From Lines to Logic:

Unraveling Euclid’s Elements

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"At the age of eleven, I began Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined there was anything so delicious in the world."

—Bertrand Russell

Inasmuch as many things, while appearing to rest on truth and to follow from scientific principles, really tend to lead one astray from the principles and deceive the more superficial minds, he has handed down methods for the discriminative understanding of these things as well, by the use of which methods we shall be able to give beginners in this study practice in the discovery of paralogisms and to avoid being misled. This treatise, by which he puts this machinery in our hands, he entitled (the book) of Pseudaria, enumerating in order their various kinds, exercising our intelligence in each case by theorems of all sorts, setting the true side by side with the false, and combining the refutation of error with practical illustration. This book then is by way of cathartic and exercise, while the Elements contain the irrefragable and complete guide to the actual scientific investigation of the subjects of geometry. Proclus (ca. 335 BC)

Chapter

1

Common notions

Continuing the journey through Euclid's Elements, one encounters the common notions—the axioms of logic upon which the entire framework of Euclidean geometry is erected. Unlike the postulates, which pertain specifically to the properties of geometric figures, the common notions are universal principles of logical inference that apply across all domains of mathematical reasoning.

Euclid enumerates a set of three common notions in Book 1, ranging from the principle of reflexivity ("Things which are equal to the same thing are also equal to one another") to the transitive property of equality ("If equals be added to equals, the wholes are equal"). These axioms may appear deceptively simple, yet they form the cornerstone of deductive reasoning, guiding the reader through a labyrinth of logical deductions with clarity and rigor.

The significance of these common notions lies in their universality—they transcend the confines of geometry, applying to all branches of mathematical inquiry. As Bertrand Russell, the eminent philosopher and mathematician, once remarked, "Euclid's common notions are not mere arbitrary assumptions but reflect the fundamental principles of logical inference that underpin all mathematical reasoning."

Russell (Russell)

As readers grapple with the implications of Euclid's common notions, they are encouraged to ponder the profound implications of these timeless axioms. Each common notion serves as a linchpin in the edifice of mathematical reasoning, anchoring the structure of Euclidean geometry in a sea of logical coherence.

Common notion 1.1.0.1

1. Things which are equal to the same thing are also equal to one another.
2. If equal be added to equals, the whole are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which conicide with one another are equal to one another.
5. The whole is greater than the part.

The common notions in Euclid's *Elements* intriguingly appear not at the outset but within the third chapter of Book I. This placement, rather than an oversight, seems a deliberate pedagogical strategy by Euclid. It ensures that readers first establish a solid understanding of specific geometric principles before grappling with these more general axioms that underpin broader logical reasoning. Such a foundation is crucial, especially considering the general applicability of these notions beyond mere geometric confines.

In historical reevaluations, some mathematicians advocate for a restructured axiomatic framework where only the first three of Euclid's original postulates are retained, expanding the list of common notions to eleven, incorporating the traditional fourth and fifth postulates. This expansion is not merely a matter of numerical reassignment but reflects a philosophical shift towards establishing a more robust logical foundation for geometry. This shift is particularly resonant in eras marked by a rigorous reexamination of mathematical proofs, such as during the Enlightenment.

John Playfair's[1] contributions exemplify this shift. In his *Elements of Geometry*, Playfair not only rephrases but repositions Euclid's postulates to highlight their universal applicability, thus aligning with Enlightenment ideals of rationalism and universality. His famous restatement of the parallel postulate—now known as Playfair's axiom, "Through a given point not on a given line, at most one line can be drawn parallel to the given line"—is celebrated for its clarity and theoretical elegance. By integrating these postulates into a broader axiomatic system, Playfair was making a profound pedagogical and philosophical statement about the nature of geometric truths. [2]

[1] ohn Playfair was a Scottish mathematician and geologist, best known for his reformulation of Euclid's Elements and his clear exposition of James Hutton's geological theories. He made significant contributions to both mathematics and the natural sciences in the late 18th and early 19th centuries.

[2] For more about Playfairs' axiums see appendix 2

Common notion 1.1.0.2

῞Α ἴσα τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστίν.

Things equal to the same thing are also equal to one another.

Here is the original Greek as written by Euclid: ῞Α ἴσα τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστίν. Below is the literal translation:

- ῞Α ἴσα (τα ισα): Things equal
- τῷ αὐτῷ (τōι αυτōι): to the same thing
- ἴσα (ισα): [are] equal
- καὶ (και): also
- ἀλλήλοις (αλλēλοις): to one another
- ἐστίν (εστιν): are

This literal translation from the original Greek, written over two millennia ago, underscores the linguistic and logical precision of Euclid's thought. Ancient Greek syntax often omits the verb "to be" in such assertions, relying on the reader's inference from context[3]. Despite such linguistic economy, the clarity of Euclid's notions endures, proving foundational to logical reasoning in geometry.

