Question 1:

a)
i.

$$g(x,y) = (2x, 7y)$$

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2x \\ 7y \end{pmatrix} \\
\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} &= \begin{pmatrix} 2x+0y \\ 0x+7y \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}
\end{aligned}$$

This is a diagonal scaling, scale by 2 in $x-$direction, 7 in $y-$direction.

ii.

$$h(x,y) = (x, 4x+y)$$

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x+y \\ 4x+y \end{pmatrix} \\
\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} &= \begin{pmatrix} x+0y \\ 4x+y \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}
\end{aligned}$$

This is a shear transformation.

iii.

$$k(x, y) = (y, x)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

$$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} 0x + y \\ x + 0y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This swaps x and y.

b)

$$g = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \qquad k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$f = k \circ h \circ g$$

First we find $h \circ g$

$$\begin{aligned}
&= h \circ g \\
&= \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix} \\
&= \begin{pmatrix} 1 \times 2 + 0 \times 0 & 1 \times 0 + 0 \times 7 \\ 4 \times 2 + 1 \times 0 & 4 \times 0 + 1 \times 7 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 \\ 8 & 7 \end{pmatrix}
\end{aligned}$$

Now we find $k \circ h \circ g$

$$\mathbf{A} = k \circ \begin{pmatrix} 2 & 0 \\ 8 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 8 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \times 2 + 1 \times 8 & 0 \times 0 + 1 \times 7 \\ 1 \times 2 + 0 \times 8 & 1 \times 0 + 0 \times 7 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 7 \\ 2 & 0 \end{pmatrix}$$

As required

c)

First we have to find the determinant of f.

To find the determiate of a matrix, we can use the formula:

$$DetA = ad - bc$$

To find the inverse of a matrix we use

$$\mathbf{A^{-1}} = \frac{1}{DetA} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$DetA = Det \begin{pmatrix} 8 & 7 \\ 2 & 0 \end{pmatrix}$$

$$= 8 \times 0 - 7 \times 2$$
$$= -14$$

As $DetA \neq 0$ f is invertable

Hence the matrix that represents f^{-1} is

$$f^{-1} = \frac{1}{DetA} \times \begin{pmatrix} 0 & -7 \\ -2 & 8 \end{pmatrix}$$

$$= \frac{-1}{14} \times \begin{pmatrix} 0 & -7 \\ -2 & 8 \end{pmatrix}$$

$$= \frac{1}{14} \times \begin{pmatrix} 0 & 7 \\ 2 & -8 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{7} & -\frac{4}{7} \end{pmatrix}$$

d)

First to find the coordinates of the point in the domain of f that is mapped to a general point (x, y) in the codomain of f.

Each point (x, y) is the image under f of the point $f^{-1}(x, y)$

$$\mathbf{A^{-1}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{7} & -\frac{4}{7} \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 0 \times x + \frac{1}{2} \times y \\ \frac{1}{7} \times x - \frac{4}{7} \times y \end{pmatrix}$$

Hence f maps the point $(\frac{y}{2}, \frac{x-4y}{7})$ to the point (x, y)

The general equation of the unit circle is

$$x^2 + y^2 = 1$$

Substitute these values into the unit circle to find the equation of the image $f(C)$

$$(\frac{y}{2}^2 + \frac{x-4y}{7}^2) = 1$$

Mulitplying out the brackets

$$\frac{y^2}{4} + \frac{x^2 - 8xy + 16y^2}{49} = 1$$

Multiplying through by 196

$$49y^2 + 4x^2 - 32xy + 64y^2 = 196$$

Combining like terms, leaves us with the equation of $f(C)$

$$\frac{1}{196}(4x^2 - 32xy + 113y^2) = 1$$

Putting this into the form $ax^2 + bxy + cy^2 = d$

$$4x^2 - 32xy + 113y^2 = 196$$

This is the equation of the image $f(C)$

where $a = 4, b = -32, c = 113, d = 196$

e)

The area of the unit circle is π the area of $f(C)$ is given by the fact that linear transformations scale the area by the absolute value of the determinant of the transformation matrix.

