Projective Geometry

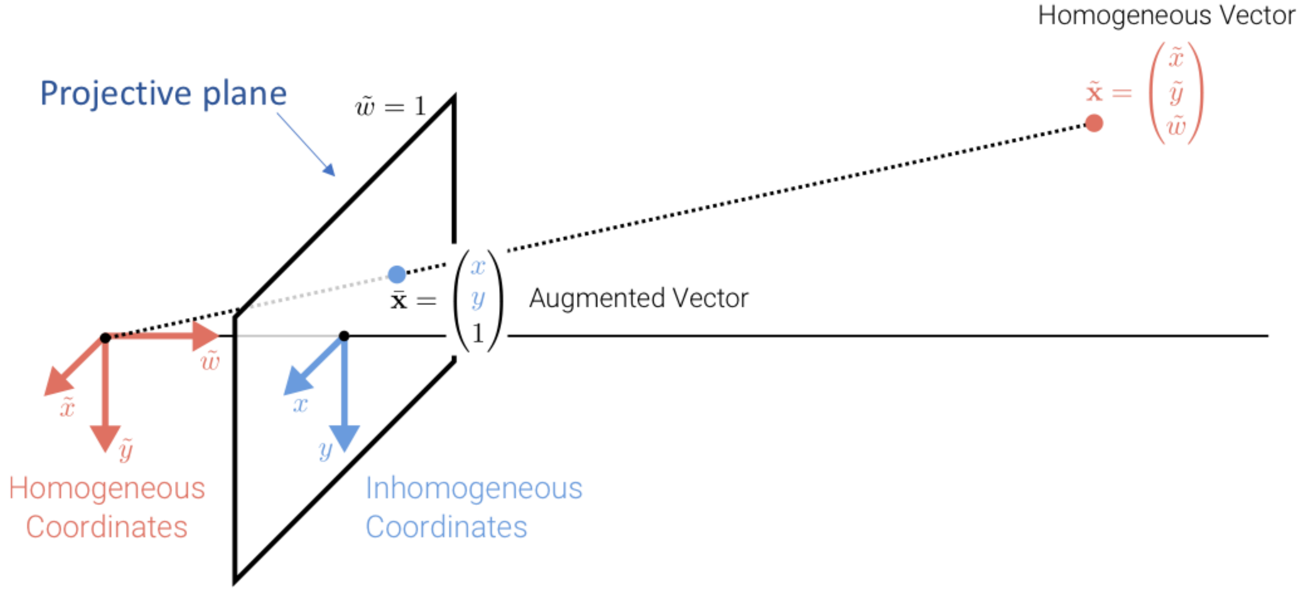
Projective Geometry: A Short Introduction

Geometry in 3D and images

3D -> 2D conversion

- Line-preserving: straight lines(3D) → straight lines(2D)
- Size of objects: inverse proportional to distance
- No angle-preserving
- Vanishing point:
  - Parallel lines are not parallel anymore
  - All mapped parallel lines intersect in a vanishing point
  - Vanishing point at infinity

The Projective Plane



The projective space $\mathcal{P}^2$, associated to the vector space $\mathbb{R}^3$, is called the projective plane. Its importance in visual computing domains is coming from the fact that the image plane of a 3D world projection can be seen as a projective plane and that relationships between images of the same 3D scene can be modeled through projective transformations.

Points and Lines

A point in $\mathcal{P}^2$ is represented by 3 homogeneous coordinates $(x_0, x_1, x_2)^\top$ defined up to scale factor.

A line in $\mathcal{P}^2$ can also be represented by 3 homogeneous coordinates $(l_0, l_1, l_2)^\top$: consider 2 points A and B of $\mathcal{P}^2$ and the line going through them. A third point C belongs to this line only if the coordinates of A, B and C are linearly dependent, i.e. their determinant vanishes

$$\begin{vmatrix} x_0^A & x_0^B & x_0^C \\ x_1^A & x_1^B & x_1^C \\ x_2^A & x_2^B & x_2^C \end{vmatrix} = 0$$

which could be rewritten as

$$l_0 x_0^C + l_1 x_1^C + l_2 x_2^C = (l_0, l_1, l_2) \cdot \begin{pmatrix} x_0^C \\ x_1^C \\ x_2^C \end{pmatrix} = L^\top \cdot \mathbf{x}^C = 0$$

where the l_is are functions of the coordinates of A and B:

$$l_0 = \begin{vmatrix} x_1^A & x_1^B \\ x_2^A & x_2^B \end{vmatrix}, \quad l_1 = -\begin{vmatrix} x_0^A & x_0^B \\ x_2^A & x_2^B \end{vmatrix}, \quad l_2 = \begin{vmatrix} x_0^A & x_0^B \\ x_1^A & x_1^B \end{vmatrix}.$$

