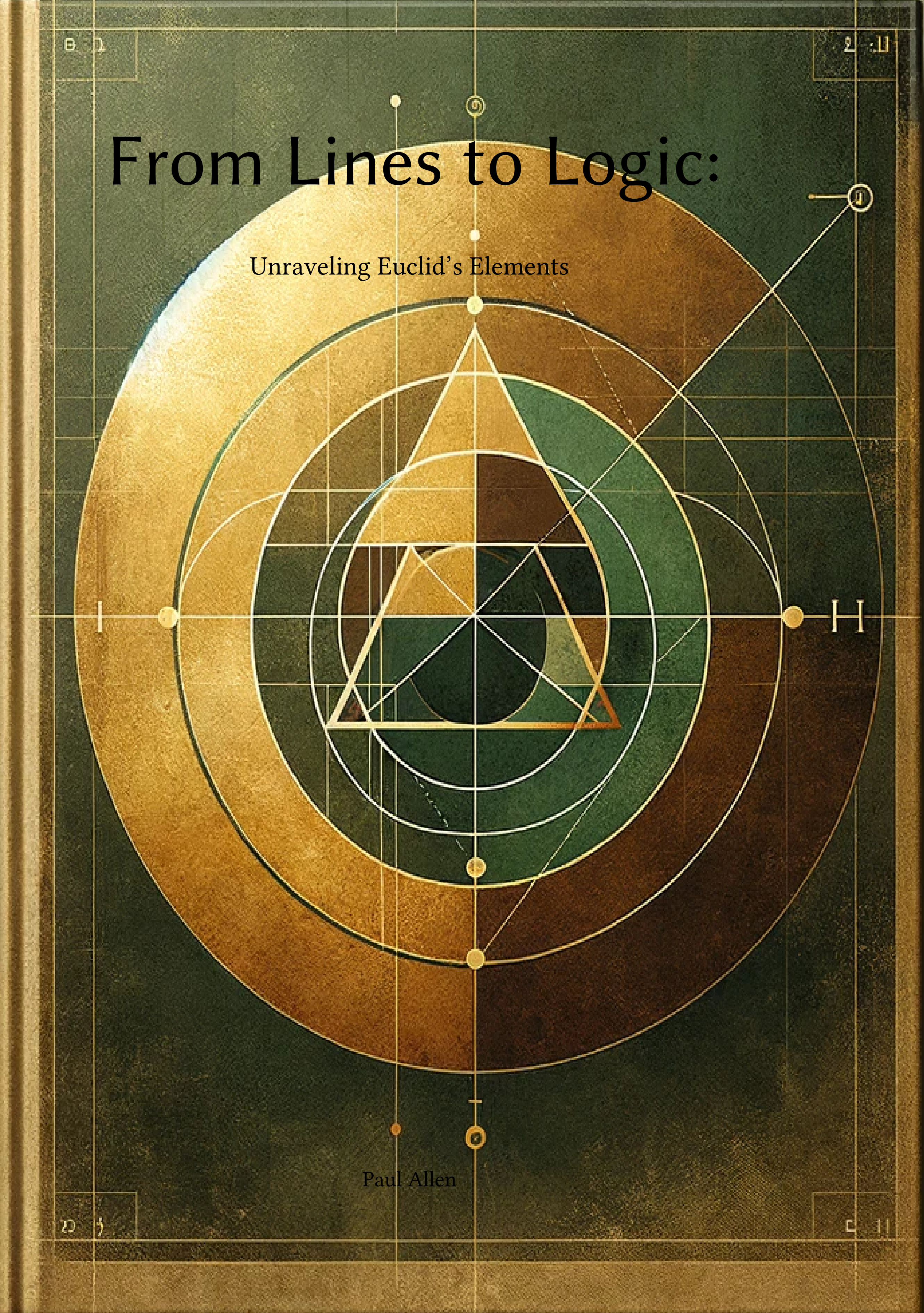
From Lines to Logic:

Unraveling Euclid's Elements



Paul Allen

"At the age of eleven, I began Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined there was anything so delicious in the world."

—Bertrand Russell

Chapter

1

Definitions

1.1

Definition 1.1.1.1

Σημεῖόν ἐστιν ὅ μέρος οὐκ ἔχει.

"A point is that which has no part."

A point, as defined by Euclid, is an entity without any dimensionality; it is the most basic element in geometry. In Euclidean geometry, points are fundamental building blocks from which all other geometric figures are constructed. Conceptually, a point has no size, no shape, and no extent,that is, it has neither length, breadth, nor thickness ; it is simply a precise location in space. **Casey (1885)**

The significance of Euclid's definition of a point lies in its foundational role in geometry. Points serve as the starting point for defining other geometric objects, such as lines, circles, and polygons. Without points, geometric reasoning and construction would be impossible.

Euclid's definition of a point has stood the test of time, forming the basis of classical geometry for over two millennia. Mathematicians and geometers have recognized the elegance and simplicity of this definition, which encapsulates the essence of spatial location without unnecessary complexity.

Many mathematicians and philosophers throughout history have refered to or acknowledged Euclid's definition of a point. For example, Rene Descartes, in his development of Cartesian coordinates, relied on the concept of points as fundamental entities in space. Additionally, mathematicians such as David Hilbert and Bertrand Russell have discussed the foundational importance of points in geometry and set theory, further reinforcing the enduring significance of Euclid's definition.

In modern mathematics, the concept of a point extends beyond Euclidean geometry to various mathematical contexts, including abstract algebra, topology, and analysis. Despite these advancements, Euclid's definition of a point remains a cornerstone of geometric reasoning and continues to inspire mathematical exploration and discovery.

1.2

Definition 1.1.2.1

Γραμμή ἐστι μῆκος ἀπλατὸν.

"A line is breadthless length."

Euclid continues to present his mathematical concepts and proofs in a rigorous and systematic manner. In the context of Definition 2 of Book 1, Euclid intended "straight line segment" to represent the shortest distance between two points in a plane. In Euclidean geometry, a straight line is defined as the path traced by a moving point that remains consistently equidistant from two fixed points. Euclid's definition was clear in its intent to describe this fundamental geometric concept, and it served as the basis for subsequent mathematical developments.

[1]

Figure 1.1: A line

[1] An infinite straight line segment.

While Euclid's geometry primarily deals with straight lines, curves and other geometric figures were considered in later mathematical developments. However, within the scope of Euclid's Elements, the focus is on straight lines and their properties, including being the shortest distance between two points. Any ambiguity regarding the definition of lines as curves is more of a modern consideration, as Euclid's approach was firmly rooted in the concept of straightness.

1.3

Definition 1.1.3.1

Οἱ τῶν γραμμῶν τετμημένοι συμπίπτουσιν ἐν τοῖς ἄκροις αὐτῶν σημείοις.

"The intersections of lines and their extremities are points."

This definition suggests that there's a relationship between certain lines and points, implying that a point can be considered an endpoint of a line. However, it doesn't provide a clear definition of what exactly "ends" are, nor does it specify how many ends a line can have. For example, while the circumference of a circle has no ends, a finite line segment has two distinct endpoints.

a: Points teminating a line segment

b: point on an intersection of two line segments

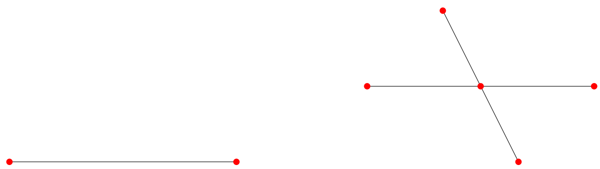


Figure 1.2: Points and intersections

1.4

Definition 1.1.4.1

Γραμμὴ δὲ ἡ μεσαία ἐνδιάμεσον ἐῶσα τὰ ἄκρα ἔχει, καλεῖται εὐθεία ἢ ὀρθὴ γραμμή, οἷον ἡ AB.

A line which lies evenly between its extreme points is called a straight or right line, such as $\overline{AB}$.

This was not as easy as it seams for Euclid to define, The truth is that Euclid was attempting the impossible. As Pfleiderer says (Scholia to Euclid), " It seems as though the notion of a straight line, owing to its simplicity, cannot be explained by any regular definition which does not introduce words already containing in themselves, by implication, the notion to be defined (such e.g. are direction, equality, uniformity or evenness of position, unswerving course), and as though it were impossible, if a person does not already know what the term straight here means, to teach it to him unless by putting before him in some way a picture or a drawing of it." **Heath (1956)**

If a point moves without changing its direction it will describe a straight (or right line). The direction in which a point moves in called its "sense." If the moving point continually changes its direction it will describe a curve; hence it follows that only one right line can be drawn between two points.

[2]

Figure 1.3: A straight line

[2] A finite stright line with distictive ends.

"If we suspend a weight by a string, the string becomes stretched, and we say it is straight, by which we mean to express that it has assumed a peculiar definite shape. If we mentally abstract from this string all thickness, we obtain the notion of the simplest of all lines, which we call a straight line." **Dodgson (1885)**

The straight line is foundational to geometry, serving as a fundamental object upon which many geometric concepts and constructions are based. Euclid's treatment of straight lines in his Elements primarily revolves around Book I, where he lays down the foundational definitions and postulates.

However, the simplicity of this definition belies the richness of the concept of a straight line. Throughout history, mathematicians have grappled with the notion of straightness, leading to various interpretations and misunderstandings.

One notable example is the parallel postulate, which states that given a line and a point not on that line, there exists exactly one line parallel to the given line through the given point. Euclid included this postulate as one of his five postulates, but its uniqueness and seemingly independent nature led mathematicians to question its validity and explore alternatives.

In the 19th century, mathematicians such as Nikolai Lobachevsky and János Bolyai challenged the assumption of Euclidean geometry by developing non-Euclidean geometries where the parallel postulate does not hold true. Their work paved the way for the development of hyperbolic geometry, where straight lines behave differently than in Euclidean geometry.

Later, in the early 20th century, Albert Einstein's theory of general relativity further revolutionized our understanding of straight lines. In the context of curved spacetime, straight lines represent the paths that objects follow in the presence of gravitational fields. These "geodesics" are not necessarily Euclidean straight lines but rather the shortest paths between points in curved spacetime.

In summary, while Euclid's definition of a straight line laid the groundwork for classical geometry, subsequent mathematicians have expanded and challenged this concept, leading to new understandings and interpretations in both Euclidean and non-Euclidean geometries, as well as in the context of modern physics.

1.5

Definition 1.1.5.1

Ἐπιφάνεια δὲ ἡ ἔχουσα μῆκος καὶ πλάτος.

surface it that which has length and breadth only.

Surface, In geometry, a two-dimensional collection of points (flat surface), a three-dimensional collection of points whose cross section is a curve (curved surface), or the boundary of any three-dimensional solid. In general, a surface is a continuous boundary dividing a three-dimensional space into two regions. For example, the surface of a sphere separates the interior from the exterior; a horizontal plane separates the half-plane above it from the half-plane below. Surfaces are often called by the names of the regions they enclose, but a surface is essentially two-dimensional and has an area, while the region it encloses is three-dimensional and has a volume. The attributes of surfaces, and in particular the idea of curvature, are investigated in differential geometry.

1.6

Definition 1.1.6.1

Τὰ πέρατα ἐπιφανείας γραμμαί εἰσι.

The extremities of a surface are lines.

Euclid, with the precision of a master craftsman, posits that the extremities of surfaces are, in essence, lines. This assertion, devoid of any flourish, cuts to the heart of geometry, establishing a foundational understanding from which complex structures are elegantly constructed. It is a statement that resonates with the clarity of a bell, inviting us to perceive the world through the lens of geometric truths.

Todhunter[3],in his role as a guide and mentor, would perhaps encourage us to visualize a vast, unbroken expanse—a canvas upon which the drama of geometry unfolds. This expanse, he might suggest, is akin to the surface of a tranquil sea, its limit defined not by the horizon but by the precise, mathematical lines that tether it to reality. These lines, invisible yet undeniable, mark the transition from the abstract to the tangible, framing our understanding of space itself.

[3] Isaac Todhunter, was a 19th century British mathematician who is renowned for, his comprehensive textbooks on subjects including, mathematics and his contributions to the fields of calculus and mathematical physics.

In this dialogue between Euclid and Todhunter, we are invited to navigate the realms of geometry with a sense of purpose and inquiry. Definition 6, in its elegant simplicity, serves as a beacon, illuminating the path toward a deeper comprehension of geometric principles. It underscores the importance of clear definitions, ensuring that each step taken on this intellectual journey is grounded in a shared understanding of foundational concepts.

1.7

Definition 1.1.7.1

Ὅταν ἡ ἐπιφάνεια ὀρθὴ γραμμὴ ἤ τις δύναμις συνάπτουσα δύο ἑκάστους τυχόντας σημείους ἐν αὐτῇ ἐν τῇ ἐπιφανείᾳ πᾶν ὅλον ἔχει, καλεῖται ἐπίπεδον.
When a surface is such that the right line joining any two arbitrary points in it lies wholly in the surface, it is called a plane.

In the context of Euclidean geometry, a plane is a fundamental concept that forms the basis for two-dimensional space. While various definitions exist, they collectively describe a plane as a flat, infinite surface devoid of thickness or curvature.

Euclid's Elements introduces a plane as a surface understood to possess two dimensions, denoted as length and breadth. However, these dimensions are not explicitly defined, leaving room for interpretation. Moreover, subsequent definitions within the Elements illustrate that a plane need not necessarily be flat, encompassing surfaces such as cones, cylinders, and spheres.

Expanding upon Euclid's foundation, modern mathematics defines a Euclidean plane as a geometric space of dimension two, symbolized as $\mathbb{E}^2$ [4] In this space, each point is determined by a pair of real numbers, providing a coordinate system to locate points on the plane. It is an affine space, meaning it includes parallel lines, and possesses metrical properties derived from a defined distance metric, allowing for the measurement of angles and the definition of circles.

[4] Represents the expected value of a random variable, or Euclidean space, or a field in a tower of fields, or the Eudoxus reals.

In geometric terms, a plane extends infinitely in all directions, with zero thickness and zero curvature. It is challenging to visualize a plane in real-life scenarios, but examples include the flat surfaces of cubes, cuboids, or sheets of paper. The position of any point on a plane can be specified using an ordered pair of coordinates, indicating its precise location relative to the origin or any chosen reference point.

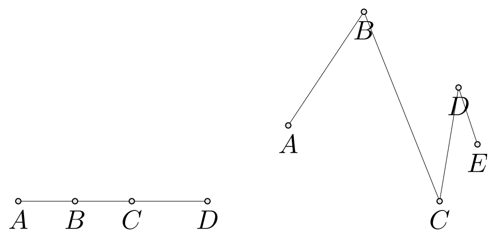
Colinear points

Points which lie on the same right line are called collinear points. A figure formed of collinear points is called a row of points

This statement is fundamental in geometry and pertains to the concept of collinearity. Let's break it down:

- Collinear Points: These are points that lie on the same straight line. In other words, if you were to draw a straight line, any points that you place on that line are collinear points. Collinear points share a common line of direction.

- Row of Points: A figure formed by collinear points is referred to as a row of points. Essentially, it's a sequence of points arranged along a straight line. It's like placing dots in a straight line; they form a row of collinear points.



a: With collinear points all the points are arranged long a straight line segment.
b: With non colinear points the points are dispersed along a line.

Figure 1.4: Coliner and non-coliner points

A straight line, by definition, extends indefinitely in both directions without curvature. $\therefore$ if points lie on the same straight line, they are collinear, regardless of whether the line is oriented horizontally, vertically, or at any angle. As long as the points can be connected by a single straight path, they are collinear. If a line is angled at some point along its path, it's still considered the same line as long as it remains straight. Any points lying on this line would still be collinear. In summary, collinear points are points that lie on the same straight line, and a row of points is formed by such collinear points. The orientation of the line, whether horizontal, vertical, or angled, doesn't affect collinearity as long as the line remains straight.

1.8

Definition 1.1.8.1

A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

To expand on Euclid's Definition VI of a plane angle from Book I of "The Elements," it's beneficial to delve deeper into its geometrical context and the historical perspective, especially considering the influence of notable mathematicians on the interpretation and understanding of Euclidean geometry.

Euclid's definition, in essence, describes a plane angle as the measure of the rotation needed to align one line with another, where both lines intersect but do not lie directly on top of one another in a straight path. This definition implicitly involves the concept of the amount of turn between the two lines, rather than the length of the lines or the distance between them.

A plane angle is fundamental to geometry because it allows for the measurement of the "opening" between two lines. This measurement is independent of the lengths of the intersecting lines but depends solely on how much one line must be rotated around the point of intersection to coincide with the other line. Angles are usually measured in degrees or radians, which quantitatively express the size of an angle.

Notable mathematician Sir Roger Penrose[5] has contributed extensively to the understanding and application of geometry in both mathematics and physics. In his work, Penrose often explores the foundational aspects of geometry and its implications in modern physics. While Penrose's work is more focused on the implications of geometry in theoretical physics and cosmology, his explorations of space, time, and the universe's geometry provide a deeper understanding of the principles that Euclid laid out millennia ago.

5 Roger Penrose is a British mathematical physicist, mathematician, and philosopher of science, awarded the Nobel Prize in Physics in 2020 for his work on black hole formation, contributing significantly to the fields of general relativity and cosmology.

[6] David Hilbert was a German mathematician, recognized as one of the most influential and universal mathematicians of the 19th and early 20th centuries, known for his foundational contributions to a variety of areas including invariant theory, algebraic number theory, and the formalization of mathematics.

Penrose, along with others like David Hilbert[6], a mathematician known for his work on the foundations of geometry, has expanded our understanding of Euclidean and non-Euclidean geometries. Hilbert's axioms, for instance, offered a more rigorous foundation for Euclidean geometry, clarifying and simplifying Euclid's original postulates and definitions.

To further understand Euclid's definition of a plane angle in the context of modern geometry, it's essential to consider the axiomatic systems that mathematicians like Hilbert and the conceptual frameworks of thinkers like Penrose have developed. These perspectives not only affirm the validity of Euclidean geometry as a mathematical model for describing space but also illuminate its limitations and the conditions under which it applies to our understanding of the physical world.

In summary, while Euclid's definition of a plane angle serves as a fundamental building block for geometry, the contributions of mathematicians like Hilbert and Penrose help us appreciate its broader implications and the evolution of geometric thought. They emphasize the importance of rigorous definitions and axioms in mathematics and the ongoing dialogue between geometry and our understanding of the universe's structure.

1.9

Definition 1.1.9.1

Ἡ προαίρεσις δύο εὐθειῶν γραμμῶν ἐκ πόντου ἑκατέρωθεν πρὸς διαφορετικὰ μέρη καλεῖται ἐυθὺ γωνία.

The inclination of two right lines extending out from one point in different directions is called a rectilineal angle.

Rectilinear angles, as defined by Euclid, refer to angles formed by straight lines.Casey 1885 This concept doesn't involve angle measurement but focuses on the angular relationship between lines.

It's important to note that in the Elements, almost all angles are rectilinear, like the illustrated angle BAC. Angles are typically named by three points, with the middle point representing the vertex of the angle. When there's no ambiguity, simply naming the angle by its vertex is sufficient, as in the example of angle A.

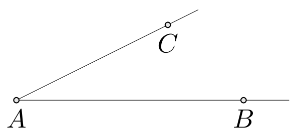


Figure 1.5: Simple angle

Language of angles

A right line drawn from the vertex and turning about it in the plane of the angle, from the position of coincidence with one leg to that of coincidence with the other, is said to turn through the angle, and the angle is the greater as the quantity of turning is the greater. Again, since the line may turn from one position to the other in either of two ways, two angles are formed by two lines drawn from a point. Thus if AB, AC be the legs, a line may turn from the position AB to the position AC in the two ways indicated by the arrows. The smaller of the angles thus formed is to be understood as the angle contained by the lines. The larger, called a re-entrant angle, seldom occurs in the "Elements."

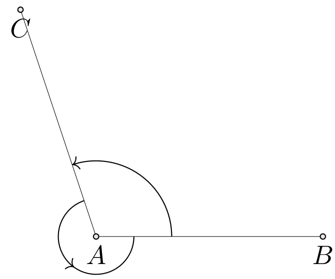


Figure 1.6: The angle

Angles as magnitudes

Regarding the treatment of angles as magnitudes by Euclid, rectilinear angles can be added together. The angle formed by joining two or more angles together is called their sum. Thus;

$$\widehat{ABC} + \widehat{PQR} = \widehat{AB'R}$$

formed by applying $\overline{QP}$ to $\overline{BC}$, so that the vertex Q shall fall on the vertex B, and $\overline{QR}$ on the opposite side of $\overline{BC}$ from $\overline{BA}$.

a: $\angle PQR$

b: $\angle ABC$

c: $\angle PQR + \angle ABC = \angle AB'R$

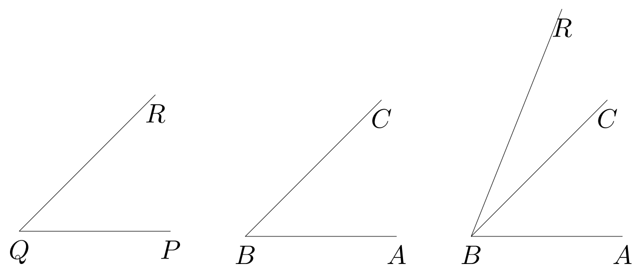


Figure 1.7: The sum of angles

However, when the sum of angles exceeds two right angles, it continues to be treated as a sum of angles rather than as an individual angle. For example, proposition I.32 demonstrates that the sum of the interior angles of a triangle equals two right angles.

It's crucial to distinguish between treating angles as magnitudes and measuring angles. In the Elements, angles themselves are the magnitudes, with measurement only done in terms of right angles, which are defined in the subsequent definition. Degree and radian measurements weren't introduced until later. In terms of degrees, a right angle is $90°$, while in radians, it's $\frac{\pi}{2}$ radians.

In ancient Greek mathematics, only positive magnitudes were considered; zero and negative magnitudes were not conceived. While this may complicate some mathematical concepts, it occasionally simplifies others. Nonetheless, the absence of zero and negative magnitudes doesn't diminish the mathematical power; any statement involving zero or negative magnitudes can be translated into one without them, albeit potentially longer and less straightforward.

In the Elements, angles are always greater than zero and less than two right angles ($180°$ or π radians), except possibly in one interpretation of proposition III.20, where the central angle of a circle could exceed two right angles.

1.10

Definition 1.1.10.1

Ορθή Γωνία

When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

In the geometry of Euclid, a right angle serves as a fundamental building block, defined with simplicity yet profound in its implications. This angle is created when a straight line stands on another straight line, making the adjacent angles equal. Such an angle is not just any angle but a right angle, marking the epitome of equality and perpendicularity in geometric relations.

[7] Although not precise enought for Euclid we wouold desribe right angle is an angle of exactly 90°, forming a perfect L shape.

