The Essential M303
Book A part 1

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5. Apologies in advance for any errors .....

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Overview

1. Induction – TMA question 2.

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2. The Euclidean Algorithm – TMA question 3.

Induction – example 1

Use mathematical induction to prove that for all integers $n \geq 1$, the number $3^{3n} - 1$ is divisible by 13.

Solution – STEP 1

Let $P(n)$ be the proposition

$$3^{3n} - 1 = 13r, \qquad \text{for some integer } r.$$

Solution – STEP 2

Then $P(1)$ is true since

$$3^{3\cdot 1} - 1 = 27 - 1 = 2 \cdot 13.$$

Solution – STEP 3

Suppose $P(k)$ is true (that is, $3^{3k} - 1 = 13s$ for some integer s) and investigate $P(k+1)$:

$$\begin{aligned} 3^{3(k+1)} - 1 &= 3^{3k} \cdot 3^3 - 1 \\ &= (13s + 1) \cdot 27 - 1 \\ &= 13(27s) + 26 = 13(27s + 2). \end{aligned}$$

which establishes $P(k+1)$.

Solution – STEP 4

The result now follows by the Principle of Mathematical Induction; p18 of Chapter 1, HB p14.

Euclidean algorithm – example 1

Use the Euclidean Algorithm to determine integers x and y such that

$$\mathrm{hcf}(350, 196) = 350x + 196y.$$

Write down the general solution of this linear Diophantine equation, and find the particular solution in which x takes its least positive value.

Solution – STEP 1

The Euclidean algorithm is described in Section 5.1 of Chapter 1. Applying it, we find

$$
\begin{aligned}
350 &= 1 \cdot 196 + 154 \\
196 &= 1 \cdot 154 + 42 \\
154 &= 3 \cdot 42 + 28 \\
42 &= 1 \cdot 28 + 14 \\
28 &= 2 \cdot 14.
\end{aligned}
$$

So $\mathrm{hcf}(350, 196) = 14$ (the last non-zero remainder) .

Solution – STEP 2

Reversing the argument:

$$\begin{aligned}
14 &= 42 - 1 \cdot 28 \\
&= 42 - (154 - 3 \cdot 42) \\
&= 4 \cdot 42 - 154 \\
&= 4(196 - 1 \cdot 154) - 154 \\
&= 4 \cdot 196 - 5 \cdot 154 \\
&= 4 \cdot 196 - 5(350 - 196) \\
&= -5 \cdot 350 + 9 \cdot 196
\end{aligned}$$

Solution – STEP 3

The general solution is given by (see Theorem 5.4 of Chapter 1, HB p16)

$$x = -5 + 14k, \; y = 9 - 25k; \; k \text{ any integer.}$$

Solution – STEP 3

The general solution is given by (see Theorem 5.4 of Chapter 1, HB p16)

$$x = -5 + 14k, \; y = 9 - 25k; \; k \text{ any integer.}$$

Note that the 14 is $196/14$ and the 25 is $350/14$.

Solution – STEP 4

Finally, the particular solution in which x takes its least positive value is the case $k = 1$, with $x = 9$ and $y = -16$.

Induction – example 2

The nth heptagonal number is given by the expression $n(5n-3)/2$. Use mathematical induction to prove that the following formula, which gives the sum of the first n heptagonal numbers, holds for all integers $n \geq 1$:

$$1 + 7 + 18 + 34 + \ldots + \frac{n(5n-3)}{2} = \frac{n(n+1)(5n-2)}{6}.$$

Solution – STEP 1

Let $P(n)$ be the proposition

$$\sum_{i=1}^{n} \frac{i(5i-3)}{2} = \frac{n(n+1)(5n-2)}{6}.$$

Solution – STEP 2

Then $P(1)$ is true since

$$\frac{1 \cdot (5 \cdot 1 - 3)}{2} = 1 = \frac{1 \cdot 2 \cdot 3}{6} = \frac{1 \cdot (1+1) \cdot (5 \cdot 1 - 2)}{6}.$$

Solution – STEP 3

Suppose $P(k)$ is true (that is, $\sum_{i=1}^{k} \frac{i(5i-3)}{2} = \frac{k(k+1)(5k-2)}{6}$) and investigate $P(k+1)$:

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{i(5i-3)}{2} &= \sum_{i=1}^{k} \frac{i(5i-3)}{2} + \frac{(k+1)(5(k+1)-3)}{2} \\
&= \frac{k(k+1)(5k-2)}{6} + \frac{(k+1)(5k+2)}{2} \\
&= \frac{(k+1)[k(5k-2)+3(5k+2)]}{6} \\
&= \frac{(k+1)(k+2)(5(k+1)-2)}{6},
\end{aligned}$$

which establishes $P(k+1)$.

Solution – STEP 4

The result now follows by the Principle of Mathematical Induction; p18 of Chapter 1, HB p14.

Euclidean algorithm – example 2

Use the Euclidean Algorithm to determine integers x and y such that

$$\mathrm{hcf}(704, 297) = 704x + 297y.$$

Write down the general solution of this linear Diophantine equation, and find the particular solution in which x takes its least positive value.

