# MST125/Specimen

**Module Examination**

**Essential mathematics 2**

There are **three sections** in this examination.

In **Section 1** you should **attempt all 18 questions**. Each question is worth 2% of the total mark. *Each question has ONE correct answer from five options.*

An incorrectly answered question will get zero marks.

Submit your answers to Section 1 using the interactive Computer-marked Examination (iCME), following the on-screen instructions. Give yourself time to check you have entered your answers correctly.

In **Section 2** you should **submit answers to all 5 questions**. Each question is worth 8% of the total mark.

In **Section 3** you should **submit answers to 2 out of the 3 questions**. Each question is worth 12% of the total mark.

Do not submit more than the required number of answers for Section 3. If you do, only the first two answers submitted for Section 3 will be marked.

For **Sections 2** and **3**:

**Include all your working**, as some marks are awarded for this.

You are expected to use the methods and results taught in MST125 rather than any alternative methods or results not covered in MST125.

Handwritten answers must be in pen, though you may draw diagrams in pencil.

Start your answer to each question on a new page, clearly indicating the number of the question.

Crossed out work will not be marked.

Follow the instructions in the online timed examination for how to submit your work.

Further information about completing and submitting your examination work is in the *Instructions and guidance for your remote examination* document on the module website.

Submit your exam using the iCMA system (iCME81). Make sure that the name of the PDF file containing your answers for Sections 2 and 3 includes your PI and the module code e.g. X1234567MST125.

# SECTION 1

*You should* ***submit answers to all questions. In this section each question is worth 2%.***

**Question 1**

What is the least residue of $6^{24}$ modulo 23 ?

**A** 1 **B** 6 **C** 9 **D** 13 **E** 17

**Question 2**

Which of the following is a value of $b$ for which the linear congruence

$$24x \equiv b \pmod{33}$$

has no solutions?

**A** 6 **B** 9 **C** 12 **D** 15 **E** 16

**Question 3**

A parabola in standard position has equation $4y^2 - 3x = 0$. What is the equation of its directrix?

**A** $x = -3$ **B** $x = -\frac{3}{4}$ **C** $x = -\frac{3}{8}$ **D** $x = -\frac{3}{16}$ **E** $x = \frac{3}{4}$

**Question 4**

What is the eccentricity of the ellipse with equation

$$\frac{x^2}{16} + \frac{y^2}{2} = 1 \ ?$$

**A** $\frac{1}{8}$ **B** $\frac{1}{2\sqrt{2}}$ **C** $\frac{\sqrt{7}}{2\sqrt{2}}$ **D** $\frac{3}{2\sqrt{2}}$ **E** $\frac{7}{8}$

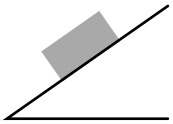
**Question 5**

Which of the following parametrisations represents the line segment that joins the points $(-1, 2)$ and $(2, 4)$ ?

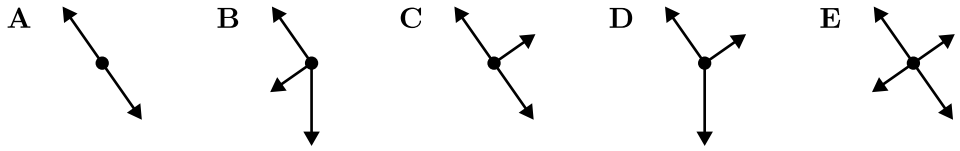
**A** $x = -1 + 3t, \quad y = 2 + 2t \quad (0 \leq t \leq 1)$
**B** $x = -1 + 3t, \quad y = 2 + 2t \quad (-1 \leq t \leq 2)$
**C** $x = 2 + 2t, \quad y = 4 - 2t \quad (0 \leq t \leq 2)$
**D** $x = 1 + 2t, \quad y = 2 + 4t \quad (0 \leq t \leq 1)$
**E** $x = 1 + 2t, \quad y = 2 + 4t \quad (-1 \leq t \leq 2)$

**Question 6**

A block rests on an inclined flat surface, as shown. The block is prevented from sliding down the slope by the friction between the block and the surface.