DUality points-lines

It could be easily seen that the line joining A and B is (by definition of cross product)

$$L = \mathbf{x}^A \times \mathbf{x}^B$$

For a point $\mathbf{x}$, if L_1 and L_2 both pass it, we have

$$\begin{cases} L_1^\top \cdot \mathbf{x} = 0 \\ L_2^\top \cdot \mathbf{x} = 0 \end{cases}$$

thus

$$\mathbf{x} = L_1 \times L_2$$

Points and lines at infinity

Points such that $x_2 = 0$ define a hyperplane of $\mathcal{P}^2$ called the line at infinity. The line at infinity contains all points at infinity.

In homogeneous coordinates, the line at infinity is $(0, 0, 1)$.

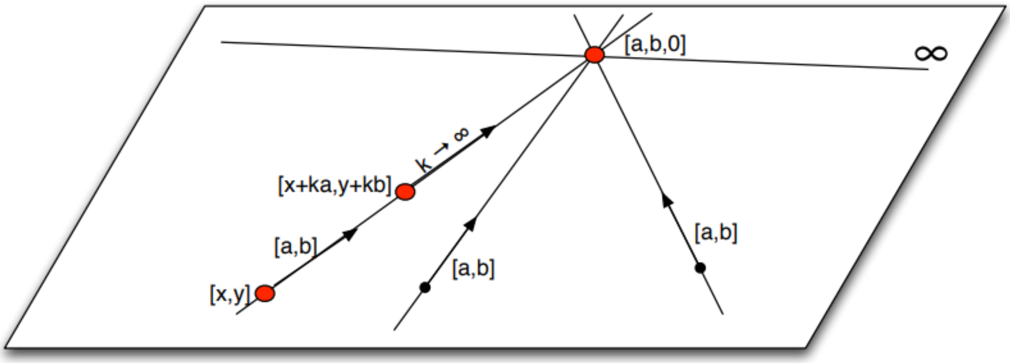
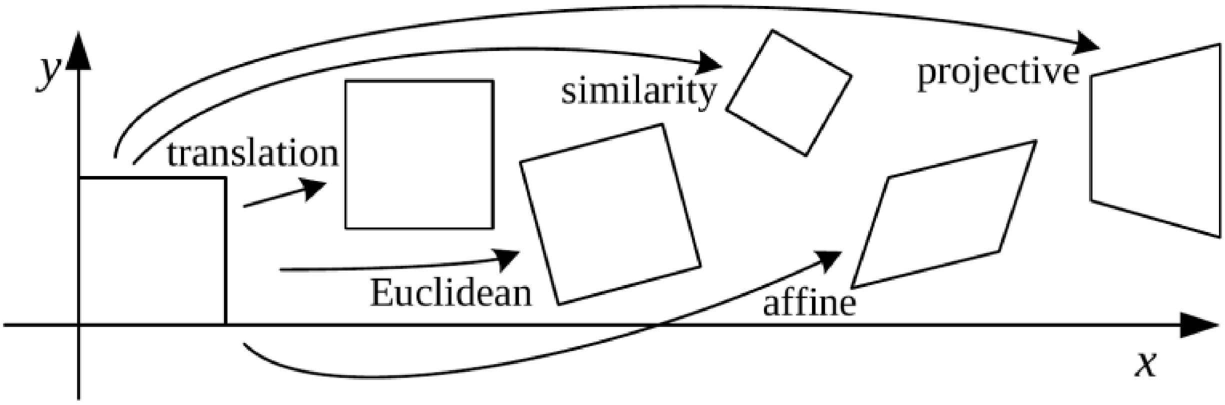


Figure 4.2: All parallel lines with direction $[a, b]$ intersect the line at infinity at the point $[x + ka, y + kb, 1]^t \sim [x/k + a, y/k + b, 1/k]^t =_{k\to\infty} [a, b, 0]^t$.

2D Transformations



1. Scaling, DoF = 2

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

2. Shearing, DoF = 1

$$\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$$

3. Rotation, DoF = 1

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

4. Translation, DoF = 2
   No possible matrix representation in 2D.

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

5. Euclidean(rigid) = rotation + translation, DoF = 3

$$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & r_3 \\ \sin\theta & \cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

6. Similarity = scaling + rotation + translation, DoF = 4

$$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & r_3 \\ s\sin\theta & s\cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

(rotation part multiplied by scale s).

