Revision of functions: one-one and onto

Please note that these resources have not been through our usual checking process and so may contain errors and/or omissions. If you see anything that you think may be incorrect, then please talk to your tutor.

Functions or maps

In M303 the words '**function**' and '**map**' are used interchangeably.
A function or map, f, from a set A to a set B is a way of associating a unique element of B with every element of A.
We write $f\colon A \to B$. The set A (the 'starting space') is called the **domain** and B (the 'finishing space') is called the **codomain**.

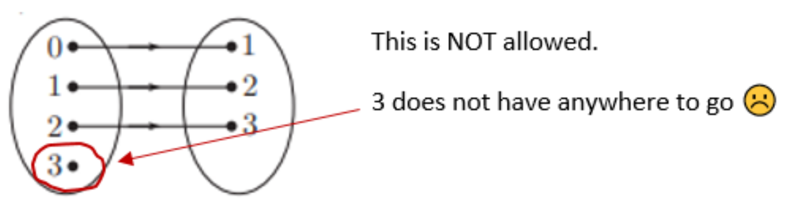
Examples of functions.

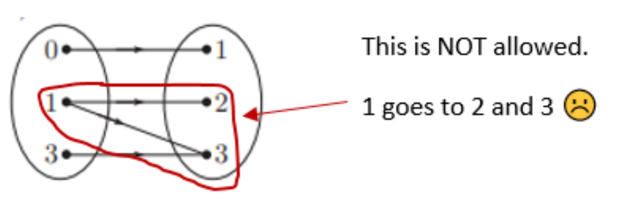
1. The map $f\colon \mathbb{Z} \to \mathbb{N}$ defined by $f(z) = z^2 + 1$.
   This function takes an element z, of $\mathbb{Z}$ (the set of all integers or whole numbers), to $z^2 + 1$.
   Since z^2 is always ≥ 0, the number $z^2 + 1$ is always ≥ 1 and so is a valid member of $\mathbb{N}$ (the set of natural numbers or counting numbers: $1, 2, 3, \ldots$).
2. The map $g\colon \{1, 2, 3\} \to \{2, 4, 6\}$ given by $g(x) = 2x$.
3. The map $h\colon \{\text{cat}, \text{dog}, \text{monkey}\} \to \{\text{apple}, \text{orange}, \text{banana}, \text{pear}\}$ given by:
   $h(\text{cat}) = h(\text{monkey}) = \text{apple}, h(\text{dog}) = \text{pear}$.

Notice that for something to be a function it only has to take every element of the starting space to a single defined member of the finishing space. We do not require the sets to have numbers in them (as the third example, involving h, shows).

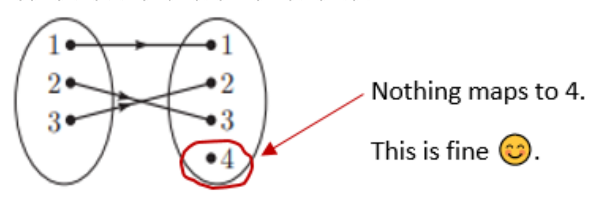
Things that are not functions.

Suppose we try to define a function $k\colon \mathbb{R} \to \mathbb{R}$ by setting $k(r) = \pm\sqrt{r}$.
This fails to be a function on two counts. Firstly, $\sqrt{-2}$ does not exist in $\mathbb{R}$ so $k(-2)$ is not defined (or, at least, is not an element of the given codomain). And secondly 4 could go to either of -2 or $+2$ so $k(4)$ is not a unique element in the codomain.
Remember, every element of the starting space must have a single, corresponding value in the finishing space.

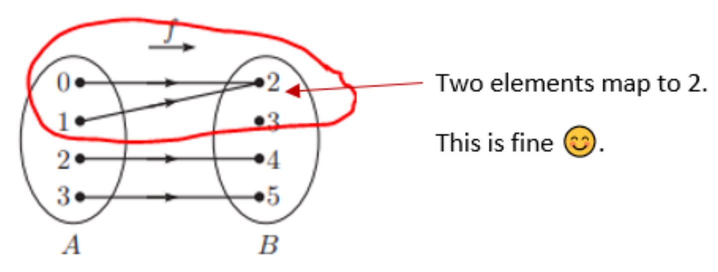




It does not matter if some elements of the finishing space have no elements that map to them. This just means that the function is not 'onto'.



