Projective Geometry: A Short Introduction

Lecture Notes

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Chapter 1

Introduction

1.1 Objective

The objective of this course is to give basic notions and intuitions on *projective geometry*. The interest of projective geometry arises in several visual computing domains, in particular computer vision modelling and computer graphics. It provides a mathematical formalism to describe the geometry of cameras and the associated transformations, hence enabling the design of computational approaches that manipulates 2D projections of 3D objects. In that respect, a fundamental aspect is the fact that objects at infinity can be represented and manipulated with projective geometry and this in contrast to the Euclidean geometry. This allows perspective deformations to be represented as projective transformations.

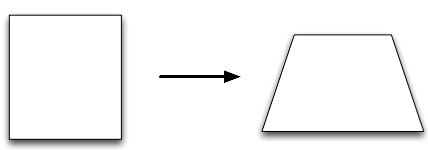


Figure 1.1: Example of perspective deformation or 2D projective transformation.

Another argument is that Euclidean geometry is sometimes difficult to use in algorithms, with particular cases arising from non-generic situations (e.g. two parallel lines never intersect) that must be identified. In contrast, projective geometry generalizes several definitions and properties, e.g. two lines always intersect (see fig. 1.2). It allows also to represent any transformation that preserves coincidence relationships in a matrix form (e.g. perspective projections) that is easier to use, in particular in computer programs.

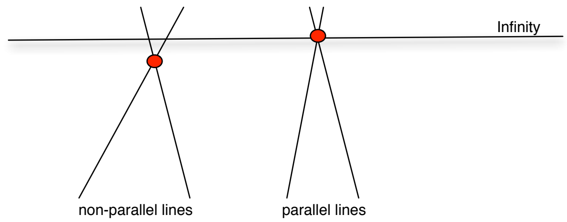


Figure 1.2: Line intersections in a projective space

1.2 Historical Background

The origins of geometry date back to Egypt and Babylon (2000 BC). It was first designed to address problems of everyday life, such as area estimations and construction, but abstract notions were missing.

- around 600 BC: The familiar form of geometry begins in Greece. First abstract notions appear, especially the notion of infinite space.
- 300 BC: **Euclide**, in the book *Elements*, introduces an axiomatic approach to geometry. From axioms, grounded on evidences or the experience, one can infer theorems. The Euclidean geometry is based on measures taken on rigid shapes, e.g. lengths and angles, hence the notion of shape invariance (under rigid motion) and also that (Euclidean) geometric properties are invariant under rigid motions.
- 15th century: the Euclidean geometry is not sufficient to model perspective deformations. Painters and architects start manipulating the notion of perspective. An open question then is "what are the properties shared by two perspective views of the same scene ?"
- 17th century: **Desargue**s (architect and engineer) describes conics as perspective deformations of the circle. He considers the point at infinity as the intersection of parallel lines.
- 18th century: **Descartes, Fermat** contrast the synthetic geometry of the Greeks, based on primitives with the analytical geometry, based instead on coordinates. Desargue's ideas are taken up by **Pascal**, among others, who however focuses on infinitesimal approaches and Cartesian coordinates. **Monge** introduces the descriptive geometry and study in particular the conservation of angles and lengths in projections.
- 19th century : **Poncelet** (a Napoleon officer) writes, in 1822, a treaty on projective properties of figures and the invariance by projection. This is the first treaty on projective geometry: a projective property is a property invariant by projection. **Chasles et Möbius** study the most general

projective transformations that transform points into points and lines into lines and preserve the cross ratio (the collineations). In 1872, **Felix Klein** proposes the Erlangen program, at the Erlangen university, within which a geometry is not defined by the objects it represents but by their transformations, hence the study of invariants for a group of transformations. This yields a hierarchy of geometries, defined as groups of transformations, where the Euclidean geometry is part of the affine geometry which is itself included into the projective geometry.

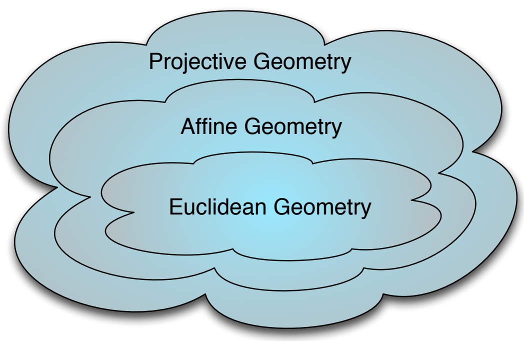


Figure 1.3: The geometry hierarchy.

1.3 Bibliography

The books below served as references for these notes. They include computer vision books that present comprehensive chapters on projective geometry.

- J.G. Semple and G.T. Kneebone, *Algebraic projective geometry*, Clarendon Press, Oxford (1952)
- R. Hartley and A. Zisserman, *Multiple View Geometry*, Cambridge University Press (2000)
- O. Faugeras and Q-T. Luong, *The Geometry of Multiple Images*, MIT Press (2001)
- D. Forsyth and J. Ponce, *Computer Vision: A Modern Approach*, Prentice Hall (2003)

Chapter 2

Projective Spaces

In this chapter, formal definitions and properties of projective spaces are given, regardless of the dimension. Specific cases such as the line and the plane are studied in subsequent chapters.

2.1 Definitions

Consider the real vector space $\mathbb{R}^{n+1}$ of dimension $n+1$. Let v be a non-zero element of $\mathbb{R}^{n+1}$ then the set $\mathcal{R}_v$ of all vectors kv, $k \in \mathbb{R}^*$ is called a ray (cf. figure 2.1).