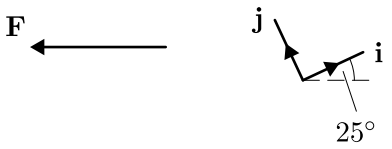


Which of the following force diagrams most accurately represents the main forces acting on the block?




**Question 7**

The force **F** acts horizontally to the left and has magnitude 8.3 N. The Cartesian unit vectors **i** and **j** have the directions shown, with **i** at an angle of 25° to the horizontal.




What is the component form of **F**, in newtons to two significant figures?

**A** $-8.0\,\mathbf{i} + 2.1\,\mathbf{j}$ **B** $-7.5\,\mathbf{i} + 3.5\,\mathbf{j}$ **C** $7.5\,\mathbf{i} - 3.5\,\mathbf{j}$
**D** $7.5\,\mathbf{i} + 3.5\,\mathbf{j}$ **E** $8.0\,\mathbf{i} + 2.1\,\mathbf{j}$

**Question 8**

Which of the following options describes the linear transformation $f(x, y) = (-y, -x)$ ?

**A** Reflection in the line $y = x$
**B** Reflection in the line $y = -x$
**C** Rotation through $\pi$ about the origin
**D** Rotation through $\pi/2$ about the origin
**E** Rotation through $-\pi/2$ about the origin

**Question 9**

Let $f(\mathbf{x}) = \mathbf{P}\mathbf{x}$ and $g(\mathbf{x}) = \mathbf{Q}\mathbf{x}$, where

$$\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}.$$

Which of the following matrices represents the composite transformation $g \circ f$ ?

**A** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **B** $\begin{pmatrix} 3 & 4 \\ 1 & -4 \end{pmatrix}$ **C** $\begin{pmatrix} 4 & 4 \\ 1 & -4 \end{pmatrix}$

**D** $\begin{pmatrix} 5 & 4 \\ 5 & -2 \end{pmatrix}$ **E** $\begin{pmatrix} 5 & 10 \\ 2 & -2 \end{pmatrix}$

**Question 10**

What is the form of the partial fraction expansion of the expression

$$\frac{4x^2 + 9x + 12}{(x-3)(x+2)^2} \ ?$$

In the options, $A$, $B$ and $C$ represent non-zero constants.

**A** $\dfrac{A}{x-3} + \dfrac{B}{x+2} + \dfrac{C}{x^2+2}$ **B** $\dfrac{A}{x-3} + \dfrac{B}{(x+2)^2}$

**C** $\dfrac{A}{x-3} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$ **D** $\dfrac{A}{x-3} + \dfrac{B}{x+2}$

**E** $\dfrac{A}{(x-3)(x+2)} + \dfrac{B}{(x+2)^2}$

**Question 11**

What is the quotient on dividing the polynomial expression $2x^2 + 5x - 3$ by the polynomial expression $x + 4$ ?

**A** $-15$ **B** $9$ **C** $x + \frac{3}{2}$ **D** $2x - 3$ **E** $2x + 3$

**Question 12**

Which of the following is the solution to the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh(2x), \text{ where } y = 1 \text{ when } x = 0?$$

**A** $y = \dfrac{1 + \cosh(2x)}{2}$ **B** $y = \dfrac{1 - \cosh(2x)}{2}$

**C** $y = \frac{1}{2}\cosh(2x)$ **D** $y = \frac{1}{2}\cosh(2x) + 1$

**E** $y = \cosh(2x) + \frac{1}{2}$

**Question 13**

What is the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\left(\frac{1}{x} + 1\right) \quad (x > 0,\, y > 0)\ ?$$

In the options, $c$ is an arbitrary constant and $A$ is an arbitrary positive constant.

