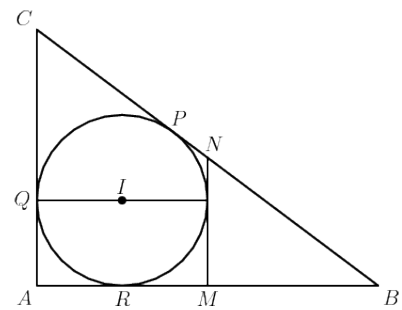
LHS Math Team

Team Contest 2012-13 Solutions

1.



Let P, Q, and R be the tangency points of the incircle with the triangle, and let M and N be the midpoints of $\overline{AB}$ and $\overline{BC}$. Then, $\overline{MN} \parallel \overline{AC}$, so since this is a right triangle, it is evident that $AR = RM = 1$ and $AM = 2$, so $AB = 4$.

By equal tangents, $CQ = CP$ and $BR = BP$. We can compute $BP = BR = RM + MB = 3$. Let $x = CQ = CP$. Then, $AC = x + 1$, $AB = 4$, and $BC = x + 3$, so by the Pythagorean theorem,

$$(x+1)^2 + 4^2 = (x+3)^2 \Rightarrow x = 2,$$

so $AC = 3$ and $BC = 5$. The area of this triangle is $3 \cdot 4/2 = \boxed{6}$.

2. In this solution, we separate digits in base 2011 representations by commas and write out the digit values in standard decimal notation.

For each n, it is clear that $P_n(2011) = 1, 1, 1, \ldots, 1_{2011}$, where there are $n+1$ 1's. From this, it is fairly easy to see that the greatest n for which a_n can be positive is 3. In addition,

$$\begin{aligned}
P_3(2011) &> 2010P_0(2011) + P_1(2011) + P_2(2011)\\
P_2(2011) &> 2010P_0(2011) + P_1(2011)\\
P_1(2011) &> P_0(2011),
\end{aligned}$$

which tells us that there is only one possible sequence $\{a_n\}$. Letting $a_3 = 1$ since $a_3 = 2$ would be too large and $a_3 = 0$ would be too small, we subtract to get

$$a_0P_0(2011) + a_1P_1(2011) + a_2P_2(2011) = 2, 0, 1, 3_{2011} - 1, 1, 1, 1_{2011} = 2009, 0, 2_{2011}.$$

Then, $a_2 = 2009$, which gives us

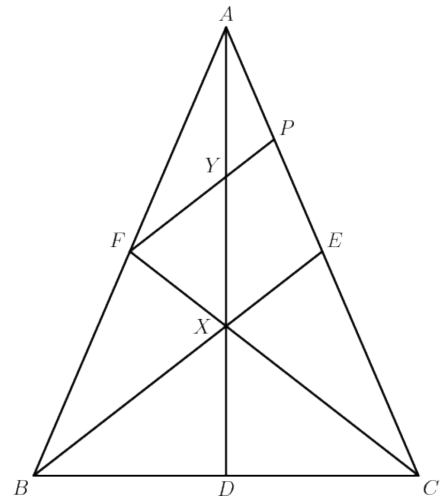
$$a_0P_0(2011) + a_1P_1(2011) = 2009, 0, 2_{2011} - 2009, 2009, 2009_{2011} = 1, 4_{2011},$$

where ω denotes 2009 in base 2011. This gives us $a_1 = 1$, and doing one last subtraction, $a_0 = 3$. In the infinite sequence, $a_n = 0$ for $n > 3$. The full sequence is $\boxed{3, 1, 2009, 1, 0, 0, 0, \ldots}$.

3. This result is $\boxed{\text{false}}$.

There are $\binom{6}{3} = 20$ ways to choose 3 courses of the 6. Suppose each student takes one of these 20 ways and consider any pair of courses. There are four courses other than these two, so there are $\binom{4}{3} = 4$ students that are in neither course. Thus, for any selection of 5 students, at least one of them must be in at least one of these two courses. Similarly, there are $\binom{4}{1} = 4$ students that take both courses, so for any selection of 5 students, at least one of them does not take both courses. Therefore, this construction provides a counterexample to the assertion.