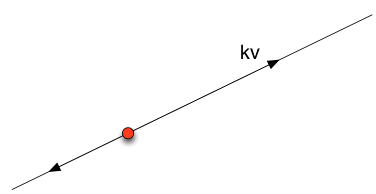


Figure 2.1: The ray $\mathcal{R}_v$ is the set of all non-zero vectors kv with direction v.

a ray $\Leftrightarrow$

Definition 2.1 (Geometric Definition) *The real projective space $\mathcal{P}^n$, of dimension n, associated to $\mathbb{R}^{n+1}$ is the set of rays of $\mathbb{R}^{n+1}$. An element of $\mathcal{P}^n$ is called a point and a set of linearly independent (respectively dependent) points of $\mathcal{P}^n$ is defined by a set of linearly independent (respectively dependent) rays.*

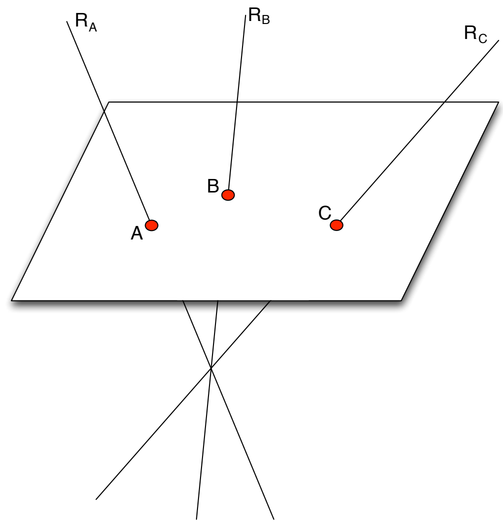


Figure 2.2: The projective space associated to $\mathbb{R}^3$ is called the projective plane $\mathcal{P}^2$.

Definition 2.2 (Algebraic Definition) *A point of a real projective space $\mathcal{P}^n$ is represented by a vector of real coordinates $X = [x_0, ..., x_n]^t$, at least one of which is non-zero. The $\{x_i\}$s are called the projective or homogeneous coordinates and two vectors X and Y represent the same point when there exists a scalar $k \in \mathbb{R}^*$ such that:*

$$x_i = k y_i \quad \forall i,$$

which we will denote by:

$$X \sim Y.$$

Hence the projective coordinates of a point are defined up to a scale factor and the correspondence between points and coordinate vectors is not one-to-one. Projective coordinates relate to a projective basis:

Definition 2.3 (Projective Basis) *A projective basis is a set of $(n+2)$ points of $\mathcal{P}^n$, no $(n+1)$ of which are linearly dependent. For example:*

associated to R^{n+1}

$$\begin{array}{ccccc} \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ . \\ \\ 0 \end{bmatrix} & ... & \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ . \\ \\ 1 \end{bmatrix} \\ A_0 & A_1 & ... & A_n & A^* \end{array}$$

is the canonical basis where the $\{A_i\}$s are called the basis points and A^* the unit point.

The relationship between projective coordinates and a projective basis is as follows.

Projective Coordinates Let $\{A_0, .., A_n, A^*\}$ be a basis of $\mathcal{P}^n$ with associated rays $\mathcal{R}_{v_i}$ and $\mathcal{R}_{v^*}$ respectively. Then for any point A of $\mathcal{P}^n$ with an associated ray $\mathcal{R}_v$, its projective coordinates $[x_0, ..., x_n]^t$ are such that:

=> n+1 DoF

$$v = x_0 v_0 + ... + x_n v_n,$$

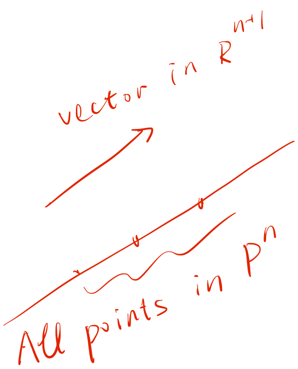
where the scales of the vectors $\{v_i\}$s associated to the $\{A_i\}$s are given by:

$$v* = v_0 + ... + v_n,$$

which determines the $\{x_i\}$s up to a scale factor.

Note on projective coordinates

To better understand the above characterization of the projective coordinates, let us consider any $(n+1)$ vectors v_i associated to the $\{A_i\}$s. By definition they form a basis of $\mathbb{R}^{n+1}$ and any vector v in this space can be uniquely decomposed as:



$$v = u_0 v_0 + ... + u_n v_n, \;\; u_i \in \mathbb{R} \; \forall i.$$

Thus v is determined by a single set of coordinates $\{u_i\}$ in the vector basis $\{v_i\}$. However the above unique decomposition with the u_i's does not transfer to the associated points A and $\{A_i\}$s of $\mathcal{P}^n$ since the correspondence between points in $\mathcal{P}^n$ and vectors in $\mathbb{R}^{n+1}$ is not one-to-one. A

For instance replacing in the decomposition u_0 by $u_0/2$ and v_0 by $2v_0$ still relates A with the $\{A_i\}$s but with a different set of $\{u_i\}$s. In order to uniquely determine the decomposition, let us consider the additional point A^* and let v^* be one of its associated vector in $\mathbb{R}^{n+1}$ then:

$$v^* = u_0^* v_0 + ... + u_n^* v_n = \overline{v}_0 + ... + \overline{v}_n. \;\; \Leftrightarrow \text{Reference point}$$

The above *scaled* vectors $\overline{v}_i$ are well defined as soon as $u_i \neq 0, \forall i$ (true by the fact that, by definition, A^* is linearly independent of any subset of n points A_i). Then, any vector v associated to A writes:

$$v = x_0 \overline{v}_0 + ... + x_n \overline{v}_n,$$

where the $\{x_i\}$s can vary only by a global scale factor function of the scales of v and v^*.

Definition 2.4 (Projective Transformations) *A matrix M of dimensions $(n+1)\times(n+1)$ such that $det(M)\neq 0$, or equivalently non-singular, defines a linear transformation from $\mathcal{P}^n$ to itself that is called a homography, a collineation or a projective transformation.*

Projective transformations are the most general transformations that preserve incidence relationships, i.e. collinearity and concurrence.

2.2 Properties

Some classical and fundamental properties of projective spaces folllow.

associated to $\mathbb{R}^m$

Theorem 2.1 *Consider m points of $\mathcal{P}^n$ that are linearly independent with $m < n$. The set of points in $\mathcal{P}^n$ that are linearly dependent on these m points form a projective space of dimension $m-1$. When this dimension is equal to 1, 2 and $n-1$, this space is called line, plane and hyperplane respectively. The set of subspaces of $\mathcal{P}^n$ with the same dimension is also a projective space.*

Examples Lines are hyperplanes of $\cancel{\mathcal{P}^2}$ P^1 and they form a projective space of dimension 2. ?

