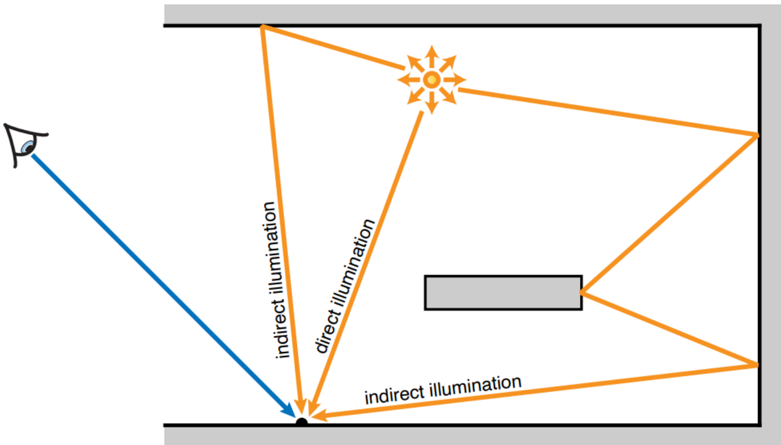
Direct Illumination

Definition

$$\underbrace{L_o\left(\mathbf{x},\vec{\omega}_o\right)}_{\text{outgoing}}=\underbrace{L_e\left(\mathbf{x},\vec{\omega}_o\right)}_{\text{emission}}+\underbrace{\int_{H^2} f_r\left(\mathbf{x},\vec{\omega}_i,\vec{\omega}_o\right)L_i\left(\mathbf{x},\vec{\omega}_i\right)\cos\theta_i d\vec{\omega}_i}_{\text{incoming}}$$

- Direct: L_i comes directly from an emitter ⇒ Length of light path = 2
- Indirect: L_i comes by bouncing off a scattering surface ⇒ Length of light path >= 2



Formulation

Let's first ignore the L_e part

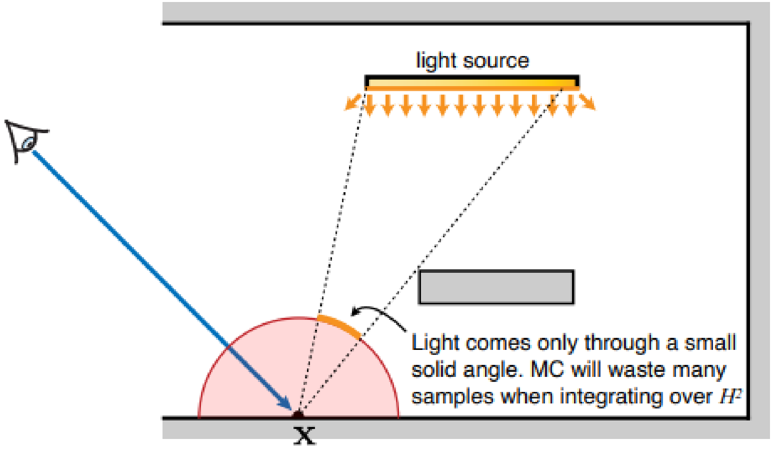
$$\int_{H^2} f_r\left(\mathbf{x},\vec{\omega}_i,\vec{\omega}_o\right)L_i\left(\mathbf{x},\vec{\omega}_i\right)\cos\theta_i d\vec{\omega}_i$$

Estimation of the integral by Monte Carlo Integration:

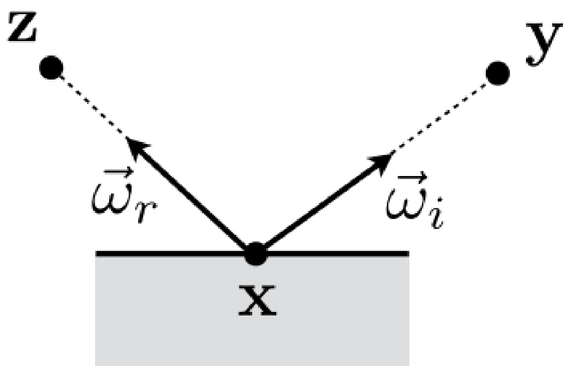
$$\left\langle L_r\left(\mathbf{x},\vec{\omega}_r\right)^N\right\rangle=\frac{1}{N}\sum_{k=1}^{N}\frac{f_r\left(\mathbf{x},\vec{\omega}_{i,k},\vec{\omega}_r\right)L_e\left(r\left(\mathbf{x},\vec{\omega}_{i,k}\right),-\vec{\omega}_{i,k}\right)\cos\theta_{i,k}d\vec{\omega}_{i,k}}{p_\Omega\left(\vec{\omega}_{i,k}\right)}$$

Here $L_i=L_e$ since we only consider light directly from the light source.

However this integral is not good to evaluate since if the light is occluded, many samples will be wasted.



So we pass to surface area form



$$
\begin{aligned}
L_i\left(\mathbf{x}, \vec{\omega}_i\right) &= L_i(\mathbf{x}, \mathbf{y}) \\
L_r\left(\mathbf{x}, \vec{\omega}_r\right) &= L_r(\mathbf{x}, \mathbf{z}) \\
f_r\left(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r\right) &= f_r(\mathbf{x}, \mathbf{y}, \mathbf{z})
\end{aligned}
$$

The Jacobian determinant of this transformation is

$$
d\vec{\omega}_i = \frac{|\cos\theta_o|}{\|\mathbf{x}-\mathbf{y}\|^2} dA
$$

Hemispherical form:

$$
L_r\left(\mathbf{x}, \vec{\omega}_r\right) = \int_{H^2} f_r\left(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r\right) L_i\left(\mathbf{x}, \vec{\omega}_i\right) \cos\theta_i d\vec{\omega}_i
$$

