MST125

Module Examination 2024

Essential mathematics 2

Monday 16 September 2024

There are **three sections** in this examination.

In **Section 1** you should **attempt all 18 questions**. Each question is worth 2% of the total mark. *Each question has ONE correct answer from five options.*

An incorrectly answered question will get zero marks.

Submit your answers to Section 1 using the interactive Computer-marked Examination (iCME), following the on-screen instructions. Give yourself time to check you have entered your answers correctly.

In **Section 2** you should **submit answers to all 5 questions**. Each question is worth 8% of the total mark.

In **Section 3** you should **submit answers to 2 out of the 3 questions**. Each question is worth 12% of the total mark.

Do not submit more than the required number of answers for Section 3. If you do, only the first two answers submitted for Section 3 will be marked.

For **Sections 2** and **3**:

Include all your working, as some marks are awarded for this.

Handwritten answers must be in pen, though you may draw diagrams in pencil.

Start your answer to each question on a new page, clearly indicating the number of the question.

Crossed out work will not be marked.

Follow the instructions in the online timed examination for how to submit your work.

Further information about completing and submitting your examination work is in the *Instructions and guidance for your remote examination* document on the module website.

Submit your exam using the iCMA system (iCME81). Make sure that the name of the PDF file containing your answers for Sections 2 and 3 includes your PI and the module code e.g. X1234567MST125.

QP ID: MST1252409I3PV1

Section 1

You should ***submit answers to all questions in this section.***
Each question is worth *2%.*

Question 1

If $a \equiv 9 \pmod{16}$ and $b \equiv 11 \pmod{16}$, which of the following is the least residue of $a \times b$ modulo 16?

- 1
- 3
- 4
- 9
- 99

Question 2

If a and n are positive integers such that $a < n$ and

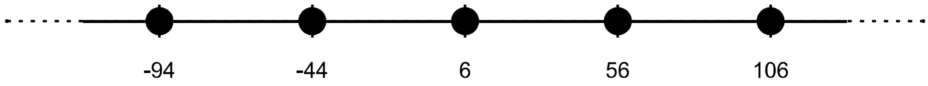
$$17a = 8n + 1,$$

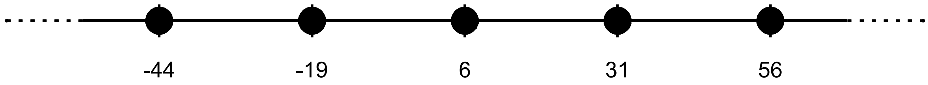
what is the multiplicative inverse of a modulo n?

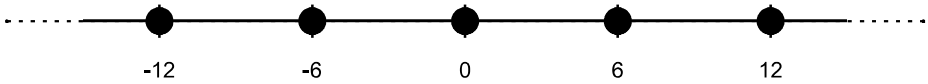
- 1
- 8
- 17
- a
- n

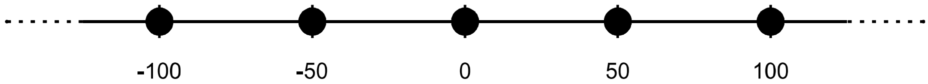
Question **3**

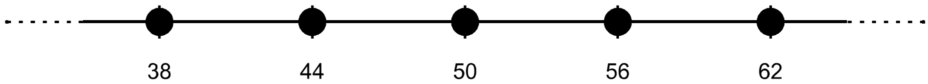
Which of the following number lines shows part of the residue class of 6 modulo 50
?

- ○ 

- ○ 

- ○ 

- ○ 

- ○ 

Question 4

Applying Euclid's algorithm to the integers a and 66 produces the following list of equations, where q is some integer:

$$a = q \times 66 + 30$$

$$66 = 2 \times 30 + 6$$

$$30 = 5 \times 6 + 0.$$

For which integers v and w does the equation

$$av + 66w = d$$

hold where d is the highest common factor of a and 66?

- $v = -2 \qquad w = 2\,q$
- $v = -2 \qquad w = 2\,q + 1$
- $v = 1 \qquad w = -q$
- $v = 2 \qquad w = 2\,q$
- $v = 2 \qquad w = 2\,q + 1$

Question 5

Which of the following is a parametrisation of the straight line passing through the points $(6, 8)$ and $(4, -10\,a)$ where a is a real number?