Theorem 2.2 (Duality) *The set of hyperplanes of a projective space $\mathcal{P}^n$ is a projective space of dimension n. Any definition, property or theorem that applies to the points of a projective space is also valid for its hyperplanes.*

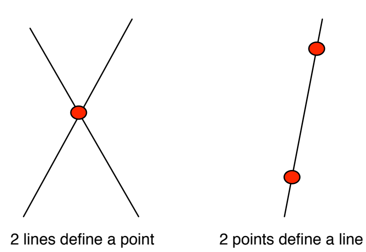


Figure 2.3: Lines and points are dual in $\mathcal{P}^2$.

Examples Points and lines are dual in the projective plane, 2 *points define a line* is dual to 2 *lines define a point*. Another interesting illustration of the duality is the Desargues' theorem (see figure 2.4) that writes:

If 2 triangles are such that the lines joining their corresponding vertices are concurrent then the points of intersections of the corresponding edges are *collinear*,

and which reciprocal is its dual (replace in the statement *lines joining* with *points of intersections of*, *vertices* with *edges* and *concurrent* with *collinear* and vice versa).

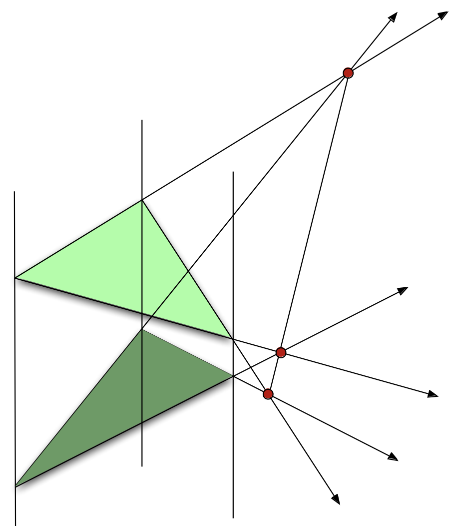


Figure 2.4: Desargues' theorem illustrated with parallel lines (hence concurrent lines in the projective sense) joining the corresponding vertices on 2 triangles.

Theorem 2.3 (Change of Basis) *Let* $\{X_0, ..., X_{n+1}\}$ *be a basis of* $\mathcal{P}^n$*, i.e. no* $(n+1)$ *of them are linearly dependent. If* $\{A_0, ..., A_{n+1} = A^*\}$ *is the canonical basis then there exists a non-singular matrix* M *of dimension* $(n+1) \times (n+1)$ *such that:*

$$M \cdot A_i = k_i X_i, k_i \in \mathbb{R}^* \quad \forall i,$$

or equivalently:

$$M \cdot A_i \sim X_i \quad \forall i.$$

2 *matrices* M *and* M' *that satisfy this property differ by a non-zero scalar factor only, which we will denote using the same notation:* $M \sim M'$*.*

Proof

The matrix M satisfies: $M \cdot A_i = k_i X_i \quad \forall i$. Stacking $A_0, ..., A_n$ into a matrix we get, by definition of the canonical basis:

$$[A_0 ... A_n] = \mathbb{I}_{n+1}$$

hence:

$$M \cdot [A_0 ... A_n] = M = [k_0 X_0 ... k_n X_N],$$

which determines M up to the scale factors $\{k_i\}$. Using this expression with A_{n+1}:

$$M \cdot A_{n+1} = [k_0 X_0 ... k_n X_N] \cdot \begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix} = [X_0 ... X_N] \cdot \begin{bmatrix} k_0 \\ . \\ . \\ . \\ k_n \end{bmatrix},$$

and since: $M \cdot A_{n+1} = k_{n+1} X_{n+1}$ we get:

$$[X_0 ... X_N] \cdot \begin{bmatrix} k_0 \\ . \\ . \\ . \\ k_n \end{bmatrix} = k_{n+1} X_{n+1},$$

that gives the $\{k_i\}$s up to a single scale factor. Note also that the $\{k_i\}$s are necessarily non-zero otherwise $(n+1)$ vectors X_i are linearly dependent by the above expression.

Thus any basis of $\mathcal{P}^n$ is related to the canonical basis by a homography. A consequence of theorem 2.3 is that:

Corollary 2.4 *Let $\{X_0, ..., X_{n+1}\}$ and $\{Y_0, ..., Y_{n+1}\}$ be 2 basis of $\mathcal{P}^n$, then there exists a non-singular matrix M of dimension $(n+1) \times (n+1)$ such that:*

$$M \cdot X_i \sim Y_i \quad \forall i,$$

where M is determined up to a scale factor.

Proof

By theorem 2.3:

$$L \cdot A_i = k_i X_i, \quad \forall i,$$

$$Q \cdot A_i = l_i Y_i, \quad \forall i.$$

Thus:

$$QL^{-1} \cdot X_i = \frac{l_i}{k_i} Y_i, \quad \forall i,$$

and the matrix $M = QL^{-1}$ is therefore such that:

$$M \cdot X_i \sim Y_i, \quad \forall i,$$

Now if there is a matrix M' such that: $M' \cdot X_i \sim Y_i, \quad \forall i$, then replacing X_i we get:

$$M'L \cdot A_i \sim Y_i, \quad \forall i,$$

and by theorem 2.3: $M'L \sim ML$, hence $M' \sim M$.

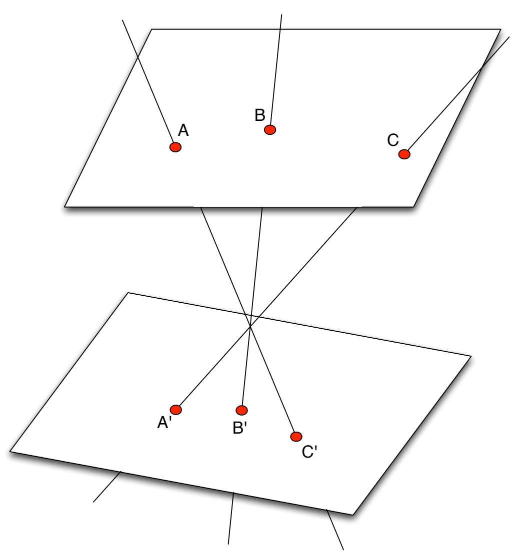


Figure 2.5: Change of basis in $\mathcal{P}^2$ or projective transformation between A, B, C and A', B', C'.

