Question 1:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Prove for $n = 4$

$$\begin{aligned} 1 + 3 + 5 + (2(4) - 1) &= 1 + 3 + 5 + 7 \\ &= 16 \\ &= 4^2 \end{aligned}$$

assume $n = k$ holds true

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

Prove for $n = k + 1$

$$\begin{aligned} 1 + 3 + 5 + (2k - 1) + (2(k + 1) - 1) &= 1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) \\ &= k^2 + (2(k + 1) - 1) \\ &= k^2 + (2k + 2 - 1) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

Thus, the statement holds true for $n = k + 1$ if it holds true for $n = k$.

Question 2:

$$5^n \text{always ends in a } 5$$

Prove for $n = 1$

$$5^1 = 5$$

assume $n = k$ holds true

$$5^k = \text{ends in a } 5$$

Prove for $n = k + 1$

$$\begin{aligned} 5^{k+1} &= 5^k \cdots 5 \\ &= \text{ends in a } 5 \cdot 5 \\ &= \text{ends in a } 5 \end{aligned}$$

Thus, the statement holds true for $n = k + 1$

Question 3:

$$7^n - 1 = 6m$$

Prove for $n = 1$

$$7^1 - 1 = 6$$

Assume $n = k$ holds true

$$7^k - 1 = 6m$$

Prove for $n = k + 1$

$$\begin{aligned} 7^{k+1} - 1 &= 7 \cdot 7^k - 1 \\ &= 7(6m + 1) - 1 \\ &= 42m + 7 - 1 \\ &= 42m + 6 \\ &= 6(7m + 1) \\ &= 6m' \end{aligned}$$

where $m' = 7m + 1$

Thus, the statement holds true for $n = k + 1$

Question 4:

$$2^n \geq n + 1$$

Prove for $n = 1$

$$2^1 \geq 1 + 1$$

$$= 2$$

assume $n = k$ holds true

$$2^k \geq k + 1$$

Prove for $n = k + 1$

$$2^{k+1} \geq k + 1 + 1$$

$$\geq k + 2$$

$$k + 1 \geq \frac{k+2}{2}$$

$$\geq \frac{k}{2} + 1$$

Thus, the statement holds true for $n = k + 1$ if it holds true for $n = k$.

Question 5:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Prove for $n = 1$

$$\begin{aligned} 1^3 &= \left(\frac{1(1+1)}{2}\right)^2 \\ &= \left(\frac{1 \cdot 2}{2}\right)^2 \\ &= 1^2 \\ &= 1 \end{aligned}$$

assume $n = k$ holds true

$$1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$$

Prove for $n = k + 1$

$$\begin{aligned} 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\ &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)(k+1)(k+1) \\ &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\ &= \left(\frac{k(k+1)}{2}\right)^2 + k^3 + 3k^2 + 3k + 1 \end{aligned}$$