Question 1:

a)
i.

$$g(x,y) = (2x, 7y)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 7y \end{pmatrix}$$

$$\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} 2x+0y \\ 0x+7y \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

This is a diagonal scaling, scale by 2 in $x-$direction, 7 in $y-$direction.

also called (2, 7)–scaling.

ii.

$$h(x,y) = (x, 4x+y)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 4x+y \end{pmatrix}$$

$$\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} x+0y \\ 4x+y \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

This is a shear transformation.

What is the shear factor? Lost 1 mark.

iii.

$$k(x,y) = (y,x)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

$$\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} 0x+y \\ x+0y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

✓

This swaps x and y.

This is a reflection along the line y = x. Lost 0.5 mark.

b)

$$g = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \qquad k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$f = k \circ h \circ g$$

First we find $h \circ g$

$$= h \circ g$$

$$= \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 2 + 0 \times 0 & 1 \times 0 + 0 \times 7 \\ 4 \times 2 + 1 \times 0 & 4 \times 0 + 1 \times 7 \end{pmatrix}$$

✓

$$= \begin{pmatrix} 2 & 0 \\ 8 & 7 \end{pmatrix}$$

Now we find $k \circ h \circ g$

$$\mathbf{A} = k \circ \begin{pmatrix} 2 & 0 \\ 8 & 7 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 8 & 7 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \times 2 + 1 \times 8 & 0 \times 0 + 1 \times 7 \\ 1 \times 2 + 0 \times 8 & 1 \times 0 + 0 \times 7 \end{pmatrix}$$
$$= \begin{pmatrix} 8 & 7 \\ 2 & 0 \end{pmatrix}$$

As required

c)

First we have to find the determinant of f.

$$DetA = Det \begin{pmatrix} 8 & 7 \\ 2 & 0 \end{pmatrix}$$
$$= 8 \times 0 - 7 \times 2$$
$$= -14$$

As $DetA \neq 0$ f is invertable

To find the determiate of a matrix, we can use the formula:

$$DetA = ad - bc$$

To find the inverse of a matrix we use

$$\mathbf{A^{-1}} = \frac{1}{DetA} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Hence the matrix that represents f^{-1} is

$$f^{-1} = \frac{1}{DetA} \times \begin{pmatrix} 0 & -7 \\ -2 & 8 \end{pmatrix}$$

$$= \frac{-1}{14} \times \begin{pmatrix} 0 & -7 \\ -2 & 8 \end{pmatrix}$$

$$= \frac{1}{14} \times \begin{pmatrix} 0 & 7 \\ 2 & -8 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{7} & -\frac{4}{7} \end{pmatrix}$$

✓

d)

First to find the coordinates of the point in the domain of f that is mapped to a general point (x, y) in the codomain of f.

Each point (x, y) is the image under f of the point $f^{-1}(x, y)$

$$\mathbf{A^{-1}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{7} & -\frac{4}{7} \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 0 \times x + \frac{1}{2} \times y \\ \frac{1}{7} \times x - \frac{4}{7} \times y \end{pmatrix}$$

✓

Hence f maps the point $(\frac{y}{2}, \frac{x-4y}{7})$ to the point (x, y)

The general equation of the unit circle is

$$x^2 + y^2 = 1$$

Substitute these values into the unit circle to find the equation of the image $f(C)$

$$\boxed{(\frac{y}{2}^2 + \frac{x-4y}{7}^2)} = 1$$

GMC: Careful with your notation. This should be

(y/2)^2 + ((x–4y)/7)^2 = 1

These are not the same.

Mulitplying out the brackets

$$\frac{y^2}{4} + \frac{x^2 - 8xy + 16y^2}{49} = 1$$

Multiplying through by 196

$$49y^2 + 4x^2 - 32xy + 64y^2 = 196$$

Combining like terms, leaves us with the equation of $f(C)$

$$\frac{1}{196}(4x^2 - 32xy + 113y^2) = 1$$

Putting this into the form $ax^2 + bxy + cy^2 = d$

$$4x^2 - 32xy + 113y^2 = 196$$

This is the equation of the image $f(C)$

where $a = 4, b = -32, c = 113, d = 196$

e)

The area of the unit circle is π the area of $f(C)$ is given by the fact that linear transformations scale the area by the absolute value of the determinant of the transformation matrix.