The area of the image $f(C)$ is

$$Area(f(C)) = |Detf| \times Area(C)$$

$$= 14 \times \pi$$

$$= 14\pi$$

Question 2:

a)
i.
The affine transformation f that maps the points $(0,0),(1,0),(0,1)$ to the points $(-3,4),(-2,4),(-3,5)$ can be represented by the matrix equation

$$f(x) = \mathbf{A}x + \mathbf{a}$$

Let, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the new vector positions.

$$\mathbf{A} = \begin{pmatrix} \mathbf{b}x - \mathbf{a}x & \mathbf{c}x - \mathbf{a}x \\ \mathbf{b}y - \mathbf{a}y & \mathbf{c}y - \mathbf{a}y \end{pmatrix}$$

and

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}x \\ \mathbf{a}y \end{pmatrix}$$

We can set up the matrices for the transformation

$$\begin{pmatrix} -2-(-3) & -3-(-3) \\ 4-4 & 5-4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence

$$f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

ii.
As the matrix is the identity matrix (f) is a translation and hence has no fixed points.

b)

The matrix to represent reflection in the line $y = -x$ is

$$R = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

To reflect in the line $y = -x + 7$, use translation h to the origin, apply R, then translate back with h^{-1}

Let $h(x, y) = (x, y - 7)$, $h^{-1}(x, y) = (x, y + 7)$

Apply the composite transformation $f(x) = h^{-1}(R(h(x)))$

Step 1: Translate down by 7

$$h(x) = \begin{pmatrix} x \\ y - 7 \end{pmatrix}$$

Step 2: Reflect in $y = -x$

$$R \times h(x) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} x \\ y - 7 \end{pmatrix} = \begin{pmatrix} -(y-7) \\ -x \end{pmatrix} = \begin{pmatrix} 7 - y \\ -x \end{pmatrix}$$

Step 3: Translate up by 7

$$f(x) = h^{-1}(R(h(x))) = \begin{pmatrix} 7 - y \\ -x + 7 \end{pmatrix}$$

Therefore, the matrix form is:

$$B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

Question 3:

a)

$$\frac{5x^3 - 11x^2 - 99x - 72}{x^2 - 3x - 18}$$

First we need to divide the polynomials using long division.

$$\begin{array}{r@{}l}
 & \;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;5x \;\; + 4 \\ \cline{2-2}
x^2 - 3x - 18 \big) & \;\; 5x^3 - 11x^2 - 99x - 72 \\
 & -5x^3 + 15x^2 + 90x \\ \cline{2-2}
 & \;\;\;\;\;\;\;\;\;\;\;\; 4x^2 \;\; - 9x - 72 \\
 & \;\;\;\;\;\;\;\; -4x^2 + 12x + 72 \\ \cline{2-2}
 & \; 3x
\end{array}$$

$$\frac{5x^3 - 11x^2 - 99x - 72}{x^2 - 3x - 18} = 5x + 4 + \frac{3x}{x^2 - 3x - 18}$$

using partial fractions on the remainder

$$\frac{3x}{x^2 - 3x - 18} = \frac{A}{x - 6} + \frac{B}{x + 3}$$

Using augmented matrix to solve for A and B

$$\begin{aligned}
3x &= A(x + 3) + B(x - 6) \\
&= Ax + 3A + Bx - 6B \\
&= (A + B)x + (3A - 6B)
\end{aligned}$$

Setting up the augmented matrix

$$\left(\begin{array}{cc|c} 3 & -6 & 0 \\ 1 & 1 & 3 \end{array}\right)$$

Subtract 3 times R2 from R1

$$\left(\begin{array}{cc|c} 0 & -9 & -9 \\ 1 & 1 & 3 \end{array}\right)$$

Divide R1 by -9

$$\left(\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 3 \end{array}\right)$$

Subtract R1 from R2

$$\left(\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 2 \end{array}\right)$$

Hence we can write

$$\frac{3x}{x^2 - 3x - 18} = \frac{2}{x-6} + \frac{1}{x+3}$$

Giving us the full expression

$$\frac{5x^3 - 11x^2 - 99x - 72}{x^2 - 3x - 18} = 5x + 4 + \frac{2}{x-6} + \frac{1}{x+3}$$

b)

Question 3 (b)

Question 3 b) Define function

–> f:(5*x^3-11*x^2-99*x-72)/(x^2-3*x-18);

$$\frac{5x^3-11x^2-99x-72}{x^2-3x-18} \tag{f}$$

Using partfrac to find the partial fraction of f

–> partfrac(f,x);

$$\frac{1}{x+3}+5x+\frac{2}{x-6}+4 \tag{\%o9}$$

c)

$$\begin{aligned}
\int \frac{5x^3 - 11x^2 - 99x - 72}{x^2 - 3x - 18}\,\mathrm{d}x &= \int \left(5x + 4 + \frac{2}{x-6} + \frac{1}{x+3}\right)\mathrm{d}x \\
&= \int 5x\,\mathrm{d}x + \int 4\,\mathrm{d}x + \int \frac{2}{x-6}\,\mathrm{d}x + \int \frac{1}{x+3}\,\mathrm{d}x \\
&= \frac{5}{2}x^2 + 4x + 2\ln|x-6| + \ln|x+3| + C \\
&= \frac{5}{2}x^2 + 4x + 2\ln(x-6) + \ln(x+3) + C
\end{aligned}$$