[7]

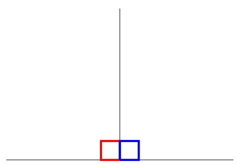


Figure 1.8: A Perpendiuular line

This definition does more than just describe; it establishes a cornerstone upon which much of Euclidean geometry rests. The right angle's properties are pivotal in constructing squares, rectangles, and understanding the principles that underpin the Pythagorean theorem. Its introduction early in Euclid's Elements is a testament to its foundational role in geometry. As we explore this concept, we see not just a definition but a gateway to understanding the spatial relationships that are central to the discipline. The clarity and precision with which Euclid delineates this term reflect a methodology that values rigor and simplicity, guiding learners from basic principles to complex constructions with logical elegance.

1.11

Definition 1.1.11.1

Οξεία Γωνία

An obtuse angle is an angle greater than a right angle.

Moving from the foundational right angle, Euclid introduces the obtuse angle—a concept that expands our geometric vocabulary and understanding. An obtuse angle is one that is greater than a right angle. This definition, though brief, opens up a new dimension of angular measurement and comparison.

[8]

[8] Again not for Euclide but, an obtuse angle is an angle greater than 90° but less than 180°.

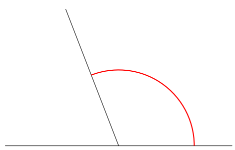


Figure 1.9: An obtuse angle

The significance of obtuse angles extends far beyond their simple definition. They play a crucial role in the classification of triangles, contributing to our understanding of the diverse geometric shapes and their properties. Obtuse angles challenge the learner to think about angles in a comparative manner, laying the groundwork for more advanced studies in trigonometry and the analysis of geometric figures.

Euclid's separate treatment of obtuse angles underscores the importance of nuanced distinctions in geometry. By differentiating between right, obtuse, and acute angles, Euclid ensures that learners grasp the full spectrum of angular possibilities, enriching their geometric comprehension and problem-solving capabilities. This approach exemplifies the educational philosophy of building knowledge step by step, ensuring a deep and lasting understanding.

1.12

Definition 1.1.12.1

Αμβλεία Γωνία

An acute angle is an angle less than a right angle.

The journey through Euclidean angles concludes with the acute angle, defined as an angle less than a right angle. This definition, while succinct, encapsulates a crucial category of angles that are omnipresent in geometric constructions and theoretical explorations.

Acute angles, with their modest measure, are indispensable in the study of triangles, polygonal forms, and the intricate relationships between geometric figures. They embody the precision and elegance of geometry, enabling the creation and analysis of a wide range of shapes and patterns. The acute angle is a testament to the diversity of geometric forms and the necessity of understanding these forms to grasp the full scope of geometric principles.

[9] once more to the layperson, an angle that is less than 90°

[9]

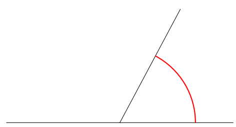


Figure 1.10: An acute angle

Euclid's separate acknowledgment of acute angles, alongside right and obtuse angles, illustrates a comprehensive approach to geometry that appreciates the variety and specificity of angular measurements. This meticulous classification enhances the learner's ability to navigate the geometric landscape, armed with a detailed understanding of angles and their significance. Through this approach, Euclid not only educates but also inspires a deeper appreciation for the beauty and logic of geometry.

1.13

Definition 1.1.13.1

"A boundary is that which is an extremity of anything."

Delving into the heart of Euclidean geometry offers a captivating journey from the simplest of concepts to the complex nature of shapes and angles. When we consider angles—be they right, acute, or obtuse—through the lens of Euclid's seminal work, "Elements," we unlock a deeper understanding and appreciation for the geometric world.

Euclid, in his wisdom, seldom approached geometry as a mere collection of measurements. For instance, while Definition XIII speaks of boundaries as the extremities of anything, it subtly lays the groundwork for all geometric exploration. It isn't directly about angles, yet it is crucial for understanding them. This definition emphasizes that geometry, at its core, is about relationships—how points connect to form lines, how lines meet to create angles, and how angles combine to shape our world.

This approach invites readers to see beyond the numbers. A right angle isn't merely $90°$; it's a cornerstone of geometric structure, creating spaces that are at once simple and infinitely complex. An obtuse angle, then, extends beyond a mere quantitative measure, challenging us to envision geometry as Euclid did: a realm where the essence of an angle is defined by its spatial harmony and discord with the figures around it.

Bringing Euclid's geometric principles to life requires us to embrace this vision, seeing angles not just as parts of geometric figures but as expressions of the fundamental properties that define our spatial reality. It's a reminder that geometry, in its truest form, is about understanding the universe's fabric, one line, and angle at a time.

This perspective transforms our approach to Euclid's "Elements," turning a study of geometry into an exploration of the world through Euclid's eyes. It's an invitation to marvel at the elegance of geometric principles and to discover the profound beauty hidden in the relationships and boundaries that shape everything from the simplest line to the most complex figures.

1.14

Definition 1.1.14.1

"A figure is that which is contained by any boundary or boundaries."

In Definition 14, Euclid extends his exploration beyond the simple existence of points and lines, venturing into the realm of plane figures. This definition, while succinct, encapsulates a profound understanding of geometrical spaces that has influenced countless mathematicians and philosophers throughout history.

Any combination of points, lines, or both that resides within a plane is classified as a **plane figure**. These figures are further categorized based on their constituents:

- **Stigmatic Figures:** Comprised entirely of points. These figures are abstract, emphasizing the concept of location without dimension.
- **Rectilineal Figures:** Formed exclusively by straight lines, showcasing geometry's inherent structure and boundary.

Mathematicians, such as David Hilbert[10] have emphasized the importance of Euclid's axiomatic approach, stating that it not only laid the groundwork for geometry but also for the axiomatic method itself. Hilbert's own work, which sought to provide a more rigorous foundation for all of mathematics, mirrors Euclid's methodological rigor, highlighting the enduring relevance of Euclidean principles.

[10] David Hilbert was a German mathematician who profoundly influenced the foundations of mathematics and geometry and is renowned for his formalization of the axiomatic system which reshaped mathematical analysis and theory.

Moreover, the classification of plane figures by Euclid offers a framework that is essential for navigating the complexity of geometric relationships. It allows mathematicians to dissect the fabric of space into comprehensible, analyzable forms.

The distinction between stigmatic and rectilineal figures, for instance, reflects a deeper philosophical inquiry into the nature of space and form, an inquiry that mathematicians like Henri Poincaré and Bernhard Riemann have further developed in their work on topology and manifold theory.

This deeper engagement with Euclid's definitions enriches our understanding of geometry as a discipline not just of measurements and calculations, but as a philosophical and logical exploration of space itself. It reminds us that the essence of geometry, as envisioned by Euclid and elaborated upon by subsequent generations of mathematicians, lies in the fundamental relationships and properties that govern the structure of our universe.

By drawing upon the insights of respected mathematicians and integrating them with Euclid's original work, we gain a more nuanced appreciation of the legacy and ongoing influence of "Elements" in the mathematical world. This approach not only honors the historical significance of Euclid's contributions but also encourages a deeper, more reflective engagement with the geometrical principles that shape our understanding of space and form.

1.15

Definition 1.1.15.1

A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;

In the pursuit of geometric clarity, Euclid presents us with a profound yet simple construction: the circle. Yet, it is within Definition 15 that we embark on a deeper exploration, delving into the subtleties of circle boundaries. A circle, as posited by Euclid, is not merely a figure but a boundary—the locus of all points equidistant from a given point, termed the center. This definition, while straightforward, belies a complex understanding of space and distance.

[11] [12] [13]

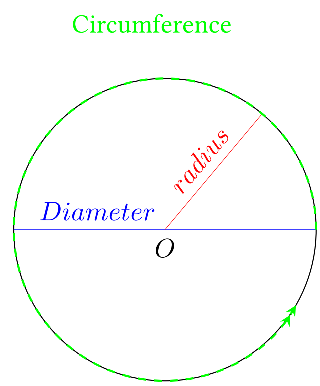


Figure 1.11: A Circle

Euclid's circle transcends its own simplicity, serving as a foundational element in the construction of geometric reality. It represents unity, perfection, and the infinite, embodying the philosophical underpinnings of geometry itself.

[11] **Radius**: The distance from the center of a circle to any point on its boundary.

[12] **Diameter**: A straight line passing through the center of a circle that connects two points on its boundary.

[13] **Circumference**: The perimeter or total distance around the outer boundary of a circle.

The circle's boundary, or circumference, becomes a powerful tool in the exploration of the geometric universe, acting as a mediator between the finite and the infinite.

The circle's inherent properties—such as the equality of radii, the significance of the diameter, and the concept of the circle as a perfect figure—reflect a deeper metaphysical order. Through Euclid's lens, the circle is not just a figure but a manifestation of geometric harmony, embodying principles that are both mathematical and philosophical.

1.16

Definition 1.1.16.1

"And the point is called the center of the circle."

Progressing to Definition 16, Euclid narrows his focus to the quintessence of circles—their centers. This pivotal concept transcends the mere identification of a fixed point within the circular boundary. It embodies the core from which all geometric properties and symmetries emanate. The center of a circle, seemingly inconspicuous, harbors a conceptual depth, serving as the crucible for the circle's harmony and equilibrium.

The elucidation of a circle's center, as meticulously presented in $Proposition III.1$, transcends a simple geometric task; it becomes a testament to the singularity and structured hierarchy within the realm of geometry. This singularity accentuates the geometric domain's precision and orderly constitution, where each figure's essence is deciphered not merely by its form but through its intrinsic attributes and interrelations.

Delving into the concept of the circle's center unveils the delicate interplay between individual components and the collective entity, echoing a recurrent theme across Euclid's expositions. It highlights that geometry, in its essence, seeks to unravel the principles orchestrating the configuration and coherence of spatial constructs.

Building upon this foundation, we discern that a circle emerges naturally as the trajectory of a point in motion, maintaining a constant separation from a stationary locus—its center. This dynamic illustrates that any given point P within the plane occupies a position relative to the circle; it is either ensconced within the circle's periphery, lies beyond its embrace, or traces its outline, contingent upon whether its distance from the center is less than, exceeds, or equals the radius, respectively. This perspective enriches our understanding, revealing that the spatial relation of points to the circle's heart governs their inclusion, exclusion, or congruence with the circle's boundary.

Casey (1885)

1.17

Definition 1.1.17.1

"A diameter of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle."

And now, Definition 17 brings our attention to the elements that define the circle's size and proportion: the radius and the diameter. These elements are not merely measurements but are fundamental to understanding the circle's geometric and symbolic significance. The radius represents the fundamental unit of measurement from the center to the boundary, symbolizing the connection between the core and the periphery.

The diameter, being twice the length of the radius, embodies the concept of duality, reflecting the balance and symmetry inherent in the circle. This balance is not just a geometric property but a philosophical ideal, mirroring the search for harmony and equilibrium in the universe.

Through these definitions, Euclid does not merely describe geometric figures but invites us on a journey through the underlying principles that shape our understanding of space. Each definition, distinct yet interconnected, weaves a narrative of geometric exploration, from the simplicity of boundaries to the depth of central points, culminating in the profound relationship between radius and diameter. This journey through Euclid's definitions reveals geometry not just as a science of measurement but as a profound reflection on the nature of reality itself.

1.18

Definition 1.1.18.1

"A semicircle is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle."

In moving towards Definition 18, we venture into the realm of the semicircle, a figure that, by its nature, embodies the dualism of geometry—being both a part and a whole, a boundary and a pathway. This delineation introduces us to a figure that is a circle halved along its diameter, yet in this division, it reveals a completeness of its own. The semicircle, straddling the domains of the finite and the infinite, serves as a poignant illustration of balance and symmetry.

Within the semicircle lies the essence of transition, where the linearity of the diameter converges with the curvature of the circle's arc, crafting a symbol of harmonic coexistence. It is here, at the juncture of the straight and the curved, that Euclid invites us to ponder the interconnectedness of geometric forms. The semicircle, thus, is not merely a segment of a circle but a standalone entity that encapsulates the principles of unity and division, offering a gateway to understanding the circle's properties through its partial representation.

The exploration of the semicircle, as presented in Euclid's geometric lexicon, extends beyond the confines of its arc and diameter, touching upon the profound interplay between space, form, and definition. It stands as a testament to the geometric axiom that even in division, there is wholeness, and in the delineation of space, there is the discovery of new dimensions of understanding.

1.19

Definition 1.1.19.1

"Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines."

a: a triangle

b: a quadrilateral, or tetragon

c: An octagon

Figure 1.12: Polygons

In Euclid's "Elements," a foundational text in the study of geometry, Definition 19 serves as a critical juncture in the exploration of geometric figures. This definition, which introduces rectilinear figures as entities enclosed by straight lines, may seem simplistic at first glance. However, it marks a significant shift in Euclidean geometry from the abstract principles of space and shape to their practical manifestations.

Euclid's approach to classification predominantly focuses on the angles of figures rather than their sides. This method is evident in his construction and naming of regular polygons in Book IV, where figures are identified by the number of their angles—such as pentagons (five-angled figures), hexagons (six-angled figures), and even pentadecagons (fifteen-angled figures). This angle-centric nomenclature is largely consistent with modern naming conventions for polygons, which also emphasize the number of angles, except in the case of "quadrilaterals," where the classification is based on sides. It's worth noting that while the names for polygons from "triangle" to "octagon" are derived from Greek, the usage of specific terms for figures with more than eight sides is uncommon in everyday practice. Additionally, the term "quadrilateral" is sometimes replaced with "tetragon," though the former is more prevalent.

Contrary to what might be implied by the straightforward definition of rectilinear figures, Definition 19 encapsulates a profound philosophical and methodological underpinning of Euclidean geometry. It represents a deliberate transition from abstract geometric concepts to their concrete counterparts, mirroring how the center of a circle acts as a fulcrum for understanding its geometric properties. The delineation of a figure by straight lines is not merely a matter of definition but a foundational principle that sets the stage for the exploration of geometric relationships and properties. In this light, Euclid's meticulous detailing of boundaries and figures underscores the rigor and systematic approach that characterizes Euclidean geometry. Every element, regardless of its apparent simplicity, is integral to the cohesive understanding of geometric principles.

This meticulous approach by Euclid, emphasizing the importance of foundational definitions and classifications, underscores the timeless relevance of "Elements" in the study of geometry. By establishing clear and precise definitions, Euclid not only facilitated a deeper understanding of geometric figures but also laid the groundwork for the systematic exploration of space and form. Definition 19, in particular, exemplifies the interplay between the abstract and the tangible, highlighting the nuanced and thoughtful methodology that defines Euclidean geometry.

1.20

Definition 1.1.20.1

"Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal."

Definition 20 classifies triangles based on their symmetries,

According to Definition 20:

A scalene triangle (C) has no symmetries. An isosceles triangle (B) has bilateral symmetry. An equilateral triangle (A) not only possesses three bilateral symmetries but also 120° rotational symmetries.

a: An equilateral triangle possesses three bilateral symmetries and also 120° rotational symmetries.

b: An isosceles triangle has bilateral symmetry.

c: A scalene triangle has no symmetries.

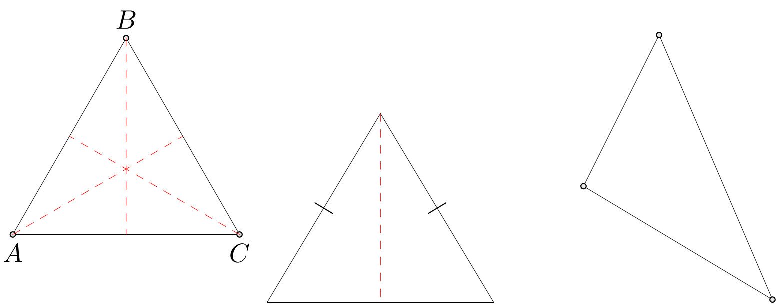


Figure 1.13: Triangles and their symmetry

It's worth noting that under this definition, an equilateral triangle is not considered an isosceles triangle. However, in Euclid's Elements, the term "isosceles triangle" is introduced in $Proposition I.5$, and later in $Books II$ and IV. The usage of "isosceles triangle" in the Elements does not exclude equilateral triangles. In modern practice, it's only necessary for at least two sides to be equal for a triangle to be classified as isosceles.

Equilateral triangles are constructed in the first proposition of the $Elements, I.1$. Additionally, an alternate characterization of isosceles triangles, namely that their base angles are equal, is demonstrated in $Propositions I.5$ and $I.6$.

1.21

Definition 1.1.21.1

"Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute."

In Euclid's Elements, Definition 21 serves as a pivotal moment where triangles are meticulously classified based on their angles, introducing readers to the rich diversity of these fundamental geometric shapes. This classification is not merely a taxonomic exercise but a profound insight into the intrinsic properties of triangles, laying the groundwork for numerous geometrical principles and propositions that follow.

Right Triangle: At the heart of this classification is the right triangle, a figure defined by the presence of a right angle, a cornerstone in Euclidean geometry. This type of triangle is emblematic of geometric rigor, embodying principles of orthogonality and symmetry. $Proposition I.17$ further illuminates this by asserting that the sum of any two angles in a triangle is less than two right angles, ensuring the uniqueness of the right angle within a triangle.

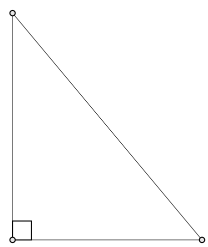


Figure 1.14: Right angled triangle
$Total angels = 2 \times 90^{\circ}$

Obtuse Triangle: The obtuse triangle, characterized by an obtuse angle, introduces the concept of angles greater than a right angle within the context of a triangle. This classification underlines a critical Euclidean theorem: a triangle cannot simultaneously house a right angle and an obtuse angle, underscoring the mutual exclusivity of these geometric figures and emphasizing the delicate balance of angles within triangles.

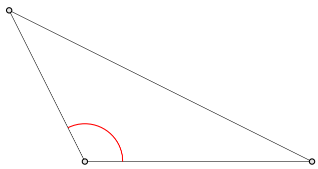


Figure 1.15: Obtuse triangle $Containsanangle > 90°$

Acute Triangle: Finally, the acute triangle, with all its angles being acute, represents the harmony and balance of smaller angles coexisting in a single shape. This classification showcases the versatility and the boundless configurations within geometric figures, highlighting Euclid's deep understanding of the interplay between angles.

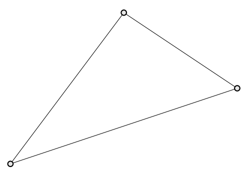


Figure 1.16: Acute triangle $Allangles < 90°$

Through these classifications, Euclid not only establishes a fundamental geometric lexicon but also sets the stage for exploring the relationships between angles and sides in triangles, a theme that permeates the Elements. This taxonomy of triangles according to their angles is not just a methodical categorization but a reflection of Euclid's broader endeavor to unveil the elegance, coherence, and profundity of the geometric world.

1.22

Definition 1.1.22.1

Περὶ τετραγώνων σχημάτων, τετράγωνος μὲν ὁ ἴσα πλευρὰς ἔχων καὶ ὀρθογώνιος, ὀρθογώνιος δὲ ὁ μὲν ὀρθογώνιος ἔχων μὴ δὲ ἴσας πλευρὰς, ῥόμβος δὲ ὁ μὲν ἴσας πλευρὰς ἔχων μὴ δὲ ὀρθογώνιος, ῥόμβοειδὴς δὲ ὁ ἀλλήλοις ἴσας ἔχων πλευρὰς καὶ γωνίας μὴ δὲ ὀρθογώνιος μηδ᾽ ἴσας πλευράς. Τραπέζιον δὲ λεγέσθω περὶ τὰς πλὴν τούτων τετραγώνους.

Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.

In this context:

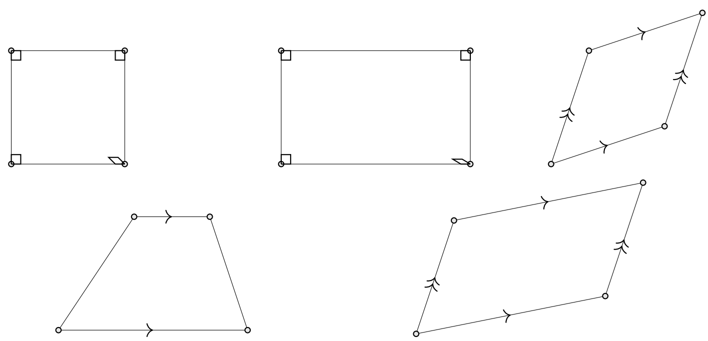
a: a square

b: an oblong, also known as a rectangle.

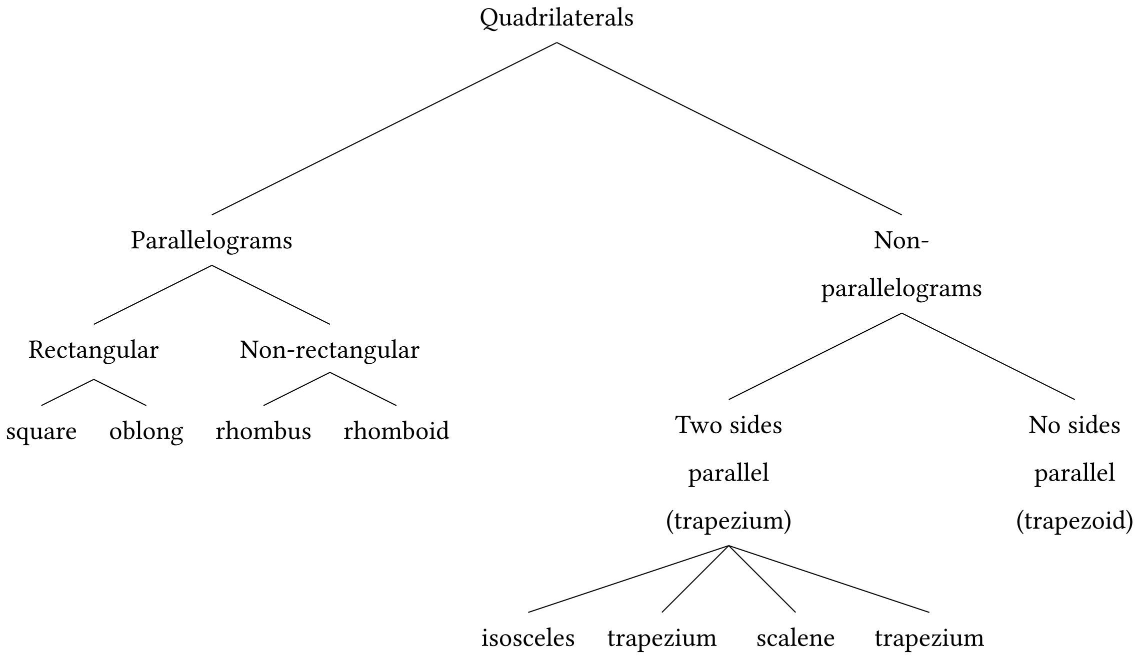
c: a rhombus.

d: a trapezium, also referred to as a trapeze or trapezoid.

e: a parallelogram, though not explicitly defined here.



