MST1252109I3PV1

MST125/Remote

Module Examination 2021

Essential mathematics 2

Tuesday 21 September 2021

There are **three sections** in this examination.

In **Section 1** you should **attempt all 18 questions**. Each question is worth 2% of the total mark. *Each question has ONE correct answer from five options.*

An incorrectly answered question will get zero marks.

Submit your answers to Section 1 using the interactive Computer-marked Examination (iCME), following the on-screen instructions. Give yourself time to check you have entered your answers correctly.

In **Section 2** you should **submit answers to all 5 questions**. Each question is worth 8% of the total mark.

In **Section 3** you should **submit answers to 2 out of the 3 questions**. Each question is worth 12% of the total mark.

For **Sections 2** and **3**, write your answers in pen, though you may draw diagrams in pencil. Start your answer to each question on a new page, clearly indicating the number of the question. Crossed out work will not be marked.

Follow the instructions in the online timed examination for how to submit your work for Sections 2 and 3.

Further information about the examination is in the *Instructions and guidance for your remote examination* document on the module website.

PLAGIARISM WARNING – the use of assessment help services and websites

The **work that you submit for any assessment/examination on any module should be your own**. Submitting work produced by or with another person, or a web service or an automated system, **as if it is your own** is cheating. It is **strictly forbidden** by the University.

You should not:

- provide any assessment question to a website, online service, social media platform or any individual or organisation, as this is an infringement of copyright.
- request answers or solutions to an assessment question on any website, via an online service or social media platform, or from any individual or organisation.
- use an automated system (other than one prescribed by the module) to obtain answers or solutions to an assessment question and submit the output as your own work.
- discuss examination questions with any other person, including your tutor.

The University actively monitors websites, online services and social media platforms for answers and solutions to assessment questions, and for assessment questions posted by students. Work submitted by students for assessment is also monitored for plagiarism.

A student who is found to have posted a question or answer to a website, online service or social media platform and/or to have used any resulting, or otherwise obtained, output as if it is their own work has committed a disciplinary offence under **Section SD 1.2** of our Code of Practice for Student Discipline. **This means the academic reputation and integrity of the University has been undermined.**

The Open University's Plagiarism policy defines plagiarism in part as:

- using text obtained from assignment writing sites, organisations or private individuals.
- obtaining work from other sources and submitting it as your own.

If it is found that you have used the services of a website, online service or social media platform, or that you have otherwise obtained the work you submit from another person, this is considered serious academic misconduct and you will be referred to the Central Disciplinary Committee for investigation.

SECTION 1

You should ***submit answers to all questions in this section. Each question is worth 2%.***

Question 1

What is the least residue of 7^{162} modulo 17 ?

A 0 **B** 1 **C** 2 **D** 15 **E** 16

Question 2

A parabola in standard position passes through the point $(3, 6)$. What is the equation of its directrix?

A $x = -6$ **B** $x = -3$

C $x = 3$ **D** $x = 6$

E It is not possible to determine the equation of the directrix from the given information.

Question 3

What is the equation of the curve described by the parametric equations

$$x = 3 + \cos t, \quad y = -5 + \sin t\,?$$

A $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$ **B** $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

C $(x-3)^2 + (y+5)^2 = 1$ **D** $(x+3)^2 - (y+5)^2 = 1$

E $x^2 + y^2 = 34$

Question 4

The magnitude F (in newtons) of a force $\mathbf{F}$ and the magnitude G (in newtons) of a force $\mathbf{G}$ satisfy the vector equation

$$F\mathbf{i} + G\mathbf{j} - 3G\mathbf{i} - 2\mathbf{j} = \mathbf{0},$$

where $\mathbf{i}$ and $\mathbf{j}$ are the Cartesian unit vectors. What is the value of F?

A -6 **B** -3 **C** -2 **D** 2 **E** 6

Question 5

A stereo rests on a horizontal surface. The normal reaction of the surface on the stereo is 13 N vertically upwards. What is the mass of the stereo, in kilograms to two significant figures? Take the magnitude of the acceleration due to gravity to be $9.8\,\mathrm{m\,s^{-2}}$.

A 0.75 **B** 1.3 **C** 13 **D** 23 **E** 127

Question 6

Which of the following is the rule of the isometry h formed from the reflection in the y-axis followed by the translation 2 units to the left?

A $h(x, y) = (-x - 2, y)$ **B** $h(x, y) = (-x + 2, y)$
C $h(x, y) = (x - 2, -y)$ **D** $h(x, y) = (y - 2, x)$
E $h(x, y) = (y + 2, x)$

Question 7

Let f and g be the linear transformations represented by the matrices

$$\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 4 & -1 \end{pmatrix}$$

respectively. What is the image of the point $(1, 0)$ under the composite transformation $g \circ f$?