7. Affine = scaling + shearing + rotation + translation, DoF = 6

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}$$

8. Projective(last elements specified by scale), DoF = 8

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Conics

A conic is a planar curve described by a second degree homogeneous (defined up to a scale factor) equation

$$ax_0^2 + bx_0x_1 + cx_1^2 + dx_0 + ex_1 + f = 0$$

where (x_0, x_1) are the affine coordinates in the plane. In homogeneous coordinates, replace x_0, x_1 with $\frac{x_0}{x_2}$ and $\frac{x_1}{x_2}$ respectively.

Using matrix notation

$$\mathbf{x}^\top \mathbf{C} \mathbf{x}$$

where

$$\mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

is called the homogeneous matrix associated to the conic.

There are 5 degrees of freedom: $\{a, b, c, d, e, f\}$ (conic defined up to scale) thus five points define a conic. For each point the conic passes though

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or

$$(x^2, xy, y^2, x, y, 1)\mathbf{c} = 0$$

stack five constraints yields

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

Dual conics

The line L tangent to $\mathbf{C}$ at point $\mathbf{x}$ on $\mathbf{C}$ is given by $L = \mathbf{C}\mathbf{x}$.

> **Proof**:
> The line $L = \mathbf{C}\mathbf{x}$ is going through $\mathbf{x}$ since $L^\top \mathbf{x} = \mathbf{x}^\top \mathbf{C}\mathbf{x} = 0$. Assume that another point $\mathbf{y}$ of L also belongs to $\mathbf{C}$ then $\mathbf{y}^\top \mathbf{C}\mathbf{y} = 0$ and $\mathbf{x}^\top \mathbf{C}\mathbf{y} = 0$ and hence any point $\mathbf{x} + k\mathbf{y}$ along the line defined by $\mathbf{x}$ and $\mathbf{y}$ belongs to $\mathbf{C}$ as well since
>
> $$(\mathbf{x} + k\mathbf{y})^\top \mathbf{C}(\mathbf{x} + k\mathbf{y}) = 0$$
>
> Thus L goes through $\mathbf{x}$ and is tangent to $\mathbf{C}$

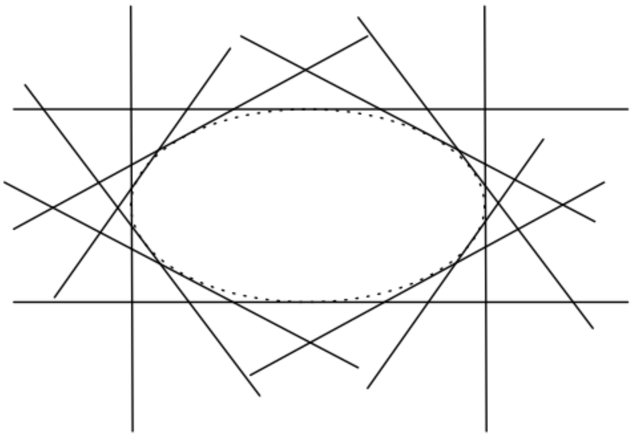
The set of lines L tangent to $\mathbf{C}$ satisfies the equation

$$L^\top \mathbf{C}^* L = 0$$

In general $\mathbf{C}^* = \mathbf{C}^{-1}$.

> **Proof**:
> Simply replace L with $\mathbf{C}\mathbf{x}$ we prove the equation

C^* is the dual conic of $\mathbf{C}$ or the conic envelop



Degenerate Conics

When the matrix $\mathbf{C}$ is singular the associated conic is said to be degenerated.

Example:
2 lines L_1 and L_2 define a degenerate conic $\mathbf{C} = L_1 \cdot L_2^\top + L_2 \cdot L_1^\top$

Projective transformations

A projectivity is an invertible mapping h form $\mathcal{P}^2$ to itself such that three points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ lie on the same line if and only if $h(\mathbf{x}_1), h(\mathbf{x}_2), h(\mathbf{x}_3)$ lie on the same line.

> Theorem:
> A mapping $h : \mathcal{P}^2 \to \mathcal{P}^2$ is a projectivity if and only if there exist a *non-singular* 3×3 matrix H such that for any point in $\mathcal{P}^2$ represented by a vector $\mathbf{x}$ it is true that $h(\mathbf{x}) = H\mathbf{x}$.

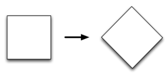
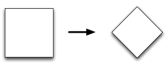
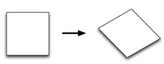
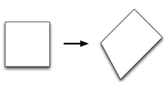
The projective transformation is thus

$$\mathbf{x}' = H\mathbf{x}$$

where

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

which is of 8 degrees of freedom (instead of 9 up to a scale factor).

