MST1252309I3PV1

# MST125

**Module Examination 2023**

**Essential mathematics 2**

**Tuesday 12 September 2023**

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There are **three sections** in this examination.

In **Section 1** you should **attempt all 18 questions**. Each question is worth 2% of the total mark. *Each question has ONE correct answer from five options.*

An incorrectly answered question will get zero marks.

Submit your answers to Section 1 using the interactive Computer-marked Examination (iCME), following the on-screen instructions. Give yourself time to check you have entered your answers correctly.

In **Section 2** you should **submit answers to all 5 questions**. Each question is worth 8% of the total mark.

In **Section 3** you should **submit answers to 2 out of the 3 questions**. Each question is worth 12% of the total mark.

Do not submit more than the required number of answers for Section 3. If you do, only the first two answers submitted for Section 3 will be marked.

For **Sections 2** and **3**:

**Include all your working**, as some marks are awarded for this.

Handwritten answers must be in pen, though you may draw diagrams in pencil.

Start your answer to each question on a new page, clearly indicating the number of the question.

Crossed out work will not be marked.

Follow the instructions in the online timed examination for how to submit your work.

Further information about completing and submitting your examination work is in the *Instructions and guidance for your remote examination* document on the module website.

Submit your exam using the iCMA system (iCME81). Make sure that the name of the PDF file containing your answers for Sections 2 and 3 includes your PI and the module code e.g. X1234567MST125.

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The **work that you submit for any assessment/examination on any module should be your own**. Submitting work produced by or with another person, or a web service or an automated system, **as if it is your own** is cheating. It is **strictly forbidden** by the University.

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- request answers or solutions to an assessment question on any website, via an online service or social media platform, or from any individual or organisation.
- use an automated system (other than one prescribed by the module) to obtain answers or solutions to an assessment question and submit the output as your own work.
- discuss examination questions with any other person, including your tutor.

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- obtaining work from other sources and submitting it as your own.

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# Section 1

*You should* ***submit answers to all questions in this section.***
***Each question is worth 2%.***

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## Question **1**

If

$$x \equiv -4 \pmod{46}$$

then $x$ satisfies which of the following congruences?

- $35x \equiv -44 \pmod{46}$
- $35x \equiv 2 \pmod{46}$
- $35x \equiv 31 \pmod{46}$
- $35x \equiv 42 \pmod{46}$
- $35x \equiv 44 \pmod{46}$

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## Question **2**

The enciphering rule of an affine cipher is

$$E(x) \equiv 19x + 23 \pmod{26}.$$

Which of the following is its deciphering rule?

- $D(y) \equiv 17(y - 23) \pmod{26}$
- $D(y) \equiv 11(y - 17) \pmod{26}$
- $D(y) \equiv 11(y - 23) \pmod{26}$
- $D(y) \equiv 11y - 23 \pmod{26}$
- $D(y) \equiv 17y - 23 \pmod{26}$

## Question 3

Which of the following is the equation of the curve given by the parametrisation

$$x = 2 - 9t, \qquad y = -6t - 11?$$

- $y = 1 - 6x$
- $y = \dfrac{2x}{3} - \dfrac{37}{3}$
- $y = \dfrac{2x}{3} + 1$
- $y = \dfrac{2x}{3} + \dfrac{194}{3}$
- $y = 54x - 119$

## Question 4

Three forces $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ act on a particle, where

$$\mathbf{F} = a\mathbf{i} + 12\mathbf{j},$$
$$\mathbf{G} = 6\mathbf{i} - 9\mathbf{j},$$
$$\mathbf{H} = -3\mathbf{i} + a\mathbf{j}.$$

For which value of $a$ are the forces in equilibrium?

- $-21$
- $-9$
- $-6$
- $-3$
- $3$

## Question 5

Suppose that $\mathbf{i}$ points horizontally to the right and $\mathbf{j}$ points vertically up. Which of the following is the correct direction of the force $\mathbf{F}$, where

$$\mathbf{F} = -8\,\mathbf{i} + 6\,\mathbf{j}?$$

The angles are given to 1 decimal place.