4.



Solution 1 (Angle Chasing): We show that if triangle XFY is isosceles, then triangle ABC is isosceles. There are three cases to consider.

Case 1: $m\angle FYX = m\angle FXY$.

Since $AF = FB$ and $AP = PE$, $\overline{FP} \parallel \overline{BE}$, so $m\angle FXY = m\angle CXD$ and $m\angle FYX = m\angle BXD$. Thus, $\overline{XD}$ is a median and an angle bisector of triangle BXC, so $\overline{XD} \perp \overline{BC}$ and $\overline{AD} \perp \overline{BC}$. Applying SAS congruence to triangles ADB and ADC, $AB = AC$.

Case 2: $m\angle YFX = m\angle YXF$.

From the same pair of parallel lines, $m\angle YFX = m\angle BXF$ and $m\angle YXF = m\angle AXF$. We are in the exact same position as in the previous case, so we go through the same work to conclude $CB = CA$.

Case 3: $m\angle XFY = m\angle XYF$.

From the same pair of parallel lines, $m\angle XFY = m\angle CXE$ and $m\angle XYF = m\angle AXE$. We conclude that $BA = BC$.

Having exhausted all possible cases, the proof is complete in this direction. All steps are reversible, so the other direction is established.

Solution 2 (Length Chasing): We start with a lemma: Two medians of a triangle have the same length if and only if the corresponding sides have the same length. WLOG, suppose the two sides are a and b. The lengths of the corresponding medians are

$$m_a = \frac{\sqrt{2b^2 + 2c^2 - a^2}}{2}, \qquad m_b = \frac{\sqrt{2a^2 + 2c^2 - b^2}}{2},$$

from which the result follows.

As before, there are three cases to consider. We note that X is the centroid of triangle ABC and Y is the centroid of triangle AFE. Furthermore, $\triangle AFE \sim \triangle ABC$ with a scale factor of 2.

Case 1: $FX = FY$.

In this case, $FY = (2/3)FP = (2/3)(1/2)BE = (1/3)BE$ and $FX = (1/3)CF$, so $BE = CF$ and $AB = AC$ by our lemma.

Case 2: $XF = XY$.

In this case, $XF = (1/3)CF$ and $AY = YX = XD = (1/3)AD$, so $CF = AD$ and $BC = BA$.

Case 3: $YF = YX$.

In this case, $YF = (1/3)BE$ and $YX = (1/3)AD$, so $BE = AD$ and $CB = CA$.

The proof is complete going in this direction and all steps are reversible.

5. We first prove the left hand side:

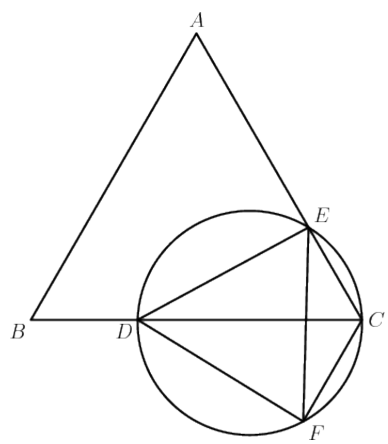
$$\begin{aligned}\frac{1}{2n+1} &< \frac{1}{2n}\\ &= \frac{1}{2}\cdot\frac{2}{3}\cdots\frac{2n-1}{2n}\\ &\le \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}.\end{aligned}$$

Now, we prove the right hand side:

$$\begin{aligned}\frac{1}{2n+1} &= \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{2n}{2n+1}\\ &> \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\cdot\frac{2n-1}{2n}\\ \frac{1}{\sqrt{2n+1}} &> \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}.\end{aligned}$$

6. Suppose $n = k^2$, where k is not necessarily an integer but n is. Then, if $\sqrt{n+2012} - \sqrt{n} < 2$, then $k^2 + 2012 < (k+2)^2 \Rightarrow k > 502$. The smallest such value for n is thus $502^2 + 1 = \boxed{252005}$.

