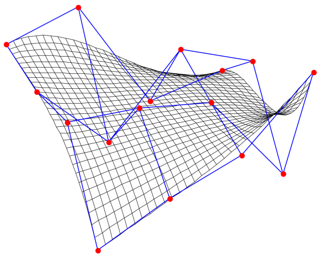
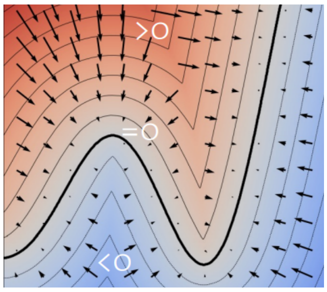
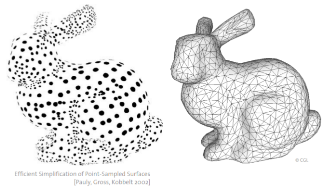
Shape Representation

We live in a 3D space and all objects are 3-dimensional. In computer graphics, 3D objects are modeled in different ways. Here below is a table containing existing models of 3D objects.

Parametric	Implicit	Discrete/Sampled
- Splines - Subdivision surface - Wikipedia	- Metaballs/Blobs - Distance fields - Procedural, CSG	- Meshes - Point set surfaces
		

Parametric

Parametric description of a shape is a mapping from $X \subseteq \mathbb{R}^m$ to $Y \subseteq \mathbb{R}^n$. We are able to query any point on the shape by choosing a point in X.

Parametric Curves

Consider a function $s : X \to Y, X \subseteq \mathbb{R}^m, Y \subseteq \mathbb{R}^n$:

- Planar curve: $m = 1, n = 2$

$$s(t) = (x(t), y(t))$$

- Space curve: $m = 1, n = 3$,

$$s(t) = (x(t), y(t), z(t))$$

Examples

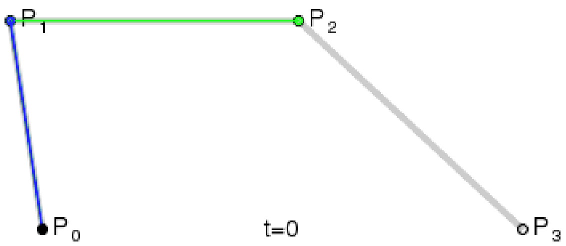
- Circle

$$\begin{aligned}\mathbf{p}(t) &: \mathbb{R} \to \mathbb{R}^2 \\ \mathbf{p}(t) &= r(\cos(t), \sin(t))\end{aligned}$$

- Bézier curve - Wikipedia

$$s(t) = \sum_{i=0}^{n} \mathbf{p}_i B_i^n(t)$$

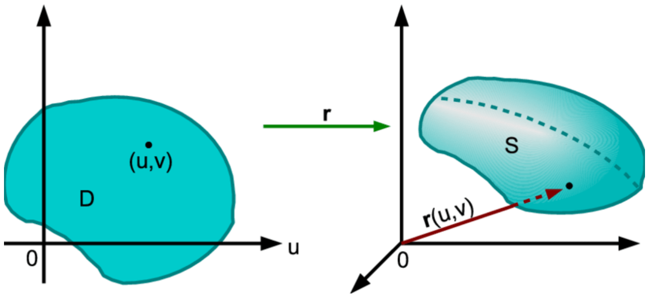
where $B_i^n = \binom{n}{i} t^i (1-t)^{n-i}$ are basis functions.



Parametric Surfaces

Surface: $m = 2, n = 3$

$$s(u, v) = (x(u, v), y(u, v), z(u, v))$$



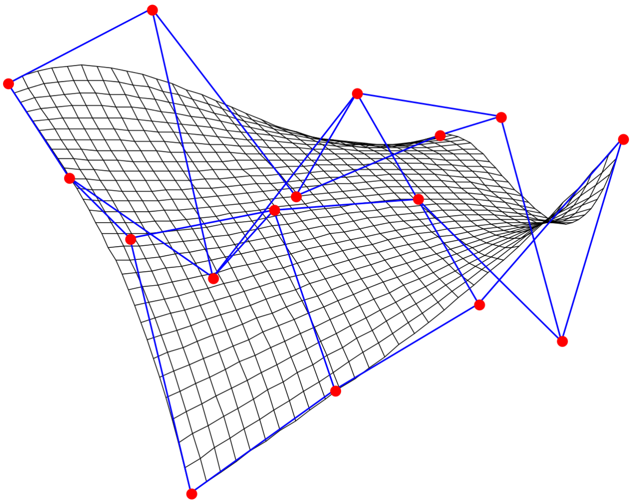
Examples

- Sphere

$$s(u, v) = r(\cos(u)\cos(v), \sin(u)\cos(v), \sin(v))$$
$$(u, v) \in [0, 2\pi) \times [-\pi/2, \pi/2]$$

- Bézier surface - Wikipedia

$$s(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{p}_{i,j} B_i^m(u) B_j^n(v)$$



Tangents and Normal

The tangents and normal at a give point on the surface are

$$
\begin{aligned}
s_u &= \frac{\partial s(u,v)}{\partial u} \\
s_v &= \frac{\partial s(u,v)}{\partial v} \\
\mathbf{n} &= \frac{s_u \times s_v}{\|s_u \times s_v\|}
\end{aligned}
$$

Please refer to any calculus book for proof.