Figure 2.5 illustrates the change of basis in $\mathcal{P}^2$. A, B, C and A', B', C' are 2 different representations of the same rays and are thus related by a homography (projective transformation). Note that this figure illustrates also the relationship between coplanar points and their images by a perspective projection.

2.3 The hyperplane at infinity

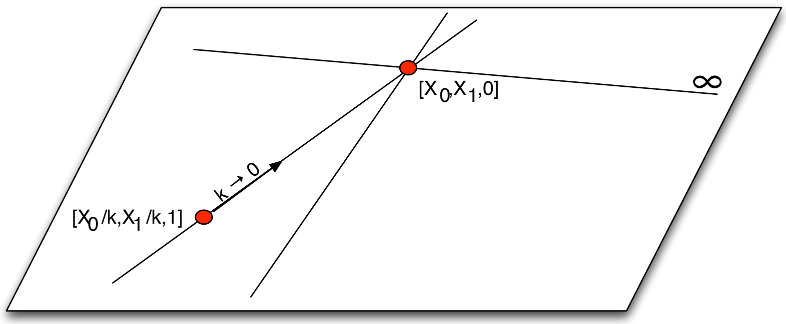


Figure 2.6: In $\mathcal{P}^2$ any line L is the hyperplane at infinity for the affine space $\mathcal{P}^2 \setminus L$. In this affine space, all lines that share the same direction are concurrent on the line at infinity.

The projective space $\mathcal{P}^n$ can also be seen as the completion of a hyperplane H of $\mathcal{P}^n$ and the set complement $\mathcal{A}^n = \mathcal{P}^n \setminus H$. $\mathcal{A}^n$ is then the affine space of dimension n (associated to the vector space $\mathbb{R}^n$) and H is its hyperplane at infinity also called ideal hyperplane. This terminology is used since H is the locus of points in $\mathcal{P}^n$ where parallel lines of $\mathcal{A}^n$ intersect.

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As an example, assume that $[x_0, ..., x_n]^t$ are the homogeneous coordinates of points in $\mathcal{P}^n$ and consider the affine space $\mathcal{A}^n$ of points with inhomogeneous coordinates (not defined up to scale factor) $[x_0, ..., x_{n-1}]^t$. Then the locus of points in $\mathcal{P}^n$ that are not reachable within $\mathcal{A}^n$ is the hyperplane with equation $x_n = 0$. To understand this, observe that there is a one-to-one mapping between points in $\mathcal{A}^n$ and points in $\mathcal{P}^n$ with homogeneous coordinates $[x_0, ..., x_{n-1}, 1]^t$. Going along the direction $[x_0, ..., x_{n-1}]^t$ in $\mathcal{A}^n$ by changing the value of k in $[x_0/k, ..., x_{n-1}/k, 1]^t$ we see that there is a point at the limit $k \to 0$, that is at infinity and not in $\mathcal{A}^n$, with homogeneous coordinates $[x_0/k, ..., x_{n-1}/k, 1]^t \sim [x_0, ..., x_{n-1}, k]^t =_{k \to 0} [x_0, ..., x_{n-1}, 0]^t$ (see Figure 2.6). This point belongs to the hyperplane at infinity (or the ideal hyperplane) associated with $\mathcal{A}^n$.

Any hyperplane H of $\mathcal{P}^n$ is thus the plane at infinity of the affine space $\mathcal{P}^n \setminus H$. Reciprocally, adding to any n-dimensional affine space $\mathcal{A}^n$ the hyperplane of its points at infinity converts it into a projective space of dimension n. This is called the projective completion of $\mathcal{A}^n$.

Chapter 3

The projective line

The space $\mathcal{P}^1$ is called the projective line. It is the completion of the affine line with a particular projective point, the point at infinity, as will be further detailed in this chapter. The projective line is useful to introduce projective notions, such as the cross-ratio, in a simple and intuitive way.

3.1 Introduction

The canonical basis of $\mathcal{P}^1$ is:

$$A_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A^* = A_1 + A_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

A point of $\mathcal{P}^1$ is represented by a vector of 2 homogeneous coordinates $X \sim [x_0, x_1] \neq [0, 0]$. Hence: $X \sim x_0 A_0 + x_1 A_1$.

Now consider the subspace of $\mathcal{P}^1$ such that $x_1 \neq 0$. This is equivalent to exclude the point A_0 and it defines the affine line $\mathcal{A}^1$. Point on $\mathcal{A}^1$ can be described by a single parameter k such that:

$$X = kA_0 + A_1,$$

where $k = x_0/x_1$ is the affine coordinate.

Figure 3.1: On the affine line, the coordinate k of C in the coordinate frame $[O, B]$ is $k = OC/OB$.

A_0 is the point at infinity, or ideal point, for the affine space $\mathcal{P}^1 \setminus A_0$.

3.2 Projective transformation of $\mathcal{P}^1$

A projective transformation of $\mathcal{P}^1$ is represented by a 2×2 non singular matrix H defined up to a scale factor:

$$H \sim \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix}.$$

- The above matrix has 3 degrees of freedom since it is defined up to a scale factor.
- From corollary 2.3 of section 2.2, it follows that 3 point correspondences, or equivalently 2 basis of $\mathcal{P}^1$, are required to estimate H.

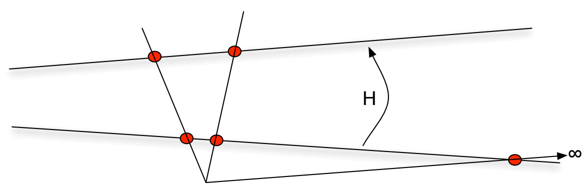


Figure 3.2: A homography in $\mathcal{P}^1$ is defined by 3 point correspondences, concerning possibly the infinite point.