Surface area form:

$$
L_r(\mathbf{x}, \mathbf{z}) = \int_A f_r(\mathbf{x}, \mathbf{y}, \mathbf{z}) L_i(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) dA(\mathbf{y})
$$

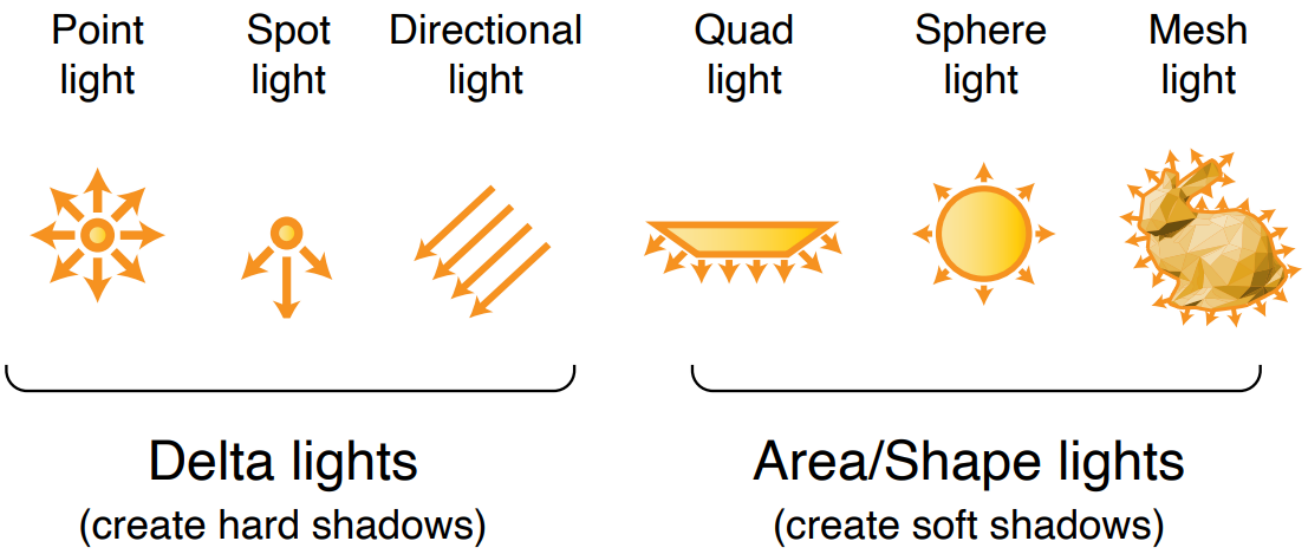
where $G(\mathbf{x}, \mathbf{y})$ is the geometry term

$$
G(\mathbf{x}, \mathbf{y}) = V(\mathbf{x}, \mathbf{y}) \frac{\overbrace{|\cos\theta_i|}^{\text{Original foreshortening term}} \quad |\cos\theta_o|}{\|\mathbf{x}-\mathbf{y}\|^2}
$$

and the $V(\mathbf{x}, \mathbf{y})$ is the visibility term

$$
V(\mathbf{x}, \mathbf{y}) = \begin{cases} 1: & \text{visible} \\ 0: & \text{not visible} \end{cases}
$$

Sampling Light Sources



Point Light

- Defined by a point $\mathbf{p}$ and a power Φ.
- Omnidirectional emission from a single point

$$L_r(\mathbf{x}, \mathbf{z}) = \frac{\Phi}{4\pi} f_r(\mathbf{x}, \mathbf{p}, \mathbf{z}) V(\mathbf{x}, \mathbf{p}) \frac{|\cos\theta_i|}{\|\mathbf{x} - \mathbf{p}\|^2}$$

Spot Light

- Defined by a point $\mathbf{p}$ and a directionally dependent radiant intensity I

$$L_r(\mathbf{x}, \mathbf{z}) = I(\mathbf{p}, \mathbf{x}) f_r(\mathbf{x}, \mathbf{p}, \mathbf{z}) V(\mathbf{x}, \mathbf{p}) \frac{|\cos\theta_i|}{\|\mathbf{x} - \mathbf{p}\|^2}$$

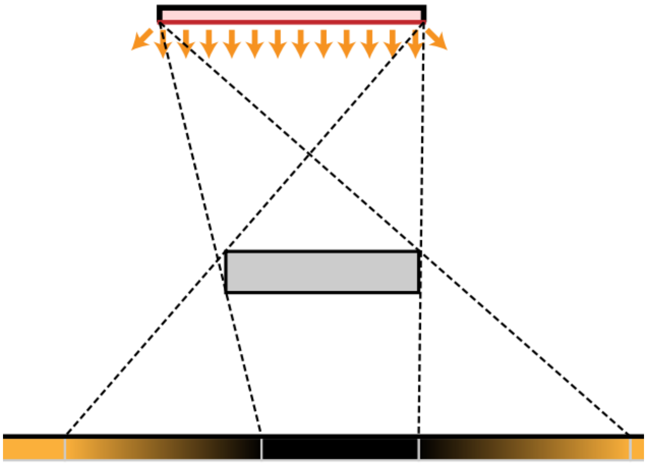
Directional Light

- Defined by a direction $\vec{\omega}$ and radiance $L_d(\vec{\omega})$ coming from direction $\vec{\omega}$

$$L_r\left(\mathbf{x}, \vec{\omega}_r\right) = f_r\left(\mathbf{x}, \vec{\omega}, \vec{\omega}_r\right) V(\mathbf{x}, \vec{\omega}) L_d(\vec{\omega}) \cos\theta$$

Note that the above light sources don't have volume, so we don't need to and can't sample them.

Quad/Area Light



An area light where every surface point emits radiance L_e. It can be uniformly sampled.