7.



Since F is chosen not to lie on $\overline{AB}$, it lies on the same side of $\overline{DE}$ as C. Furthermore, $m\angle DFE = m\angle DCE = 60^\circ$, so quadrilateral $DECF$ is cyclic. By Ptolemy's theorem,

$$(DE)(CF) + (DF)(CE) = (DC)(EF).$$

Triangle DEF is equilateral, so $DE = EF = FD$. Furthermore, since $BD = CE$, $CD = 1 - CE$. Finally, $CF = 11/31$ is given. Substituting all of these relations, we get

$$11/31 + CE = 1 - CE \Rightarrow CE = \boxed{10/31}.$$

8. **Solution 1:** Since $|z_1| = 1$, $z_1^{-1} = \overline{z_1}$, and similarly for z_2 and z_3. We have the equations

$$\begin{aligned}z_1\overline{z_1} &= 1\\ z_2\overline{z_2} &= 1\\ z_3\overline{z_3} &= 1\\ z_1\overline{z_2} + z_2\overline{z_3} + z_3\overline{z_1} &= 1.\end{aligned}$$

Taking the conjugate of the last equation, we get $\overline{z_1}z_2 + \overline{z_2}z_3 + \overline{z_3}z_1 = 1$. Adding all of these equations together, we get

$$\begin{aligned}
5 &= z_1\overline{z_1} + z_2\overline{z_2} + z_3\overline{z_3} + z_1\overline{z_2} + z_2\overline{z_3} + z_3\overline{z_1} + \overline{z_1}z_2 + \overline{z_2}z_3 + \overline{z_3}z_1 \\
&= (z_1 + z_2 + z_3)(\overline{z_1} + \overline{z_2} + \overline{z_3}) \\
&= |z_1 + z_2 + z_3|^2.
\end{aligned}$$

Thus, $|z_1 + z_2 + z_3| = \boxed{\sqrt{5}}$.

Solution 2: Let $a_n = z_n/z_{n+1}$ for $n = 1, 2, 3$, where $z_4 = z_1$. Then,

$$\begin{aligned}
a_1 + a_2 + a_3 &= 1 \\
a_1a_2 + a_2a_3 + a_3a_1 &= 1 \\
a_1a_2a_3 &= 1,
\end{aligned}$$

where the second equation comes from the conjugate step from the previous solution. Thus, a_1, a_2, and a_3 satisfy $x^3 - x^2 + x - 1 = 0$, which has roots 1 and $\pm i$. The equations given are invariant of rotation in the complex plane, so WLOG, suppose $z_1 = 1$. If $a_1 = 1$, then $z_2 = 1$ and $z_3 = \pm i$. If $a_1 = \pm i$, then $z_2 = \mp i$ and $z_3 = \mp i$. In both cases, $|z_1 + z_2 + z_3| = \boxed{\sqrt{5}}$.

9. We can factor this quadratic as $(\lfloor x \rfloor - 1)(\lfloor x \rfloor - 4) = 0$, so $\lfloor x \rfloor = 1$ or $\lfloor x \rfloor = 4$. From the definition, it follows that the solution set is $[1, 2) \cup [4, 5)$.

10. Triangles XYZ and XPZ share the height from Z to $\overline{XY}$. Thus, the ratio of their areas is the ratio of their bases:
$$\frac{[XPZ]}{[XYZ]} = \frac{XP}{XY} = \frac{10}{61} \Rightarrow [XPZ] = \frac{10}{61} \cdot 2013 = \boxed{330}.$$

11. Let x be the number of times Amazon increments the price. The number of books sold is $575 - 25x$ and the price per book is $170 + 10x$. The total revenue is $(170 + 10x)(575 - 25x) = 250(17 + x)(23 - x)$. By AM-GM,
$$\sqrt{(17 + x)(23 - x)} \le \frac{(17 + x) + (23 - x)}{2} = 20,$$
with equality only when $17 + x = 23 - x \Rightarrow x = 3$. The corresponding price is \$200.