- To the $\mathbf{left}$ and $\mathbf{down}$ at an angle of $\mathbf{36.9}^\circ$ to the horizontal
- To the $\mathbf{left}$ and $\mathbf{up}$ at an angle of $\mathbf{36.9}^\circ$ to the horizontal
- To the $\mathbf{right}$ and $\mathbf{up}$ at an angle of $\mathbf{36.9}^\circ$ to the horizontal
- To the $\mathbf{left}$ and $\mathbf{up}$ at an angle of $\mathbf{53.1}^\circ$ to the horizontal
- To the $\mathbf{right}$ and $\mathbf{down}$ at an angle of $\mathbf{53.1}^\circ$ to the horizontal

## Question 6

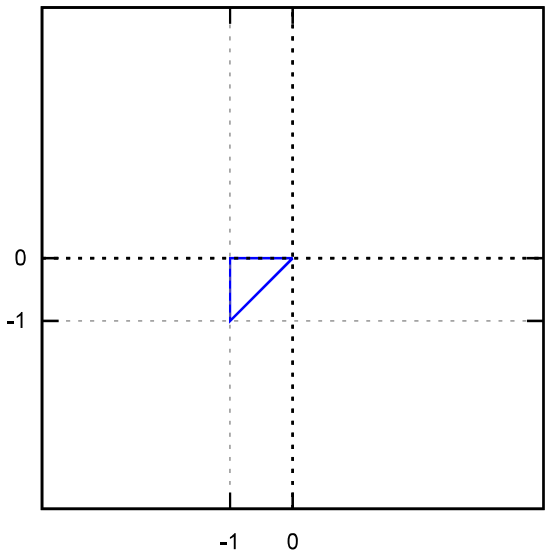
Which of the statements below is true for the linear transformation

$$f(\mathbf{x}) = \begin{pmatrix} 6 & 1 \\ -2 & 4 \end{pmatrix} \mathbf{x}\,?$$

- $f$ is not one-to-one
- $f$ preserves orientation and scales areas by the factor $\mathbf{22}$
- $f$ preserves orientation and scales areas by the factor $\mathbf{26}$
- $f$ reverses orientation and scales areas by the factor $\mathbf{22}$
- $f$ reverses orientation and scales areas by the factor $\mathbf{26}$

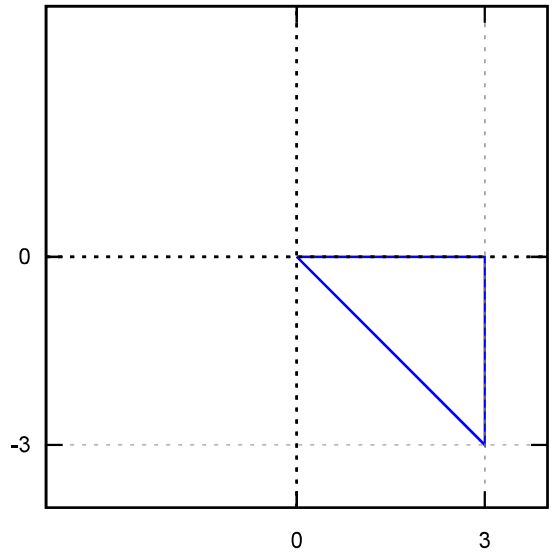
## Question 7

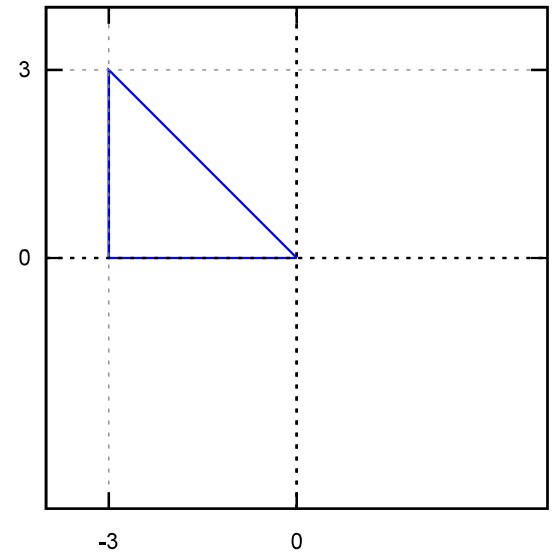
A triangle is shown below.