Sphere Light

Typically defined using a point $\mathbf{p}$, radius r and emitted power Φ.

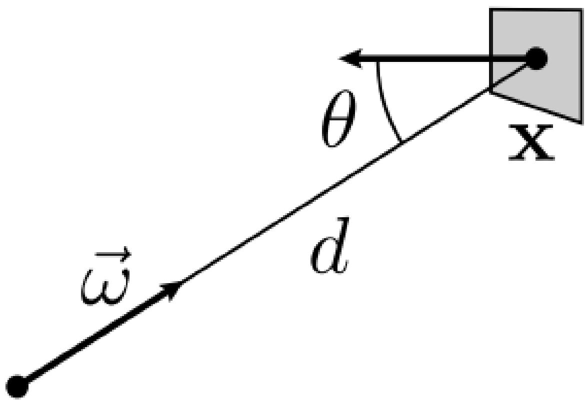
Sample Method

- Uniformly sample sphere area → more than half of the samples not visible
- Uniformly sample area of the visible spherical cap
- Uniformly sample solid angle subtended by the sphere

Remark

make sure to convert the PDF into the measure of the integral

$$
\begin{aligned}
p_A(\mathbf{x}) &= \frac{\cos\theta}{d^2} p_\Omega(\vec{\omega}) \\
p_\Omega(\vec{\omega}) &= \frac{d^2}{\cos\theta} p_A(\mathbf{x})
\end{aligned}
$$



Mesh Light

- An emissive mesh where every surface point emits radiance L_e

- Preprocess
  build a discrete PDF p_Δ for choosing polygons(triangles) proportional to their area

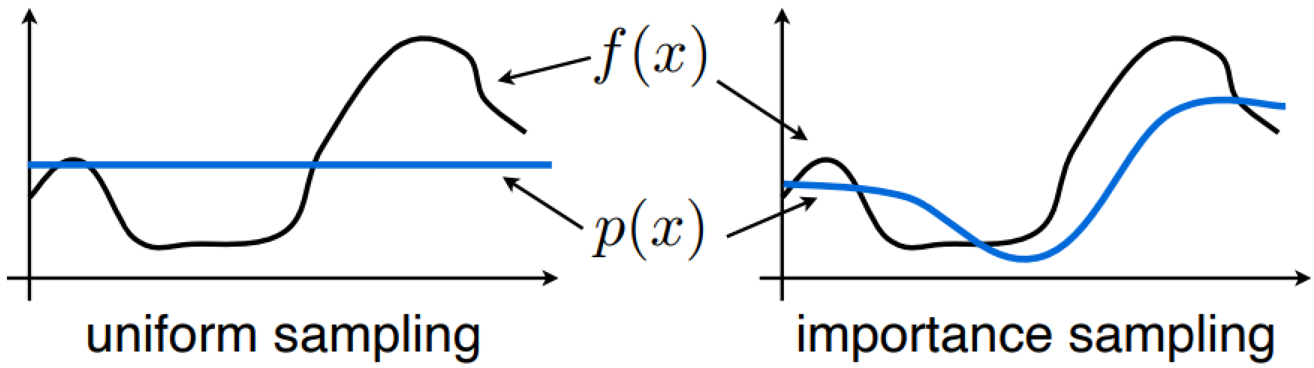
$$p_\Delta(i) = \frac{A(i)}{\sum A(k)}$$

- Run-time
  - sample a polygon i and a point $\mathbf{x}$ on i
  - compute the PDF of choosing the point $\mathbf{x}$

$$p_A(\mathbf{x}) = p_\Delta(i) p_A(\mathbf{x}|i) = \frac{1}{\sum A(k)}$$

Importance Sampling

We could place samples intelligently to reduce variance.



$$L_r\left(\mathbf{x}, \vec{\omega}_r\right) = \int_{H^2} f_r\left(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r\right) L_i\left(\mathbf{x}, \vec{\omega}_i\right) \cos\theta_i d\vec{\omega}_i$$

Three terms could be used for importance sampling:

- $\cos$ term
- BRDF
- Incident radiance

$\cos$ term

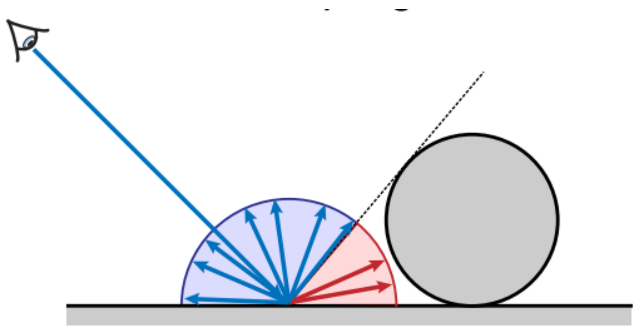
For ambient occlusion

$$L_r(\mathbf{x}) = \frac{\rho}{\pi} \int_{H^2} V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

By Monte Carlo Integration

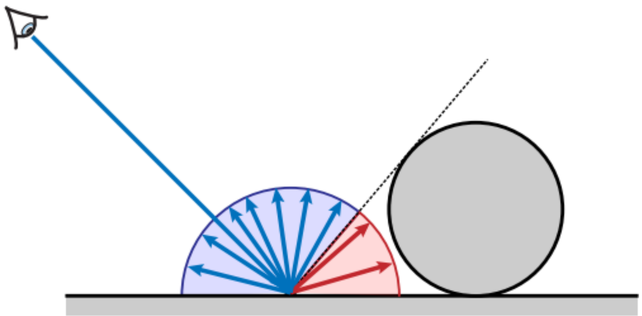
$$L_r(\mathbf{x}) = \frac{\rho}{\pi N} \sum_{k=1}^{N} \frac{V(\mathbf{x}, \vec{\omega}_{i,k}) \cos\theta_{i,k}}{p(\vec{\omega}_{i,k})}$$