    Alternatively, to maximize the quadratic $(17 + x)(23 - x)$, we can expand this as $-x^2 + 6x + 391$. The vertex of the graph is where this quadratic attains its maximum, and is located at $x = -b/2a$. Here, $b = 6$ and $a = -1$, corresponding to $x = 3$. We finish as above.

12. **Solution 1 (Modular Arithmetic):** We first note that clearly, x is nonnegative. Otherwise, the left hand side is a terminating decimal whereas the right hand side is not, so the two cannot be equal.

    For $x \ge 2$, taking the equation modulo 4, we get $0 + 1 \equiv -1 \pmod 4$, a contradiction. Thus, $x = 0$ or $x = 1$. Plugging these in, $\boxed{1}$ is the only solution.

    Solution 2 (Bounding): For $x = 2$, we note that $5^{2x} = 625 > 243 = 3^{2x+1}$. For each increment of x, 5^{2x} is multiplied by 25 and 3^{2x} is multiplied by 9, so $5^{2x} > 3^{2x+1}$ for $x \ge 2$. Thus, in order to satisfy the given equation, $x < 2$. Testing $x = 0$ and $x = 1$, we get the unique solution $x = \boxed{1}$.

13. In total, there are $\binom{n}{2} = \frac{n(n-1)}{2}$ matches. Each match has a winner, so there are $\frac{n(n-1)}{2}$ wins distributed among the n players. By the Pigeonhole Principle, there is some player with at least $\frac{n(n-1)/2}{n} = \frac{n-1}{2}$ wins.

14. We claim the number of pairs of consecutive integers in the set $\{10^n, 10^n+1, \ldots, 2\cdot 10^n\}$ for which no carrying is required when adding the two is equal to $1+5+5^2+\cdots+5^n = (5^{n+1}-1)/4$. For $n=3$, this gives us an answer of $\boxed{156}$. We proceed by induction.

    For $n=0$, this is clearly true, as there is only one pair $(1,2)$, and the addition of 1 and 2 requires no carrying. Suppose the result is true for $n=k$. For $n=k+1$, we have two cases depending on the last digit of the smaller number.

    Case 1: The last digit is not a 9.

    In order for no carrying to occur, every digit is doubled, except the last digit, which has 1 added to it. However, this does not change whether or not carrying occurs. Thus, each digit must range from 0 through 4, except the first digit, which must be 1. This case thus contributes 5^{k+1} numbers.

    Case 2: The last digit is a 9.

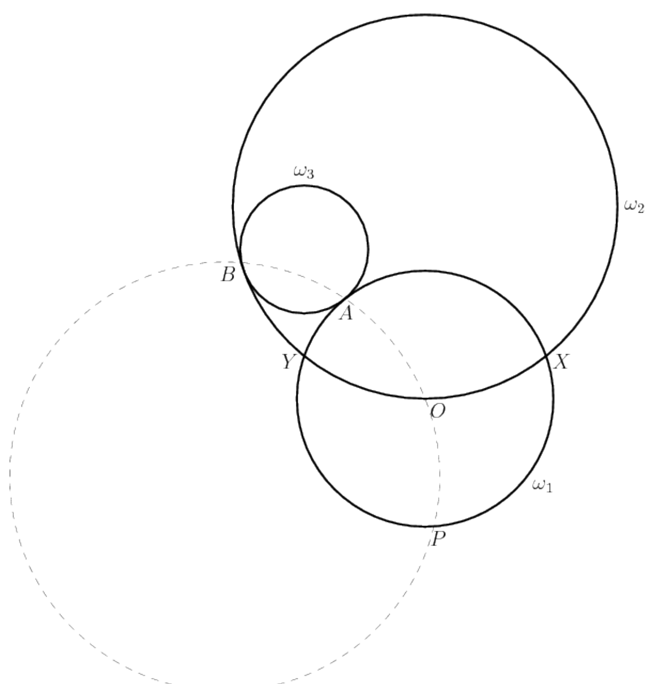
    The greater number has a last digit of 0, so no carrying occurs in the units digit. Consider the numbers with the last digit truncated. These two numbers will be consecutive integers of length $k+1$ instead of $k+2$. Using the inductive hypothesis, there are $(5^{k+1}-1)/4$ numbers satisfying the given constraints in this case.

