# The Calculus Bee at MIT

## Sample problems

*Problem contributors.*

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### Instructions

Enclosed are three integration problems, three short-answer problems, and one proof-level problem. These are not official and they will not appear on any CalcBee contest but they should give some insights into the difficulty level of the problems on the contest.

**Integration solutions.**

We will ignore the integration constants.

1. *Problem*: Compute $\displaystyle\int_1^{\ln 2015} (1 + \ln|x|)\ \mathrm{d}x$.

   *Solution*: Note that $\displaystyle\int (1 + \ln|x|)\ \mathrm{d}x = x + \int \ln|x|\ \mathrm{d}x$.

   By integration by parts using $u = \ln|x|$ and $\mathrm{d}v = \mathrm{d}x$, we find

$$\begin{aligned}\int \ln|x|\ \mathrm{d}x &= x\ln|x| - \int \mathrm{d}x \\ &= x\ln x.\end{aligned}$$

   Thus, our answer is

$$2015\ln 2015 - 1\ln 1 = \boxed{2015\ln 2015}.$$

2. *Problem*: Find $\displaystyle\int \sinh\left(\sqrt{x}\right)\ \mathrm{d}x$.

   *Solution*: Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sinh\left(\sqrt{x}\right)\right) = \frac{\cosh(\sqrt{x})}{2\sqrt{x}},$$

   so $\sqrt{x}\sinh(\sqrt{x})$, $\sqrt{x}\cosh(\sqrt{x})$, $\sinh(\sqrt{x})$, and $\cosh(\sqrt{x})$ seem helpful. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\sinh\left(\sqrt{x}\right)\right) &= \frac{\sinh(\sqrt{x})}{2\sqrt{x}} + \sqrt{x}\cosh(\sqrt{x}) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\cosh\left(\sqrt{x}\right)\right) &= \frac{\cosh(\sqrt{x})}{2\sqrt{x}} + \sqrt{x}\sinh(\sqrt{x}) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\sinh(\sqrt{x})\right) &= \frac{\cosh(\sqrt{x})}{2\sqrt{x}} \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\cosh(\sqrt{x})\right) &= \frac{\sinh(\sqrt{x})}{2\sqrt{x}}\end{aligned}$$

   Then, we realize that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\cosh(\sqrt{x})\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\sinh(\sqrt{x})\right)\ \mathrm{d}x = \frac{\sinh(\sqrt{x})}{2}$. Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(2\sqrt{x}\cosh(\sqrt{x}) - 2\sinh(\sqrt{x})\right) &= \sinh(\sqrt{x}) \\ \Longrightarrow \int \sinh(\sqrt{x})\,\mathrm{d}x &= \boxed{2\sqrt{x}\cosh(\sqrt{x}) - 2\sinh(\sqrt{x})}.\end{aligned}$$

3. *(Hard.)* Find $\displaystyle\int \frac{1 - x^2}{1 + 3x^2 + x^4}\,\mathrm{d}x$.

*Solution*: Notice that the degree of the denominator is double the degree of the numerator, so there is a possibility that the answer is of the form $\arctan(f(x))$, where $f(x)$ is a rational function. So we try it out. If this is the case, we have

$$\frac{f'(x)}{1+[f(x)]^2} = \frac{1-x^2}{1+3x^2+x^4}.$$

Then, let $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with degree $a$ and $b$, respectively. Then,

$$f'(x) = \frac{q(x)p'(x) - p(x)q'(x)}{[q(x)]^2},$$

where the numerator has degree less than or equal to $a+b-1$ and the denominator has degree $2b$. Therefore, $b = 2$ and $a = 1$. Then,

$$f(x) = \frac{rx+s}{tx^2+ux+v} \implies f'(x) = \frac{(rv-su) - 2stx - rtx^2}{t^2x^4 + 2tux^3 + (2tv+u^2)x^2 + 2uvx + v^2}.$$

Then, the derivative of $\arctan(f(x))$ is

$$\frac{f'(x)}{1+[f(x)]^2} = \frac{\frac{(rv-su)-2stx-rtx^2}{t^2x^4+2tux^3+(2tv+u^2)x^2+2uvx+v^2}}{1+\left(\frac{rx+s}{tx^2+ux+v}\right)^2}.$$

We find that $p(x) = x$ and $q(x) = 1 + x^2$, so our answer is

$$\boxed{\arctan\left(\frac{x}{1+x^2}\right)}.$$

**Short-answer problems.**

1. *Problem*: A car has instantaneous velocity function $v(t) = \frac{1}{2t-3}$. Given that the position at time $t = 0$ is 0, what is the position at time $t = 2$? If indeterminate, say so.

   *Bogus Solution*: Since $v(t) = \frac{1}{2t-3}$, we have $x(t) = \frac{\ln|2t-3|}{2} - \frac{\ln 3}{2}$. Thus, the answer is $x(2) = -\frac{\ln 3}{2}$.

   *Real Solution*: The above integral cannot be taken because the velocity is undefined at $t = \frac{3}{2}$. Hence, the answer is $\boxed{\text{indeterminate}}$.

2. *Problem*: Consider the polynomial $f(x) = ax^2 + 20x + 15$. If $f'(t) = f''(t)$, then $t$ is an integer. Compute the sum of all possible values of $a$.

   *Solution*: Since $f'(t) = 2at + 20$ and $f''(t) = 2a$, we have

   $$f'(t) = f''(t) \iff 2ax + 20 = 2a \iff x = 1 - \frac{10}{a}.$$

   Hence, the set of all possible values of $a$ is the set of all integer divisors of $a$, and the sum of these values is $\boxed{0}$.

3. *Problem*: Find the volume of the solid resulting from the rotation of the region enclosed by $y = x^m$ and $y = x^n$ about the line $x = 1$.

   *Solution*: The radius of each large disk is $1 - y^{\frac{1}{\min(m,n)}}$. The radius of each small disk is $1 - y^{\frac{1}{\max(m,n)}}$. Hence, the answer is

$$\int_0^1 \pi \cdot \left(\left(1 - y^{\frac{1}{\min(m,n)}}\right)^2 - \left(1 - y^{\frac{1}{\max(m,n)}}\right)^2\right) \, \mathrm{d}y.$$

   Let $a = \frac{1}{\min(m,n)}$ and $b = \frac{1}{\max(m,n)}$. Then, our answer is

$$\begin{aligned} \int_0^1 \pi \cdot \left((1 - y^a)^2 - \left(1 - y^b\right)^2\right) \, \mathrm{d}y &= \pi \cdot \int_0^1 \left(1 - 2y^a + y^{2a} - 1 + 2y^b - y^{2b}\right) \, \mathrm{d}y \\ &= \pi \cdot \left[-\frac{2y^{a+1}}{a+1} + \frac{y^{2a+1}}{2a+1} + \frac{2y^{b+1}}{b+1} - \frac{y^{2b+1}}{2b+1}\right]_0^1 \\ &= \boxed{\pi \cdot \left(\frac{1}{2a+1} + \frac{2}{b+1} - \frac{2}{a+1} - \frac{1}{2b+1}\right)}. \end{aligned}$$

**Proof problem.**

1. *Problem*: Compute, with proof, the limit

$$\lim_{n\to\infty} \sqrt[n]{\ln\left|1 + \left(\frac{1}{n \cdot \log n}\right)^k\right|}.$$

   *Solution*:

$$\forall\, x \geq 0 \;:\; x - \frac{x^2}{2} \leq \ln(1 + x) \leq x.$$

   Hence, by the squeeze theorem, the answer is $\boxed{1}$ for all $k$. □