Which of the following shows the image of this triangle under the linear transformation represented by the matrix

$$\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}?$$

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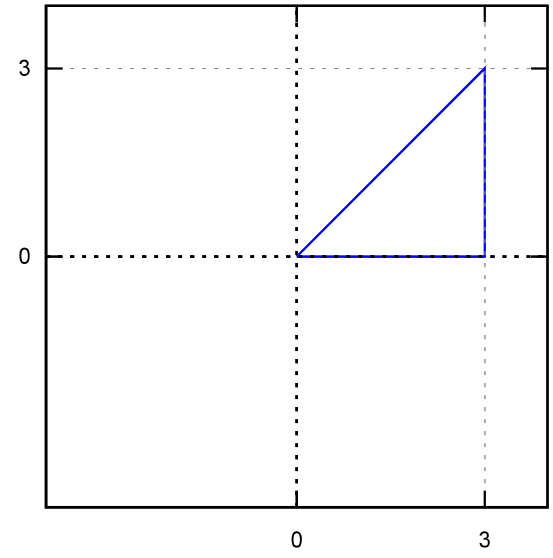
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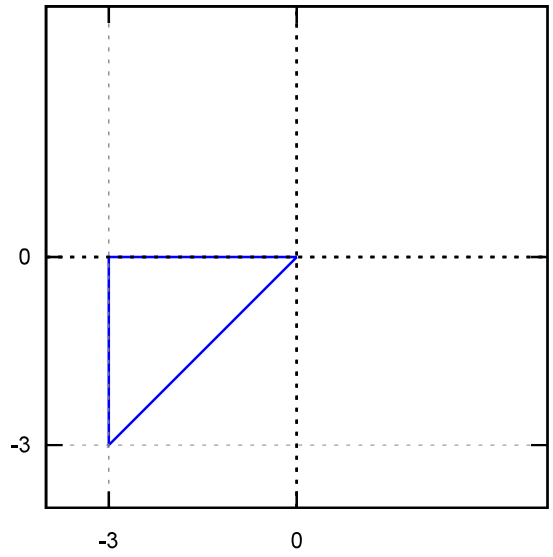
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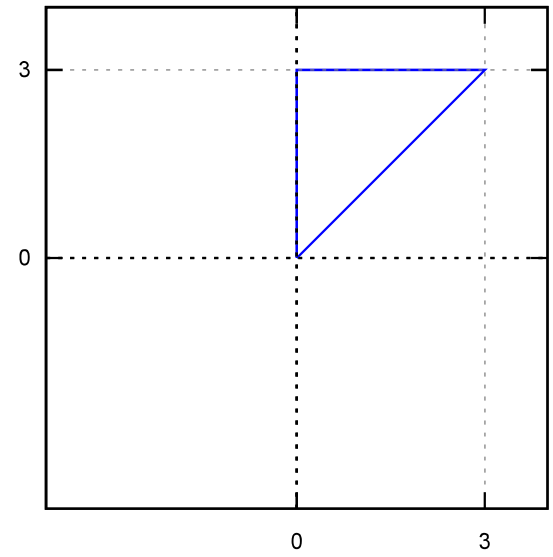



## Question 8

Let $f$ be a vertical shear with shear factor $9$ and $g$ be a horizontal shear with shear factor $4$.

Which of the following matrices represents the linear transformation $g \circ f$?