    Adding these together, we have

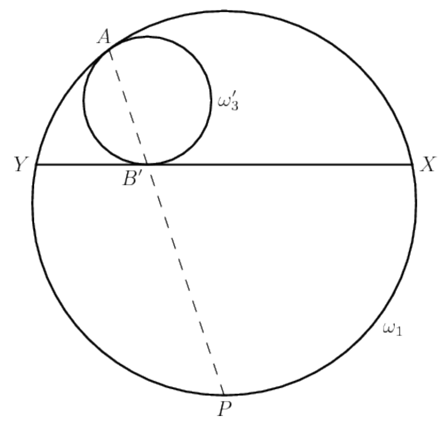
$$5^{k+1} + \frac{5^{k+1}-1}{4} = \frac{4\cdot 5^{k+1} + 5^{k+1} - 1}{4} = \frac{5^{k+2}-1}{4},$$

    completing the inductive step, so the proof is complete.

15. We claim that the point P is the midpoint of the arc XY opposite ω_2, where X and Y are the intersection points of ω_1 with ω_2.

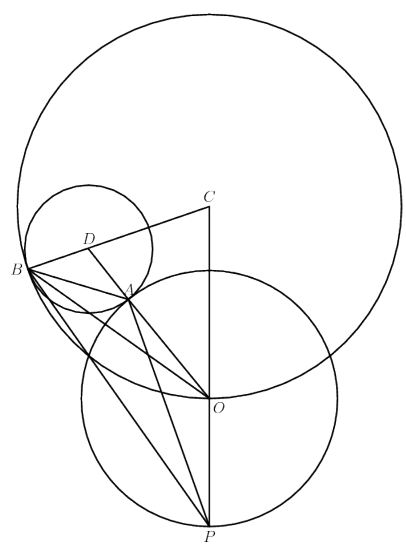


Solution 1 (Inversive): We invert about circle ω_2. The circle ω_2 maps to the line $\overleftrightarrow{XY}$. It is clear that inversion preserves tangency, so the circle ω_3 maps to a circle ω_3' that is internally tangent to ω_1 and tangent to $\overline{XY}$. The circle through A, B, and O is the line through A and the tangency point K of ω_3' with $\overline{XY}$. It suffices to prove that this line always passes through the midpoint P of arc $\overline{XY}$ opposite point A.



A homothety centered at A taking ω_3' to ω_1 takes $\overline{XY}$ to a segment parallel to itself and tangent to ω_1. The tangency point is thus the midpoint of the arc XY opposite A, and since homothety preserves tangency, this is always the image of point B', so points A, B', and P are collinear, as desired.

Solution 2 (Angle Chasing): We wish to show that quadrilateral $POAB$ is cyclic.



Let C and D be the centers of ω_2 and ω_3, and let $m\angle DBA = x$ and $m\angle ABD = y$.

Since $CB = CO$, $m\angle BOC = x + y$. and $m\angle BCO = 180 - 2x - 2y$. In addition, since $DB = DA$, $m\angle BAD = x$ and $m\angle ODC = 2x$. Thus, $m\angle COD = 2y$. We have $OA = OP$, so $m\angle OAP = m\angle OPA = y$. Since B and P lie on the same side of $\overline{OA}$ and $m\angle ABO = m\angle APO = y$, quadrilateral $POAB$ is cyclic, as desired.

16. It is evident that to avoid any intersections, the points must be connected in order. Thus, this is equivalent to determining the minimum necessary number of swaps of adjacent elements for sorting a list of numbers from 1 to 2013. For sorting the numbers from 1 to n, we claim that we need $\binom{n}{2}$ swaps to guarantee that this is the case. For $n = 2013$, this gives us an answer of $\boxed{2013 \cdot 2012/2}$ or $\boxed{2025078}$. We first demonstrate that no initial configuration can be worse. Then, we demonstrate that we cannot guarantee any better.