Question 4:

$$f(x) = \frac{2-x}{x^2+21}$$

a)
The domain of f is all real numbers $\mathbb{R}$, since the denominator $x^2 + 21$ is never zero.

For the intercepts;

To find the x-intercept, set $f(x) = 0$

$$\begin{aligned} 0 &= \frac{2-x}{x^2+21} \\ 2-x &= 0 \\ x &= 2 \end{aligned}$$

So the x-intercept is at

$$(2, 0)$$

To find the y-intercept, set $x = 0$

$$\begin{aligned} f(0) &= \frac{2-0}{0^2+21} \\ &= \frac{2}{21} \end{aligned}$$

So the y-intercept is at

$$(0, \frac{2}{21})$$

b)
For the stationary points, we need to find the derivative of f and set it to zero.

$$f(x) = \frac{2-x}{x^2+21}$$

Using the quotient rule, where $u = 2 - x$ and $v = x^2 + 21$

$$\begin{aligned} f'(x) &= \frac{(v \times u' - u \times v')}{v^2} \\ &= \frac{((x^2+21)(-1) - (2-x)(2x))}{(x^2+21)^2} \\ &= \frac{-x^2 - 21 - (4x - 2x^2)}{(x^2+21)^2} \\ &= \frac{-x^2 - 21 - 4x + 2x^2}{(x^2+21)^2} \\ &= \frac{x^2 - 4x - 21}{(x^2+21)^2} \end{aligned}$$

Setting the numerator to zero for stationary points

$$x^2 - 4x - 21 = 0$$

$$(x+3)(x-7) = 0$$

$$x = -3 \quad \text{or} \quad x = 7$$

So the stationary points are at

$(-3, f(-3))$ and $(7, f(7))$

Calculating $f(-3)$

$$\begin{aligned} f(-3) &= \frac{2-(-3)}{(-3)^2+21} \\ &= \frac{5}{9+21} \\ &= \frac{5}{30} \\ &= \frac{1}{6} \end{aligned}$$

So the stationary point is at

$$(-3, \frac{1}{6})$$

Calculating $f(7)$

$$\begin{aligned} f(7) &= \frac{2-7}{7^2+21} \\ &= \frac{-5}{49+21} \\ &= \frac{-5}{70} \\ &= -\frac{1}{14} \end{aligned}$$

So the stationary point is at

$$(7, -\frac{1}{14})$$

The stationary points are

$$(-3, \frac{1}{6}) \text{ and } (7, -\frac{1}{14})$$

c)

Using a table of signs to determine the nature of the stationary points:

Interval	$(-\infty, -3)$	-3	$(-3, 7)$	7	$(7, \infty)$
$x-7$	$-$	$-$	$-$	0	$+$
$x+3$	$-$	0	$+$	$+$	$+$
$(x-7)(x+3)$	$+$	0	$-$	0	$+$

This is showing that the curve is increasing on the interval $(\infty, -3)$ the turning point at $x = -3$ the decreasing on the interval $(-3, 7)$ and then a second turning point at $x = 7$ the the curve is increasing on the interval $(7, \infty)$.

d)

The horizontal asymptote is found by looking at the limit of $f(x)$ as x approaches infinity.

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{2-x}{x^2+21} \\ &= \lim_{x\to\infty} \frac{-x}{x^2} \\ &= \lim_{x\to\infty} -\frac{1}{x} \\ &= 0\end{aligned}$$