- $\begin{pmatrix} -35 & -4 \\ 9 & 1 \end{pmatrix}$
- $\begin{pmatrix} -35 & 4 \\ -9 & 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & 4 \\ -9 & -35 \end{pmatrix}$
- $\begin{pmatrix} 1 & 9 \\ -4 & -35 \end{pmatrix}$

## Question 9

Using the substitution $x = \cosh(3u)$ transforms the integral

$$\int \frac{x}{\sqrt{x^2 - 1}}\,\mathrm{d}x \qquad (x \geq 1)$$

into which of the following integrals?

- $\displaystyle\int \cosh(3\,u)\,\mathrm{d}u$
- $\displaystyle\int 3\,\cosh(3\,u)\,\mathrm{d}u$
- $\displaystyle\int \frac{\cosh(3\,u)}{3\,\sinh^2(3\,u)}\,\mathrm{d}u$
- $\displaystyle\int \frac{\cosh(3\,u)}{\sinh(3\,u)}\,\mathrm{d}u$
- $\displaystyle\int 3\,\cosh(3\,u)\,\sinh(3\,u)\,\mathrm{d}u$

## Question **10**

If $p(x)$ is a polynomial of degree less than $3$ then the proper rational expression

$$\frac{p(x)}{(9x-8)\,(2x^2+10x+25)}$$

has a partial fraction expansion of which of the following forms?

- $\dfrac{A}{9x-8}+\dfrac{B}{2x^2+10x+25}$
- $\dfrac{A+Bx}{9x-8}+\dfrac{B}{2x^2+10x+25}$
- $\dfrac{A}{9x-8}+\dfrac{B}{2x-5}+\dfrac{C}{x-5}$
- $\dfrac{A}{9x-8}+\dfrac{B}{2x+5}+\dfrac{C}{x+5}$
- $\dfrac{A}{9x-8}+\dfrac{Bx+C}{2x^2+10x+25}$

## Question 11

Which of the following gives the remainder on dividing

$x^3 - 2x^2 - 12x + c$

by

$x^2 + 3x + 3$,

where $c$ is a real number?

- $c + 12$
- $c + 15$
- $x - 5$
- $-5x^2 - 15x + c$
- $x^3 - 6x + c + 6$

## Question 12

Which of the following describes the first order differential equation

$$\frac{dy}{dx} = 4\sin(8x)y^2 + 2\sin(8x)?$$

- Directly integrable
- Separable
- Linear homogeneous
- Linear non-homogeneous
- None of the above

## Question **13**

For which of the following differential equations could
we find a general solution using the integrating factor method with integrating factor
$\exp\left(\frac{3\,\cosh(8\,x)}{8}\right)$?

- $y\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{3\,\sinh(8\,x)}{64}y=\cosh(8\,x)$
- $\frac{\mathrm{d}y}{\mathrm{d}x}+3\,\sinh(8\,x)y=\tanh(8\,x)$
- $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{3\,\sinh(8\,x)}{8}y=\sinh(8\,x)$
- $\frac{\mathrm{d}y}{\mathrm{d}x}+3\,\sinh(8\,x)y=\sinh(8\,y)$
- $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{3\,\cosh(8\,x)}{8}y=8\,x^3$

## Question **14**

Differentiating the equation

$$-3\,y^6-5\,y=2\,x^2+6\,x$$

with respect to $x$ shows that it is a solution in implicit form of which of the following differential equations?

- $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{4\,x+11}{18\,y^5}$
- $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{18\,y^5+5}{4\,x+6}$
- $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{18\,y^5+4\,x+6}{5}$
- $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{4\,x+6}{18\,y^5+5}$
- $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{28\,x^3+126\,x^2}{18\,y^7+105\,y^2}$

## Question **15**

Suppose that

- $P(x)$ is the statement '$x \geq 9$',
- $Q(x)$ is the statement '$x \leq -2$',
- $R(x)$ is the statement '$(x-9)(x+2) \geq 0$'.

Which of the following statements is true?

- $R(x) \Leftrightarrow P(x)$
- $R(x) \Rightarrow P(x)$
- $P(x) \Rightarrow R(x)$
- $P(x) \Rightarrow Q(x)$
- $R(x) \Rightarrow Q(x)$

## Question **16**

Consider the statement:

> There do not exist integers $m$ and $n$ with no common divisors such that $\sqrt{30} = \frac{m}{n}$.