For the first part, we proceed by induction. For $n = 1$, it is clear that we do not need to make any swaps. Suppose the result holds for $n = k$. For $n = k + 1$, we do this sort in two parts. We first move 1 to the front of the list. After this point, we run the sort on the rest of the list, which has k items. It takes at most k swaps to move 1 to the front. By the inductive hypothesis, it takes at most $\binom{k}{2} = k(k-1)/2$ steps to sort the rest of the list. Thus, it takes at most $k + k(k-1)/2 = k(k+1)/2 = \binom{k+1}{2}$ steps to sort any given list.

Now, consider the initial list $(n, n-1, \ldots, 1)$, i.e. the sorted list in reverse order. Each swap can only correct the relative ordering of a single pair of numbers. There are $\binom{n}{2}$ pairs of numbers that are initially in the incorrect relative order, so it will take at least that many swaps to sort this list correctly. This completes the proof.

17. If the last term does not start in the middle of this string, it must contain it entirely. The smallest value of n for which this is the case is 20133102. We show that we can do better.

    To do this, we can either add numbers before and add numbers after to create two consecutive integers. To add numbers to the front, we can take a chunk from the end and move it to the beginning. We see that the digit 2 is repeated, so we can use a chunk of the end of "013310". It is easy to see that to minimize n, we want to use the string "10". Thus, we have 10/20133102/0134. The answer is $\boxed{1020134}$.

18. Associate each square of the chessboard with a point in the set $\{(x, y) | 1 \le x, y \le 2013\}$ in the obvious way. Since the 2013 rooks are not attacking each other, each row from 1 through 2013 is occupied exactly once and each column from 1 through 2013 is occupied exactly once. The sum of the coordinates is thus $2(1 + 2 + \cdots + 2013) = 2013 \cdot 2014$.

    Each step changes the sum of the coordinates by 1. Thus, after 2013 steps, the sum of the coordinates will be odd. In particular, it will not be $2013 \cdot 2014$, so some rook will be attacking another rook.

19. Each prime factor p appears once for each multiple of p. Thus, the number of times p is represented is $\lfloor 40/p \rfloor$, so the contribution to the sum is $p\lfloor 40/p \rfloor$. Summing this up for all $p < 40$, we get

$$2(20) + 3(13) + 5(8) + 7(5) + 11(3) + \cdots + 31(1) + 37(1) = \boxed{418}.$$

20. Substituting this condition, we want to show that

$$a + b + c - \left(\frac{a}{2(b+c)} + \frac{b}{2(c+a)} + \frac{c}{2(a+b)} \right) < a^4 + b^4 + c^4 + 2ab^2 + 2bc^2 + 2ca^2.$$

    By Nesbitt's inequality, the expression in parentheses is at least $3/4$ with equality at $a = b = c$, so it suffices to show

$$a + b + c - \frac{3}{4} \le a^4 + b^4 + c^4 + 2ab^2 + 2bc^2 + 2ca^2$$

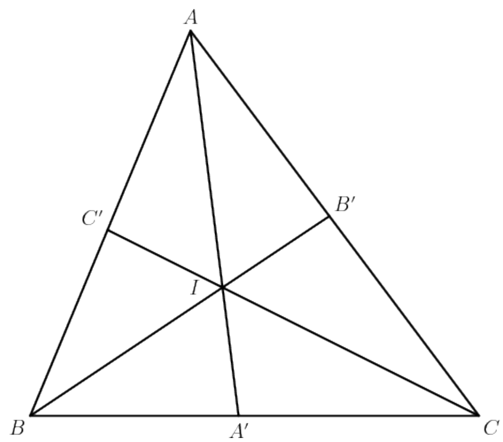
    without equality at $a = b = c$. This rearranges to

$$(a + b^2 - 1/2)^2 + (b + c^2 - 1/2)^2 + (c + a^2 - 1/2)^2 \ge 0,$$

    so the nonstrict inequality is proven. Plugging in $a = b = c = 1/3$, we see that equality is avoided, so the inequality is strict.