If we uniformly sample the hemisphere



$$p(\vec{\omega}_{i,k}) = \frac{1}{2\pi}$$
$$L_r(\mathbf{x}) \simeq \frac{2\rho}{N}\sum_{k=1}^{N} V(\mathbf{x}, \vec{\omega}_{i,k}) \cos\theta_{i,k}$$

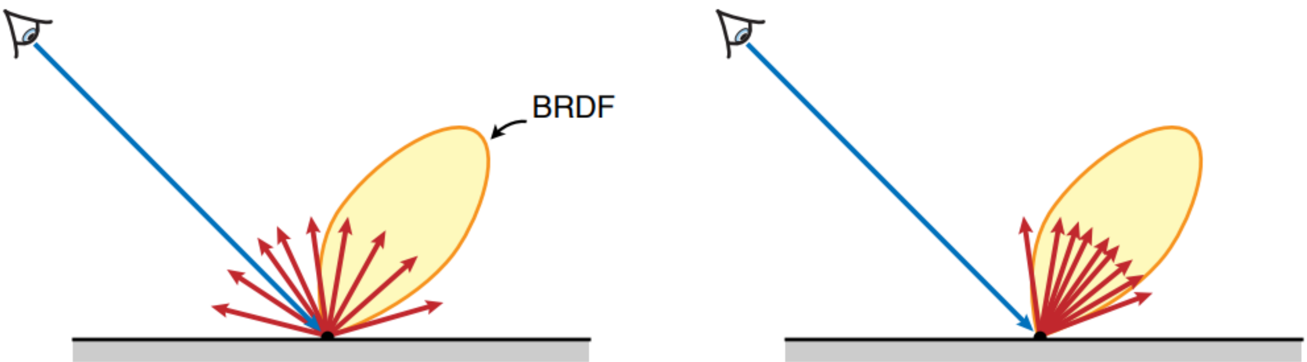
If we use cosine weighted importance sampling



$$p(\vec{\omega}_{i,k}) = \frac{\cos\theta_{i,k}}{\pi}$$
$$L_r(\mathbf{x}) \simeq \frac{\rho}{N}\sum_{k=1}^{N} V(\mathbf{x}, \vec{\omega}_{i,k})$$

BRDF

Importance sampling the BRDF



For Normalized Phong model, we could use the normalized Phong-like $\cos^2 \alpha$ lobe

$$p(\theta, \phi) = \frac{\alpha + 2}{2\pi} \cos^{\alpha}\theta$$
$$(\theta, \phi) = \left(\cos^{-1}\left((1-\xi_1)^{\frac{1}{\alpha+2}}\right), 2\pi\xi_2\right)$$

For BRDFs with multiple lobes:

- Probabilistically choose a lobe, e.g. proportional to the coefficient
- Sample a direction using the lobe

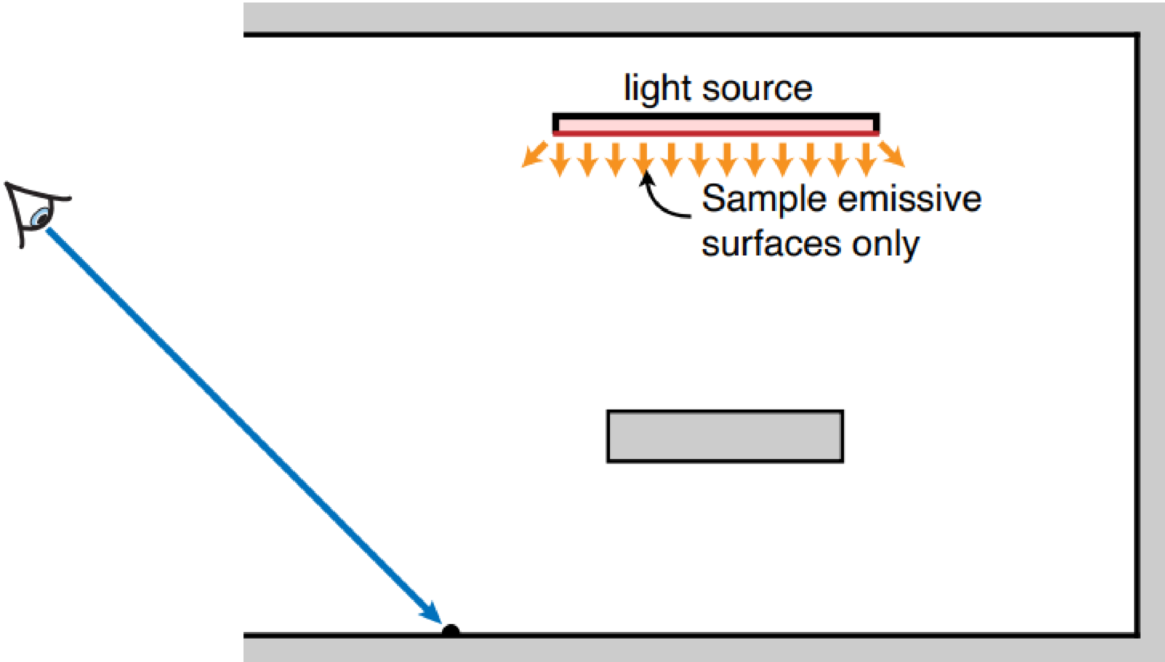
$$p_\Omega(\omega) = p_1(l)p_\Omega(\omega|l)$$

Example:

- Blend Microfacet model
  - k_d diffuse coefficient
  - k_s specular/glossy coefficient
- Disney BSDF