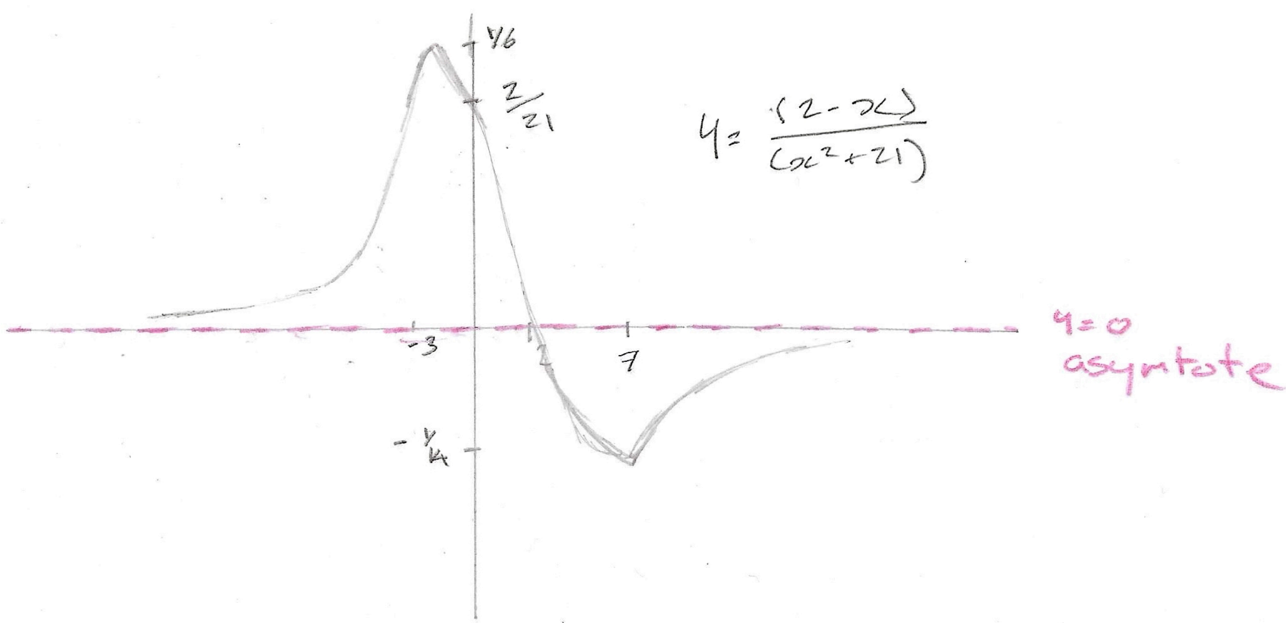
So the horizontal asymptote is at $y = 0$.

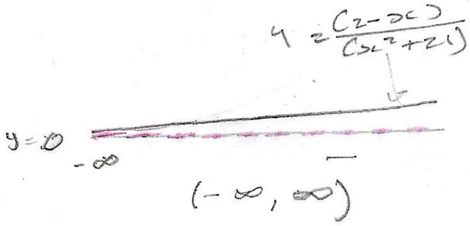
There is no vertical asymtope since the denominator $x^2 + 21$ is never zero for any real x.

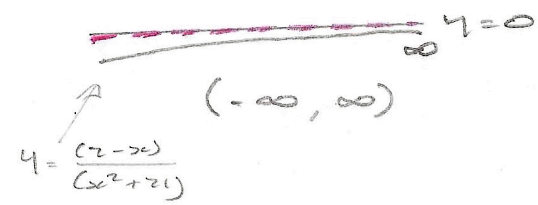
e)
$f(x)$ is neither odd or even, since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$.

f)

Question 4 (g)


1/6
2/21
y = (2 - x) / (x^2 + 21)
-3
2
7
-1/4
y = 0
asymptote


y = (2 - x) / (x^2 + 21)
y = 0
-∞
(-∞, ∞)


y = 0
∞
(-∞, ∞)
y = (2 - x) / (x^2 + 21)

Question 5:

$$\int e^x \cosh^3(e^x)\,\mathrm{d}x$$

a)

$$\cosh^2(x) = \frac{\cosh(2x)+1}{2}$$

$$cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$sinh(x) = \frac{e^x - e^{-x}}{2}$$

Using the substitution $u = e^x$, then $du = e^x\,\mathrm{d}x$.