The area of the image $f(C)$ is

$$Area(f(C)) = |Detf| \times Area(C)$$

$$= 14 \times \pi$$

$$= 14\pi$$

Q1: 14/15

Question 2:

a)
i.
The affine transformation f that maps the points $(0,0),(1,0),(0,1)$ to the points $(-3,4),(-2,4),(-3,5)$ can be represented by the matrix equation

$$f(x) = \mathbf{A}x + \mathbf{a}$$

Let, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the new vector positions.

$$\mathbf{A} = \begin{pmatrix} \mathbf{b}x - \mathbf{a}x & \mathbf{c}x - \mathbf{a}x \\ \mathbf{b}y - \mathbf{a}y & \mathbf{c}y - \mathbf{a}y \end{pmatrix}$$

and

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}x \\ \mathbf{a}y \end{pmatrix}$$

We can set up the matrices for the transformation

$$\begin{pmatrix} -2-(-3) & -3-(-3) \\ 4-4 & 5-4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence

$$f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

Alternatively, use the definition of fixed points and show that it won't have a solution.

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

ii.
As the matrix is the identity matrix (f) is a translation and hence has no fixed points.

b)

The matrix to represent reflection in the line $y = -x$ is

$$R = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

To reflect in the line $y = -x + 7$, use translation h to the origin, apply R, then translate back with h^{-1}

Let $h(x, y) = (x, y - 7)$, $h^{-1}(x, y) = (x, y + 7)$ ✓

Apply the composite transformation $f(x) = h^{-1}(R(h(x)))$

Step 1: Translate down by 7

$$h(x) = \begin{pmatrix} x \\ y - 7 \end{pmatrix}$$ ✓

Step 2: Reflect in $y = -x$

$$R \times h(x) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} x \\ y - 7 \end{pmatrix} = \begin{pmatrix} -(y - 7) \\ -x \end{pmatrix} = \begin{pmatrix} 7 - y \\ -x \end{pmatrix}$$

Step 3: Translate up by 7

$$f(x) = h^{-1}(R(h(x))) = \begin{pmatrix} 7 - y \\ -x + 7 \end{pmatrix}$$ ✓

Therefore, the matrix form is:

$$B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$ ✓

Q2:
10/10

Well done on
Question 2!

Question 3:

a)

$$\frac{5x^3 - 11x^2 - 99x - 72}{x^2 - 3x - 18}$$

First we need to divide the polynomials using long division.

$$\begin{array}{r l}
 & \phantom{5x^3 - 11x^2 -{}} 5x \quad + 4 \\
x^2 - 3x - 18 \big) & 5x^3 - 11x^2 - 99x - 72 \\
 & -5x^3 + 15x^2 + 90x \\
\hline
 & \phantom{-5x^3 +{}} 4x^2 \quad - 9x - 72 \\
 & -4x^2 + 12x + 72 \\
\hline
 & \phantom{-5x^3 + 15x^2 + {}} 3x
\end{array}$$

$$\frac{5x^3 - 11x^2 - 99x - 72}{x^2 - 3x - 18} = 5x + 4 + \frac{3x}{x^2 - 3x - 18}$$

using partial fractions on the remainder

$$\frac{3x}{x^2 - 3x - 18} = \frac{A}{x - 6} + \frac{B}{x + 3}$$

Using augmented matrix to solve for A and B

$$\begin{aligned}
3x &= A(x + 3) + B(x - 6) \\
&= Ax + 3A + Bx - 6B \\
&= (A + B)x + (3A - 6B)
\end{aligned}$$

Setting up the augmented matrix

$$\left(\begin{array}{cc|c} 3 & -6 & 0 \\ 1 & 1 & 3 \end{array}\right)$$

Subtract 3 times R2 from R1

$$\left(\begin{array}{cc|c} 0 & -9 & -9 \\ 1 & 1 & 3 \end{array}\right)$$

Divide R1 by -9

$$\left(\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 3 \end{array}\right)$$

Subtract R1 from R2

$$\left(\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 2 \end{array}\right)$$

This is an interesting method. Why did you use an augmented matrix considering you are only solving a system of 2 linear equations?