- $x = 2\,t + 6, \qquad y = (-10\,a - 4)\,t + 8.$
- $x = (-10\,a - 4)\,t + 4, \qquad y = 2\,t - 10\,a.$
- $x = 2\,t + 8, \qquad y = (10\,a + 8)\,t + 4.$
- $x = -2\,t + 6, \qquad y = (-10\,a - 8)\,t + 8.$
- $x = 2\,t + 4, \qquad y = 2\,t - 10\,a.$

Question 6

Suppose that the equation

$$-81\,x^2+B\,x\,y-36\,y^2+D\,x+E\,y+F=0,$$

where B, D, E and F are constants, represents a non-degenerate conic. If $B>-108$, what type of conic could this be?

- A parabola
- A parabola or a hyperbola
- An ellipse
- An ellipse or a parabola
- An ellipse, a parabola or a hyperbola

Question 7

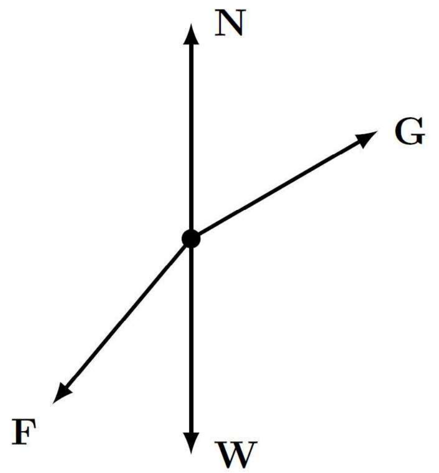
Suppose that a force **F** can be written in component form as $\mathbf{F}=-4\,\mathbf{i}-3\,\mathbf{j}$ where **i** and **j** are the Cartesian unit vectors with **i** parallel to the horizontal and **j** pointing vertically up.

What is the acute angle in degrees to 2 significant figures between **F** and the horizontal?

- $0.64°$
- $37°$
- $41°$
- $49°$
- $53°$

Question **8**

A box of mass 2 kg rests on a horizontal smooth surface. A force diagram showing all the forces acting on the box is shown below:



where **W** is the weight of the box, **N** is the normal reaction of the surface on the box, and **F** and **G** are two additional forces acting on the box.

Given that the box is in equilibrium and

$\mathbf{F} = -4\,\mathbf{i} - 12\,\mathbf{j}$

$\mathbf{G} = 4\,\mathbf{i} + 7\,\mathbf{j}$

$\mathbf{W} = -2g\,\mathbf{j}$

where g is the acceleration due to gravity (in m s^{-2}) and the Cartesian unit vectors **i** and **j** are chosen so that **i** is horizontal and **j** points vertically up, which of the following gives the normal reaction **N** of the surface on the box?

- $5\,\mathbf{j}$
- $(-2\,g - 5)\,\mathbf{j}$
- $-2\,g\,\mathbf{j}$
- $2\,g\,\mathbf{j}$
- $(2\,g + 5)\,\mathbf{j}$

Question **9**

Which of the following gives $(g \circ f)(x, y)$ where

$$f(x, y) = (-2\,x + 7\,y, -8\,x + 4\,y), \qquad g(x, y) = (-7\,y, 3\,x)\,?$$

- $(14\,x - 49\,y, -24\,x + 12\,y)$
- $(56\,x - 28\,y, -6\,x + 21\,y)$
- $(-24\,x + 12\,y, 14\,x - 49\,y)$
- $(21\,x + 14\,y, 12\,x + 56\,y)$
- $(-6\,x + 21\,y, 56\,x - 28\,y)$

Question **10**

What type of linear transformation is represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}?$$

- a horizontal shear
- a vertical shear
- a rotation
- a reflection
- a flattening

Question **11**

If $p(x)$ is a polynomial of degree less than 3 then the proper rational expression

$$\frac{p(x)}{(-4\,x-9)\,(4\,x^2+32\,x+73)}$$

has a partial fraction expansion of which of the following forms, where $A,$ B and C are constants?

- $\dfrac{A}{-4\,x-9}+\dfrac{B}{4\,x^2+32\,x+73}$
- $\dfrac{A+Bx}{-4\,x-9}+\dfrac{B}{4\,x^2+32\,x+73}$
- $\dfrac{A}{-4\,x-9}+\dfrac{B}{2\,x-73}+\dfrac{C}{2\,x-1}$
- $\dfrac{A}{-4\,x-9}+\dfrac{B}{2\,x+73}+\dfrac{C}{2\,x+1}$
- $\dfrac{A}{-4\,x-9}+\dfrac{B\,x+C}{4\,x^2+32\,x+73}$

Question **12**

Consider the initial value problem

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=-4\,x,$ where $y=-9$ when $x=0$.