Among these figures, Euclid primarily utilizes the concept of a square. The other figure names might have been common during Euclid's time, inherited from earlier versions of the Elements, or possibly introduced later.



Euclid extensively employs the concept of parallelograms or parallelogrammic areas without providing a formal definition. It's evident that he refers to quadrilaterals with parallel opposite sides, encompassing rhombi and rhomboids as special cases. Additionally, instead of "oblong," Euclid employs the term "rectangle" or "rectangular parallelogram," which encompasses both squares and oblongs.

Squares and oblongs are defined to have right angles, meaning all four angles are right angles. While these definitions may seem brief, their intended meaning can be inferred from their usage. For instance, Proposition I.46 constructs a square, ensuring that all four angles are right angles, not just one of them.

Euclid might have considered quadrilaterals as less fundamental or as variations of more basic shapes. Triangles, for example, are the simplest polygon, and all other polygons can be divided into triangles. Circles hold a unique place in geometry, being shapes of constant distance from a center point, and they were of particular interest in the mathematics and philosophy of ancient Greece. This might explain why Euclid gave more foundational importance to circles and triangles, exploring their properties in greater detail through multiple definitions.

As Euclid has not yet defined parallel lines and does not anywhere define a parallelogram, he is not in a position to make the more elaborate classification of quadrilaterals attributed by Proclus to Posidonius and appearing also in Heron's Definitions.

Definition 22, by detailing the triangle as a three-sided figure, and extending the classification to polygons with an increasing number of sides, does more than categorize; it unveils a hierarchy and methodology in approaching geometric analysis. In this context, Euclid's focus on the number of angles or sides isn't merely a classification system but a reflection of the inherent complexity and diversity within geometrical forms. This systematization reveals an elegant universe of geometry where forms are understood not just by their appearance but by their fundamental characteristics.

1.23

Definition 1.1.23.1

"Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction."

Euclid's Definition 23 subtly yet profoundly articulates the notion of parallel lines as straight lines that, lying in the same plane and being extended indefinitely in both directions, do not meet. This definition serves not merely as a description but as a foundational axiom that undergirds the geometric construct of parallelism, setting a stage for the exploration of lines, angles, and shapes that form the corpus of Euclidean geometry.

[12]

[12] Parralel lines are denoted by arrow headds along the line

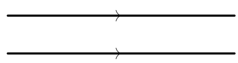


Figure 1.18: Parallel lines

The essence of this definition encapsulates a pivotal geometric principle—the condition under which two lines, though infinitely extended, are deemed parallel. It presupposes an understanding of infinity, a concept that, for the ancient Greeks, was as much philosophical as it was mathematical, imbuing the definition with layers of interpretative complexity.

This definition is inherently linked to Euclid's controversial fifth postulate, the parallel postulate, which posits that given a line and a point not on it, there is exactly one line through the point that does not intersect the original line. The intricacies of this postulate, along with attempts to derive it from Euclid's first four axioms, have sparked extensive debate and led to the development of non-Euclidean geometries, thereby highlighting the profound impact of Definition 23 on the evolution of mathematical thought.

Indeed, $Proposition I.31$ in "The Elements" offers a practical demonstration of drawing a line parallel to a given line through a specified point, thus not only affirming the conceptual validity of parallel lines within Euclidean geometry but also confirming their geometric construction and existence.

Furthermore, the distinction between lines that do not meet and those that are parallel is nuanced, as observed by Geminus. This distinction is crucial for understanding the geometric and philosophical depths of parallelism. For instance, a curve and its asymptote do not intersect yet are not parallel in the Euclidean sense, underscoring the specificity required in defining parallel lines.

Taylor (1792) Proclus, in his commentary , elaborates on the nature of parallel lines, emphasizing that parallelism is characterized not merely by the non-intersection of lines but by the equality of perpendicular distances across the lines. This clarification not only enriches our comprehension of parallel lines but also ties the concept to the practicalities of measuring distances and areas, further cementing the foundational role of parallel lines in the architecture of Euclidean geometry.

In summary, Definition 23, while seemingly straightforward, opens up a vast terrain of geometric exploration and philosophical inquiry. Its examination, enriched by the contributions of mathematicians and scholars such as Geminus and Proclus, offers profound insights into the nature of space, the concept of infinity, and the intricacies of geometric definitions. This definition, therefore, stands not only as a testament to the enduring legacy of Euclid's geometric principles but also as a beacon guiding the ongoing dialogue between mathematics and the quest to understand the universe's fundamental structures.

Chapter

2

Postulates

Postulate 2.2.0.1

Let the following be postulated: to draw a straight line from any point to any point.

Although it is a direct translation from the Greek, it would be more accurate to say 'from every point to every point', just like in *Postulate 3*, where the first words are 'with every centre and distance'.

The first idea presented here suggests that whenever we have two points, like A and B, we can always draw a straight line connecting them We denoted this as $\overline{AB}$. This construction uses a straightedge, as does another method discussed in a later postulate.

13

Figure 2.1: The line

[13] The diagram illustrates a straight line drawn from point A to point B, with both points highlighted in red.

Even though it's not directly stated, there's only one straight line possible between these two points. Euclid assumes this uniqueness as part of the *Postulate*.This implicit assumption of uniqueness underpins much of Euclidean geometry, reinforcing the foundational nature of *Postulate 1*, though it would have been clearer if he would have mentioned it explicitly.However, even Proclus agrees, based on further writings of Euclid, such as *Proposition I.4*, that there must only be one straight line from any point to any point. Proclus elaborates on this in his commentary by referencing Euclid's *Proposition I.4*, where the construction of congruent triangles presupposes the uniqueness of the line connecting two points.

In the latter part of Euclid's Elements, dealing with solid geometry, the two points mentioned in the *Postulate* can be any pair in space. *Proposition XI.1* asserts that if a part of a line lies within a plane, then the entirety of the line does as well. In the sections on plane geometry, it is implied that $\overline{AB}$, connecting points A and B, lies within the plane under discussion. This extension of principles from plane to solid geometry underscores Euclid's systematic approach, demonstrating that fundamental truths apply irrespective of the spatial context.

Postulate 2.2.0.2

To produce a finite straight line continuously in a straight line.

Again we have a translation issue with the *Postulate*, Heath has decided to use the word finite here instead of other words like limited. For example, the term 'finite', when applied to a straight line, might not adequately convey what modern mathematicians of the time termed 'rectilinear segments'—that is, a straight line defined by two extremities. **Heath (1956)**

14

Figure 2.2: Produce a straight line

[14] This diagram depicts a straight line segment CD, with points A and B marked in red on the segment, demonstrating the concept of producing a straight line from point C to D.

Here we have the second chance to use a straightedge, namely, to extend (produce) a given $\overline{AB}$ to C and D. This *Postulate* does not say how far a straight line can be extended. Sometimes it is used so that the extension equals some other straight line. Other times it is extended arbitrarily far. Heath describes this *Postulate* with reference to *Postulate 1* to emphasise this unique property of a straight line; **Heath (1956)**

"Just as Post. 1 asserting the possibility of drawing a straight line from any one point to another must be held to declare at the same time that the straight line so drawn is unique, so Post. 2 maintaining the possibility of producing a finite straight line (a " rectilinear segment") continuously in a straight line must also be held to assert that the straight line can only be produced in one way at either end, or that the produced part in either direction is unique; in other words, that two straight lines cannot have a common segment."

In Euclid's works, much is left for the reader to infer from seemingly simple statements. These texts suggest not only that we can extend a line to points arbitrarily far from one another, as long as they reside on the same straight line, but also that such a line is inherently unique, and no two straight lines can share a common segment. This distinction is crucial and must be understood from the outset, as early as *Proposition I.* It's implied that two straight lines cannot share identical segments. Proclus[2] highlights this necessity, noting that if it were not so, lines $\overline{AC}$ and $\overline{BC}$ might intersect before reaching point C, sharing a portion of their lengths. This would imply that the resulting triangle formed by these lines with $\overline{AB}$ would not be equilateral, contradicting the fundamental properties of Euclidean geometry .

[2] Proclus Diadochus (412–485 CE), often referred to simply as Proclus, was a Greek Neoplatonist philosopher and mathematician. Notable for his influential commentaries on Plato's and Euclid's works, Proclus sought to harmonize philosophical and mathematical truths, significantly impacting the medieval understanding of Euclidean geometry.

Proclus (1987)

As with the first *Postulate*, it is implicitly assumed in the books on plane geometry that when a straight line is extended, it remains in the plane of discussion. The first *Proposition* on solid geometry, *proposition XI.1*, claims that a straight line can't be only partly in a plane. The central step in the proof of that *Proposition* is to show that a straight line cannot be extended in two ways, that is, there is only one continuation of a straight line. The proof is hardly convincing. Rather, this *Postulate* should include a clause to that effect.

Postulate 2.2.0.3

To describe a circle with any center and radius.

In his Elements, Euclid outlines several methods of construction fundamental to geometry, one of which is the drawing of a circle with a compass. This construction, detailed as the third postulate, enables the creation of a circle given two points: one for the center and another on the circumference. The compass used in this method can have an arbitrarily large radius and does not maintain its radius once removed from the page, reflecting Euclid's concept of a collapsing compass.

In order to construct a circle certain characteristics are needed:

1. a point A designated as the center of the circle,
2. another point B located on the circumference of the circle,
3. a plane within which these two points exist.

[15]

Figure 2.3: Produce a straight line

[15] This diagram depicts a straight line segment CD, with points A and B marked in red on the segment, demonstrating the concept of producing a straight line from point C to D.

Euclid's description of a circle in *Definitions I.15* and *Definition I.16* is notable for its simplicity and depth: a circle is a plane figure with all radii equal, extending from the center to the circumference. This definition underpins many of the *Propositions* within Elements, requiring the geometer to employ just a collapsing compass and a straightedge to validate theorems. The notion of the collapsing compass, a tool that resets its radius each time it is lifted, is pivotal in ensuring that geometric constructions rely solely on given points and distances, reinforcing the accuracy and replicability of geometric principles.

This seemingly simple Postulate underpins much of our daily life, often taken for granted. Consider architecture: the need to draw accurate diagrams is imperative to solving force equations that building components must withstand. The use of a compass, a tool perfected by Euclid, demonstrates his rigor and the detailed thought that goes into each construction. Since the compass does not retain its set radius once lifted from the paper, accurately planning the sequence of circles and arcs drawn is crucial.

Thomas Heath provides valuable insight into the language used in the *Postulates*, highlighting a shift from the passive 'a circle can be drawn' in the original text to the more active 'to describe' in Proclus's interpretation, aligning it with the first two postulates. Interestingly, the Greeks did not have a specific term for 'radius'; instead, they described it indirectly as 'a straight line drawn from the center' . This absence of direct terminology reflects a broader flexibility in Euclid's definitions, allowing for circles of any size, from infinitesimally small to indefinitely large. This notion suggests a continuum of sizes, hinting at a boundless but finite space—a concept that would later influence other mathematical fields.

Heath (1956)

The philosophical implications of Euclidean geometry, particularly the circle, extend beyond mere mathematical interest. Shortly after Euclid, the Neoplatonist movement, spearheaded by philosophers like Plotinus,[16]began to explore geometry's metaphysical aspects. Plotinus viewed the emanation of all existence from a singular source, the One or the Good, through the lens of geometric symmetry, harmony, and proportion—ideas central to the understanding of circles and their inherent properties. Plotinus' concept of emanation, where the One is at the top, followed by the Intellect (Nous), and then the Soul (Psyche), can be metaphorically represented by concentric circles, with the One being the innermost circle. Each level of reality emanates from the One, just as circles might ripple outwards from a point. This emanation is not a diminishment but an overflowing of abundance, where each level participates in the perfection of the One in a way akin to geometric principles of symmetry and harmony.

[16] Plotinus, a philosopher and a wirter on metaphysical systems, in the 3rd century CE, was the founder of Neoplatonism, a school of thought that sought to interpret and synthesize the ideas of Plato.

By examining these elements together, from the technical aspects of Euclidean constructions to their philosophical resonance, we can gain a comprehensive understanding of the profound impact of Euclidean geometry on both mathematics and philosophy. This approach not only elucidates the geometric principles but also connects them to broader metaphysical theories, underscoring the enduring relevance of Euclid's work.

Postulate 2.2.0.4

That all right angles equal one another.

In defining a right angle, it's evident that the two angles formed at the intersection of a perpendicular line, as such:

[17] $\angle ACD \cong \angle BCD$

[17]

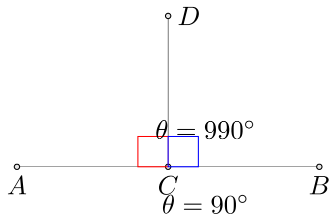


Figure 2.4: Diagram showing two right angles formed by perpendicular lines AC and BC at point C. Both angles ACD and BCD are marked to indicate they are right angles, denoted by $\theta = 90°$.

Heath (1956)

This Postulate asserts the essential truth that a right angle is a determinate magnitude , this concept is crucial. *Postulate 4* states: "All right angles are equal to one another." Euclid's reliance on such specific definitions ensures that the conclusions drawn from his axioms and postulates are logically sound and universally applicable within the framework of classical geometry. This exactness is what makes the Elements a seminal work in the logical presentation of mathematical proofs and theorems. The specificity in stating that certain angles are equal to two right angles—or any other exact angle measurement—is critical for the development of logical proofs throughout the work. This precision in describing angles is deeply connected to Euclid's foundational postulates, particularly *Postulate 4*, which establishes the equality of all right angles. By doing so, it provides a universal reference for measuring other angles.

This postulate states that an angle formed at the intersection of one perpendicular line, like $\angle ACD$, is equal to an angle formed at the intersection of any other perpendicular line, such as $\angle EGH$.

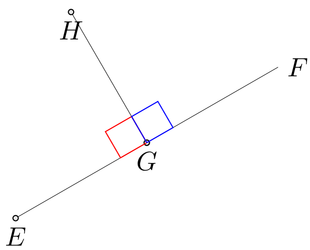


Figure 2.5: in the document we will use the notation; $A + B = 2 \times 90^{\circ}$

This kind of specific statement is a fundamental building block in geometric proofs because it leverages the established truths (axioms and postulates) to make broader conclusions. The only angle measurements considered in the Elements are in terms of right angles. For example, in *Proposition I.17*, it's demonstrated that the sum of two angles is always less than two right angles. Another instance is found in the proof of *Proposition II.9*, where two angles are proven to each be half of a right angle, hence they are congruent. Furthermore, in *Proposition III.16*, this postulate is invoked to argue that the sum of two angles cannot be less than two right angles while also being equal to two other right angles.

This approach underlines the rigor of Euclidean geometry, where every assertion, no matter how simple it seems, is tightly bound to the system's logical structure. It ensures that every geometric statement can be verified based on universally accepted truths, maintaining the integrity and internal consistency of mathematical arguments.

In a broader sense, a determinate magnitude in Euclidean geometry can refer to any precisely defined length, area, volume, or angular measure. Euclid's reliance on such specific definitions ensures that the conclusions drawn from his *Axioms* and *Postulates* are logically sound and universally applicable within the framework of classical geometry. This exactness is what makes the Elements a seminal work in the logical presentation of mathematical proofs and theorems. This is a demonstration of Euclid's use of right angles in his geometric proofs.[3]

3 For a deeper historical insight into how these principles were applied in ancient Greek mathematics and their influence on later scholars, see references such as Heath's translation of the *Elements*, which provides extensive commentary and analysis.

Postulate 2.2.0.5

That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

This postulate, less self-evident compared to Euclid's other four, has been extensively scrutinized and many have attempted to derive it from more "obvious" postulates. While ancient Greek mathematicians like Thales and Pythagoras laid foundations that contributed to Euclid's axiomatic approach, explicit efforts to address Euclid's fifth postulate are scarcely documented. Post-Euclid, the obsession to prove this fifth postulate using simpler axioms failed repeatedly, often falling into the trap of *petitio principii*[18]—assuming the truth of what it tried to establish. Ptolemy's attempts to prove Proposition I.29 without using Postulate 5 inadvertently resulted in deducing the postulate from his proof.

[18] Petitio principii, or "begging the question," is a logical fallacy where the conclusion is assumed in the premises, inadvertently using what needs to be proven as the proof itself.

The premise of this postulate is illustrated in the following diagram; if:

$$\angle ABE + \angle BED < 2 \times 90^\circ$$

then lines $\overline{AC}$ and $\overline{DF}$, when extended towards points A and D respectively, will intersect.

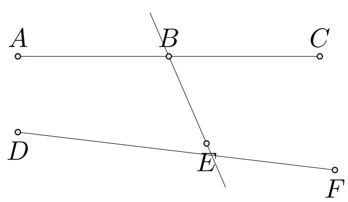


Figure 2.6: Illustration of non-parallel lines meeting

Commonly known as the "parallel postulate," it is pivotal in proving properties of parallel lines, extensively explored up to *Proposition I.31*.

If instead:

$$\angle ABE + \angle BED = 2 \times 90^\circ$$

lines are parallel, as shown below.

2

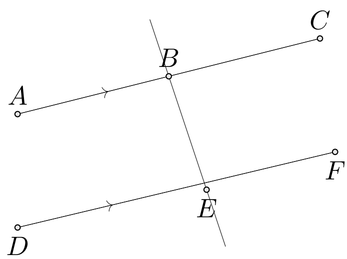


Figure 2.7: Illustration of parallel lines

2 We use $>$ to indicate the lines are parallel

In the early 19th century, mathematicians such as Bolyai, Lobachevsky, and Gauss explored non-Euclidean geometries, including hyperbolic and elliptic, which, though diverging from Euclidean principles, proved internally consistent and practically applicable.

The parallel postulate is essential for deriving the principles of Euclidean geometry, and understanding non-Euclidean geometries provides valuable insight. Although Euclid does not explicitly employ this postulate until *Proposition I.29*, the subsequent propositions heavily rely on it. Additional commentary on this postulate is provided in *Propositions I.29* and *I.30*.

Chapter

3

Propositions

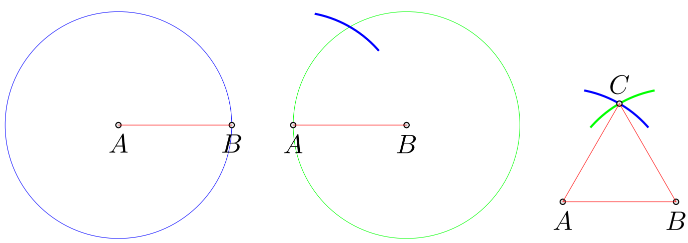
Construction 3.3.0.1

To construct an equilateral triangle on a given finite straight line $\overline{AB}$.

Let $\overline{AB}$ be the given finite straight line. *I.1*

Figure 3.1: Equilateral triangle construction

1. It is required to construct an equilateral triangle on the straight line $\overline{AB}$.
2. Describe the $\mathscr{C}(A; AB)$.
3. Then, describe the $\mathscr{C}(A; BA)$.
4. Join the straight lines $\overline{CA}$ and $\overline{CB}$ from the point C at which the circles cut one another to the points A and B. *I.Post.3, I.Post.1*



a: Step 1

b: Step 2

c: Step 3

Figure 3.2: Equilateral triangle construction

Proof. • Since the point A is the center of the $\mathscr{C}(A; AB)$, $\therefore overline AC$ equals $\overline{AB}$. Again, since the point B is the center of the $\mathscr{C}(A; BA)$ $\therefore \overline{BC}$ equals $\overline{BA}$. *I.Def.15*

- Things which equal the same thing also equal one another $\therefore \overline{AC}$ also equals$\overline{BC}$. *C.N.1*

Conclusion:

- $\therefore$, the three straight lines $\overline{AC}$, $\overline{AB}$, and $\overline{BC}$ equal one another.
- $\therefore$, $\widehat{ABC}$ is equilateral, and it has been constructed on the given finite straight line $\overline{AB}$. *I.Def.20*

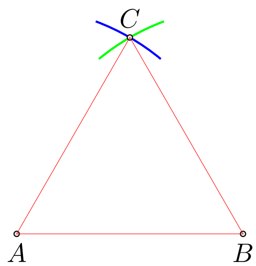


Figure 3.3: Equilateral triangle construction

The construction has been successfully completed. Q.E.F

- The construction relies on the principles of circle description and intersection, line segment joining, and the properties of circles with given centers. I.Post.1, I.Post.3, I.Def.15
- The Equilateral Triangle is characterized by the equality of its three sides ($\overline{AC}$, $\overline{AB}$, $\overline{BC}$). *I.Def.20*

Construction 3.3.0.2

To place a straight line equal to a given straight line with one end at a given point A.

Let $\overline{BC}$ be the given straight line.

Con

1. Given a point A and the given straight finite line$\overline{BC}$.

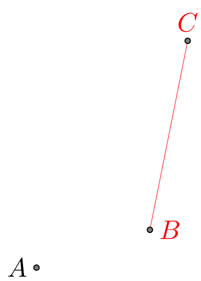


Figure 3.4: step1

2. Join the straight line $\overline{AB}$ from point A to point B, and construct the equilateral $\widehat{DAB}$ on it. Post.1, I.1

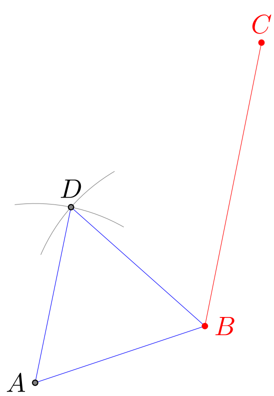
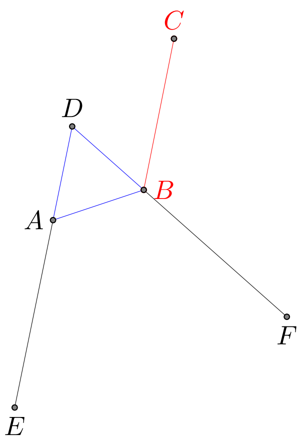


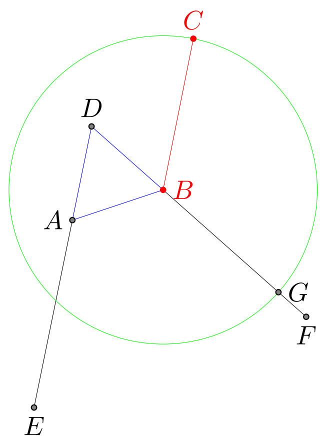
Figure 3.5: step 2

3. Produce the straight lines $\overline{AE}$ and $\overline{BF}$ in a straight line with $\overline{DA}$ and $\overline{DB}$.

4. Describe $\mathscr{C}(B; BC)$, getting point G, the intersect of this circle with $\overline{BF}$.

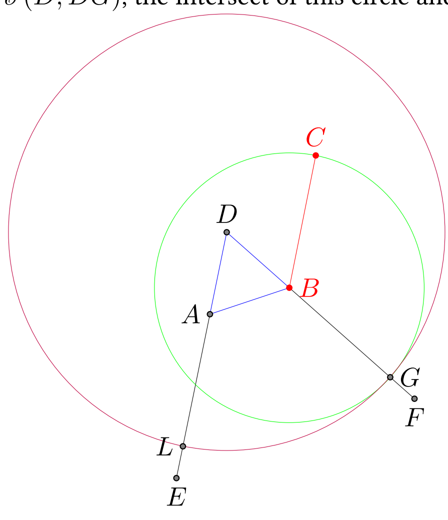


(a) Produce the line



(b) draw a circle

5. Now, describe $\mathscr{C}(D; DG)$, the intersect of this circle and $\overline{AE}$ gives point L.



(c) intersection

Figure 3.6: something

Post.2,Post.3

Proof. • Since the point B is the center of the circle with radius BC, $\therefore \overline{BC}$ equals $\overline{BG}$. Again, since the point D is the center of the circle with radius DG, $\therefore \overline{DL}$ equals $\overline{DG}$. I.Def.15

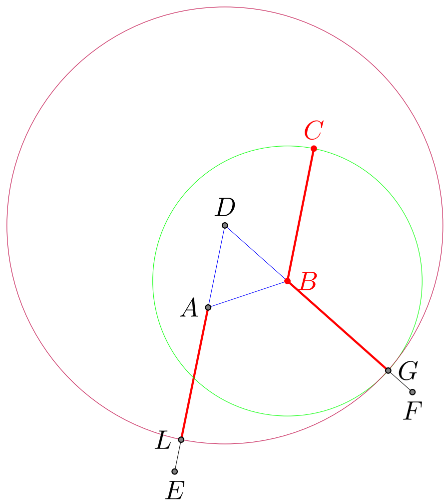


Figure 3.7: end

- In these, $\overline{DA}$ equals $\overline{DB}$, $\therefore$ the remainder AL equals the remainder BG. C.N.3

- But $\overline{BC}$ was also proved equal to $\overline{BG}$, $\therefore$ each of the straight lines $\overline{AL}$ and $\overline{BC}$ equals $\overline{BG}$. And things which equal the same thing also equal one another $\therefore \overline{AL}$ also equals $\overline{BC}$.

Conclusion:

- $\therefore$ the straight line $\overline{AL}$ equal to the given straight line $\overline{BC}$ has been placed with one end at the given point A.

- The construction has been successfully completed.

Q.E.F

Construction 3.3.0.3

To cut off from the greater of two given unequal straight lines $\overline{AB}$ and $\overline{CD}$ (with $\overline{AB}$ being the greater) a straight line equal to the less $\overline{CD}$.

Given: $\overline{AB}$ and $\overline{CD}$, where $\overline{AB}$ is greater.

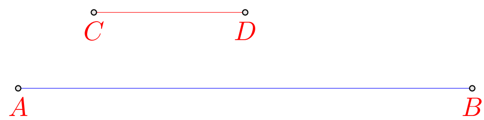


Figure 3.8: Cut off a segment from a larger segment

Construction:

1. Place $\overline{CD}$ at point A and describe $\mathscr{C}(A; AD)$. I.2, I.Post.3

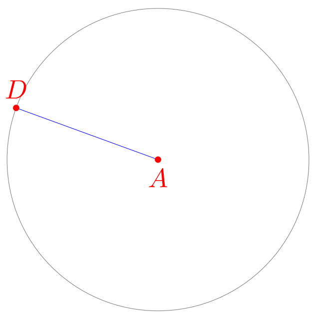


Figure 3.9: Cut off a segment from a larger segment: step 1

2. draw the segment $\overline{AB}$ and mark the intersect with $\mathscr{C}(A; AD)$, E

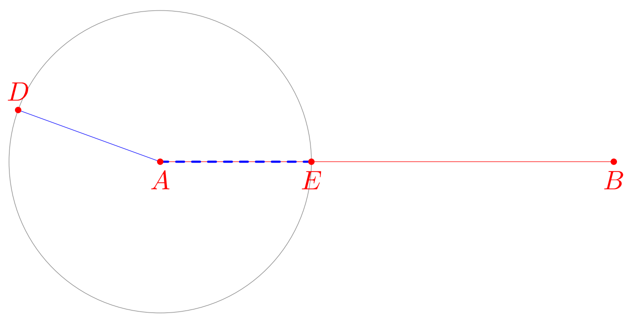


Figure 3.10: Cut off a segment from a larger segment: step 2

Proof.

- Since the point A is the center of $\mathscr{C}(A; AD)$, $\therefore$

$$\overline{AE} = \overline{AD}$$

I.Def.15

and

$$\overline{CD} = \overline{AD}$$

$\therefore$ each of the straight lines $\overline{AE}$ and $\overline{CD}$ equals $\overline{AD}$, so that

$$\overline{AE} = \overline{CD}$$

C.N.1

Conclusion:

$\therefore$ given the two straight lines $\overline{AB}$ and $\overline{CD}$, $\overline{AE}$ has been cut off from $\overline{AB}$ (the greater) equal to $\overline{CD}$ (the less).

The construction has been successfully completed.

Q.E.F

Theorem 3.3.0.1

If two triangles have two sides equal to two sides respectively and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

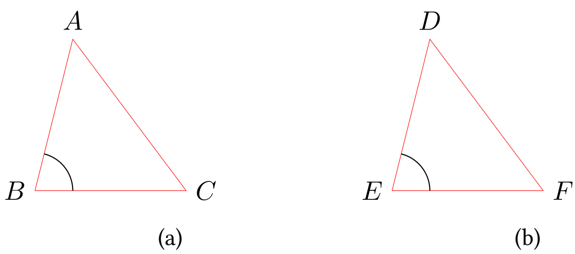


Figure 3.11: Side angle side theorem

Proof. Given two $\widehat{ABC}$ and $\widehat{DEF}$, where side $\overline{AB}$ is equal to side $\overline{DE}$, side $\overline{AC}$ is equal to side $\overline{DF}$, and $\angle BAC$ is equal to $\angle EDF$.

- **Lemma 1.** *If two triangles have two sides equal to two sides respectively, and have the included angle equal, then the triangles are congruent.*
- Proof: By side-side-angle congruence criterion.

  Lemma 2. *If $\widehat{ABC}$ is superimposed onto $\widehat{DEF}$, such that point A coincides with point D and $\overline{AB}$ coincides with $\overline{DE}$, then point B coincides with point E.*
- Proof: By definition of superposition, and the fact that $\overline{AB}$ equals $\overline{DE}$.

  Lemma 3. *If $\overline{AB}$ coincides with $\overline{DE}$, then $\overline{AC}$ coincides with $\overline{DF}$.*

- Proof: Since $\angle BAC$ equals $\angle EDF$, by definition of coinciding angles.

Lemma 4. *If B coincides with E, then $\overline{BC}$ coincides with $\overline{EF}$ and equals it.*

- Proof: By definition of coinciding points, and the fact that $\overline{AB}$ equals $\overline{DE}$. C.N.4

Lemma 5. *The whole $\widehat{ABC}$ coincides with the whole $\widehat{DEF}$ and equals it.*

- Proof: By Lemma 1.4.2, Lemma 1.4.4, and Lemma 1.4.5 C.N.4

Lemma 6. *The remaining angles also coincide and are equal.*

- Proof: By Lemma 1.4.6 and Lemma 14.2

Lemma 7. *If two triangles have two sides equal to two sides respectively and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.*

- Proof: By Lemma 5, Lemma 6, and the initial assumption.

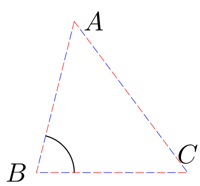


Figure 3.12: SAS conclusion

Conclusion: The theorem is proved.

Q.E.F

Theorem 3.3.0.2

In isosceles triangles, the angles at the base equal one another, and if the equal straight lines are produced further, then the angles under the base equal one another.

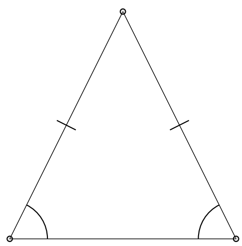


Figure 3.13: Isosceles Triangle Theorem

Let $\widehat{ABC}$ be an isosceles triangle with side $\overline{AB}$ equal to side $\overline{AC}$, and let the straight lines $\overline{BD}$ and $\overline{CE}$ be produced further in a straight line with $\overline{AB}$ and $\overline{AC}$. I.Def.20, I.Post.2

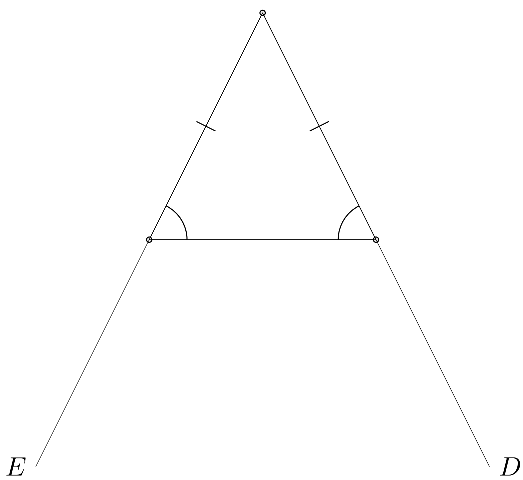


Figure 3.14: Isosceles Triangle Theorem; step 1

. Definitions and Postulates:

- An isosceles triangle is a triangle with two sides of equal length. I.Def.20
- A straight line can be produced indefinitely. I.Post.2
- Given two points, a straight line can be drawn between them. I.Post.1

Lemma 8. *If two sides and the included angle of one triangle are equal to the corresponding sides and angle of another triangle, then the two triangles are congruent.* *I.3*

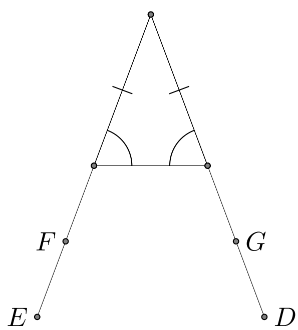
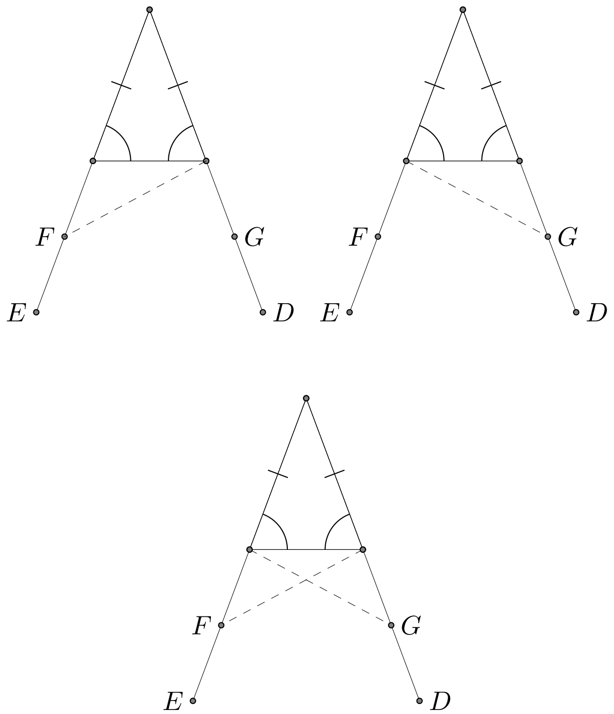


Figure 3.15: Isosceles Triangle Theorem; step 2

Construction 3.3.0.4

1. Take an arbitrary point F on $\overline{BD}$.
2. Cut off $\overline{AG}$ from $\overline{AE}$, where $\overline{AG} > \overline{AF}$.
3. Join FC and GB. I.3, I.Post.1



a: Join $\overline{FC}$

b: Join $\overline{GB}$

c: $\overline{FC}$ $\overline{GB}$

Figure 3.16: Isosceles Triangle Theorem: step 3

Proof.

- Since $\overline{AF} = \overline{AG}$ and $\overline{AB} = \overline{AC}$, $\widehat{FAG}$ and $\widehat{CAB}$ are congruent by SSS (side-side-side) criterion.

$$\Rightarrow \angle FAG = \angle CAB$$

- $FC = GB$ (Base angles in congruent triangles are equal). I.4

$$\Rightarrow \widehat{A}FC \cong \widehat{A}GB$$

$$\Rightarrow \angle ACF = \angle ABG, \quad \angle AFC = \angle AGB$$

- $BF = CG$. C.N.3

$$\Rightarrow BF + FC = CG + GB$$

$$\Rightarrow \widehat{B}FC \cong \widehat{C}GB$$

$$\Rightarrow \angle BFC = \angle CGB, \quad \angle BCF = \angle CBG$$

- Combining the results:

$$\Rightarrow \angle ACF = \angle ABG$$

$$\Rightarrow \angle AFC = \angle AGB$$

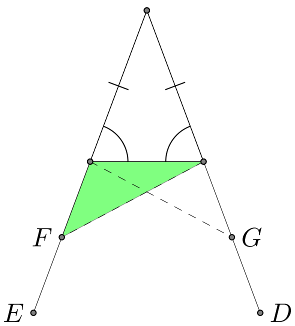
$$\Rightarrow \angle BFC = \angle CGB$$

$$\Rightarrow \angle BCF = \angle CBG$$

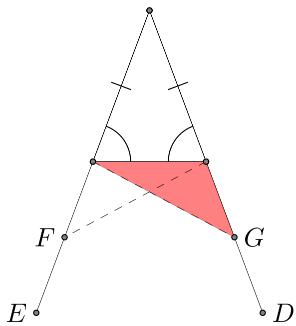
- Since $\angle ACF = \angle ABG$ and $\angle CBG = \angle BCF$, BC is parallel to FG. I.Post.1

- Since $\angle ABC$ and $\angle ACB$ are corresponding angles when BC is parallel to FG, $\angle ABC = \angle ACB$.

- Also, $\angle FBC = \angle GCB$ as proved earlier.



(a)



(b)

Figure 3.17: Isosceles Triangle Theorem: step 4

$\therefore$ in isosceles $\widehat{ABC}$, the angles at the base ($\angle ABC$ and $\angle ACB$) are equal, and if the equal sides are produced further, the angles under the base ($\angle FBC$ and $\angle GCB$) are equal.

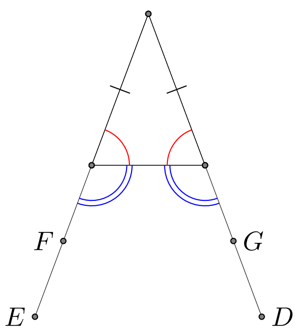


Figure 3.18: Isosceles Triangle Theorem: conclusion

Q.E.F

Theorem 3.3.0.3

If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

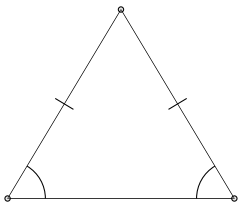


Figure 3.19

Given:

- If $\overline{AB}$ does not equal $\overline{AC}$, then one of them is greater. C.N
- If $\overline{AB}$ is greater, cut off $\overline{DB}$ from $\overline{AB}$ (the greater) equal to $\overline{AC}$ (the less), and join $\overline{DC}$. I.3, I.Post.1

Construction 3.3.0.5

1. **Assumption:** Suppose $\overline{AB}$ is greater than $\overline{AC}$.
2. Cut off $\overline{DB}$ from $\overline{AB}$ such that $\overline{DB} = \overline{AC}$, and join $\overline{DC}$.

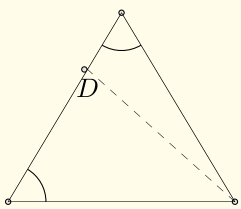


Figure 3.20

Proof. **Proof by Contradiction:**

- Since $\overline{DB} = \overline{AC}$ and $\overline{BC}$ is common $\therefore \widehat{DBC} \cong \widehat{ACB}$ by SSS (side-side-side) criterion. [$\Rightarrow \overline{DC} = \overline{AB}$ and $\angle DBC = \angle ACB$]

- This implies the base $\overline{DC}$ equals the base $\overline{AB}$, and the $\widehat{DBC}$ equals $\widehat{ACB}$, which is absurd since the less ($\widehat{DBC}$) equals the greater ($\widehat{ACB}$).

$\therefore$ the assumption that $\overline{AB}$ is greater than $\overline{AC}$ is false.

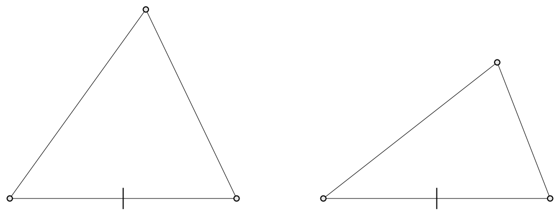
- Hence, $\overline{AB}$ is not unequal to $\overline{AC}$, it, $\therefore$ equals it.

Conclusion:

$\therefore$ if in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another. Q.E.F

Theorem 3.3.0.4

Given two straight lines constructed from the ends of a straight line and meeting at a point, it is not possible to construct two other straight lines on the same side of the original line, meeting in another point, and equal to the first two, namely each equal to that from the same end.



(a) (b)

Figure 3.21

- On the straight line $\overline{AB}$ draw two lines from A and B respectivly to a point C.

- Attempt to construct two other straight lines $\overline{ED}$ and $\overline{FD}$ on the straight line $\overline{EF}$, were $\overline{EF} = \overline{AB}$, meeting in another point D, and make them equal to $\overline{AC}$ and $\overline{BC}$, respectively.

1. **Assumption:** Suppose $\overline{AB}$ is greater than $\overline{AC}$.

2. Assume the contrary, that the construction is possible.

3. Lay $\overline{BC}$ over $\overline{EF}$.

4. Join $\overline{AD}$.

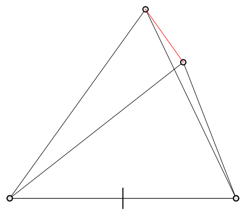


Figure 3.22

Proof. **Proof by Contradiction:**

- A straight line can be drawn between any two points. $\therefore$ $\overline{AD}$ can be drawn joining points A and D. I.Post.1

- If equals are added to equals, the wholes are equal. If equals are subtracted from equals, the remainders are equal. C.N.5

- Since $\overline{BA}$ equals $\overline{BD}$, and $\overline{BC}$ is common, $\angle ACD$ equals $\angle ADC$. C.N.5

- **Angle Comparison:** $\angle ADC$ is greater than $\angle DCB$.

- **Contradiction:** $\angle DCB$ is equal to $\angle CDB$ (by the given construction) but is also shown to be much greater than $\angle DCB$, leading to a contradiction.

  Conclusion: The assumption that the construction is possible leads to a contradiction.

$\therefore$ the original statement is proven—given two straight lines constructed from the ends of a straight line and meeting at a point, it is not possible to construct two other straight lines on the same side of the original line, meeting in another point, and equal to the first two.

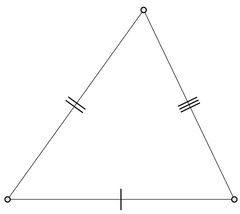
Q.E.F

Theorem 3.3.0.5

If two triangles ABC and DEF have the two sides AB and AC equal to the two sides DE and DF respectively, namely AB = DE and AC = DF, and the base BC equal to the base EF, then the angle BAC equals the angle EDF.

Given: $\widehat{ABC}$ and $\widehat{DEF}$ with $\overline{AB} = \overline{DE}$, $\overline{AC} = \overline{DF}$, and $\overline{BC} = \overline{EF}$.

To Prove: $\angle BAC = \angle EDF$.



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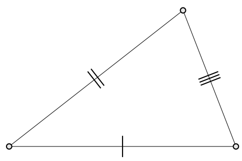


(b)

Figure 3.23

Proof. **Proof by Construction:**

1. Apply $\widehat{ABC}$ to $\widehat{DEF}$ by placing point B on point E and aligning $\overline{BC}$ with $\overline{EF}$.

2. Since $\overline{BC}$ coincides with $\overline{EF}$, point C also coincides with point F.

3. If $\overline{BA}$ and $\overline{AC}$ do not coincide with $\overline{ED}$ and $\overline{DF}$, then, two other straight lines would have been constructed on the same side of $\overline{AB}$ and meeting in a point, equal to $\overline{BA}$ and $\overline{AC}$, respectively. This is a contradiction. I.7, C.N.4

$\therefore$ $\overline{BA}$ and $\overline{AC}$ coincide with $\overline{ED}$ and $\overline{DF}$.

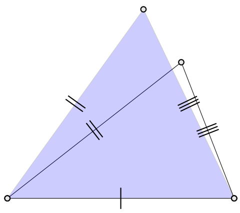


Figure 3.24

Conclusion:

- The base $\overline{BC}$ coinciding with $\overline{EF}$ implies that sides $\overline{BA}$ and $\overline{AC}$ coincide with $\overline{ED}$ and $\overline{DF}$ respectively.
- Thus, $\angle BAC$ coincides with $\angle EDF$, and they are equal.

$\therefore$ if two triangles have two sides equal to two sides respectively, and also have the base equal to the base, then the angles contained by the equal sides are also equal.

Q.E.F

Construction 3.3.0.6

It is required to bisect $\angle BAC$.

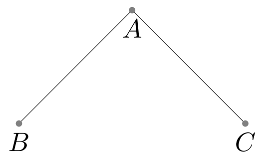


Figure 3.25

Proof by Construction:

1. Take an arbitrary point D on $\overline{AB}$.

2. Cut off $\overline{AE}$ from $\overline{AC}$ equal to $\overline{AD}$, and join $\overline{DE}$. I.3

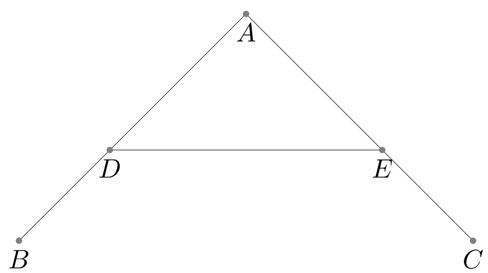


Figure 3.26

3. Construct the equilateral $\triangle DEF$ on DE. I.1

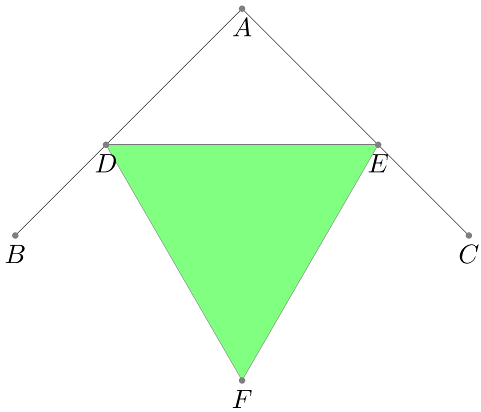


Figure 3.27

4. Join AF.

5. Then, the $\angle BAC$ is bisected by the straight line $\overline{AF}$.

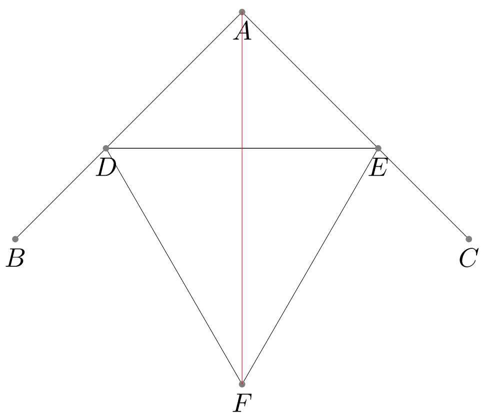


Figure 3.28

Proof.

Lemma 9. *since $AD = AE$ and AF is common, the two sides AD and AF are equal to the two sides EA and AF respectively.* *C.N.1,I.Post.1*

Lemma 10. *Since $\overline{DF} = \overline{EF}$, the angles $\angle DAF$ and $\angle EAF$ are equal.* *Def.20,I.8*

Conclusion:

$\therefore$ the given rectilinear angle $\angle BAC$ is bisected by the straight line AF. I.8

Q.E.F

Construction 3.3.0.7

Let$\overline{AB}$ be the given finite straight line. It is required to bisect the finite straight line $\overline{AB}$.

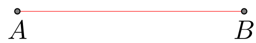


Figure 3.29

Construction:

1. Construct an equilateral $\widehat{ABC}$ on the straight line $\overline{AB}$. I.1

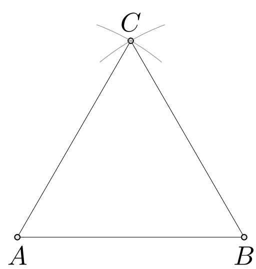


Figure 3.30

2. Bisect the $\angle ACB$ by the straight line $\overline{CE}$. I.9

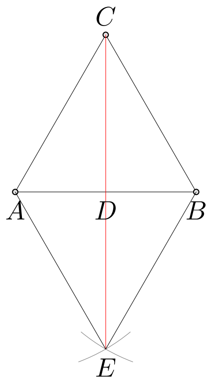


Figure 3.31

Proof.

- Let

$$\overline{CA} = \overline{CB}$$

- and $\overline{CD}$ is common to both $\widehat{ACD}$ and $\widehat{BCD}$. I.Def.20

- By Side-Angle-Side (SAS) congruence:

$$\widehat{ACD} \cong \widehat{BCD}$$

- $\therefore$ the corresponding parts are equal:

$$\overline{AD} = \overline{BD}$$

I.4

- Thus, the straight line $\overline{AB}$ is bisected at the point D.

- Hence, the construction is successful, and the proof is complete.

Q.E.F

Construction 3.3.0.8

To draw a straight line at right angles to a given straight line from a given point on it.

Let $\overline{AB}$ be the given straight line, and C the given point on it.

Figure 3.32

1. Take an arbitrary point D on $\overline{AC}$.

2. Make $\overline{CE}$ equal to $\overline{CD}$.

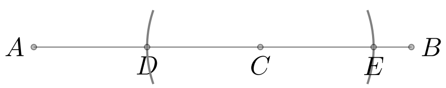


Figure 3.33

3. Construct the equilateral $\widehat{FDE}$ on $\overline{DE}$.

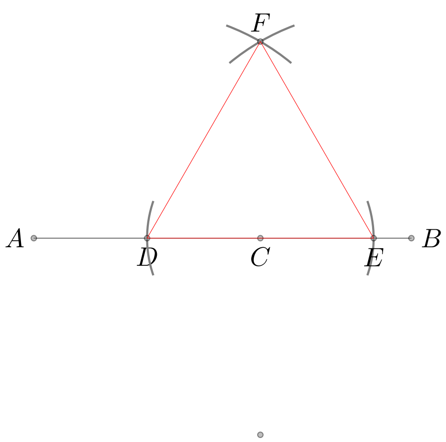


Figure 3.34

4. Join $\overline{CF}$.

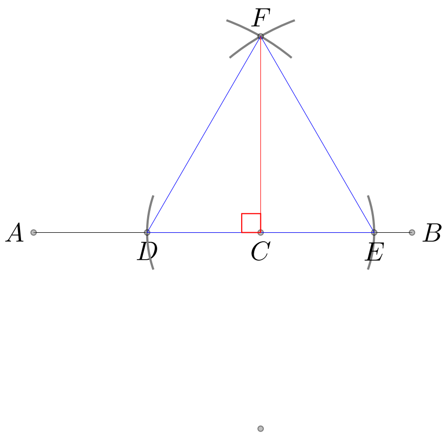


Figure 3.35

Lemma 11. *The straight line $\overline{CF}$ has been drawn at right angles to the given straight line $\overline{AB}$ from the given point C on it.*

Proof.

- Since

$$\overline{CD} = \overline{CE}$$

- and $\overline{CF}$ is common, $\therefore$ the two sides $\overline{CD}$ and $\overline{CF}$ equal the two sides $\overline{CE}$ and $\overline{CF}$ respectively,

- and the base

$$\overline{DF} = \overline{EF}$$

$\therefore$

$$\angle DCF = \angle ECF$$

- and they are adjacent angles. I.Def.20, I.8

- When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right. $\therefore$ each of $\angle DCF$ and $\angle FCE$ is right. I.Def.10

$\therefore$ the straight line $\overline{CF}$ has been drawn at right angles to the given straight line $\overline{AB}$ from the given point C on it.

Q.E.F

Construction 3.3.0.9

To draw a straight line perpendicular to a given infinite straight line from a given point not on it.

Let $\overline{AB}$ be the given infinite straight line, and C the given point not on it. It is required to draw a straight line perpendicular to the given infinite straight line $\overline{AB}$ from the given point C.

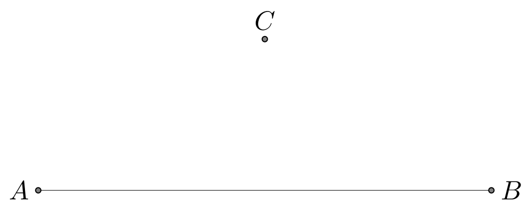


Figure 3.36

1. Take an arbitrary point D on the other side of the straight line $\overline{AB}$.

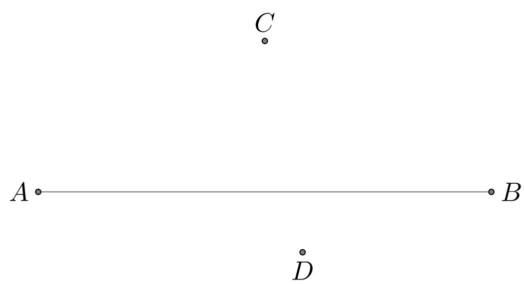


Figure 3.37

2. Describe $\mathscr{C}(C; CD)$.

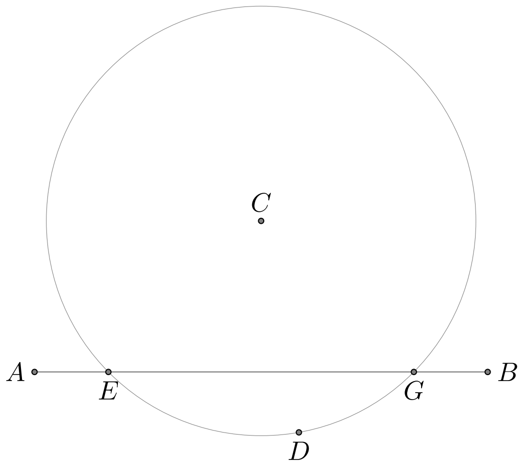


Figure 3.38

3. Bisect the straight line $\overline{EG}$ at H, and join the straight lines $\overline{CG}$, $\overline{CH}$, and $\overline{CE}$. I.10

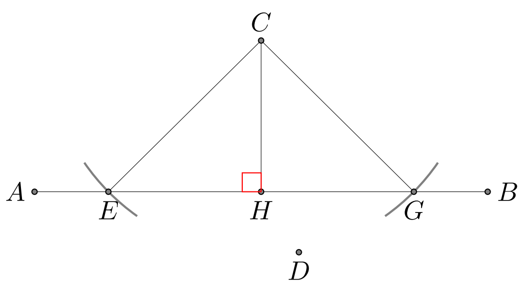


Figure 3.39

Proof.

- Since

$$\overline{GH} = \overline{HE}$$

- and $\overline{HC}$ is common, $\therefore$ the two sides $\overline{GH}$ and $\overline{HC}$ equal the two sides $\overline{EH}$ and $\overline{HC}$ respectively

- and the bases

$$\overline{CG} = \overline{CE}$$

$\therefore$

$$\angle CHG = \angle EHC$$

and they are adjacent angles.

- When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands. I.Def.10

$\therefore$ $\overline{CH}$ has been drawn perpendicular to the given infinite straight line $\overline{AB}$ from the given point C not on it.

Q.E.F

Theorem 3.3.0.6

If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

Let any straight line $\overline{AB}$ standing on the straight line $\overline{CD}$ make $\angle CBA$ and $\angle ABD$. I say that either $\angle CBA$ and $\angle ABD$ are two right angles or their sum equals two right angles.

Proof.

- If,

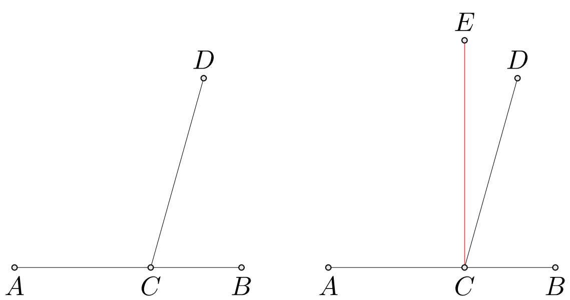
$$\angle CBA = \angle ABD$$

  then they are two right angles. I.Def.10

- If

$$\angle CBA \neq \angle ABD$$

  draw $\overline{BE}$ from B perpendicular to $\overline{CD}$. I.11



(a) (b)

Figure 3.40

- Now

$$\angle CBE = \angle CBA + \angle ABE$$

and

$$\angle EBD = \angle DBE + \angle EBA$$

I.Def.10

$\therefore$

$$\angle CBE + \angle EBD = \angle CBA + \angle ABE + \angle DBE + \angle EBA$$

C.N.2

- Also,

$$\angle DBA = \angle DBE + \angle EBA$$

and

$$\angle ABC = \angle ABE + \angle CBA$$

I.Def.10

$\therefore$

$$\angle DBA + \angle ABC = \angle DBE + \angle EBA + \angle ABE + \angle CBA$$

C.N.2

- Since

$$\angle CBE + \angle EBD = \angle DBA + \angle ABC$$

and $\angle CBE$ and $\angle EBD$ are two right angles,

- then

$$\angle DBA + \angle ABC = 2 \times 90^\circ$$

C.N.1

- Thus, if a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

Q.E.F

Theorem 3.3.0.7

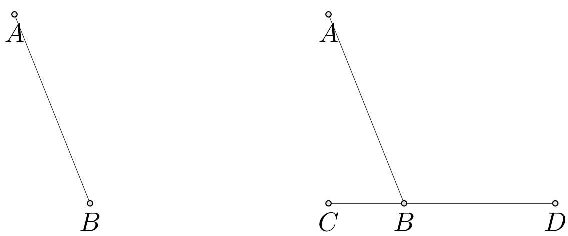
If, with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

Given: If with any straight line $\overline{AB}$, and at a point B on it, two straight lines $\overline{BC}$ and $\overline{BD}$ not lying on the same side make;

$$\angle ABC + \angle ABD = 2 \times 90^\circ$$

To prove: The two straight lines $\overline{CB}$ and $\overline{BD}$ are in a straight line with one another.

With any straight line $\overline{AB}$, and at the point B on it, let the two straight lines $\overline{BC}$ and $\overline{BD}$ not lying on the same side make the sum of the adjacent $\angle ABC$ and $\angle ABD$ equal to two right angles. I say that $\overline{BD}$ is in a straight line with $\overline{CB}$.



(a) (b)

Figure 3.41

Lemma 12. *If a straight line (in this case, $\overline{BD}$) falling on two straight lines (in this case, $\overline{AB}$ and $\overline{BE}$) makes the interior angles on the same side less than two right angles, then the two straight lines (AB and BE) produced indefinitely meet on that side on which are the angles less than the two right angles.* *I.Post.2*

Lemma 13. *If a straight line stands on a straight line, then it makes either two right angles or angles less than two right angles.* *I.Post.4*

Assume for contradiction: $\overline{BD}$ is not in a straight line with $\overline{CB}$.

By Lemma 1 Produce $\overline{BE}$ in a straight line with $\overline{CB}$. I.Post.2

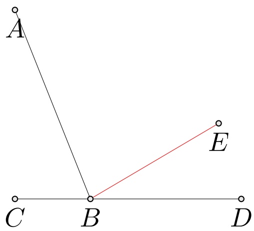


Figure 3.42

Proof.

By Lemma 2 Since, AB stands on BE,

$$\angle ABC + \angle ABE = 2 \times 90°$$

I.Post.4,

Given:

$$\angle ABC + \angle ABD = 2 \times 90°$$

$\therefore$

$$\angle CBA + \angle ABE = \angle CBA + \angle ABD$$

By Lemma 3 Subtract angle CBA from each side:

$$\angle ABE = \angle ABD$$

C.N.3

Contradiction: The less ($\angle ABE$) equals the greater ($\angle ABD$), which is impossible.

$\therefore$

$$\overline{BE} \neq \overline{CB}$$

Similarly, it can be proven that no other straight line except $\overline{BD}$ is in line with $\overline{CB}$.

Thus, $\overline{CB}$ is in a straight line with $\overline{BD}$.

Conclusion If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

Q.E.F

Theorem 3.3.0.8

If two straight lines cut one another, then they make the vertical angles equal to one another.

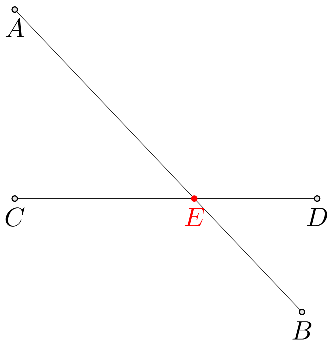


Figure 3.43

Let the straight lines $\overline{AB}$ and $\overline{CD}$ cut one another at the point E.

Proof.

Lemma 14.

$$\angle CEA = \angle DEB \text{ and } \angle BEC = \angle AED$$

.Proof: By the angle sum property

$$\angle CEA + \angle AED = 2 \times 90^\circ$$

I.13

Similarly,

$$\angle AED + \angle DEB = 2 \times 90^\circ$$

Lemma 15.

$$\angle CEA + \angle AED = \angle AED + \angle DEB$$

- By previous claims,

$$\angle CEA + \angle AED = \angle AED + \angle DEB$$

- Subtracting $\angle AED$ from both sides,

$$\angle CEA = \angle DEB$$

Extension of lemma 0.1.3 Similarly, $\angle BEC = \angle AED$.

$\therefore$ if two straight lines cut one another, then they make the vertical angles equal to one another.

Q.E.F

Theorem 3.3.0.9

In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

Let $\widehat{ABC}$ be a triangle, and let one side $\overline{BC}$ be produced to D. We want to show that the exterior $\angle ACD$ is greater than either of the interior and opposite $\angle CBA$ and $\angle BAC$.

Construction

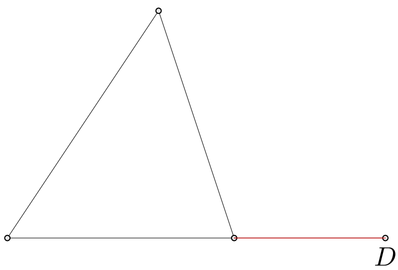


Figure 3.44

1. Produce $\overline{AC}$ to G

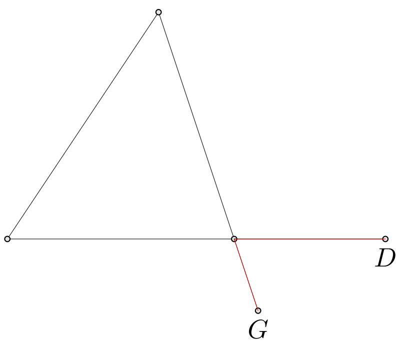


Figure 3.45

2. Bisect $\overline{AC}$, and produce $\overline{BE}$ to F, were $\overline{BE} = \overline{EF}$.

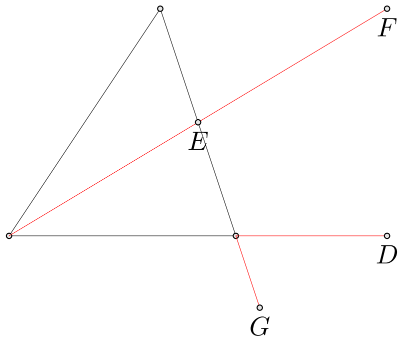


Figure 3.46

3. Draw $\overline{FC}$

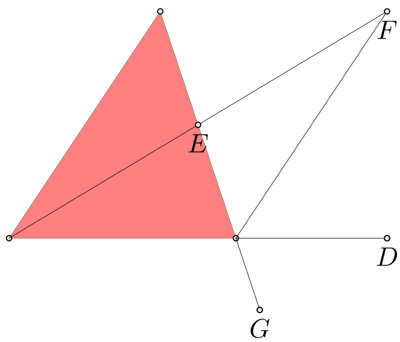


Figure 3.47

Proof. • Since

$$\overline{AE} = \overline{EC} \text{ and } \overline{BE} = \overline{EF}$$

$$\widehat{ABE} \cong \widehat{CFE}$$

I.15,I.4

$\therefore$

$$\overline{AB} = \overline{FC} \text{ and } \angle BAE = \angle ECF$$

. C.N.5

- But

$$\angle ECD > \angle ECF$$

$\therefore$,

$$\angle ACD > \angle BAE$$

I.15

Similarly: If $\overline{BC}$ is bisected, then $\angle BCG$, that is, $\angle ACD$, can also be proved to be greater than $\angle ABC$.

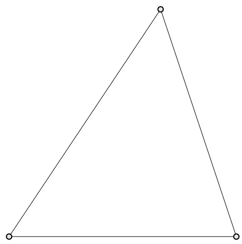
$\therefore$ in any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

Q.E.F

Theorem 3.3.0.10

In any triangle, the sum of any two angles is less than two right angles.

Let ABC be a triangle.



Produce $\overline{BC}$ to D. I.Post.2

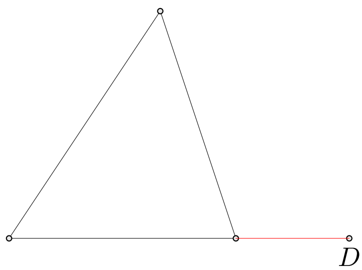


Figure 3.48

Proof.

- Since the $\angle ACD$ is an exterior angle of the $\widehat{ABC}$, $\therefore$ it is greater than the interior and opposite $\angle ABC$. Add $\angle ACB$ to each.

$$\angle ACD + \angle ACB > \angle ABC + \angle BCA$$

I.16, C.N.2

- But

$$\angle ACD + \angle ACB = 2 \times 90^\circ$$

$\therefore$

$$\angle ABC + \angle BCA < 2 \times 90^\circ$$

I.13

- Similarly, we can prove that

$$\angle BAC \angle ACB < 2 \times 90^\circ,$$

  and so the sum of the angles $\angle CAB$ and $\angle ABC$ as well.

$\therefore$,in any triangle, the sum of any two angles is less than two right angles.

Q.E.F

Theorem 3.3.0.11

In any triangle, the angle opposite the greater side is greater.

Let $\widehat{ABC}$ be a triangle having the side $\overline{AC}$ greater than $\overline{AB}$.

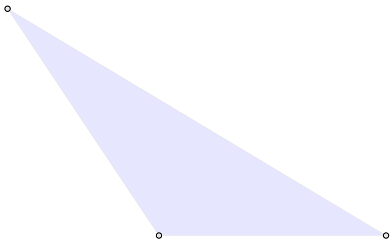


Figure 3.49

Since $\overline{AC}$ is greater than $\overline{AB}$, make $\overline{AD}$ equal to $\overline{AB}$, and join $\overline{BD}$. I.3, I.Post.1

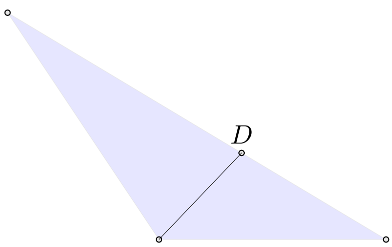


Figure 3.50

Proof.

- Since $\angle ADB$ is an exterior angle of the $\widehat{BCD}$,

$\therefore$ it is greater than the interior and opposite $\angle DCB$. I.16

- But the $\angle ADB$ equals $\angle ABD$, since the side $\overline{AB}$ equals $\overline{AD}$,

$\therefore$ $\angle ABD > \angle ACB$

$\therefore$ $\angle ABC \gg \angle ACB$. I.5

$\therefore$ in any triangle, the angle opposite the greater side is greater.

Q.E.F

Theorem 3.3.0.12

In any triangle the side opposite the greater angle is greater.

Let $\triangle ABC$ have $\angle ABC > \angle BCA$.

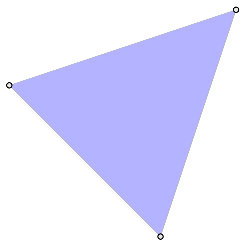


Figure 3.51

Lemma 16. *If*

$$\overline{AC} = \overline{AB}$$

then

$$\angle ABC = \angle ACB$$

I.5

Lemma 17. *If*

$$\overline{AC} < \overline{AB}$$

then

$$\angle ABC < \angle ACB$$

I.18

Proof.

- Let $\widehat{ABC}$ be a triangle with

$$\angle ABC > \angle BCA$$

- **Assumption:** Suppose, for the sake of contradiction, that side

$$\overline{AC} \not> \overline{AB}$$

.

- This implies either

$$\overline{AC} = \overline{AB}$$

or

$$\overline{AC} < \overline{AB}$$

- **Case 1** If

$$\overline{AC} = \overline{AB},$$

then

$$\angle ABC = \angle ACB$$

which contradicts the assumption that

$$\angle ABC > \angle BCA$$

I.15

- **Case 2** If

$$\overline{AC} < \overline{AB}$$

then

$$\angle ABC < \angle ACB$$

I.18

which also contradicts the assumption.

- **Conclusion:** Since both cases lead to contradictions, our assumption that side

$$\overline{AC} \ngtr \overline{AB}$$

is false.

$\therefore$ side

$$\overline{AC} > \overline{AB}$$

In any triangle, the side opposite the greater angle is greater.

Thus, we have shown that in any triangle, the side opposite the greater angle is greater.

Q.E.F

Theorem 3.3.0.13

In any triangle the sum of any two sides is greater than the remaining one.

Given $\widehat{ABC}$

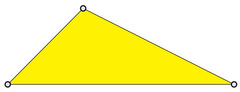


Figure 3.52

Construction

1. Produce $\overline{BA}$ through to the point D. I.Post.2, I.3

2. Make $\overline{DA}$ equal to $\overline{CA}$.

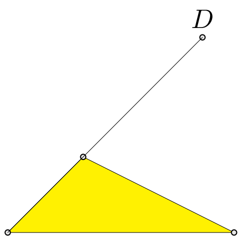


Figure 3.53

3. Join $\overline{DC}$. I.Post.1

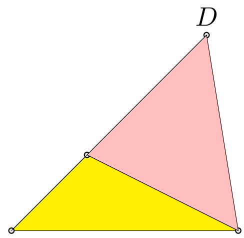


Figure 3.54

Proof.

- Since

$$\overline{DA} = \overline{AC}$$

$$\angle ADC = \angle ACD$$

I.5, C.N.5

- Since $\triangle DCB$ has

$$\angle BCD > \angle BDC$$

the side opposite the greater angle is greater.

$\therefore$

$$\overline{DB} > \overline{BC}$$

I.19

Conclusion:

– Since

$$\overline{DA} = \overline{AC}$$

$$\overline{DB} > \overline{BC}$$

– By the triangle inequality theorem,

$$\overline{BA} + \overline{AC} > \overline{BC}$$

Similarly:

- Apply the same reasoning to the other sides:

$$\overline{AB} + \overline{BC} > \overline{CA}$$

$$\overline{BC} + \overline{CA} > \overline{AB}$$

$\therefore$ In any $\widehat{ABC}$, the sum of any two sides is greater than the remaining one.

- The statement has been demonstrated and proven.

Q.E.F

Theorem 3.3.0.14

If from the ends of one of the sides of a triangle, two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle. Additionally, the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

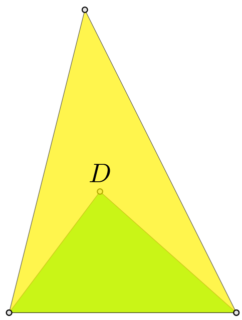


Figure 3.55

Let $\triangle ABC$ be a triangle, and from the ends B and C of one of the sides $\overline{BC}$, let the two straight lines $\overline{BD}$ and$\overline{DC}$ be constructed meeting within the triangle.

Draw $\overline{BD}$ through to E. I.Post.2

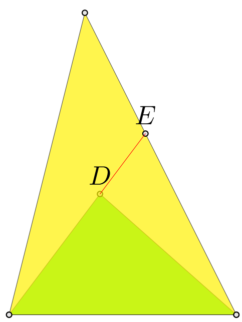


Figure 3.56

Proof.

- Since in any triangle, the sum of two sides is greater than the remaining one, $\therefore$ in the $\triangle ABE$,

$$\overline{AB} + \overline{AE} > \overline{BE}$$

I.20

- Add $\overline{EC}$ to each,

$$\overline{BA} + \overline{AC} > \overline{BE} + \overline{EC}$$

C.N.1

- Again, since in $\widehat{CED}$,

$$\overline{CE} + \overline{ED} > \overline{CD}$$

add $\overline{DB}$ to each.

$\therefore$

$$\overline{CE} + \overline{EB} > \overline{CD} + \overline{DB}.$$

I.20, C.N.1

- But,

$$\overline{BA} + \overline{AC} > \overline{BE} + \overline{EC}$$

$\therefore$

$$\overline{BA} + \overline{AC} >> \overline{BD} + \overline{DC}.$$

C.N.1

- Again, since in any triangle, the exterior angle is greater than the interior and opposite angle, $\therefore$ in $\widehat{CDE}$,

$$\angle BDC > \angle CED$$

I.16

- For the same reason, moreover, in $\widehat{ABE}$,

$$\angle CEB > \angle BAC$$

- But

$$\angle BDC > \angle CEB$$

$\therefore$

$$\angle BDC >> \angle BAC$$

$\therefore$ if from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

Q.E.F

3.1

Theorem 3.3.1.1

To construct a triangle out of three straight lines which equal three given straight lines, it is necessary that the sum of any two of the straight lines should be greater than the remaining one.

Construction 3.3.1.1

1. Let $A = 4$, $B = 5$, and $C = 6$ be the three given straight lines. Construct a triangle with sides equal to A, B, and C if $A + B > C$, $A + C > B$, and $B + C > A$. I.20

Figure 3.57: Triangle Construction Step:1

2. Set out a straight line $\overline{DE}$, terminated at D but of infinite length in the direction of E. Make

$$\overline{DF} = A$$

$$\overline{FG} = B$$

$$\overline{GH} = C$$

I.Post.3 I.Post.1

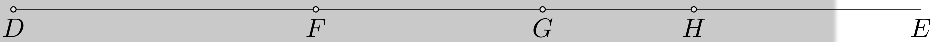


Figure 3.58: Triangle Construction Step:2

3. Describe $\mathscr{C}(F; FD)$.

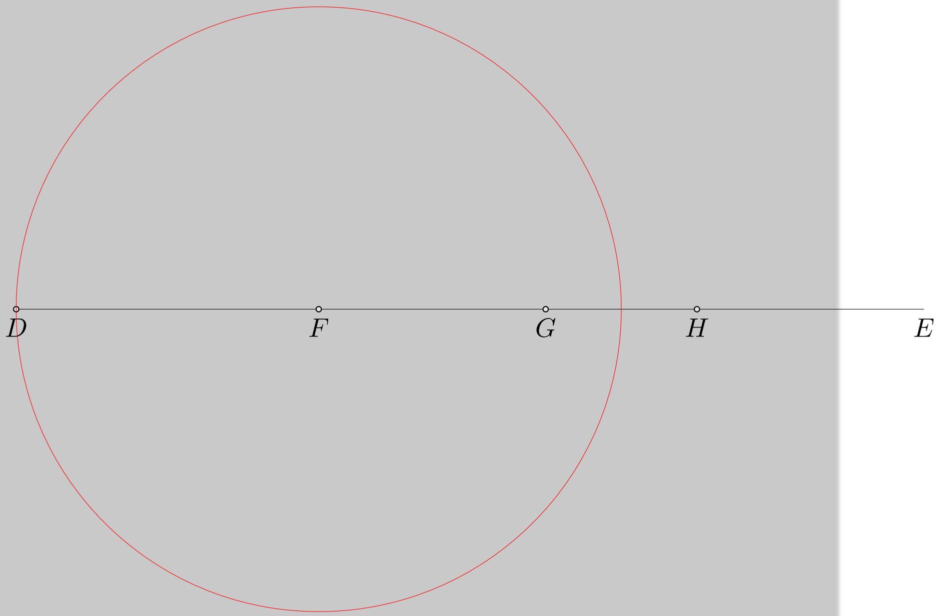


Figure 3.59: Triangle Construction Step:3

4. Again, describe $\mathscr{C}(G; GH)$.
5. Find the intersection of $\mathscr{C}(F; FD)$ and $\mathscr{C}(G; GH)$, K.

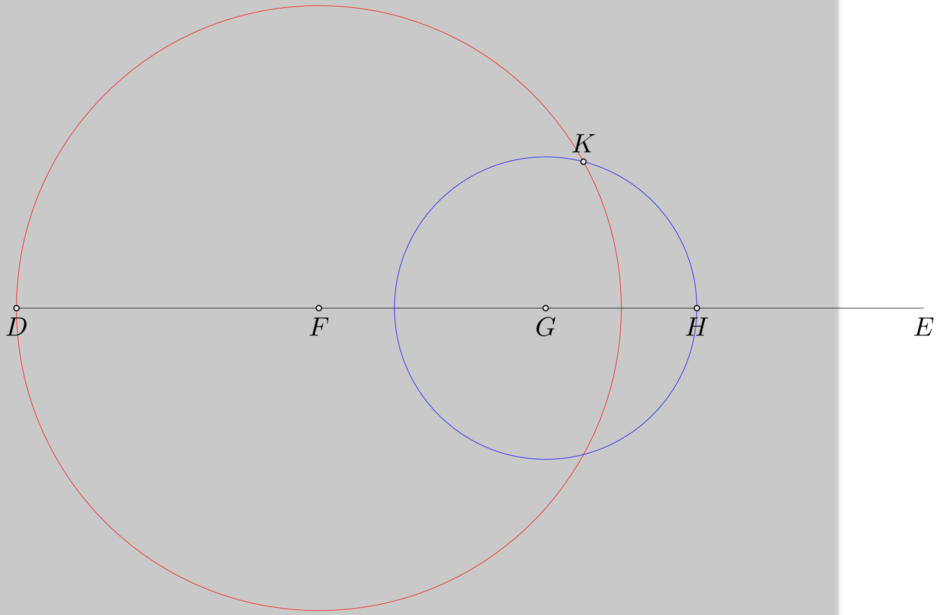


Figure 3.60: Triangle Construction Step:4

6. Join $\overline{KF}$ and $\overline{KG}$. I.Post.3, I,Post.1

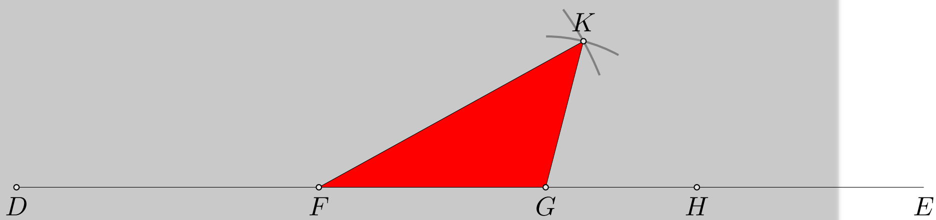


Figure 3.61: Triangle Construction Step:5

Claim: The $\widehat{KFG}$ has been constructed out of three straight lines equal to A, B, and C.

Proof.

- Since the point F is the center of $\mathscr{C}FD$,

$\therefore$

$$\overline{FD} = \overline{FK}$$

But

$$\overline{FD} = A$$

$\therefore$

$$\overline{KF} = A$$

I.Def.16 C.N.1

- Again, since the point G is the center of $\mathscr{G}GH$,

$\therefore$

$$\overline{GH} = \overline{GK}$$

But

$$\overline{GH} = C$$

$\therefore$

$$\overline{KG} = C$$

- And

$$\overline{FG} = B$$

$\therefore$ the three straight lines $\overline{KF}$, $\overline{FG}$, and $\overline{GK}$ equal the three straight lines A, B, and C.

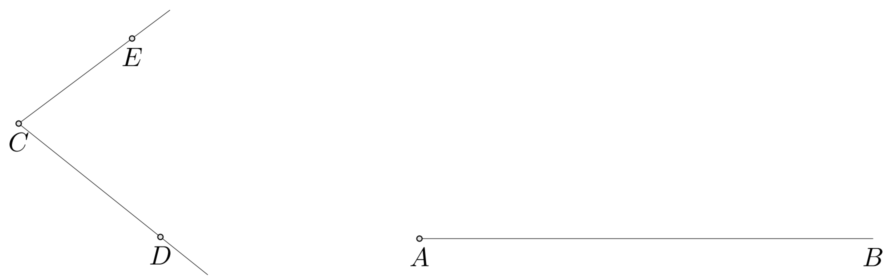
$\therefore$ out of the three straight lines $\overline{KF}$, $\overline{FG}$, and $\overline{GK}$, which equal the three given straight lines A, B, and C, the $\widehat{KFG}$ has been constructed.

Q.E.F.

Construction 3.3.1.2

To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it, let $\angle DCE$ be the given rectilinear angle, $\overline{AB}$ the given straight line, and A the point on it. Construct a rectilinear angle $\angle KAG$ on the given straight line $\overline{AB}$ and at the point A on it such that $\angle DCE = \angle FAG$.

Construction:



(a) (b)

Figure 3.62

1. Draw the segment $\overline{ED}$. Post.1

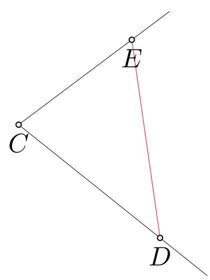


Figure 3.63

2. Draw the given straight line $\overline{AB}$ and mark the point A.

3. On $\overline{AB}$ mark the points G, H and I so that,

$$\overline{AG} = \overline{CD}$$

$$\overline{GH} = \overline{ED}$$

$$\overline{AI} = \overline{CE}$$

I.22

Figure 3.64

4. Mark the intersect of $\mathscr{C}(A; AG)$ and $\mathscr{C}(H; HI)$, K. I.22

5. Construct a triangle $\widehat{AKH}$ such that:

$$\overline{AK} = \overline{ED}$$

$$\overline{AH} = \overline{CD}$$

$$\overline{KH} = \overline{CE}$$

I.22

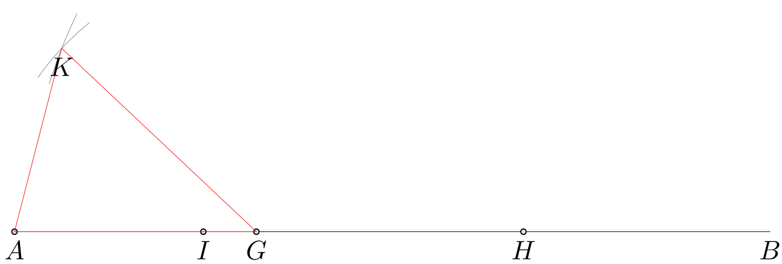


Figure 3.65

Proof.

- By construction,

$$\angle DCE = \angle KAG$$

$\therefore$ a rectilinear $\angle KAG$ has been constructed on the given straight line $\overline{AB}$ and at the point A on it, equal to the given rectilinear $\angle DCE$. I.8

Q.E.F

Theorem 3.3.1.2

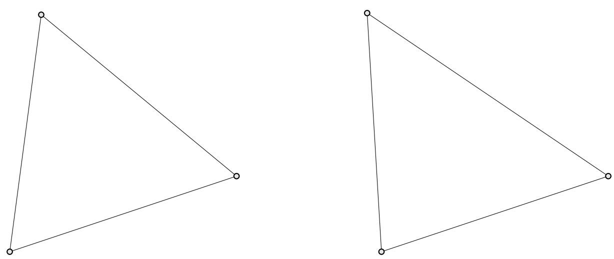
If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.

Let $\widehat{ABC}$ and $\widehat{DEF}$, such that

$$\overline{AB} = \overline{DE}$$

$$\overline{AC} = \overline{DF}$$

$$\angle CAB > \angle FDE$$



(a) (b)

Figure 3.66

Construction:

1. Construct $\angle EDG$ equal to $\angle BAC$ at point D on $\overline{DE}$. I.23, I.3, I.Post.1
2. Make $\overline{DG}$ equal to either $\overline{AC}$ or $\overline{DF}$.
3. Join $\overline{EG}$ and $\overline{FG}$.

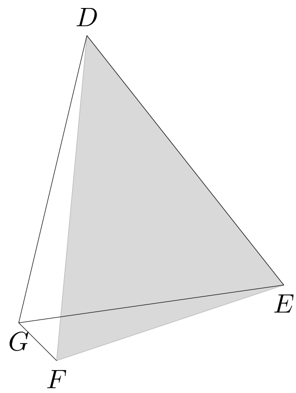


Figure 3.67

Proof. • If two angles of a triangle are unequal, then the sides opposite these angles are also unequal. I.23

- Since

$$\overline{AB} = \overline{DE}$$

and

$$\overline{AC} = \overline{DG}$$

  - The two sides $\overline{BA}$ and $\overline{AC}$ equal the two sides $\overline{ED}$ and $\overline{DG}$ respectively. I.4

  - Also, $\angle BAC$ equals $\angle EDG$.

$\therefore$ the base $\overline{BC}$ equals the base $\overline{EG}$. C.N.1

- Since $\overline{DF}$ equals $\overline{DG}$, $\angle DGF$ equals $\angle DFG$.

$\therefore$

$$\angle DFG > \angle EGF$$

I.5

Thus , $\widehat{EFG}$, the side $\overline{EG}$ opposite the greater angle is greater than $\overline{EF}$. I.19

- Since

$$\overline{EG} = \overline{BC}$$

it follows that

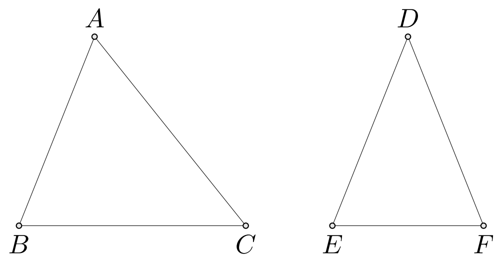
$$\overline{BC} > \overline{EF}$$

$\therefore$ if two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base $\overline{BC}$ greater than the base $\overline{EF}$.

Q.E.F

Theorem 3.3.1.3

If two triangles ABC and DEF have two sides AB and AC equal to two sides DE and DF, respectively (AB = DE and AC = DF), and the base BC is greater than the base EF (BC > EF), then the angle BAC is greater than the angle EDF.



(a) (b)

Figure 3.68

To Prove:

$$\angle BAC > \angle EDF$$

Proof.

- Assume, for the sake of contradiction, that

$$\angle BAC = \angle EDF$$

  - If

$$\angle BAC = \angle EDF$$

    then by the Angle-Side-Angle (ASA) congruence,

$$\widehat{ABC} \cong \widehat{DEF}$$

- This implies

$$\overline{BC} = \angle EF$$

but this contradicts the given condition

$$\overline{BC} > \overline{EF}$$

I.4

- Next, assume that

$$\angle BAC < \angle EDF$$

- If

$$\angle BAC < \angle EDF$$

then by the Angle-Side-Angle (ASA) inequality,

$$\widehat{ABC} \cong \widehat{DEF}$$

- This implies

$$\overline{BC} < \overline{EF}$$

but this contradicts the given condition

$$\overline{BC} > \overline{EF}$$

I.24

- Since $\angle BAC$ cannot be equal to or less than $\angle EDF$, the only remaining possibility is that

$$\angle BAC > \angle EDF$$

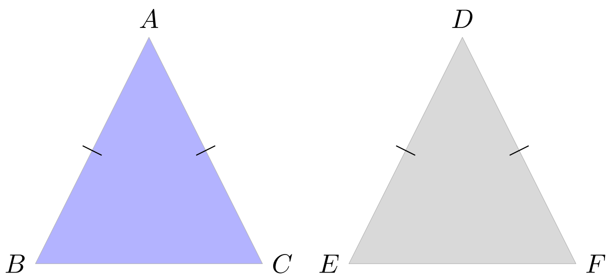
$\therefore$ if two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have one of the angles contained by the equal straight lines greater than the other.

Conclusion: The statement is proved by contradiction, and the desired inequality $\angle BAC > \angle EDF$ holds.

Q.E.F

Theorem 3.3.1.4

(SAS Congruence Criterion): If $\widehat{ABC}$ and $\widehat{DEF}$ have $\angle ABC$ and $\angle BCA$ equal to $\angle DEF$ and $\angle EFD$, respectively, and one side $\overline{BC}$ equal to $\overline{EF}$, then the triangles are congruent. This means that $\overline{AB}$ equals $\overline{DE}$, $\overline{AC}$ equals $\overline{DF}$, and $\angle BAC$ equals $\angle EDF$.



(a) (b)

Figure 3.69: SAS Congruence

Given:

- $\triangle ABC$ and $\triangle DEF$ such that:

$$\angle ABC = \angle DEF$$

$$\angle BCA = \angle EFD$$

$$\overline{BC} = \overline{EF}$$

- **To Prove:**

$$\overline{AB} = \overline{DE}$$

$$\overline{AC} = \overline{DF}$$

$$\angle BAC = \angle EDF$$

Proof. **Case 1:** Suppose $AB \neq DE$.

- By the Segment Equality Postulate, one must be greater. I.Post.1
- Assume $AB > DE$.
- Construct $BG = DE$ and join GC.

$$\widehat{GBC} \cong \widehat{DEF}$$

I.4

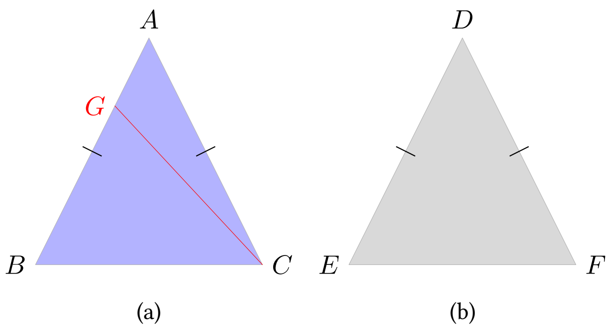


Figure 3.70: SAS Congruence.2

$\therefore$,

$$\angle GCB = \angle DFE$$

- But, (given).

$$\angle DFE = \angle ACB$$

- Hence,

$$\angle GCB = \angle BAC$$

- This is impossible, as the lesser angle cannot equal the greater angle. Thus,

$$AB = DE$$

Case 2: $AB = DE$.

- Given that $BC = EF$,
- By Side-Side-Side (SSS) congruence:

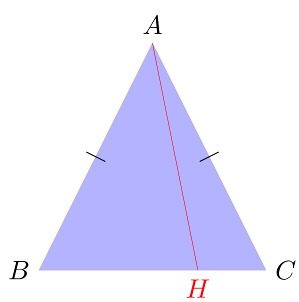
$$\widehat{ABC} \cong \widehat{DEF}$$

C.N.1

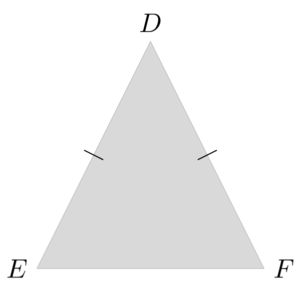
$\therefore$, $AC = DF$ and $\angle BAC = \angle EDF$.

Case 3: Suppose $BC \neq EF$.

- By the Segment Equality Postulate, one must be greater. I.Post.1
- Assume $BC > EF$.
- Construct $BH = EF$ and join AH.



(a)



(b)

Figure 3.71

- By Side-Angle-Side (SAS) congruence: I.Post.4

$$\triangle ABH \cong \triangle DEF$$

$\therefore$

$$\angle BHA = \angle EFD$$

- But,(given),

$$\angle EFD = \angle BCA$$

- Hence,

$$\angle BHA = \angle BCA$$

- This is impossible, as the exterior angle of a triangle cannot be equal to its interior opposite angle.
- Thus, $BC = EF$.

Conclusion:

- From both cases, it's evident that:

$$AB = DE$$

$$AC = DF$$

$$\angle BAC = \angle EDF$$

- Thus, if two triangles have two angles equal to two angles respectively, and one side equal to one side, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.

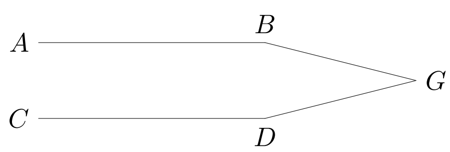
Q.E.F

Theorem 3.3.1.5

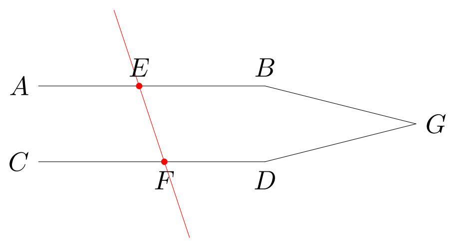
If a straight line $\overline{EF}$ intersects two straight lines $\overline{AB}$ and $\overline{CD}$, and the alternate $\angle AEF$ and $\angle EFD$ are equal, then $\overline{AB}$ is parallel to $\overline{CD}$.

To Prove: $\overline{AB}$ is parallel to $\overline{CD}$.

Proof. Assume, for the sake of contradiction, that $\overline{AB}$ and $\overline{CD}$ when produced meet in the direction of B and D at a point G.



- In $\widehat{GEF}$, the exterior $\angle AEF$ equals the interior and opposite $\angle EFG$ (by exterior angle theorem).



- This is impossible, as the exterior angle cannot be equal to the interior and opposite angle in a triangle. I.16

$\therefore$ $\overline{AB}$ and $\overline{CD}$ when produced do not meet in the direction of B and D. Similarly, it can be proved that they do not meet towards A and C.

Since straight lines that do not meet in either direction are parallel, it follows that $\overline{AB}$ is parallel to $\overline{CD}$. I.Def.23

Conclusion: If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

Q.E.F

Theorem 3.3.1.6

If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

Given: Straight line $\overline{EF}$ falls on the two straight lines $\overline{AB}$ and $\overline{CD}$, and the exterior $\angle EGB$ is equal to the interior and opposite $\angle GHD$, or the sum of the interior $\angle BGH$ and $\angle GHD$ is equal to two right angles.

Proof.

- Assume: $\overline{AB} \nparallel \overline{CD}$.
- By the definition of parallel lines, $if \overline{AB} \nparallel \overline{CD}$, then there exists a transversal $\overline{EF}$ that intersects $\overline{AB}$ and $\overline{CD}$. I.28

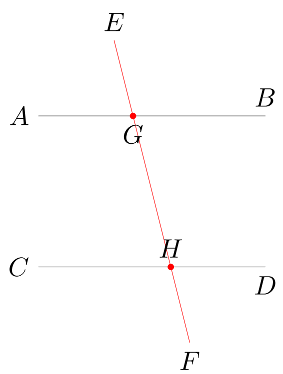


Figure 3.72

- By the exterior angle theorem,

$$\angle EGB = \angle GHD$$

I.15

- By the alternate interior angles theorem, since

  $$\angle EGB = \angle GHD$$

  and

  $$\angle EGB = \angle AGH$$

  then

  $$\angle AGH equals \angle GHD$$

C.N.1, I.27

- By the alternate interior angles theorem, since

  $$\angle AGH = \angle GHD$$

  $$\overline{AB} \parallel \overline{CD}$$

I.13

- By the definition of parallel lines, if

  $$\overline{AB} \parallel \overline{CD}$$

  then the assumption in step 1 is false. C.N.1

$\therefore \overline{AB} \parallel to \overline{CD}$

- Next, assume again: $\overline{AB} \nparallel \overline{CD}$

- By the sum of interior angles on the same side theorem,

$$\angle BGH + \angle GHD = 2 \times 90^\circ$$

I.Post.4

- By the sum of interior angles on the same side theorem,

$$\angle AGH + \angle BGH + 2 \times 90^\circ$$

C.N.3, I.27

- Subtract $\angle BGH$ from both sides of the equation in above, yielding

$$\angle AGH = \angle GHD$$

- By the alternate interior angles theorem, since

$$\angle AGH = \angle GHD$$

$$\overline{AB} \parallel \overline{CD}$$

I.13

- By the definition of parallel lines, if

$$\overline{AB} \parallel \overline{CD}$$

then the assumption above is also false. C.N.1

$\therefore$

$$\overline{AB} \parallel \overline{CD}$$

- Since assuming

  $$\overline{AB} \nparallel \overline{CD}$$

  leads to a contradiction in both cases, it must be true that

  $$\overline{AB} \parallel \overline{CD}$$

$\therefore$ if a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

Q.E.F

Theorem 3.3.1.7

In a system of parallel straight lines, if a straight line $\overline{EF}$ intersects the lines $\overline{AB}$ and $\overline{CD}$, then the following statements hold true:

1. Alternate $\angle AGH$ and $\angle GHD$ are equal.
2. Exterior $\angle EGB$ is equal to the interior and opposite $\angle GHD$.
3. The sum of the interior angles on the same side, namely $\angle BGH$ and $\angle GHD$, is equal to two right angles.

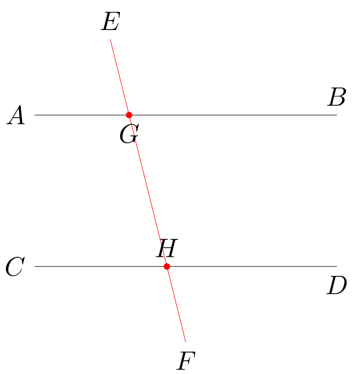


Figure 3.73: Alternate angles

Proof. Proof of Statement 1:

Suppose, for the sake of contradiction, that,

$$\angle AGH \neq \angle GHD$$

- Without loss of generality, assume:

$$\angle AGH > \angle GHD$$

- Adding $\angle BGH$ to both sides, we get,

$$\angle AGH + \angle BGH > \angle BGH + \angle GHD$$

- However, by the Angle Sum Property,

$$\angle AGH + \angle BGH = 2 \times 90°$$

- This implies that, I.13

$$\angle BGH + \angle GHD < 2 \times 90°$$

Now, by the Parallel Postulate, extended lines produced from angles less than two right angles meet. This contradicts the fact that $\overline{AB}$ and $\overline{CD}$, if produced indefinitely, should meet, but they are parallel by hypothesis

$\therefore$ our assumption that,

$$\angle AGH \neq \angle GHD$$

must be false, and thus, Post.5

$$\angle AGH = \angle GHD$$

Proof of Statement 2:

- Since

$$\angle AGH = \angle GHD$$

- (proved above) and

$$\angle AGH = \angle EGB$$

- (given), it follows that

$$\angle EGB = \angle GHD$$

Proof of Statement 3:

- Adding angle $\angle BGH$ to both sides of the equation

$$\angle EGB = \angle GHD$$

- We obtain

$$\angle EGB + \angle BGH = \angle BGH + \angle GHD$$

- By the Angle Sum Property,

$$\angle EGB + \angle BGH = 2 \times 90^\circ$$

- $\therefore$ I.13

$$\angle BGH + \angle GHD = 2 \times 90^\circ$$

Hence, we have established that a straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

Q.E.F

Theorem 3.3.1.8

If two straight lines, such as $\overline{AB}$ and $\overline{CD}$, are each parallel to a third straight line $\overline{EF}$, then $\overline{AB}$ is also parallel to $\overline{CD}$.

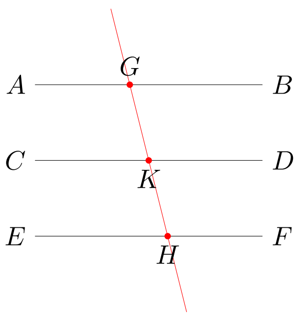


Figure 3.74

Proof. *Given:* Straight lines $\overline{AB}$ and $\overline{CD}$ are each parallel to $\overline{EF}$.

To prove:

$$\overline{AB} \parallel \overline{CD}$$

- Let the straight line $\overline{GH}$ fall upon $\overline{AB}$ and $\overline{CD}$. Since $\overline{GK}$ falls on the parallel straight lines $\overline{AB}$ and $\overline{EF}$, it follows from the Corresponding Angles Postulate that

$$\angle AGK = \angle GHF$$

I.29

- Again, since $\overline{GH}$ falls on the parallel straight lines $\overline{EF}$ and $\overline{CD}$, the Corresponding Angles Postulate implies that the

$$\angle GHF = \angle GKD$$

By the Corresponding Angles Postulate, we have

$$\angle AGK = \angle GHF$$

and

$$\angle GHF = \angle GKD$$

$\therefore$

$$\angle AGK = \angle GKD$$

and they are alternate angles. I.29, C.N.1

- Thus, by the Alternate Interior Angles Theorem, it is concluded that

$$\overline{AB} \parallel \overline{CD}$$

I.29

$\therefore$ straight lines parallel to the same straight line are also parallel to one another.

Q.E.F

Theorem 3.3.1.9

Given a point A and a line $\overline{BC}$ there exists a unique line passing through A that is parallel to $\overline{BC}$.

To prove; To draw a straight line through a given point A parallel to a given straight line $\overline{BC}$.

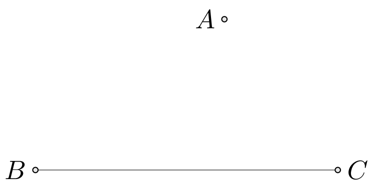


Figure 3.75

Proof. **Proof by construction:**

1. Take a point D at random on $\overline{BC}$.
2. Join $\overline{AD}$.
3. Construct $\angle DAE$ equal to $\angle ADC$ on the straight line $\overline{DA}$ at point A.
4. Produce the straight line $\overline{AF}$ in a straight line with $\overline{EA}$. I.Post.1,I.23,I.Post.2

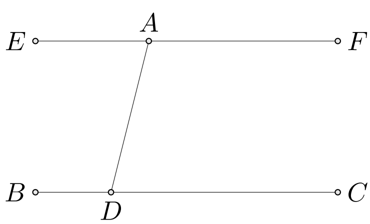


Figure 3.76

Justification: Since the straight line $\overline{AD}$ falling on the two straight lines $\overline{BC}$

and $\overline{EF}$ makes the alternate $\angle EAD$ and $\angle ADC$ equal to one another, $\therefore$ $\overline{EAF}$ is parallel to $\overline{BC}$. $\therefore$ the straight line $\overline{EAF}$ has been drawn through the given point A parallel to the given straight line $\overline{BC}$. I.27

Q.E.F

Theorem 3.3.1.10

In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.

Given: $\widehat{ABC}$, and side $\overline{BC}$ produced to point D.

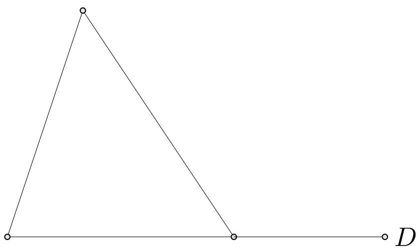


Figure 3.77

To prove:

1. The exterior $\angle ACD$ equals the sum of the two interior and opposite $\angle CAB$ and $\angle ABC$.
2. The sum of the three interior $\angle ABC$, $\angle BCA$, and $\angle CAB$ equals two right angles.

Proof.

- Draw $\overline{CE}$ through point C parallel to line $\overline{AB}$.

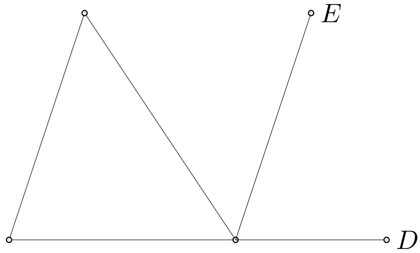


Figure 3.78

- Since

$$\overline{AB} \parallel \overline{CE}$$

and $\overline{AC}$ falls upon them, the alternate

$$\angle BAC = \angle ACE$$

I.29

- Since

$$\overline{AB} \parallel \overline{CE}$$

and line $\overline{BD}$ falls upon them, the exterior

$$\angle ECD = \angle ABC$$

I.29

- By combining both statements, we conclude that the whole

$$\angle ACD = \angle BAC + \angle CAB$$

C.N.2

- Now, add $\angle ACB$ to each side.

$$\angle ACD + \angle ACB = \angle ABC + \angle BAC + \angle CAB$$

I.13, C.N.1

- But,

$$\angle ACD + \angle ACB = 2 \times 90^\circ$$

$\therefore$

$$\angle ABC + \angle BCA + \angle CAB = 2 \times 90^\circ$$

I.13

Hence, in any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.

Q.E.F

3.2

Theorem 3.3.2.1

Straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.

Proof.

- Let $\overline{AB}$ and $\overline{CD}$ be equal and parallel, and let the straight lines $\overline{AC}$ and $\overline{BD}$ join them at their ends in the same directions. I say that $\overline{AC}$ and $\overline{BD}$ are also equal and parallel.
- Join BC. I.33,I.Post.1

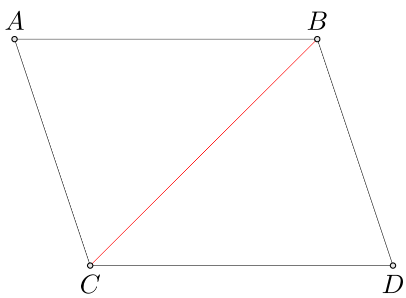


Figure 3.79

- Since

$$\overline{AB} \parallel \overline{CD}$$

and BC falls upon them,

$\therefore$ the alternate angles

$$\angle ABC = \angle BCD$$

I.29

- Since

$$\overline{AB} = \overline{CD}$$

and $\overline{BC}$ is common, the two sides $\overline{AB}$ and $\overline{BC}$ equal the two sides $\overline{DC}$ and $\overline{CB}$, and

$$\angle ABC = \angle BCD$$

$\therefore$ the base

$$\overline{AC} = \overline{BD}$$

the triangle

$$\widehat{ABC} = \widehat{DCB}$$

and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

$\therefore$ the angle

$$\angle ACB = \angle CBD$$

I.4

- Since the straight line $\overline{BC}$ falling on the two straight lines $\overline{AC}$ and $\overline{BD}$ makes the alternate angles equal to one another,

$\therefore$

$$\overline{AC} \parallel \overline{BD}$$

I.27

And it was also proved equal to it. $\therefore$ straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.

Q.E.F

Theorem 3.3.2.2

In a parallelogram, opposite sides and angles are equal, and the diameter bisects the area.

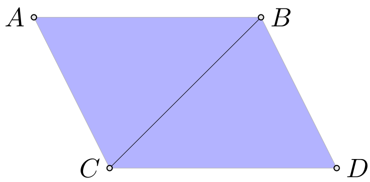


Figure 3.80

Given: Parallelogram $ACDB$ with diameter BC.

Proof. **1. Opposite Angles are Equal** I.34, I.29

- Since $\overline{AB} \parallel \overline{CD}$ and $\overline{BC}$ is a transversal line, we have

$$\angle ABC = \angle BCD$$

I.34

- Similarly, since $\overline{AC} \parallel \overline{BD}$ and $\overline{BC}$ is a transversal line,we have

$$\angle ACB = \angle CBD$$

I.29

2. Triangle Equality I.26

- Consider $\widehat{ABC}$ and $\widehat{DCB}$
- They have

$$\angle ABC = \angle BDC$$

$$\angle ACB = \angle CBD$$

and $\overline{BC}$ in common.

$\therefore$ by SAS criterion,

$$\widehat{ABC} \cong \widehat{DCB}$$

- This implies

$$\overline{AB} = \overline{CD}$$

$$\overline{AC} = \overline{BD}$$

$$\angle BAC = \angle CDB$$

3. Sum of Angles C.N.2

- $\angle ABC = \angle BCD$ and $\angle CBD = \angle ACB$,
- imply that the sum of angles $\angle ABD = \angle ACD$.

4. Conclusion - Opposite Sides and Angles:

- In parallelogram $ACDB$,
- opposite sides

$$\overline{AB} = \overline{CD}$$

and

$$\overline{AC} = \overline{BD}$$

- and opposite angles

$$\angle ABC = \angle BDC$$

and

$$\angle ACB = \angle CBD$$

5. Diameter Bisects the Parallelogram I.4

- Since

$$\overline{AB} = \overline{CD}$$

  $\overline{BC}$ is common, and

$$\angle ABC = \angle BCD$$

-

$$\widehat{ABC} \cong \widehat{BCD}$$

- This implies

$$\overline{AC} = \overline{DB}$$

$$\widehat{ABC} = \widehat{DCB}$$

6. Conclusion - Diameter Bisects the Area:

$\therefore$ the diameter $\overline{BC}$ bisects the parallelogram $ACDB$.

Q.E.F

Theorem 3.3.2.3

Parallelograms on the same base and in the same parallels are equal to one another.

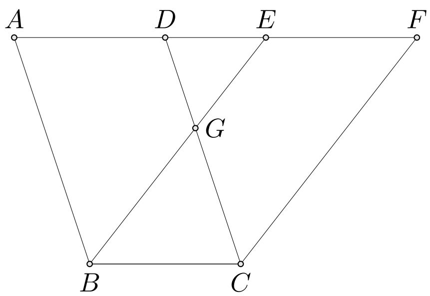


Figure 3.81

Given: Parallelograms $ABCD$ and $EBCF$ on the same base $\overline{BC}$ and in the same parallels $\overline{AF}$ and $\overline{BC}$.

Proof. 1. **Parallelogram Properties** I.35, I.34

- Since $ABCD$ is a parallelogram,

$$\overline{AD} = \overline{BC}$$

I.35

- Similarly, for $EBCF$,

$$\overline{EF} = \overline{BC}$$

I.34

2. **Triangle Equality** C.N.1, C.N.2, I.4

- Since

  $$\overline{AD} = \overline{BC}$$

  and

  $$\overline{EF} = \overline{BC}$$

  we can conclude that

  $$\overline{AD} = \overline{EF}$$

- Since $\overline{DE}$ is common, we have

  $$\overline{AE} = \overline{DF}$$

  C.N.1

- Also,

  $$\overline{AB} = \overline{DC}$$

  I.34

$\therefore$ the two sides $\overline{EA}$ and $\overline{AB}$ equal the two sides $\overline{FD}$ and $\overline{DC}$, respectively, and $\angle FDC = \angle EAB$. I.4

- Thus, the base

  $$\overline{EB} = \overline{FC}$$

  and

  $$\widehat{EAB} = \widehat{FDC}$$

  C.N.2

3. **Trapezium Equality** C.N.3

   - Subtract $\widehat{DGE}$ from each side.

   - Then the trapezium $ABGD$ equals the trapezium $EGCF$.

4. **Whole Parallelogram Equality.** C.N.2

   - Add the $\widehat{GBC}$ to each side.
   - Then the whole parallelogram $ABCD$ equals the whole parallelogram $EBCF$.

Conclusion:

$\therefore$ parallelograms on the same base and in the same parallels equal one another.

Q.E.F

Theorem 3.3.2.4

Triangles which are on the same base and in the same parallels are equal to one another.

Let $\widehat{ABC}$ and $\widehat{DBC}$ be triangles on the same base $\overline{BC}$ and in the same parallels $\overline{AD}$ and $\overline{BC}$.

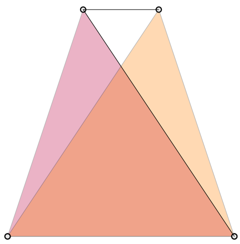


Figure 3.82

I say that $\widehat{ABC}$ equals $\widehat{DBC}$.

Produce $\overline{AD}$ in both directions to E and F. Draw $\overline{BE}$ through B parallel to $\overline{CA}$, and draw $\overline{CF}$ through C parallel to $\overline{BD}$. I.Post.2, I.31

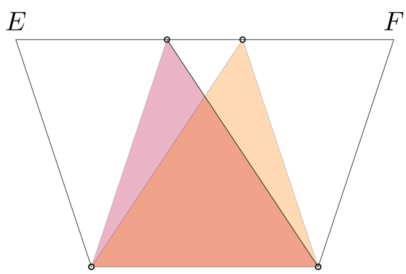


Figure 3.83

Proof.

- Then each of the figures $EBCA$ and $DBCF$ is a parallelogram, and they are equal, for they are on the same base $\overline{BC}$ and in the same parallels $\overline{BC}$ and $\overline{EF}$. I.35

- Moreover, the$\widehat{ABC}$ is half of the parallelogram $EBCA$, for the diameter $\overline{AB}$ bisects it.
- And the $\widehat{DBC}$ is half of the parallelogram $DBCF$, for the diameter $\overline{DC}$ bisects it. I.34

$\therefore$ the

$$\widehat{ABC} = \widehat{DBC}$$

C.N.1

$\therefore$ triangles which are on the same base and in the same parallels equal one another.

Q.E.F

Theorem 3.3.2.5

Triangles which are on equal bases and in the same parallels are equal to one another.

Let $\widehat{ABC}$ and $\widehat{DEF}$ be triangles on equal bases $\overline{BC}$ and $\overline{EF}$ and in the same parallels $\overline{BF}$ and $\overline{AD}$.

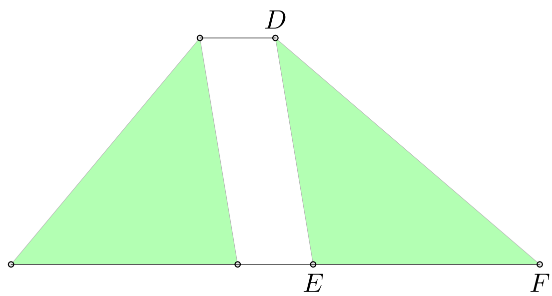


Figure 3.84

I say that the $\widehat{ABC}$ equals the $\widehat{DEF}$.

Produce $\overline{AD}$ in both directions to G and H. Draw $\overline{BG}$ through B parallel to $\overline{CA}$, and draw $\overline{FH}$ through F parallel to $\overline{DE}$. I.Post.2, I.31

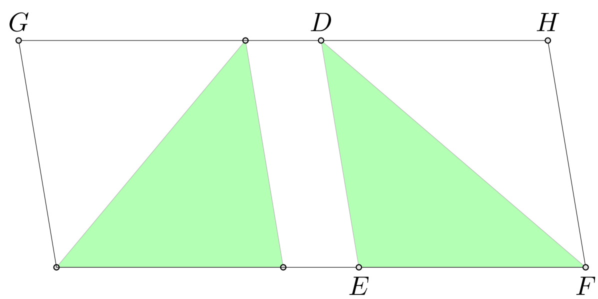


Figure 3.85

Proof. • Then each of the figures $GBCA$ and $DEFH$ is a parallelogram, and

$GBCA$ equals $DEFH$, for they are on equal bases $\overline{BC}$ and $\overline{EF}$ and in the same parallels $\overline{BF}$ and $\overline{GH}$. I.36

- Moreover, the $\widehat{ABC}$ is half of the parallelogram $GBCA$, for the diameter $\overline{AB}$ bisects it.

- And the $\widehat{FED}$ is half of the parallelogram $DEFH$, for the diameter $\overline{DF}$ bisects it. I.35

$\therefore$ the

$$\widehat{ABC} = \widehat{DEF}$$

C.N.1

$\therefore$ triangles which are on equal bases and in the same parallels equal one another.

Q.E.F

Theorem 3.3.2.6

Equal triangles which are on the same base and on the same side are also in the same parallels.

Given: Let $\widehat{ABC}$ and $\widehat{DBC}$ be equal triangles which are on the same base $\overline{BC}$ and on the same side of it.

Join $\overline{AD}$. I.Post.1

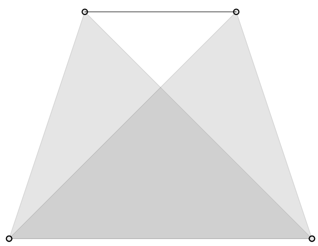


Figure 3.86

To Prove:

$$\overline{AD} \parallel \overline{BC}$$

Construction: If not, draw $\overline{AE}$ through the point A parallel to the straight line $\overline{BC}$, and join$\overline{EC}$.

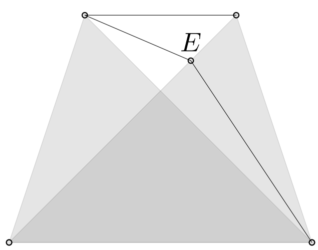


Figure 3.87

Proof. **Proof by Contradiction:**

∴

$$\widehat{ABC} = \widehat{EBC}$$

for it is on the same base $\overline{BC}$ with it and in the same parallels. I.37

- But

$$\widehat{ABC} = \widehat{DBC}$$

(by hypothesis)

∴

$$\widehat{DBC} = \widehat{EBC}$$

the greater equals the less, which is impossible. C.N.1

∴

$$\overline{AE} \parallel \overline{BC}$$

- Similarly, we can prove that neither is any other straight line except$\overline{AD}$.

∴

$$\overline{AD} \parallel \overline{BC}$$

∴ equal triangles which are on the same base and on the same side are also in the same parallels.

Q.E.F

Theorem 3.3.2.7

Equal triangles which are on equal bases and on the same side are also in the same parallels.

Given: Let $\widehat{ABC}$ and $\widehat{CDE}$ be equal triangles on equal bases $\overline{BC}$ and $\overline{CE}$ and on the same side.

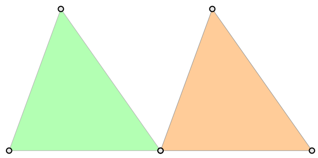


Figure 3.88

To Prove: They are also in the same parallels.

Construction: Join AD. I say that AD is $\parallel$ to BE. I.Post.1

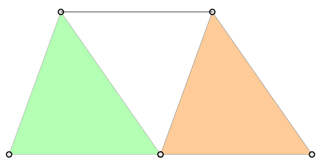


Figure 3.89

Proof. **Proof by Contradiction:**

If not, draw AF through A $\parallel$ to BE, and join FE. I.31, I.Post.1

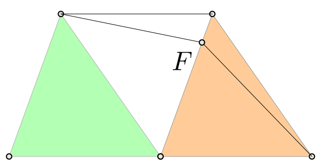


Figure 3.90

$\therefore$

$$\widehat{ABC} = \widehat{FCE}$$

for they are on equal bases $\overline{BC}$ and $\overline{CE}$ and in the same parallels $\overline{BE}$ and $\overline{AF}$. I.38

- But,

$$\widehat{ABC} = \widehat{DCE}$$

$\therefore$

$$\widehat{DCE} = \widehat{FCE}$$

the greater equals the less, which is impossible. C.N.1

$\therefore$

$$\overline{AF} \nparallel \overline{BE}$$

- Similarly, we can prove that neither is any other straight line except $\overline{AD}$.

$\therefore$

$$\overline{AD} \parallel \overline{BE}$$

$\therefore$ equal triangles which are on equal bases and on the same side are also in the same parallels.

Q.E.F

Theorem 3.3.2.8

In any parallelogram, the complements of the parallelograms about the diameter equal one another.

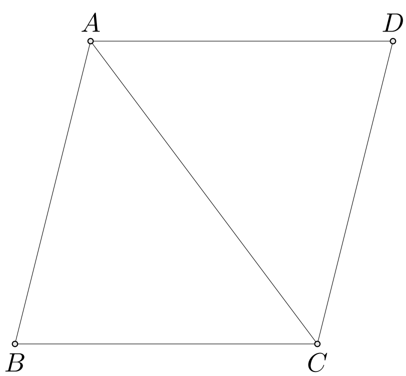


Figure 3.91

Proof.

- In parallelogram $AEKH$, since $\overline{AK}$ is its diameter:

$$\widehat{AEK} = \widehat{AHK}$$

  Similarly,

$$\widehat{KFC} = \widehat{KGC}$$

I.34,C.N.2

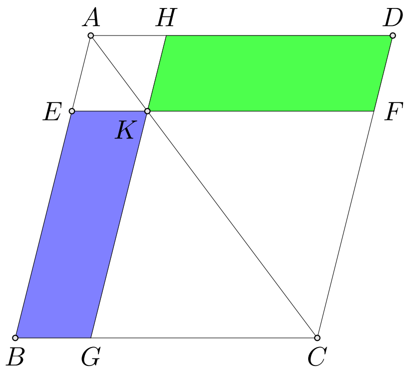


Figure 3.92

- Since

$$\widehat{AEK} = \widehat{AHK}$$

I.34 and

$$\widehat{KFC} = \widehat{KGC}$$

C.N.2

∴

$$\widehat{AEK} + \widehat{KGC} = \widehat{AHK} + \widehat{KFC}$$

C.N.3

- Since the whole

$$\widehat{ABC} = \widehat{ADC}$$

∴ the remaining complement $BGKE$ equals the remaining complement $KFDH$.

Q.E.F

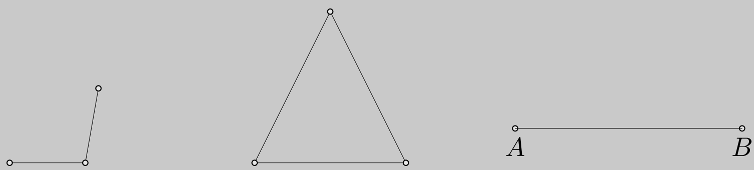
Construction 3.3.2.1

To a given straight line in a given rectilinear angle, to apply a parallelogram equal to a given triangle.

Given a straight line $\overline{AB}$, a rectilinear angle D, and a triangle C, the goal is to apply a parallelogram equal to triangle C to the straight line $\overline{AB}$ in an angle equal to D.

Construction 3.3.2.2

1. Construct the parallelogram $BEFG$ equal to triangle C in the $\angle EBG$ which equals D, and let it be placed so that $\overline{BE}$ is in a straight line with $\overline{AB}$.



(a) Given angle D (b) Given triangle C (c) given $\overline{AB}$

Figure 3.93: Given Prop 44

Construct a polygon equal to C; I.42

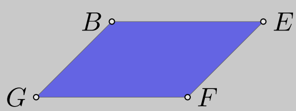


Figure 3.94: Polygon $BEFG$

2. Produce $\overline{FG}$ through to H, and draw $\overline{AH}$ through $A \parallel$ to either $\overline{BG}$ or $\overline{EF}$. Join$\overline{HB}$. I.42

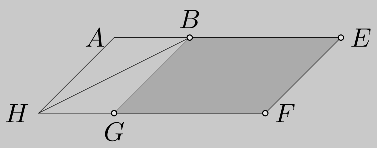


Figure 3.95

3. By the construction, $\overline{HF}$ falls upon the parallels $\overline{AH}$ and $\overline{EF}$. ∴, the sum of $\angle AHF$ and $\angle HFE$ equals two right angles. I.Post.2
4. Since $\overline{HF}$ falls on parallels, the sum of $\angle BHG$ and $\angle GFE$ is less than two right angles. Straight lines produced indefinitely from angles less than two right angles meet, so$\overline{HB}$ and $\overline{FE}$ will meet. I.31, I.Post.1
5. Let them be produced and meet at K. Draw $\overline{KL}$ through the point $K \parallel$ to either $\overline{EA}$ or $\overline{FH}$. Produce $\overline{HA}$ and $\overline{GB}$ to the points L and M. I.31

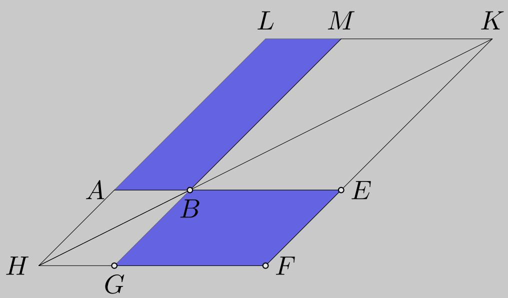


Figure 3.96

Proof. • Then $HLKF$ is a parallelogram, $\overline{HK}$ is its diameter, and $AHGB$ and $MBEK$ are parallelograms, and $LABM$ and $BGFE$ are the complements about HK. ∴, $LABM$ equals $BGFE$. I.43

- But $BGFE$ equals triangle C, ∴ $LABM$ also equals C. C.N.1

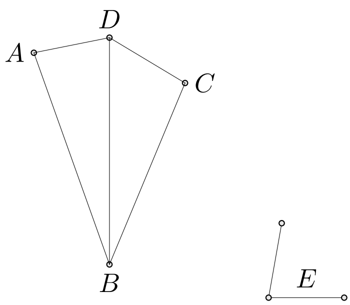
- Since $\angle GBE$ equals $\angle ABM$, while $\angle GBE$ equals D, ∴ $\angle ABM$ also equals the $\angle D$. I.15, C.N.1

$\therefore$ the parallelogram $LABM$ equal to the given triangle C has been applied to the given straight line $\overline{AB}$, in $\angle ABM$ which equals D.

Q.E.F

Construction 3.3.2.3

To construct a parallelogram equal to a given rectilinear figure $ABCD$ in a given rectilinear angle E.



(a) Rectilier shape $ABCD$ (b) Rectiliear angle E

Figure 3.97

Given:

- Rectilinear figure $ABCD$.
- Rectilinear angle E.
- A straight line can be drawn between any two points, join $\overline{BD}$. I.Post.1

Construction 3.3.2.4

1. A straight line can be drawn between any two points, join $\overline{BD}$. I.Post.1
2. Construct the parallelogram $FGHK$ equal $\widehat{ABD}$ in $\angle HKF$ which equals E. I.42

3. Construct the parallelogram $GLMN$ equal to $\widehat{DBC}$ to $\overline{GH}$ in $\angle GHM$ which equals E. I.44

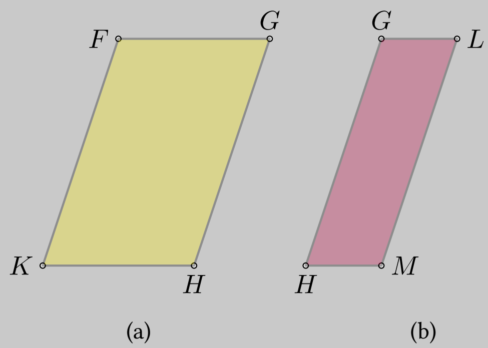


Figure 3.98

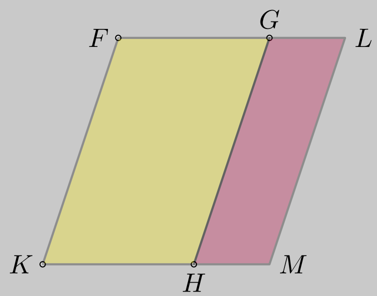


Figure 3.99

Proof. 1. Since angle E equals both $\angle HKF$ and $\angle GHM$, $\angle HKF$ equals $\angle GHM$. C.N.1

2. Add $\angle KHG$ to both sides.

$\therefore$

$$\angle FKH + \angle KHG = \angle KHG + \angle GHM$$

C.N.2

3.

$$\angle FKH + \angle KHG 2 \times 90^\circ$$

$\therefore$

$$\angle KHG + \angle GHM = 2 \times 90^\circ$$

I.29, C.N.1

4. With a straight line $\overline{GH}$, and at point H on it, two straight lines $\overline{KH}$ and $\overline{HM}$ not lying on the same side make the adjacent angles together equal to two right angles.

$\therefore$ $\overline{KH}$ is in a straight line with $\overline{HM}$. I.29

5. Since line $\overline{HG}$ falls upon the parallels $\overline{KM}$ and $\overline{FG}$, the alternate $\angle MHG$ and $\angle HGF$ are equal. I.29

6. Add $\angle HGL$ to both sides,

$$\angle MHG + \angle HGL = \angle HGF + \angle HGL$$

C.N.2

7.

$$\angle MHG + \angle HGL = 2 \times 90^\circ$$

$\therefore$

$$\angle HGF + \angle HGL = 2 \times 90^\circ$$

I.29, C.N.1, I.14

8. $\overline{FG}$ is in a straight line with $\overline{GL}$. I.34, I.30

9. Since

$$\overline{FK} \# \overline{HG}$$

and

$$\overline{HG} \# \overline{ML}$$

$\therefore$

$$\overline{KF} \# \overline{ML}$$

and the straight lines $\overline{KM}$ and $\overline{FL}$ join them at their ends.

$\therefore$

$$\overline{KM} \# \overline{FL}$$

I.33, C.N.1

10. Since $\widehat{ABD}$ equals parallelogram $FKHG$, and $\widehat{DBC}$ equals parallelogram $GHML$, the whole rectilinear figure $ABCD$ equals the whole parallelogram $KFLM$. C.N.2

$\therefore$ the parallelogram $KFLM$ has been constructed equal to the given rectilinear figure $ABCD$ in the$\angle FKM$, which equals the given angle E.

Q.E.F

Construction 3.3.2.5

To describe a square on a given straight line.

Given: A straight line $\overline{AB}$.

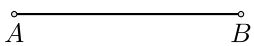


Figure 3.100: Given Line $\overline{AB}$

Construction 3.3.2.6

1. Draw $\overline{AC}$ at right angles to the straight line $\overline{AB}$ from point A on it.
2. Make
$$\overline{AD} = \overline{AB}$$
3. Draw $\overline{DE}$ through point D parallel to $\overline{AB}$.
4. Draw $\overline{BE}$ through point B parallel to $\overline{AD}$. I.11,I.3,I.31

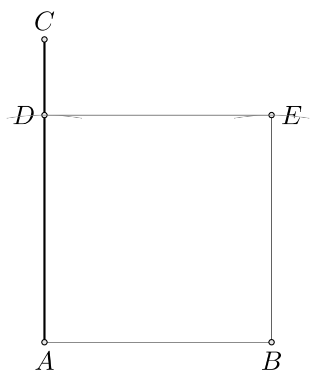


Figure 3.101: Parallelogram

Proof.

Lemma 1. *$ADEB$ is a parallelogram.*

- *Proof: By construction,*

$$\overline{DE} \parallel \overline{AB}$$

and

$$\overline{BE} \parallel \overline{AD}$$

thus $ADEB$ is a parallelogram.

Lemma 2. *$ADEB$ is equilateral.*

- *Proof: Since*

$$\overline{AB} = \overline{AD}$$

and

$$\overline{AB} = \overline{DE}$$

(as opposite sides of a parallelogram are equal),

$$\overline{AB} = \overline{AD} = \overline{DE}$$

Thus, $ADEB$ is equilateral.

Lemma 3. *$ADEB$ is right-angled.*

- *Proof:*

$$\angle BAD + \angle ADE = 2 \times 90^\circ$$

Since $\overline{AD}$ falls on parallels $\overline{AB}$ and $\overline{DE}$

$$\angle BAD = 90^\circ$$

Given:

$$\rightarrow \angle ADE = 90^\circ$$

I.29,I.Post.4,I.34

Now, consider the parallelogram $ADEB$.

Similarly, using the property of parallelograms, we can show that $\angle ABE$ and $\angle BED$ are also right angles.

Conclusion: Since $ADEB$ is both equilateral and right-angled, it is a square, and it is described on the straight line $\overline{AB}$.

$\therefore$ the square on the given $\overline{AB}$ is successfully constructed and proven. I.Def.22

Q.E.F

Theorem 3.3.2.9

In a right-angled triangle, the square of the length of the side opposite the right angle is equal to the sum of the squares of the lengths of the other two sides.

Proof.

Lemma 4. *The square described on the side of a right triangle opposite the right angle is equal to the sum of the squares described on the other two sides.* I.46

Lemma 5. *If a straight line falling on two straight lines makes the alternate angles equal to one another, the straight lines are parallel to one another.* I.31

Lemma 6. *A straight line can be drawn from any point to any other point.* I.Post.1

Lemma 7. *Parallels are lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.*I.Def.22,I.14

Construction 3.3.2.7

1. Describe the square $BDEC$ on side $\overline{BC}$.
2. Describe the squares $GFBA$ and $HACK$ on sides $\overline{BA}$ and $\overline{AC}$
3. Draw $\overline{AL}$ through A parallel to either $\overline{BD}$ or$\overline{CE}$.
4. Join $\overline{AD}$ and $\overline{FC}$. I.Def.23

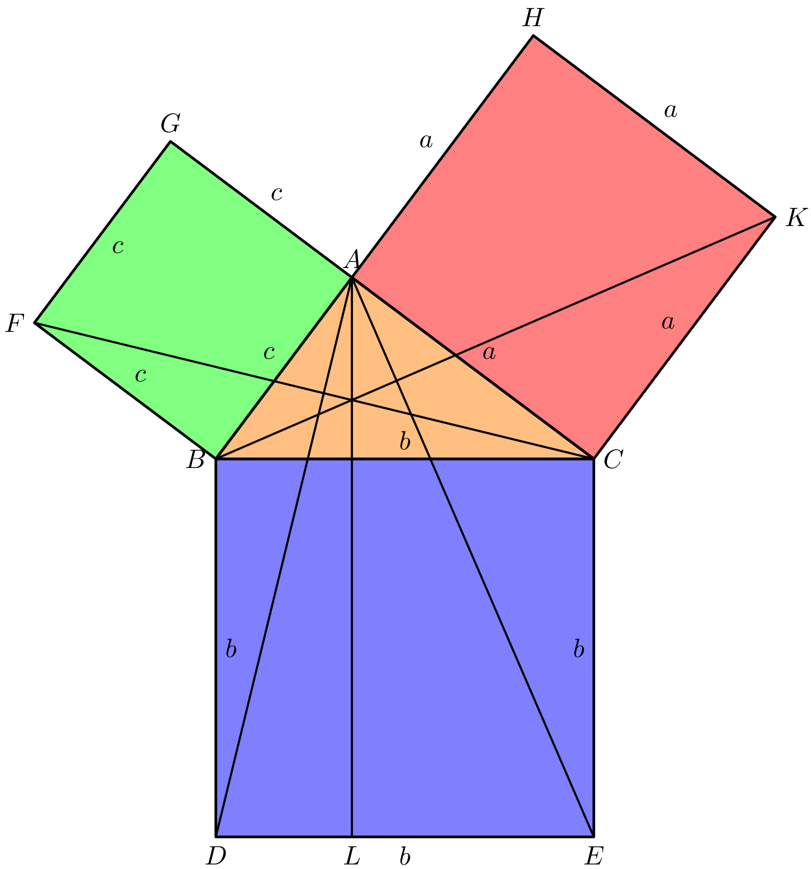


Figure 3.102

- In parallelograms, the opposite sides and angles are equal.hfillI.47
- If a straight line intersects two other straight lines, and if the interior angles on the same side of the intersecting line are supplementary, then the two straight lines are parallel to each other.hfillI.14
- Parallels are lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.hfillI.Def.22
- If a straight line falling on two straight lines makes the exterior angle equal

to the interior and opposite angle on the same side, the straight lines are parallel to one another.hfillI.Post.42

- If equals are added to equals, the wholes are equal.hfillC.N.2

  –

  $$\angle DBC = \angle FBA = 2 \times 90^{\circ}$$

  – add $\angle ABC$ to each, thus,

  $$\angle DBA = \angle FBC$$

  – Since

  $$\overline{DB} = \overline{BC}$$

  and

  $$\overline{FB} = \overline{BA}$$

  by side-angle-side equality, triangle

  $$\angle ABD = \triangle FBC$$

  – Parallelogram

  $$BL = 2 \times \{triangle ABD$$

  and square

  $$GB = 2 \times \triangle FBC$$

  $\therefore$ parallelogram BL equals square GB.

  – Similarly, parallelogram CL equals square HC.

– Hence, the whole square

$$BDEC = GFBS + HACK$$

C.N.2

$\therefore$ in right-angled triangles, the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Q.E.F

Theorem 3.3.2.10

If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right.

To prove:

$$\angle BAC$$

is a right angle.

Construction: Draw $\overline{AD}$ from point A perpendicular to side $\overline{AC}$. Make $\overline{AD}$ equal to $\overline{BA}$ and join $\overline{DC}$. I.11, I.3,I.Post.1

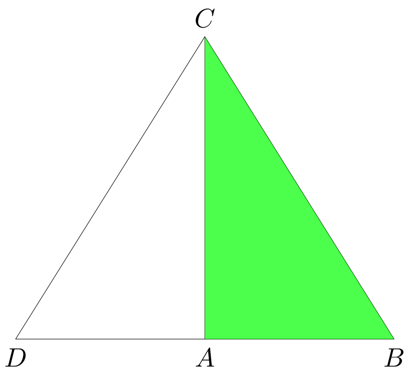


Figure 3.103

Proof.

- Since $\overline{DA}$ equals $\overline{AB}$ (by construction), the square on $\overline{DA}$ equals the square on $\overline{AB}$.

  Addition of Squares: Add the square on $\overline{AC}$ to both sides of the equation. Now, the sum of the squares on $\overline{DA}$ and $\overline{AC}$ equals the sum of the squares on $\overline{BA}$ and $\overline{AC}$. C.N.2

  Claim: The square on $\overline{DC}$ equals the sum of the squares on $\overline{DA}$ and $\overline{AC}$.

- Since $\angle DAC$ is right (by construction), the square on $\overline{DC}$ equals the sum of the squares on $\overline{DA}$ and $\overline{AC}$ (by the Pythagorean Theorem). I.47, C.N.1

Conclusion: Combining the two claims, the square on $\overline{DC}$ equals the square on $\overline{BC}$, and $\therefore$, $\overline{DC}$ equals $\overline{BC}$.

Congruence: Since

$$\overline{DA} = \overline{AB}$$

$\overline{AC}$ is common, and

$$\overline{DC} = \overline{BC}$$

and

$$\widehat{DAC} \cong \widehat{BAC}$$

Conclusion: Since $\angle DAC$ is a right angle, $\angle BAC$ is also a right angle (corresponding parts of congruent triangles are equal).

$\therefore$ If in a triangle, the square on one side equals the sum of the squares on the remaining two sides, then the angle contained by the remaining two sides is right.

Q.E.F

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