Pros:

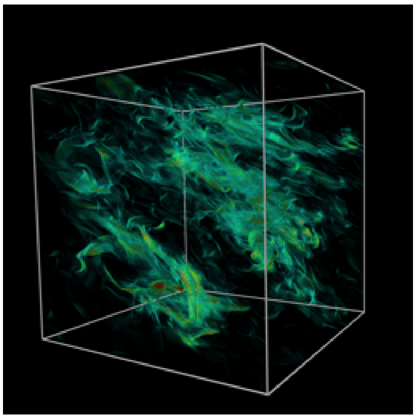
- Easy to generate points on the curve/surface
- Analytic formulas for derivatives

Cons:

- Hard to determine inside/outside
- Hard to determine if a point is on the curve/surface

Volumetric Density

This function gives the density of any point in the space: $m = 3, n = 1$:



You can not really say this representation is really a shape, but it's really useful later in volumetric rendering(participating media).

Pros:

- Easy to generate points on the curve/surface
- Analytic formulas for derivatives

Cons:

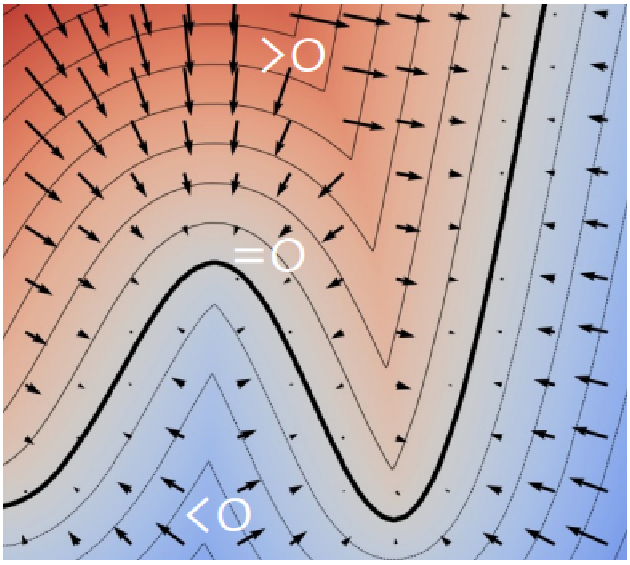
- Hard to determine inside/outside
- Hard to determine if a point is on the curve/surface

Implicit

Signed Distance Function

A surface is represented as the kernel/zero level set of a scalar function $f : \mathbb{R}^m \to \mathbb{R}$

- Curve in 2D: $S = \{x \in \mathbb{R}^2 | f(x) = 0\}$
- Surface in 3D: $S = \{x \in \mathbb{R}^3 | f(x) = 0\}$



The normal vector to the surface(curve) is given by the gradient of the implicit function

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)^\top$$

Boolean Set Operations

- Union

$$\bigcup_i f_i(x) = \min f_i(x)$$

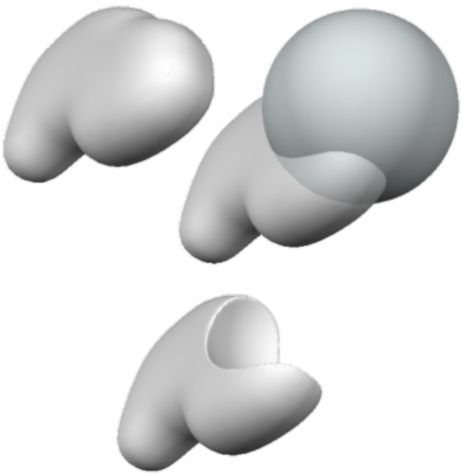
- Intersection

$$\bigcap_i f_i(x) = \max f_i(x)$$

- Subtraction

$$h = \max(f, -g)$$

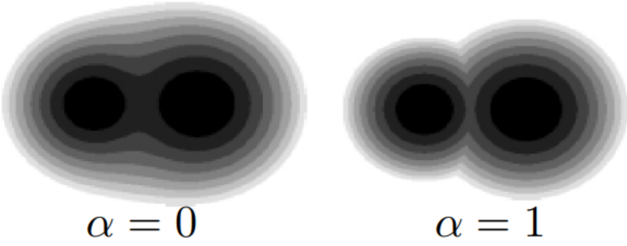
	$f > 0$	$f < 0$
$g > 0$	$h > 0$	$h < 0$
$g < 0$	$h > 0$	$h > 0$



