From Lines to Logic:

Unraveling Euclid's Elements

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"At the age of eleven, I began Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined there was anything so delicious in the world."

—Bertrand Russell

Inasmuch as many things, while appearing to rest on truth and to follow from scientific principles, really tend to lead one astray from the principles and deceive the more superficial minds, he has handed down methods for the discriminative understanding of these things as well, by the use of which methods we shall be able to give beginners in this study practice in the discovery of paralogisms and to avoid being misled. This treatise, by which he puts this machinery in our hands, he entitled (the book) of Pseudaria, enumerating in order their various kinds, exercising our intelligence in each case by theorems of all sorts, setting the true side by side with the false, and combining the refutation of error with practical illustration. This book then is by way of cathartic and exercise, while the Elements contain the irrefragable and complete guide to the actual scientific investigation of the subjects of geometry. Proclus (ca. 335 BC)

Chapter

1

Postulates

Postulate 1.1.0.1

Let the following be postulated: to draw a straight line from any point to any point.

Although it is a direct translation from the Greek, it would be more accurate to say 'from every point to every point', just like in *Postulate 3*, where the first words are 'with every centre and distance'.

The first idea presented here suggests that whenever we have two points, like A and B, we can always draw a straight line connecting them We denoted this as $\overline{AB}$. This construction uses a straightedge, as does another method discussed in a later postulate.

[1]

Figure 1.1: The line

[1] The diagram illustrates a straight line drawn from point A to point B, with both points highlighted in red.

Even though it's not directly stated, there's only one straight line possible between these two points. Euclid assumes this uniqueness as part of the *Postulate*.This implicit assumption of uniqueness underpins much of Euclidean geometry, reinforcing the foundational nature of *Postulate 1*, though it would have been clearer if he would have mentioned it explicitly.However, even Proclus agrees, based on further writings of Euclid, such as *Proposition I.4*, that there must only be one straight line from any point to any point. Proclus elaborates on this in his commentary by referencing Euclid's *Proposition I.4*, where the construction of congruent triangles presupposes the uniqueness of the line connecting two points.

In the latter part of Euclid's Elements, dealing with solid geometry, the two points mentioned in the *Postulate* can be any pair in space. *Proposition XI.1* asserts that if a part of a line lies within a plane, then the entirety of the line does as well. In the sections on plane geometry, it is implied that $\overline{AB}$, connecting points A and B, lies within the plane under discussion. This extension of principles from plane to solid geometry underscores Euclid's systematic approach, demonstrating that fundamental truths apply irrespective of the spatial context.

Postulate 1.1.0.2

To produce a finite straight line continuously in a straight line.

Again we have a translation issue with the *Postulate*, Heath has decided to use the word finite here instead of other words like limited. For example, the term 'finite', when applied to a straight line, might not adequately convey what modern mathematicians of the time termed 'rectilinear segments'—that is, a straight line defined by two extremities. **Heath (Heath)**

[2]

Figure 1.2: Produce a straight line

[2] This diagram depicts a straight line segment CD, with points A and B marked in red on the segment, demonstrating the concept of producing a straight line from point C to D.

Here we have the second chance to use a straightedge, namely, to extend (produce) a given $\overline{AB}$ to C and D. This *Postulate* does not say how far a straight line can be extended. Sometimes it is used so that the extension equals some other straight line. Other times it is extended arbitrarily far. Heath describes this *Postulate* with reference to *Postulate 1* to emphasise this unique property of a straight line; **Heath (Heath)**

"Just as Post. 1 asserting the possibility of drawing a straight line from any one point to another must be held to declare at the same time that the straight line so drawn is unique, so Post. 2 maintaining the possibility of producing a finite straight line (a " rectilinear segment") continuously in a straight line must also be held to assert that the straight line can only be produced in one way at either end, or that the produced part in either direction is unique; in other words, that two straight lines cannot have a common segment."

In Euclid's works, much is left for the reader to infer from seemingly simple statements. These texts suggest not only that we can extend a line to points arbitrarily far from one another, as long as they reside on the same straight line, but also that such a line is inherently unique, and no two straight lines can share a common segment. This distinction is crucial and must be understood from the outset, as early as *Proposition I.* It's implied that two straight lines cannot share identical segments. Proclus[2] highlights this necessity, noting that if it were not so, lines $\overline{AC}$ and $\overline{BC}$ might intersect before reaching point C, sharing a portion of their lengths. This would imply that the resulting triangle formed by these lines with $\overline{AB}$ would not be equilateral, contradicting the fundamental properties of Euclidean geometry .

[2] Proclus Diadochus (412–485 CE), often referred to simply as Proclus, was a Greek Neoplatonist philosopher and mathematician. Notable for his influential commentaries on Plato's and Euclid's works, Proclus sought to harmonize philosophical and mathematical truths, significantly impacting the medieval understanding of Euclidean geometry.

Proclus1 (Proclus1)

As with the first *Postulate*, it is implicitly assumed in the books on plane geometry that when a straight line is extended, it remains in the plane of discussion. The first *Proposition* on solid geometry, *proposition XI.1*, claims that a straight line can't be only partly in a plane. The central step in the proof of that *Proposition* is to show that a straight line cannot be extended in two ways, that is, there is only one continuation of a straight line. The proof is hardly convincing. Rather, this *Postulate* should include a clause to that effect.

Postulate 1.1.0.3

To describe a circle with any center and radius.

In his Elements, Euclid outlines several methods of construction fundamental to geometry, one of which is the drawing of a circle with a compass. This construction, detailed as the third postulate, enables the creation of a circle given two points: one for the center and another on the circumference. The compass used in this method can have an arbitrarily large radius and does not maintain its radius once removed from the page, reflecting Euclid's concept of a collapsing compass.

In order to construct a circle certain characteristics are needed:

1. a point A designated as the center of the circle,
2. another point B located on the circumference of the circle,
3. a plane within which these two points exist.

[3]

Figure 1.3: Produce a straight line

[3] This diagram depicts a straight line segment CD, with points A and B marked in red on the segment, demonstrating the concept of producing a straight line from point C to D.

Euclid's description of a circle in *Definitions I.15* and *Definition I.16* is notable for its simplicity and depth: a circle is a plane figure with all radii equal, extending from the center to the circumference. This definition underpins many of the *Propositions* within Elements, requiring the geometer to employ just a collapsing compass and a straightedge to validate theorems. The notion of the collapsing compass, a tool that resets its radius each time it is lifted, is pivotal in ensuring that geometric constructions rely solely on given points and distances, reinforcing the accuracy and replicability of geometric principles.

This seemingly simple Postulate underpins much of our daily life, often taken for granted. Consider architecture: the need to draw accurate diagrams is imperative to solving force equations that building components must withstand. The use of a compass, a tool perfected by Euclid, demonstrates his rigor and the detailed thought that goes into each construction. Since the compass does not retain its set radius once lifted from the paper, accurately planning the sequence of circles and arcs drawn is crucial.

Thomas Heath provides valuable insight into the language used in the *Postulates*, highlighting a shift from the passive 'a circle can be drawn' in the original text to the more active 'to describe' in Proclus's interpretation, aligning it with the first two postulates. Interestingly, the Greeks did not have a specific term for 'radius'; instead, they described it indirectly as 'a straight line drawn from the center' . This absence of direct terminology reflects a broader flexibility in Euclid's definitions, allowing for circles of any size, from infinitesimally small to indefinitely large. This notion suggests a continuum of sizes, hinting at a boundless but finite space—a concept that would later influence other mathematical fields.

Heath (Heath)

The philosophical implications of Euclidean geometry, particularly the circle, extend beyond mere mathematical interest. Shortly after Euclid, the Neoplatonist movement, spearheaded by philosophers like Plotinus,[4]began to explore geometry's metaphysical aspects. Plotinus viewed the emanation of all existence from a singular source, the One or the Good, through the lens of geometric symmetry, harmony, and proportion—ideas central to the understanding of circles and their inherent properties. Plotinus' concept of emanation, where the One is at the top, followed by the Intellect (Nous), and then the Soul (Psyche), can be metaphorically represented by concentric circles, with the One being the innermost circle. Each level of reality emanates from the One, just as circles might ripple outwards from a point. This emanation is not a diminishment but an overflowing of abundance, where each level participates in the perfection of the One in a way akin to geometric principles of symmetry and harmony.

[4] Plotinus, a philosopher and a wirter on metaphysical systems, in the 3rd century CE, was the founder of Neoplatonism, a school of thought that sought to interpret and synthesize the ideas of Plato.

By examining these elements together, from the technical aspects of Euclidean constructions to their philosophical resonance, we can gain a comprehensive understanding of the profound impact of Euclidean geometry on both mathematics and philosophy. This approach not only elucidates the geometric principles but also connects them to broader metaphysical theories, underscoring the enduring relevance of Euclid's work.

Postulate 1.1.0.4

That all right angles equal one another.

In defining a right angle, it's evident that the two angles formed at the intersection of a perpendicular line, as such:

[5] $\angle ACD \cong \angle BCD$

[5]

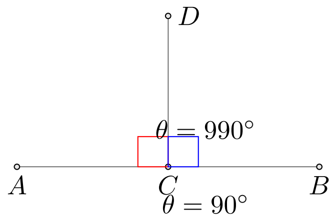

Figure 1.4: Diagram showing two right angles formed by perpendicular lines AC and BC at point C. Both angles ACD and BCD are marked to indicate they are right angles, denoted by $\theta = 90°$.

Heath (Heath)

This Postulate asserts the essential truth that a right angle is a determinate magnitude , this concept is crucial. *Postulate 4* states: "All right angles are equal to one another." Euclid's reliance on such specific definitions ensures that the conclusions drawn from his axioms and postulates are logically sound and universally applicable within the framework of classical geometry. This exactness is what makes the Elements a seminal work in the logical presentation of mathematical proofs and theorems. The specificity in stating that certain angles are equal to two right angles—or any other exact angle measurement—is critical for the development of logical proofs throughout the work. This precision in describing angles is deeply connected to Euclid's foundational postulates, particularly *Postulate 4*, which establishes the equality of all right angles. By doing so, it provides a universal reference for measuring other angles.

This postulate states that an angle formed at the intersection of one perpendicular line, like $\angle ACD$, is equal to an angle formed at the intersection of any other perpendicular line, such as $\angle EGH$.

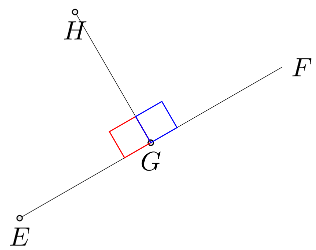

Figure 1.5: in the document we will use the notation; $A + B = 2 \times 90^{\circ}$

This kind of specific statement is a fundamental building block in geometric proofs because it leverages the established truths (axioms and postulates) to make broader conclusions. The only angle measurements considered in the Elements are in terms of right angles. For example, in *Proposition I.17*, it's demonstrated that the sum of two angles is always less than two right angles. Another instance is found in the proof of *Proposition II.9*, where two angles are proven to each be half of a right angle, hence they are congruent. Furthermore, in *Proposition III.16*, this postulate is invoked to argue that the sum of two angles cannot be less than two right angles while also being equal to two other right angles.

This approach underlines the rigor of Euclidean geometry, where every assertion, no matter how simple it seems, is tightly bound to the system's logical structure. It ensures that every geometric statement can be verified based on universally accepted truths, maintaining the integrity and internal consistency of mathematical arguments.

In a broader sense, a determinate magnitude in Euclidean geometry can refer to any precisely defined length, area, volume, or angular measure. Euclid's reliance on such specific definitions ensures that the conclusions drawn from his *Axioms* and *Postulates* are logically sound and universally applicable within the framework of classical geometry. This exactness is what makes the Elements a seminal work in the logical presentation of mathematical proofs and theorems. This is a demonstration

[3] For a deeper historical insight into how these principles were applied in ancient Greek mathematics and their influence on later scholars, see references such as Heath's translation of the *Elements*, which provides extensive commentary and analysis.

of Euclid's use of right angles in his geometric proofs.[3]

Postulate 1.1.0.5

That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

This postulate, less self-evident compared to Euclid's other four, has been extensively scrutinized and many have attempted to derive it from more "obvious" postulates. While ancient Greek mathematicians like Thales and Pythagoras laid foundations that contributed to Euclid's axiomatic approach, explicit efforts to address Euclid's fifth postulate are scarcely documented. Post-Euclid, the obsession to prove this fifth postulate using simpler axioms failed repeatedly, often falling into the trap of *petitio principii*[6]—assuming the truth of what it tried to establish. Ptolemy's attempts to prove Proposition I.29 without using Postulate 5 inadvertently resulted in deducing the postulate from his proof.

[6] Petitio principii, or "begging the question," is a logical fallacy where the conclusion is assumed in the premises, inadvertently using what needs to be proven as the proof itself.

The premise of this postulate is illustrated in the following diagram; if:

$$\angle ABE + \angle BED < 2 \times 90^\circ$$

then lines $\overline{AC}$ and $\overline{DF}$, when extended towards points A and D respectively, will intersect.

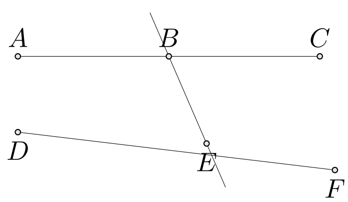

Figure 1.6: Illustration of non-parallel lines meeting

Commonly known as the "parallel postulate," it is pivotal in proving properties of parallel lines, extensively explored up to *Proposition I.31*.

If instead:

$$\angle ABE + \angle BED = 2 \times 90^\circ$$

lines are parallel, as shown below.

[2]

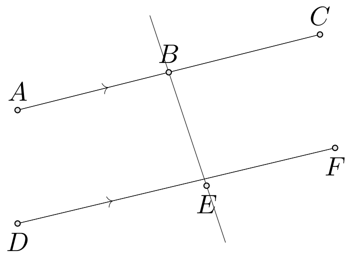

Figure 1.7: Illustration of parallel lines

[2] We use $>$ to indicate the lines are parallel

In the early 19th century, mathematicians such as Bolyai, Lobachevsky, and Gauss explored non-Euclidean geometries, including hyperbolic and elliptic, which, though diverging from Euclidean principles, proved internally consistent and practically applicable.

The parallel postulate is essential for deriving the principles of Euclidean geometry, and understanding non-Euclidean geometries provides valuable insight. Although Euclid does not explicitly employ this postulate until *Proposition I.29*, the subsequent propositions heavily rely on it. Additional commentary on this postulate is provided in *Propositions I.29* and *I.30*.