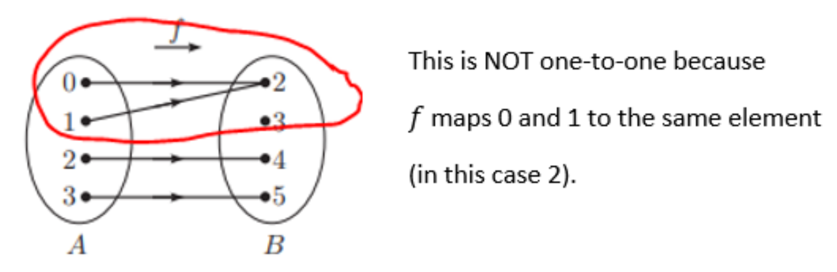
It does not matter if two elements of the starting space map to the same elements of the finishing space. This just means that the function is not 'one-to-one'.

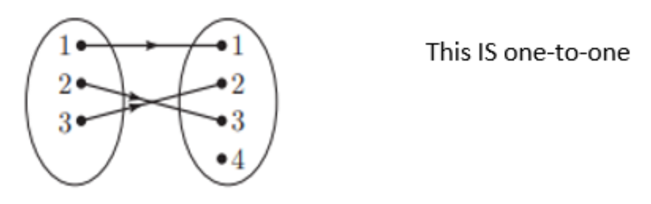


One-to-one and onto

One-to-one functions or maps.

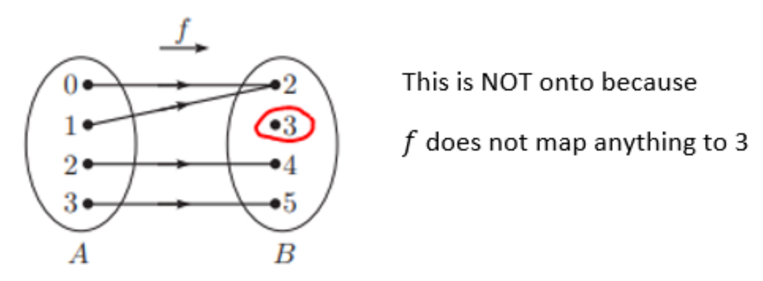
A map or function, f, is **one-to-one** if each element in the image of f is the image of exactly one element in the domain of f. In other words, if $f(a) = f(b)$ for some a and b in the domain of the function f, then $a = b$. This is the same as saying that no two elements of the domain are mapped to the same element of the image.

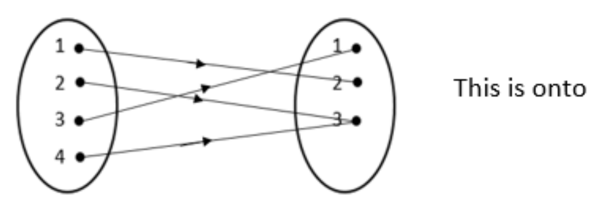




Onto functions or maps.

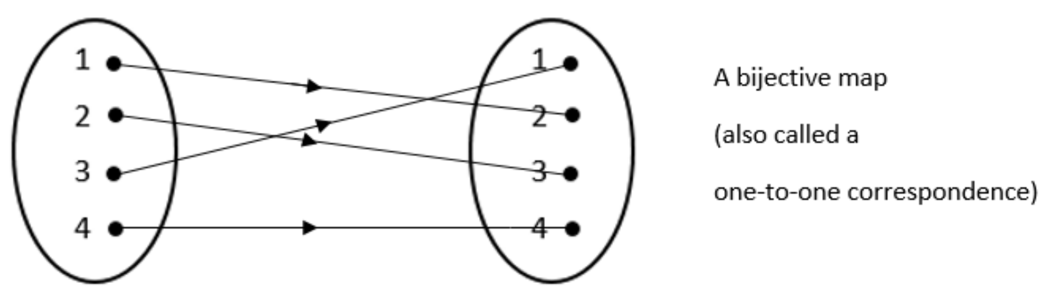
A map or function, f, from a set A to a set B is **onto** if each element in B is the image of at least one element from A. In other words, for each $b \in B$ we can find an $a \in A$ such that $b = f(a)$. This is the same as saying that every element in B is the image of an element in A and that the image set, $f(A)$, which is defined to be the set $\{f(a) : a \in A\}$, is equal to the codomain B.