Transf. group	Dof	Matrix	Deformation	Invariants
Euclidean	3	$\begin{bmatrix} \cos\theta & -\sin\theta & T_0 \\ \sin\theta & \cos\theta & T_1 \\ 0 & 0 & 1 \end{bmatrix}$		length, area
Isometry	4	$\begin{bmatrix} \epsilon\cos\theta & -\sin\theta & T_0 \\ \epsilon\sin\theta & \cos\theta & T_1 \\ 0 & 0 & 1 \end{bmatrix}$		length ratio, angle
Affine	6	$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}$		parallelism, area ratio, length ratio on a line, linear vector combinations
Projective	8	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		incidence, collinearity, concurrence, cross-ratio

To determine a projective transformation given points before and after transformations, we need 4 points for an exact solution for H. Since

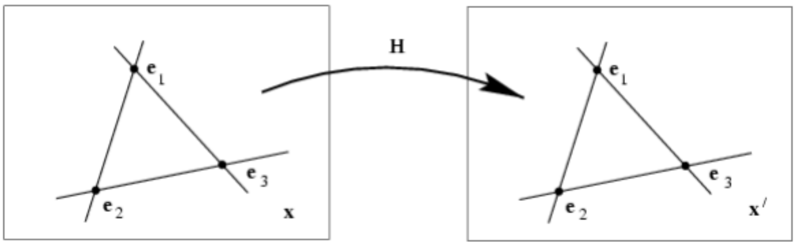
$$\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

we have 2 independent equations for one point. As we have 8 Dof, we need at least 4 points to determine H. If more points are observed, we have not exact solution, because measurements are inexact (noise in measurement).

Transformation of 2D points, lines and conics

- Points: $\mathbf{x}' = H\mathbf{x}$
- Lines: $L' = H^{-\top}L$
- Conics: $\mathbf{C}' = H^{-\top}\mathbf{C}H^{-1}$
- Dual conics: $\mathbf{C}'^{*} = H\mathbf{C}^{*}H^{\top}$

Fixed points and lines



- Eigenvectors of H are fixed points
- Eigenvectors of $H^{-\top}$ are fixed lines

Line at infinity

The line at infinity L_∞ is a fixed line under a projective transformation H if and only if $H = H_A$ is an affinity.

$$L'_\infty = H_A^{-\top} L_\infty = \begin{bmatrix} A^{-\top} & 0 \\ -\mathbf{t}A^{-\top} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = L_\infty$$

Circular Points and Its Conic Dual

Two *circular points* are defined to be

$$I = (1, i, 0)^\top$$
$$J = (1, -i, 0)^\top$$

which satisfies

$$x_1^2 + x_2^2 = 0$$

Circular points algebraically codes orthogonal directions

$$I = (1, 0, 0)^\top + i(0, 1, 0)^\top$$

The conic dual to the circular points is

$$\mathbf{C}^*_\infty = IJ^\top + JI^\top = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The dual conic $\mathbf{C}^*_\infty$ is fixed conic under the projective transformation H if and only if $H = H_S$ is a similarity.

$$\mathbf{C}^*_\infty = H_S \mathbf{C}^*_\infty H_S^\top$$

Angles

Let $l(l_1, l_2, l_3)$ and $m(m_1, m_2, m_3)$ be two lines, and they form an angle θ. Their Euclidean angle is

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

The projective angle is

$$\cos\theta = \frac{l^\top \mathbf{C}^*_\infty m}{\sqrt{(l^\top \mathbf{C}^*_\infty l)(m^\top \mathbf{C}^*_\infty m)}}$$

Direct Linear Transformation(DLT)

Let $\mathbf{x}'_i = H\mathbf{x}_i$, then

$$\mathbf{x}'_i \times H\mathbf{x}_i = 0$$

Denote $\mathbf{x}'_i = (x'_i, y'_i, w'_i)$ and

$$H = \begin{pmatrix} \mathbf{h}_1^\top \\ \mathbf{h}_2^\top \\ \mathbf{h}_3^\top \end{pmatrix}$$