**A** $y = \ln x + x + c$ **B** $y = Axe^x$ **C** $y = A\ln x + c$
**D** $y = \dfrac{Ae^x}{x}$ **E** $y = x\ln x + c$

**Question 14**

Which of the following is an integrating factor $p(x)$ for the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = \frac{1}{x} \quad (x > 0)\ ?$$

**A** $p(x) = \ln x$ **B** $p(x) = e^{x^2}$ **C** $p(x) = e^x$
**D** $p(x) = x$ **E** $p(x) = x^2$

**Question 15**

Let $P$ be the following statement:

If $n$ is a multiple of 6, then $n$ is even.

Which of the following statements is the contrapositive of $P$ ?

**A** If $n$ is even, then $n$ is a multiple of 6.
**B** If $n$ is odd, then $n$ is a multiple of 6.
**C** If $n$ is odd, then $n$ is not a multiple of 6.
**D** If $n$ is not a multiple of 6, then $n$ is even.
**E** If $n$ is not a multiple of 6, then $n$ is odd.

**Question 16**

An object moving along a straight line slows down at the rate of $3\,\mathrm{m\,s^{-2}}$ and comes to a halt after 12 seconds. What is the distance travelled by the object during this time?

**A** 18 m **B** 24 m **C** 36 m **D** 204 m **E** 216 m

**Question 17**

Which of the following numbers is an eigenvalue of the matrix $\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$ corresponding to the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ?

**A** $-1$ **B** $1$ **C** $2$ **D** $3$ **E** $4$

**Question 18**

The matrix $\begin{pmatrix} 10 & 14 \\ 5 & 1 \end{pmatrix}$ has eigenvalues $-4$ and $15$ and corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 14 \\ 5 \end{pmatrix}$, respectively. What is the general solution of the following system of differential equations?

$$\dot{x} = 10x + 14y$$
$$\dot{y} = 5x + y$$

In the options, $C$ and $D$ are arbitrary constants.

**A** $x = -4Ce^{t} + 15De^{14t}$, $y = -4Ce^{-t} + 15De^{5t}$

**B** $x = -Ce^{-4t} + 5De^{15t}$, $y = Ce^{-4t} + 14De^{15t}$

**C** $x = Ce^{-4t} + 14De^{15t}$, $y = -Ce^{-4t} + 5De^{15t}$

**D** $x = 4Ce^{t} + 15De^{14t}$, $y = -4Ce^{-t} - 15De^{5t}$

**E** $x = 14Ce^{-4t} + De^{15t}$, $y = 5Ce^{-4t} - De^{15t}$

**TURN OVER FOR SECTION 2**

# SECTION 2

*You should **submit answers to all questions**, write in **pen** and start your answer to each question on a new page.*

***Include all your working**, as most marks are awarded for this. Answers without appropriate supporting working as directed by the question **will not be given credit**.*

***Each question is worth 8%.***

**Question 19**

(a) Use Euclid's algorithm to find the highest common factor $d$ of 87 and 38. [3]

(b) Hence find two integers $v$ and $w$ such that $87v + 38w = d$. State the values of $v$ and $w$. [5]

**Question 20**

Using the substitution $x = 3\cosh u$, or otherwise, find the integral $\displaystyle\int \frac{x^2}{\sqrt{x^2 - 9}}\,\mathrm{d}x$. [8]

**Question 21**

Let $P(n)$ be the following statement:

$(n + 1)! > 3^n$.

(a) Determine whether $P(n)$ is true or false for each of $n = 3$ and $n = 4$. [2]

(b) Use mathematical induction to show that $P(n)$ is true for all $n \geq 4$. [6]

**Question 22**

A rugby ball is kicked into the air from the ground. When it leaves the ground, it is travelling at a speed of $10\,\mathrm{m\,s^{-1}}$ at an angle of $60°$ to the horizontal.
Take the $x$-axis to point horizontally in the direction of motion, and the $y$-axis to point vertically upwards, with the origin at the starting point of the ball. Assume that there are no forces from the air on the ball.