What is the value of y when $x=-3$?

- $-9\,e^{12}$
- -27
- -18
- 144
- 159

Question 13

Separating the variables in the differential equation

$$\frac{dy}{dx} = \frac{\sin(5\,x)}{y^3}$$

leads to which of the following equations?

- $\int y^3 \, dy = \int \sin(5x) \, dx$
- $\int y^3 \, dy = \int \frac{1}{\sin(5x)} \, dx$
- $\int dy = \int \frac{\sin(5x)}{y^3} \, dx$
- $\int \frac{1}{y^3} \, dy = \int \sin(5x) \, dx$
- $\int \frac{1}{y^3} \, dy = \int \frac{1}{\sin(5x)} \, dx$

Question **14**

The following argument is correct.

Suppose that there exists $x > 3$ such that $x^5 - 15\,x^3 + 167 = 0$.

Let $y = x - 3$. Then $y > 0$ since $x > 3$, and we have

$$
\begin{aligned}
0 &= x^5 - 15\,x^3 + 167 \\
&= (y+3)^5 - 15\,(y+3)^3 + 167 \\
&= y^5 + 15\,y^4 + 75\,y^3 + 135\,y^2 + 5 \\
&> 0,
\end{aligned}
$$

since $y > 0$.

This is a contradiction as 0 is not greater than 0.

What conclusion can be drawn from it?

- For every $x \leq 3$, $x^5 - 15\,x^3 + 167 = 0$.
- For every $x > 3$, $x^5 - 15\,x^3 + 167 = 0$.
- There exists $x \leq 3$ such that $x^5 - 15\,x^3 + 167 = 0$.
- There does not exist $x > 3$ such that $x^5 - 15\,x^3 + 167 = 0$.
- There does not exist $x \leq 3$ such that $x^5 - 15\,x^3 + 167 = 0$.

Question 15

The acceleration of a particle at time t is given by

$$\mathbf{a} = \cosh(9t)\ \mathbf{i} + 12t\ \mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denote the Cartesian unit vectors.

Given that when $t = 0$ the particle has velocity $3\,\mathbf{i} + 2\,\mathbf{j}$, what is its velocity at a general time t?

- $\left(-\frac{1}{9}\sinh(9\,t) + 3\right)\ \mathbf{i} + 2\,\mathbf{j} + 6\,t^2\ \mathbf{k}$
- $\left(\frac{1}{9}\sinh(9\,t) + 3\right)\ \mathbf{i} + 2\,\mathbf{j} + 6\,t^2\ \mathbf{k}$
- $\frac{1}{9}\sinh(9\,t)\ \mathbf{i} + 6\,t^2\ \mathbf{k}$
- $9\sinh(9\,t)\ \mathbf{i} + 12\,\mathbf{k}$
- $(9\sinh(9\,t) + 3)\ \mathbf{i} + 2\,\mathbf{j} + 12\,\mathbf{k}$

Question 16

A 2×2 matrix has trace 5 and determinant 4. What are its eigenvalues?

- -4 and -1
- 1 and 4
- $-\frac{\sqrt{41} - 5}{2}$ and $\frac{\sqrt{41} + 5}{2}$
- $-\frac{\sqrt{41} + 5}{2}$ and $\frac{\sqrt{41} - 5}{2}$
- $2 - \mathrm{i}$ and $\mathrm{i} + 2$

Question **17**

The $\mathbf{2} \times \mathbf{2}$ lower triangular matrix $\mathbf{A}$ has eigenvalues $\boldsymbol{\lambda}$ and $-\mathbf{8}$ (with $\boldsymbol{\lambda} \neq -\mathbf{8}$).

An eigenvector corresponding to the eigenvalue $\boldsymbol{\lambda}$ is $\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}$. Which of the following could be the matrix $\mathbf{A}$?

- $\begin{pmatrix} \lambda & 0 \\ 2 & -8 \end{pmatrix}$
- $\begin{pmatrix} \lambda & -8 \\ 0 & 2 \end{pmatrix}$
- $\begin{pmatrix} -8 & \lambda \\ 2 & 0 \end{pmatrix}$
- $\begin{pmatrix} -8 & 0 \\ 2 & \lambda \end{pmatrix}$
- $\begin{pmatrix} -8 & 2 \\ 0 & \lambda \end{pmatrix}$

Question **18**

If

$$\mathbf{A} = \begin{pmatrix} 5 & 3 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} -4 & 0 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} -1 & -3 \\ 2 & 5 \end{pmatrix},$$

then which of the following is equal to $\mathbf{A}^{89}$?