    Note: Equality is avoided even without considering Nesbitt's, so there is a lot of room for error.

21.



Let $AB = c$, $AC = b$, and $BC = a$. By the Angle Bisector theorem, $A'C = \dfrac{ab}{b+c}$. Applying the Angle Bisector theorem again,

$$\begin{aligned}\frac{IA}{AA'} &= \frac{b}{b + \frac{ab}{b+c}} \\ &= \frac{b+c}{a+b+c}.\end{aligned}$$

By symmetry,

$$\begin{aligned}\frac{IB}{BB'} &= \frac{a+c}{a+b+c} \\ \frac{IC}{CC'} &= \frac{a+b}{a+b+c},\end{aligned}$$

so

$$\frac{IA \cdot IB \cdot IC}{AA' \cdot BB' \cdot CC'} = \frac{(a+b)(b+c)(c+a)}{(a+b+c)^3}.$$

By AM-GM, we know that

$$\frac{\frac{a+b}{a+b+c} + \frac{b+c}{a+b+c} + \frac{c+a}{a+b+c}}{3} \geq \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{(a+b+c)^3}}.$$

The LHS equals $2/3$. Taking the 3rd power of both sides, we get

$$\frac{(a+b)(b+c)(c+a)}{(a+b+c)^3} \leq \boxed{\frac{8}{27}}.$$

This is attained when $a = b = c$, i.e. when ABC is an equilateral triangle.

22. We claim that for all $\boxed{N \geq 3}$, this is possible. Clearly with 1 or 2 coins we cannot determine the relative weight of the fake coin. With 1 coin we do not have a real coin to compare it against, and with 2 coins, there is no way to determine which one is real and which one is fake.

With 3 coins, the following process suffices: Weigh two of the coins. If they are the same weight, they must both be real and we can determine the relative weight of the fake coin by weighing it against one of the two real coins. If the scale does not balance, then the last coin must be real. We weigh that coin against one of the coins that is currently on the scale. If it does not balance, then we have found the fake coin, and know its relative weight. If the scale balances now, we know that both of these coins are real, and since we already weighed the fake coin against a real coin, we know its relative weight.

With 4 coins, we do the following: Weigh two of the coins against the other two coins. Then weigh two coins that were on the same side. If they weigh the same, they must both be real, and from the first weighing we know the relative weight of the fake coin, though not which one it is. If they have different weights, we know one of them must be fake, and since we weighed them against two real coins, we know the relative weight of the fake coin.

If the number of coins is a multiple of 3 or 4 coins, say $3n$ or $4n$, we can group the coins into groups of n, and then determine the relative weight of the fake coin using the same processes as above, treating a group of n coins as one.

The processes explained above motivate the following result: If the number of coins is between $3n$ and $4n$ for any n, we can determine the relative weight of the fake coin with a combination of the above two processes.

Say the number of coins is between $3a$ and $4a$ (inclusive) for some a, we can divide the coins into two groups of size a and one group of size b, where $a \leq b \leq 2a$. First we weigh the two groups of size a against each other. If they have the same weight, we know the fake coin must be in the group of size b and all $2a$ coins we have must be real. We then weigh the group of size b against b of the real coins. We know the fake coin is in the group of size b, so we can determine its relative weight. If the two groups of size a have different weights, the fake coin must be among them. We weigh one of the groups of a against a real coins taken from the group of size b. If they have the same weight then, the fake coin must have been in the other group of size a, and since we weighed that group against the other group of size a, we know the relative weight of the fake coin. If the two do not have the same weight, we know the relative weight of the fake coin since we already know that the coins taken from the group of size b must have all been real.

Every positive integer is between $3n$ and $4n$ for some n except for 1, 2, and 5. We already know that with 1 or 2 coins it is impossible to determine the relative weight of the fake coin, so 5 coins is the only case we have left to check.