Smooth Set Operations

In many cases, smooth blending is desired (PDF) Blending Operations for the Functionally Based Constructive Geometry (researchgate.net)

$$f \cup g = \frac{1}{1+\alpha}\left(f + g - \sqrt{f^2 + g^2 - 2\alpha f g}\right)$$
$$f \cap g = \frac{1}{1+\alpha}\left(f + g + \sqrt{f^2 + g^2 - 2\alpha f g}\right)$$



For $\alpha = 1$, this is equivalent to $\min$ and $\max$:

$$\lim_{\alpha \to 1} f \cup g = \frac{1}{2}\left(f + g - \sqrt{(f-g)^2}\right) = \frac{f+g}{2} - \frac{|f-g|}{2} = \min(f, g)$$
$$\lim_{\alpha \to 1} f \cap g = \frac{1}{2}\left(f + g + \sqrt{(f-g)^2}\right) = \frac{f+g}{2} + \frac{|f-g|}{2} = \max(f, g)$$

Metaballs / Blobs

An analytical sphere in 3D can be implicitly represented by

$$f(\mathbf{p}) = \|\mathbf{p}\|^2 - r^2$$

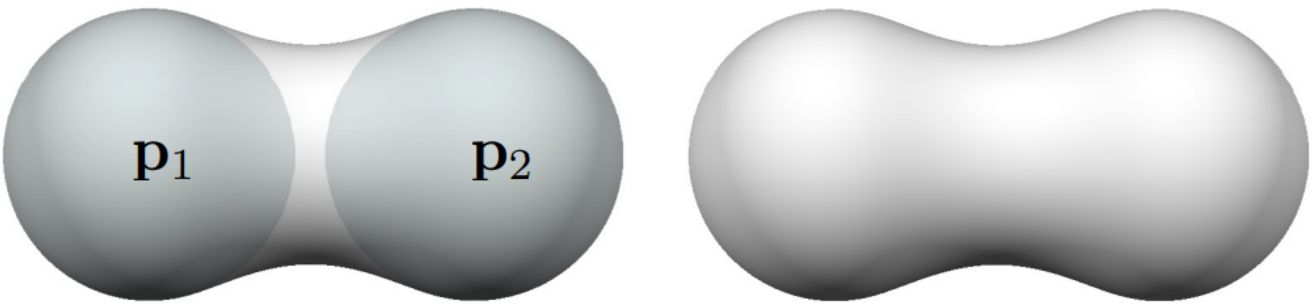
which is equivalent to

$$f(\mathbf{p}) = e^{-\|\mathbf{p}\|^2/r^2}$$

at e^{-1}.

With smooth fall-off functions, adding implicit functions generates a blend:

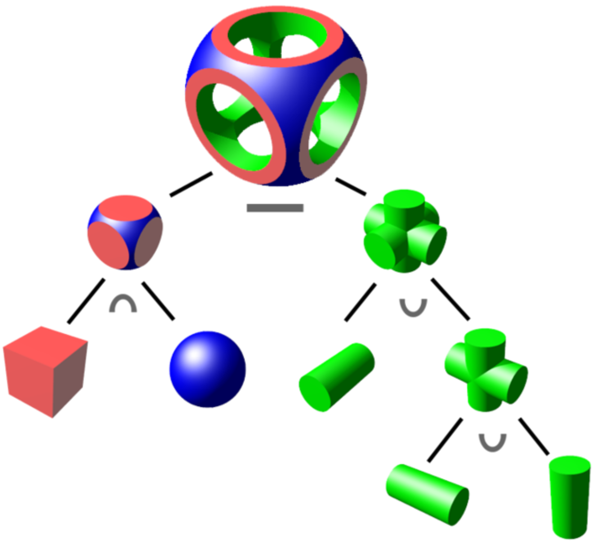
$$f(\mathbf{p}) = e^{-\|\mathbf{p}-\mathbf{p}_1\|^2} + e^{-\|\mathbf{p}-\mathbf{p}_2\|^2}$$



Set operations by simple addition/subtraction.