- $\begin{pmatrix} 5 & 3 \\ -2 & -1 \end{pmatrix}^{89} \begin{pmatrix} -4 & 0 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ -2 & -1 \end{pmatrix}^{89}$
- $\begin{pmatrix} 5 & 3 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} -4 & 0 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} -1 & -3 \\ 2 & 5 \end{pmatrix}$
- $\begin{pmatrix} 5 & 3 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} (-4)^{89} & 0 \\ 0 & (-5)^{89} \end{pmatrix} \begin{pmatrix} -1 & -3 \\ 2 & 5 \end{pmatrix}$
- $\begin{pmatrix} 5^{89} & 3 \\ -2 & (-1)^{89} \end{pmatrix} \begin{pmatrix} (-4)^{89} & 0 \\ 0 & (-5)^{89} \end{pmatrix} \begin{pmatrix} (-1)^{89} & -3 \\ 2 & 5^{89} \end{pmatrix}$
- $\begin{pmatrix} 5^{89} & 3^{89} \\ (-2)^{89} & (-1)^{89} \end{pmatrix} \begin{pmatrix} (-4)^{89} & 0 \\ 0 & (-5)^{89} \end{pmatrix} \begin{pmatrix} (-1)^{89} & (-3)^{89} \\ 2^{89} & 5^{89} \end{pmatrix}$

SECTION 2

*You should **attempt all questions**, write in **pen** and start your answer to each question on a new page.*

***Include all your working**, as some marks are awarded for this. **Each question is worth 8%**.*

Question 19 *(Unit 4)*

An ellipse in standard position has vertices at $(\pm 4, 0)$ and foci at $(\pm\frac{2}{3}\sqrt{35}, 0)$.

(a) Find the eccentricity of the ellipse. [2]

(b) Hence show that its equation is

$$\frac{x^2}{16} + \frac{9y^2}{4} = 1.$$ [4]

(c) The ellipse can be obtained by scaling the circle with equation $x^2 + y^2 = 16$. Find the scaling required. [2]

Question 20 *(Unit 6)*

(a) Write down the matrix of the linear transformation f that maps $(1, 0)$ to $(2, 2)$ and $(0, 1)$ to $(-1, 3)$. [1]

(b) Show that f is invertible and find the matrix of f^{-1}. [2]

(c) A triangle T has vertices at $(2, 0)$, $(0, 2)$ and $(0, 0)$. Find the area of $f(T)$. [2]

(d) Let g be the affine transformation given by

$$g(\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Find the rule of $f^{-1} \circ g$ in the form

$$(f^{-1} \circ g)(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$$

where $\mathbf{A}$ is a 2×2 matrix and $\mathbf{a}$ is a 2×1 column vector.

Simplify your answer as much as possible. [3]

Question 21 *(Unit 7)*

(a) Given that

$$(a - 1)^3 = a^3 - 3a^2 + 3a - 1,$$

show that

$$\sinh^6 x = \cosh^6 x - 3\cosh^4 x + 3\cosh^2 x - 1.$$ [2]

(b) Hence find the integral

$$\int \sinh^7 x \cosh^m x \, dx,$$

where m is a positive integer. [6]

Question 22 *(Units 7 and 8)*

The rational expression $\dfrac{1}{(3x+2)(x+1)}$ can be written as

$$\frac{1}{(3x+2)(x+1)} = \frac{A}{3x+2} - \frac{1}{x+1},$$

where A is constant.

(a) Determine the value of A and hence find the integral

$$\int \frac{1}{(3x+2)(x+1)}\,\mathrm{d}x\,.$$ *[4]*

(b) Use your answer to part (a) to show that

$$p(x) = \frac{3x+2}{x+1}$$

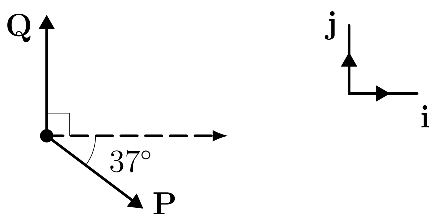
is an integrating factor for the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + 2 - \frac{y}{(3x+2)(x+1)} \qquad (x > -\tfrac{2}{3}).$$ *[4]*

Question 23 *(Unit 10)*

A particle of mass 2.4 kg is acted on by two forces $\mathbf{P}$ and $\mathbf{Q}$, and no other forces, causing it to accelerate in a straight line perpendicular to $\mathbf{Q}$ and at an angle of 37° to $\mathbf{P}$, as shown below. The path of the particle is shown as a dashed line in the diagram.