Solution – STEP 1

The Euclidean algorithm is described in Section 5.1 of Chapter 1. Applying it, we find

$$\begin{aligned} 704 &= 2 \cdot 297 + 110 \\ 297 &= 2 \cdot 110 + 77 \\ 110 &= 1 \cdot 77 + 33 \\ 77 &= 2 \cdot 33 + 11 \\ 33 &= 3 \cdot 11. \end{aligned}$$

So $\mathrm{hcf}(704, 297) = 11$ (the last non-zero remainder) .

Solution – STEP 2

Reversing the argument:

$$\begin{aligned}
11 &= 77 - 2 \cdot 33 \\
&= 77 - 2(110 - 77) \\
&= 3 \cdot 77 - 2 \cdot 110 \\
&= 3(297 - 2 \cdot 110) - 2 \cdot 110 \\
&= 3 \cdot 297 - 8 \cdot 110 \\
&= 3 \cdot 297 - 8(704 - 2 \cdot 297) \\
&= -8 \cdot 704 + 19 \cdot 297
\end{aligned}$$

Solution – STEP 3

The general solution is given by (see Theorem 5.4 of Chapter 1, HB p16)

$$x = -8 + 27k, \; y = 19 - 64k; \; k \text{ any integer.}$$

Solution – STEP 3

The general solution is given by (see Theorem 5.4 of Chapter 1, HB p16)

$$x = -8 + 27k, \;\; y = 19 - 64k; \;\; k \text{ any integer.}$$

Note that the 27 is $297/11$ and the 64 is $704/11$.

Solution – STEP 4

Finally, the particular solution in which x takes its least positive value is the case $k = 1$, with $x = 19$ and $y = -45$.

Induction – example 3

Use mathematical induction to prove that the following formula holds for all integers $n \geq 1$:

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} \dots \frac{2n+1}{n^2 \times (n+1)^2} = 1 - \frac{1}{(n+1)^2}.$$

Solution – STEP 1

Let $P(n)$ be the proposition

$$\sum_{i=1}^{n} \frac{2i+1}{i^2(i+1)^2} = 1 - \frac{1}{(n+1)^2}.$$

Solution – STEP 2

Then $P(1)$ is true since

$$\frac{3}{1 \cdot 4} = 1 - \frac{1}{4}.$$

Solution – STEP 3

Suppose $P(k)$ is true (that is, $\sum_{i=1}^{k} \frac{2i+1}{i^2(i+1)^2} = 1 - \frac{1}{(k+1)^2}$) and investigate $P(k+1)$:

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{2i+1}{i^2(i+1)^2} &= \sum_{i=1}^{k} \frac{2i+1}{i^2(i+1)^2} + \frac{2k+3}{(k+1)^2(k+2)^2} \\
&= 1 - \frac{1}{(k+1)^2} + \frac{2k+3}{(k+1)^2(k+2)^2} \\
&= 1 - \frac{(k+2)^2 - 2k - 3}{(k+1)^2(k+2)^2} = 1 - \frac{1}{(k+2)^2},
\end{aligned}$$

which establishes $P(k+1)$.

Solution – STEP 4

The result now follows by the Principle of Mathematical Induction; p18 of Chapter 1, HB p14.

Euclidean algorithm – example 3

Use the Euclidean Algorithm to determine integers x and y such that

$$\mathrm{hcf}(247, 156) = 247x + 156y.$$

Write down the general solution of this linear Diophantine equation, and find the particular solution in which x takes its least positive value.

Solution – STEP 1

The Euclidean algorithm is described in Section 5.1 of Chapter 1. Applying it, we find

$$\begin{aligned} 247 &= 1 \cdot 156 + 91 \\ 156 &= 1 \cdot 91 + 65 \\ 91 &= 1 \cdot 65 + 26 \\ 65 &= 2 \cdot 26 + 13 \\ 26 &= 2 \cdot 13. \end{aligned}$$

So $\mathrm{hcf}(247, 156) = 13$ (the last non-zero remainder) .

Solution – STEP 2

Reversing the argument:

$$\begin{aligned}
13 &= 65 - 2 \cdot 26 \\
&= 65 - 2(91 - 65) \\
&= 3 \cdot 65 - 2 \cdot 91 \\
&= 3(156 - 91) - 2 \cdot 91 \\
&= 3 \cdot 156 - 5 \cdot 91 \\
&= 3 \cdot 156 - 5(247 - 156) \\
&= 8 \cdot 156 - 5 \cdot 247
\end{aligned}$$

Solution – STEP 3

The general solution is given by (see Theorem 5.4 of Chapter 1, HB p16)

$$= -5 + 12k, \ y = 8 - 19k; \ k \text{ any integer.}$$

Solution – STEP 3

The general solution is given by (see Theorem 5.4 of Chapter 1, HB p16)

$$= -5 + 12k,\ \ y = 8 - 19k;\ \ k \text{ any integer.}$$

Note that the 12 is $156/13$ and the 19 is $247/13$.

Solution – STEP 4

Finally, the particular solution in which x takes its least positive value is the case $k = 1$, with $x = 7$ and $y = -11$.

Thanks for listening!!!