A $(-1, 1)$ **B** $(1, -2)$ **C** $(1, 5)$ **D** $(2, 5)$ **E** $(6, -1)$

Question 8

Which option describes the linear transformation represented by the matrix $\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$?

A preserves orientation
B scales areas by factor $\frac{1}{2}$
C has an image set that is a line
D does not fix any points
E is a scaling

Question 9

Which of the following functions has all of the following features?

- One negative x-intercept
- One positive y-intercept
- One vertical asymptote
- One horizontal asymptote

A $f(x) = \dfrac{3 - x}{x + 2}$ **B** $f(x) = \dfrac{3 - x}{x^2 + 2}$ **C** $f(x) = \dfrac{x + 2}{3 - x}$

D $f(x) = \dfrac{x + 2}{x^2 + 3}$ **E** $f(x) = \dfrac{x^2 + 2}{3 - x}$

Question 10

A rational expression $f(x)$ has a partial fraction expansion of the form

$$\frac{A}{2x+7}+\frac{B}{(2x+7)^2}$$

where A and B are integers.

What is the form of

$$\int f(x)\,\mathrm{d}x\,?$$

In the options, c is an arbitrary constant.

A $\dfrac{A}{2}\ln|2x+7|-\dfrac{B}{2(2x+7)}+c$ **B** $\dfrac{A}{2}\ln|2x+7|-\dfrac{2B}{(2x+7)^2}+c$

C $2A\ln|2x+7|-\dfrac{2B}{2x+7}+c$ **D** $\dfrac{A-2B}{2}\ln|2x+7|+c$

E $-\dfrac{A}{(2x+7)^2}-\dfrac{2B}{(2x+7)^3}+c$

Question 11

Which of the following is *not* a particular solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=3\cosh(3x)\,?$$

A $y=\dfrac{\sinh(3x)+1}{3}$ **B** $y=\dfrac{6\sinh(3x)-1}{6}$

C $y=\dfrac{2\sinh(3x)-1}{2}$ **D** $y=\sinh(3x)$

E $y=\sinh(3x)+1$

Question 12

Which of the following differential equations is separable?

A $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\cos(ty)$ **B** $\dfrac{\mathrm{d}y}{\mathrm{d}t}=t\cos(ty)$ **C** $\dfrac{\mathrm{d}y}{\mathrm{d}t}=t+\cos y$

D $\dfrac{\mathrm{d}y}{\mathrm{d}t}-1=t\cos y$ **E** $\dfrac{\mathrm{d}y}{\mathrm{d}t}-t=t\cos y$

Question 13

Which of the following is an integrating factor $p(x)$ for the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y}{x} + x \quad (x > 0)\,?$$

A $p(x) = -\dfrac{2}{x}$ **B** $p(x) = \dfrac{1}{x^2}$ **C** $p(x) = e^{-x^2}$

D $p(x) = e^{1/x^2}$ **E** $p(x) = e^{-2/x}$

Question 14

What is the negation of the following statement?

For all $n \in \mathbb{N}$, the integer $n^2 - n + 1$ is odd.

A For all $n \in \mathbb{N}$, the integer $n^2 - n + 1$ is even.

B For all $n^2 - n + 1 \in \mathbb{N}$, the integer n is even.

C For all $n^2 - n + 1 \in \mathbb{N}$, the integer n is odd.

D There is an $n \in \mathbb{N}$ such that the integer $n^2 - n + 1$ is odd.

E There is an $n \in \mathbb{N}$ such that the integer $n^2 - n + 1$ is even.

Question 15

An object falls from rest under gravity alone. How long, in seconds to two significant figures, does it take to hit the ground 15 m below? Take the acceleration due to gravity to be $9.8\,\mathrm{m\,s^{-1}}$.

A 0.30 **B** 1.2 **C** 1.5 **D** 1.7 **E** 3.1

Question 16

The velocity of a particle is given in terms of the time t by

$$\mathbf{v} = \sin t\,\mathbf{i} - 2\cos(2t)\,\mathbf{j} + t^2\,\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the Cartesian unit vectors. What is the acceleration of the particle when $t = \pi$?