A proof by contradiction for this statement would begin with which of the following assumptions?

- For every $m$ and $n$ with no common divisors, $\sqrt{30} \neq \frac{m}{n}$.
- There do not exist integers $m$ and $n$ with a common divisor such that $\sqrt{30} \neq \frac{m}{n}$.
- There exist integers $m$ and $n$ with a common divisor such that $\sqrt{30} \neq \frac{m}{n}$.
- There exist integers $m$ and $n$ with no common divisors such that $\sqrt{30} = \frac{m}{n}$.
- There exist integers $m$ and $n$ with no common divisors such that $\sqrt{30} \neq \frac{m}{n}$.

## Question 17

The position of a particle is given in terms of the time $t$ by

$$\mathbf{r} = 5\,\sin(2t)\,\mathbf{i} + 2e^{4t}\,\mathbf{j} + \left(2t^4 + t^2\right)\mathbf{k},$$

where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are the Cartesian unit vectors. What is the velocity of the particle at time $t$?

- $10\,\cos(2t)\,\mathbf{i} + 8e^{4t}\,\mathbf{j} + \left(8t^3 + 2t\right)\mathbf{k}$
- $5\,\cos(2t)\,\mathbf{i} + 2e^{4t}\,\mathbf{j} + \left(2t^3 + t\right)\mathbf{k}$
- $-10\,\cos(2t)\,\mathbf{i} + 8e^{4t}\,\mathbf{j} + \left(8t^3 + 2t\right)\mathbf{k}$
- $\left(-\dfrac{5\,\cos(2t)}{2}\right)\mathbf{i} + \dfrac{e^{4t}}{2}\,\mathbf{j} + \left(\dfrac{2t^5}{5} + \dfrac{t^3}{3}\right)\mathbf{k}$
- $10\,\cos(2t)\,\mathbf{i} + 2e^{4t}\,\mathbf{j} + \left(2t^3 + t\right)\mathbf{k}$

## Question 18

One of the eigenvalues of the matrix

$$\begin{pmatrix} -1 & 4 \\ 49 & -1 \end{pmatrix}$$

is $-15$. If $\begin{pmatrix} x \\ y \end{pmatrix}$ is the corresponding eigenvector then which of the following equations holds?

- $49x - y = 0$
- $64x - y = 0$
- $-4x + y = 0$
- $7x + 2y = 0$
- $-x + 4y = 0$

# SECTION 2

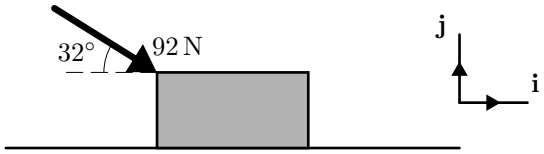
*You should **attempt all questions**, write in **pen** and start your answer to each question on a new page.*

***Include all your working**, as some marks are awarded for this. **Each question is worth 8%.***

**Question 19** (*Unit 5*)

A box of mass 6.5 kg remains at rest on a rough floor even though it is pushed by a force of magnitude 92 N acting at an angle of 32° to the horizontal, as shown below.

Model the box as a particle. Take the magnitude of the acceleration due to gravity to be $g = 9.8\,\mathrm{m\,s^{-2}}$.




(a) The forces acting on the box are its weight **W**, the pushing force **P**, the normal reaction **N** of the floor on the box and the friction force **F**. Draw a force diagram to represent these four forces, labelling them appropriately and indicating their directions by marking the sizes of suitable angles. [*2*]

(b) Find expressions for the component forms of the four forces, in terms of unknown magnitudes where appropriate, taking the Cartesian unit vectors **i** and **j** to point horizontally and vertically, respectively, in the directions shown above. [*3*]

(c) Find the magnitude of the normal reaction and the magnitude of the friction force, in each case in newtons to two significant figures. [*3*]

