant figures, assuming the pulley is a model pulley. [*2*]

Question 25

(a) Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^3(2x+1)\sin(x^2+x)}{2-y^3e^y} \qquad (y > 1).$$

Without solving the differential equation, show that

$$\frac{1}{y^2} + e^y = \cos(x^2+x)$$

is a solution in implicit form. [*8*]

(b) (i) Write down a differential equation that has the following as a particular solution:

$$y = (x+1)^{-3} + \cos(3x) + e^{-x} + 7.$$ [*2*]

(ii) Find an initial condition for your differential equation that is satisfied by the particular solution in part (b)(i). [*2*]

Question 26

(a) Prove that the following statement is true for all real numbers x by using a sequence of equivalences:

$$(x+3)(3x+3) \leq (2x+3)^2.$$ [*3*]

(b) Use mathematical induction to prove that the following statement is true for the matrix $\mathbf{M} = \begin{pmatrix} 2 & -2 \\ 0 & -1 \end{pmatrix}$:

$$\mathbf{M}^n = \begin{pmatrix} 2^n & \frac{2}{3}((-1)^n - 2^n) \\ 0 & (-1)^n \end{pmatrix} \text{ for all } n \in \mathbb{N}.$$ [*9*]

[END OF QUESTION PAPER]