Part A

Question 1:

Question 2:

$$G = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$$

and $+_G$ be

$$(a + a') + (b + b')\sqrt{-5}$$

and a sunset of G

$$H = \{a + b\sqrt{-5} : a, b \in 4\mathbb{Z}\}$$

a)
i.
The identity element, e, of G is, $0 + 0\sqrt{-5}$.

ii.
An element of H that is not the identity is, $4 + 8\sqrt{-5}$.

iii.
An element of G that is not in H is, $3 + 5\sqrt{-5}$.

iv.
The sum, under $+_G$, of these two elements are,

$$\begin{aligned}(4 + 8\sqrt{-5}) + (3 + 5\sqrt{-5}) &= (4 + 3) + (8 + 5)\sqrt{-5} \\ &= 7 + 13\sqrt{-5}\end{aligned}$$

b)
Closure:

Let,

$$x = a + b\sqrt{-5} \quad \text{and } y = a' + b'\sqrt{-5}$$

then

$$x + y = (a + a') + (b + b')\sqrt{-5}$$

and as $\mathbb{Z}$ is closed under addition $a + a' \in \mathbb{Z}$ and $b + b' \in \mathbb{Z}$, Hence, $x + y \in G$.

Associativity:

Let,

$$x = a + b\sqrt{-5}, \quad y = a' + b'\sqrt{-5}, \quad z = a'' + b''\sqrt{-5}$$

then

$$(x + y) + z = ((a + a') + a'') + ((b + b') + b'')\sqrt{-5}$$

$$x + (y + z) = (a + (a' + a'')) + (b + (b' + b''))\sqrt{-5}$$

As addition in $\mathbb{Z}$ is associative,

$$(a + a') + a'' = a + (a' + a'')$$

and

$$(b + b') + b'' = b + (b' + b'')$$

Thus,

$$(x + y) + z = x + (y + z)$$

Identity element:

Let,

$$e = 0 + 0\sqrt{-5}$$

and

$$x = a + b\sqrt{-5}$$

then

$$\begin{aligned} x + e &= (a + 0) + (b + 0)\sqrt{-5} \\ &= a + b\sqrt{-5} \\ &= x \end{aligned}$$

Inverse element:

Let,

$$x = a + b\sqrt{-5}$$

then

$$\begin{aligned} x + (-x) &= (a + -a) + (b + -b)\sqrt{-5} \\ &= 0 + 0\sqrt{-5} \\ &= e \end{aligned}$$

Hence $(G, +_G)$ is a group.

c)
Using Proposition 2.5; Subgroup criteria, HB Chapter 5 p26.

$$x, y \in H \implies x - y \in H$$

Let,

$$x = 4m + 4n\sqrt{-5} \quad \text{and } y = 4p + 4q\sqrt{-5}$$

then

$$\begin{aligned} x - y &= (4m - 4p) + (4n - 4q)\sqrt{-5} \\ &= 4(m - p) + 4(n - q)\sqrt{-5} \end{aligned}$$

As $m - p \in \mathbb{Z}$ and $n - q \in \mathbb{Z}$, then $x - y \in H$. Hence, H is a subgroup of G.

d)
The quotient order of the group $\frac{G}{H}$ is 16.

e)
The order of the element $1+\sqrt{-5}+H \in \frac{G}{H}$ is 4.

Question 3:

$$M = \begin{pmatrix} a, 0 \\ b, c \end{pmatrix} : a, b, c \in \mathbb{Z}_5$$

a)
The additive inverse is

$$\begin{pmatrix} -a, -0 \\ -b, -c \end{pmatrix}$$

b)
Let

$$\varphi : M \to \mathbb{Z}_5$$

be the map that takes a matrix in M to the lower left entry in M.

Proof. To show that φ is a homomorphism, let

$$X = \begin{pmatrix} a, 0 \\ b, c \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} d, 0 \\ e, f \end{pmatrix}$$

then

$$\begin{aligned} \varphi(X + Y) &= \varphi \begin{pmatrix} a + d, 0 + 0 \\ b + e, c + f \end{pmatrix} \\ &= b + e \end{aligned}$$

and

$$\varphi(X) + \phi(Y) = b + e$$

Thus,

$$\varphi(X + Y) = \varphi(X) + \varphi(Y)$$

Hence, φ is a homomorphism. □

c)
The kernel of φ is,

$$\begin{pmatrix} a, 0 \\ b, c \end{pmatrix} : a, c \in \mathbb{Z}, b \in 5\mathbb{Z}$$

d)
A group that is isomorphic to the quotient group $M/Ker(\varphi)$ is $\mathbb{Z}_5$

Question 4:

$$G = \mathbb{Z}_{101} \times \mathbb{Z}_{50}$$

a)
As 101 and 50 are coprime (Corollary 2.12, HB Chapter 6, pg.32),

$$\mathbb{Z}_{101} \times \mathbb{Z}_{50} \cong \mathbb{Z}_{5050}$$

The order of G is $101 \times 50 = 5050$.

i.
The order of the element $(25, 1)$ is;

$$\frac{101}{\text{hcf}(101, 25)} = 101$$
$$\frac{50}{\text{hcf}(50, 1)} = 50$$

Thus, the order of (25,1) is

$$\text{lcm}(101, 50) = 5050$$

Hence $(25, 1)$ is a generator of G.

ii.
The order of the element $(1, 25)$ is;

$$\frac{101}{\text{hcf}(101, 1)} = 101$$
$$\frac{50}{\text{hcf}(50, 25)} = 2$$

Thus, the order of (1,25) is

$$\text{lcm}(101, 2) = 202$$

Hence $(1, 25)$ is not a generator of G.

b)
i.
First group

$$\mathbb{Z}_{600} \times \mathbb{Z}_{49}$$

The order of this group is $600 \times 49 = 29400$.