**Question 20** (*Unit 6*)

One of the following transformations of the plane is linear, and the other is not:

$$f(x, y) = (x - y, 2x + 2y)$$

$$g(x, y) = (x - 2, x - 2y).$$

(a) Which of the transformations is not linear? Justify your answer. [*1*]

(b) Write down the matrix of the linear transformation. [*2*]

(c) Show that this linear transformation is invertible and write down the matrix of its inverse transformation. [*3*]

(d) A circle $\mathcal{C}$ has area $\frac{1}{2}\pi$. Find the area enclosed by the image of $\mathcal{C}$ under the linear transformation. [*2*]

**Question 21** (*Unit 8*)

In an experiment the volume $V$ of liquid in a container is observed to decrease with time $t$ (in seconds since the experiment started) according to the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{k}{V} \qquad (V > 0,\, t \geq 0),$$

where $k$ is a positive constant.

(a) Solve the differential equation to show that its general solution is

$$V = \sqrt{C - 2kt},$$

where $C$ is a constant. [3]

(b) At the start of the experiment the volume of the liquid is $V_0$. Find the particular solution that describes the volume as a function of time in terms of $V_0$. [2]

(c) It is found that $k = V_0^2$. Find the time at which the volume of the liquid decreases to a third of its starting volume. [3]

**Question 22** (*Unit 9*)

(a) Write down the converse of the following statement about positive integers $m$ and $n$:

> If $m$ and $n$ are both multiples of 11, then $3m - 2n$ is a multiple of 11. [1]

(b) Of the statement given in part (a) and its converse, one is true and the other is false.

(i) Give a counter-example to the false statement. [3]

(ii) Prove the true statement. [4]

**Question 23** (*Unit 11*)

The $2 \times 2$ matrix $\mathbf{A}$ has eigenvectors $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$, with corresponding eigenvalues 2 and $-1$, respectively.

(a) Write down a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{PDP^{-1}}$, and calculate $\mathbf{P^{-1}}$. [2]

(b) Hence find the matrix $\mathbf{A}^n$, where $n \in \mathbb{N}$, in the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c$ and $d$ are expressions in $n$. [4]

(c) Find the matrix $\mathbf{A}$. [2]

# SECTION 3

*You should* ***submit answers to two questions in this section****. If you submit more, only the first two answers in your submission will be marked. Write in* ***pen*** *and start your answer to each question on a new page.*

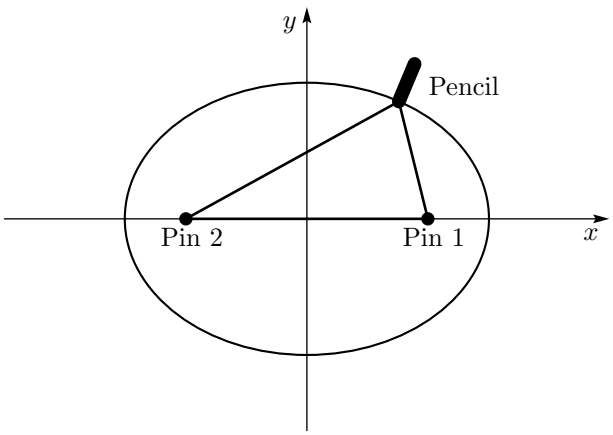
***Include all your working****, as most marks are awarded for this. Answers without appropriate supporting working as directed by the question* ***will not be given credit****.*

**Each question is worth 12%.**

**Question 24** *(Unit 4 and Unit 6)*

An ellipse is drawn using the gardener's method as follows. Two pins are stuck on a corkboard through a sheet of paper, and a loop of string is placed around them. A pencil is used to hold the loop taut and carefully moved around to mark the ellipse. Axes are drawn so that the ellipse is in standard position, with the $x$-axis lying on the line that passes through the two pins, as shown in the diagram below.

The $y$-intercepts of the ellipse are $\pm\sqrt{11}$ and its eccentricity is $e = \dfrac{\sqrt{5}}{7}$.




