Faculty of Science, Technology, Engineering and Mathematics

M303 Further pure mathematics

M303

TMA 02 2025J

Mainly covers Book B *Groups* **Cut-off date 16 December 2025**

You will find instructions for completing TMAs in the Assessment resources area of the M303 website. Please remind yourself of these instructions before beginning work on each TMA. You can submit each TMA electronically by using the University's online TMA/EMA service or by post together with completed TMA form (PT3). In both cases the University must receive it by the cut-off dated given above unless you have agreed an extention in advance with your tutor.

Half of each TMA is formative and the other half summative. The formative questions are designed to *extend* your understanding of the topics, while the summative ones place more emphasis on *assessing* your understanding of the topics. Both types of question will be marked by your tutor, but (unlike the summative questions) marks for the formative questions do not count towards your final grade. You should submit your solutions to both sets of questions to your tutor by the cut-off date.

The marks allocated to each part of a question are indicated in the margin.

Although many of the questions on these assignments require numerical answers, the questions invariably carry *method marks.*

The words listed below, when used in questions, should be interpreted as indicated.

prove, show, explain, justify	Clear reasoning and explanation for all steps are called for.
determine, find, devise, calculate, compute	An indication of the method used and all working in arriving at an answer should be given.
deduce	Clear explanation of how one result follows from another is required.
solve	Working must be shown. A numerical answer alone is not sufficient.
evaluate, give, write down, list	Answer suffices. No explanation need be given.
hence	No marks will be awarded for any alternative method.

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<u>PLAGIARISM WARNING</u> – the use of assessment help services and websites

The **work that you submit for any assessment/examination on any module should be your own**. Submitting work produced by or with another person, or a web service or an automated system, **as if it is your own** is cheating. It is **strictly forbidden** by the University.

You should not:

- provide any assessment question to a website, online service, social media platform or any individual or organisation, as this is an infringement of copyright
- request answers or solutions to an assessment question on any website, via an online service or social media platform, or from any individual or organisation
- use an automated system (other than one prescribed by the module) to obtain answers or solutions to an assessment question and submit the output as your own work
- discuss examination questions with any other person, including your tutor.

The University actively monitors websites, online services and social media platforms for answers and solutions to assessment questions, and for assessment questions posted by students. Work submitted by students for assessment is also monitored for plagiarism.

A student who is found to have posted a question or answer to a website, online service or social media platform and/or to have used any resulting, or otherwise obtained, output as if it is their own work has committed a disciplinary offence under our Code of Practice for Student Discipline. **This means the academic reputation and integrity of the University has been undermined.**

The Open University's Academic Conduct Policy defines plagiarism in part as:

- using text obtained from assignment writing sites, organisations or private individuals
- obtaining work from other sources and submitting it as your own.

If it is found that you have used the services of a website, online service or social media platform, or that you have otherwise obtained the work you submit from another person, this is considered serious academic misconduct and you will be referred to the Central Disciplinary Committee for investigation.

This TMA is based on Book B *Groups*, except for the formative Question 8, which covers the start of Book C.

Part A is **summative**. These questions assess your knowledge of the module. You must not discuss these questions in the forums, and your tutor is not allowed to help you directly.

Part B is **formative**. You should choose (at most) two formative questions to submit to your tutor for marking.

Doing the formative questions will help you to understand key concepts covered in the text. You are encouraged to discuss these questions in the module forums and with your tutor (who is allowed to help you).

Question 8 is designed to help you with TMA 03. Questions 9, 10 and 11 cover material that is more central to the module, while Question 12 is designed to expand your understanding beyond the core material.

Your tutor will mark all the summative answers that you submit and the first two of any formative answers that you submit. If you have attempted at least one formative question then when your TMA is returned you will be sent worked solutions to all the formative questions. Note that only your marks for Part A count towards your overall continuous assessment score.

Part A Summative

Question 1 – 2 marks

This question has two parts. Each part requires you to use the M303 website. If, for any reason, you cannot access the website, then contact your tutor.

(a) Look at the **Week 6 (8–14 November)** section of the M303 Study Calendar.

(i) Two sections of Chapter 6 are described as core. Write down the section numbers of these two core sections of Chapter 6.

(ii) One section of Chapter 6 is described as non-essential. Write down the section number of this non-essential section of Chapter 6. [*1*]

(b) Attempt Practice Quiz 2, and submit your attempt. In your TMA, include a screenshot or photo of the page that appears when you have submitted your attempt.

(It does not matter what your score is. Full marks for this question part can be obtained for any *attempt* at the quiz. If you cannot access the quiz for any reason, then contact your tutor.) [*1*]

Question 2 – 10 marks

This M303 question tests your understanding of material covered in Chapter 5.

