Counting and Probability

1) **[1]** At lunch, you can get 3 different entrees, 2 different drinks and 2 different desserts. How many different lunches can you have if each lunch must consist of one entree, one drink, and one dessert?

2) **[2]** In how many ways can you choose three groups of three people from nine people, assuming each person can only be assigned to one group?

3) **[1]** In how many ways can I select a president and two assistants from a group of 20 people?

4) **[2]** How many paths are possible between the points (2, -5) and (6, 4) if these paths are the shortest length possible and must lie along grid lines?

5) **[2]** A traffic light runs repeatedly through the following cycle: green for 30 seconds, then yellow for 3 seconds, and then red for 30 seconds. Erica picks a random three-second time interval to watch the light. What is the probability that the color changes while she is watching?

6) **[2]** Three vertices of a unit cube are chosen at random. What is the probability that they form a triangle an area greater than ½?

7) **[2]** Suppose I have a bag with 12 Magic-8 Balls in it. Some of the Magic-8 Balls actually have a 3 on them, and the rest have the conventional 8 on them. If the expected value of the number shown on a ball randomly drawn from the bag is 4.25, then how many Magic-8 Balls have actually have an 8?

8) **[3]** In how many ways can we pick a group of 3 different numbers from the group 1, 2, 3, … 500 such that one number is the average of the other two? Assume order does not matter.

9) **[2]** Determine the number of distinct arrangements of the letters of the word SYMMETRICAL.

Geometry

10) **[1]** How many degrees are in the interior angle of a regular polygon with 40 sides?

11) **[½ each]** Consider a triangle with side lengths 13, 14, and 15. Find each of the following:

   a) Perimeter

b) Area

c) Inradius (radius of inscribed circle)

d) Circumradius (radius of circumscribed circle)

12) **[1]** What is the maximum area a triangle can have if two of its sides have lengths 6 and 7?

13) **[1]** Find the coordinates of the foci of the equation $4x^2+9y^2=36$.

14) **[2]** Name 3 of the several special points of a triangle that lie on the Euler line.

15) **[3]** A hexagon inscribed in a circle has three consecutive sides, each of length 3, and three consecutive sides, each of length 5. The chord of the circle that divides the hexagon into two trapezoids, one with three sides, each of length 3, and the other with three sides, each of length 5, has length equal to m/n, where m and n are relatively prime positive integers. Find m+n.

16) **[3]** Medians XA and YB of ΔXYZ are perpendicular. If m<XYZ = 90° and XY = 8, find YZ.

17) **[2]** Square A'B'C'D' is the image of a dilation of square ABCD about the origin with scale factor of 3. Suppose AB = 3, A has coordinates (3, 10), B has coordinates (6,10), and the sides of the square are parallel to the coordinate axes. Find BC' and CD', respectively, in simplest radical form.

18) **[2]** Find the radius of a circle that is inscribed in a rhombus that has diagonals of length 16 and 30. Express your answer as a fraction in simplest form.

19) **[1 each]** A cyclic quadrilateral has side lengths 6, 9, 13, and 8. Find each of the following:

a) Area in simplest radical form

b) The product of the lengths of its diagonals

c) The sine of the angle between the sides of length 6 and 8

Number Theory

20) **[1]** How many factors of 420 are there?

21) **[1]** How many numbers less than 91 are relatively prime to it?

22) **[2]** Convert 256_7 to base 11.

23) **[2]** Give the prime factorization of 40216.

24) **[3]** Convert 14.38_{10} to base 3.

25) **[1]** How many numbers less than 120 are relatively prime to 15?

26) **[2]** What is 2^{2016} (mod 100)?

27) **[3]** What famous conjecture states that all even numbers greater than or equal to 4 can be expressed as the sum of two not necessarily distinct primes?

Algebra

28) **[1]** What is the solution to the equation $3x - 4 = x + 14$?

29) **[1]** Solve for x and y:
$$3x + 7y = 47$$
$$x + 4y = 24$$

30) **[2]** Solve for x, y, and z:
$$3x + y - z = 2$$
$$x + 4y + 3z = 10$$
$$2x - y + z = -3$$

31) **[2]** Solve the quadratic $3x^2 + 8x + 5 = 0$.

32) **[2]** Suppose I am running a bookshop. At $8 a book, I sell 40 of them. For every dollar that I increase my price by, I sell two less books. What price should I sell my books at to maximize my revenue?

33) **[3]** Evaluate $\sum_{i=0}^{\infty} \frac{3^i}{4^i}$.

34) **[2]** If $x+y = 18$ and and $x^2+y^2=170$, find x^3+y^3.

35) **[3]** Describe all solutions for $\cos^2(x)+\cos(x)=\sin^2(x)$ in radians.

36) **[2]** Let $a_1, a_2, \ldots$ be a sequence for which $a_1 = 2$, $a_2 = 3$, and $a_n = a_{n-1}/a_{n-2}$ for each positive integer n ≥ 3. What is the product of a_{2015}, a_{2016}, and a_{2017}?

37) **[2]** Find all ordered pairs of positive integers (x,y) that satisfy both $y \leq 5-x$ and $y \geq \log_2 x$.

Statistics

Questions 38-43 are about the following set of numbers:

{ 13, 11, 6, 8, 7, 4, 8, 8, 10, 10, 8, 4, 8, 15, 4 }

38) **[1]** What is the range of the set?

39) **[1]** What is the mode?

40) **[1]** What is the median?

41) **[1]** What is the 3rd quartile minus the first quartile?

42) **[2]** To the nearest whole number, what is the standard deviation of the set?

43) **[2]** What is the number of distinct two-element subsets of the set with relatively prime elements?

44) **[2]** What kind of error occurs when the null hypothesis is true and it is incorrectly rejected, known as a false positive?

45) **[3]** A weighted coin with a 25% chance to draw heads is tossed 400 times. What is the standard deviation of the number of heads face-up in simplest radical form?

46) **[1 each]** Give the name of the sampling method that each of the following situations demonstrates:

   a) A lottery is held with 500 contestants. Each contestant writes his or her name on a slip of paper and is placed into a jar and one is drawn. Each slip has an equal probability of being drawn.

   b) All members of a population of blobfish are separated into groups based solely on their size. Certain groups are chosen at random and only blobfish in these groups are surveyed.

   c) All members of a population of blobfish are separated into groups based solely on their size. Certain individuals from each group are chosen at random and are surveyed.

Arithmetic

47) **[1]** What is the sum of the first 1000 even numbers?

48) **[1]** How many four-digit palindromes are divisible by 3?

49) **[2]** What is the smallest five-digit palindrome divisible by 13?

50) **[1]** Legend has it that as a young child, Gauss was disruptive during class, so the teacher sent him to a corner and told him to add up the first 100 natural numbers. Before Gauss even got to the seat, he already got the answer. The teacher went home, added up all the numbers from 1 to 100, and saw that Gauss was correct. What is this sum?

51) **[3]** The sum of three numbers is 56 and their product is 4200. What is the value of the square of the largest number minus the sum of the squares of the smaller two numbers?

52) **[2]** Find the product of the first 10 terms in the Fibonacci Sequence. Assume the Fibonacci Sequence starts, from the first term, with 1, 1, 2, 3, and so on.

53) **[2]** What is the sum of the two numbers whose squares differ by 321?

"Proof Questions"

1) **[4 for best answer]** What is the numerical value of the summation of the positive integers 9 and 10 over the field of real numbers? Provide a thorough explanation to support your answer.

2) **[30]** Prove that the real part of every non-trivial zero of the analytic continuation of the function $\zeta(s)$ is 1/2.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$