# MST124 Tutorial

Unit 4

Trigonometry

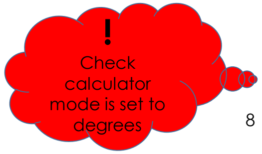
1

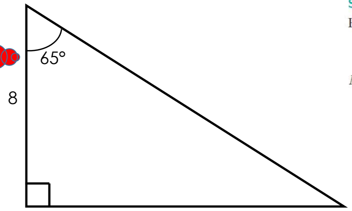
## Learning Objectives
## Section 1

- use radians as a measure of angle, and convert between radians and degrees
- use sine, cosine and tangent to find unknown side lengths and angles of right-angled triangles

2

The right-angled triangle drawn below (which is not drawn to scale) has an angle of 65°.
The shorter of the sides adjacent to this angle has length 8.
What is the length, to one decimal place, of the hypotenuse?



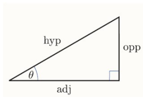




**Sine, cosine and tangent** Handbook p4

For an acute angle $\theta$,

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}.$$

*Mnemonic*: SOH CAH TOA.




**A** 3.4 **B** 18.9 **C** 17.1 **D** 8.8 **E** 9.1

Let $h$ be the hypotenuse

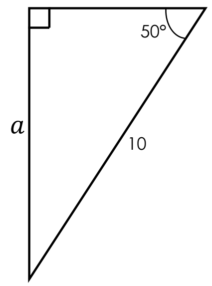
Then $\cos 65° = \frac{8}{h}$

$h = \frac{8}{\cos 65°} = 18.92 \ldots$

So, the length of the hypotenuse is 18.9 (1 d.p.)

3

The right-angled triangle drawn below (which is not drawn to scale) has an angle of 50°.
The hypotenuse has length 10.
What is the length, to one decimal place, of the side labelled $a$, opposite to the angle of size 50° in the diagram?




$\sin 50° = \frac{a}{10}$

$a = 10 \sin 50° = 7.66 \cdots$

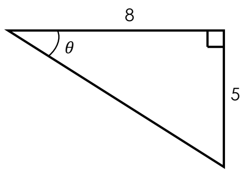
$a = 7.7$(to 1d.p.)

**A** 6.4 **B** 7.6 **C** 13.1 **D** 7.7 **E** 8.7

4

The two shorter sides of the right-angled triangle drawn below (which is not drawn to scale) are 5 and 8 respectively.
What is the size, in radians to 3 decimal places, of the angle marked $\theta$, opposite the side of length 5.

! Check calculator mode is set to radians.




**A** 0.675 **B** 0.585 **C** 0.830 **D** 0.800 **E** 0.559

$$\tan\theta = \frac{5}{8}$$
$$\theta = \tan^{-1}\left(\frac{5}{8}\right)$$
$$= 0.5585\cdots$$

So, the size of the angle $\theta$, in radians, is 0.559 (3d.p.)

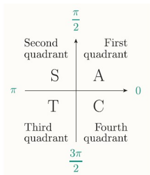
5

# Learning Objectives
## Section 2

- define the sine, cosine and tangent of any angle
- sketch the graphs of sine, cosine and tangent
- use the ASTC diagram or graphs to find values of sine, cosine and tangent
- define inverse sine, inverse cosine and inverse tangent for any angle
- solve trigonometric equations using the ASTC diagram or graphs

6

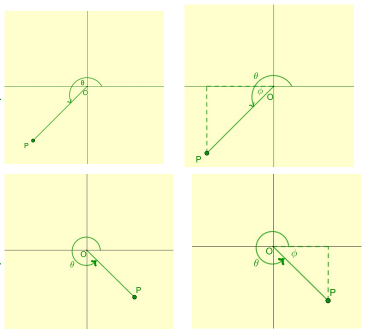
Find all solutions between 0 and $2\pi$ of the equation $\sin\theta = -\frac{1}{\sqrt{2}}$




From the ASTC diagram, $\sin\theta$ is negative in the third and fourth quadrants.

Third quadrant

Fourth quadrant



$$\theta = \phi + \pi$$
$$= \frac{5\pi}{4}$$

$$\theta = 2\pi - \phi$$
$$= \frac{7\pi}{4}$$

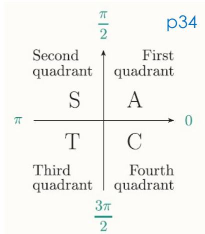
The solutions are $\theta = \frac{5\pi}{4}$ and $\frac{7\pi}{4}$

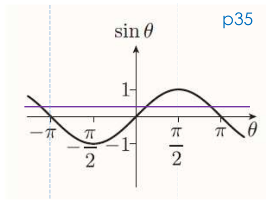
$\sin\phi = \frac{1}{\sqrt{2}}$ $\phi = \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

7

How many values of $\theta$ in radians between $-\pi$ and $\frac{\pi}{2}$ satisfy $\sin\theta = 0.3$ ?

**A** 0 **B** 1 **C** 2 **D** 3 **E** 4

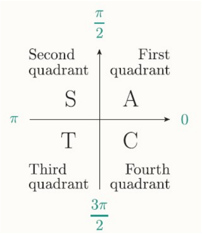


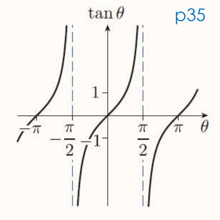

8

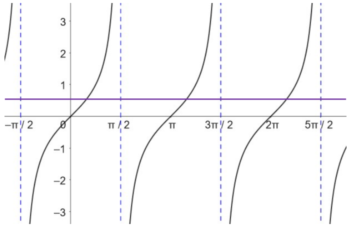
How many values of $\theta$ in radians between $-\frac{\pi}{2}$ and $\frac{5\pi}{2}$ satisfy $\tan\theta = 0.6$ ?

**A** 0 **B** 1 **C** 2 **D** 3 **E** 4




p35

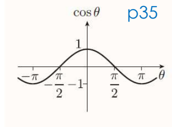


9

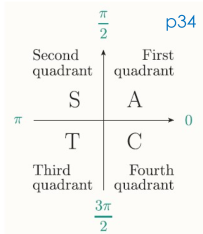
How many values of $\theta$ in radians between $-2\pi$ and $\frac{3\pi}{2}$ satisfy $\cos\theta = -0.5$ ?

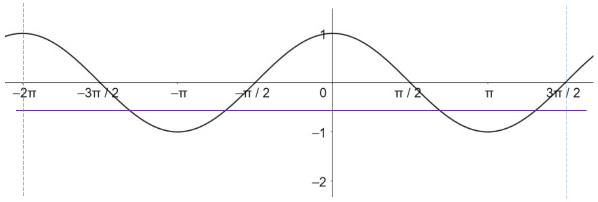
p35




**A** 0 **B** 1 **C** 2 **D** 3 **E** 4

p34



10

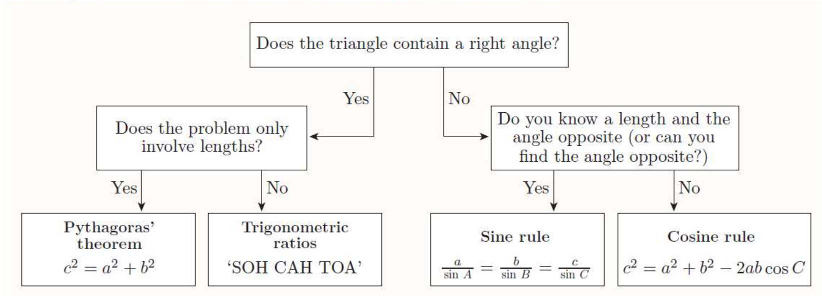
# Learning Objectives
## Section 3

- find unknown side lengths and angles in triangles using the sine rule and cosine rule
- calculate the area of a triangle from two side lengths and the angle between them

11

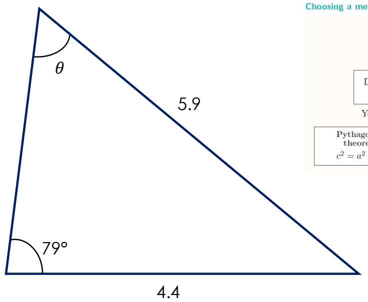
**Choosing a method for finding unknown sides and angles in triangles**

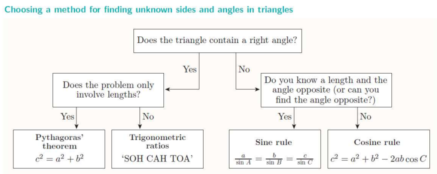
Handbook p37




12

The triangle below has sides of lengths 4.4 and 5.9. The angle opposite the side of length 5.9 is 79°.
What is the size, to the nearest degree, of the angle marked $\theta$, opposite the side of length 4.4.




$$\frac{\sin\theta}{4.4} = \frac{\sin 79°}{5.9}$$

$$\theta = \sin^{-1}\left(\frac{4.4\sin 79°}{5.9}\right)$$

$$= 47.059\cdots \text{or } 132.940\cdots$$

The second value can be rejected as the sum of the angles in the triangle would then be greater than 180.

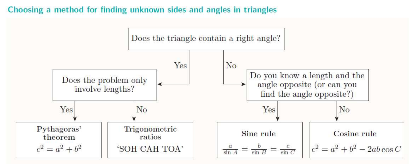
$\theta = 47°$ (to 1 d.p.)

**A** 48° **B** 43° **C** 47° **D** 42° **E** 38 °

13

The triangle below, which is not drawn to scale, has sides of lengths 3, 5 and 7.

What is the size, to the nearest degree, of the angle marked $\theta$, opposite the side of length 7?




$$c^2 = a^2 + b^2 - 2ab\cos C$$

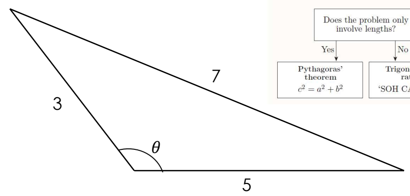
$$2ab\cos C = a^2 + b^2 - c^2$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{25 + 9 - 49}{30}$$

$$= \frac{1}{2}$$

$$C = \cos^{-1}\left(\frac{1}{2}\right) = 120°$$

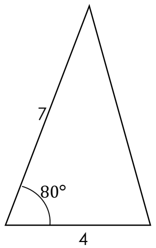



**A** 120° **B** 102° **C** 60° **D** 115° **E** 111 °

14

The triangle below, which is not drawn to scale, has two sides of lengths 4 and 7 and the angle between them is 80°.

What is the length, to one decimal place, of the third side of the triangle?




$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 16 + 49 - 56\cos 80° = 55.275\cdots$$

$$c = \sqrt{55.275\cdots} = 7.434\cdots$$

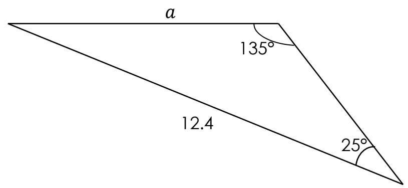
So, the length, to one decimal place, of the third side of the triangle is 7.4

**A** 7.0 **B** 7.4 **C** 8.4 **D** 13.8 **E** 9.9

15

The triangle below has angles of size 135° and 25°. The side opposite the angle of size 135° has length 12.4.

What is the length, to one decimal place, of the side, labelled as $a$, opposite the angle of size 25°?




$$\frac{a}{\sin 25°} = \frac{12.4}{\sin 135°}$$

$$a = \frac{12.4\sin 25°}{\sin 135°} = 7.411\cdots$$

$a = 7.4$(1d.p.)

**A** 20.7 **B** 7.1 **C** 2.7 **D** 7.2 **E** 7.4

16

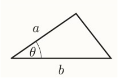
The triangle below, which is not drawn to scale, has two sides of lengths 11.7 and 12.1 and the angle between them is 40°.

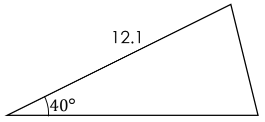
What is the area of the triangle, to one decimal place?

**Area of a triangle** p37

For a triangle with an angle $\theta$ between sides of lengths $a$ and $b$,

area $= \frac{1}{2}ab\sin\theta$.




Area of the triangle is $\frac{1}{2} \times 12.1 \times 11.7 \sin 40° = 45.499\cdots$

$= 45.5 \text{ (1d. p.)}$

**A** 45.5 **B** 8.1 **C** 66.4 **D** 6.7 **E** 91.0

17

# Learning Objectives
# Section 4

- define the cosecant, secant and cotangent of any angle
- understand and use trigonometric identities.

18

## Trigonometric identities

Handbook p5

**tan, cosec, sec, cot in terms of sin and cos**

$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$

**Pythagorean identities**

$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \cot^2\theta + 1 = \operatorname{cosec}^2\theta$

**Symmetry identities**

$\sin(-\theta) = -\sin\theta \qquad \sin(\theta + 2\pi) = \sin\theta \qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

$\cos(-\theta) = \cos\theta \qquad \cos(\theta + 2\pi) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\tan(-\theta) = -\tan\theta \qquad \tan(\theta + \pi) = \tan\theta$

**Angle sum and angle difference identities**

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

**Double-angle and half-angle identities**

$\sin(2\theta) = 2\sin\theta\cos\theta$

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \qquad \text{so} \quad \sin^2\theta = \tfrac{1}{2}(1 - \cos(2\theta)) \\ &= 2\cos^2\theta - 1 \qquad \text{so} \quad \cos^2\theta = \tfrac{1}{2}(1 + \cos(2\theta))\end{aligned}$$

$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$

**Sum to product identities**

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \qquad \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \qquad \cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

**Product to sum identities**

$\sin A\sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$

$\cos A\cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$

$\sin A\cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$

19

If $\sin\theta \neq 0$ and $\cos\theta \neq 0$ , which of the following is equivalent to $\frac{\operatorname{cosec}\theta}{\cot\theta}$ ?

**tan, cosec, sec, cot in terms of sin and cos** p5

$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$

**A** $\cos\theta$ **B** $\sin\theta$ **C** $\sec\theta$ **D** $\tan\theta$ **E** $\sec\theta\tan\theta$

$$\begin{aligned}\frac{\operatorname{cosec}\theta}{\cot\theta} &= \frac{1}{\sin\theta} \div \frac{\cos\theta}{\sin\theta} \\ &= \frac{1}{\sin\theta}\frac{\sin\theta}{\cos\theta} \\ &= \frac{1}{\cos\theta} \\ &= \sec\theta\end{aligned}$$

20

Which of the following is equivalent to $2\,\mathrm{cosec}\,2\theta\cos\theta$ ?

**Double-angle and half-angle identities** p5

$\sin(2\theta) = 2\sin\theta\cos\theta$

**A** $\cot\theta$ **B** $\sin\theta$ **C** $\sec\theta$ **D** $\tan\theta$ **E** $\mathrm{cosec}\,\theta$

$$2\,\mathrm{cosec}\,2\theta\cos\theta = \frac{2\cos\theta}{\sin 2\theta}$$

$$= \frac{2\cos\theta}{2\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta}$$

$$= \mathrm{cosec}\,\theta$$

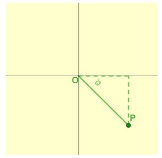
21

Given that $-\pi < \theta < \pi$ such that $\mathrm{cosec}\,\theta = -\sqrt{2}$ and $\tan\theta = -1$ , find the exact value of $\theta$ and justify your answer.

$$\mathrm{cosec}\,\theta = -\sqrt{2}$$

$$\frac{1}{\sin\theta} = -\sqrt{2}$$

$$\sin\theta = -\frac{1}{\sqrt{2}}$$




Since $\sin\theta$ and $\tan\theta$ are both negative, $\theta$ must be in the fourth quadrant

Referring to the diagram, $\phi = \tan^{-1}(1) = \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

So $\theta = -\frac{\pi}{4}$

22

Given that $\theta \in \left(0, \frac{\pi}{2}\right)$ and $\sin\theta = \frac{2}{5}$, use appropriate trigonometric formulas to find the exact values of:

(i) $\cos 2\theta$ (ii) $\cos\theta$ (iii) $\sin 2\theta$

(i) $\cos(2\theta) = 1 - 2\sin^2\theta = 1 - 2\left(\frac{2}{5}\right)^2 = 1 - 2\left(\frac{4}{25}\right) = \frac{25}{25} - \frac{8}{25} = \frac{17}{25}$

(ii) $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta)) = \frac{1}{2}\left(1 + \frac{17}{25}\right) = \frac{1}{2}\left(\frac{25+17}{25}\right) = \frac{42}{50} = \frac{21}{25} \therefore \cos\theta = \pm\frac{\sqrt{21}}{5}$

$\theta$ is in $\left(0, \frac{\pi}{2}\right) \therefore \cos\theta > 0 \therefore \cos\theta = \frac{\sqrt{21}}{5}$

(iii) $\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{2}{5}\right)\left(\frac{\sqrt{21}}{5}\right) = \frac{4}{25}\sqrt{21}$

23

Use a trigonometric identity to find the exact value of $\cos\frac{\pi}{12}$.

Hence, find the exact value of $\sin\frac{\pi}{24}$.

**Angle sum and angle difference identities**

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

**Special angles**

| $\theta$ in radians | $\theta$ in degrees | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|---|
| $0$ | $0°$ | $0$ | $1$ | $0$ |
| $\frac{\pi}{6}$ | $30°$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| $\frac{\pi}{4}$ | $45°$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | $1$ |
| $\frac{\pi}{3}$ | $60°$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| $\frac{\pi}{2}$ | $90°$ | $1$ | $0$ | – |
| $\pi$ | $180°$ | $0$ | $-1$ | $0$ |

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{1}{2}\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$

24

Use a trigonometric identity to find the exact value of $\cos\frac{\pi}{12}$.

Hence, find the exact value of $\sin\frac{\pi}{24}$.

$$\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$$

**Double-angle and half-angle identities**

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$ so $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$

$= 2\cos^2\theta - 1$ so $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

Notice that $\frac{\pi}{12} = \frac{2\pi}{24} = 2\left(\frac{\pi}{24}\right)$ So, if $\theta = \frac{\pi}{24}$ then $2\theta = \frac{\pi}{12}$

So, we know the exact value of $\cos 2\theta$ and we want to work out the exact value of $\sin\theta$

The identity that connects these is $\cos 2\theta = 1 - 2\sin^2\theta$

$$2\sin^2\frac{\pi}{24} = 1 - \cos\frac{\pi}{12}$$

$$= 1 - \frac{1+\sqrt{3}}{2\sqrt{2}}$$

$$\sin\frac{\pi}{24} = \sqrt{\frac{1}{2}\left(1 - \frac{1+\sqrt{3}}{2\sqrt{2}}\right)} = \sqrt{\frac{2\sqrt{2}-1-\sqrt{3}}{4\sqrt{2}}}$$

25

Find all the solutions in the interval $[0, 2\pi]$ of the equation $\sin 3\theta\cos 2\theta - \sin 2\theta\cos 3\theta = 0$

**Angle sum and angle difference identities**

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

Using the identity $\sin(A-B) = \sin A\cos B - \cos A\sin B$ with $A = 3\theta, B = 2\theta$

$$\sin 3\theta\cos 2\theta - \sin 2\theta\cos 3\theta = \sin(3\theta - 2\theta) = \sin\theta = 0$$

In the interval $[0, 2\pi]$, $\sin\theta = 0$ when $\theta = 0, \pi, 2\pi$

So, the solutions in the interval $[0, 2\pi]$ of the equation $\sin 3\theta\cos 2\theta - \sin 2\theta\cos 3\theta = 0$ are $\theta = 0, \pi, 2\pi$

26

Find all the solutions in the interval $(0, \pi)$ of the equation $\sqrt{3}\sin 2\theta + 2\sin^2\theta = 1$

$$\sqrt{3}\sin 2\theta + 2\sin^2\theta = 1$$

$$\sqrt{3}\sin 2\theta = 1 - 2\sin^2\theta$$

Using the identity $\cos 2\theta = 1 - 2\sin^2\theta$

$$\sqrt{3}\sin 2\theta = \cos 2\theta$$

$\frac{\sin 2\theta}{\cos 2\theta} = \frac{1}{\sqrt{3}}$ (assuming $\cos 2\theta \neq 0$)

$$\tan 2\theta = \frac{1}{\sqrt{3}}$$

$$2\theta = \frac{\pi}{6}, 2\theta = \frac{7\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{7\pi}{12}$$

**Special angles**

| $\theta$ in radians | $\theta$ in degrees | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|---|
| $0$ | $0°$ | $0$ | $1$ | $0$ |
| $\frac{\pi}{6}$ | $30°$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| $\frac{\pi}{4}$ | $45°$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | $1$ |
| $\frac{\pi}{3}$ | $60°$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| $\frac{\pi}{2}$ | $90°$ | $1$ | $0$ | – |
| $\pi$ | $180°$ | $0$ | $-1$ | $0$ |

27

By starting with the identity $\sin^2\theta + \cos^2\theta = 1$ ,
obtain an identity linking $\tan^2\theta$ and $\sec^2\theta$ .

Dividing through by $\cos^2\theta$ (assuming $\cos\theta \neq 0$)

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

28

Find all the solutions between 0° and 360° of the equation $\sec^2\theta + \tan\theta = 3$

$\tan^2\theta + 1 = \sec^2\theta$

$\sec^2\theta + \tan\theta = 3$

$\tan^2\theta + 1 + \tan\theta = 3$

$\tan^2\theta + \tan\theta - 2 = 0$

$(\tan\theta + 2)(\tan\theta - 1) = 0$

$\tan\theta = -2$ or $\tan\theta = 1$

$\theta = 116.56\cdots, 296.56\cdots \qquad 45, 225$

the solutions between 0° and 360° of the equation $\sec^2\theta + \tan\theta = 3$ to one decimal place, are

$\theta = 45°, 116.6°, 225°, 296.6°$

29