Then

$$H\mathbf{x}_i = \begin{pmatrix} \mathbf{h}_1^\top \mathbf{x}_i \\ \mathbf{h}_2^\top \mathbf{x}_i \\ \mathbf{h}_3^\top \mathbf{x}_i \end{pmatrix}$$

and

$$\mathbf{x}'_i \times H\mathbf{x}_i = \begin{pmatrix} y'_i \mathbf{h}_3^\top \mathbf{x}_i - w'_i \mathbf{h}_2^\top \mathbf{x}_i \\ w'_i \mathbf{h}_1^\top \mathbf{x}_i - x'_i \mathbf{h}_3^\top \mathbf{x}_i \\ x'_i \mathbf{h}_2^\top \mathbf{x}_i - y'_i \mathbf{h}_1^\top \mathbf{x}_i \end{pmatrix}$$

which is equivalent to

$$\begin{bmatrix} \mathbf{0}^\top & -w'_i \mathbf{x}_i^\top & y'_i \mathbf{x}_i^\top \\ w'_i \mathbf{x}_i \top & \mathbf{0}^\top & -x'_i \mathbf{x}_i^\top \\ -y'_i \mathbf{x}_i^\top & x'_i \mathbf{x}_i^\top & \mathbf{0}^\top \end{bmatrix} \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{pmatrix} = \mathbf{0}$$

Since only 2 out 3 equations are linearly independent, we could drop the third row(only if $w'_i \neq 0$).

Denote

$$\begin{bmatrix} \mathbf{0}^\top & -w'_i \mathbf{x}_i^\top & y'_i \mathbf{x}_i^\top \\ w'_i \mathbf{x}_i \top & \mathbf{0}^\top & -x'_i \mathbf{x}_i^\top \end{bmatrix} = A_i$$

stack A_is to get

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

and solve $A\mathbf{h} = 0$ for H, where

$$\mathbf{h} = \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{pmatrix}$$

A could be 8×9 or 12×9, but of rank 8.

If we have more than 4 point pairs, we get an overdetermined equation. No exact solution exist because of inexact measurement. To find approximate solution, we could

- Add additional constraint needed to avoid 0, e.g. $\|\mathbf{h}\| = 1$
- If $A\mathbf{h} = 0$ not possible, try to minimize $\|A\mathbf{h}\|$

> **DLT algorithm**
> Objective:
> Given $n \geq 4$ 2D to 2D point correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{x}_i'\}$, determine the 2D homography matrix H such that $\mathbf{x}_i' = H\mathbf{x}_i$
> Algorithm:
> 1. For each correspondence $\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$ compute A_i. Usually only two first rows needed.
> 2. Assemble n 2×9 matrices A_i into a single $2n \times 9$ matrix A.
> 3. Obtain SVD of A. Solution for $\mathbf{h}$ is the last column of V.
> 4. Determine H from $\mathbf{h}$.

Previously we say if $A\mathbf{h} = 0$ is not possible, we try to minimize $\|A\mathbf{h}\|$. There are several ways we can minimize the term, in different definitions of cost functions.

Algebraic Distance

Define:

- $\mathbf{e} = A\mathbf{h}$, the residual vector
- $\mathbf{e}_i = A_i\mathbf{h}$, the partial vector for each $\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$
- $d_{\text{alg}}^2(\mathbf{x}_1, \mathbf{x}_2) = a_1^2 + a_2^2$ where $\mathbf{a} = (a_1, a_2, a_3)^\top = \mathbf{x}_1 \times \mathbf{x}_2$

Thus

$$d_{\text{alg}}^2(\mathbf{x}_i', H\mathbf{x}_i) = \|\mathbf{e}_i\|^2 = \left\|\begin{bmatrix} 0^\top & -w_i'\mathbf{x}_i^\top & y_i'\mathbf{x}_i^\top \\ w_i'\mathbf{x}_i^\top & 0^\top & -x_i'\mathbf{x}_i^\top \end{bmatrix}\mathbf{h}\right\|^2$$

and

$$\sum_i d_{\text{alg}}^2(\mathbf{x}_i', H\mathbf{x}_i) = \sum_i \|\mathbf{e}_i\|^2 = \|A\mathbf{h}\|^2 = \|\mathbf{e}\|^2$$

Algebraic distance is not geometrically/statistically meaningful, but given good normalization it works fine and is very fast (use for initialization).

It could be easily seen that DLT minimizes $\|A\mathbf{h}\|$.