$$\int e^x \cosh^3(e^x)\,\mathrm{d}x = \int \cosh^3(u)\,\mathrm{d}u$$

$$= \int (\cosh^2(u)\cosh(u))\,\mathrm{d}(u)$$

using half variable identiy

$$= \int \frac{1}{2}(\cosh(2u)+1)\cosh(u)\,\mathrm{d}u$$

$$= \frac{1}{2}\int \cosh(2u)\cosh(u)\,\mathrm{d}u + \frac{1}{2}\int \cosh(u)\,\mathrm{d}u$$

$$= \frac{1}{2}\int \frac{e^{-3u}}{4}\,\mathrm{d}u + \frac{1}{2}\int \frac{e^{-u}}{4}\,\mathrm{d}u + \frac{1}{2}\int \frac{e^{u}}{4}\,\mathrm{d}u + \frac{1}{2}\int \frac{e^{3u}}{4}\,\mathrm{d}u + \frac{1}{2}\int \cosh(u)\,\mathrm{d}u$$

$$= \frac{1}{8}\left(\int e^{-3u}\,\mathrm{d}u + \int e^{-u}\,\mathrm{d}u + \int e^{u}\,\mathrm{d}u + \int e^{3u}\,\mathrm{d}u\right) + \frac{1}{2}\int \cosh(u)\,\mathrm{d}u$$

$$= \frac{1}{8}\left(-\frac{1}{3}e^{-3u} - e^{-u} + e^{u} + \frac{1}{3}e^{3u}\right) + \frac{1}{2}\sinh(u) + C$$

$$= \frac{1}{8}\left(\frac{2}{3}sinh(3u) + 2\sinh(u)\right) + \frac{1}{2}\sinh(u) + C$$

$$= \frac{1}{12}\left(sinh(3u) + 9\sinh(u)\right) + C$$

Substituting back for $u = e^x$

$$= \frac{1}{12}\left(sinh(3e^x) + 9\sinh(e^x)\right) + C$$

b)

i.

Using $\cos A \cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$

$$\begin{aligned}\cos(2x)\cos(5x) &= \frac{1}{2}\cos(2x+5x) + \frac{1}{2}\cos(2x-5x)\\ &= \frac{1}{2}\cos(7x) + \frac{1}{2}\cos(-3x)\end{aligned}$$

Using the fact that $\cos(-x) = \cos(x)$

$$= \frac{1}{2}(\cos(7x) + \cos(3x)) \qquad \textit{As required}$$

ii.

$$\int \sin^2(x)\cos(5x)$$

$$\begin{aligned}\int \sin^2(x)\cos(5x)\,\mathrm{d}x &= \int \frac{1}{2}(1-\cos(2x))cos(5x)\,\mathrm{d}x\\ &= \frac{1}{2}\int cos(5x)\,\mathrm{d}(x) - \frac{1}{2}\int \cos(2x)\cos(5x)\,\mathrm{d}x\end{aligned}$$

Using the result from part (b)

$$\begin{aligned}&= \frac{1}{2}\int cos(5x)\,\mathrm{d}x - \frac{1}{2}\int \left(\frac{1}{2}(\cos(7x)+\cos(3x))\right)\mathrm{d}x\\ &= \frac{1}{2}\int cos(5x)\,\mathrm{d}x - \frac{1}{4}\int \cos(7x)\,\mathrm{d}x - \frac{1}{4}\int \cos(3x)\,\mathrm{d}x\\ &= \frac{1}{2}\times\frac{1}{5}\sin(5x) - \frac{1}{4}\times\frac{1}{7}\sin(7x) - \frac{1}{4}\times\frac{1}{3}\sin(3x) + C\\ &= \frac{1}{10}\sin(5x) - \frac{1}{28}\sin(7x) - \frac{1}{12}\sin(3x) + C\end{aligned}$$

Question 6:

a)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t^7}{(t^8+32)^{\frac{3}{5}}}$$

This is a directly integrable first order differential equation. To solve it, we integrate both sides with respect to t.

b)

$$\int \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int \frac{t^7}{(t^8+32)^{\frac{3}{5}}}\,\mathrm{d}t$$

$$x(t) = \int \frac{t^7}{(t^8+32)^{\frac{3}{5}}}\,\mathrm{d}t$$

Using the substitution $u = t^8 + 32$, then $du = 8t^7\,\mathrm{d}t$

$$= \frac{1}{8}\int \frac{1}{u^{\frac{3}{5}}}\,\mathrm{d}u$$

$$= \frac{1}{8} \times \frac{u^{\frac{2}{5}}}{\frac{2}{5}} + C$$

$$= \frac{5}{16}(t^8+32)^{\frac{2}{5}} + C$$

c)

To find the particular solution that satisfies $x(0) = 1$.

$$
\begin{aligned}
x(0) &= \frac{5}{16}(0^8 + 32)^{\frac{2}{5}} + C \\
1 &= \frac{5}{16}(32)^{\frac{2}{5}} + C \\
&= \frac{5}{16} \times 4 + C \\
&= \frac{5}{4} + C \\
C &= 1 - \frac{5}{4} \\
&= -\frac{1}{4}
\end{aligned}
$$