(a) Find the $x$-intercepts of the ellipse and hence show that its equation is

$$\frac{4x^2}{49} + \frac{y^2}{11} = 1\,. \qquad [3]$$

(b) Find the coordinates of the pins and the length of the loop of string. [3]

(c) Find the directrices of the ellipse. [1]

(d) Find a parametrisation of the part of the ellipse that lies to left of the $y$-axis. [2]

(e) (i) Give a geometric description of a linear transformation $f$ that maps the unit circle $\mathcal{C}$ to this ellipse.

(ii) Hence write down the matrix that represents $f$, and use it to find the area enclosed by the ellipse. [3]

**Question 25** (*Unit 7*)

(a) (i) Show that

$$\sinh^4 x = \cosh^4 x - 2\cosh^2 x + 1. \qquad [2]$$

(ii) Hence find the following integral

$$\int \sinh^5 x \cosh^m x \, \mathrm{d}x,$$

where $m > 1$ is an integer. [6]

(b) The information below about a rational function $f$ was obtained by following the steps of the graph-sketching strategy.

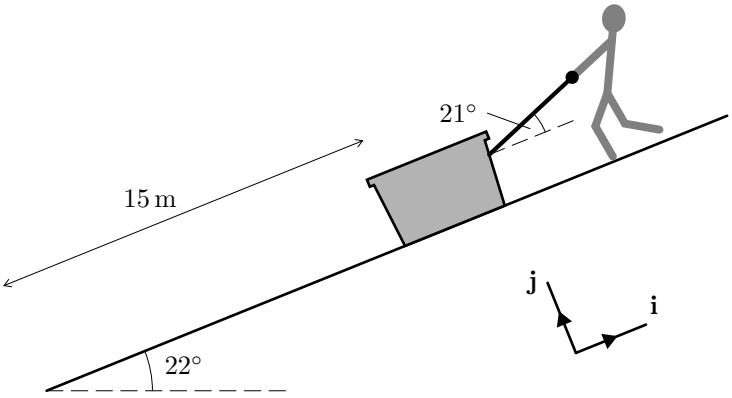
- The domain of $f$ is $(-\infty, \infty)$.
- The graph of $f$ has only one $y$-intercept, namely 2, and has no $x$-intercepts.
- $f$ is increasing on the interval $(-\infty, 4)$, and decreasing on the interval $(4, \infty)$.
- $f$ has a stationary point, which is a local maximum, at $(4, 7)$.
- The graph of $f$ has one asymptote: $y = 1$.
- $f$ is neither even nor odd.

Sketch the graph of $f$. [4]

**Question 26** (*Unit 10*)

A full crate of mass 170 kg is initially held stationary 15 metres up a slope inclined at 22° to the horizontal, by an attached rope tied to a post. A person unties the rope, and the crate starts to accelerate down the slope, with the person continually pulling the rope as hard as they can with a force of 130 N at an angle of 21° to the slope, as shown below. The coefficient of sliding friction between the slope and the crate is 0.29.

Model the crate as a particle and the rope as a model string, and take the magnitude of the acceleration due to gravity to be $g = 9.8\,\mathrm{m\,s^{-2}}$.




(a) State the four forces acting on the crate while it is sliding down the slope, and draw a force diagram to represent them, labelling each force appropriately and indicating the directions of the forces by marking the sizes of suitable angles. [*4*]

(b) Find expressions for the component forms of the four forces, in terms of unknown magnitudes where appropriate, taking the Cartesian unit vectors **i** and **j** to point parallel and perpendicular, respectively, to the slope, in the directions shown above. [*3*]

(c) Write down the vector equation obtained by applying Newton's second law of motion to the crate. Hence find the magnitude of the acceleration of the crate, in $\mathrm{m\,s^{-2}}$ to two significant figures. [*3*]

(d) The person can run down the slope at a maximum speed of $3.2\,\mathrm{m\,s^{-1}}$. Determine whether they have to release the crate before it reaches the bottom of the slope. [*2*]

**[END OF QUESTION PAPER]**