Geometric Distance

Define:

- $\mathbf{x}$, measured coordinates
- $\hat{\mathbf{x}}$, estimated coordinates
- $\overline{\mathbf{x}}$, true coordinates
- $d(\cdot, \cdot)$, Euclidean distance (in an image)

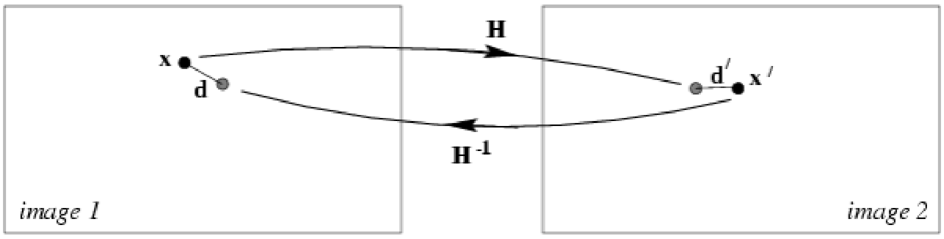
Then define

- Error in one image

$$\hat{H} = \arg\min_H \sum_i d^2(\mathbf{x}_i', H\overline{\mathbf{x}}_i)$$

- Symmetric transfer error

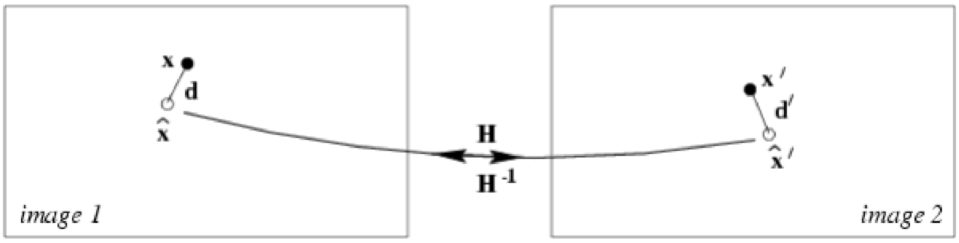
$$\hat{H} = \arg\min_H \sum_i [d^2(\mathbf{x}_i, H^{-1}\mathbf{x}_i') + d^2(\mathbf{x}_i', H\mathbf{x}_i)]$$



- Reprojection error

$$\left(\hat{H}, \hat{\mathbf{x}}_i, \hat{\mathbf{x}}_i'\right) = \underset{H, \hat{\mathbf{x}}_i, \hat{\mathbf{x}}_i'}{\arg\min} \sum_i d\left(\mathbf{x}_i, \hat{\mathbf{x}}_\mathrm{i}\right)^2 + d\left(\mathbf{x}_i', \hat{\mathbf{x}}'\right)^2$$

$$\text{subject to } \hat{\mathbf{x}}_i' = \hat{H}\hat{\mathbf{x}}_i$$



Comparison of Geometric and Algebraic Distances

Denote $\mathbf{x}_i' = (x_i', y_i', w_i')$, $\hat{\mathbf{x}} = (\hat{x}_i', \hat{y}_i', \hat{w}_i') = H\overline{\mathbf{x}}$. And we have (you might need to deduce this)

$$A_i\mathbf{h} = \mathbf{e}_i = \begin{pmatrix} y_i'\hat{w}_i' - w_i'\hat{y}_i' \\ w_i'\hat{x}_i' - x_i'\hat{w}_i' \end{pmatrix}$$

Then the algebraic distance is

$$d_{\text{alg}}^2(\mathbf{x}_i', \hat{\mathbf{x}}_i') = \left(y_i'\hat{w}_i' - w_i'\hat{y}_i'\right)^2 + \left(w_i'\hat{x}_i' - x_i'\hat{w}_i'\right)^2$$

and the Euclidean distance is

$$d^2(\mathbf{x}_i', \hat{\mathbf{x}}_i') = \left(y_i'/w_i' - \hat{y}_i'/\hat{w}_i'\right)^2 + \left(\hat{x}_i'/\hat{w}_i' - x_i'/w_i'\right)^2 = \frac{d_{\text{alg}}^2(\mathbf{x}_i', \hat{\mathbf{x}}_i')}{(w_i'\hat{w}_i')^2}$$

$w_i' = 1$ typically, and $\hat{w}_i' = \mathbf{x}_i^\top \mathbf{h}_3$, but for affinities $\hat{w}_i' = 1$, too. Thus for affinities DLT can minimize geometric distance.

Statistical Cost FUnction and MLE

Assume zero-mean isotropic Gaussian noise

$$P(x) = \frac{1}{2\pi\sigma^2} e^{-d^2(\mathbf{x}, \overline{\mathbf{x}})/(2\sigma^2)}$$

The pdf of error in one image is

$$P(\{\mathbf{x}_i'\}|H) = \prod_i \frac{1}{2\pi\sigma^2} e^{-d^2(\mathbf{x}_i', H\overline{\mathbf{x}}_i)/(2\sigma^2)}$$

The log-likelihood is then

$$\text{LL} = -\frac{1}{2\sigma^2} \sum_i d^2(\mathbf{x}_i', H\overline{\mathbf{x}}_i) + \text{const}$$

MLE is thus equivalent to minimization of geometric distance.

