MST1252206F1PV1

MST125

Module Examination 2022

Essential mathematics 2

Thursday 9 June 2022

There are **three sections** in this examination.

In **Section 1** you should **attempt <u>all</u> 18 questions**. Each question is worth 2% of the total mark. *Each question has ONE correct answer from five options.*

An incorrectly answered question will get zero marks.

Submit your answers to Section 1 using the interactive Computer-marked Examination (iCME), following the on-screen instructions. Give yourself time to check you have entered your answers correctly.

In **Section 2** you should **submit answers to <u>all</u> 5 questions**. Each question is worth 8% of the total mark.

In **Section 3** you should **submit answers to <u>2 out of the 3</u> questions**. Each question is worth 12% of the total mark.

Do not submit more than the required number of answers for Section 3. If you do, only the first two answers submitted for Section 3 will be marked.

For **Sections 2** and **3**:

Include all your working, as some marks are awarded for this.

Handwritten answers must be in pen, though you may draw diagrams in pencil.

Start your answer to each question on a new page, clearly indicating the number of the question.

Crossed out work will not be marked.

Follow the instructions in the online timed examination for how to submit your work.

Further information about completing and submitting your examination work is in the *Instructions and guidance for your remote examination* document on the module website.

Submit your exam using the iCMA system (iCME81). Make sure that the name of the PDF file containing your answers for Sections 2 and 3 includes your PI and the module code e.g. X1234567MST125.

<u>PLAGIARISM WARNING</u> – the use of assessment help services and websites

The **work that you submit for any assessment/examination on any module should be your own**. Submitting work produced by or with another person, or a web service or an automated system, **as if it is your own** is cheating. It is **strictly forbidden** by the University.

You should not:

- provide any assessment question to a website, online service, social media platform or any individual or organisation, as this is an infringement of copyright.
- request answers or solutions to an assessment question on any website, via an online service or social media platform, or from any individual or organisation.
- use an automated system (other than one prescribed by the module) to obtain answers or solutions to an assessment question and submit the output as your own work.
- discuss examination questions with any other person, including your tutor.

The University actively monitors websites, online services and social media platforms for answers and solutions to assessment questions, and for assessment questions posted by students. Work submitted by students for assessment is also monitored for plagiarism.

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The Open University's Plagiarism policy defines plagiarism in part as:

- using text obtained from assignment writing sites, organisations or private individuals.
- obtaining work from other sources and submitting it as your own.

If it is found that you have used the services of a website, online service or social media platform, or that you have otherwise obtained the work you submit from another person, this is considered serious academic misconduct and you will be referred to the Central Disciplinary Committee for investigation.

SECTION 1

You should ***submit answers to all questions in this section.***
Each question is worth 2%.

Question 1

What is the least residue of $16\,000\,012 \times 48\,000\,003$ modulo 16?

A 4 **B** 8 **C** 10 **D** 12 **E** 36

Question 2

What is the least residue of 7^p modulo p, where $p > 7$ is a prime?

A 0 **B** 1 **C** 7 **D** p **E** 49

Question 3

Which of the following is a value of a for which the linear congruence

$$ax \equiv 7 \pmod{12}$$

has solutions?

A 2 **B** 3 **C** 4 **D** 5 **E** 6

Question 4

Which of the following is a parametrisation of the straight line that passes through $(-2, 3)$ and $(1, 9)$?

A $x = -2 + t,\ y = 3 - 6t$

B $x = -2 + 3t,\ y = 3 + 6t$

C $x = 2 - t,\ y = 3 + 2t$

D $x = 2 + t,\ y = -3 - 2t$

E $x = t,\ y = 3 + 3t$

Question 5

Which of the following could be the equation of a hyperbola in standard position with directrices $x = \pm\frac{4}{3}$ and eccentricity $\frac{3}{2}$?

A $\dfrac{x^2}{\left(\frac{9}{16}\right)} - \dfrac{y^2}{5} = 1$

B $\dfrac{x^2}{2} - \dfrac{y^2}{5} = 1$

C $\dfrac{x^2}{3} - \dfrac{y^2}{5} = 1$

D $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$

E $\dfrac{x^2}{9} - \dfrac{y^2}{5} = 1$

Question 6

A crate of mass 7.3 kg rests on level ground. What is the magnitude of the normal reaction of the ground on the crate, in newtons to 2 significant figures? Take the magnitude of the acceleration due to gravity to be $9.8\,\mathrm{m\,s^{-2}}$.

A 0.74 **B** 1.3 **C** 71 **D** 71.54 **E** 72

Question 7

Three forces $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ are in equilibrium. The component form of $\mathbf{F}$ is $3\mathbf{i}+\mathbf{j}$ and the component form of $\mathbf{G}$ is $-4\mathbf{i}+5\mathbf{j}$. What is the component form of $\mathbf{H}$?

A $-7\mathbf{i}+4\mathbf{j}$ **B** $-\mathbf{i}+6\mathbf{j}$ **C** $\mathbf{i}+6\mathbf{j}$

D $\mathbf{i}-6\mathbf{j}$ **E** $7\mathbf{i}-4\mathbf{j}$

Question 8

A circle has area 4π. What is the area of the image of the circle under the linear transformation represented by the matrix

$$\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}?$$

A -12π **B** -4π **C** $\frac{4}{3}\pi$ **D** 4π **E** 12π

Question 9

Which of the following describes the linear transformation with matrix

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}?$$

A The reflection in the line through the origin with angle of inclination 15°.

B The reflection in the line through the origin with angle of inclination 30°.

C The reflection in the line through the origin with angle of inclination 165°.

D The rotation through 15° about the origin.

E The rotation through 30° about the origin.

Question 10

A rational expression $f(x)$ has a partial fraction expansion of the form

$$\frac{A}{x+4} - \frac{2}{2x+1},$$

where A is an integer.

What is the form of

$$\int f(x)\,\mathrm{d}x\,?$$

In the options, c is an arbitrary constant.

A $\ln\dfrac{|x+4|^A}{|2x+1|} + c$

B $\ln\dfrac{|x+4|^A}{2|2x+1|} + c$

C $A\ln\dfrac{|x+4|}{|2x+1|} + c$

D $A\ln\dfrac{|x+4|}{2|2x+1|} + c$

E $\ln\left(A|x+4| - |2x+1|\right)+c$

Question 11

Which of the following is *not* a particular solution of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2\sinh(t) + 3t^2\,?$$

A $x = 2(\cosh(t) - t) + t^3$

B $x = 2(\cosh(t) - 1) + t^3$

C $x = 2\cosh(t) + t^3$

D $x = 2\cosh(t) + t^3 - 3$

E $x = 3 + 2\cosh(t) + t^3$

Question 12

Which of the following is an integrating factor $p(x)$ for the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x - 3x^2 y?$$

A $\cos x$

B $\dfrac{\cos x}{3x^2}$

C $\exp(\cos x)$

D $\exp(3x^2)$

E $\exp(x^3)$

Question 13

Let $P(x)$ be the variable proposition

$$x^2 > x.$$

Which of the following statements is true?

A NOT $P(-1)$ **B** $P(-0.5)$ **C** $P(0.5)$

D $P(1)$ **E** NOT $P(2)$

Question 14

What is the negation of the following statement, where n is an integer?

n is prime or $n \leq 6$.

A n is prime and $n \leq 6$.

B n is prime and $n > 6$.

C n is not prime or $n > 6$.

D n is not prime and $n > 6$.

E n is not prime and $n \leq 6$.

Question 15

A particle moves along a straight line with constant acceleration. Its initial velocity is $5\,\mathrm{m\,s^{-1}}$ and its velocity after 3 seconds is $15\,\mathrm{m\,s^{-1}}$. What is the distance travelled by the particle, in metres?

A 5 **B** 8 **C** 10 **D** 30 **E** 60

Question 16

A crate of mass 14 kg, initially at rest, is being pushed along a straight line by a resultant horizontal force of 7 N. What is the speed of the crate after 4 seconds, in $\mathrm{m\,s^{-1}}$?

A 2 **B** 7 **C** 8 **D** 14 **E** 10

Question 17

The velocity of a particle is given in terms of the time t by

$$\mathbf{v} = \cos t\,\mathbf{i} - 3\sin(5t)\,\mathbf{j} + t^4\,\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the Cartesian unit vectors. What is the acceleration of the particle when $t = \pi$?

A $-15\,\mathbf{j} + 4\pi^3\,\mathbf{k}$ **B** $-\frac{3}{5}\mathbf{j} + \frac{\pi^5}{5}\mathbf{k}$ **C** $\frac{3}{5}\mathbf{j} + \frac{\pi^5}{5}\mathbf{k}$

D $\frac{3\sqrt{2}}{2}\mathbf{j} + 4\pi^3\,\mathbf{k}$ **E** $15\,\mathbf{j} + 4\pi^3\,\mathbf{k}$

Question 18

A 2×2 matrix has characteristic equation

$$\lambda^2 + a\lambda - 2a^2 = 0,$$

where a is a non-zero real number. What are its eigenvalues?

A $-a$ and $-2a$ **B** $-a$ and $2a$ **C** a and $-2a$

D a and $-2a^2$ **E** $2a$ and a^2

SECTION 2

You should ***submit answers to all questions in this section****, write in* ***pen*** *and start your answer to each question on a new page.*

Include all your working*, as most marks are awarded for this. Answers without appropriate supporting working as directed by the question* ***will not be given credit****.*

Each question is worth 8%.

Question 19

An ellipse in standard position has vertices $(0, \pm 2)$ on the y-axis and eccentricity $\dfrac{\sqrt{2}}{2}$.

(a) Find the vertices of the ellipse on the x-axis, giving their exact coordinates. *[3]*

(b) Hence find the equation of the ellipse. *[2]*

(c) Find the foci and the directrices of the ellipse. *[2]*

(d) The ellipse is translated 1 unit to the right. Find a parametrisation of the translated ellipse. *[1]*

Question 20

(a) Complete the square in the quadratic $x^2 + 4x + 5$. *[1]*

(b) Hence, using a trigonometric substitution, find the integral

$$\int \frac{x+2}{\sqrt{x^2+4x+5}}\,\mathrm{d}x.$$

Give your answer in terms of trigonometric and inverse trigonometric functions. *[7]*

Question 21

A container of hot liquid is placed in a room in which the air temperature is 10°C. The temperature T of the liquid (in degrees Celsius), at time t (in minutes) after the container was placed in the room, can be modelled by the differential equation

$$\frac{dT}{dt} = -0.009(T - 10) \qquad (T > 10).$$

(a) Solve this differential equation to show that the general solution is

$$T = 10 + Ae^{-0.009t},$$

where A is an arbitrary constant. [3]

(b) The liquid initially has a temperature of 90°C. Find the particular solution that describes the temperature of the liquid as a function of the time since it was placed in the room. [2]

(c) Calculate how many minutes the liquid takes to cool to 50°C, giving your answer to 2 significant figures. [3]

Question 22

(a) Write down the converse of the following statement about positive integers m and n.

If 7 divides each of m and n, then 7 divides $2m - n$. [1]

(b) Of the statement given in part (a) and its converse, one is true and the other is false.

(i) Give a counter-example to the false statement. [2]

(ii) Prove the true statement. [5]

Question 23

(a) Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}$. Without using the characteristic equation of $\mathbf{A}$, show that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$, and find the corresponding eigenvalue. [1]

(b) A 2×2 matrix $\mathbf{B}$ has eigenvectors $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ with corresponding eigenvalues 1 and 5 respectively.

(i) Write down a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{P}$ such that $\mathbf{B} = \mathbf{PDP^{-1}}$, and calculate $\mathbf{P^{-1}}$. [2]

(ii) Hence find the matrix $\mathbf{B}^n$, where $n \in \mathbb{N}$, in the form

$$\mathbf{B}^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c and d are expressions in n. [5]

SECTION 3

*You should **submit answers to two questions in this section**. If you submit more, only the first two answers in your submission will be marked. Write in **pen** and start your answer to each question on a new page.*

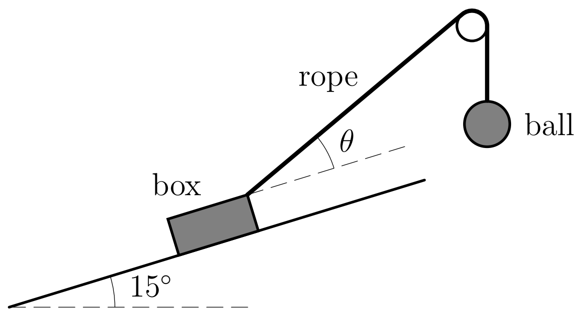
***Include all your working**, as most marks are awarded for this. Answers without appropriate supporting working as directed by the question **will not be given credit**.*

Each question is worth 12%.

Question 24

A box of mass 8 kg rests on a rough ramp inclined at an angle of 15° to the horizontal. A rope attached to the box runs at an angle of θ to the ramp and passes over a pulley suspended above the top of the ramp, as shown in the diagram below. A metal ball hangs from the other end of the rope. The rope remains taut.

The normal reaction of the ramp on the box is 10 newtons, and the coefficient of static friction is 0.28. The box is at rest, but is on the point of slipping up the ramp.



Take the magnitude of the acceleration due to gravity to be $9.8\,\mathrm{m\,s^{-2}}$. Model the box and the ball as particles, the rope as a model string and the pulley as a model pulley.

In your response to this question, underline vectors to distinguish them from scalar quantities. If the magnitude of a vector is unknown, use the vector letter to represent the magnitude. For example, write the magnitude of a vector $\underline{\text{A}}$ as A.

(a) (i) State the four forces that act on the *box*. [*2*]

(ii) Draw a force diagram that represents the forces acting on the box, including the angles that show their directions and labelling the forces clearly. Take the unit vector $\mathbf{i}$ to point parallel to and up the ramp and the unit vector $\mathbf{j}$ to point perpendicular to the ramp in an upward direction. Mark these unit vectors on your force diagram. [*2*]

(iii) Find expressions (in terms of $\mathbf{i}$ and $\mathbf{j}$) for the component forms of the four forces acting on the box. [*2*]

(iv) Write down the vector equation obtained by applying the equilibrium condition to the box. Hence find the value of θ and the magnitude of the tension in the rope, both to 2 significant figures. *[4]*

(b) (i) Draw a force diagram that represents the forces acting on the *ball*, labelling the forces clearly. *[1]*

(ii) Hence find the mass of the ball, in kilograms to two significant figures. *[1]*

Question 25

This question concerns transformations of the plane.

(a) Let f be the horizontal shear with shear factor 3. Let $\mathcal{H}$ be the hyperbola with equation

$$\frac{x^2}{9} - \frac{y^2}{4} = 1.$$

(i) Write down the matrix that represents f. *[1]*

(ii) Find the matrix that represents f^{-1}. *[1]*

(iii) Use your solution to part (a)(ii) to find the equation of the image $f(\mathcal{H})$ of $\mathcal{H}$ under f in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$ *[3]*

(iv) Hence show that $f(\mathcal{H})$ is also a hyperbola. *[1]*

(b) (i) Write down the matrix that represents the rotation g through $\pi/3$ about the origin. *[1]*

(ii) Write down the rule of the translation h that maps the point $(2, 0)$ to the origin. *[1]*

(iii) Use your solutions to parts (b)(i) and (b)(ii) to show that the rotation j through $\pi/3$ about the point $(2, 0)$ is given by

$$j(\mathbf{x}) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix},$$

where $\mathbf{x}$ is the position vector of the point (x, y). *[4]*

Question 26

(a) A rational function g has *derivative*

$$g'(x) = \frac{4(x-2)}{(x+3)^3}.$$

(i) Find the x-coordinate(s) of the stationary point(s) of g. [*1*]

(ii) Construct a table of signs for g'. [*3*]

(iii) State the interval(s) on which g is increasing and the interval(s) on which g is decreasing. [*1*]

(iv) Determine the nature of the stationary point(s). [*1*]

(b) Determine whether the function

$$h(x) = \frac{2}{x(3-x^2)}$$

is even, odd or neither. [*1*]

(c) The information below about a rational function f was obtained by following the steps of the graph-sketching strategy. Sketch the graph of f.

- The domain of f is $(-\infty, 3) \cup (3, \infty)$.
- The graph of f has one x-intercept, namely 0, and one y-intercept, namely 0.
- f is decreasing on the intervals $(-\infty, -3)$ and $(3, \infty)$, and increasing on the interval $(-3, 3)$.
- f has a stationary point, which is a local minimum, at $(-3, -1)$.
- The graph of f has two asymptotes: $x = 3$ and $y = 0$.
- f is neither even nor odd. [*5*]

[END OF QUESTION PAPER]