Question 1:

a)
If if the following statments are true:

If it is Robin's birthday, then Robin eats cake. Robin is eating cake.

Let P be the statement "It is Robin's birthday" and Q be the statement "Robin eats cake". The first statement can be written as $P \implies Q$ and the second statement can be written as Q.

The statment;

It is Robin's birthday.

This Statement Q is stand alone and does not nesecerily imply that P is true. Robin could be eating cake for any number of reasons. This is an example of the formal falacy afferming the consequences.

b)

For all positive integers $n,$ We have $5^n \leq (6+1)^6$

This can be proved false with;

$$5^7 = 78125 > 7^6 = 117649$$

Question 2:

a)

$$y = \frac{4x-1}{3x-4}$$

Given $x \neq \frac{4}{3}$ and $y \neq \frac{4}{3}$
When $x = \frac{4y-1}{3y-4}$

$$y = \frac{4x-1}{3x-4}$$

$$= \frac{4\left(\frac{4y-1}{3y-4}\right)-1}{3\left(\frac{4y-1}{3y-4}\right)-4}$$

Distributing the 4 and 3

$$= \frac{\frac{16y-4}{3y-4}-1}{\frac{12y-3}{3y-4}-4}$$

Using the common denominator of $3y-4$

$$= \frac{\frac{16y-4-(3y-4)}{3y-4}}{\frac{12y-3-4(3y-4)}{3y-4}}$$

Simplifying the numerator and denominator

$$= \frac{\frac{16y-4-3y+4}{3y-4}}{\frac{12y-3-12y+16}{3y-4}}$$

$$= \frac{\frac{13y}{3y-4}}{\frac{13}{3y-4}}$$

Cancelling the common factor of $3y-4$

$$= \frac{13y}{13}$$

$$= y$$

Assume $x = \frac{4y-1}{3y-4}$

Then $y = \frac{4x-1}{3x-4}$

$$x = \frac{4y-1}{3y-4}$$

$$= \frac{4\left(\frac{4x-1}{3x-4}\right)-1}{3\left(\frac{4x-1}{3x-4}\right)-4}$$

Distributing the 4 and 3

$$= \frac{\frac{16x-4}{3x-4}-1}{\frac{12x-3}{3x-4}-4}$$

Using the common denominator of $3x-4$

$$= \frac{\frac{16x-4-(3x-4)}{3x-4}}{\frac{12x-3-4(3x-4)}{3x-4}}$$

Simplifying the numerator and denominator

$$= \frac{\frac{16x-4-3x+4}{3x-4}}{\frac{12x-3-12x+16}{3x-4}}$$

$$= \frac{\frac{13x}{3x-4}}{\frac{13}{3x-4}}$$

Cancelling the common factor of $3x-4$

$$= \frac{13x}{13}$$

$$= x$$

Thus the function is it's own inverse. Hence,

$$y = \frac{4x-1}{3x-4} \text{if and only if} x = \frac{4y-1}{3y-4}$$

for all real numbers such that $y \neq \frac{4}{3}$ and $x \neq \frac{4}{3}$.

b)

To prove

$n+1$ is even if and only if $2(n+1)$ is a multiple of 4

Assume $n+1$ is even, therefore it can be writen as $n+1=2k$ for some integer k.

It follows that we can write;

$$2(n+3) = 2(n+1) + 4$$

Substituting $n+1=2k$

$$= 2(2k) + 4$$

$$= 4k + 4$$

$$= 4(k+1)$$

and thus a multiple of 4

Conversly, assume $2(n+1)$ is a multiple of 4, therefore it can be written as $2(n+1)=4k$ for some integer k.

$$2(n+3) = 4k$$

Dividing both sides by 2

$$n+3 = 2k$$

Rearranging gives

$$n = 2k - 3$$

$$n+1 = 2k - 2$$

$$= 2(k-1)$$

and thus an even number

Question 3:

a)

$$(3n)! \geq (n!)^3, \text{ for all } n \in \mathbb{N}$$

Prooof by induction.

Base case: $n = 1$

$$(3 \cdot 1)! = 3! = 6$$

$$(1!)^3 = 1^3 = 1$$

Thus

$$6 \geq 1 \text{ is true.}$$

Inductive step: Assume $(3n)! \geq (n!)^3$ is true for some $n \in \mathbb{N}$.

We need to show that $(3(n+1))! \geq ((n+1)!)^3$.

Since

$$\begin{aligned} 3(k+1)! &= 3(k+1)(k!) \\ &= (3k+3)(3k+2)(3k+1)(3k!) \end{aligned}$$

and

$$\begin{aligned} ((k+1)!)^3 &= ((k+1)(k!))^3 \\ &= (k+1)^3(k!)^3 \end{aligned}$$

Now using our assumption for the inductive step, we have to show;

$$(3k+3)(3k+2)(3k+1) \geq (k+1)^3$$

Expanding the LHS:

$$(9k^2 + 9k + 6k + 6)(3k+1) \geq (k+1)^3$$

$$27k^3 + 54k^2 + 33k + 6 \geq (k+1)^3$$

Expanding the RHS:

$$\begin{aligned}27k^3 + 54k^2 + 33k + 6 &\geq (K+1)(k+1)(k+1)\\ &\geq (k^2+2k+2)(k+1)\\ &\geq k^3+3k^2+3k+1\end{aligned}$$

Reagrranging

$$26k^3 + 51k^2 + 30k + 5 \geq 0$$

This is true for all $k \in \mathbb{N}$

Question 4:

a)
Prove that no such value of x exists such that x is a real posive number.

$$\frac{7x}{x+3} \leq \frac{x-3}{7x}$$

Assume that x is a positive real number.

$$frac7xx+3 \leq \frac{x-3}{7x}$$

Cross multiplying gives

$$(7x)(7x) \leq (x-3)(x+3)$$

Expanding both sides

$$49x^2 \leq x^2 - 9$$

Rearranging gives

$$48x^2 + 9 \leq 0$$

This is not possible as $48x^2$ is always positive for all real numbers x and 9 is a positive constant.

Thus, we have a contradiction.

We can conclude that no such value of x exists such that x is a positive real number.

b)
Prove that:

If $n^3 + 2n^2$ is not a multiple of 16, then n is odd.

Let us consider the contraposition of this stament;

If n is even, then $n^3 + 2n^2$ is a multiple of 16.

Assume n is even, therefore it can be written as $n = 2k$ for some integer k.

$$\begin{aligned} n^3 + 2n^2 &= (2k)^3 + 2(2k)^2 \\ &= 8k^3 + 2(4k^2) \\ &= 8k^3 + 8k^2 \\ &= 8(k^3 + k^2) \\ &= 8k^2(k+1) \\ &= (8k)(k(k+1)) \end{aligned}$$

As $k(k+1)$ is even, we can write it as $2l$ for some integer l.

$$\begin{aligned} &= (8k)(2l) \\ &= 16kl \end{aligned}$$

Hence a multiple of 16

Thus by proof by contraposition:

If $n^3 + 2n^2$ is not a multiple of 16, then n is odd.

Question 5:

a)

$$x_0 = 0\,\mathrm{m}, \quad x_1 = 300\,\mathrm{m}, \quad v_0 = 0\,\mathrm{m/s}, \quad \mathrm{a} = \mathrm{g} = 9.8\,\mathrm{m/s}$$

Using $x = v_0 t + \frac{1}{2}at^2$ to find t

$$x_1 = v_0 t + \frac{1}{2}at^2$$

$$300 = \frac{1}{2}gt^2$$

$$600 = gt^2$$

$$t^2 = \frac{600}{g}$$

$$t = \sqrt{\frac{600}{g}}$$

$$= \sqrt{\frac{600}{9.8}}$$

$$= \qquad\qquad = v_0 + at$$

$$= 0 + g\sqrt{\frac{600}{g}}$$

$$= g\sqrt{\frac{600}{g}}$$

$$= \sqrt{600g}$$

$$= \sqrt{600 \times 9.8}$$

$$= 77.46\ldots$$

$$= 77\,\mathrm{m\,s^{-1}} \qquad \textit{to 2 s.f}$$

b)

inset graph here

Question 6:

$$v_0 = 0\,\text{meter/s}, \quad x_0 = 0\,\text{m}, \quad v_1 = 9\,\text{m}\,\text{s}^{-1}, \quad x_1 = 30\,\text{m}$$

a)
Insert diagram here

Using

$$v_1 = v_0 + at$$

$$9 = at$$

and

$$x_1 = v_0 t + \frac{1}{2}at^2$$

$$30 = \frac{1}{2}at^2$$

$$60 = at^2$$

Substituting $at = 9$

$$60 = 9t$$

$$t = \frac{60}{9}$$

$$= \frac{20}{3}$$

$$= 6.67\,\text{s}$$

and

$$a = \frac{9}{t}$$

$$a = \frac{9}{\frac{20}{3}}$$

$$a = \frac{27}{20}$$

$$= 1.35\,\text{m}\,\text{s}^{-2}$$

b)

$$\begin{aligned}
\mathbf{F} &= \mu|N| \\
\mathbf{N} &= |N| \\
\mathbf{W} &= -\sin(30)mg - \cos(30)mg \\
F_i &= \sin(30)mg - \mathbf{F} \\
&= \sin(30)mg - \mu|N| \\
N_j &= \cos(30)mg
\end{aligned}$$

thus

$$F = \sin 30mg - \mu\cos(30)mg$$

And using $F = ma$

$$\begin{aligned}
ma &= \sin(30)mg - \mu\cos(30)mg \\
&\qquad\qquad \textit{Divinding through by } m \\
a &= \sin(30)g - \mu\cos(30)g \\
1.35 &= \sin(30)g - \mu\cos(30)g \\
&\qquad\qquad \textit{Rearranging} \\
\mu\cos(30)g &= \sin(30)g - 1.35 \\
\mu &= \frac{\sin(30)g - 1.35}{\cos(30)g} \\
&= \frac{\sin(30)9.8 - 1.35}{\cos(30)9.8} \\
&= \frac{4.9 - 1.35}{8.487} \\
&= \frac{3.55}{8.487} \\
&= 0.418\ldots \\
&= 0.42 \\
&\qquad\qquad \textit{to 2 s.f.}
\end{aligned}$$

Question 7:

a)
The vector expression for the acceleration is

$$\mathbf{a} = -g\mathbf{j}$$

b)
The initial velocity vector is

$$\mathbf{v}_0 = 12\cos(50^\circ)\,\mathbf{i} + 12\sin(50^\circ)\,\mathbf{j}$$

Integrating the acceleration vector to find the velocity vector:

$$\begin{aligned}\mathbf{v}(t) &= \int \mathbf{a}\ \mathrm{d}t \\ &= \int -g\mathbf{j}\ \mathrm{d}t \\ &= -gt\,\mathbf{j} + \mathbf{C}_1\end{aligned}$$

Using the initial velocity to find the constant of integration:

$$\mathbf{v}(0) = \mathbf{v}_0 \Rightarrow \mathbf{C}_1 = \mathbf{v}_0$$

Thus, the velocity vector is:

$$\mathbf{v}(t) = \mathbf{v}_0 - gt\,\mathbf{j}$$

Integrating the velocity vector to find the position vector:

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)\ \mathrm{d}t \\ &= \int (\mathbf{v}_0 - gt\,\mathbf{j})\,\mathrm{d}t \\ &= \mathbf{v}_0 t - \frac{1}{2}gt^2\,\mathbf{j} + \mathbf{C}_2\end{aligned}$$

Taking the initial position as the origin:

$$\mathbf{r}(0) = \mathbf{0} \Rightarrow \mathbf{C}_2 = \mathbf{0}$$

Therefore

$$\mathbf{r}(t) = \mathbf{v}_0 t - \frac{1}{2}gt^2\,\mathbf{j}$$

Substituting the expression for $\mathbf{v}_0$:

$$\mathbf{r} = (12t\cos(50^\circ))\,\mathbf{i} + \left(12t\sin(50^\circ) - \frac{1}{2}gt^2\right)\mathbf{j}$$ *(as required)*

c)

i.

Given the position vector $\mathbf{r}$ is from the orign, we want to find the time t for which the $\mathbf{j}$-component is -1.5, that is,

$$12t\sin(50^\circ) - \frac{1}{2}gt^2 = -1.5$$

Rewriting

$$-\frac{1}{2}gt^2 + 12t\sin(50^\circ) + 1.5 = 0$$

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^{2-4ac}}}{2a}$$

This is a quadratic equation in t:

$$t = \frac{-12\sin(50) \pm \sqrt{(12\sin(50)^2 - 4(-\frac{1}{2}g)(1.5))}}{2(-\frac{1}{2}g)}$$

$$= \frac{-12\sin(50) \pm \sqrt{(12\sin(50))^2 + \frac{147}{5}}}{-g}$$

$$= -0.151\ldots \text{ and } 2.027\ldots$$

since we can reject the negaive value for time, we have

$$= 2.027\ldots$$

$$= 2.0\,\mathrm{s}$$

to 2 s.f.

ii.
The horizonal distance traveled by the ball is given by the **i**-component of the position vector **r** at time t:

$$\mathbf{r_i} = 12t\cos(50)\mathbf{i}$$

Substituting $t = 2.027\ldots$

$$= 12(2.027\ldots)\cos(50)\mathbf{i}$$

$$= 15.635\ldots$$

$$= 16\,\mathrm{m}$$

to 2 s.f.

Question 8:

$$\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 18 & 2 \end{pmatrix}$$

a)
The determinant of matrix $\mathbf{A}$ is given by:

$$\det(\mathbf{A}) = 5 \cdot 2 - 6 \cdot 18$$

$$= 10 - 108$$

$$= -98$$

The trace of matrix $\mathbf{A}$ is given by the sum of the diagonal elements:

$$\mathrm{tr}(\mathbf{A}) = 5 + 2$$

$$= 7$$

The characteristic equation of a 2×2 matrix $\mathbf{A}$ is given by:

$$\lambda^2 - (tr\mathbf{A})\lambda + \det \mathbf{A} = 0$$

where λ is the eigenvalue.