The pdf of error in both images is

$$P(\{\mathbf{x}_i'\}|H) = \prod_i \frac{1}{2\pi\sigma^2} e^{-(d(\mathbf{x}_i,\hat{\mathbf{x}}_i)^2 + d(\mathbf{x}_i',\hat{\mathbf{x}}')^2)/(2\sigma^2)}$$

MLE is equivalent to the minimization of reprojection error.

Mahalonobis distance

This is the general Gaussian case. Observations are now not independent. Measurement X with covariance matrix Σ.

$$\|X - \overline{X}\|_\Sigma^2 = (X - \overline{X})^\top \Sigma^{-1} (X - \overline{X})$$

Projective 3D Space

Points and Planes

The homogeneous representation of points in $\mathcal{P}^3$ is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

and a plane is represented by

$$\pi = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix}$$

The point $\mathbf{x}$ lies on the plane if and only if $\pi^\top \mathbf{x} = 0$, and the plane π goes through the point $\mathbf{x}$ if and only if $\pi^\top \mathbf{x} = 0$.

Determine a Plane from Points

Three points determine a plane

$$\begin{bmatrix} \mathbf{x}_1^\top \\ \mathbf{x}_2^\top \\ \mathbf{x}_3^\top \end{bmatrix} \pi = 0$$

Determine a Point from Planes

Three planes determine a point

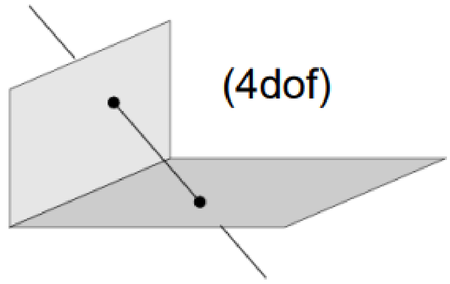
$$\begin{bmatrix} \pi_1^\top \\ \pi_2^\top \\ \pi_3^\top \end{bmatrix} \mathbf{x} = 0$$

Let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ be a span of a plane π, then

$$\pi^\top [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 0$$

Lines

Lines are represented by its span:



$$W = \begin{bmatrix} A^\top \\ B^\top \end{bmatrix}, \quad \lambda A + \mu B$$

The dual representation of a line is

$$W^* = \begin{bmatrix} P^\top \\ Q^\top \end{bmatrix}, \quad \lambda P + \mu Q$$

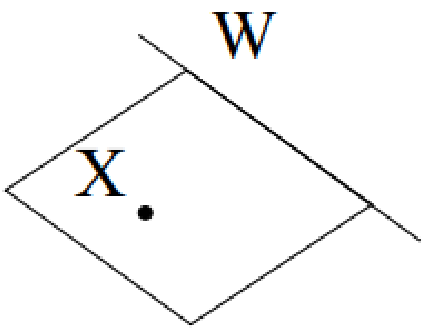
Example: X-axis

$$W = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad W^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Points, Lines and Planes

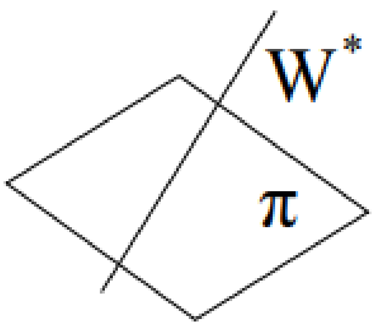
A line and a Point => A plane

$$M = \begin{bmatrix} W \\ \mathbf{x}^\top \end{bmatrix} \quad M\pi = 0$$



A Line and a Plane => A Point

$$M = \begin{bmatrix} W^* \\ \pi^\top \end{bmatrix} \quad M\mathbf{x} = 0$$



Quadrics and Dual Quadrics

A quadric Q satisfies

$$\mathbf{x}^\top Q\mathbf{x} = 0$$

where Q is a 4×4 symmetric matrix.