Incident Radiance

For direct illumination we can explicitly sample emissive surfaces



$$L_r(\mathbf{x},\mathbf{z}) = \int_{A_e} f_r(\mathbf{x},\mathbf{y},\mathbf{z})L_e(\mathbf{y},\mathbf{x})V(\mathbf{x},\mathbf{y})\frac{|\cos\theta_i|\,|\cos\theta_o|}{\|\mathbf{x}-\mathbf{y}\|^2}dA(\mathbf{y})$$

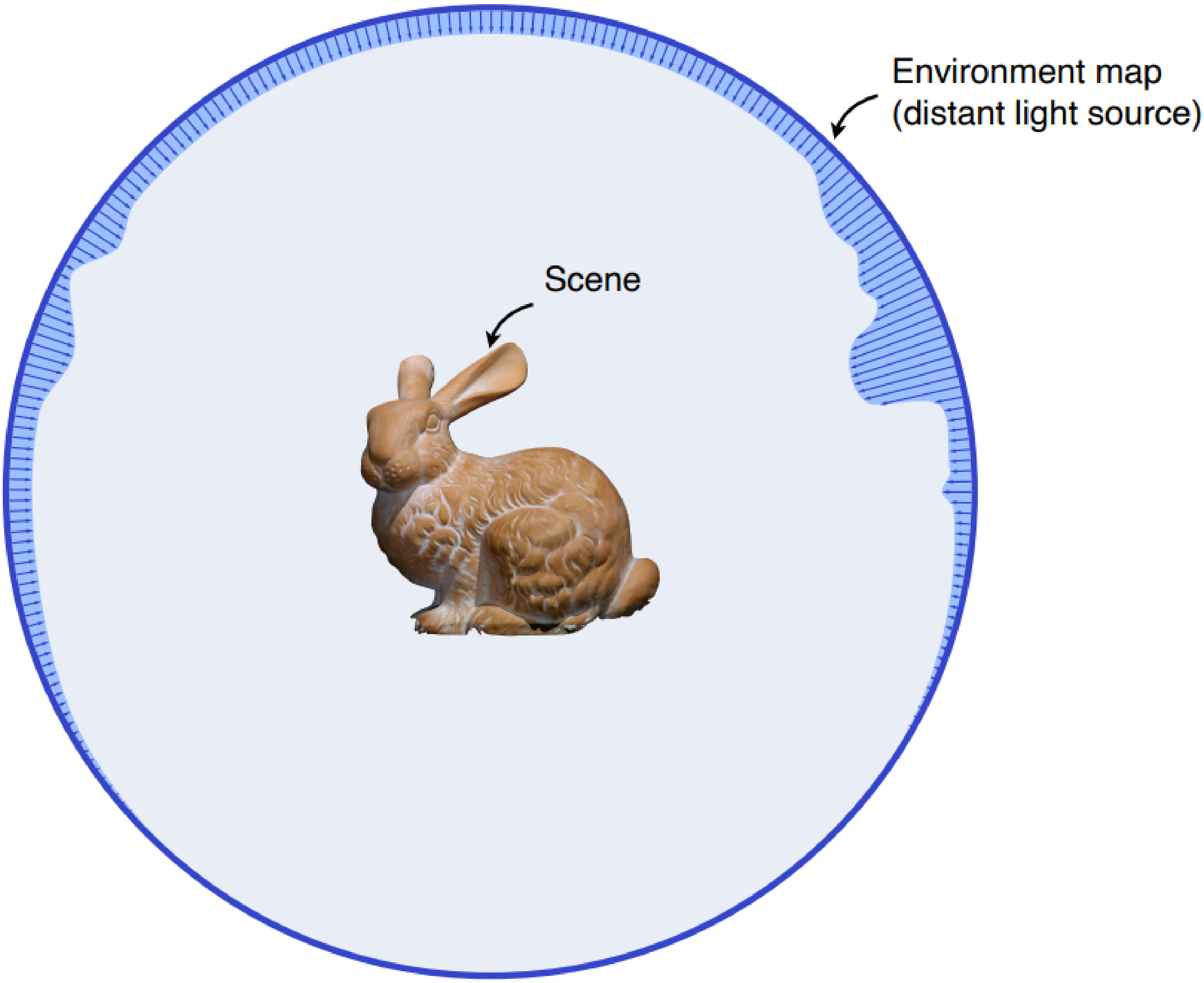
Integrate over emissive surfaces only

We could combine multiple sampling strategies using multiple importance sampling.

Environment Lighting

The environment of a scene is represented with one or more images. The images “wrap” the virtual scene serving as a distant sources of illumination.

$$\begin{aligned} L_r(\mathbf{x},\vec{\omega}_r) &= \int_\Omega f_r(\vec{\omega}_i,\vec{\omega}_r)L_i(\mathbf{x},\vec{\omega}_i)\cos\theta_i d\vec{\omega}_i \\ &= \int_\Omega f_r(\vec{\omega}_i,\vec{\omega}_r)L_{\text{env}}(\vec{\omega}_i)V(\mathbf{x},\vec{\omega}_i)\cos\theta_i d\vec{\omega}_i \end{aligned}$$