Procedural Implicits

CSG (Constructive solid geometry - Wikipedia)



Pros:

- Easy to determine inside/outside
- Easy to determine if a point is on the curve/surface

Cons:

- Hard to generate points on the curve/surface
- Does not lend itself to (real-time) rendering

Discrete/Sampled

Point Set Surfaces

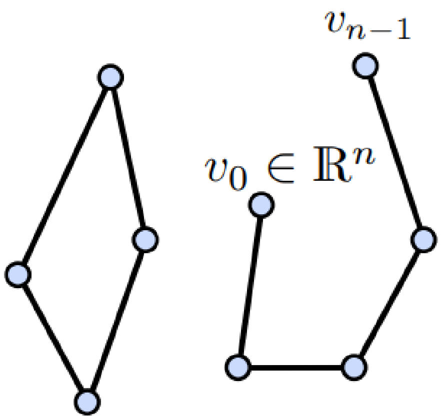
This notion is not explained in details in the lecture for now. Please refer to points_set_vis01.pdf (tau.ac.il)

Polygonal Meshes

Polygonal Meshes are the most used representations in CG. It's a boundary representation of an object

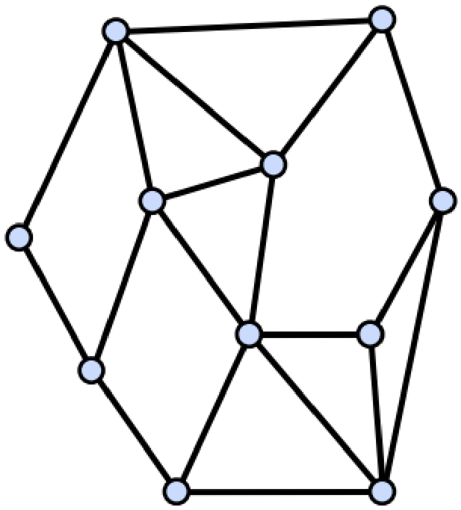
- Piecewise linear approximation
  - Error is $O(h^2)$ where h is the sampling difference.
- Arbitrary topology
- Piecewise smooth surfaces
- Efficient rendering

Polygon



- vertices: $v_0, v_1, \dots$
- edges: $(v_0, v_1), \dots, (v_{n-2}, v_{n-1})$
- closed: $v_0 = v_{n-1}$
- Planar: all vertices on a plane
- Simple: not self-intersecting

Polygonal mesh



- A finite set M of closed simple polygons Q_i is a polygonal mesh.
- The intersection of two polygons in M is either empty, a vertex or and edge.

$$M =< \underbrace{V}_{\text{vertices}}, \underbrace{E}_{\text{edges}}, \underbrace{F}_{\text{faces}} >$$

- Every edge belongs to at least one polygon.
- Each Q_i defines a face of the polygonal mesh

Vertex degree or valence: number of incident edges
Boundary: the set of all edges that belong to only on polygon:
Triangle meshes:

- connectivity: vertices, edges, triangles
- geometry: vertex positions
- cannot easily remove triangles(create holes)

Manifold

A surface is a closed 2-manifold if it is everywhere locally homeomorphic to a disk

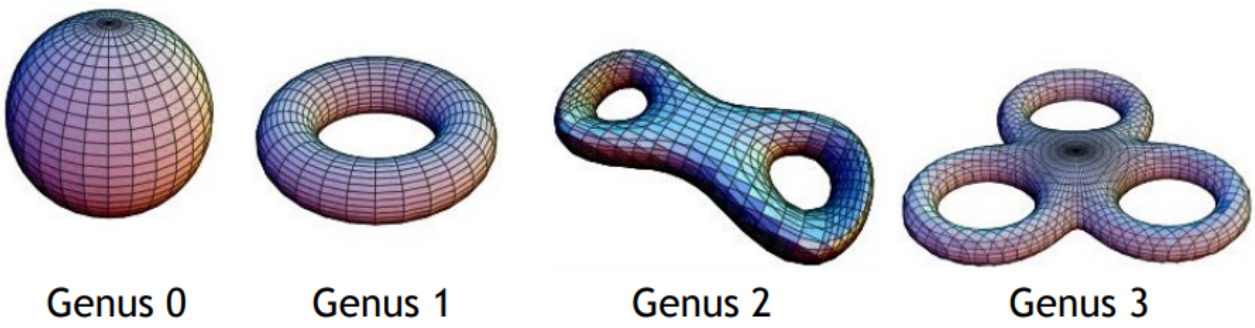
- at most 2 faces sharing an edge
  - Boundary edges have on incident face
  - Inner edges have two incident faces
- If closed and not intersecting, a manifold divides the space into inside and outside(watertight)
- A closed manifold polygonal mesh is called polyhedron

A polygonal mesh is manifold if

- every vertex has 1 connected ring or half ring of faces around it
- every edge is shared by at most faces

Global topology of meshes

- Genus: $\frac{1}{2} \times$ the maximal number of closed paths that do not disconnect the graph
  - Informally, the number of handles (donut holes)



Euler-Poincaré Formula
The sum

$$\chi(M) = v - e + f$$

is *constant* for a a given surface topology, no matter which (manifold) mesh we choose
– v = number of vertices
– e = number of edges
– f = number of faces

Triangulation

Polygonal mesh where every face is a triangle

- Simplifies data structures
- Simplifies algorithms
- Simplifies rendering
- Any polygon can be triangulated