The restriction of the projective transformation H to the affine space $\mathcal{A}^1$ is a transformation M that does not affect the point at infinity, i.e. $M \cdot A_0 \sim A_0$. Hence M is of the form:

$$M \sim \begin{bmatrix} m_1 & m_2 \\ 0 & 1 \end{bmatrix},$$

where m_1 is a scale factor and m_2 a translation parameter. 2 point correspondences are sufficient to estimate M.

3.3 The cross-ratio

The cross-ratio, also called double ratio ("bi-rapport" in French), is the fundamental invariant of $\mathcal{P}^1$, that is to say a quantity that is preserved by projective transformation. It is the projective equivalent to the Euclidean distance with rigid transformations. Let A, B, C, D be 4 points on the projective line then their cross-ratio writes:

$$\{A, B; C, D\} = \frac{|AC||BD|}{|BC||AD|},$$

where $|AB| = \det \begin{vmatrix} x_0^A & x_0^B \\ x_1^A & x_1^B \end{vmatrix} = (x_0^A x_1^B) - (x_1^A x_0^B)$.

Some remarks are in order:

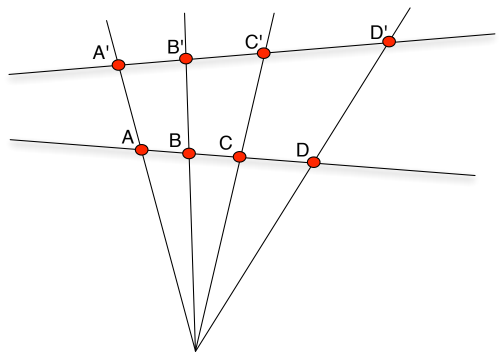


Figure 3.3: A, B, C, D and A', B', C', D' are related by a projective transformation, hence their cross-ratios are equal.

1. The cross-ratio is independent of the basis chosen for $\mathcal{P}^1$.
2. On the affine line, $A = [k_a, 1]$, $B = [k_b, 1]$, *etc.* and the cross-ratio becomes:
$$\{A, B; C, D\} = \frac{(k_A - k_C)(k_B - k_D)}{(k_B - k_C)(k_A - k_D)},$$
hence in the Euclidean space:
$$\{A, B; C, D\} = \frac{d_{AC} d_{BD}}{d_{BC} d_{AD}},$$
with $d_{\{\}}$ being the Euclidean distance between 2 points.
3. $$\{\infty, B; C, D\} = \frac{(k_B - k_D)}{(k_B - k_C)},$$
$$\{A, \infty; C, D\} = \frac{(k_A - k_C)}{(k_A - k_D)},$$
$$\ldots$$
4. By permuting the points A, B, C, D, 24 quadruplets can be formed. These quadruplets define only 6 different values of the cross-ratio: $\rho, \frac{1}{\rho}, 1 - \rho, \frac{1}{1-\rho}, \frac{\rho}{\rho-1}, \frac{\rho-1}{\rho}$.
5. Let A_0, A_1, A_2 be a basis of $\mathcal{P}^1$ and $[x_0, x_1]$ the coordinates of P in this basis, then:
$$\{A_0, A_1; A_2, P\} = \frac{x_0}{x_1}.$$

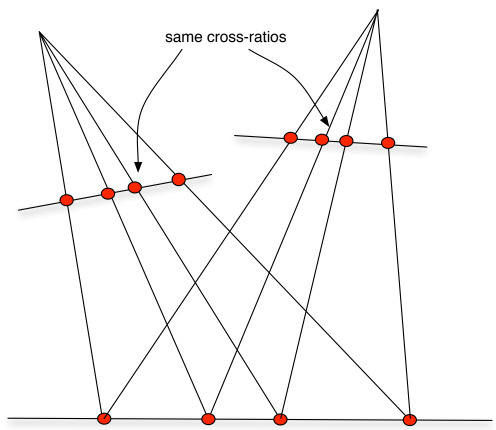


Figure 3.4: Sets of 4 colinear points present the same cross-ratios by perspective projections.

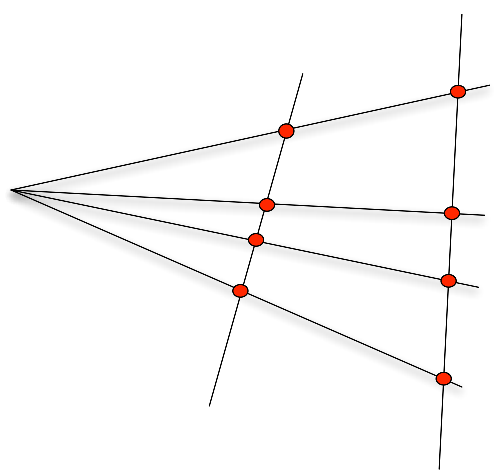


Figure 3.5: The cross-ratio of 4 concurrent lines (a pencil of lines) is the cross-ratio of any 4 intersecting colinear points.

Chapter 4

The projective plane

The space $\mathcal{P}^2$ is called the projective plane. Its importance in visual computing domains is coming from the fact that the image plane of a 3D world projection can be seen as a projective plane and that relationships between images of the same 3D scene can be modeled through projective transformations. As an illustration of this principle, and following Desargues in the 16th century, perspective deformations of the circle can be described with projective transformations of $\mathcal{P}^2$.

4.1 Points and lines

A point in $\mathcal{P}^2$ is represented by 3 homogeneous coordinates $[x_0, x_1, x_2]^t$ defined up to scale factor. Consider 2 points A and B of $\mathcal{P}^2$ and the line going through them. A third point C belongs to this line only if the coordinates of A, B and C are linearly dependent, i.e their determinant vanishes:

$$\begin{vmatrix} x_0^A & x_0^B & x_0^C \\ x_1^A & x_1^B & x_1^C \\ x_2^A & x_2^B & x_2^C \end{vmatrix} = 0.$$

The above expression can be rewritten as:

$$l_0 x_0^C + l_1 x_1^C + l_2 x_2^C = [l_0, l_1, l_2] \cdot \begin{bmatrix} x_0^C \\ x_1^C \\ x_2^C \end{bmatrix} = L^t \cdot C = 0, \qquad (4.1)$$

where the l_is are functions of the coordinates of A and B:

$$l_0 = \begin{vmatrix} x_1^A & x_1^B \\ x_2^A & x_2^B \end{vmatrix}, \ l_1 = -\begin{vmatrix} x_0^A & x_0^B \\ x_2^A & x_2^B \end{vmatrix}, \ l_2 = \begin{vmatrix} x_0^A & x_0^B \\ x_1^A & x_1^B \end{vmatrix}.$$

The equation 4.1 is satisfied by all points $C = [x_0^C, x_1^C, x_2^C]^t$ that belongs to the line going through A and B. Reciprocally, an expression of this type where at least one l_i is non-zero is the equation of a line L. The l_is, as the x_is, are defined up to a scale factor in this homogeneous equation. They are the projective coordinates of the line L going through A and B.