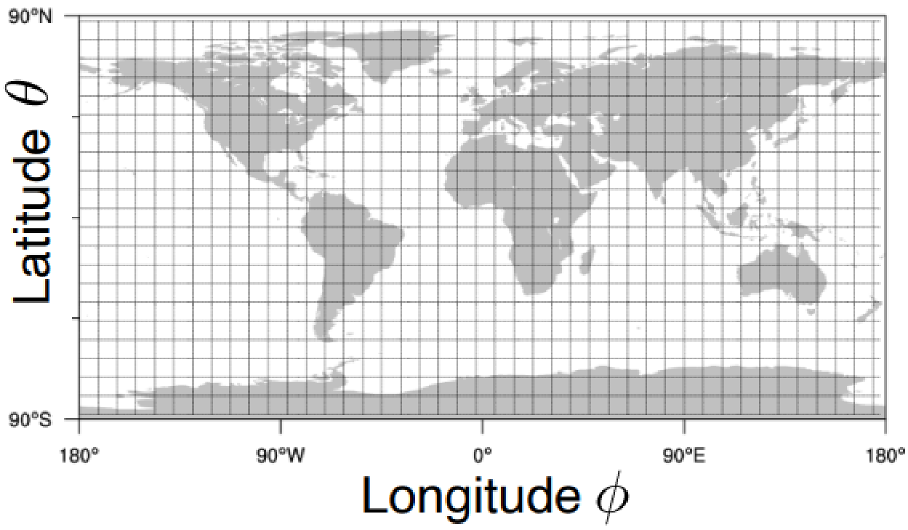
We could model the environment using

- Latitude/longitude map
- Cube map(sky box)
- Octahedron map
- HEALPix

Importance Sampling L_{env}

Marginal/Conditional CDF Method

Assuming we are using the lat/long parameterization



and we reduce the variance by making

$$p(\theta, \phi) \propto L_{\text{env}}(\theta, \phi) \sin\theta$$

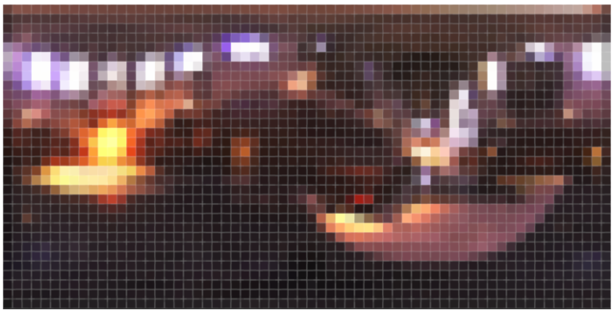
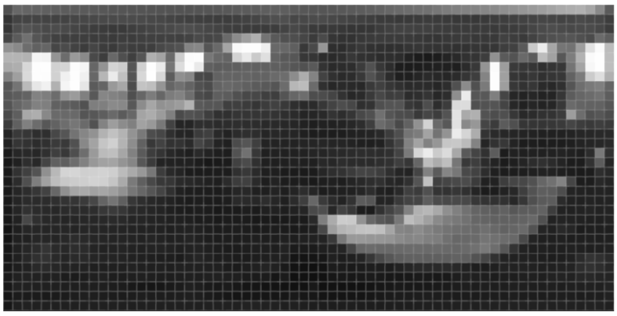
The $\sin$ term is because if we want to integrate over S^2, we have

$$\begin{aligned}\int_{S^2} f(\vec{\omega}) d\vec{\omega} &= \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) \sin\theta d\theta d\phi \\ &\approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(\theta_i, \phi_i) \sin\theta_i}{p(\theta_i, \phi_i)}\end{aligned}$$

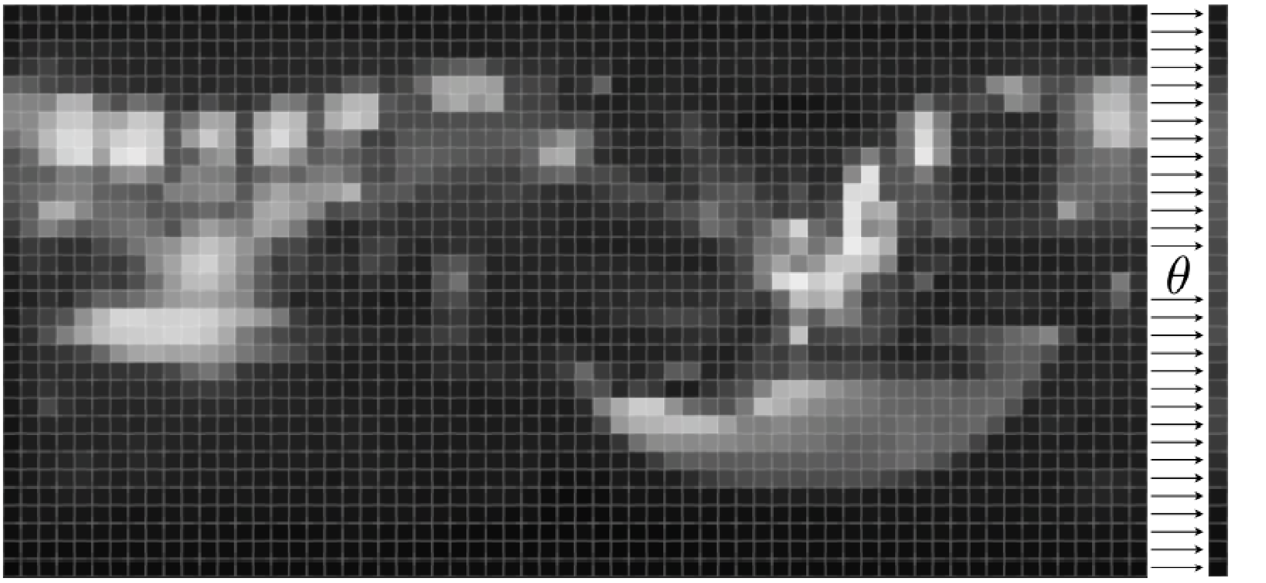
the $\sin$ term will effectively cancel out.

Now we can draw samples from the joint PDF $p(\theta, \phi) \propto L_{\text{env}}(\theta, \phi) \sin\theta$.

