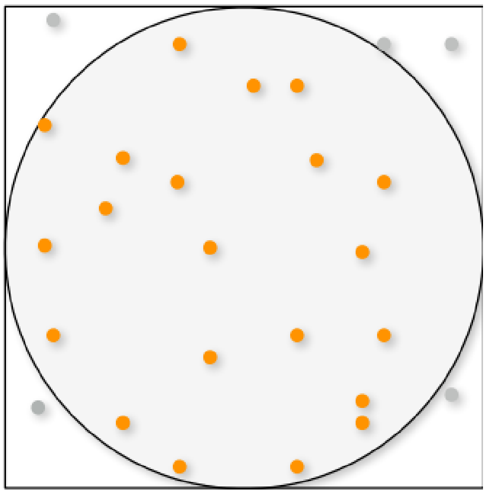
Sampling

We now have the great Monte Carlo integration method. To apply it, we need a way to sample points for integration.

Rejection Sampling

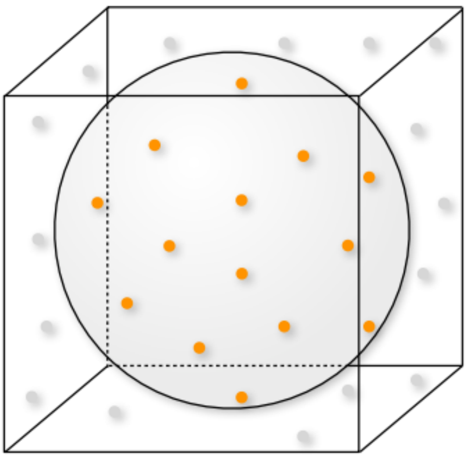
Disk



C++

```
Vec2 v;
do
{
        v.x = 1-2*drand48();
        v.y = 1-2*drand48();
} while(v.length2() > 1)
```

Sphere

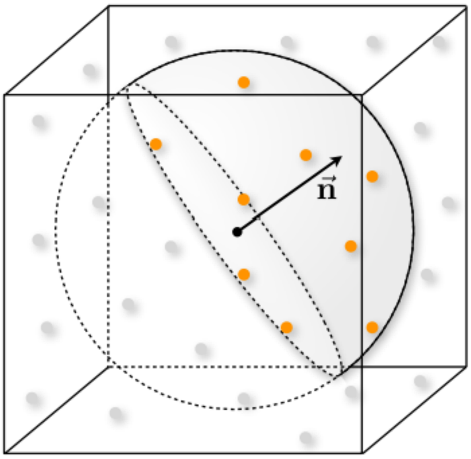


C++

```
Vector3D v;
do
{
        v.x = 1-2*drand48();
        v.y = 1-2*drand48();
        v.z = 1-2*drand48();
} while(v.length2() > 1)

// Project onto sphere
v /= v.length();
```

Hemisphere



C++

```
Vector3D v;
do
{
        v.x = 1-2*drand48();
        v.y = 1-2*drand48();
        v.z = 1-2*drand48();
} while(v.length2() > 1)

// flip to proper hemisphere
if (dot(v,n) < 0) v = -v;
// Project onto hemisphere
v /= v.length();
```

Pros:

- Flexible

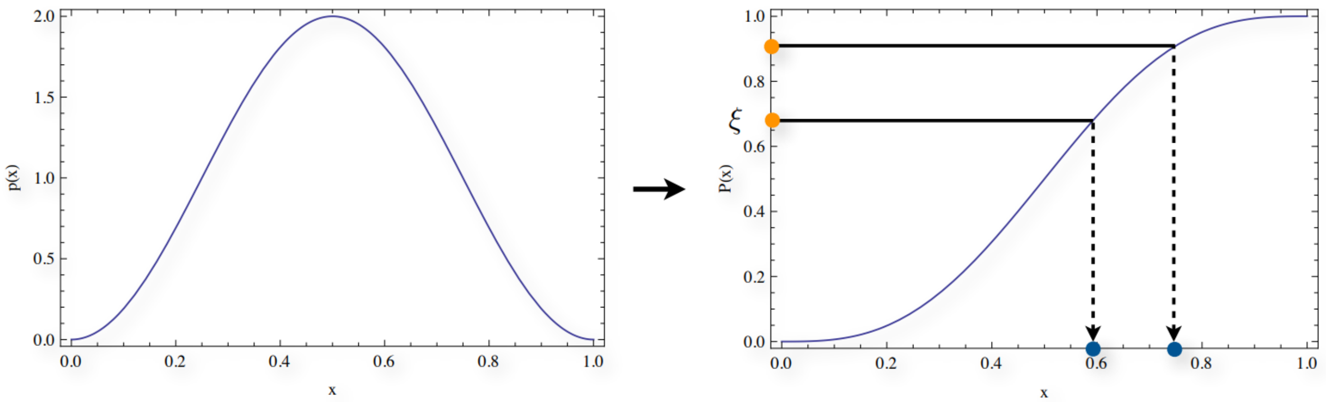
Cons:

- Inefficient
- Difficult/impossible to combine with stratification or quasi-Monte Carlo

Sampling Arbitrary Distributions

The Inversion Method

1. Compute the CDF $P(x) = \int_0^x p(t)\,dt$.
2. Compute the inverse function $P^{-1}(x)$.
3. Obtain a uniformly distributed random number ξ.
4. Compute $X_i = P^{-1}(\xi)$.