[3] For a deeper dive into the translation nuances and broader implications of the Common Notions, refer to Appendix 3.

Composed in the thriving intellectual climate of Hellenistic Alexandria, Euclid's work synthesizes and systematizes earlier mathematical theories. This environment stimulated the cross-pollination of ideas and nurtured the development of tools not only for geometry but also for a structured approach to scientific inquiry.[4] The applicability of C.N.1 extends beyond geometry, underpinning algebraic operations and supporting arguments in calculus. For instance, when proving the equality of two expressions in algebra, we frequently invoke this principle to justify equivalence derived from a common equality.

[4] See the recommended readings for detailed discussions on Hellenistic influences on Euclidean geometry.

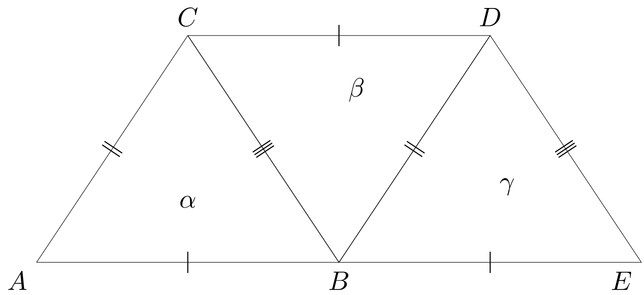


Figure 1.1: This diagram demonstrates that if $\alpha \cong \beta$ and Triangle $\beta \cong \gamma$, then α must $\cong \gamma$ according to Common Notion 1.

Euclid's Elements has had a profound influence on mathematics and has been applied directly in modern geometry, demonstrating its enduring relevance. His logical constructs and axiomatic methodology have been foundational not only in mathematics but also in shaping methodologies across scientific and philosophical inquiries.

Aristotle emphasized that axioms are self-evident truths that cannot be demonstrated. The use of the term 'things' in this axiom is pivotal, as it broadens its applicability across various fields, representing space, numbers, time, and speed depending on the discipline. The principle articulated in C.N.1 has been a fundamental aspect of mathematical proofs throughout history. For example, it is integral to arguments in algebra and calculus, where it underlies operations involving equalities and equivalences. Its logical simplicity makes it a crucial tool not only for theoretical mathematics but also for practical applications in engineering and physics. The notion that 'things equal to the same thing are also equal to one another' enables a transition from intuitive geometric truths to formal algebraic statements, bridging ancient methodologies with modern mathematical practice.

Each Common Notion serves as a cornerstone of logical reasoning in geometry, establishing foundational principles for complex propositions. This method exemplifies the precision and discipline of classical deductive reasoning.

Common notion 1.1.0.3

If equal be added to equals, the whole are equal.

Common notion 1.1.0.4

If equals be subtracted from equals, the remainders are equal.

In our exploration of Euclid's *Elements*, we've already delved into the subtleties of *Common Notion 1* (C.N.1) and now we shall look at *C.N.2 and C.N.3*;

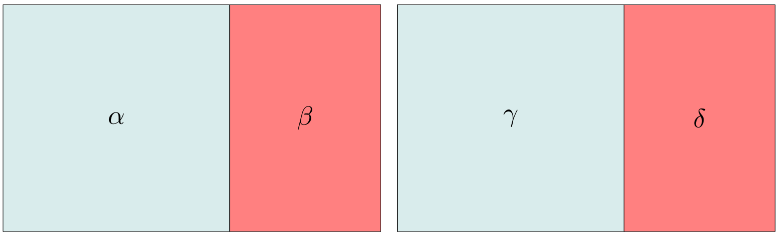


Figure 1.2: Illustration of Common Notion 2

a: $\alpha + \beta$: Addition of areas.
b: $\gamma + \delta$: Addition of areas.

$$\alpha = \gamma$$
$$\beta = \delta$$
$$\therefore \alpha + \beta = \gamma + \delta$$

We can use the same reasoning for *Common Notion 3*.

Historical studies suggest that later mathematicians sought to expand upon Euclid's original set of axioms, proposing additional ones that might seem intuitive but were not explicitly stated by Euclid. Here is a brief look at these proposed axioms:

a If equals be added to unequals, the wholes are unequal.

b If equals be subtracted from unequals, the remainders are unequal.

c Things which are double of the same thing are equal to one another.

d Things which are halves of the same thing are equal to one another.

e If unequals be added to equals, the difference between the wholes is equal to the difference between the added parts.

f If equals be added to unequals, the difference between the wholes is equal to the difference between the original unequals.

Upon close inspection, it becomes clear why Euclid may have chosen to omit these from his Common Notions. The language of 'unequals' used in propositions (a) and (b) introduces a level of ambiguity absent in Euclid's precise formulations. For instance, Euclid's Proposition I.17 relies on C.N.1 to argue that when an angle is added to both a greater and a lesser angle, the resulting sums reflect their original relationships—greater and lesser, respectively. The additional axioms proposed above could imply contradictory outcomes, thus they are not suitable for inclusion.

As for (c) and (d), their redundancy might be the reason behind their exclusion. These axioms assert equality relations among multiples and fractions of the same quantity, something that would be inherently understood within the framework of Euclid's other axioms and propositions. Essentially, if two quantities are equal to a third, they are equal to each other, a principle already covered under other axioms.

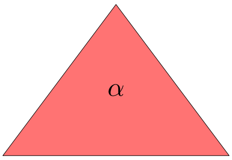
Lastly, (e) and (f) were analyzed by the philosopher Proclus, who suggested that these follow logically from existing axioms, making their explicit mention unnecessary. Proclus' commentary supports the notion that Euclid's original axioms were sufficiently comprehensive to derive further logical conclusions without additional elaboration.

In conclusion, while these additional axioms offer interesting insights into logical extensions, Euclid's original choices reflect a preference for essential, universally applicable principles that elegantly support the structure of his geometric proofs.

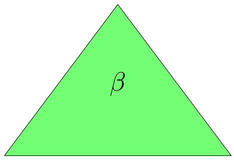
Common notion 1.1.0.5

Τὰ ἐφαρμόζοντα ἀλλήλοις ἴσα ἐστίν. (Ta epharmozonta allēlois isa estin)

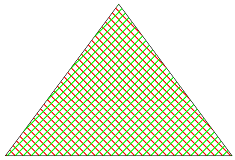
Things which coincide with one another are equal to one another.



(a) Triangle α



(b) Triangle β



c: Superimposition showing α coincides exactly with β

Figure 1.3: Illustration of Common Notion 4: Geometric coincidence implies equality.

As α fits exactly on top of β without any overlap, this demonstrates that $\alpha \cong \beta$, showing geometric congruence as per Common Notion 4.

Linguistic Analysis: The term εφαρμόζω (epharmozō), meaning "to fit exactly" or "to coincide," is crucial in understanding the precision of Euclidean geometry. This term is used in different voices to express variations in meaning:

1. **Passive Voice:** εφαρμόζεσθαι (epharmozesthai) - "to be applied to," suggesting a possible but not guaranteed exact fit.

2. **Active Voice:** εφαρμόζω (epharmozō) - implies a perfect alignment without overhang or shortfall.

Furthermore, the prepositions and cases used with εφαρμόζω affect its interpretation:

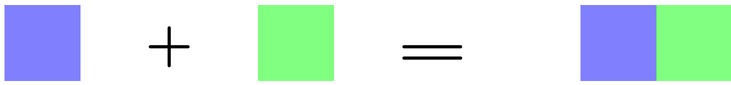
- **With πρός (pros) and Accusative:** Indicates dynamic motion towards alignment.

- **With Dative:** Implies a static, existing state of application, common in the writings of Pappus.

These subtleties enhance our understanding of Euclid's methods and the precise nature of geometric proofs in the *Elements*.

Common notion 1.1.0.6

The whole is greater than the part.



(a) Part (b) Part (c) Whole

Figure 1.4: Visualization of Euclid's Common Notion 5

At first glance, the statement that the whole is greater than the part might seem straightforward, almost too self-evident to warrant explicit mention. However, this simplicity is precisely why it is foundational not only in axiomatic geometry and logical proofs but also in numerous other fields beyond pure mathematics.

Consider biology, for instance. The approach to studying the cells that make up the brain—simple firings of synapses—is markedly different from studying the brain itself, capable of language, imagination, and dreams. Zooming out further, the entire body emerges as a system capable of turning those dreams into reality. This illustrates how the properties of the whole cannot be deduced merely by examining its parts.

Similarly, in systems theory, this principle finds its echo in the idea that a system can exhibit properties and behaviors that its individual components do not possess. This notion is crucial in understanding complex systems in both natural and artificial contexts.

Philosophically, this principle touches on concepts of unity and integrity, suggesting a way of viewing objects and entities not merely as assemblies of parts but as cohesive wholes. This perspective is also vital in education, where understanding individual elements of a subject is essential for grasping the discipline as a whole.

Although Proclus includes this axiom using the same argument as for the previous, but others disagree. Tannery for example objected due to the language used in I.6 namely 'the triangle DBC will be equal to the triangle ACB, the less to the greater:

which is absurd.', and no other direct or even indirect references seem to be in book I. So the liklyhood is that it was added later despite Proclus' inclusion.