(a) Find the velocity of the ball when it leaves the ground, in component form. [2]

(b) Find an expression in component form for the position $\mathbf{r}$ (in metres) of the ball in terms of the time $t$ (in seconds) since it left the ground. Denote the magnitude of the acceleration due to gravity by $g$. [6]

**Question 23**

Express the matrix $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ in the form $\mathbf{PDP}^{-1}$, where $\mathbf{D}$ and $\mathbf{P}$ are $2 \times 2$ matrices, and $\mathbf{D}$ is a diagonal matrix. Evaluate $\mathbf{P}^{-1}$ as part of your answer. [*8*]

**TURN OVER FOR SECTION 3**

# SECTION 3

*You should **submit answers to two questions**. If you submit more, only the first two answers in your submission will be marked.*

*Write in **pen** and start your answer to each question on a new page.*

***Include all your working**, as most marks are awarded for this. Answers without appropriate supporting working as directed by the question **will not be given credit**.*

***Each question is worth 12%.***

### Question 24

Let $f$ be the linear transformation that maps the point $(1, 0)$ to the point $(2, 8)$, and the point $(0, 1)$ to the point $(1, 3)$.

(a) Write down the matrix that represents $f$. [*1*]

(b) Show that $f$ is invertible, and find the matrix that represents $f^{-1}$. [*2*]

(c) Find the equation of the image $f(\mathcal{C})$ of the unit circle $\mathcal{C}$, in the form

$$ax^2 + bxy + cy^2 = d,$$

where $a, b, c$ and $d$ are integers whose values you should find. [*5*]

(d) Check the equation that you found in part (c) by substituting into it the coordinates of a known point on $f(\mathcal{C})$. [*2*]

(e) Let $g$ be the translation by 1 unit to the right and 2 units down. Find the rule of the affine transformation $f \circ g$ in the form

$$(f \circ g)(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a},$$

where $\mathbf{A}$ is a $2 \times 2$ matrix and $\mathbf{a}$ is a 2-dimensional vector. [*2*]

### Question 25

Consider the function $f(x) = \dfrac{x}{x^2 - 1}$.

(a) Find its domain and intercepts. [*2*]

(b) Find $f'(x)$. [*2*]

(c) Construct a table of signs for $f'(x)$, determine the intervals on which $f$ is increasing or decreasing and find any stationary points and their natures. [*3*]

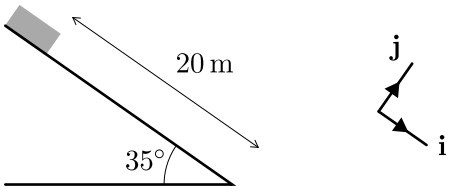
(d) Find the asymptotes of $f$. [*2*]

(e) Determine whether $f$ is an even or odd function, or neither. [*1*]

(f) Sketch the graph of $f$. [*2*]

**Question 26**

A box is released from rest at the top of a flat rough slope of length 20 metres that is inclined at an angle of 35° to the horizontal. It slides under gravity to the bottom of the slope. The coefficient of sliding friction between the box and the slope is 0.17. Take the magnitude of the acceleration due to gravity to be $g = 9.8\,\mathrm{m\,s^{-2}}$. Model the box as a particle.




(a) State the three forces acting on the box during its motion, and draw a force diagram representing these forces, labelling them clearly. [3]

(b) Find expressions for the component forms of the three forces, in terms of the mass $m$ (in kg) of the box and any unknown magnitude(s) where appropriate. In doing this, take the Cartesian unit vectors **i** and **j** to point parallel and perpendicular to the slope in the directions shown above. [2]

(c) Hence or otherwise find the magnitude of the acceleration of the box, in $\mathrm{m\,s^{-2}}$ to two significant figures. [5]

(d) Find the time that the box takes to reach the bottom of the slope, in seconds to two significant figures. [2]

**[END OF QUESTION PAPER]**