Example:
Sample with pdf

$$
\begin{aligned}
p(y) &= 2y, \quad y \in [0,1] \\
P(y) &= y^2 \\
P^{-1}(\xi) &= \sqrt{\xi}
\end{aligned}
$$

Thus when we sample from a uniform distribution from a uniform distribution $\xi \sim \mathrm{uniform}(0,1)$, we get the corresponding sample $y_i = \sqrt{\xi_i}$.

Sampling 2D Distributions

Draw samples (X, Y) from a 2D distribution $p(x, y)$:

- If $p(x, y)$ is separable, i.e., $p(x, y) = p_x(x)p_y(y)$, we can independently sample $p_x(x)$ and $p_y(y)$.
- Otherwise, compute the marginal density function:

$$
p(x) = \int p(x, y)\,dy
$$

- and the conditional density:

$$
p(y|x) = \frac{p(x, y)}{p(x)}
$$

- Procedure: first sample $X_i \sim p(x)$, then sample $Y_i \sim p(y|x = X_i)$.

Area-preserving Sampling

Usually if we want to sample a point on a surface or in a volume, the PDF is simply 1 divided by the surface area or the volume. If we want to keep this property that the point is sampled uniformly in the space, we can

1. Define the desired probability density of samples in a convenient coordinate system
2. Find (another) coordinate system for a convenient parameterization of the samples
3. Relate the PDFs in the two systems

- Requires computing the determinant of the Jacobian
4. Compute marginal and conditional 1D PDFs
5. Sample 1D PDFs using the inversion method

Uniform(Area-Preserving) Sampling of a Disk

- pdf for uniform sampling on a disk in cartesian system

$$p_c(x,y) = \begin{cases} \frac{1}{\pi} & x^2+y^2<1 \\ 0 & \text{otherwise} \end{cases}$$

- Sample in polar coordinates r, θ where:

$$x = r\cos\theta, \quad y = r\sin\theta$$

- Relate the PDFs

$$p_p(r,\theta) = |J| p_c(x,y) = \frac{r}{\pi}$$

- Compute marginal and conditional 1D PDFs:
  - Marginal PDF:

$$p(r) = \int_0^{2\pi} p_p(r,\theta)\, d\theta = 2r$$

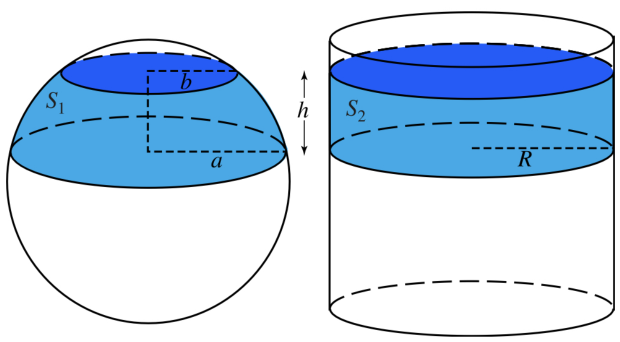
  - Conditional PDF:

$$p(\theta|r) = \frac{p_p(r,\theta)}{p(r)} = \frac{1}{2\pi}$$

- Using inversion method we have

$$r = \sqrt{\xi_1}, \quad \theta = 2\pi\xi_2$$

Uniform(Area Preserving) Sampling a Unit Sphere/ Hemisphere/ Sphere Cap
Archimedes' hat box theorem



Enclose a sphere in a cylinder and cut out a spherical segment by slicing twice perpendicularly to the cylinder's axis. Then the lateral surface area of the spherical segment S_1 is equal to the lateral surface area cut out of the cylinder S_2 by the same slicing planes, i.e.,

$$S_1 = S_2 = 2\pi Rh$$

Using this property, we could sample a point on the cylinder and find a bijective mapping between cylinder and sphere, then map the point on the cylinder to the sphere. This could be easily done by using the same z coordinate.

The probability in cylindrical coordinates when sampling a cylinder is

$$p(\phi, z) = \frac{1}{2\pi} \cdot \frac{1}{z_2 - z_1}$$

and we could easily transform a square to this

$$\begin{aligned} \phi &= 2\pi\xi_1 \\ z &= (z_2 - z_1)\xi_2 \end{aligned}$$

The corresponding spherical coordinates is

$$\begin{aligned} \theta &= \arccos z \\ \phi &= \phi \end{aligned}$$

To sample on hemisphere and sphere cap, simply change the range of z to match the height.

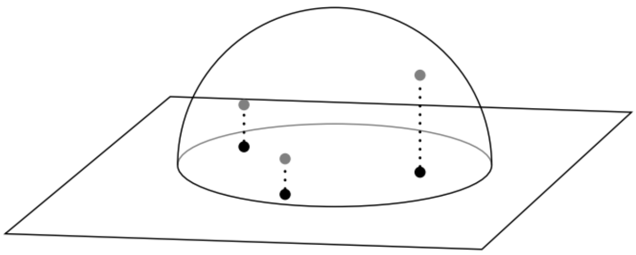
Cosine-weighted Hemispherical Sampling

Generating points uniformly on the disc, and then project these points to the surface of the hemisphere produces the desired distribution.