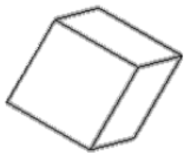
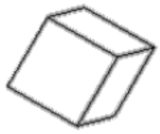
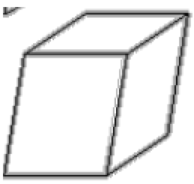
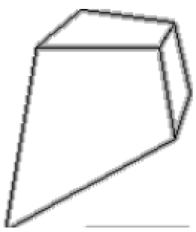
- 9 Dof, i.e. 9 points define a quadric
- tangent plane of a quadric at $\mathbf{x}$ is $\pi = Q\mathbf{x}$

The dual quadric is Q^* and in general $Q^* = Q^{-1}$, and it satisfies

$$\pi^\top Q^* \pi = 0$$

Transformation of 3D Points, Planes and Quadrics

- Points: $\mathbf{x}' = H\mathbf{x}$
- Planes: $\pi' = H^{-\top}\pi$
- Quadrics: $Q' = H^{-\top} Q H^{-1}$
- Dual conics: $Q'^* = HQ^*H^\top$

Trans.	Dof	Matrix	Deformation	Invariants
Euclidean	6	$\begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$		Volume
Similarity	7	$\begin{bmatrix} sR & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$		Angles, ratios of length. The absolute conic Ω_∞
Affine	12	$\begin{bmatrix} A & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids, the plane at infinity π_∞
Projective	15	$\begin{bmatrix} A & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix}$		Intersection and tagency

The Plane at Infinity

The plane at infinity π_∞ is a fixed plane under a projective transformation H if and only if $H = H_A$ is an affinity.

$$\pi'_\infty = H_A^{-\top}\pi_\infty = \begin{bmatrix} A^{-\top} & \mathbf{0} \\ -\mathbf{t}^\top A^{-\top} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \pi_\infty$$

- canonical position $\pi_\infty = (0,0,0,1)^\top$

- contains directions $D = (x_1, x_2, x_3, 0)^\top$
- two planes are parallel $\iff$ line of intersection in π_∞
- parallel lines $\iff$ point of intersection in π_∞

The Absolute Conic

The absolute conic Ω_∞ is a (point) conic on π_∞. In metric frame

$$\begin{cases} x_1^2 + x_2^2 + x_3^2 & = 0 \\ x_4 & = 0 \end{cases}$$

The absolute conic Ω_∞ is a fixed conic under the projective transformation H if and only is H is a similarity.

The absolute dual quadric is

$$\Omega_\infty^* = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix}$$

The absolute conic Ω_∞^* is a fixed conic under the projective transformation H if and only is H is a similarity.

Angles

$$\cos\theta = \frac{\pi_1^\top \Omega_\infty^* \pi_2}{\sqrt{(\pi_1^\top \Omega_\infty^* \pi_1)(\pi_2^\top \Omega_\infty^* \pi_2)}}$$

Action of Projective Camera on Points and Lines

Denote $\mathbf{x}_w$ world coordinates and $\mathbf{x}_p$ the picture coordinates.

Projection of Points

$$\mathbf{x}_p = P\mathbf{x}_w = P \begin{bmatrix} R^\top & -R^\top \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \mathbf{x}_w$$

Forward Projection of Lines

Lines ⇒ Lines

$$P(A + \mu B) = PA + \mu PB = \mathbf{a} + \mu \mathbf{b}$$

Back-projection of lines

Lines ⇒ Planes

$$\pi = P^\top l$$

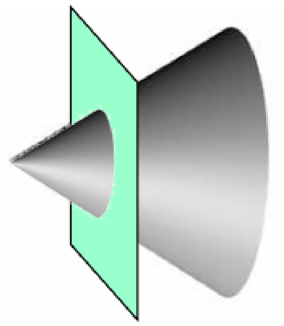
Proof:

$$\pi^\top \mathbf{x}_w = l^\top P \mathbf{x}_w = l^\top \mathbf{x}_p$$

$l^\top \mathbf{x}_p = 0$ as long as $\pi^\top \mathbf{x}_w = 0$, i.e. points on plane π after projection must be on line l.

Action of projective camera on conics and quadrics

Back-projection to COne

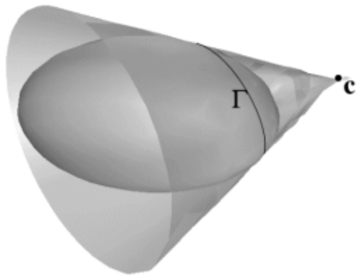


$$Q = P^\top \mathbf{C} P$$

Proof:

$$\mathbf{x}_p^\top \mathbf{C} \mathbf{x}_p = \mathbf{x}_w^\top P^\top \mathbf{C} P \mathbf{x}_w = \mathbf{x}_w^\top Q \mathbf{x}_w$$

Forward-Projection to Quadric



$$\mathbf{C}^* = P Q^* P^\top$$

Proof:

$$\pi^\top Q^* \pi = l^\top P Q^* P^\top l = l^\top C^* l$$