Let G be the set given by $G = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$. Let $+_G$ be the binary operation on G given by

$$(a + b\sqrt{-5}) +_G (a' + b'\sqrt{-5}) = (a + a') + (b + b')\sqrt{-5},$$

for $a, a', b, b' \in \mathbb{Z}$. You may assume that $+_G$ is commutative (in other words, you may assume that

$$(a + b\sqrt{-5}) +_G (a' + b'\sqrt{-5}) = (a' + b'\sqrt{-5}) +_G (a + b\sqrt{-5}).$$

Let $H = \{a + b\sqrt{-5} : a, b \in 4\mathbb{Z}\}$ where $4\mathbb{Z} = \{4n : n \in \mathbb{Z}\}$ is the set of multiples of 4.

(a) Write down the following. (Remember that you do not need to provide any explanation or proof to get full marks if a questions says 'write down'.)

(i) The identity element of $(G, +_G)$.

(ii) An element of H that is not the identity.

(iii) An element of G that is not in H.

(iv) The sum (under $+_G$) of your answers to parts (a)(ii) and (a)(iii). [*2*]

(b) Show that $(G, +_G)$ is a group. [*4*]

(c) By using the subgroup criterion (in additive form, Proposition 2.5), or otherwise, show that H is a subgroup of $(G, +_G)$. [*2*]

(d) Since $+_G$ is commutative, we know that G is abelian and so all subgroups are normal. In particular, H is normal and we can form the quotient group G/H. Write down the order of the quotient group G/H. [*1*]

(e) Write down the order of the element $1 + \sqrt{-5} + H \in G/H$. (Remember that you do not need to provide any explanation or proof to get full marks if a questions says 'write down'.) [*1*]

Question 3 – 6 marks

This M303 question tests your understanding of material covered in Chapter 5.

Let M be the group defined as follows. The underlying set of M is the set of all 2×2 matrices with coefficients in $\mathbb{Z}_5$, and a zero in the top right-hand corner. That is

$$M = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{Z}_5 \right\}.$$

Examples of elements of M include

$A = \begin{pmatrix} 3 & 0 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix}$.

The operation is $+$ where $+$ is the usual matrix addition. So, for example,

$A + B = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}$ because
$3 + 4 \equiv 2 \pmod 5$; $0 + 0 \equiv 0 \pmod 5$; $2 + 2 \equiv 4 \pmod 5$; $4 + 3 \equiv 2 \pmod 5$.

You may assume that this is a group. You do not have to prove this.

(Remember that you do not need to provide any explanation or proof to get full marks if a questions says 'write down'.)

(a) Write down the additive inverse of the group element $\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$. *[1]*

(b) Let $\varphi \colon M \to \mathbb{Z}_5$ be the map that takes a matrix in M to the lower left entry in M. In other words, φ is the map defined by

$$\varphi\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\right) = b.$$

Prove that φ is a homomorphism from M to the group $\mathbb{Z}_5$ (where the operation in the group $\mathbb{Z}_5$ is the usual addition of integers modulo 5). *[3]*

(c) Write down a non-zero element of the kernel of φ. *[1]*

(d) Write down a group that is isomorphic to the quotient group $M/\mathrm{Ker}(\varphi)$. *[1]*

Hint: Theorem 4.13 of Chapter 5 is helpful for this question part.

Question 4 – 12 marks

This M303 question tests your understanding of material covered in Chapter 6.

(a) Let $G = \mathbb{Z}_{101} \times \mathbb{Z}_{50}$. Note that 101 is prime.

(i) Show that $(25, 1)$ generates G. *[2]*

(ii) Show that $(1, 25)$ does not generate G. *[2]*

(b) *For this part of the question you may find Exercise 2.13 and Exercise 2.14 of Chapter 6 helpful. (Remember that you do not need to provide any explanation or proof to get full marks if a questions says 'write down'.)*

Consider the following abelian groups of order 29400:

$$\mathbb{Z}_{600} \times \mathbb{Z}_{49}, \quad \mathbb{Z}_2 \times \mathbb{Z}_{12} \times \mathbb{Z}_{175} \times \mathbb{Z}_7, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{1225}.$$

(i) For each of these groups, write down a direct product of cyclic groups of prime power order that is isomorphic to the group. *[3]*

(ii) Determine which, if any, of the groups is cyclic. [3]

(c) Write down two different (non-isomorphic) non-cyclic abelian groups of order 1144. [2]

Question 5 – 12 marks

This M303 question tests your understanding of material covered in Chapters 6 and 7.