Lines (hyperplanes) of $\mathcal{P}^2$ form therefore a projective space of dimension 2 that is dual to the point space. This is illustrated with equation 4.1 that is satisfied by all the points along L given its coordinates or by all the lines going through C given its coordinates (see figure 4.1).

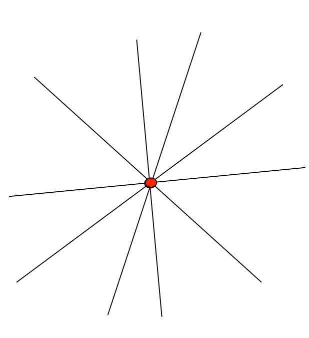


Figure 4.1: Equation 4.1 can describe all the lines going through a point or, in a dual way, all the points along a line.

Exercises

1. What is the point X intersection of the lines L_1 and L_2 ?
   Using equation 4.1 we have: $L_1^t \cdot X = 0$ and $L_2^t \cdot X = 0$. Hence: $X \sim L_1 \times L_2$, where $\times$ denotes the cross product.

2. What is the line L going through the points X_1 and X_2 ?
   By duality: $L \sim X_1 \times X_2$,

4.2 Line at infinity

Consider points in $\mathbb{R}^2$ with coordinates $[a, b]^t$. There is a one-to-one mapping between these points and points in $\mathcal{P}^2$ with inhomogeneous coordinates (i.e. not defined up to a scale factor) $[x_0/x_2, x_1/x_2, 1]^t$, $x_2 \neq 0$. Points such that $x_2 = 0$ define then a hyperplane of $\mathcal{P}^2$ called the line at infinity or the ideal line associated to the *affine* subspace of points in $\mathcal{P}^2$ with inhomogeneous coordinates $[a, b, 1]^t$.

To better understand what composes this line let us consider 2 lines of $\mathcal{P}^2$ of the form: $l_0x_0 + l_1x_1 + l_2x_2 = 0$, and $l_0x_0 + l_1x_1 + l_2'x_2 = 0$.

Using their homogeneous representations $L_1 = [l_0, l_1, l_2]^t$ and $L_2 = [l_0, l_1, l_2']^t$ the point at the intersection of these lines is then:

$$X \sim L_1 \times L_2 = [l_1, -l_0, 0]^t.$$

This point belongs to the line with equation $x_2 = 0$. This line is not part of the affine subspace of points with coordinates $[a, b, 1]^t$ but at *infinity* with respect to that space. All the lines of the form $L = [l_0, l_1, l_2]^t$, with l_0 and l_1 fixed, are concurrent on the line at infinity at the point $X = [l_1, -l_0, 0]^t$. $[l_1, -l_0]^t$ is their direction in the affine subspace.

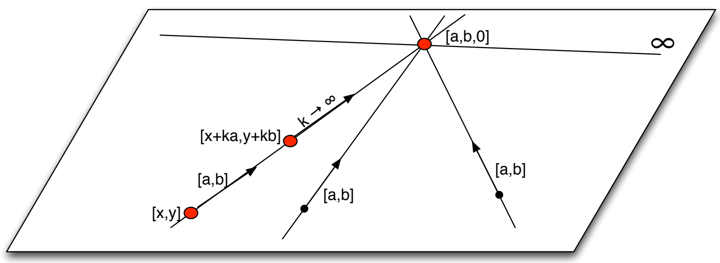


Figure 4.2: All parallel lines with direction $[a, b]$ intersect the line at infinity at the point $[x + ka, y + kb, 1]^t \sim [x/k + a, y/k + b, 1/k]^t =_{k\to\infty} [a, b, 0]^t$.

The projective plane $\mathcal{P}^2$ is the completion of an affine space of dimension 2 with the line at infinity. Note that parallelism is therefore an affine notion since the line at infinity must be identified in $\mathcal{P}^2$ in order to define directions of lines in $\mathcal{P}^2$ (i.e. their intersections with the line at infinity).

4.3 Homographies

The linear transformations of $\mathcal{P}^2$ to itself defined (up to a scale factor) by non-singular 3×3 matrices are called homographies, collineation or projective transformation of $\mathcal{P}^2$.

$$H \sim \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}.$$

A basis of $\mathcal{P}^2$ is defined by 4, non collinear points, and a homography is therefore determined by 4 point correspondences solving for 8 degrees of freedom (the h_is minus a scale factor). Homographies transform points, lines and pencils of lines into points lines and pencils of lines, respectively, and they preserve the cross-ratio.

Exercises

1. How can we determine a homography H given 4 point correspondences ?

2. If H is a homography that transforms points then the associated transformation for lines is: H^{-t}.
   Since incidence relationships are preserved, points X that belong to a line L still belong to a line after transformation. Denote X' and L' the transformed point and line then $L'^t \cdot X' = 0$ thus $L'^t \cdot H \cdot X = 0$ which implies that $L' \sim H^{-t} \cdot L$ since $L^t \cdot X = 0$.

4.4 Conics

A conic is a planar curve described by a second degree homogeneous (defined up to a scale factor) equation:

$$ax_0^2 + bx_0x_1 + cx_1^2 + dx_0 + ex_1 + f = 0.$$

where $[x_0, x_1]$ are the affine coordinates in the plane. Using matrix notation:

$$[x_0, x_1, 1] \cdot \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ 1 \end{bmatrix} = 0.$$

And:

$$[kx_0, kx_1, k] \cdot \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \cdot \begin{bmatrix} kx_0 \\ kx_1 \\ k \end{bmatrix} = X^t \cdot C \cdot X = 0, \ \forall k \in \mathbb{R}^*,$$

where X is the vector of homogeneous coordinates associated to points in the plane and C is the homogeneous matrix associated to the conic. A non degenerate conic has therefore 5 degrees of freedom and is defined by 5 points. Any point transformation H deforms the conic C and yields a new conic $C' \sim H^{-1} \cdot C \cdot H^{-t}$ since $X^t \cdot C \cdot X = 0$ and $X' = H \cdot X$ gives $X'^t \cdot \underbrace{H^{-1} \cdot C \cdot H^{-t}} \cdot X' = 0$.

Hence, in the projective plane, there is no distinction between conics that are simply projective deformations of the circle through projective transformations H of the plane. However, in the affine plane and as shown in figure 4.3, a conic can be classified with respect to its incidence with the line at infinity associated to the affine coordinate system $[x_0, x_1]$ in the projective plane.

Dual conics

The line L tangent to a point X on a conic C can be expressed as: $L = C \cdot X$.

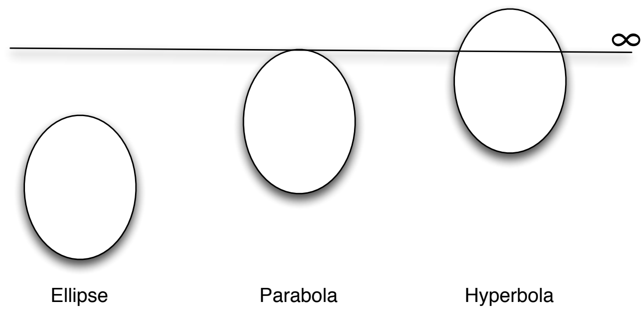


Figure 4.3: The affine classification of conics with respect to their incidences with the line at infinity.

Proof

The line $L = C \cdot X$ is going through X since $L^t \cdot X = X^t \cdot C \cdot X = 0$. Assume that another point Y of L also belongs to C then: $Y^t \cdot C \cdot Y = 0$ and $X^t \cdot C \cdot Y = 0$ and hence any point $X + kY$ along the line defined by X and Y belongs to C as well since: $(X + kY)^t \cdot C \cdot (X + kY) = X^t \cdot C \cdot X + X^t \cdot C \cdot kY + kY^t \cdot C \cdot X + kY^t \cdot C \cdot kY = 0$. Thus L goes through X and is tangent to C.

The set of lines L tangent to C satisfies the equation: $L^t \cdot C^{-1} \cdot L = 0$ (since replacing L with $C \cdot X$ we get $X^t \cdot C^t \cdot C^{-1} \cdot C \cdot X = 0$). C^{-1} is the dual conic of C or the conic envelop (see figure 4.4).

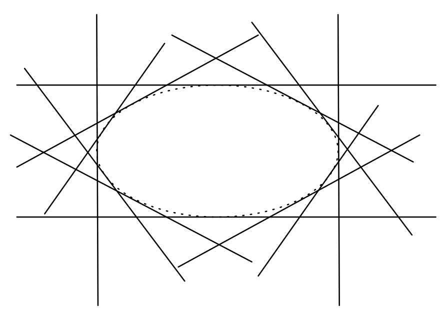


Figure 4.4: The dual conic of lines.

Degenerate conics

When the matrix C is singular the associated conic is said to be degenerated.

For instance 2 lines L_1 and L_2 define a degenerate conic $C = L_1 \cdot L_2^t + L_2 \cdot L_1^t$.

4.5 Affine transformations

The group of affine transformations is a sub-group of the projective transformation group. Affine transformations of the plane are defined, up to a scale factor, by non singular matrices of the form:

$$A \sim \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using the inhomogeneous representation $\bar{X} = [x_0, x_1]^t$, which is valid in the affine plane, we can see that the image $\bar{X}'$ of $\bar{X}$ by A writes:

$$\bar{X}' = \begin{bmatrix} a_1 & a_2 \\ a_4 & a_5 \end{bmatrix} \cdot \bar{X} + \begin{bmatrix} a_3 \\ a_6 \end{bmatrix},$$

where the 2×2 matrix of the left term is of rank 2. This is the usual representation for affine transformations of the plane.

The group of affine transformations of the plane is the group of transformations that leaves the line at infinity globally invariant, i.e. points on the line at infinity are transformed into points on the line at infinity (easily checked with: $A \cdot [x_0, x_1, 0]^t = [x_0', x_1', 0]^t$). This means that parallelism (lines that intersect at infinity) is an affine property that is preserved by affine transformation (see figure 4.5).

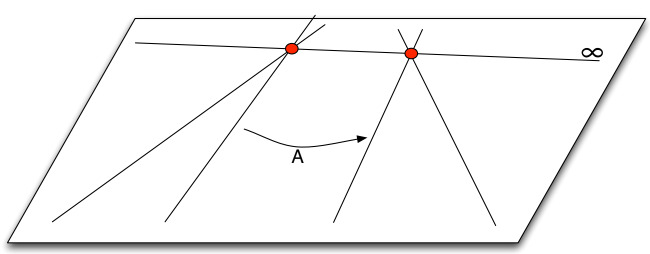


Figure 4.5: Affine transformations preserve the line at infinity and therefore parallelism.