Bijective maps

A **one-to-one correspondence** is a function or map that is BOTH one-to-one and onto. In M303 (in line with many other texts) we call a one-to-one correspondence a **bijection.**
A **bijective map** is BOTH one-to-one and onto. Remember, bijection is just another word for one-to-one correspondence. A map is a bijection if it is BOTH one-to-one and onto.



Determining whether a map is one-to-one, onto, and/or bijective

To show that a map is one-to-one.

To show that $f\colon A \to B$ is one-to-one we can do the following.
1. Assume that $f(a) = f(b)$ for some $a, b \in A$.
2. Show that this means that $a = b$.

Example.

Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = n + 303$. Show that f is one-to-one.

Solution.

Assume that $f(a) = f(b)$ for some $a, b \in \mathbb{Z}$. Then we have:
$f(a) = f(b)$, that is $a + 303 = b + 303$, which implies that $a = b$. Therefore f is one-to-one.

To show that a map is not one-to-one.

To show that $f\colon A \to B$ is one-to-one we can do the following. Find two different elements in A, say $a \neq b$ which are such that $f(a) = f(b)$.

Example.

Let $f\colon \mathbb{Z} \to \mathbb{N} \cup \{0\}$ be given by $f(z) = z^2$. Show that f is not one-to-one.

Solution.

Since -2 and 2 are in $\mathbb{Z}$, and $-2 \neq 2$ but $f(-2) = f(2) = 4$, the map f is not one-to-one.

To show that a map is onto.

To show that $f\colon A \to B$ is onto we can do the following.
1. Assume that $b \in B$ is an arbitrary element of B.
2. Find an $a \in A$ which is such that $f(a) = b$.

Example.

Let $f\colon \mathbb{Q} \to \mathbb{Q}$ be given by $f(q) = \frac{q}{303}$. Show that f is onto.

Solution.

Assume that $b \in Q$. We need to find an $a \in Q$ with $f(a) = b$. Let $a = 303 \times b$.
Then $f(a) = f(303 \times b) = \frac{303b}{303} = b$. Therefore f is onto.

To show that a map is not onto.

To show that $f\colon A \to B$ is not onto we can do the following. Find an element $b \in B$ which is not equal to $f(a)$ for any $a \in A$.

Example.

Let $f\colon \{1, 2, 3\} \to \{2, 4, 6, 8\}$ be given by $f(a) = 2a$ for each $a \in \{1, 2, 3\}$. Show that f is not onto.

Solution.

By inspection (this just means 'by looking') we can see that no element of a is mapped onto the element $8 \in B$. Therefore, f is not onto.

To show that a map is a bijection.

We need to show that f is BOTH one-to-one and onto.

Example.

Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be given by $f(z) = z - 303$. Show that f is a bijection.

Solution.

1. Show that f is one-to-one. Assume that $f(a) = f(b)$. Then we have $a - 303 = b - 303$ which implies that $a = b$. Therefore, f is one-to-one.
2. Show that f is onto. Assume that $b \in \mathbb{Z}$. Then by inspection we see that $f(b + 303) = (b + 303) - 303 = b$ and so f in onto.

Therefore, f is both one-to-one and onto and so f is a bijection.

To show that a map is not a bijection.

We can either show that f is not one-to-one or we can show that f is not onto: it only needs to fail one of these conditions to fail to be a bijection.

Example.

Show that $f\colon \mathbb{Z} \to \mathbb{Z}$ given by $f(z) = z^2 + 1$ is not a bijection.

Solution one.

Note that $f(-2) = f(2) = 5$. Therefore, since $-2 \neq 2$, we have that f is not one-to-one. Therefore, f is not a bijection.

Solution two.

Note that $z^2 + 1 > 0$ for all $z \in \mathbb{Z}$. Therefore no number can mapped to -1 by f and so f is not onto. Therefore, f is not a bijection.

Practice questions

This quiz contains some questions to give you practice at the above concepts. Each question has several variants, and you can take the quiz as many times as you wish.

One-to-one and onto practice