Hence, the characteristic equation of matrix $\mathbf{A}$ is:

$$\lambda^2 - 7\lambda - 98 = 0$$

$$(\lambda - 14)(\lambda + 7) = 0$$

Hence the eigenvalues are:

$$\lambda_1 = 14$$

and

$$\lambda_2 = -7$$

The corresponding eigenvectors can be found by solving the equation:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

$$\begin{pmatrix} 5-\lambda & 6 \\ 18 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For

$$\lambda_1 = 14:$$

$$\begin{pmatrix} 5-14 & 6 \\ 18 & 2-14 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -9 & 6 \\ 18 & -12 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives the system of equations:

$$-9x + 6y = 0$$

$$18x - 12y = 0$$

Hence

$$-9x + 6y = 18x - 12y$$

Rearranging gives

$$-27x + 18y = 0$$

$$18y = 27x$$

$$2y = 3x$$

This gives us the eigenvector:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

For

$$\lambda_2 = -7:$$

$$\begin{pmatrix} 5-(-7) & 6 \\ 18 & 2-(-7) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 & 6 \\ 18 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives the system of equations:

$$12x + 6y = 0$$

$$18x + 9y = 0$$

Hence

$$12x + 6y = 18x + 9y$$

Rearranging gives

$$-6x - 3y = 0$$

$$-3y = 6x$$

$$-y = 2x$$

This gives us the eigenvector:

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

b)

We can express

$$\mathbf{A} = \mathbf{PDP}^{-1}$$

Where $\mathbf{P}$ is $\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$

and $\mathbf{D}$ is $\begin{pmatrix} 14 & 0 \\ 0 & -7 \end{pmatrix}$

and $\mathbf{P}^{-1}$ is the inverse of $\mathbf{P}$; $\begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{pmatrix}$

The inverse of a 2×2 matrix $\mathbf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by:

$$\mathbf{P}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where $ad - bc$ is the determinant of $\mathbf{P}$.

Hense, we can write:

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 14 & 0 \\ 0 & -7 \end{pmatrix} \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{pmatrix}$$

c)

$$\mathbf{A}^5 = \mathbf{P}\mathbf{D}^5\mathbf{P}^{-1}$$

$$\begin{aligned}
\mathbf{A}^5 &= \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 14^5 & 0 \\ 0 & (-7)^5 \end{pmatrix} \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 537824 & 0 \\ 0 & -16807 \end{pmatrix} \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{pmatrix} \\
&= \begin{pmatrix} 2 \cdot 537824 + -1 \cdot 0 & 2 \cdot 0 + -1 \cdot -16807 \\ 3 \cdot 537824 + 2 \cdot 0 & 3 \cdot 0 + 2 \cdot -16807 \end{pmatrix} \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{pmatrix} \\
&= \begin{pmatrix} 1075648 & 16807 \\ 1613472 & -33614 \end{pmatrix} \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1075648 \cdot 2}{7} + \frac{16807 \cdot -3}{7} & \frac{1075648 \cdot 1}{7} + \frac{16807 \cdot 2}{7} \\ \frac{1613472 \cdot 2}{7} + \frac{-33614 \cdot -3}{7} & \frac{1613472 \cdot 1}{7} + \frac{-33614 \cdot 2}{7} \end{pmatrix} \\
&= \begin{pmatrix} \frac{2151296 - 50421}{7} & \frac{1075648 + 33614}{7} \\ \frac{3226944 + 100842}{7} & \frac{1613472 - 67228}{7} \end{pmatrix} \\
&= \begin{pmatrix} \frac{2100875}{7} & \frac{1109262}{7} \\ \frac{3327786}{7} & \frac{1546244}{7} \end{pmatrix} \\
&= \begin{pmatrix} 300125 & 158466 \\ 475397 & 220892 \end{pmatrix}
\end{aligned}$$

d)
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e)

$$\dot{x} = 5x + 6y$$
$$\dot{y} = 18x + 2y$$

The general soution of a system of linear differential equations is given by:

$$\mathbf{x} = e^{\lambda_1 t}\mathbf{v_1} + e^{\lambda_2 t}\mathbf{v_2}$$

where λ_1 and λ_2 are the eigenvalues, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are the corresponding eigenvectors.

$$\mathbf{x} = Ce^{14t}\begin{pmatrix}2\\3\end{pmatrix} + De^{-7t}\begin{pmatrix}-1\\2\end{pmatrix}$$

hence we can write:

$$x = 2Ce^{14t} - De^{-7t}$$
$$y = 3Ce^{14t} + 2De^{-7t}$$

where C and D are constants determined by initial conditions.