A $\mathbf{i} + 2\pi\,\mathbf{k}$ **B** $\mathbf{i} + \dfrac{\pi^3}{3}\,\mathbf{k}$ **C** $-\mathbf{i} + 2\pi\,\mathbf{k}$

D $-\mathbf{i} + \dfrac{\pi^3}{3}\,\mathbf{k}$ **E** $4\,\mathbf{j} + 2\pi\,\mathbf{k}$

Question 17

An object is acted on by two forces, $10\,\mathbf{i} + 3\,\mathbf{j}$ and $5\,\mathbf{i} - 3\,\mathbf{j}$ (with magnitudes in newtons). The object has an acceleration of $2\,\mathrm{m\,s^{-2}}$ in the direction of the resultant force. What is the mass of the object in kilograms to two significant figures?

A 0.13 **B** 0.51 **C** 1.7 **D** 2.1 **E** 7.5

Question 18

A 2×2 matrix has determinant 9 and a repeated positive eigenvalue. What is this eigenvalue?

A -9 **B** -3 **C** $\frac{1}{3}$ **D** 3 **E** 9

SECTION 2

*You should **submit answers to all questions in this section**, write in **pen** and start your answer to each question on a new page.*

***Include all your working**, as most marks are awarded for this. Answers without appropriate supporting working as directed by the question **will not be given credit**.*

Each question is worth 8%.

Question 19

In each of parts (a) and (b), either solve the congruence, giving the solution as a least residue, or show that it has no solutions.

(a) $33x \equiv 21 \pmod{117}$

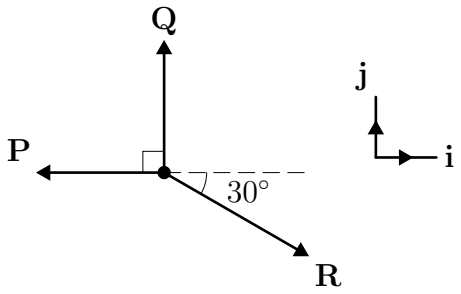
(b) $33x \equiv 22 \pmod{117}$ [*8*]

Question 20

A particle, which remains at rest, is acted on by three forces, **P**, **Q** and **R**, and no others, as shown in the diagram below.

The force **P** acts horizontally to the left, the force **Q** acts vertically upwards and the force **R** acts downwards and to the right at an angle of 30° to the horizontal. The magnitude of **P** is 52 N. Let the magnitudes of **Q** and **R** in newtons be Q and R respectively.

Take the Cartesian unit vectors **i** and **j** to be in the opposite direction to **P** and in the same direction as **Q**, respectively.



(a) Find expressions for the component forms of the three forces **P**, **Q** and **R**. [*4*]

(b) Hence or otherwise find Q to two significant figures. [*4*]

Question 21

(a) Use suitable sum and difference identities for hyperbolic functions to show that

$$\sinh(5x)\cosh(3x) = \frac{1}{2}(\sinh(8x) + \sinh(2x)).$$ [4]

(b) Hence, or otherwise, find

$$\int \sinh(5x)\cosh(3x)\,\mathrm{d}x.$$ [4]

Question 22

In each of the following statements, n represents a positive integer. One of the statements is true, the other is false.

(I) For all n, the integer $9^n + 7$ is a multiple of 8.

(II) For all n, the integer $9^n + 6$ is a multiple of 5.

(a) Write down which statement is false and give a counter-example to show that it is false. [2]

(b) Prove the other statement by mathematical induction. [6]

Question 23

(a) Let $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 12 & -1 \end{pmatrix}$.

Without using the characteristic equation of $\mathbf{A}$, show that $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$, and find the corresponding eigenvalue. [2]

(b) A matrix $\mathbf{B}$ has eigenvectors $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, with corresponding eigenvalues 15 and 10 respectively.

(i) Write down a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{P}$ such that $\mathbf{B} = \mathbf{PDP^{-1}}$, and calculate $\mathbf{P^{-1}}$. [2]

(ii) Hence show that

$$\mathbf{B}^4 = \begin{pmatrix} 18125 & -16250 \\ -16250 & 42500 \end{pmatrix}.$$ [4]

SECTION 3

*You should **submit answers to two questions in this section**. If you submit more, only the first two answers in your submission will be marked.*

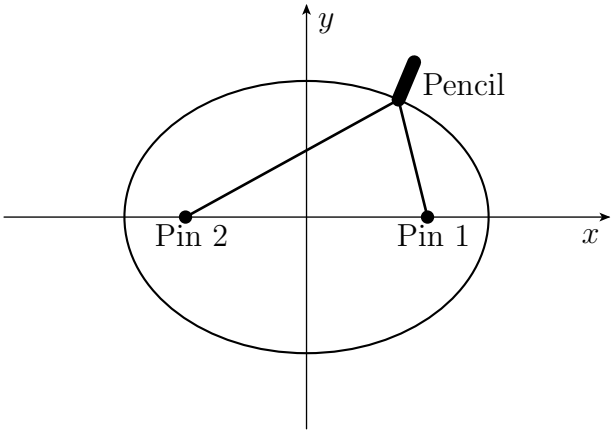
*Write in **pen** and start your answer to each question on a new page.*

***Include all your working**, as most marks are awarded for this. Answers without appropriate supporting working as directed by the question **will not be given credit**.*

Each question is worth 12%.