The probability under spherical coordinates is

$$p(\theta, \phi) = \begin{cases} \frac{\cos\theta}{\pi}, & \text{on hemisphere} \\ 0, & \text{else} \end{cases}$$

If you compare the form with sampling on a disk, we could easily conclude that it's the same distribution.



Beckmann Distribution

The cosine weighted Beckmann distribution is

$$p(\theta, \phi) = D(\theta)\cos\theta = \frac{e^{-\tan^2\theta/\alpha^2}}{\pi\alpha^2\cos^4\theta}\cos\theta$$

The CDF is

$$\begin{aligned} P(\Theta, \Phi) &= \int_0^\Theta \int_0^\Phi p(\theta, \phi) \sin\theta \, d\theta \, d\phi \\ &= \int_0^\Theta \int_0^\Phi \frac{e^{-\tan^2\theta/\alpha^2}}{\pi\alpha^2\cos^3} \sin\theta \, d\theta \, d\phi \\ &= \frac{\Phi}{2\pi} \int_0^\Theta e^{-\tan^2\theta/\alpha^2} \, d\left(\frac{\tan^2\theta}{\alpha^2}\right) \\ &= \frac{\Phi}{2\pi}[1 - e^{-\tan^2\Theta/\alpha^2}] = F_1(\Phi)F_2(\Theta) \end{aligned}$$

where

$$\begin{aligned} F_1(\Phi) &= \frac{\Phi}{2\pi} \\ F_2(\Theta) &= 1 - e^{-\tan^2\Theta/\alpha^2} \end{aligned}$$

which means we can sample ϕ, θ independently. Inverse the function we have

$$\phi = 2\pi\xi_1$$
$$\theta = \arctan\sqrt{-\alpha^2 \ln(1-\xi_2)}$$

Useful Transformations

Target space	Density	Domain	Transformation
Radius R disk	$p(r,\theta) = \frac{1}{\pi R^2}$	$\theta \in [0, 2\pi]$ $r \in [0, R]$	$\theta = 2\pi u$ $r = R\sqrt{v}$
Sector of radius R disk	$p(r,\theta) = \frac{2}{(\theta_2 - \theta_1)(r_2^2 - r_1^2)}$	$\theta \in [\theta_1, \theta_2]$ $r \in [r_1, r_2]$	$\theta = \theta_1 + u(\theta_2 - \theta_1)$ $r = \sqrt{r_1^2 + v(r_2^2 - r_1^2)}$
Phong density exponent n	$p(\theta,\phi) = \frac{n+1}{2\pi}\cos^n\theta$	$\theta \in \left[0, \frac{\pi}{2}\right]$ $\phi \in [0, 2\pi]$	$\theta = \arccos\left((1-u)^{1/(n+1)}\right)$ $\phi = 2\pi v$
Separated triangle filter	$p(x, y)(1 - \lvert x\rvert)(1 - \lvert y\rvert)$	$x \in [-1, 1]$ $y \in [-1, 1]$	$x = \begin{cases} 1 - \sqrt{2(1-u)} & \text{if } u \geq 0.5 \\ -1 + \sqrt{2u} & \text{if } u < 0.5 \end{cases}$ $y = \begin{cases} 1 - \sqrt{2(1-v)} & \text{if } v \geq 0.5 \\ -1 + \sqrt{2v} & \text{if } v < 0.5 \end{cases}$
Triangle with vertices a_0, a_1, a_2	$p(a) = \frac{1}{\text{area}}$	$s \in [0, 1]$ $t \in [0, 1 - s]$	$s = 1 - \sqrt{1-u}$ $t = (1-s)v$ $a = a_0 + s(a_1 - a_0) + t(a_2 - a_0)$
Surface of unit sphere	$p(\theta,\phi) = \frac{1}{4\pi}$	$\theta \in [0, \pi]$ $\phi \in [0, 2\pi]$	$\theta = \arccos(1 - 2u)$ $\phi = 2\pi v$
Sector on surface of unit sphere	$p(\theta,\phi) = \frac{1}{(\phi_2 - \phi_1)(\cos\theta_1 - \cos\theta_2)}$	$\theta \in [\theta_1, \theta_2]$ $\phi \in [\phi_1, \phi_2]$	$\theta = \arccos[\cos\theta_1 + u(\cos\theta_2 - \cos\theta_1)]$ $\phi = \phi_1 + v(\phi_2 - \phi_1)$
Interior of radius R sphere	$p = \frac{3}{4\pi R^3}$	$\theta \in [0, \pi]$ $\phi \in [0, 2\pi]$ $R \in [0, R]$	$\theta = \arccos(1 - 2u)$ $\phi = 2\pi v$ $r = w^{1/3}R$

[a] The symbols u, v, and w represent instances of uniformly distributed random variables ranging over [0, 1].