Using Corollary 2.12 and Theorem 2.13, HB Chapter 6, p32,

$$\mathbb{Z}_{600} \cong \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$$
$$\mathbb{Z}_{49} \cong \mathbb{Z}_{49}$$

Hence,

$$\mathbb{Z}_{600} \times \mathbb{Z}_{49} \cong \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_{25} \times \mathbb{Z}_{49}$$

Second group

$$\mathbb{Z}_2 \times \mathbb{Z}_{12} \times \mathbb{Z}_{175} \times \mathbb{Z}_7$$

The order of this group is $2 \times 12 \times 175 \times 7 = 29400$.

Using Corollary 2.12 and Theorem 2.13, HB Chapter 6, p32,

$$\mathbb{Z}_2 \cong \mathbb{Z}_2$$
$$\mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$$
$$\mathbb{Z}_{175} \cong \mathbb{Z}_{25} \times \mathbb{Z}_7$$
$$\mathbb{Z}_7 \cong \mathbb{Z}_7$$

Therefore a decomposition as a direct product of cyclic groups of prime power order is,

$$\mathbb{Z}_2 \times (\mathbb{Z}_4 \times \mathbb{Z}_3) \times (\mathbb{Z}_{25} \times \mathbb{Z}_7) \times \mathbb{Z}_7 \cong$$
$$\mathbb{Z}_2 \times \mathbb{Z}_4 \cong \mathbb{Z}_3$$

Thus (By Theorem 2.13, HB Chapter 6, p32),

$$\mathbb{Z}_2 \times \mathbb{Z}_{12} \times \mathbb{Z}_{175} \times \mathbb{Z}_7 \cong \mathbb{Z}_{14} \times \mathbb{Z}_{2100}$$

Third group

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{1225}$$

The order of this group is $2 \times 2 \times 6 \times 1225 = 29400$.

Using Corollary 2.12 and Theorem 2.13, HB Chapter 6, p32,

$$\mathbb{Z}_2 \cong \mathbb{Z}_2$$
$$\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$$
$$\mathbb{Z}_{1225} \cong \mathbb{Z}_{25} \times \mathbb{Z}_{49}$$

Thus (By Theorem 2.13, HB Chapter 6, p32),

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{1225} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{25} \times \mathbb{Z}_{49}$$

ii.

By Theorem 2.11, HB Chapter 6, p32, a direct product is cyclic if and only if the component orders are coprime.

- First group; $\mathbb{Z}_{600} \times \mathbb{Z}_{49}$; $\mathrm{hcf}(600, 49) = 1$ so it is cyclic.
- Second group; $\mathbb{Z}_2 \times \mathbb{Z}_{12} \times \mathbb{Z}_{175} \times \mathbb{Z}_7$; $\mathrm{hcf}(2, 12) = 2$ so it is not cyclic.
- Third group; $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{1225}$; $\mathrm{hcf}(2, 6) = 2$ so it is not cyclic.

Therefore only the group $\mathbb{Z}_{600} \times \mathbb{Z}_{49}$ is cyclic.

c)

Since

$$1144 = 2^3 \times 11 \times 13$$

and, Using Corollary 2.12, HB Chapter 6, p32,

$$\mathbb{Z}_{11} \times \mathbb{Z}_{13} \cong \mathbb{Z}_{143}$$

Two possible (non-isomorphic) non-cyclic abelian groups of order 1144 are,

$$\mathbb{Z}_{1144} \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_{143}$$
$$\mathbb{Z}_{1144} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{143}$$

They are non-isomorphic because their 2-power components differ.

Question 5:

$$D_{50} = \langle r, s \mid r^{50} = s^2 = e, sr = r^{49}s \rangle$$

a)
Let $H = \{r^{2i}s^j : i = 0, \ldots, 24, j = 0, 1\}$

Using Proposition 2.4, HB p25.

Proof.
a. Non-empty:

Let $i = 0, j = 0$

$$r^0 s^0 = e \in H$$

Thus, H is non-empty.

b. $x, y \in H \implies x^{-1}y \in H$:

Let

$$x = r^{2a}s^b \text{ and } y = r^{2c}s^d, \quad a, c \in \{0, \ldots, 24\}, \quad b, d \in \{0, 1\}$$

then

$$x^{-1} = r^{-2a}s^b$$

since $s^{-1} = s$

$$x^{-1}y = r^{-2a}s^b \cdot r^{2c}s^d$$

$$= r^{-2a} \cdot s^{b+d} \cdot r^{2c}$$

Let $j = 0$,

$$x^{-1}y = r^{-2a} \cdot r^{2c}$$

$$= r^{2(c-a)} \in H$$

Let $j = 1$,

$$x^{-1}y = r^{-2a}s \cdot r^{2c}s$$

$$= r^{-2a} \cdot r^{2c} \cdot s^2$$

As $s^2 = e$,

$$= r^{2(c-a)} \in H$$

Hence, H is a subgroup of D_{50}.

$\square$

Part B