Let D_{50} be the dihedral group

$$D_{50} = \langle r, s \mid r^{50} = s^2 = e,\ sr = r^{49}s \rangle.$$

You may assume that each element can be expressed uniquely in the standard form $r^i s^j$ for $i = 0, \ldots, 49$, $j = 0, 1$, and that $sr^{-i} = r^i s$.

(a) Using the subgroup criterion (or otherwise), show that $H = \{r^{2i}s^j : i = 0, \ldots, 24,\ j = 0, 1\}$ is a subgroup of D_{50}. [3]

(b) By considering the index of H (or otherwise), show that the subgroup H in part (a) is a normal subgroup of D_{50}. [1]

(c) Let K be the subgroup generated by the element r^{25}. By showing that $|K| = 2$, deduce that $K \cong \mathbb{Z}_2$. [1]

(d) By writing sr^{25} in standard form (or otherwise), prove that K is contained in the centre of D_{50}, and show that K is normal in D_{50}. [3]

(e) Show that $H \cap K = \{e\}$. [1]

(f) Hence, by using Proposition 1.9 of Chapter 6 (or otherwise), deduce that D_{50} is isomorphic to the direct product $H \times K$. [3]

Question 6 – 5 marks

This M303 question tests your understanding of material covered in Chapter 8.

Let G be a group of order $48668 = 4 \times 23^3$.

(a) For each prime, p, dividing $|G|$, write down the orders of the Sylow p-subgroups of G. (Remember that you do not need to provide any explanation or proof to get full marks if a questions says 'write down'.) [1]

(b) For each prime, p, dividing $|G|$, use the Sylow theorems to determine the possible numbers of Sylow p-subgroups. [4]

Question 7 – 3 marks

This M303 question tests your understanding of material covered in book B.

Let q be a prime, with $q \neq 29$. Consider the following 'proof' that any group of order $29q$ is cyclic.

(Incorrect) 'proof'

Line 1 Let $q \neq 29$ be a prime number, and let G be a group of order $29q$.

Line 2 Then G has an element, say g_{29}, of order 29 and an element, say g_q, of order q by Cauchy's theorem (Thm 3.16, HB p34).

Line 3 Let H be the subgroup generated by g_{29} and let K be the subgroup generated by g_q. The subgroup H has order 29 and K has order q.

Line 4 Since H has order 29, it is cyclic and isomorphic to $\mathbb{Z}_{29}$.

Line 5 Since K has prime order q, it is isomorphic to the cyclic group $\mathbb{Z}_q$.

Line 6 Since $K \cong \mathbb{Z}_q$ is cyclic, it is abelian.

Line 7 Since H and K are both cyclic subgroups of G, they are both normal subgroups of G.

Line 8 The subgroups H and K have coprime orders, and $|G| = |\mathbb{Z}_{29}| \times |\mathbb{Z}_q|$.

Line 9 Therefore $G \cong H \times K$ (by Theorem 2.3 of Chapter 8), where H and K are cyclic and so G is also cyclic.

(a) Write down a counter-example to this 'proof'. (That is, write down – by name or presentation – a non-cyclic group that has order equal to $29q$ for a prime $q \neq 29$. (Remember that you do not need to provide any explanation or proof to get full marks if a questions says 'write down'.)) [*1*]

(b) Line 9 of the 'proof' would be true *if* the first 8 lines were all correct. They are not all correct. One of them is incorrect. Write down the line number of the first incorrect line, and explain briefly why the line is incorrect. [*2*]

Part B Formative

Question 8 – 9 marks

This M303 question concerns Chapter 9 of Book C, and relates to Question 1 in TMA 03.

(a) Suppose that a positive integer n is such that $4n + \sigma(n)$ is divisible by 5.

(i) Show that n cannot be prime. [*2*]

(ii) If $n = p^2$, where p is a prime number, show that $p \equiv 4 \pmod 5$. [*2*]

(iii) Write down an example of a number $n > 1$ such that $4n + \sigma(n)$ is divisible by 5. [*1*]

(b) Determine the pairs of distinct primes p, q for which $pq + \sigma(pq)$ is even. [*2*]

(c) Suppose that an odd positive integer m is such that $\tau(m)$ divides m. Show that m is a square of an integer. [*2*]

Question 9 – 8 marks

This M303 question will help your understanding of material covered in Chapter 5.

Let $C[0,1]$ be the set of all continuous functions from $[0,1]$ to $\mathbb{R}$. Define an operation $+$ on this set by

$f + g$ is the function from $[0,1]$ to $\mathbb{R}$ given by $(f+g)(x) = f(x) + g(x)$.

For example, if $f\colon [0,1] \to \mathbb{R}$ is the continuous function such that $x \mapsto 2x$, and $g\colon [0,1] \to \mathbb{R}$ is the constant function such that $x \mapsto -3$, then $f + g\colon [0,1] \to \mathbb{R}$ is the function such that $x \mapsto 2x - 3$.

Show that $(C[0,1], +)$ is a group. You may assume that addition in $\mathbb{R}$ is associative. [*8*]

Question 10 – 9 marks

This M303 question is intended to offer some practice with quotient groups. The notion of quotient is important, and is generalised for rings and fields in Book E. Quotient groups are covered in Chapter 5, but note that more details on quotients of dihedral groups are given in Chapter 7.

The dihedral group of order 24 has presentation

$$D_{12} = \langle r, s \mid r^{12} = s^2 = e,\ sr = r^{11}s \rangle.$$

You may assume that the elements of D_{12} can be written uniquely in the standard form $r^i s^j$ for $i = 0, 1, \ldots, 11$, $j = 0, 1$.

(a) Write down the elements of a cyclic subgroup H of D_{12} of order 6. [*1*]

(b) Prove that H is a normal subgroup of D_{12}. [*4*]

(c) (i) List the elements of each of the four cosets H, rH, sH and rsH of the quotient group D_{12}/H, and write down the Cayley table of D_{12}/H. [*3*]

(ii) Is D_{12}/H cyclic? Justify your answer briefly. [*1*]

Question 11 – 16 marks

This M303 question involves many of the fundamental concepts from Book B. You might find that you are able to start this question soon after starting Book B, but will not be able to finish it until later in your study of the book.

Consider the groups

$$\mathrm{Dic}_6, \quad \mathbb{Z}_{12}, \quad \mathbb{Z}_3 \times \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, \quad \mathbb{Z}_2 \times \mathbb{Z}_6, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_4.$$

(a) Write down the two pairs of isomorphic groups in this list. [*2*]

(b) Complete the table below (using the groups in the list above). A copy of the table is provided on page 11 of this booklet for you to print off and fill in if you wish.

As an example, the two groups $\mathbb{Z}_{15}$ and $\mathbb{Z}_2 \times \mathbb{Z}_{10}$ have been put into the table below. They are not included in the list of groups that you are asked to work with. [*12*]

Group	Order of group	Isomorphic groups	Orders of elements	Cyclic? Abelian?
$\mathbb{Z}_{15}$	15	$\mathbb{Z}_3 \times \mathbb{Z}_5$	1, 3, 5, 15	Cyclic
$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	20	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$	1, 2, 5, 10	Abelian, non-cyclic
Dic_6				
$\mathbb{Z}_3 \times \mathbb{Z}_4$				
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$				

(c) Write down any groups from the table that have a subgroup isomorphic to $\mathbb{Z}_4$. [*2*]

Question 12 – 8 marks

This M303 question is intended to deepen your understanding of direct products.

(a) The dihedral group D_{10} is given by

$$D_{10} = \{r, s\colon\ r^{10} = s^2 = e,\ sr = r^9 s\}.$$

The conjugacy classes of D_{10} are

$$\{e\},\ \{r^5\},\ \{r,\, r^9\},\ \{r^2,\, r^8\},\ \{r^3,\, r^7\},\ \{r^4,\, r^6\},$$
$$\{s,\, r^2s,\, r^4s,\, r^6s,\, r^8s\},\ \{rs,\, r^3s,\, r^5s,\, r^7s,\, r^9s\}.$$

For each of the following pairs H, K of subgroups of D_{10}, determine whether D_{10} is the internal direct product of H and K. Justify your answer fully. You can assume that H and K are subgroups of D_{10}.

(i) $H = \{e, r^2, r^4, r^6, r^8\}$, $K = \{e, r^5, s, r^5s\}$

(ii) $H = \{e, s, r^2, r^4, r^6, r^8, r^2s, r^4s, r^6s, r^8s\}$, $K = \{e, r^5\}$ [4]

(b) Let p and q be distinct primes.

(i) Is $\mathbb{Z}_{p^2q}$ isomorphic to $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$? Justify your answer.

(ii) Is $\mathbb{Z}_{p^2q}$ isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$? Justify your answer. [4]

You may like to use this table for your answer to Question 11.

Group	Order of group	Isomorphic groups	Orders of elements	Cyclic? Abelian?
$\mathbb{Z}_{15}$	15	$\mathbb{Z}_3 \times \mathbb{Z}_5$	1, 3, 5, 15	Cyclic
$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	20	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$	1, 2, 5, 10	Abelian, non-cyclic
Dic_6				
$\mathbb{Z}_3 \times \mathbb{Z}_4$				
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$				