Hence

$$
x(t) = \frac{5}{16}(t^8 + 32)^{\frac{2}{5}} - \frac{1}{4}
$$

Question 7:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\sqrt{1-y^2}}{t}, (t > 0, -1 < y < 1)$$

a)
This is a separable differential equation. We can separate the variables and integrate both sides.

b)

$$\int \frac{1}{\sqrt{1-y^2}}\,\mathrm{d}y = \int \frac{1}{t}\,\mathrm{d}t$$

$$\sin^{-1}(y) = \ln(t) + C$$

Exponentiating both sides

$$y = \sin(\ln(t) + C)$$

Question 8:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = x^5 sinhx, (x > 0)$$

a)

This is a first order linear differential equation. We can rewrite it in standard form and find an integrating factor.

The integrating factor is given by

$$p(x) = \exp\left(\int g(x)\,\mathrm{d}x\right)$$

The general solution for afirst order differential equation is

$$y = \frac{1}{p(x)}\left(\int p(x)h(x)\right)$$

b)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = x^5 sinh(x)$$

Divide by x

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{4}{x}y = x^4 sinh(x)$$

This is in the form $\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y = h(x)$

Where

$$p(x) = \exp\left(\int \frac{4}{x}\,\mathrm{d}x\right)$$

$$= \exp(4\ln(x))$$

$$= x^4$$

Using this to write the general solution

$$y = \frac{1}{x^{-4}}\left(\int x^{-4}(x^4\sinh(x))\,\mathrm{d}x\right) \quad = x^4\left(\int \sinh(x)\,\mathrm{d}x\right)$$

Integrating $\sinh(x)$

$$= x^4(\cosh(x) + C)$$

Question 9:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{10000}(100 - y)\text{Where}y(0) = 30$$

a)

This is a first order linear differential equation. We can separate the variables and integrate both sides.

$$\int \frac{1}{100 - y}\,\mathrm{d}y = \int \frac{1}{10000}\,\mathrm{d}t$$

Let $u = 100 - y$ and $du = -\,\mathrm{d}y$

$$-\int \frac{1}{u}\,\mathrm{d}u = \frac{1}{10000}\int \mathrm{d}t$$

$$-\ln|u| = \frac{t}{10000} + C$$

Substituting back for $u = 100 - y$

$$-\ln(100 - y) = \frac{t}{10000} + C$$

Exponentiating both sides

$$100 - y = e^{-\frac{t}{10000} - C}$$

$$100 - y = Ae^{-\frac{t}{10000}} \qquad \textit{where } A = e^{-C}$$

Rearranging gives us the general solution

$$y = 100 - Ae^{-\frac{t}{10000}}$$

b)
To find the particular solution that satisfies $y(0) = 30$.

$$y(0) = 100 - Ae^{-\frac{0}{10000}}$$

$$30 = 100 - A$$

$$A = 70$$

So the particular solution is

$$y = 100 - 70e^{-\frac{t}{10000}}$$

This can be simplified to

$$y = 100 - 70e^{-\frac{t}{10000}}$$

c)
After 600 s;

$$\begin{aligned} y(600) &= 100 - 70e^{-\frac{600}{10000}} \\ &= 100 - 70e^{-\frac{3}{50}} \\ &= 100 - 70 \times e^{-0.06} \\ &= 100 - 70 \times 0.9417\ldots \\ &= 100 - 65.9235\ldots \\ &= 34.0764\ldots \\ &= 34\,\mathrm{kg} \end{aligned}$$

To 2 s.f

d)
As $t \to \infty$, the term $e^{-\frac{t}{10000}}$ approaches zero, so

$$y(t) \to 100 - 70 \times 0$$
$$= 100$$

So the limiting value of y as $t \to \infty$ is $100\,\mathrm{kg}$.

e)

Question 9 (e)

Question 9 e)

(%i5) f:'diff(y,t) = (1/(10000))*(100-y);

$$\frac{d}{dt}y = \frac{100-y}{10000} \tag{f}$$

(%i11) ode2(f,y,t);

$$y = \%e^{-\frac{t}{10000}}\left(100\%e^{\frac{t}{10000}} + \%c\right) \tag{\%o11}$$

(%i12) sol: ic1(%,y=30,t=0);

$$y = \%e^{-\frac{t}{10000}}\left(100\%e^{\frac{t}{10000}} - 70\right) \tag{sol}$$

Question 10: