Lagrange Interpolation.

Let $f(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$. and $y_i = f(x_i)$.

Then, $f(x) = y_0 \cdot l_0(x) + y_1 \, l_1(x) + \cdots + y_{n-1} \cdot l_{n-1}(x)$ where $l_k(x_j) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$

$$\Rightarrow l_k(x) = \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i} = \frac{(x - x_1)}{x_k - x_1} \cdot \frac{x - x_2}{x_k - x_2} \cdots \frac{x - x_i}{x_k - x_i} \cdots$$

$$\underbrace{\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}}_{} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

called Vandermonde matrix V

and $\det V = \prod_{1 \le j < k \le n} (x_k - x_j)$

$\rightarrow \deg h = \max\{\deg f, \deg g\}$

Let $h_1(x) = f(x) + g(x)$.

Then, $h(0) = f(0) + g(0) \Rightarrow s_0 + s_1$

$h(1) = f(1) + g(1)$

$\vdots$

$h(n-1) = f(n-1) + g(n-1)$

Let $h_2(x) = f(x) \cdot g(x)$

Since $\deg h_2 = \deg f + \deg g = 2n-2$, we need $2n-2$ distinct points to recover h_2.

Think thing