Instead (and probably simpler), use the substitution method or equate the coefficients of the equation you found at the end of page 8.

Hence we can write

$$\frac{3x}{x^2-3x-18} = \frac{2}{x-6} + \frac{1}{x+3}$$

Giving us the full expression

$$\frac{5x^3-11x^2-99x-72}{x^2-3x-18} = 5x+4+\frac{2}{x-6}+\frac{1}{x+3}$$

✓

b)

Question 3 (b)

Question 3 b) Define function

–> f:(5*x^3-11*x^2-99*x-72)/(x^2-3*x-18);

$$\frac{5x^3 - 11x^2 - 99x - 72}{x^2 - 3x - 18} \tag{f}$$

Using partfrac to find the partial fraction of f

–> partfrac(f,x);

$$\frac{1}{x+3} + 5x + \frac{2}{x-6} + 4 \tag{\%o9}$$

c)

$$\begin{aligned}\int \frac{5x^3-11x^2-99x-72}{x^2-3x-18}\,\mathrm{d}x &= \int \left(5x+4+\frac{2}{x-6}+\frac{1}{x+3}\right)\mathrm{d}x\\ &= \int 5x\,\mathrm{d}x + \int 4\,\mathrm{d}x + \int \frac{2}{x-6}\,\mathrm{d}x + \int \frac{1}{x+3}\,\mathrm{d}x\\ &= \frac{5}{2}x^2+4x+2\ln|x-6|+\ln|x+3|+C \quad ✓\\ &= \frac{5}{2}x^2+4x+2\ln(x-6)+\ln(x+3)+C\end{aligned}$$

No information is given about the x, so don't drop the modulus symbols to ensure you are taking the ln of a positive number.

Extension: express the ln terms as a single ln.

Q3: 15/15

Question 4:

$$f(x) = \frac{2-x}{x^2+21}$$

a)
The domain of f is all real numbers $\mathbb{R}$, since the denominator $x^2 + 21$ is never zero.

For the intercepts;

To find the x-intercept, set $f(x) = 0$

$$\begin{aligned} 0 &= \frac{2-x}{x^2+21} \\ 2-x &= 0 \\ x &= 2 \end{aligned}$$

So the x-intercept is at

$$(2, 0)$$

To find the y-intercept, set $x = 0$

$$\begin{aligned} f(0) &= \frac{2-0}{0^2+21} \\ &= \frac{2}{21} \end{aligned}$$

So the y-intercept is at

$$(0, \frac{2}{21})$$

b)
For the stationary points, we need to find the derivative of f and set it to zero.

$$f(x) = \frac{2-x}{x^2+21}$$

Using the quotient rule, where $u = 2 - x$ and $v = x^2 + 21$

$$\begin{aligned}
f'(x) &= \frac{(v \times u' - u \times v')}{v^2} \\
&= \frac{((x^2+21)(-1) - (2-x)(2x))}{(x^2+21)^2} \\
&= \frac{-x^2 - 21 - (4x - 2x^2)}{(x^2+21)^2} \\
&= \frac{-x^2 - 21 - 4x + 2x^2}{(x^2+21)^2} \\
&= \frac{x^2 - 4x - 21}{(x^2+21)^2}
\end{aligned}$$

Setting the numerator to zero for stationary points

$$\begin{aligned}
x^2 - 4x - 21 &= 0 \\
(x+3)(x-7) &= 0
\end{aligned}$$

$$x = -3 \quad \text{or} \quad x = 7$$

So the stationary points are at

$(-3, f(-3))$ and $(7, f(7))$

Calculating $f(-3)$

$$\begin{aligned}
f(-3) &= \frac{2-(-3)}{(-3)^2+21} \\
&= \frac{5}{9+21} \\
&= \frac{5}{30} \\
&= \frac{1}{6}
\end{aligned}$$

So the stationary point is at

$$(-3, \frac{1}{6})$$

Calculating $f(7)$

$$\begin{aligned} f(7) &= \frac{2-7}{7^2+21} \\ &= \frac{-5}{49+21} \\ &= \frac{-5}{70} \\ &= -\frac{1}{14} \end{aligned}$$

So the stationary point is at

$$(7, -\frac{1}{14})$$

The stationary points are

$$(-3, \frac{1}{6}) \text{ and } (7, -\frac{1}{14})$$

c)

Using a table of signs to determine the nature of the stationary points:

Interval	$(-\infty, -3)$	-3	$(-3, 7)$	7	$(7, \infty)$
$x-7$	$-$	$-$	$-$	0	$+$
$x+3$	$-$	0	$+$	$+$	$+$
$(x-7)(x+3)$	$+$	0	$-$	0	$+$

GMC: What about the $(x^2 + 21)^2$. You need to explain clearly why you didn't consider this factor.

This is showing that the curve is increasing on the interval $(\infty, -3)$ the turning point at $x = -3$ the decreasing on the interval $(-3, 7)$ and then a second turning point at $x = 7$ the the curve is increasing on the interval $(7, \infty)$.

Then classify what type of turning points they are. Lost 1 mark.

d)

The horizontal asymptote is found by looking at the limit of $f(x)$ as x approaches infinity.

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{2-x}{x^2+21} \\ &= \lim_{x\to\infty} \frac{-x}{x^2} \\ &= \lim_{x\to\infty} -\frac{1}{x} \\ &= 0\end{aligned}$$

So the horizontal asymptote is at $y = 0$.

There is no vertical asymtope since the denominator $x^2 + 21$ is never zero for any real x.

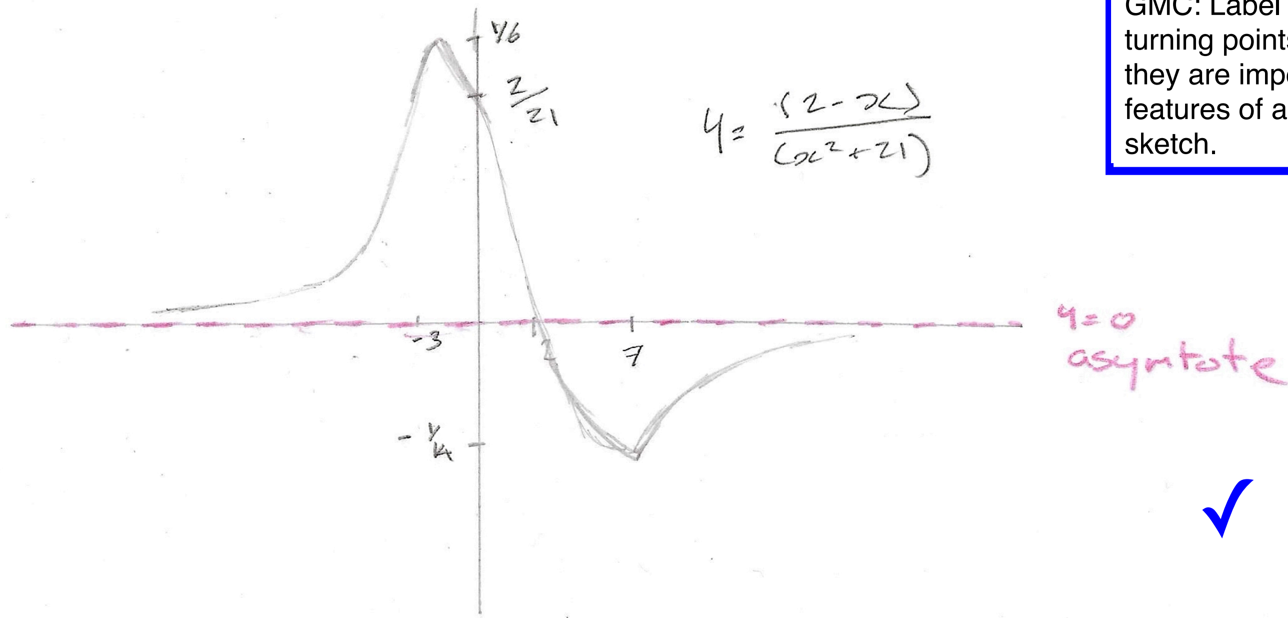
e)
$f(x)$ is neither odd or even, since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$.

GMC: Show why these two are not equal.
Alternatively, use specific values, say f(2) and f(–2), to show that it is neither even nor odd.

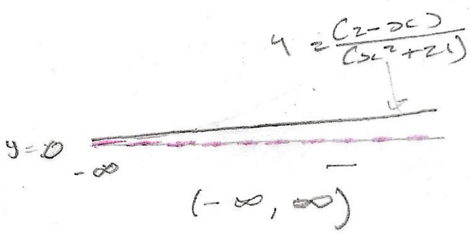
f)

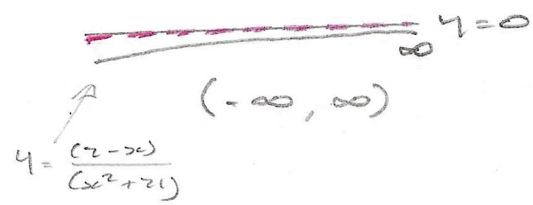
Question 4 (g)

GMC: Label the turning points as they are important features of a sketch.



✓





Q4: 14/15

Question 5:

$$\int e^x \cosh^3(e^x)\,\mathrm{d}x$$

a)

Using the substitution $u = e^x$, then $du = e^x\,\mathrm{d}x$.

$$\cosh^2(x) = \frac{\cosh(2x)+1}{2}$$

$$cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\int e^x \cosh^3(e^x)\,\mathrm{d}x = \int \cosh^3(u)\,\mathrm{d}u$$

$$= \int (\cosh^2(u)\cosh(u))\,\mathrm{d}(u)$$

using half variable identiy

Alternatively (and probably, more efficient solution), use the identity $\cosh^2 u - \sinh^2 u = 1$.

$$= \int \frac{1}{2}(\cosh(2u)+1)\cosh(u)\,\mathrm{d}u$$

$$= \frac{1}{2}\int \boxed{\cosh(2u)\cosh(u)\,\mathrm{d}}u + \frac{1}{2}\int \cosh(u)\,\mathrm{d}u$$

Or you can use the products to sums formulas instead of the definition of cosh u.

$$= \frac{1}{2}\int \frac{e^{-3u}}{4}\,\mathrm{d}u + \frac{1}{2}\int \frac{e^{-u}}{4}\,\mathrm{d}u + \frac{1}{2}\int \frac{e^{u}}{4}\,\mathrm{d}u + \frac{1}{2}\int \frac{e^{3u}}{4}\,\mathrm{d}u + \frac{1}{2}\int \cosh(u)\,\mathrm{d}u$$

$$= \frac{1}{8}\left(\int e^{-3u}\,\mathrm{d}u + \int e^{-u}\,\mathrm{d}u + \int e^{u}\,\mathrm{d}u + \int e^{3u}\,\mathrm{d}u\right) + \frac{1}{2}\int \cosh(u)\,\mathrm{d}u$$

$$= \frac{1}{8}\left(-\frac{1}{3}e^{-3u} - e^{-u} + e^{u} + \frac{1}{3}e^{3u}\right) + \frac{1}{2}\sinh(u) + C$$

$$= \frac{1}{8}\left(\frac{2}{3}sinh(3u) + 2\sinh(u)\right) + \frac{1}{2}\sinh(u) + C$$

$$= \frac{1}{12}\left(sinh(3u) + 9\sinh(u)\right) + C$$

Substituting back for $u = e^x$

$$= \frac{1}{12}\left(sinh(3e^x) + 9\sinh(e^x)\right) + C$$

b)

i.

Using $\cos A \cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$

$$\cos(2x)\cos(5x) = \frac{1}{2}\cos\boxed{2x+5x)} + \frac{1}{2}\cos\boxed{2x-5x)}$$

$$= \frac{1}{2}\cos(7x) + \frac{1}{2}\cos(-3x)$$

Note however that the instructions indicate to use sum and difference identities, and not the product to sum identities. Start with the RHS instead. Lost 3 marks.

Using the fact that $\cos(-x) = \cos(x)$

$$= \frac{1}{2}(\cos(7x) + \cos(3x)) \qquad \textit{As required}$$

The sum and difference identities for trigonometric functions give

$$\cos(5x+2x) = \cos 5x \cos 2x - \sin 5x \sin 2x$$
$$\cos(5x-2x) = \cos 5x \cos 2x + \sin 5x \sin 2x.$$

Here's the start of the solution. See if you can complete it.

ii.

$$\int \sin^2(x)\cos(5x)$$

$$\int \sin^2(x)\cos(5x)\,\mathrm{d}x = \int \frac{1}{2}(1-\cos(2x))cos(5x)\,\mathrm{d}x$$

$$= \frac{1}{2}\int cos(5x)\,\mathrm{d}(x) - \frac{1}{2}\int \cos(2x)\cos(5x)\,\mathrm{d}x$$ ✓

Using the result from part (b)

$$= \frac{1}{2}\int cos(5x)\,\mathrm{d}x - \frac{1}{2}\int \left(\frac{1}{2}(\cos(7x)+\cos(3x))\right)\mathrm{d}x$$

$$= \frac{1}{2}\int cos(5x)\,\mathrm{d}x - \frac{1}{4}\int \cos(7x)\,\mathrm{d}x - \frac{1}{4}\int \cos(3x)\,\mathrm{d}x$$

$$= \frac{1}{2}\times\frac{1}{5}\sin(5x) - \frac{1}{4}\times\frac{1}{7}\sin(7x) - \frac{1}{4}\times\frac{1}{3}\sin(3x) + C$$ ✓

$$= \frac{1}{10}\sin(5x) - \frac{1}{28}\sin(7x) - \frac{1}{12}\sin(3x) + C$$

Q5: 12/15

Question 6:

a)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t^7}{(t^8+32)^{\frac{3}{5}}}$$

This is a directly integrable first order differential equation. To solve it, we integrate both sides with respect to t.

b)

$$\int \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int \frac{t^7}{(t^8+32)^{\frac{3}{5}}}\,\mathrm{d}t$$

$$x(t) = \int \frac{t^7}{(t^8+32)^{\frac{3}{5}}}\,\mathrm{d}t$$

Using the substitution $u = t^8 + 32$, then $du = 8t^7\,\mathrm{d}t$

$$= \frac{1}{8}\int \frac{1}{u^{\frac{3}{5}}}\,\mathrm{d}u$$

$$= \frac{1}{8} \times \frac{u^{\frac{2}{5}}}{\frac{2}{5}} + C$$

$$= \frac{5}{16}(t^8+32)^{\frac{2}{5}} + C$$

c)

To find the particular solution that satisfies $x(0) = 1$.

$$
\begin{aligned}
x(0) &= \frac{5}{16}(0^8 + 32)^{\frac{2}{5}} + C \\
1 &= \frac{5}{16}(32)^{\frac{2}{5}} + C \\
&= \frac{5}{16} \times 4 + C \\
&= \frac{5}{4} + C \\
C &= 1 - \frac{5}{4} \\
&= -\frac{1}{4}
\end{aligned}
$$

Hence

$$
x(t) = \frac{5}{16}(t^8 + 32)^{\frac{2}{5}} - \frac{1}{4}
$$

Q6:
5/5

Question 7:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\sqrt{1-y^2}}{t}, (t > 0, -1 < y < 1)$$

a)

This is a separable differential equation. We can separate the variables and integrate both sides.

✓

b)

$$\int \frac{1}{\sqrt{1-y^2}}\,\mathrm{d}y = \int \frac{1}{t}\,\mathrm{d}t$$

$$\sin^{-1}(y) = \boxed{\ln(t)} + C$$

Explain why you dropped the modulus symbols.

Exponentiating both sides

✓

$$y = \sin(\ln(t) + C)$$

Q7:
5/5

Question 8:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = x^5 sinhx, (x > 0)$$

a)

This is a first order linear differential equation. We can rewrite it in standard form and find an integrating factor.

✓

The integrating factor is given by

$$p(x) = \exp\left(\int g(x)\,\mathrm{d}x\right)$$

The general solution for afirst order differential equation is

$$y = \frac{1}{p(x)}\left(\int p(x)h(x)\right)$$

b)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = x^5 sinh(x)$$

Divide by x

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{4}{x}y = x^4 sinh(x)$$

This is in the form $\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y = h(x)$

Where

$$p(x) = \exp\left(\int \boxed{\frac{4}{x}\,\mathrm{d}x}\right)$$

This should be –4/x, which means the integrating factor is x^{-4}

Lost 1 mark.

$$= \exp(4\ln(x))$$

$$= x^4$$

Using this to write the general solution

$$y = \boxed{\frac{1}{x^{-4}}}\left(\int x^{-4}(x^4\sinh(x))\,\mathrm{d}x\right) \quad = x^4\left(\int \sinh(x)\,\mathrm{d}x\right)$$

However, you have the correct integrating factor here.

Integrating $\sinh(x)$

$$= x^4(\cosh(x) + C)$$

✓

Q8:
4/5

Question 9:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{10000}(100 - y)\text{Where}y(0) = 30$$

a)
This is a first order linear differential equation. We can separate the variables and integrate both sides.

$$\int \frac{1}{100 - y}\,\mathrm{d}y = \int \frac{1}{10000}\,\mathrm{d}t$$

Let $u = 100 - y$ and $du = -\,\mathrm{d}y$

$$-\int \frac{1}{u}\,\mathrm{d}u = \frac{1}{10000}\int \mathrm{d}t$$

$$-\ln|u| = \frac{t}{10000} + C$$

Substituting back for $u = 100 - y$

$$-\ln(100 - y) = \frac{t}{10000} + C$$

GMC: Similarly, explain why you dropped the modulus.

Exponentiating both sides

$$100 - y = e^{-\frac{t}{10000} - C}$$

$$100 - y = Ae^{-\frac{t}{10000}} \qquad \textit{where } A = e^{-C}$$

Rearranging gives us the general solution

$$y = 100 - Ae^{-\frac{t}{10000}}$$

b)
To find the particular solution that satisfies $y(0) = 30$.

$$y(0) = 100 - Ae^{-\frac{0}{10000}}$$

$$30 = 100 - A$$

$$A = 70$$

So the particular solution is

$$y = 100 - 70e^{-\frac{t}{10000}}$$

This can be simplified to

$$y = 100 - 70e^{-\frac{t}{10000}}$$

c)
After 600 s;

$$\begin{aligned} y(600) &= 100 - 70e^{-\frac{600}{10000}} \\ &= 100 - 70e^{-\frac{3}{50}} \\ &= 100 - 70 \times e^{-0.06} \\ &= 100 - 70 \times 0.9417\ldots \\ &= 100 - 65.9235\ldots \\ &= 34.0764\ldots \\ &= 34\,\text{kg} \end{aligned}$$

To 2 s.f

d)

As $t \to \infty$, the term $e^{-\frac{t}{10000}}$ approaches zero, so

$$y(t) \to 100 - 70 \times 0$$

$$= 100$$

So the limiting value of y as $t \to \infty$ is $100\,\mathrm{kg}$.

e)

Question 9 (e)

Question 9 e)

(%i5) f:'diff(y,t) = (1/(10000))*(100-y);

$$\frac{d}{dt}y = \frac{100-y}{10000} \tag{f}$$

(%i11) ode2(f,y,t);

$$y = \%e^{-\frac{t}{10000}}\left(100\%e^{\frac{t}{10000}} + \%c\right) \tag{\%o11}$$

(%i12) sol: ic1(%,y=30,t=0);

$$y = \%e^{-\frac{t}{10000}}\left(100\%e^{\frac{t}{10000}} - 70\right) \tag{sol}$$

✓

Question 10: Your solutions are generally clearly written and easy to follow. However, there were places where you didn't fully explain your methods (e.g., not explaining how the modulus symbols were dropped). See Questions 4 and 9. Lost 1 mark.

Q9: 10/10

Q10: 4/5

TMA: 93/100

Question 10: