From Lines to Logic:

Unraveling Euclid’s Elements

Paul Allen

"At the age of eleven, I began Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined there was anything so delicious in the world."

—Bertrand Russell

Inasmuch as many things, while appearing to rest on truth and to follow from scientific principles, really tend to lead one astray from the principles and deceive the more superficial minds, he has handed down methods for the discriminative understanding of these things as well, by the use of which methods we shall be able to give beginners in this study practice in the discovery of paralogisms and to avoid being misled. This treatise, by which he puts this machinery in our hands, he entitled (the book) of Pseudaria, enumerating in order their various kinds, exercising our intelligence in each case by theorems of all sorts, setting the true side by side with the false, and combining the refutation of error with practical illustration. This book then is by way of cathartic and exercise, while the Elements contain the irrefragable and complete guide to the actual scientific investigation of the subjects of geometry. Proclus (ca. 335 BC)

Chapter

1

Proposition

Construction 1.1.0.1

To construct an equilateral triangle on a given finite straight line $\overline{AB}$.

Construction 1.1.0.2

Let $\overline{AB}$ be the given finite straight line. I.1

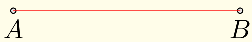
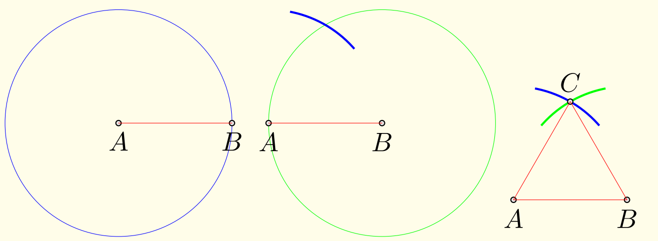


Figure 1.1: Equilateral triangle construction

1. It is required to construct an equilateral triangle on the straight line $\overline{AB}$.
2. Describe the $\mathscr{C}(A; AB)$.
3. Then, describe the $\mathscr{C}(A; BA)$.
4. Join the straight lines $\overline{CA}$ and $\overline{CB}$ from the point C at which the circles cut one another to the points A and B. I.Post.3, I.Post.1



(a) Step 1 (b) Step 2 (c) Step 3

Figure 1.2: Equilateral triangle construction

Proof.

- Since the point A is the center of the $\mathscr{C}(A; AB)$, $\therefore overlineAC$ equals $\overline{AB}$. Again, since the point B is the center of the $\mathscr{C}(A; BA)$ $\therefore \overline{BC}$ equals $\overline{BA}$. I.Def.15

- Things which equal the same thing also equal one another $\therefore \overline{AC}$ also equals$\overline{BC}$. C.N.1

Conclusion:

- $\therefore$, the three straight lines $\overline{AC}$, $\overline{AB}$, and $\overline{BC}$ equal one another.
- $\therefore$, $\widehat{ABC}$ is equilateral, and it has been constructed on the given finite straight line $\overline{AB}$. I.Def.20

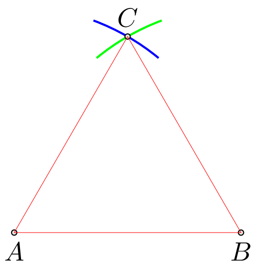


Figure 1.3: Equilateral triangle construction

The construction has been successfully completed. Q.E.F

- The construction relies on the principles of circle description and intersection, line segment joining, and the properties of circles with given centers. I.Post.1, I.Post.3, I.Def.15
- The Equilateral Triangle is characterized by the equality of its three sides ($\overline{AC}$, $\overline{AB}$, $\overline{BC}$). I.Def.20

Construction 1.1.0.3

To place a straight line equal to a given straight line with one end at a given point A.

Let $\overline{BC}$ be the given straight line.

Con

1. Given a point A and the given straight finite line$\overline{BC}$.

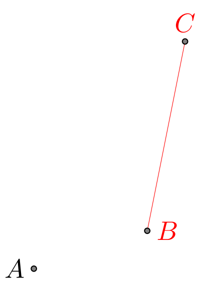


Figure 1.4: step1

2. Join the straight line $\overline{AB}$ from point A to point B, and construct the equilateral $\widehat{DAB}$ on it. Post.1, I.1

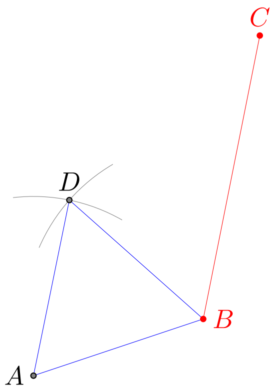
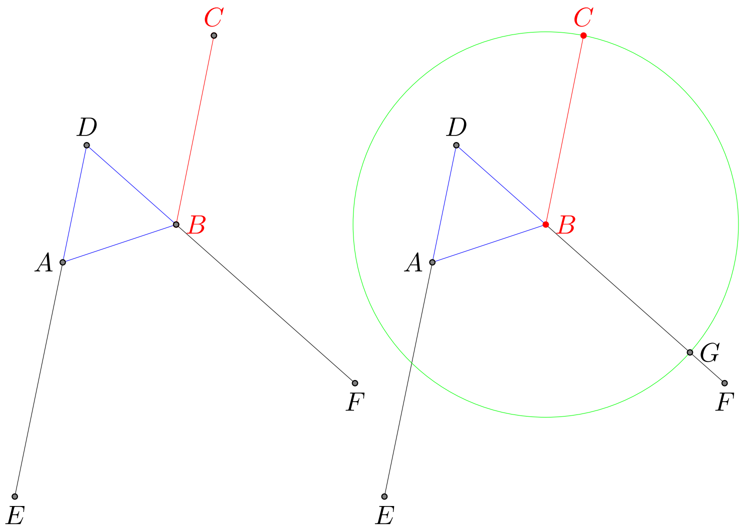


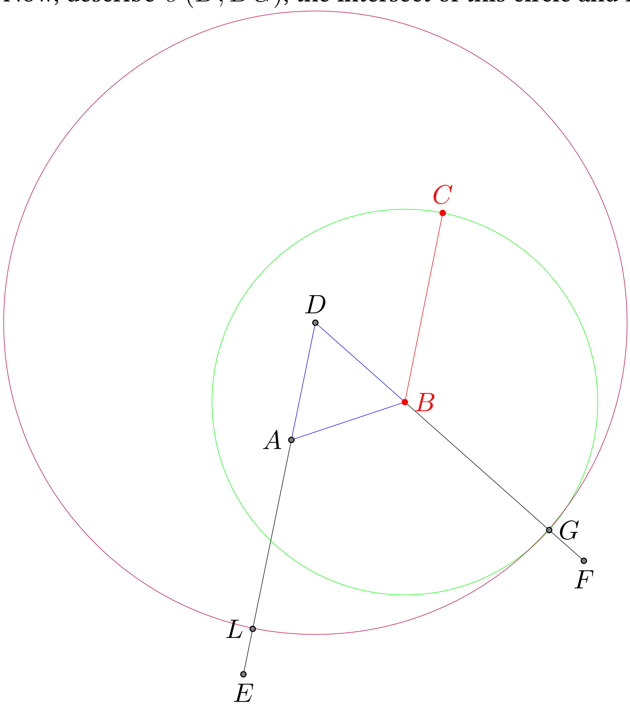
Figure 1.5: step 2

3. Produce the straight lines $\overline{AE}$ and $\overline{BF}$ in a straight line with $\overline{DA}$ and $\overline{DB}$.
4. Describe $\mathscr{C}(B; BC)$, getting point G, the intersect of this circle with $\overline{BF}$.



(a) Produce the line (b) draw a circle

5. Now, describe $\mathscr{C}(D; DG)$, the intersect of this circle and $\overline{AE}$ gives point L.



(c) intersection

Figure 1.6: something

Post.2,Post.3

Proof.

- Since the point B is the center of the circle with radius BC, $\therefore$ $\overline{BC}$ equals $\overline{BG}$. Again, since the point D is the center of the circle with radius DG, $\therefore$ $\overline{DL}$ equals $\overline{DG}$. I.Def.15

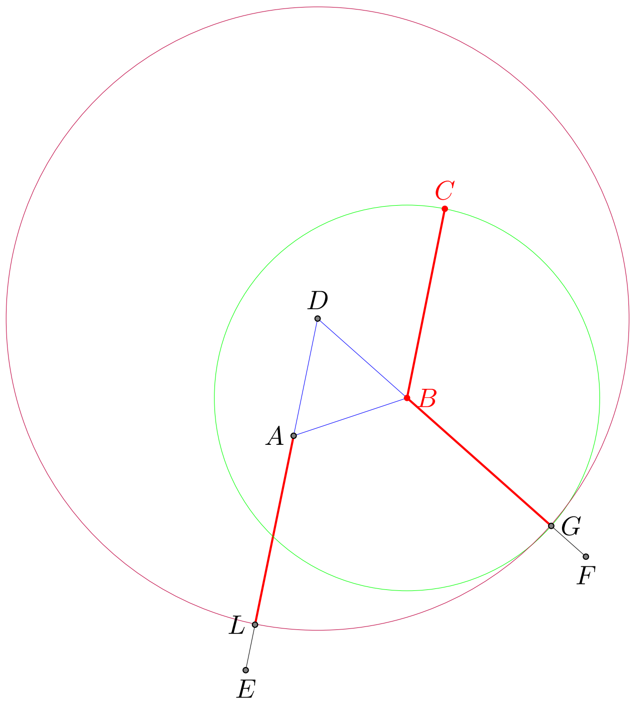


Figure 1.7: end

- In these, $\overline{DA}$ equals $\overline{DB}$, $\therefore$ the remainder AL equals the remainder BG. C.N.3

- But $\overline{BC}$ was also proved equal to $\overline{BG}$, $\therefore$ each of the straight lines $\overline{AL}$ and $\overline{BC}$ equals $\overline{BG}$. And things which equal the same thing also equal one another $\therefore$ $\overline{AL}$ also equals $\overline{BC}$.

Conclusion:

- $\therefore$ the straight line $\overline{AL}$ equal to the given straight line $\overline{BC}$ has been placed with one end at the given point A.

- The construction has been successfully completed.

Q.E.F

Construction 1.1.0.4

To cut off from the greater of two given unequal straight lines $\overline{AB}$ and $\overline{CD}$ (with $\overline{AB}$ being the greater) a straight line equal to the less $\overline{CD}$.

Given: $\overline{AB}$ and $\overline{CD}$, where $\overline{AB}$ is greater.

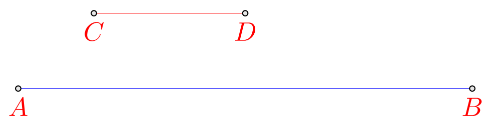


Figure 1.8: Cut off a segment from a larger segment

Construction:

1. Place $\overline{CD}$ at point A and describe $\mathscr{C}(A; AD)$. I.2, I.Post.3

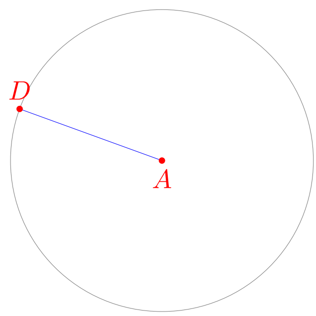


Figure 1.9: Cut off a segment from a larger segment: step 1

2. draw the segment $\overline{AB}$ and mark the intersect with $\mathscr{C}(A; AD)$, E

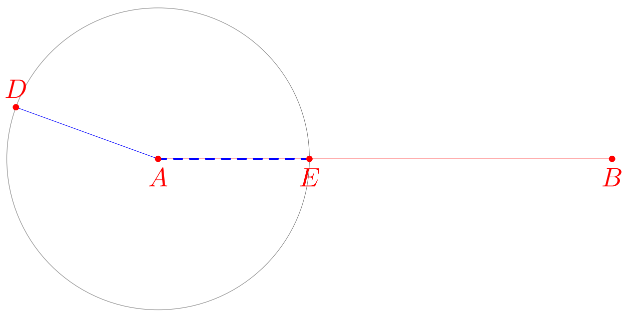


Figure 1.10: Cut off a segment from a larger segment: step 2

Proof.

- Since the point A is the center of $\mathscr{C}(A; AD)$, $\therefore$

$$\overline{AE} = \overline{AD}$$

I.Def.15

and

$$\overline{CD} = \overline{AD}$$

$\therefore$ each of the straight lines $\overline{AE}$ and $\overline{CD}$ equals $\overline{AD}$, so that

$$\overline{AE} = \overline{CD}$$

C.N.1

Conclusion:

$\therefore$ given the two straight lines $\overline{AB}$ and $\overline{CD}$, $\overline{AE}$ has been cut off from $\overline{AB}$ (the greater) equal to $\overline{CD}$ (the less).

The construction has been successfully completed.

Q.E.F

Theorem 1.1.0.1

If two triangles have two sides equal to two sides respectively and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

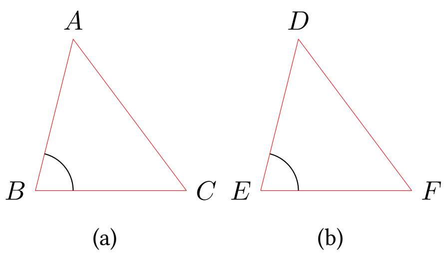


Figure 1.11: Side angle side theorem

Proof. Given two $\widehat{ABC}$ and $\widehat{DEF}$, where side $\overline{AB}$ is equal to side $\overline{DE}$, side $\overline{AC}$ is equal to side $\overline{DF}$, and $\angle BAC$ is equal to $\angle EDF$.

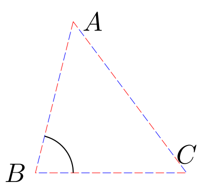


Figure 1.12: SAS conclusion

Conclusion: The theorem is proved.

Q.E.F

Theorem 1.1.0.2

In isosceles triangles, the angles at the base equal one another, and if the equal straight lines are produced further, then the angles under the base equal one another.

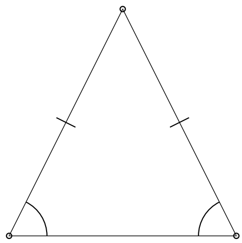


Figure 1.13: Isosceles Triangle Theorem

Let $\widehat{ABC}$ be an isosceles triangle with side $\overline{AB}$ equal to side $\overline{AC}$, and let the straight lines $\overline{BD}$ and $\overline{CE}$ be produced further in a straight line with $\overline{AB}$ and $\overline{AC}$. I.Def.20, I.Post.2

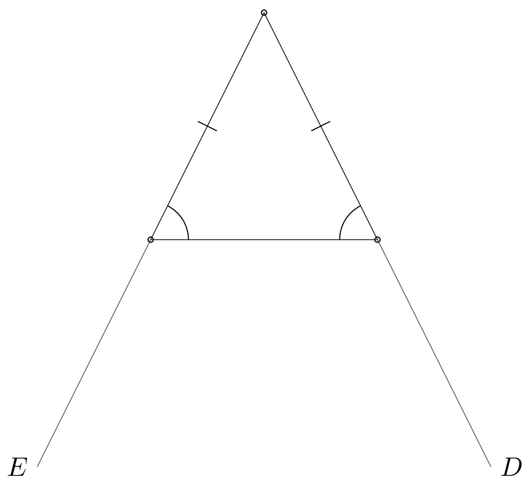


Figure 1.14: Isosceles Triangle Theorem; step 1

. Definitions and Postulates:

- An isosceles triangle is a triangle with two sides of equal length. I.Def.20
- A straight line can be produced indefinitely. I.Post.2
- Given two points, a straight line can be drawn between them. I.Post.1

Lemma 1.1. *If two sides and the included angle of one triangle are equal to the corresponding sides and angle of another triangle, then the two triangles are congruent.* *I.3*

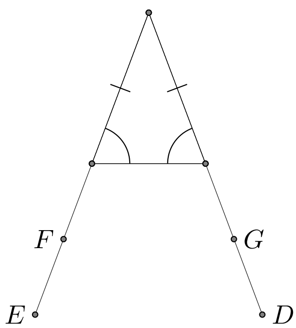


Figure 1.15: Isosceles Triangle Theorem; step 2

Construction 1.1.0.5

1. Take an arbitrary point F on $\overline{BD}$.
2. Cut off $\overline{AG}$ from $\overline{AE}$, where $\overline{AG} > \overline{AF}$.
3. Join FC and GB. I.3, I.Post.1

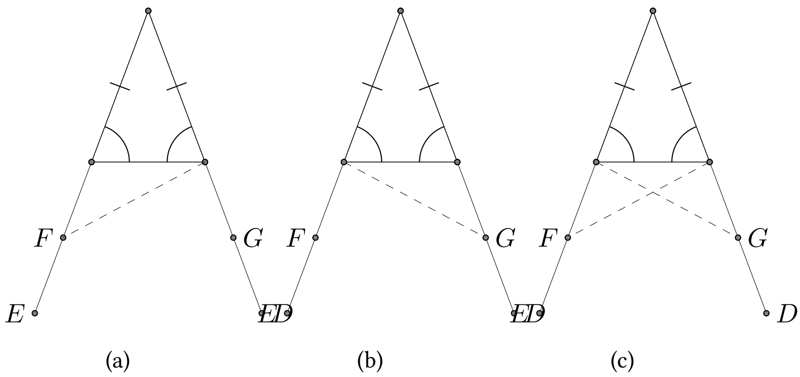


Figure 1.16: Isosceles Triangle Theorem: step 3

Proof.

- Since $\overline{AF} = \overline{AG}$ and $\overline{AB} = \overline{AC}$, $\widehat{FAG}$ and $\widehat{CAB}$ are congruent by SSS (side-side-side) criterion.

$$\Rightarrow \angle FAG = \angle CAB$$

- $FC = GB$ (Base angles in congruent triangles are equal). I.4

$$\Rightarrow \widehat{A}FC \cong \widehat{A}GB$$

$$\Rightarrow \angle ACF = \angle ABG, \quad \angle AFC = \angle AGB$$

- $BF = CG$. C.N.3

$$\Rightarrow BF + FC = CG + GB$$

$$\Rightarrow \widehat{B}FC \cong \widehat{C}GB$$

$$\Rightarrow \angle BFC = \angle CGB, \quad \angle BCF = \angle CBG$$

- Combining the results:

$$\Rightarrow \angle ACF = \angle ABG$$

$$\Rightarrow \angle AFC = \angle AGB$$

$$\Rightarrow \angle BFC = \angle CGB$$

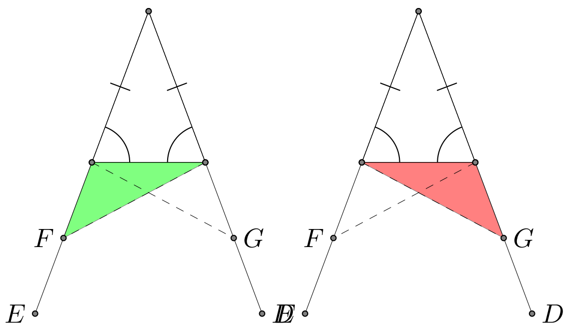
$$\Rightarrow \angle BCF = \angle CBG$$

- Since $\angle ACF = \angle ABG$ and $\angle CBG = \angle BCF$, BC is parallel to FG. I.Post.1

- Since $\angle ABC$ and $\angle ACB$ are corresponding angles when BC is parallel to

$FG, \angle ABC = \angle ACB.$

- Also, $\angle FBC = \angle GCB$ as proved earlier.



(a) (b)

Figure 1.17: Isosceles Triangle Theorem: step 4

$\therefore$ in isosceles $\widehat{ABC}$, the angles at the base ($\angle ABC$ and $\angle ACB$) are equal, and if the equal sides are produced further, the angles under the base ($\angle FBC$ and $\angle GCB$) are equal.

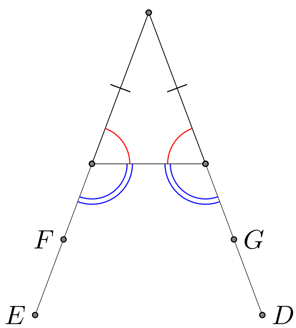


Figure 1.18: Isosceles Triangle Theorem: conclusion

Q.E.F

Theorem 1.1.0.3

If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

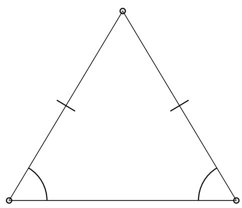


Figure 1.19

Given:

- If $\overline{AB}$ does not equal $\overline{AC}$, then one of them is greater. C.N
- If $\overline{AB}$ is greater, cut off $\overline{DB}$ from $\overline{AB}$ (the greater) equal to $\overline{AC}$ (the less), and join $\overline{DC}$. I.3, I.Post.1

Construction 1.1.0.6

1. **Assumption:** Suppose $\overline{AB}$ is greater than $\overline{AC}$.
2. Cut off $\overline{DB}$ from $\overline{AB}$ such that $\overline{DB} = \overline{AC}$, and join $\overline{DC}$.

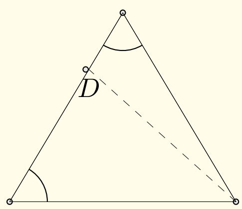


Figure 1.20

Proof. **Proof by Contradiction:**

- Since $\overline{DB} = \overline{AC}$ and $\overline{BC}$ is common $\therefore \widehat{DBC} \cong \widehat{ACB}$ by SSS (side-side-side) criterion. [$\Rightarrow \overline{DC} = \overline{AB}$ and $\angle DBC = \angle ACB$]

- This implies the base $\overline{DC}$ equals the base $\overline{AB}$, and the $\widehat{DBC}$ equals $\widehat{ACB}$, which is absurd since the less ($\widehat{DBC}$) equals the greater ($\widehat{ACB}$).

$\therefore$ the assumption that $\overline{AB}$ is greater than $\overline{AC}$ is false.

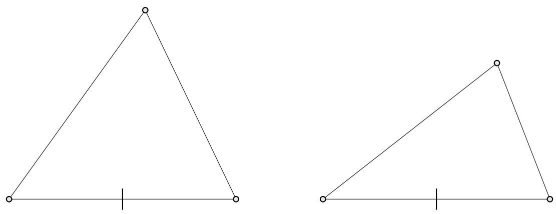
- Hence, $\overline{AB}$ is not unequal to $\overline{AC}$, it, $\therefore$ equals it.

Conclusion:

$\therefore$ if in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another. Q.E.F

Theorem 1.1.0.4

Given two straight lines constructed from the ends of a straight line and meeting at a point, it is not possible to construct two other straight lines on the same side of the original line, meeting in another point, and equal to the first two, namely each equal to that from the same end.



(a) (b)

Figure 1.21

- On the straight line $\overline{AB}$ draw two lines from A and B respectivly to a point C.
- Attempt to construct two other straight lines $\overline{ED}$ and $\overline{FD}$ on the straight line $\overline{EF}$, were $\overline{EF} = \overline{AB}$, meeting in another point D, and make them equal to $\overline{AC}$ and $\overline{BC}$, respectively.

Construction 1.1.0.7

1. **Assumption:** Suppose $\overline{AB}$ is greater than $\overline{AC}$.
2. Assume the contrary, that the construction is possible.
3. Lay $\overline{BC}$ over $\overline{EF}$.
4. Join $\overline{AD}$.

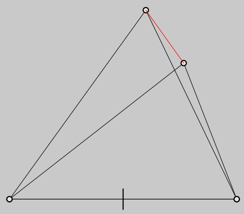


Figure 1.22

Proof. **Proof by Contradiction:**

- A straight line can be drawn between any two points. $\therefore$ $\overline{AD}$ can be drawn joining points A and D. I.Post.1

- If equals are added to equals, the wholes are equal. If equals are subtracted from equals, the remainders are equal. C.N.5

- Since $\overline{BA}$ equals $\overline{BD}$, and $\overline{BC}$ is common, $\angle ACD$ equals $\angle ADC$. C.N.5

- **Angle Comparison:** $\angle ADC$ is greater than $\angle DCB$.

- **Contradiction:** $\angle DCB$ is equal to $\angle CDB$ (by the given construction) but is also shown to be much greater than $\angle DCB$, leading to a contradiction.

  Conclusion: The assumption that the construction is possible leads to a contradiction.

$\therefore$ the original statement is proven—given two straight lines constructed from the ends of a straight line and meeting at a point, it is not possible to construct two other straight lines on the same side of the original line, meeting in another point, and equal to the first two.

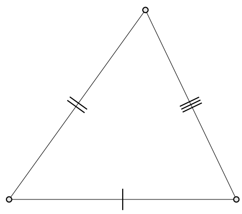
Q.E.F

Theorem 1.1.0.5

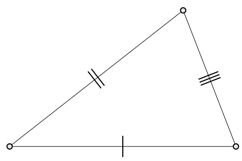
If two triangles ABC and DEF have the two sides AB and AC equal to the two sides DE and DF respectively, namely AB = DE and AC = DF, and the base BC equal to the base EF, then the angle BAC equals the angle EDF.

Given: $\widehat{ABC}$ and $\widehat{DEF}$ with $\overline{AB} = \overline{DE}$, $\overline{AC} = \overline{DF}$, and $\overline{BC} = \overline{EF}$.

To Prove: $\angle BAC = \angle EDF$.



(a)



(b)

Figure 1.23

Proof. **Proof by Construction:**

1. Apply $\widehat{ABC}$ to $\widehat{DEF}$ by placing point B on point E and aligning $\overline{BC}$ with $\overline{EF}$.

2. Since $\overline{BC}$ coincides with $\overline{EF}$, point C also coincides with point F.

3. If $\overline{BA}$ and $\overline{AC}$ do not coincide with $\overline{ED}$ and $\overline{DF}$, then, two other straight lines would have been constructed on the same side of $\overline{AB}$ and meeting in a point, equal to $\overline{BA}$ and $\overline{AC}$, respectively. This is a contradiction. I.7, C.N.4

$\therefore$ $\overline{BA}$ and $\overline{AC}$ coincide with $\overline{ED}$ and $\overline{DF}$.

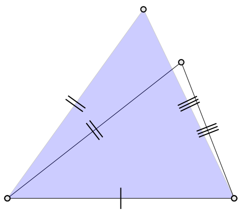


Figure 1.24

Conclusion:

- The base $\overline{BC}$ coinciding with $\overline{EF}$ implies that sides $\overline{BA}$ and $\overline{AC}$ coincide with $\overline{ED}$ and $\overline{DF}$ respectively.
- Thus, $\angle BAC$ coincides with $\angle EDF$, and they are equal.

$\therefore$ if two triangles have two sides equal to two sides respectively, and also have the base equal to the base, then the angles contained by the equal sides are also equal.

Q.E.F

Construction 1.1.0.8

It is required to bisect $\angle BAC$.

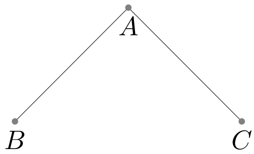


Figure 1.25

Proof by Construction:

1. Take an arbitrary point D on $\overline{AB}$.

2. Cut off $\overline{AE}$ from $\overline{AC}$ equal to $\overline{AD}$, and join $\overline{DE}$. I.3

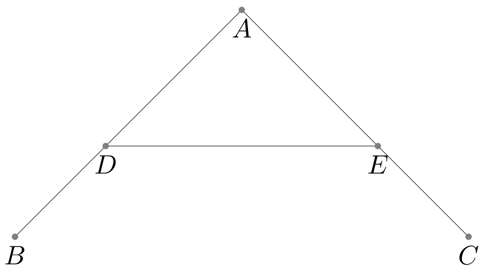


Figure 1.26

3. Construct the equilateral $\triangle DEF$ on DE. I.1

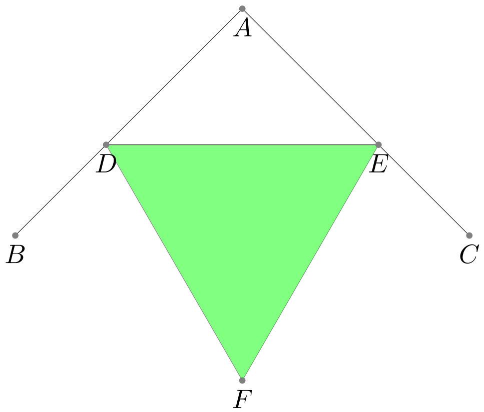


Figure 1.27

4. Join AF.

5. Then, the $\angle BAC$ is bisected by the straight line $\overline{AF}$.

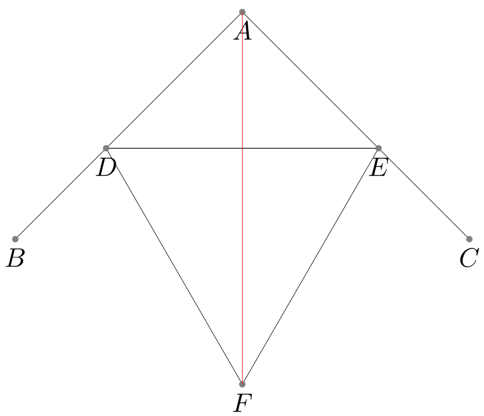


Figure 1.28

Proof.

Lemma 1.2. *since $AD = AE$ and AF is common, the two sides AD and AF are equal to the two sides EA and AF respectively.* *C.N.1,I.Post.1*

Lemma 1.3. *Since $\overline{DF} = \overline{EF}$, the angles $\angle DAF$ and $\angle EAF$ are equal.* *Def.20,I.8*

Conclusion:

$\therefore$ the given rectilinear angle $\angle BAC$ is bisected by the straight line AF. I.8

Q.E.F

Construction 1.1.0.9

Let$\overline{AB}$ be the given finite straight line. It is required to bisect the finite straight line $\overline{AB}$.

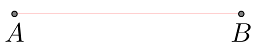


Figure 1.29

Construction:

1. Construct an equilateral $\widehat{ABC}$ on the straight line $\overline{AB}$. I.1

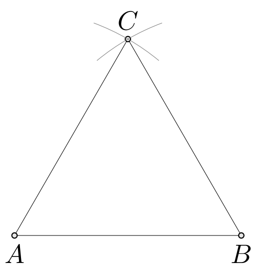


Figure 1.30

2. Bisect the $\angle ACB$ by the straight line $\overline{CE}$. I.9

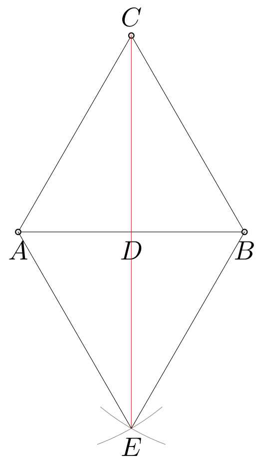


Figure 1.31

Proof.

- Let

$$\overline{CA} = \overline{CB}$$

- and $\overline{CD}$ is common to both $\widehat{ACD}$ and $\widehat{BCD}$. I.Def.20

- By Side-Angle-Side (SAS) congruence:

$$\widehat{ACD} \cong \widehat{BCD}$$

- $\therefore$ the corresponding parts are equal:

$$\overline{AD} = \overline{BD}$$

I.4

- Thus, the straight line $\overline{AB}$ is bisected at the point D.

- Hence, the construction is successful, and the proof is complete.

Q.E.F

Construction 1.1.0.10

To draw a straight line at right angles to a given straight line from a given point on it.

Let $\overline{AB}$ be the given straight line, and C the given point on it.

Figure 1.32

1. Take an arbitrary point D on $\overline{AC}$.
2. Make $\overline{CE}$ equal to $\overline{CD}$.

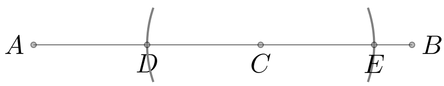


Figure 1.33

3. Construct the equilateral $\widehat{FDE}$ on $\overline{DE}$.

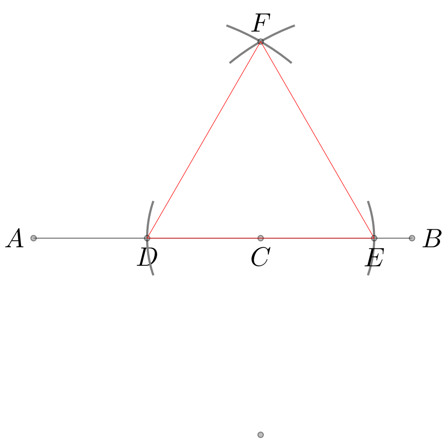


Figure 1.34

4. Join $\overline{CF}$.

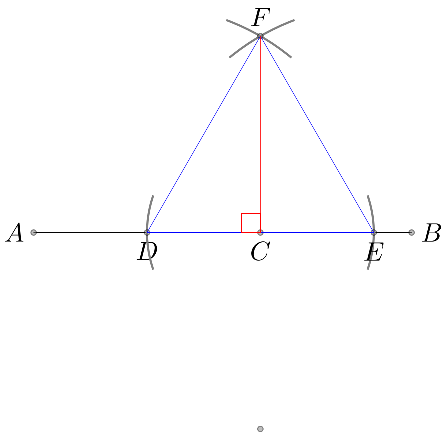


Figure 1.35

Lemma 1.4. *The straight line $\overline{CF}$ has been drawn at right angles to the given straight line $\overline{AB}$ from the given point C on it.*

Proof.

- Since

$$\overline{CD} = \overline{CE}$$

- and $\overline{CF}$ is common, $\therefore$ the two sides $\overline{CD}$ and $\overline{CF}$ equal the two sides $\overline{CE}$ and $\overline{CF}$ respectively,

- and the base

$$\overline{DF} = \overline{EF}$$

$\therefore$

$$\angle DCF = \angle ECF$$

- and they are adjacent angles. I.Def.20, I.8

- When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right. $\therefore$ each of $\angle DCF$ and $\angle FCE$ is right. I.Def.10

$\therefore$ the straight line $\overline{CF}$ has been drawn at right angles to the given straight line $\overline{AB}$ from the given point C on it.

Q.E.F

Construction 1.1.0.11

To draw a straight line perpendicular to a given infinite straight line from a given point not on it.

Let $\overline{AB}$ be the given infinite straight line, and C the given point not on it. It is required to draw a straight line perpendicular to the given infinite straight line $\overline{AB}$ from the given point C.

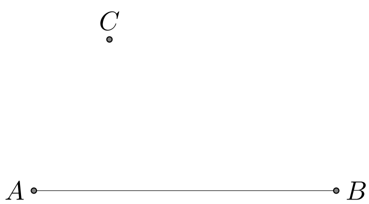


Figure 1.36

Construction 1.1.0.12

1. Take an arbitrary point D on the other side of the straight line $\overline{AB}$.

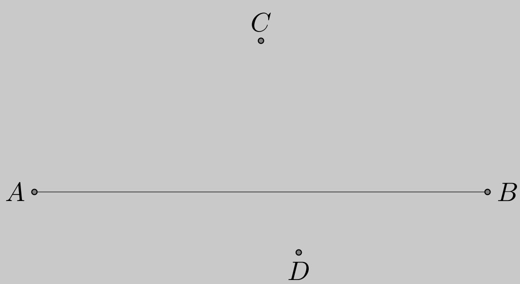


Figure 1.37

2. Describe $\mathscr{C}(C; CD)$.

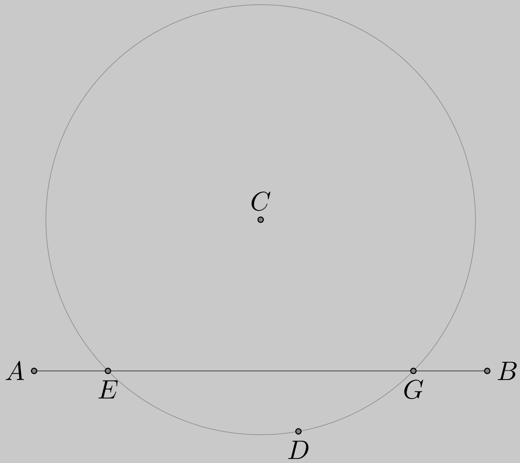


Figure 1.38

3. Bisect the straight line $\overline{EG}$ at H, and join the straight lines $\overline{CG}$, $\overline{CH}$, and $\overline{CE}$. I.10

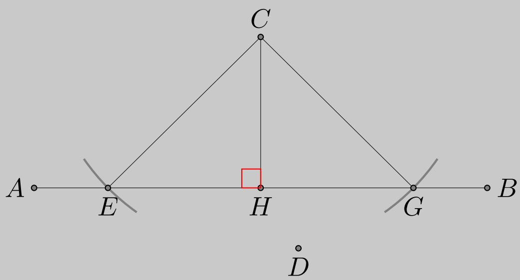


Figure 1.39

Proof.

- Since

$$\overline{GH} = \overline{HE}$$

- and $\overline{HC}$ is common, $\therefore$ the two sides $\overline{GH}$ and $\overline{HC}$ equal the two sides $\overline{EH}$

and $\overline{HC}$ respectively

- and the bases

$$\overline{CG} = \overline{CE}$$

$\therefore$

$$\angle CHG = \angle EHC$$

and they are adjacent angles.

- When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands. I.Def.10

$\therefore$ $\overline{CH}$ has been drawn perpendicular to the given infinite straight line $\overline{AB}$ from the given point C not on it.

Q.E.F

Theorem 1.1.0.6

If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

Let any straight line $\overline{AB}$ standing on the straight line $\overline{CD}$ make $\angle CBA$ and $\angle ABD$. I say that either $\angle CBA$ and $\angle ABD$ are two right angles or their sum equals two right angles.

Proof.

- If,

$$\angle CBA = \angle ABD$$

then they are two right angles. I.Def.10

- If

$$\angle CBA \neq \angle ABD$$

draw $\overline{BE}$ from B perpendicular to $\overline{CD}$. I.11

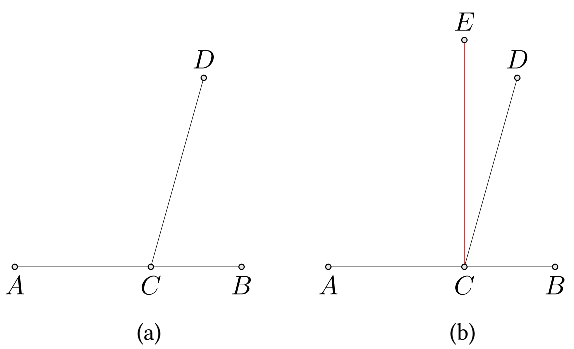


Figure 1.40

- Now

$$\angle CBE = \angle CBA + \angle ABE$$

and

$$\angle EBD = \angle DBE + \angle EBA$$

I.Def.10

$\therefore$

$$\angle CBE + \angle EBD = \angle CBA + \angle ABE + \angle DBE + \angle EBA$$

C.N.2

- Also,

$$\angle DBA = \angle DBE + \angle EBA$$

and

$$\angle ABC = \angle ABE + \angle CBA$$

I.Def.10

$\therefore$

$$\angle DBA + \angle ABC = \angle DBE + \angle EBA + \angle ABE + \angle CBA$$

C.N.2

- Since

$$\angle CBE + \angle EBD = \angle DBA + \angle ABC$$

and $\angle CBE$ and $\angle EBD$ are two right angles,

- then

$$\angle DBA + \angle ABC = 2 \times 90^\circ$$

C.N.1

- Thus, if a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

Q.E.F

Theorem 1.1.0.7

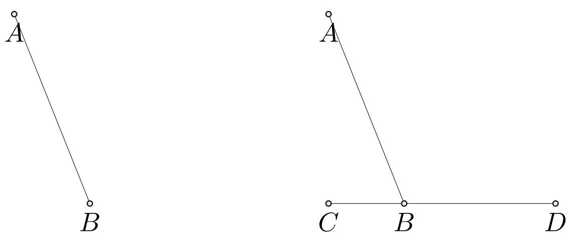
If, with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

Given: If with any straight line $\overline{AB}$, and at a point B on it, two straight lines $\overline{BC}$ and $\overline{BD}$ not lying on the same side make;

$$\angle ABC + \angle ABD = 2 \times 90^\circ$$

To prove: The two straight lines $\overline{CB}$ and $\overline{BD}$ are in a straight line with one another.

With any straight line $\overline{AB}$, and at the point B on it, let the two straight lines $\overline{BC}$ and $\overline{BD}$ not lying on the same side make the sum of the adjacent $\angle ABC$ and $\angle ABD$ equal to two right angles. I say that $\overline{BD}$ is in a straight line with $\overline{CB}$.



(a) (b)

Figure 1.41

Lemma 1.5. *If a straight line (in this case, $\overline{BD}$) falling on two straight lines (in this case, $\overline{AB}$ and $\overline{BE}$) makes the interior angles on the same side less than two right angles, then the two straight lines (AB and BE) produced indefinitely meet on that side on which are the angles less than the two right angles.* *I.Post.2*

Lemma 1.6. *If a straight line stands on a straight line, then it makes either two right angles or angles less than two right angles.* *I.Post.4*

Assume for contradiction: $\overline{BD}$ is not in a straight line with $\overline{CB}$.

By Lemma 1 Produce $\overline{BE}$ in a straight line with $\overline{CB}$. I.Post.2

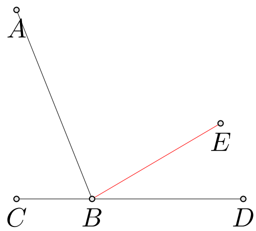


Figure 1.42

Proof.

By Lemma 2 Since, AB stands on BE,

$$\angle ABC + \angle ABE = 2 \times 90^\circ$$

I.Post.4,

Given:

$$\angle ABC + \angle ABD = 2 \times 90^\circ$$

$\therefore$

$$\angle CBA + \angle ABE = \angle CBA + \angle ABD$$

By Lemma 3 Subtract angle CBA from each side:

$$\angle ABE = \angle ABD$$

C.N.3

Contradiction: The less ($\angle ABE$) equals the greater ($\angle ABD$), which is impossible.

$\therefore$

$$\overline{BE} \neq \overline{CB}$$

Similarly, it can be proven that no other straight line except $\overline{BD}$ is in line with $\overline{CB}$.

Thus, $\overline{CB}$ is in a straight line with $\overline{BD}$.

Conclusion If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

Q.E.F

Theorem 1.1.0.8

If two straight lines cut one another, then they make the vertical angles equal to one another.

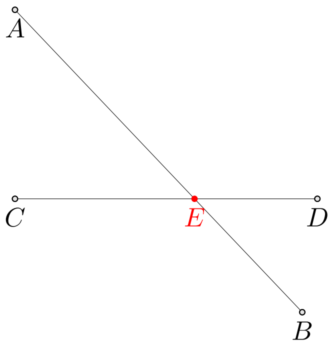


Figure 1.43

Let the straight lines $\overline{AB}$ and $\overline{CD}$ cut one another at the point E.

Proof.

Lemma 1.7.

$$\angle CEA = \angle DEB \text{ and } \angle BEC = \angle AED$$

.Proof: By the angle sum property

$$\angle CEA + \angle AED = 2 \times 90^\circ$$

I.13

Similarly,

$$\angle AED + \angle DEB = 2 \times 90^\circ$$

Lemma 1.8.

$$\angle CEA + \angle AED = \angle AED + \angle DEB$$

- By previous claims,

$$\angle CEA + \angle AED = \angle AED + \angle DEB$$

- Subtracting $\angle AED$ from both sides,

$$\angle CEA = \angle DEB$$

Extension of lemma 0.1.3 Similarly, $\angle BEC = \angle AED$.

$\therefore$ if two straight lines cut one another, then they make the vertical angles equal to one another.

Q.E.F

Theorem 1.1.0.9

In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

Theorem 1.1. *Let $\widehat{ABC}$ be a triangle, and let one side $\overline{BC}$ be produced to D. We want to show that the exterior $\angle ACD$ is greater than either of the interior and opposite $\angle CBA$ and $\angle BAC$.*

Construction

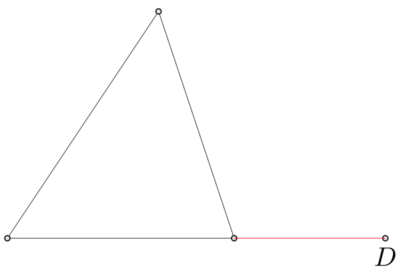


Figure 1.44

1. Produce $\overline{AC}$ to G

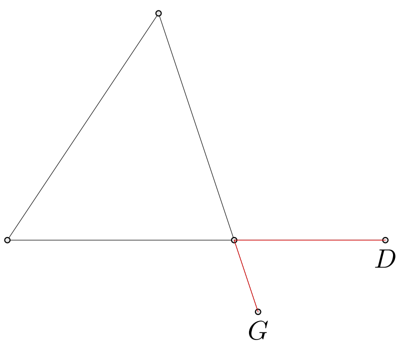


Figure 1.45

2. Bisect $\overline{AC}$, and produce $\overline{BE}$ to F, were $\overline{BE} = \overline{EF}$.

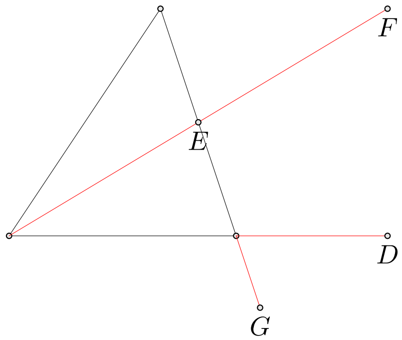


Figure 1.46

3. Draw $\overline{FC}$

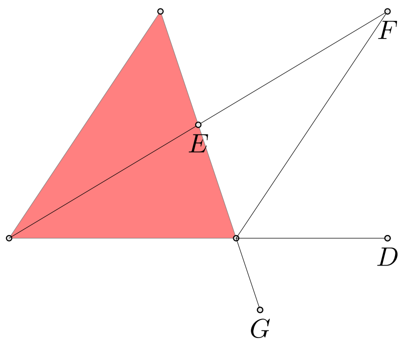


Figure 1.47

Proof. • Since

$$\overline{AE} = \overline{EC} \text{ and } \overline{BE} = \overline{EF}$$

$$\widehat{ABE} \cong \widehat{CFE}$$

I.15,I.4

$\therefore$

$$\overline{AB} = \overline{FC} \text{ and } \angle BAE = \angle ECF$$

. C.N.5

- But

$$\angle ECD > \angle ECF$$

$\therefore$,

$$\angle ACD > \angle BAE$$

I.15

Similarly: If $\overline{BC}$ is bisected, then $\angle BCG$, that is, $\angle ACD$, can also be proved to be greater than $\angle ABC$.

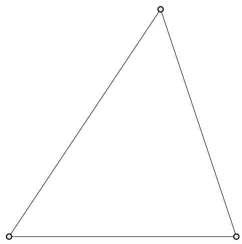
$\therefore$ in any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

Q.E.F

Theorem 1.1.0.10

In any triangle, the sum of any two angles is less than two right angles.

Let ABC be a triangle.



Produce $\overline{BC}$ to D. I.Post.2

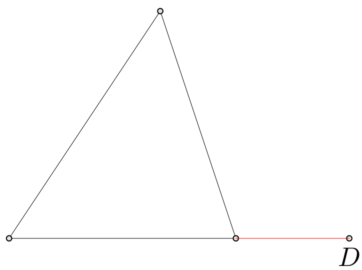


Figure 1.48

Proof.

- Since the $\angle ACD$ is an exterior angle of the $\widehat{ABC}$, $\therefore$ it is greater than the interior and opposite $\angle ABC$. Add $\angle ACB$ to each.

$$\angle ACD + \angle ACB > \angle ABC + \angle BCA$$

I.16, C.N.2

- But

$$\angle ACD + \angle ACB = 2 \times 90^\circ$$

$\therefore$

$$\angle ABC + \angle BCA < 2 \times 90^\circ$$

I.13

- Similarly, we can prove that

$$\angle BAC \angle ACB < 2 \times 90^\circ,$$

  and so the sum of the angles $\angle CAB$ and $\angle ABC$ as well.

$\therefore$,in any triangle, the sum of any two angles is less than two right angles.

Q.E.F

Theorem 1.1.0.11

In any triangle, the angle opposite the greater side is greater.

Let $\widehat{ABC}$ be a triangle having the side $\overline{AC}$ greater than $\overline{AB}$.

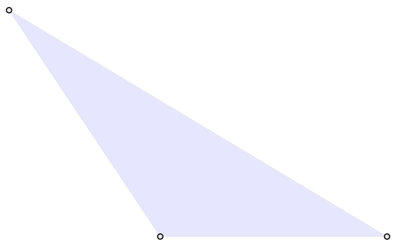


Figure 1.49

Since $\overline{AC}$ is greater than $\overline{AB}$, make $\overline{AD}$ equal to $\overline{AB}$, and join $\overline{BD}$. I.3, I.Post.1

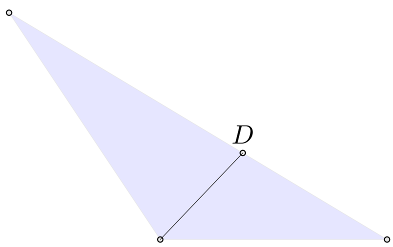


Figure 1.50

Proof.

- Since $\angle ADB$ is an exterior angle of the $\widehat{BCD}$,

$\therefore$ it is greater than the interior and opposite $\angle DCB$. I.16

- But the $\angle ADB$ equals $\angle ABD$, since the side $\overline{AB}$ equals $\overline{AD}$,

$\therefore$ $\angle ABD > \angle ACB$

$\therefore$ $\angle ABC \gg \angle ACB$. I.5

$\therefore$ in any triangle, the angle opposite the greater side is greater.

Q.E.F

Theorem 1.1.0.12

In any triangle the side opposite the greater angle is greater.

Let $\triangle ABC$ have $\angle ABC > \angle BCA$.

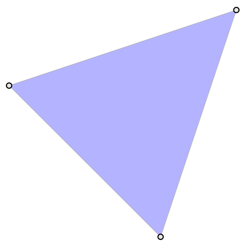


Figure 1.51

Lemma 1.9. *If*

$$\overline{AC} = \overline{AB}$$

then

$$\angle ABC = \angle ACB$$

I.5

Lemma 1.10. *If*

$$\overline{AC} < \overline{AB}$$

then

$$\angle ABC < \angle ACB$$

I.18

Proof.

- Let $\widehat{ABC}$ be a triangle with

$$\angle ABC > \angle BCA$$

- **Assumption:** Suppose, for the sake of contradiction, that side

$$\overline{AC} \not> \overline{AB}$$

.

- This implies either

$$\overline{AC} = \overline{AB}$$

or

$$\overline{AC} < \overline{AB}$$

- **Case 1** If

$$\overline{AC} = \overline{AB},$$

then

$$\angle ABC = \angle ACB$$

which contradicts the assumption that

$$\angle ABC > \angle BCA$$

I.15

- **Case 2** If

$$\overline{AC} < \overline{AB}$$

then

$$\angle ABC < \angle ACB$$

I.18

which also contradicts the assumption.

- **Conclusion:** Since both cases lead to contradictions, our assumption that side

$$\overline{AC} \ngtr \overline{AB}$$

is false.

$\therefore$ side

$$\overline{AC} > \overline{AB}$$

In any triangle, the side opposite the greater angle is greater.

Thus, we have shown that in any triangle, the side opposite the greater angle is greater.

Q.E.F

Theorem 1.1.0.13

In any triangle the sum of any two sides is greater than the remaining one.

Given $\widehat{ABC}$

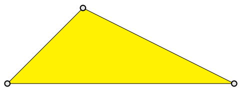


Figure 1.52

Construction

1. Produce $\overline{BA}$ through to the point D. I.Post.2, I.3

2. Make $\overline{DA}$ equal to $\overline{CA}$.

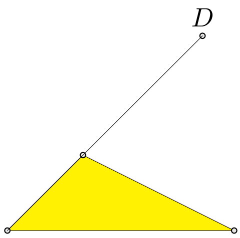


Figure 1.53

3. Join $\overline{DC}$. I.Post.1

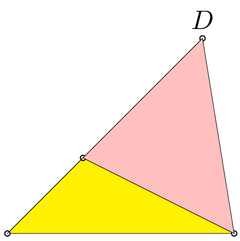


Figure 1.54

Proof.

- Since

$$\overline{DA} = \overline{AC}$$

$$\angle ADC = \angle ACD$$

I.5, C.N.5

- Since $\triangle DCB$ has

$$\angle BCD > \angle BDC$$

the side opposite the greater angle is greater.

$\therefore$

$$\overline{DB} > \overline{BC}$$

I.19

Conclusion:

- Since

$$\overline{DA} = \overline{AC}$$

$$\overline{DB} > \overline{BC}$$

- By the triangle inequality theorem,

$$\overline{BA} + \overline{AC} > \overline{BC}$$

Similarly:

- Apply the same reasoning to the other sides:

$$\overline{AB} + \overline{BC} > \overline{CA}$$

$$\overline{BC} + \overline{CA} > \overline{AB}$$

$\therefore$ In any $\widehat{ABC}$, the sum of any two sides is greater than the remaining one.

- The statement has been demonstrated and proven.

Q.E.F

Theorem 1.1.0.14

If from the ends of one of the sides of a triangle, two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle. Additionally, the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

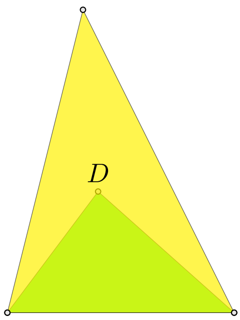


Figure 1.55

Let $\triangle ABC$ be a triangle, and from the ends B and C of one of the sides $\overline{BC}$, let the two straight lines $\overline{BD}$ and$\overline{DC}$ be constructed meeting within the triangle.

Draw $\overline{BD}$ through to E. I.Post.2

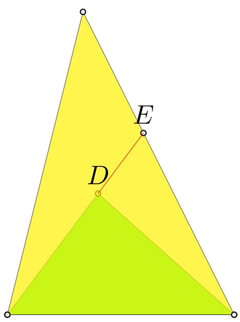


Figure 1.56

Proof.

- Since in any triangle, the sum of two sides is greater than the remaining one, $\therefore$ in the $\triangle ABE$,

$$\overline{AB} + \overline{AE} > \overline{BE}$$

I.20

- Add $\overline{EC}$ to each,

$$\overline{BA} + \overline{AC} > \overline{BE} + \overline{EC}$$

C.N.1

- Again, since in $\widehat{CED}$,

$$\overline{CE} + \overline{ED} > \overline{CD}$$

  add $\overline{DB}$ to each.

$\therefore$

$$\overline{CE} + \overline{EB} > \overline{CD} + \overline{DB}.$$

I.20, C.N.1

- But,

$$\overline{BA} + \overline{AC} > \overline{BE} + \overline{EC}$$

$\therefore$

$$\overline{BA} + \overline{AC} >> \overline{BD} + \overline{DC}.$$

C.N.1

- Again, since in any triangle, the exterior angle is greater than the interior and opposite angle, $\therefore$ in $\widehat{CDE}$,

$$\angle BDC > \angle CED$$

I.16

- For the same reason, moreover, in $\widehat{ABE}$,

$$\angle CEB > \angle BAC$$

- But

$$\angle BDC > \angle CEB$$

$\therefore$

$$\angle BDC >> \angle BAC$$

$\therefore$ if from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

Q.E.F

1.1

Theorem 1.1.1.1

To construct a triangle out of three straight lines which equal three given straight lines, it is necessary that the sum of any two of the straight lines should be greater than the remaining one.

Construction 1.1.1.1

1. Let $A = 4$, $B = 5$, and $C = 6$ be the three given straight lines. Construct a triangle with sides equal to A, B, and C if $A + B > C$, $A + C > B$, and $B + C > A$. I.20

Figure 1.57: Triangle Construction Step:1

2. Set out a straight line $\overline{DE}$, terminated at D but of infinite length in the direction of E. Make

$$\overline{DF} = A$$

$$\overline{FG} = B$$

$$\overline{GH} = C$$

I.Post.3 I.Post.1

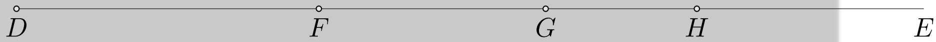


Figure 1.58: Triangle Construction Step:2

3. Describe $\mathscr{C}(F; FD)$.

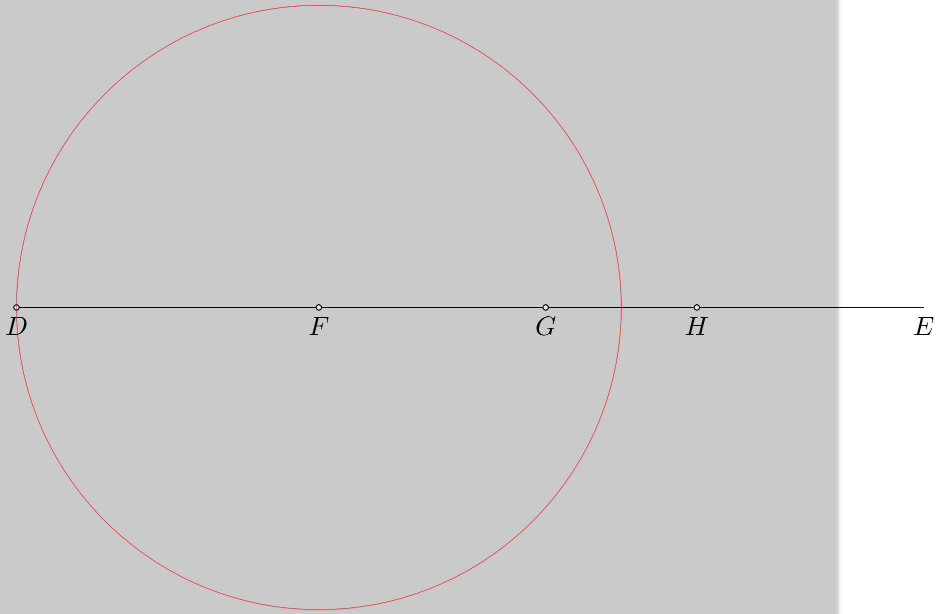


Figure 1.59: Triangle Construction Step:3

4. Again, describe $\mathscr{C}(G; GH)$.
5. Find the intersection of $\mathscr{C}(F; FD)$ and $\mathscr{C}(G; GH)$, K.

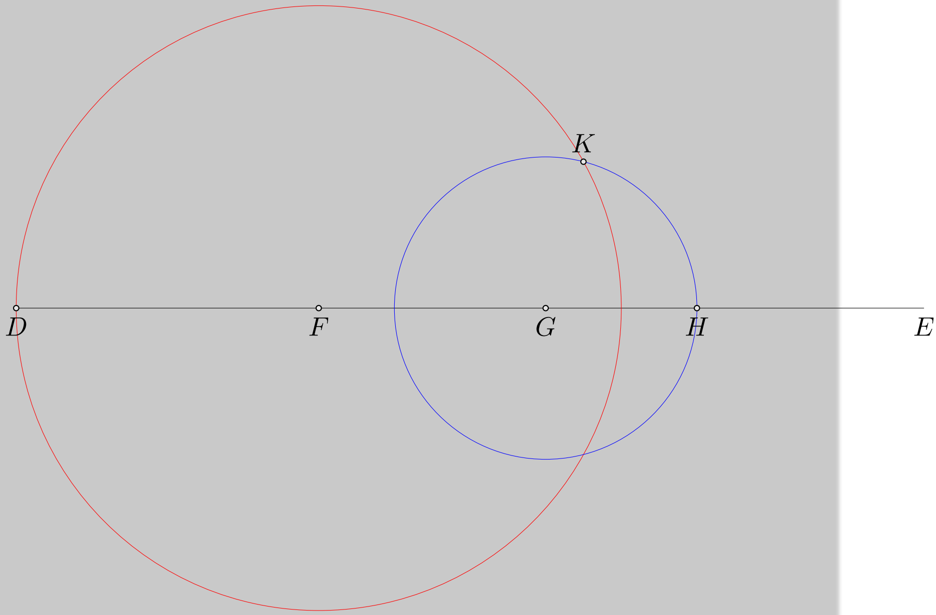


Figure 1.60: Triangle Construction Step:4

6. Join $\overline{KF}$ and $\overline{KG}$. I.Post.3, I,Post.1

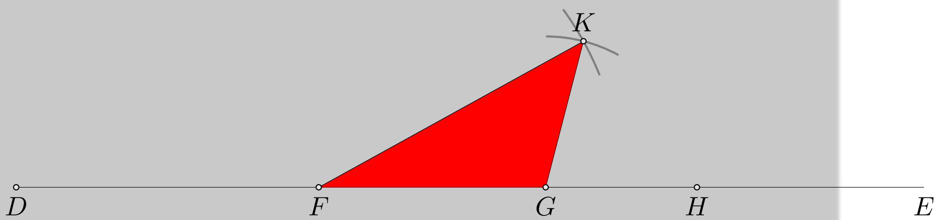


Figure 1.61: Triangle Construction Step:5

Claim: The $\widehat{KFG}$ has been constructed out of three straight lines equal to A, B, and C.

Proof.

- Since the point F is the center of $\mathscr{C}FD$,

$\therefore$

$$\overline{FD} = \overline{FK}$$

But

$$\overline{FD} = A$$

$\therefore$

$$\overline{KF} = A$$

I.Def.16 C.N.1

- Again, since the point G is the center of $\mathscr{G}GH$,

$\therefore$

$$\overline{GH} = \overline{GK}$$

But

$$\overline{GH} = C$$

$\therefore$

$$\overline{KG} = C$$

- And

$$\overline{FG} = B$$

$\therefore$ the three straight lines $\overline{KF}$, $\overline{FG}$, and $\overline{GK}$ equal the three straight lines A, B, and C.

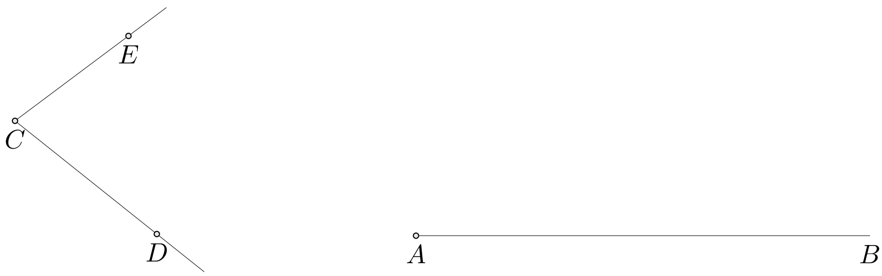
$\therefore$ out of the three straight lines $\overline{KF}$, $\overline{FG}$, and $\overline{GK}$, which equal the three given straight lines A, B, and C, the $\widehat{KFG}$ has been constructed.

Q.E.F.

Construction 1.1.1.2

To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it, let $\angle DCE$ be the given rectilinear angle, $\overline{AB}$ the given straight line, and A the point on it. Construct a rectilinear angle $\angle KAG$ on the given straight line $\overline{AB}$ and at the point A on it such that $\angle DCE = \angle FAG$.

Construction:



(a) (b)

Figure 1.62

1. Draw the segment $\overline{ED}$. Post.1

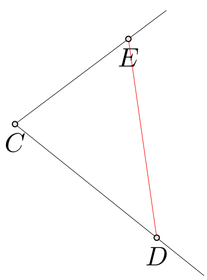


Figure 1.63

2. Draw the given straight line $\overline{AB}$ and mark the point A.

3. On $\overline{AB}$ mark the points G, H and I so that,

$$\overline{AG} = \overline{CD}$$

$$\overline{GH} = \overline{ED}$$

$$\overline{AI} = \overline{CE}$$

I.22

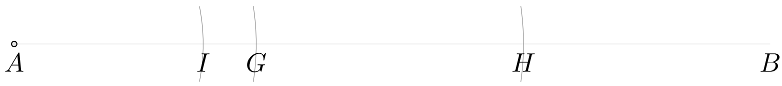


Figure 1.64

4. Mark the intersect of $\mathscr{C}(A; AG)$ and $\mathscr{C}(H; HI)$, K. I.22

5. Construct a triangle $\widehat{AKH}$ such that:

$$\overline{AK} = \overline{ED}$$

$$\overline{AH} = \overline{CD}$$

$$\overline{KH} = \overline{CE}$$

I.22

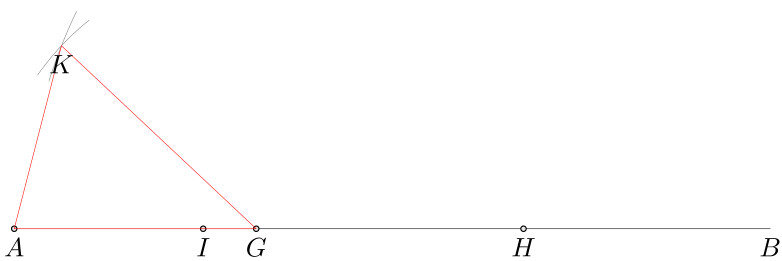


Figure 1.65

Proof.

- By construction,

$$\angle DCE = \angle KAG$$

$\therefore$ a rectilinear $\angle KAG$ has been constructed on the given straight line $\overline{AB}$ and at the point A on it, equal to the given rectilinear $\angle DCE$. I.8

Q.E.F

Theorem 1.1.1.2

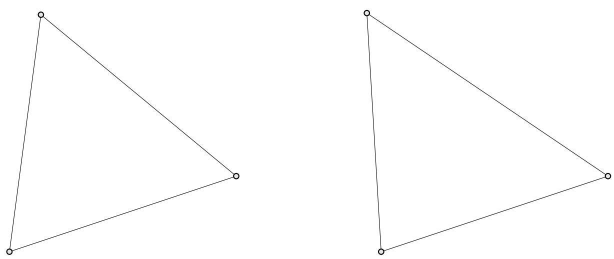
If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.

Let $\overset{\frown}{ABC}$ and $\overset{\frown}{DEF}$, such that

$$\overline{AB} = \overline{DE}$$

$$\overline{AC} = \overline{DF}$$

$$\angle CAB > \angle FDE$$



(a) (b)

Figure 1.66

Construction:

1. Construct $\angle EDG$ equal to $\angle BAC$ at point D on $\overline{DE}$. I.23, I.3, I.Post.1
2. Make $\overline{DG}$ equal to either $\overline{AC}$ or $\overline{DF}$.
3. Join $\overline{EG}$ and $\overline{FG}$.

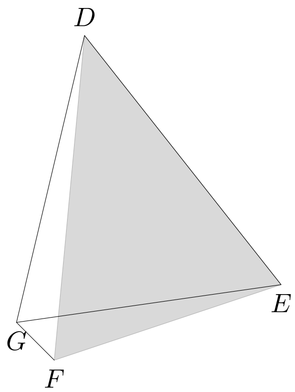


Figure 1.67

Proof.

- If two angles of a triangle are unequal, then the sides opposite these angles are also unequal. I.23
- Since

$$\overline{AB} = \overline{DE}$$

and

$$\overline{AC} = \overline{DG}$$

  - The two sides $\overline{BA}$ and $\overline{AC}$ equal the two sides $\overline{ED}$ and $\overline{DG}$ respectively. I.4
  - Also, $\angle BAC$ equals $\angle EDG$.

$\therefore$ the base $\overline{BC}$ equals the base $\overline{EG}$. C.N.1

- Since $\overline{DF}$ equals $\overline{DG}$, $\angle DGF$ equals $\angle DFG$.

$\therefore$

$$\angle DFG > \angle EGF$$

I.5

Thus , $\widehat{EFG}$, the side $\overline{EG}$ opposite the greater angle is greater than $\overline{EF}$. I.19

- Since
$$\overline{EG} = \overline{BC}$$
it follows that
$$\overline{BC} > \overline{EF}$$

$\therefore$ if two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base $\overline{BC}$ greater than the base $\overline{EF}$.

Q.E.F

Theorem 1.1.1.3

If two triangles ABC and DEF have two sides AB and AC equal to two sides DE and DF, respectively (AB = DE and AC = DF), and the base BC is greater than the base EF (BC > EF), then the angle BAC is greater than the angle EDF.

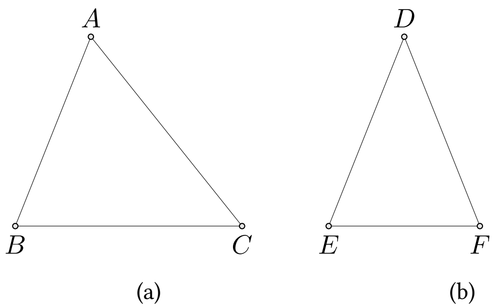


Figure 1.68

To Prove:

$$\angle BAC > \angle EDF$$

Proof.

- Assume, for the sake of contradiction, that

$$\angle BAC = \angle EDF$$

  - If

$$\angle BAC = \angle EDF$$

    then by the Angle-Side-Angle (ASA) congruence,

$$\widehat{ABC} \cong \widehat{DEF}$$

  - This implies

$$\overline{BC} = \angle EF$$

    but this contradicts the given condition

$$\overline{BC} > \overline{EF}$$

I.4

- Next, assume that

$$\angle BAC < \angle EDF$$

  - If

$$\angle BAC < \angle EDF$$

    then by the Angle-Side-Angle (ASA) inequality,

$$\widehat{ABC} \cong \widehat{DEF}$$

  - This implies

$$\overline{BC} < \overline{EF}$$

    but this contradicts the given condition

$$\overline{BC} > \overline{EF}$$

I.24

- Since $\angle BAC$ cannot be equal to or less than $\angle EDF$, the only remaining possibility is that

$$\angle BAC > \angle EDF$$

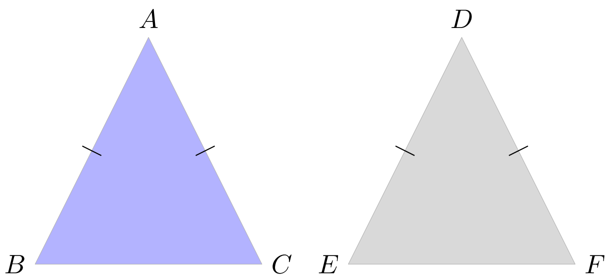
$\therefore$ if two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have one of the angles contained by the equal straight lines greater than the other.

Conclusion: The statement is proved by contradiction, and the desired inequality $\angle BAC > \angle EDF$ holds.

Q.E.F

Theorem 1.1.1.4

(SAS Congruence Criterion): If $\widehat{ABC}$ and $\widehat{DEF}$ have $\angle ABC$ and $\angle BCA$ equal to $\angle DEF$ and $\angle EFD$, respectively, and one side $\overline{BC}$ equal to $\overline{EF}$, then the triangles are congruent. This means that $\overline{AB}$ equals $\overline{DE}$, $\overline{AC}$ equals $\overline{DF}$, and $\angle BAC$ equals $\angle EDF$.



(a) (b)

Figure 1.69: SAS Congruence

Given:

- $\triangle ABC$ and $\triangle DEF$ such that:

$$\angle ABC = \angle DEF$$

$$\angle BCA = \angle EFD$$

$$\overline{BC} = \overline{EF}$$

- **To Prove:**

$$\overline{AB} = \overline{DE}$$

$$\overline{AC} = \overline{DF}$$

$$\angle BAC = \angle EDF$$

Proof. **Case 1:** Suppose $AB \neq DE$.

- By the Segment Equality Postulate, one must be greater. I.Post.1
- Assume $AB > DE$.
- Construct $BG = DE$ and join GC.

$$\widehat{GBC} \cong \widehat{DEF}$$

I.4

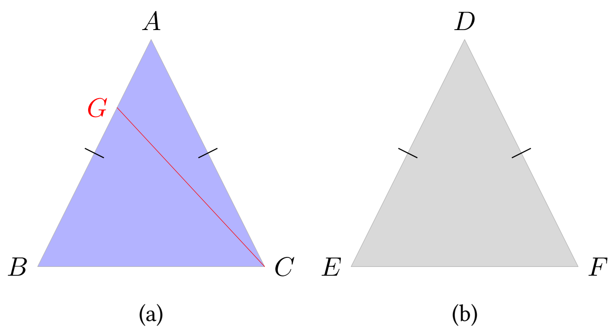


Figure 1.70: SAS Congruence.2

$\therefore$,

$$\angle GCB = \angle DFE$$

- But, (given).

$$\angle DFE = \angle ACB$$

- Hence,

$$\angle GCB = \angle BAC$$

- This is impossible, as the lesser angle cannot equal the greater angle. Thus,

$$AB = DE$$

Case 2: $AB = DE$.

- Given that $BC = EF$,

- By Side-Side-Side (SSS) congruence:

$$\widehat{ABC} \cong \widehat{DEF}$$

C.N.1

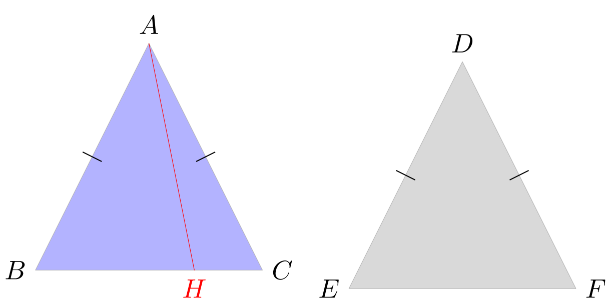
$\therefore$, $AC = DF$ and $\angle BAC = \angle EDF$.

Case 3: Suppose $BC \neq EF$.

- By the Segment Equality Postulate, one must be greater. I.Post.1

- Assume $BC > EF$.

- Construct $BH = EF$ and join AH.



(a) (b)

Figure 1.71

- By Side-Angle-Side (SAS) congruence: I.Post.4

$$\triangle ABH \cong \triangle DEF$$

$\therefore$

$$\angle BHA = \angle EFD$$

- But,(given),

$$\angle EFD = \angle BCA$$

- Hence,

$$\angle BHA = \angle BCA$$

- This is impossible, as the exterior angle of a triangle cannot be equal to its interior opposite angle.
- Thus, $BC = EF$.

Conclusion:

- From both cases, it's evident that:

$$AB = DE$$

$$AC = DF$$

$$\angle BAC = \angle EDF$$

- Thus, if two triangles have two angles equal to two angles respectively, and one side equal to one side, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.

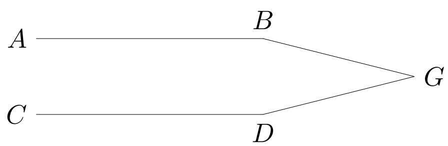
Q.E.F

Theorem 1.1.1.5

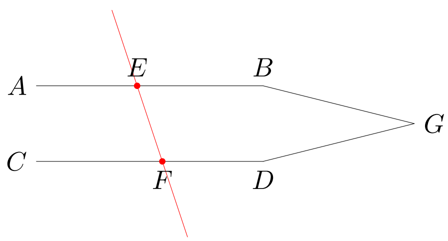
If a straight line $\overline{EF}$ intersects two straight lines $\overline{AB}$ and $\overline{CD}$, and the alternate $\angle AEF$ and $\angle EFD$ are equal, then $\overline{AB}$ is parallel to $\overline{CD}$.

To Prove: $\overline{AB}$ is parallel to $\overline{CD}$.

Proof. Assume, for the sake of contradiction, that $\overline{AB}$ and $\overline{CD}$ when produced meet in the direction of B and D at a point G.



- In $\widehat{GEF}$, the exterior $\angle AEF$ equals the interior and opposite $\angle EFG$ (by exterior angle theorem).



- This is impossible, as the exterior angle cannot be equal to the interior and opposite angle in a triangle. I.16

$\therefore$ $\overline{AB}$ and $\overline{CD}$ when produced do not meet in the direction of B and D. Similarly, it can be proved that they do not meet towards A and C.

Since straight lines that do not meet in either direction are parallel, it follows that $\overline{AB}$ is parallel to $\overline{CD}$. I.Def.23

Conclusion: If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

Q.E.F

Theorem 1.1.1.6

If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

Given: Straight line$\overline{EF}$ falls on the two straight lines $\overline{AB}$ and $\overline{CD}$, and the exterior $\angle EGB$ is equal to the interior and opposite $\angle GHD$, or the sum of the interior $\angle BGH$ and $\angle GHD$ is equal to two right angles.

Proof.

- Assume: $\overline{AB} \nparallel \overline{CD}$.
- By the definition of parallel lines, $if \overline{AB} \nparallel \overline{CD}$, then there exists a transversal $\overline{EF}$ that intersects $\overline{AB}$ and $\overline{CD}$. I.28

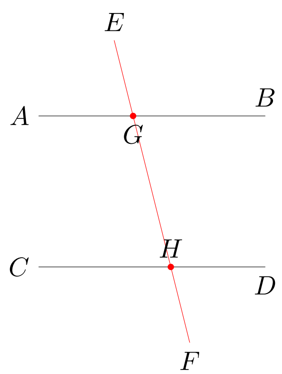


Figure 1.72

- By the exterior angle theorem,

$$\angle EGB = \angle GHD$$

I.15

- By the alternate interior angles theorem, since

$$\angle EGB = \angle GHD$$

and

$$\angle EGB = \angle AGH$$

then

$$\angle AGH equals \angle GHD$$

C.N.1, I.27

- By the alternate interior angles theorem, since

$$\angle AGH = \angle GHD$$

$$\overline{AB} \parallel \overline{CD}$$

I.13

- By the definition of parallel lines, if

$$\overline{AB} \parallel \overline{CD}$$

then the assumption in step 1 is false. C.N.1

$\therefore \overline{AB} \parallel to \overline{CD}$

- Next, assume again: $\overline{AB} \nparallel \overline{CD}$

- By the sum of interior angles on the same side theorem,

$$\angle BGH + \angle GHD = 2 \times 90°$$

I.Post.4

- By the sum of interior angles on the same side theorem,

$$\angle AGH + \angle BGH + 2 \times 90°$$

C.N.3, I.27

- Subtract $\angle BGH$ from both sides of the equation in above, yielding

$$\angle AGH = \angle GHD$$

- By the alternate interior angles theorem, since

$$\angle AGH = \angle GHD$$

$$\overline{AB} \parallel \overline{CD}$$

I.13

- By the definition of parallel lines, if

$$\overline{AB} \parallel \overline{CD}$$

  then the assumption above is also false. C.N.1

$\therefore$

$$\overline{AB} \parallel \overline{CD}$$

- Since assuming

$$\overline{AB} \nparallel \overline{CD}$$

leads to a contradiction in both cases, it must be true that

$$\overline{AB} \parallel \overline{CD}$$

$\therefore$ if a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

Q.E.F

Theorem 1.1.1.7

In a system of parallel straight lines, if a straight line $\overline{EF}$ intersects the lines $\overline{AB}$ and $\overline{CD}$, then the following statements hold true:

1. Alternate $\angle AGH$ and $\angle GHD$ are equal.
2. Exterior $\angle EGB$ is equal to the interior and opposite $\angle GHD$.
3. The sum of the interior angles on the same side, namely $\angle BGH$ and $\angle GHD$, is equal to two right angles.

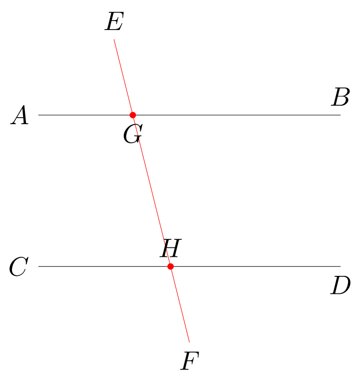


Figure 1.73: Alternate angles

Proof. Proof of Statement 1:

Suppose, for the sake of contradiction, that,

$$\angle AGH \neq \angle GHD$$

- Without loss of generality, assume:

$$\angle AGH > \angle GHD$$

- Adding $\angle BGH$ to both sides, we get,

$$\angle AGH + \angle BGH > \angle BGH + \angle GHD$$

- However, by the Angle Sum Property,

$$\angle AGH + \angle BGH = 2 \times 90^\circ$$

- This implies that, I.13

$$\angle BGH + \angle GHD < 2 \times 90^\circ$$

Now, by the Parallel Postulate, extended lines produced from angles less than two right angles meet. This contradicts the fact that $\overline{AB}$ and $\overline{CD}$, if produced indefinitely, should meet, but they are parallel by hypothesis

$\therefore$ our assumption that,

$$\angle AGH \neq \angle GHD$$

must be false, and thus, Post.5

$$\angle AGH = \angle GHD$$

Proof of Statement 2:

- Since

$$\angle AGH = \angle GHD$$

- (proved above) and

$$\angle AGH = \angle EGB$$

- (given), it follows that

$$\angle EGB = \angle GHD$$

Proof of Statement 3:

- Adding angle $\angle BGH$ to both sides of the equation

$$\angle EGB = \angle GHD$$

- We obtain

$$\angle EGB + \angle BGH = \angle BGH + \angle GHD$$

- By the Angle Sum Property,

$$\angle EGB + \angle BGH = 2 \times 90^\circ$$

- $\therefore$ I.13

$$\angle BGH + \angle GHD = 2 \times 90^\circ$$

Hence, we have established that a straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

Q.E.F

Theorem 1.1.1.8

If two straight lines, such as $\overline{AB}$ and $\overline{CD}$, are each parallel to a third straight line $\overline{EF}$, then $\overline{AB}$ is also parallel to $\overline{CD}$.

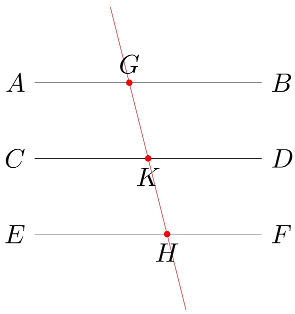


Figure 1.74

Proof. *Given:* Straight lines $\overline{AB}$ and $\overline{CD}$ are each parallel to $\overline{EF}$.

To prove:

$$\overline{AB} \parallel \overline{CD}$$

- Let the straight line $\overline{GH}$ fall upon $\overline{AB}$ and $\overline{CD}$. Since $\overline{GK}$ falls on the parallel straight lines $\overline{AB}$ and $\overline{EF}$, it follows from the Corresponding Angles Postulate that

$$\angle AGK = \angle GHF$$

I.29

- Again, since $\overline{GH}$ falls on the parallel straight lines $\overline{EF}$ and $\overline{CD}$, the Corresponding Angles Postulate implies that the

$$\angle GHF = \angle GKD$$

By the Corresponding Angles Postulate, we have

$$\angle AGK = \angle GHF$$

and

$$\angle GHF = \angle GKD$$

$\therefore$

$$\angle AGK = \angle GKD$$

and they are alternate angles. I.29, C.N.1

- Thus, by the Alternate Interior Angles Theorem, it is concluded that

$$\overline{AB} \parallel \overline{CD}$$

I.29

$\therefore$ straight lines parallel to the same straight line are also parallel to one another.

Q.E.F

Theorem 1.1.1.9

Given a point A and a line $\overline{BC}$ there exists a unique line passing through A that is parallel to $\overline{BC}$.

To prove; To draw a straight line through a given point A parallel to a given straight line $\overline{BC}$.

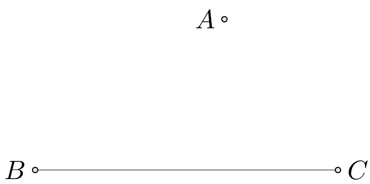


Figure 1.75

Proof. **Proof by construction:**

1. Take a point D at random on $\overline{BC}$.
2. Join $\overline{AD}$.
3. Construct $\angle DAE$ equal to $\angle ADC$ on the straight line $\overline{DA}$ at point A.
4. Produce the straight line $\overline{AF}$ in a straight line with $\overline{EA}$. I.Post.1,I.23,I.Post.2

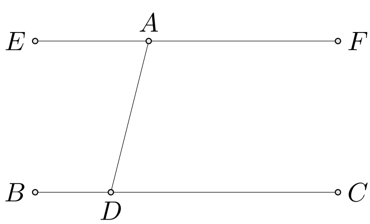


Figure 1.76

Justification: Since the straight line $\overline{AD}$ falling on the two straight lines $\overline{BC}$

and $\overline{EF}$ makes the alternate $\angle EAD$ and $\angle ADC$ equal to one another, $\therefore$ $\overline{EAF}$ is parallel to $\overline{BC}$. $\therefore$ the straight line $\overline{EAF}$ has been drawn through the given point A parallel to the given straight line $\overline{BC}$. I.27

Q.E.F

Theorem 1.1.1.10

In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.

Given: $\widehat{ABC}$, and side $\overline{BC}$ produced to point D.

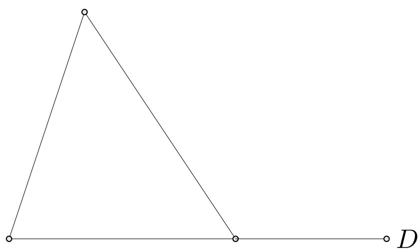


Figure 1.77

To prove:

1. The exterior $\angle ACD$ equals the sum of the two interior and opposite $\angle CAB$ and $\angle ABC$.
2. The sum of the three interior $\angle ABC$, $\angle BCA$, and $\angle CAB$ equals two right angles.

Proof.

- Draw $\overline{CE}$ through point C parallel to line $\overline{AB}$.

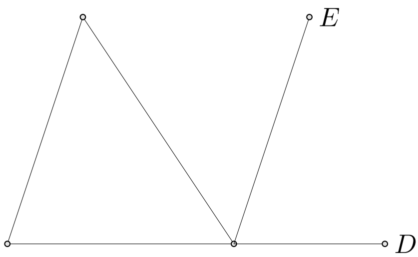


Figure 1.78

- Since

$$\overline{AB} \parallel \overline{CE}$$

and $\overline{AC}$ falls upon them, the alternate

$$\angle BAC = \angle ACE$$

I.29

- Since

$$\overline{AB} \parallel \overline{CE}$$

and line $\overline{BD}$ falls upon them, the exterior

$$\angle ECD = \angle ABC$$

I.29

- By combining both statements, we conclude that the whole

$$\angle ACD = \angle BAC + \angle CAB$$

C.N.2

- Now, add $\angle ACB$ to each side.

$$\angle ACD + \angle ACB = \angle ABC + \angle BAC + \angle CAB$$

I.13, C.N.1

- But,

$$\angle ACD + \angle ACB = 2 \times 90^\circ$$

$\therefore$

$$\angle ABC + \angle BCA + \angle CAB = 2 \times 90^\circ$$

I.13

Hence, in any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.

Q.E.F

1.2

Theorem 1.1.2.1

Straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.

Proof.

- Let $\overline{AB}$ and $\overline{CD}$ be equal and parallel, and let the straight lines $\overline{AC}$ and $\overline{BD}$ join them at their ends in the same directions. I say that $\overline{AC}$ and $\overline{BD}$ are also equal and parallel.
- Join BC. I.33,I.Post.1

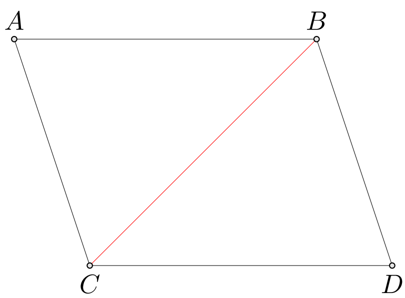


Figure 1.79

- Since

$$\overline{AB} \parallel \overline{CD}$$

and BC falls upon them,

$\therefore$ the alternate angles

$$\angle ABC = \angle BCD$$

I.29

- Since

$$\overline{AB} = \overline{CD}$$

and $\overline{BC}$ is common, the two sides $\overline{AB}$ and $\overline{BC}$ equal the two sides $\overline{DC}$ and $\overline{CB}$, and

$$\angle ABC = \angle BCD$$

$\therefore$ the base

$$\overline{AC} = \overline{BD}$$

the triangle

$$\widehat{ABC} = \widehat{DCB}$$

and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

$\therefore$ the angle

$$\angle ACB = \angle CBD$$

I.4

- Since the straight line $\overline{BC}$ falling on the two straight lines $\overline{AC}$ and $\overline{BD}$ makes the alternate angles equal to one another,

$\therefore$

$$\overline{AC} \parallel \overline{BD}$$

I.27

And it was also proved equal to it. $\therefore$ straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.

Q.E.F

Theorem 1.1.2.2

In a parallelogram, opposite sides and angles are equal, and the diameter bisects the area.

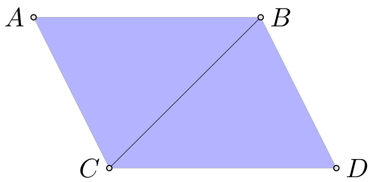


Figure 1.80

Given: Parallelogram $ACDB$ with diameter BC.

Proof. **1. Opposite Angles are Equal** I.34, I.29

- Since $\overline{AB} \parallel \overline{CD}$ and $\overline{BC}$ is a transversal line, we have

$$\angle ABC = \angle BCD$$

I.34

- Similarly, since $\overline{AC} \parallel \overline{BD}$ and $\overline{BC}$ is a transversal line,we have

$$\angle ACB = \angle CBD$$

I.29

2. Triangle Equality I.26

- Consider $\widehat{ABC}$ and $\widehat{DCB}$
- They have

$$\angle ABC = \angle BDC$$

$$\angle ACB = \angle CBD$$

and $\overline{BC}$ in common.

$\therefore$ by SAS criterion,

$$\widehat{ABC} \cong \widehat{DCB}$$

- This implies

$$\overline{AB} = \overline{CD}$$

$$\overline{AC} = \overline{BD}$$

$$\angle BAC = \angle CDB$$

3. Sum of Angles C.N.2

- $\angle ABC = \angle BCD$ and $\angle CBD = \angle ACB$,
- imply that the sum of angles $\angle ABD = \angle ACD$.

4. Conclusion - Opposite Sides and Angles:

- In parallelogram $ACDB$,
- opposite sides

$$\overline{AB} = \overline{CD}$$

and

$$\overline{AC} = \overline{BD}$$

- and opposite angles

$$\angle ABC = \angle BDC$$

and

$$\angle ACB = \angle CBD$$

5. Diameter Bisects the Parallelogram I.4

- Since

$$\overline{AB} = \overline{CD}$$

$\overline{BC}$ is common, and

$$\angle ABC = \angle BCD$$

-

$$\widehat{ABC} \cong \widehat{BCD}$$

- This implies

$$\overline{AC} = \overline{DB}$$

$$\widehat{ABC} = \widehat{DCB}$$

6. Conclusion - Diameter Bisects the Area:

$\therefore$ the diameter $\overline{BC}$ bisects the parallelogram $ACDB$.

Q.E.F

Theorem 1.1.2.3

Parallelograms on the same base and in the same parallels are equal to one another.

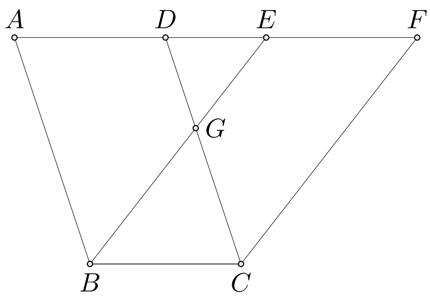


Figure 1.81

Given: Parallelograms $ABCD$ and $EBCF$ on the same base $\overline{BC}$ and in the same parallels $\overline{AF}$ and $\overline{BC}$.

Proof. 1. **Parallelogram Properties** I.35, I.34

- Since $ABCD$ is a parallelogram,

$$\overline{AD} = \overline{BC}$$

I.35

- Similarly, for $EBCF$,

$$\overline{EF} = \overline{BC}$$

I.34

2. **Triangle Equality** C.N.1, C.N.2, I.4

- Since

$$\overline{AD} = \overline{BC}$$

and

$$\overline{EF} = \overline{BC}$$

we can conclude that

$$\overline{AD} = \overline{EF}$$

- Since $\overline{DE}$ is common, we have

$$\overline{AE} = \overline{DF}$$

C.N.1

- Also,

$$\overline{AB} = \overline{DC}$$

I.34

$\therefore$ the two sides $\overline{EA}$ and $\overline{AB}$ equal the two sides $\overline{FD}$ and $\overline{DC}$, respectively, and $\angle FDC = \angle EAB$. I.4

- Thus, the base

$$\overline{EB} = \overline{FC}$$

and

$$\widehat{EAB} = \widehat{FDC}$$

C.N.2

3. **Trapezium Equality** C.N.3

- Subtract $\widehat{DGE}$ from each side.

- Then the trapezium $ABGD$ equals the trapezium $EGCF$.

4. **Whole Parallelogram Equality.** C.N.2

   - Add the $\widehat{GBC}$ to each side.
   - Then the whole parallelogram $ABCD$ equals the whole parallelogram $EBCF$.

Conclusion:

$\therefore$ parallelograms on the same base and in the same parallels equal one another.

Q.E.F

Theorem 1.1.2.4

Triangles which are on the same base and in the same parallels are equal to one another.

Let $\widehat{ABC}$ and $\widehat{DBC}$ be triangles on the same base $\overline{BC}$ and in the same parallels $\overline{AD}$ and $\overline{BC}$.

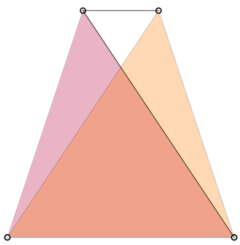


Figure 1.82

I say that $\widehat{ABC}$ equals $\widehat{DBC}$.

Produce $\overline{AD}$ in both directions to E and F. Draw $\overline{BE}$ through B parallel to $\overline{CA}$, and draw $\overline{CF}$ through C parallel to $\overline{BD}$. I.Post.2, I.31

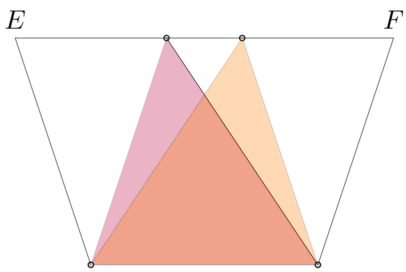


Figure 1.83

Proof.

- Then each of the figures $EBCA$ and $DBCF$ is a parallelogram, and they are equal, for they are on the same base $\overline{BC}$ and in the same parallels $\overline{BC}$ and $\overline{EF}$. I.35

- Moreover, the $\widehat{ABC}$ is half of the parallelogram $EBCA$, for the diameter $\overline{AB}$ bisects it.
- And the $\widehat{DBC}$ is half of the parallelogram $DBCF$, for the diameter $\overline{DC}$ bisects it. I.34

$\therefore$ the

$$\widehat{ABC} = \widehat{DBC}$$

C.N.1

$\therefore$ triangles which are on the same base and in the same parallels equal one another.

Q.E.F

Theorem 1.2. *Triangles which are on equal bases and in the same parallels are equal to one another.*

Let $\widehat{ABC}$ and $\widehat{DEF}$ be triangles on equal bases $\overline{BC}$ and $\overline{EF}$ and in the same parallels $\overline{BF}$ and $\overline{AD}$.

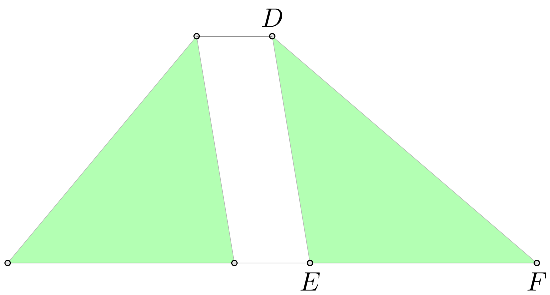


Figure 1.84

I say that the $\widehat{ABC}$ equals the $\widehat{DEF}$.

Produce $\overline{AD}$ in both directions to G and H. Draw $\overline{BG}$ through B parallel to $\overline{CA}$, and draw $\overline{FH}$ through F parallel to $\overline{DE}$. I.Post.2, I.31

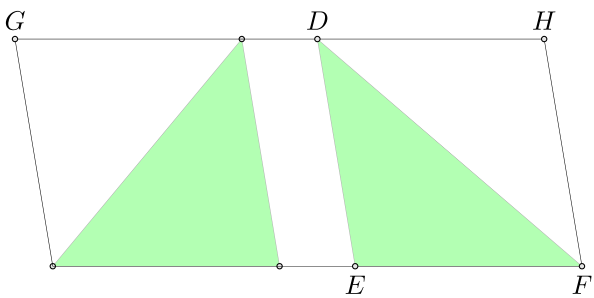


Figure 1.85

Proof.

- Then each of the figures $GBCA$ and $DEFH$ is a parallelogram, and $GBCA$ equals $DEFH$, for they are on equal bases $\overline{BC}$ and $\overline{EF}$ and in the same parallels $\overline{BF}$ and $\overline{GH}$. I.36

- Moreover, the $\widehat{ABC}$ is half of the parallelogram $GBCA$, for the diameter $\overline{AB}$ bisects it.
- And the $\widehat{FED}$ is half of the parallelogram $DEFH$, for the diameter $\overline{DF}$ bisects it. I.35

$\therefore$ the

$$\widehat{ABC} = \widehat{DEF}$$

C.N.1

$\therefore$ triangles which are on equal bases and in the same parallels equal one another.

Q.E.F

Theorem 1.1.2.5

Equal triangles which are on the same base and on the same side are also in the same parallels.

Given: Let $\widehat{ABC}$ and $\widehat{DBC}$ be equal triangles which are on the same base $\overline{BC}$ and on the same side of it.

Join $\overline{AD}$. I.Post.1

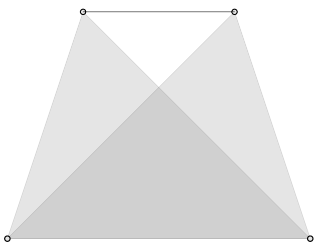


Figure 1.86

To Prove:

$$\overline{AD} \parallel \overline{BC}$$

Construction: If not, draw $\overline{AE}$ through the point A parallel to the straight line $\overline{BC}$, and join$\overline{EC}$.

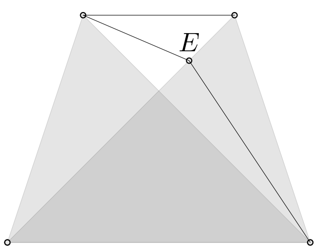


Figure 1.87

Proof. **Proof by Contradiction:**

∴

$$\widehat{ABC} = \widehat{EBC}$$

for it is on the same base $\overline{BC}$ with it and in the same parallels. I.37

- But

$$\widehat{ABC} = \widehat{DBC}$$

(by hypothesis)

∴

$$\widehat{DBC} = \widehat{EBC}$$

the greater equals the less, which is impossible. C.N.1

∴

$$\overline{AE} \parallel \overline{BC}$$

- Similarly, we can prove that neither is any other straight line except$\overline{AD}$.

∴

$$\overline{AD} \parallel \overline{BC}$$

∴ equal triangles which are on the same base and on the same side are also in the same parallels.

Q.E.F

Theorem 1.1.2.6

Equal triangles which are on equal bases and on the same side are also in the same parallels.

Given: Let $\widehat{ABC}$ and $\widehat{CDE}$ be equal triangles on equal bases $\overline{BC}$ and $\overline{CE}$ and on the same side.

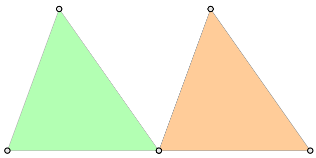


Figure 1.88

To Prove: They are also in the same parallels.

Construction: Join AD. I say that AD is $\parallel$ to BE. I.Post.1

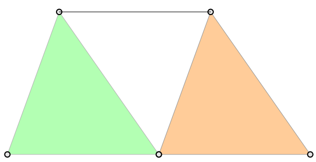


Figure 1.89

Proof. **Proof by Contradiction:**

If not, draw AF through A $\parallel$ to BE, and join FE. I.31, I.Post.1

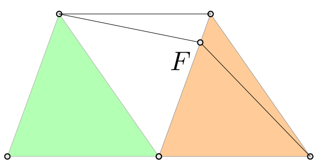


Figure 1.90

$\therefore$

$$\widehat{ABC} = \widehat{FCE}$$

for they are on equal bases $\overline{BC}$ and $\overline{CE}$ and in the same parallels $\overline{BE}$ and $\overline{AF}$. I.38

- But,

$$\widehat{ABC} = \widehat{DCE}$$

$\therefore$

$$\widehat{DCE} = \widehat{FCE}$$

the greater equals the less, which is impossible. C.N.1

$\therefore$

$$\overline{AF} \nparallel \overline{BE}$$

- Similarly, we can prove that neither is any other straight line except $\overline{AD}$.

$\therefore$

$$\overline{AD} \parallel \overline{BE}$$

$\therefore$ equal triangles which are on equal bases and on the same side are also in the same parallels.

Q.E.F

Theorem 1.1.2.7

In any parallelogram, the complements of the parallelograms about the diameter equal one another.

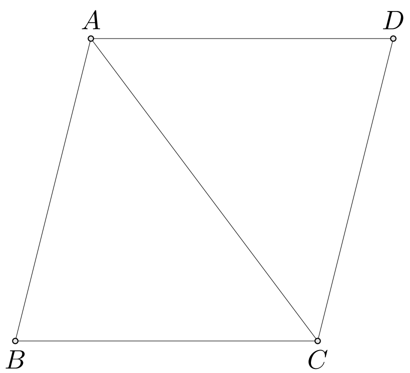


Figure 1.91

Proof.

- In parallelogram $AEKH$, since $\overline{AK}$ is its diameter:

$$\widehat{AEK} = \widehat{AHK}$$

Similarly,

$$\widehat{KFC} = \widehat{KGC}$$

I.34,C.N.2

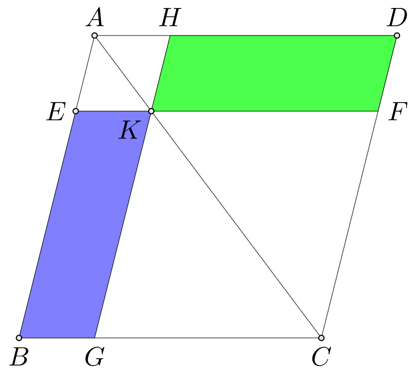


Figure 1.92

- Since

$$\widehat{AEK} = \widehat{AHK}$$

I.34 and

$$\widehat{KFC} = \widehat{KGC}$$

C.N.2

$\therefore$

$$\widehat{AEK} + \widehat{KGC} = \widehat{AHK} + \widehat{KFC}$$

C.N.3

- Since the whole

$$\widehat{ABC} = \widehat{ADC}$$

$\therefore$ the remaining complement $BGKE$ equals the remaining complement $KFDH$.

Q.E.F

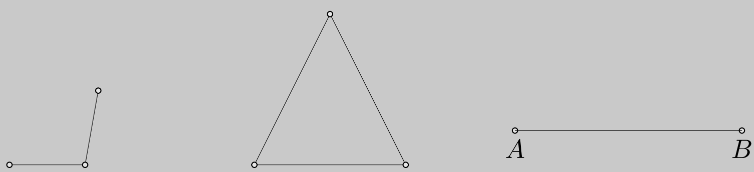
Construction 1.1.2.1

To a given straight line in a given rectilinear angle, to apply a parallelogram equal to a given triangle.

Given a straight line $\overline{AB}$, a rectilinear angle D, and a triangle C, the goal is to apply a parallelogram equal to triangle C to the straight line $\overline{AB}$ in an angle equal to D.

Construction 1.1.2.2

1. Construct the parallelogram $BEFG$ equal to triangle C in the $\angle EBG$ which equals D, and let it be placed so that $\overline{BE}$ is in a straight line with $\overline{AB}$.



(a) Given angle D (b) Given triangle C (c) given $\overline{AB}$

Figure 1.93: Given Prop 44

Construct a polygon equal to C; I.42

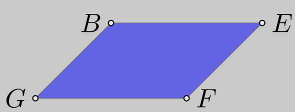


Figure 1.94: Polygon $BEFG$

2. Produce $\overline{FG}$ through to H, and draw $\overline{AH}$ through A $\parallel$ to either $\overline{BG}$ or $\overline{EF}$. Join$\overline{HB}$. I.42

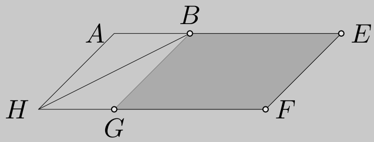


Figure 1.95

3. By the construction, $\overline{HF}$ falls upon the parallels $\overline{AH}$ and $\overline{EF}$. $\therefore$, the sum of $\angle AHF$ and $\angle HFE$ equals two right angles. I.Post.2
4. Since $\overline{HF}$ falls on parallels, the sum of $\angle BHG$ and $\angle GFE$ is less than two right angles. Straight lines produced indefinitely from angles less than two right angles meet, so$\overline{HB}$ and $\overline{FE}$ will meet. I.31, I.Post.1
5. Let them be produced and meet at K. Draw $\overline{KL}$ through the point K $\parallel$ to either $\overline{EA}$ or $\overline{FH}$. Produce $\overline{HA}$ and $\overline{GB}$ to the points L and M. I.31

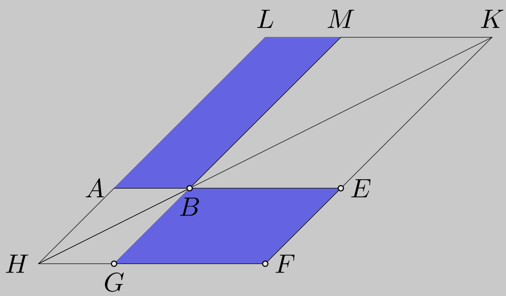


Figure 1.96

Proof. • Then $HLKF$ is a parallelogram, $\overline{HK}$ is its diameter, and $AHGB$ and $MBEK$ are parallelograms, and $LABM$ and $BGFE$ are the complements about HK. $\therefore$, $LABM$ equals $BGFE$. I.43

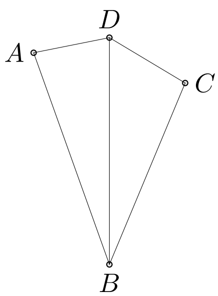
- But $BGFE$ equals triangle C, $\therefore$ $LABM$ also equals C. C.N.1
- Since $\angle GBE$ equals $\angle ABM$, while $\angle GBE$ equals D, $\therefore$ $\angle ABM$ also equals the $\angle$ D. I.15, C.N.1

$\therefore$ the parallelogram $LABM$ equal to the given triangle C has been applied to the given straight line $\overline{AB}$, in $\angle ABM$ which equals D.

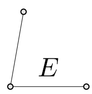
Q.E.F

Construction 1.1.2.3

To construct a parallelogram equal to a given rectilinear figure $ABCD$ in a given rectilinear angle E.



(a) Rectilier shape $ABCD$



(b) Rectiliear angle E

Figure 1.97

Given:

- Rectilinear figure $ABCD$.
- Rectilinear angle E.
- A straight line can be drawn between any two points, join $\overline{BD}$. I.Post.1

Construction 1.1.2.4

1. A straight line can be drawn between any two points, join $\overline{BD}$. I.Post.1
2. Construct the parallelogram $FGHK$ equal $\widehat{ABD}$ in $\angle HKF$ which equals E. I.42

3. Construct the parallelogram $GLMN$ equal to $\widehat{DBC}$ to $\overline{GH}$ in $\angle GHM$ which equals E. I.44

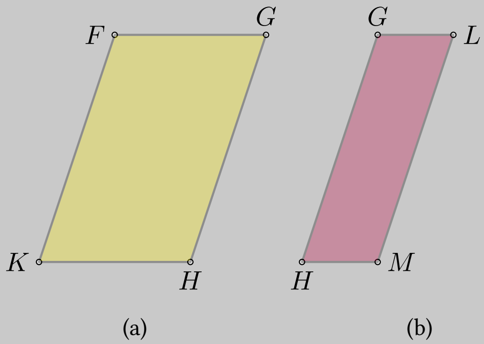


Figure 1.98

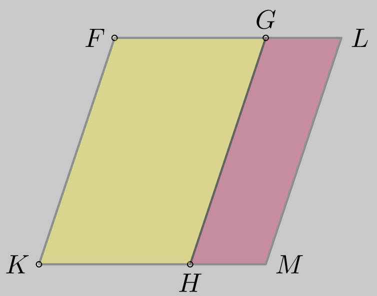


Figure 1.99

Proof. 1. Since angle E equals both $\angle HKF$ and $\angle GHM$, $\angle HKF$ equals $\angle GHM$. C.N.1

2. Add $\angle KHG$ to both sides.

$\therefore$

$$\angle FKH + \angle KHG = \angle KHG + \angle GHM$$

C.N.2

3.

$$\angle FKH + \angle KHG 2 \times 90^\circ$$

$\therefore$

$$\angle KHG + \angle GHM = 2 \times 90^\circ$$

I.29, C.N.1

4. With a straight line $\overline{GH}$, and at point H on it, two straight lines $\overline{KH}$ and $\overline{HM}$ not lying on the same side make the adjacent angles together equal to two right angles.

$\therefore$ $\overline{KH}$ is in a straight line with $\overline{HM}$. I.29

5. Since line $\overline{HG}$ falls upon the parallels $\overline{KM}$ and $\overline{FG}$, the alternate $\angle MHG$ and $\angle HGF$ are equal. I.29

6. Add $\angle HGL$ to both sides,

$$\angle MHG + \angle HGL = \angle HGF + \angle HGL$$

C.N.2

7.

$$\angle MHG + \angle HGL = 2 \times 90^\circ$$

$\therefore$

$$\angle HGF + \angle HGL = 2 \times 90^\circ$$

I.29, C.N.1, I.14

8. $\overline{FG}$ is in a straight line with $\overline{GL}$. I.34, I.30

9. Since

$$\overline{FK} \# \overline{HG}$$

and

$$\overline{HG} \# \overline{ML}$$

$\therefore$

$$\overline{KF} \# \overline{ML}$$

and the straight lines $\overline{KM}$ and $\overline{FL}$ join them at their ends.

∴

$$\overline{KM} \nparallel \overline{FL}$$

I.33, C.N.1

10. Since $\widehat{ABD}$ equals parallelogram $FKHG$, and $\widehat{DBC}$ equals parallelogram $GHML$, the whole rectilinear figure $ABCD$ equals the whole parallelogram $KFLM$. C.N.2

∴ the parallelogram $KFLM$ has been constructed equal to the given rectilinear figure $ABCD$ in the $\angle FKM$, which equals the given angle E.

Q.E.F

Construction 1.1.2.5

To describe a square on a given straight line.

Given: A straight line $\overline{AB}$.

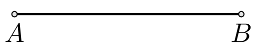


Figure 1.100: Given Line $\overline{AB}$

Construction 1.1.2.6

1. Draw $\overline{AC}$ at right angles to the straight line $\overline{AB}$ from point A on it.
2. Make

$$\overline{AD} = \overline{AB}$$

3. Draw $\overline{DE}$ through point D parallel to $\overline{AB}$.
4. Draw $\overline{BE}$ through point B parallel to $\overline{AD}$. I.11,I.3,I.31

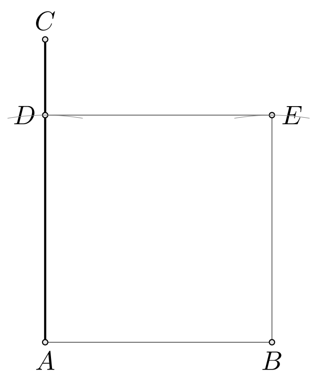


Figure 1.101: Parallelogram

Proof.

Lemma 1.11. *$ADEB$ is a parallelogram.*

- *Proof: By construction,*

$$\overline{DE} \parallel \overline{AB}$$

and

$$\overline{BE} \parallel \overline{AD}$$

thus $ADEB$ is a parallelogram.

Lemma 1.12. *$ADEB$ is equilateral.*

- *Proof: Since*

$$\overline{AB} = \overline{AD}$$

and

$$\overline{AB} = \overline{DE}$$

(as opposite sides of a parallelogram are equal),

$$\overline{AB} = \overline{AD} = \overline{DE}$$

Thus, $ADEB$ is equilateral.

Lemma 1.13. *$ADEB$ is right-angled.*

- *Proof:*

$$\angle BAD + \angle ADE = 2 \times 90°$$

Since $\overline{AD}$ falls on parallels $\overline{AB}$ and $\overline{DE}$

$$\angle BAD = 90°$$

Given:

$$\to \angle ADE = 90°$$

I.29,I.Post.4,I.34

Now, consider the parallelogram $ADEB$.

Similarly, using the property of parallelograms, we can show that $\angle ABE$ and $\angle BED$ are also right angles.

Conclusion: Since $ADEB$ is both equilateral and right-angled, it is a square, and it is described on the straight line $\overline{AB}$.

$\therefore$ the square on the given $\overline{AB}$ is successfully constructed and proven. I.Def.22

Q.E.F

Theorem 1.1.2.8

In a right-angled triangle, the square of the length of the side opposite the right angle is equal to the sum of the squares of the lengths of the other two sides.

Proof.

Lemma 1.14. *The square described on the side of a right triangle opposite the right angle is equal to the sum of the squares described on the other two sides.* I.46

Lemma 1.15. *If a straight line falling on two straight lines makes the alternate angles equal to one another, the straight lines are parallel to one another.* I.31

Lemma 1.16. *A straight line can be drawn from any point to any other point.*I.Post.1

Lemma 1.17. *Parallels are lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.*I.Def.22,I.14

Construction 1.1.2.7

1. Describe the square $BDEC$ on side $\overline{BC}$.
2. Describe the squares $GFBA$ and $HACK$ on sides $\overline{BA}$ and $\overline{AC}$
3. Draw $\overline{AL}$ through A parallel to either $\overline{BD}$ or$\overline{CE}$.
4. Join $\overline{AD}$ and $\overline{FC}$. I.Def.23

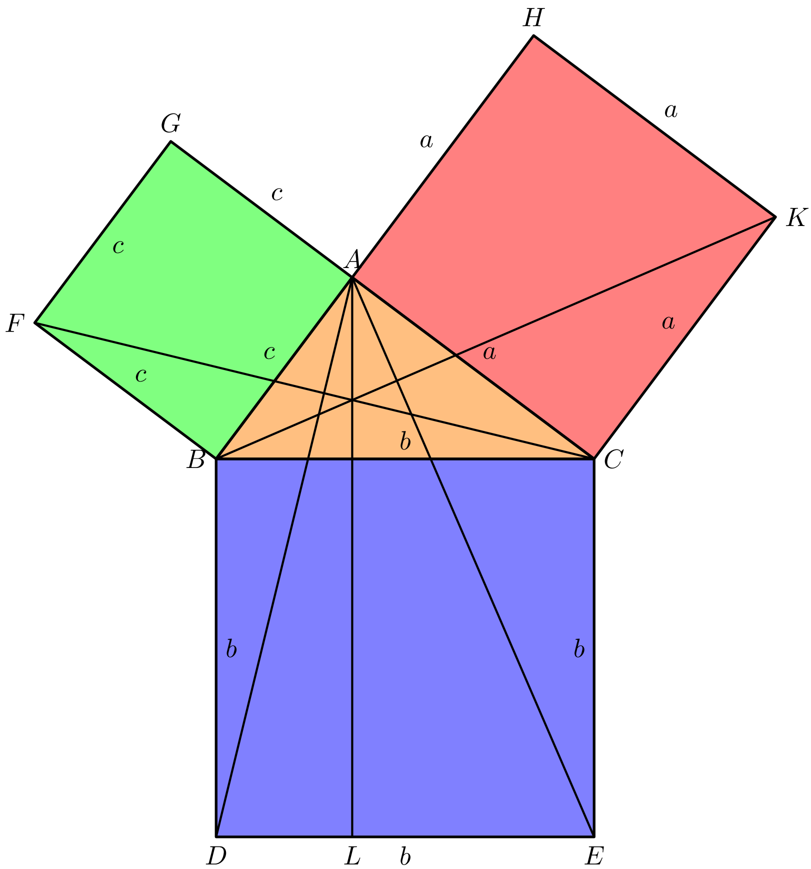


Figure 1.102

- In parallelograms, the opposite sides and angles are equal.hfillI.47
- If a straight line intersects two other straight lines, and if the interior angles on the same side of the intersecting line are supplementary, then the two straight lines are parallel to each other.hfillI.14
- Parallels are lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.hfillI.Def.22
- If a straight line falling on two straight lines makes the exterior angle equal

to the interior and opposite angle on the same side, the straight lines are parallel to one another.hfillI.Post.42

- If equals are added to equals, the wholes are equal.hfillC.N.2

  -
$$\angle DBC = \angle FBA = 2 \times 90^\circ$$

  - add $\angle ABC$ to each, thus,
$$\angle DBA = \angle FBC$$

  - Since
$$\overline{DB} = \overline{BC}$$
and
$$\overline{FB} = \overline{BA}$$
by side-angle-side equality, triangle
$$\angle ABD = \triangle FBC$$

  - Parallelogram
$$BL = 2 \times \{triangle ABD$$
and square
$$GB = 2 \times \triangle FBC$$

  ∴ parallelogram BL equals square GB.

  - Similarly, parallelogram CL equals square HC.

– Hence, the whole square

$$BDEC = GFBS + HACK$$

C.N.2

$\therefore$ in right-angled triangles, the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Q.E.F

Theorem 1.1.2.9

If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right.

To prove:

$$\angle BAC$$

is a right angle.

Construction: Draw $\overline{AD}$ from point A perpendicular to side $\overline{AC}$. Make $\overline{AD}$ equal to $\overline{BA}$ and join $\overline{DC}$. I.11, I.3,I.Post.1

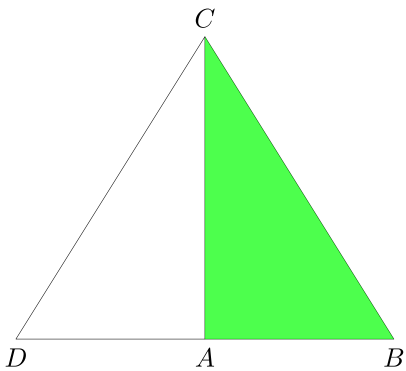


Figure 1.103

Proof. • Since $\overline{DA}$ equals $\overline{AB}$ (by construction), the square on $\overline{DA}$ equals the square on $\overline{AB}$.

Addition of Squares: Add the square on $\overline{AC}$ to both sides of the equation. Now, the sum of the squares on $\overline{DA}$ and $\overline{AC}$ equals the sum of the squares on $\overline{BA}$ and $\overline{AC}$. C.N.2

Claim: The square on $\overline{DC}$ equals the sum of the squares on $\overline{DA}$ and $\overline{AC}$.

- Since $\angle DAC$ is right (by construction), the square on $\overline{DC}$ equals the sum of the squares on $\overline{DA}$ and $\overline{AC}$ (by the Pythagorean Theorem). I.47, C.N.1

Conclusion: Combining the two claims, the square on $\overline{DC}$ equals the square on $\overline{BC}$, and $\therefore$, $\overline{DC}$ equals $\overline{BC}$.

Congruence: Since

$$\overline{DA} = \overline{AB}$$

$\overline{AC}$ is common, and

$$\overline{DC} = \overline{BC}$$

and

$$\widehat{DAC} \cong \widehat{BAC}$$

Conclusion: Since $\angle DAC$ is a right angle, $\angle BAC$ is also a right angle (corresponding parts of congruent triangles are equal).

$\therefore$ If in a triangle, the square on one side equals the sum of the squares on the remaining two sides, then the angle contained by the remaining two sides is right.

Q.E.F