1. Create scalar version $L'(\theta, \phi)$ of $L_{\text{env}}(\theta, \phi) \sin\theta$.

Substeps	
Scalar version(avg, max)	 θ ϕ
Multiplied by $\sin\theta$	 θ ϕ

2. Marginalization



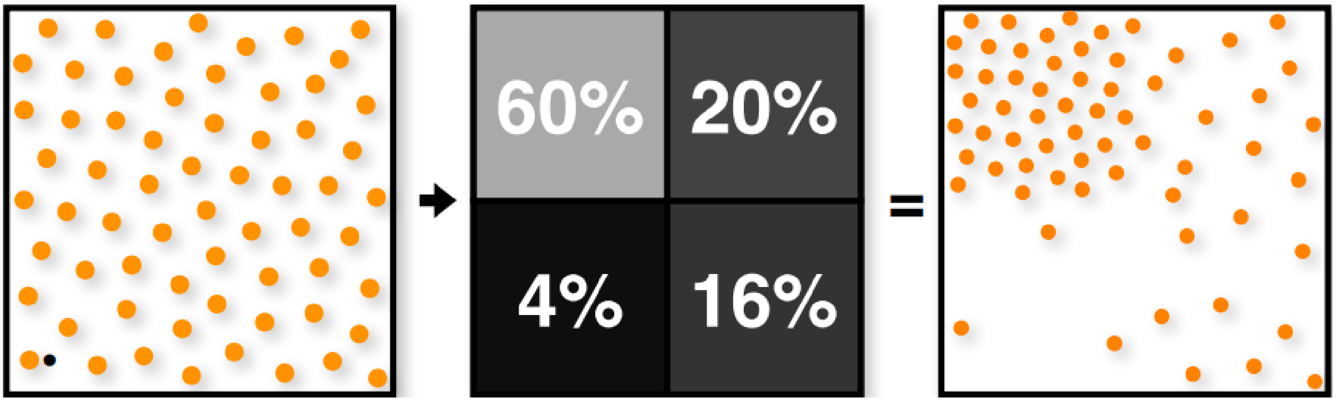
3. Conditional PDFs
   Normalize each row to get the conditional PDF.