Question 24

An ellipse is drawn using the gardener's method as follows. Two pins are stuck on a corkboard through a sheet of paper at a distance of 4 cm from one another and joined by a length of string. A pencil is used to hold the string taut and carefully moved around to mark the ellipse. The length of the taut string is 5 cm. Assume that axes are drawn so that the ellipse is in standard position with the major axis lying on the line that passes through the two pins, as shown in the diagram below.



(a) Find the coordinates of the foci. By considering the length of the string, or otherwise, find the x-intercepts and the eccentricity of the ellipse. [3]

(b) Hence find the y-intercepts and show that the equation of the ellipse is

$$\frac{4x^2}{25} + \frac{4y^2}{9} = 1\,.$$ [3]

(c) Find the directrices of the ellipse. [1]

(d) Use the focus-directrix definition of an ellipse to show that if $P = (x, y)$ is a point on the ellipse and F_1 is the focus that lies to the right of the y-axis, then

$$PF_1 = \frac{5}{2} - \frac{4}{5}x\,.$$ [2]

(e) (i) Give a geometric description of the linear transformation f that maps the unit circle $\mathcal{C}$ to this ellipse.

(ii) Hence write down the matrix that represents f, and use it to find the area enclosed by the ellipse. [3]

Question 25

Consider the function $f(x) = \dfrac{2-x}{x^2+32}$.

(a) Find the domain of f and the x- and y-intercepts, if any. [*1*]

(b) Show that

$$f'(x) = \frac{(x+4)(x-8)}{(x^2+32)^2}.$$ [*2*]

(c) Find the coordinates of any stationary points of f. [*1*]

(d) Construct a table of signs for $f'(x)$. [*2*]

(e) Determine the intervals on which f is increasing and the intervals on which it is decreasing. [*1*]

(f) Determine the nature(s) of the stationary point(s). [*1*]

(g) Find the equations of the asymptotes of f. [*1*]

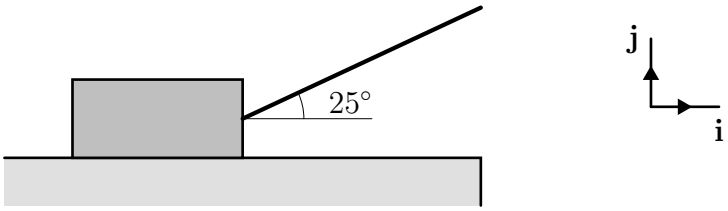
(h) Determine whether f is an even or odd function, or neither. [*1*]

(i) Sketch the graph of f. [*2*]

Justify your answers fully.

Question 26

A child pulls a string attached to a toy of mass 0.4 kg on a low horizontal table. The toy moves in a straight line towards the table edge with a constant acceleration of magnitude $1.5\,\mathrm{m\,s^{-2}}$. The string is taut and makes an angle of 25° with the horizontal, as shown in the diagram below.



The coefficient of sliding friction between the toy and the table is 0.1. Take the magnitude of the acceleration due to gravity to be $9.8\,\mathrm{m\,s^{-2}}$. Model the toy as a particle and the string as a model string.

In your response to this question, underline vectors to distinguish them from scalar quantities. Where the magnitude of a vector is unknown, use the vector letter to represent the magnitude: for example, write the magnitude of a vector $\underline{\mathrm{A}}$ as A.

(a) (i) State the four forces that act on the toy. [*2*]

(ii) Draw a force diagram that represents the four forces, labelling the forces clearly. [*2*]

(iii) Take the Cartesian unit vectors **i** and **j** to point horizontally in the direction of motion of the toy and vertically upwards respectively, as shown above. Write down expressions (in terms of **i** and **j**) for the component forms of the four forces acting on the toy. [*2*]

(iv) Write down the vector equation obtained by applying Newton's second law of motion to the toy. Hence, or otherwise, find the magnitude of the pulling force exerted by the child, to two significant figures. [*4*]

(b) The toy is initially at rest, and it falls off the table after 1.3 seconds. Calculate, to two significant figures, the initial distance of the toy from the table edge. [*2*]

[END OF QUESTION PAPER]