Let the directions of the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ be as shown, with $\mathbf{i}$ parallel to the path of the particle.



(a) The particle starts from rest and travels 9 metres in 1.5 seconds. Show that its acceleration has magnitude $8\,\mathrm{m\,s^{-2}}$. *[2]*

(b) Write down expressions for the component forms of the forces $\mathbf{P}$ and $\mathbf{Q}$, denoting their magnitudes by P and Q, respectively. *[2]*

(c) Write down the vector equation obtained by applying Newton's second law of motion to the particle. Hence, find the magnitudes of the forces $\mathbf{P}$ and $\mathbf{Q}$, in newtons to two significant figures. *[4]*

SECTION 3

*You should **submit answers to two questions in this section**. If you submit more, only the first two answers in your submission will be marked. Write in **pen** and start your answer to each question on a new page.*

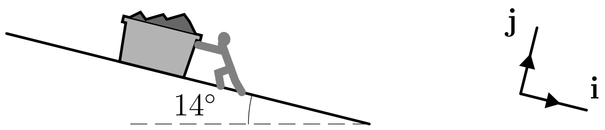
***Include all your working**, as most marks are awarded for this. Answers without appropriate supporting working as directed by the question **will not be given credit**.*

Each question is worth 12%.

Question 24 (*Unit 5*)

A full crate of mass 60 kg lies on a rough slope inclined at 14° to the horizontal. The crate is pushed by an upwards force parallel to the slope, as shown below, and remains at rest. It is not necessarily on the point of slipping either up or down the slope.

Model the crate as a particle, and take the magnitude of the acceleration due to gravity to be $g = 9.8\,\mathrm{m\,s^{-2}}$. Give your answers to parts (c) and (d) to two significant figures.



Take the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ to point parallel and perpendicular, respectively, to the slope, in the directions shown above.

Two of the forces acting on the crate are the pushing force $\mathbf{P}$, and the friction force $\mathbf{F}$ between the crate and the slope. Let F be the magnitude of $\mathbf{F}$.

(a) State the other two forces acting on the crate. Draw a force diagram to represent all four forces acting on the crate, labelling them appropriately and indicating their directions by marking the sizes of suitable angles. [*3*]

(b) Write down expressions for the component forms of the four forces, in terms of unknown magnitudes of forces, and F, where appropriate. [*3*]

(c) The coefficient of static friction between the crate and the slope is 0.16. Find the magnitude of $\mathbf{P}$ if the crate is on the point of slipping up the slope. [*4*]

(d) Find the magnitude and direction of $\mathbf{F}$ if the magnitude of $\mathbf{P}$ is half of that found in part (c). [*2*]

Question 25 *(Unit 7)*

(a) Find the quotient and remainder on dividing $x^3 - 2x^2 - 11x + c$ by $x^2 + 3x + 4$, where c is a real number. *[4]*

(b) Determine whether each of the following functions are even, odd or neither

(i) $f(x) = \dfrac{x^4 + 5}{(1 - x)}$ *[1]*

(ii) $g(x) = \sin^{77} x$. *[1]*

(iii) $h(x) = \cos^m x$, where m is a positive integer. *[1]*

(c) The information below about a rational function f was obtained by following the steps of the graph-sketching strategy.

- The domain of f is $(-\infty, -2) \cup (-2, \infty)$.
- The graph of f has only one x-intercept, namely -1, and only one y-intercept, namely $-\frac{1}{2}$.
- f is increasing on the intervals $(-\infty, -6)$, and decreasing on the interval $(-6, -2)$ and $(-2, \infty)$.
- f has a stationary point, which is a local maximum, at $\left(-6, -\frac{1}{4}\right)$.
- The graph of f has two asymptotes: $x = -2$ and $y = -1$.
- f is neither even nor odd.

Sketch the graph of f. *[5]*

Question 26 *(Unit 9)*

(a) Prove that the following statement is true for all real numbers x by using a sequence of equivalences:

$$x^8 \geq 10x^4 - 25.$$ *[3]*

(b) Use mathematical induction to prove that

$$1 \times 6 + 3 \times 9 + \cdots + (2n - 3) \times 3n = \frac{(n - 1)(4n^2 + n - 6)}{2},$$

for all integers $n \geq 2$. *[9]*

[END OF QUESTION PAPER]