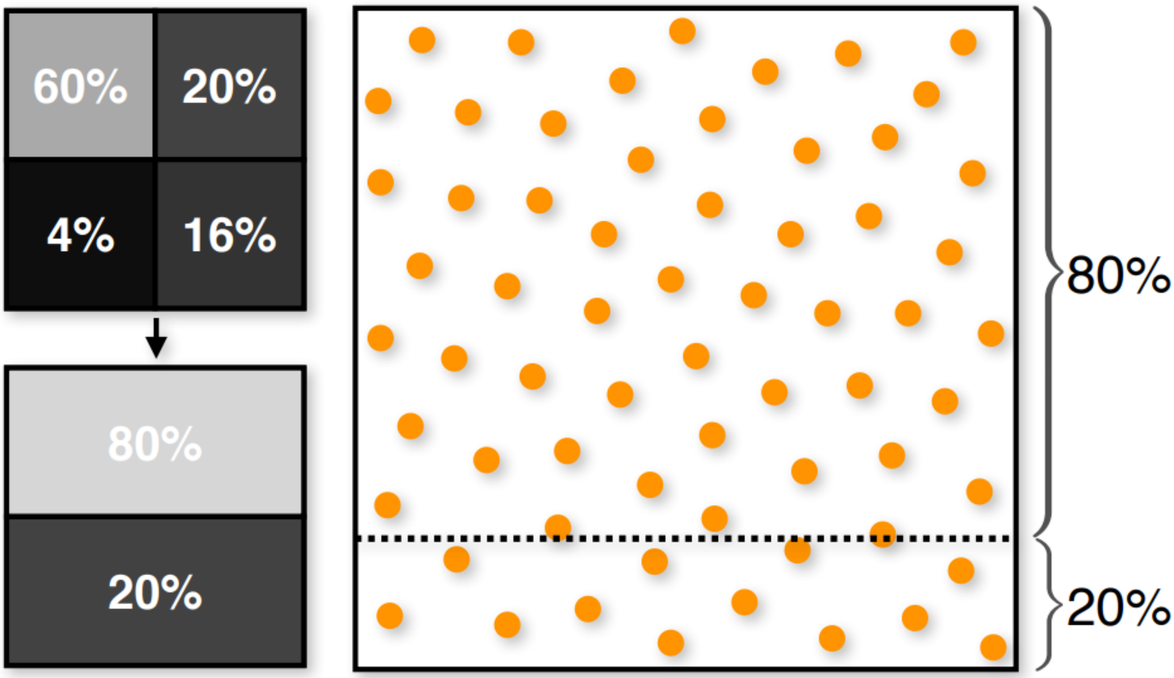
4. Sampling

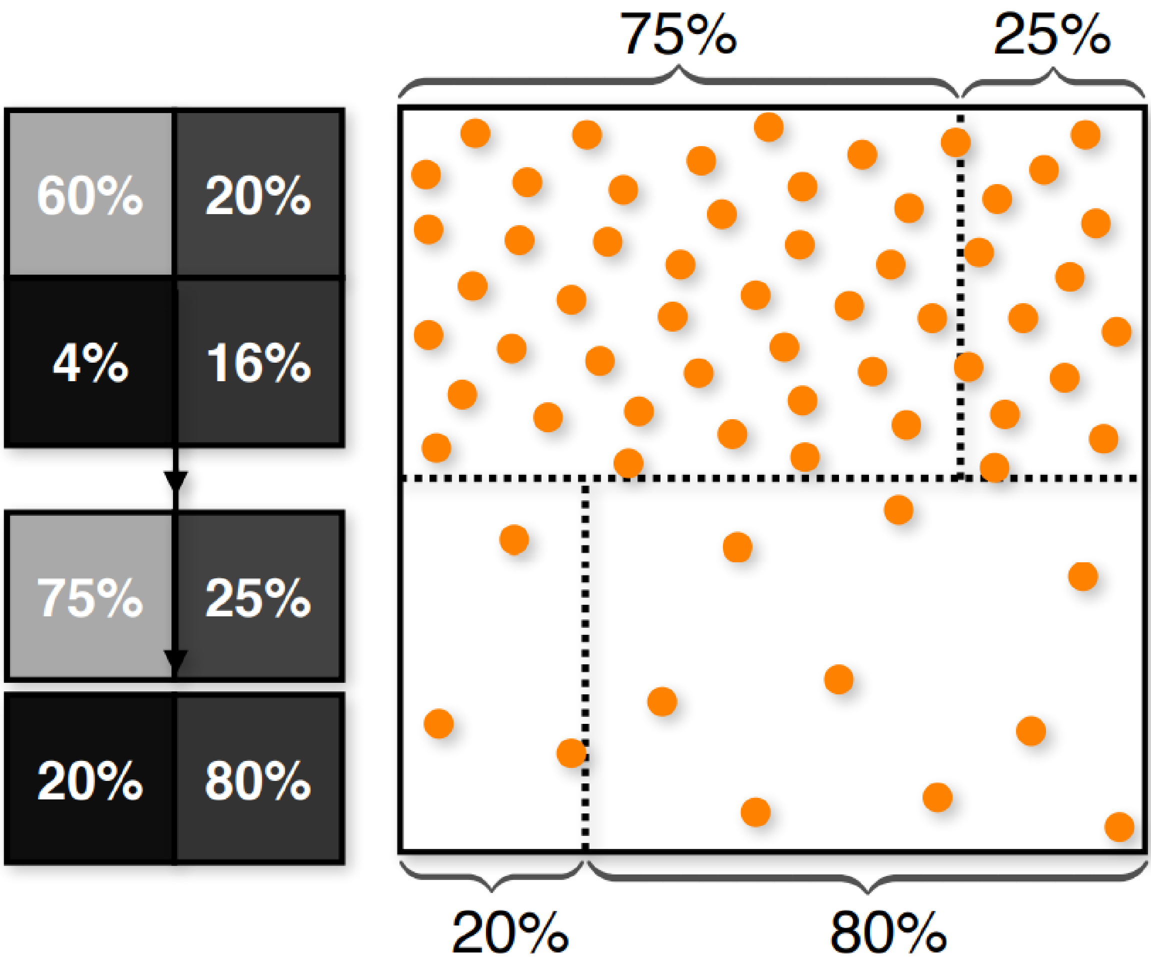
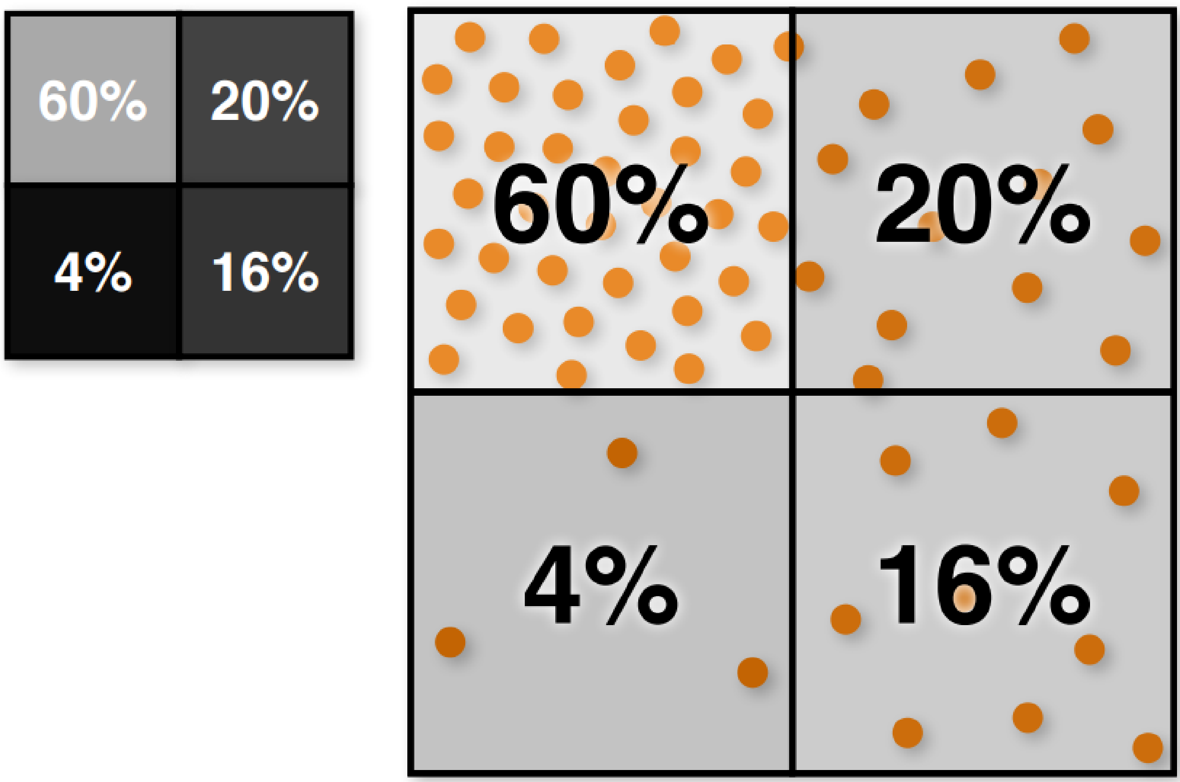
Hierarchical Warping Method

`Input`:

- a point set
- hierarchical representation of density function



steps	representations
1	

steps	representations
2	
3	 repeat on each quadrant recursively