4.6 Euclidean transformations

Restricting further the group of transformations in the plane we can consider the linear transformations defined by matrices of the form:

$$\begin{bmatrix} \cos\theta & -\sin\theta & T_0 \\ \sin\theta & \cos\theta & T_1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using the inhomogeneous representation $\bar{X} = [x_0, x_1]^t$ the image X' of X by an Euclidean transformation becomes:

$$\bar{X}' = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right| \cdot \bar{X} + \begin{bmatrix} T_0 \\ T_1 \end{bmatrix}.$$

The above expression is a rigid transformation in the plane composed of a rotation with angle θ and of a translation $[T_0, T_1]$. Euclidean transformations are affine transformations and as such they preserve the line at infinity. More precisely Euclidean transformations preserve two particular points I and J on the line at infinity called the absolut points[1] with their set being called the absolut and with coordinates $[1, \pm i, 0]$ where $i = \sqrt{-1}$.

$$\begin{bmatrix} \cos\theta & -\sin\theta & T_0 \\ \sin\theta & \cos\theta & T_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta - i\sin\theta \\ \sin\theta + i\cos\theta \\ 0 \end{bmatrix} = \begin{bmatrix} \exp -i\theta \\ i\exp -i\theta \\ 0 \end{bmatrix} = \exp -i\theta \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix},$$

and:

$$\begin{bmatrix} \cos\theta & -\sin\theta & T_0 \\ \sin\theta & \cos\theta & T_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \left[\begin{array}{c} 1 \\ -i \\ 0 \end{array} \right| = \begin{bmatrix} \exp i\theta \\ -i\exp i\theta \\ 0 \end{bmatrix} = \exp i\theta \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}.$$

Stratification of geometry

In the projective plane $\mathcal{P}^2$ the affine plane is defined by the identification of the line at infinity. Similarly, the Euclidean plane is defined by further identifying the absolut points on the line at infinity. This stratification of geometry from projective to Euclidean is well known in computer vision where it was introduced by Faugeras for the reconstruction of 3D scenes from images.

The preservation of the absolut allows to define invariant quantities with the euclidean transformations. These Euclidean properties include angles and Euclidean distances.

Angles

The angle between two lines L_1 and L_2 can be defined using the absolut points. This is known as the Laguerre's Formula and it writes:

$$\alpha = \frac{1}{2i} \log\left(\{P_1, P_2; I, J\}\right),$$

[1] Also the circular or cyclic points.

where α is the angle between L_1 and L_2 and P_1, P_2 are the intersections of these lines with the ideal line (see figure 4.6).

Examples:

1. Assume L_1 and L_2 are parallel then $P_1 = P_2$ and $\alpha = \frac{1}{2i}\log 1 = 0$.

2. Assume L_1 and L_2 are orthogonal and assume further that their intersection points on the ideal line are $[0,1,0]$ and $[1,0,0]$ then:

$$\alpha = \frac{1}{2i}\log\frac{\begin{vmatrix} 0 & 1 \\ 1 & i \end{vmatrix}\begin{vmatrix} 1 & 1 \\ 0 & -i \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 0 & i \end{vmatrix}\begin{vmatrix} 0 & 1 \\ 1 & -i \end{vmatrix}} = \frac{1}{2i}\log(-1) = \frac{1}{2i}\log\exp(i\pi) = \frac{\pi}{2}$$

.

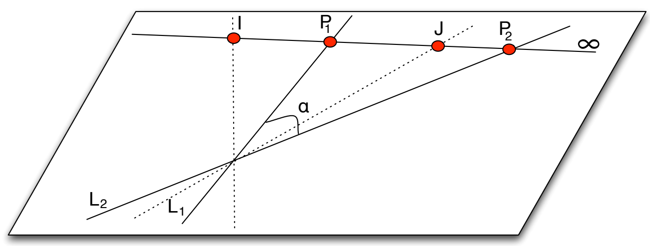


Figure 4.6: The angle between L_1 and L_2 is defined using the cross-ratio between their intersections at infinity and the absolut points I and J.

4.7 Particular transformations

Isometries

Isometries of the plane are linear transformations defined by matrices of the form:

$$\begin{bmatrix} \epsilon\cos\theta & -\sin\theta & T_0 \\ \epsilon\sin\theta & \cos\theta & T_1 \\ 0 & 0 & 1 \end{bmatrix}, \epsilon = \pm 1.$$

They preserve lengths and angles however while direct isometries ($\epsilon = 1$) preserve the orientation, i.e. the absolut points are transformed into themselves, indirect or opposite isometries ($\epsilon = -1$) reverse the orientation, i.e. I becomes J and vice versa.

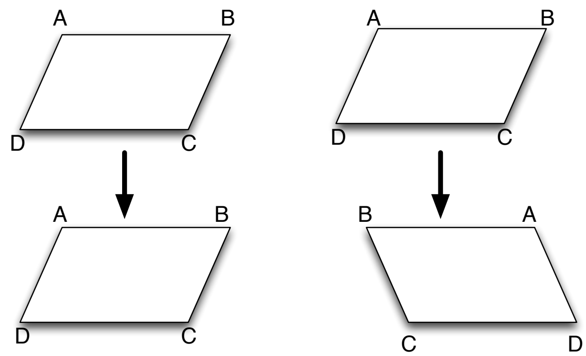


Figure 4.7: Direct (left) and indirect (right) isometries.

Similitudes

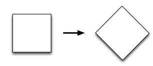
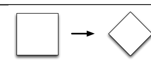
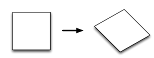
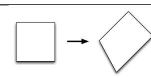
Similitudes of the plane are linear transformations defined by matrices of the form:

$$\begin{bmatrix} s\cos\theta & -s\sin\theta & T_0 \\ s\sin\theta & s\cos\theta & T_1 \\ 0 & 0 & 1 \end{bmatrix}.$$

They preserve the absolut points but not lengths. As a result shapes are preserved by similarity transformations but not scales.

4.8 Transformation hierarchy

In the table below, each group of transformations is included in the following group. Invariants are specific to each group.

Transf. group	Dof	Matrix	Deformation	Invariants
Euclidean	3	$\begin{bmatrix} \cos\theta & -\sin\theta & T_0 \\ \sin\theta & \cos\theta & T_1 \\ 0 & 0 & 1 \end{bmatrix}$		length, area
Isometry	4	$\begin{bmatrix} \epsilon\cos\theta & -\sin\theta & T_0 \\ \epsilon\sin\theta & \cos\theta & T_1 \\ 0 & 0 & 1 \end{bmatrix}$		length ratio, angle, absolut
Affine	6	$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}$		parallelism, area ratio, length ratio on a line, linear vector combinations
Projective	8	$\begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$		incidence, collinearity, concurrence, cross-ratio