With 5 coins, the following process suffices: Weigh two of the coins against the other two coins. If they have the same weight, we know the fifth coin must be fake and we can determine its relative weight by weighing it against one of the other real coins. If the two groups have different weights, one of them must be fake, and we can determine the relative weight of the fake coin in the same way that we did with 4 coins.

23. There are 8 planes corresponding to the faces of the octahedron. In addition, there are 3 planes passing through the center and 4 of the 6 vertices, so the total is $\boxed{11}$.

    To see that there are no more, we can overcount and then correct for the overcounting. To form a plane, we pick 3 of the 6 vertices, which can be done in $\binom{6}{3} = 20$ ways. However, each of the planes through the center and 4 vertices is counted 4 times (once for each selection of 3 of the 4 vertices), and we only want to count them once. Thus, we must subtract 3 for each of these 3, bringing our count to $20 - 3(3) = 11$. These two counts coincide, so we know we did not miss any cases.

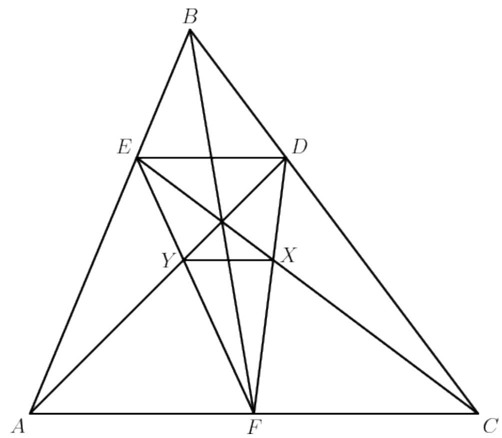
24. We compute finite differences:

$$\begin{array}{r|ccccccc}
P(x) & -2 & & -1 & & 1 & & 2 \\
\Delta P(x) & & 1 & & 2 & & 1 & \\
\Delta^2 P(x) & & & 1 & & -1 & & \\
\Delta^3 P(x) & & & & -2 & & &
\end{array}$$

Since the polynomial is cubic, after three iterations, we must have a constant sequence. Thus, we can extend the last row outwards and backtrack up the pyramid:

$$\begin{array}{r|ccccccccc}
P(x) & -2 & & -1 & & 1 & & 2 & & \boxed{0} \\
\Delta P(x) & & 1 & & 2 & & 1 & & -2 & \\
\Delta^2 P(x) & & & 1 & & -1 & & -3 & & \\
\Delta^3 P(x) & & & & -2 & & -2 & & &
\end{array}$$

25.



Solution 1 (Projective): Since $\overline{BF}$, $\overline{AY}$ and $\overline{CX}$ all intersect at one point, triangles FYX and BAC must be projective images of each other. By Desargues' theorem, the intersections of $\overline{FY}$ with $\overline{AB}$, $\overline{FX}$ with $\overline{BC}$, and $\overline{XY}$ with $\overline{AC}$ must be collinear. We see that $\overline{FY}$ and $\overline{AB}$ intersect at E and $\overline{FX}$ and $\overline{BC}$ intersect at D. Therefore, $\overline{XY}$ must intersect $\overline{AC}$ at the intersection of $\overline{ED}$ and $\overline{AC}$. However, we know that $\overline{ED}$ is parallel to $\overline{AC}$, so $\overline{XY}$ is also parallel to these two lines.

Solution 2 (Similar Triangles): From the parallel segments, $BE/EA = BD/DC$. By Ceva's theorem, $(BE/EA)(AF/FC)(CD/DB) = 1$, so $AF = FC$. For a triangle ABC, let h^A_{ABC} denote the length of the altitude from A in triangle ABC.

Triangles YED and YFA are similar, so

$$\frac{h^Y_{YED}}{h^Y_{YFA}} = \frac{ED}{FA}.$$

Similarly, triangles XED and XCF are similar, so

$$\frac{h^X_{XED}}{h^X_{XCF}} = \frac{ED}{CF}.$$

Since $FA = CF$,

$$\frac{h^Y_{YED}}{h^Y_{YFA}} = \frac{h^X_{XED}}{h^X_{XCF}}.$$

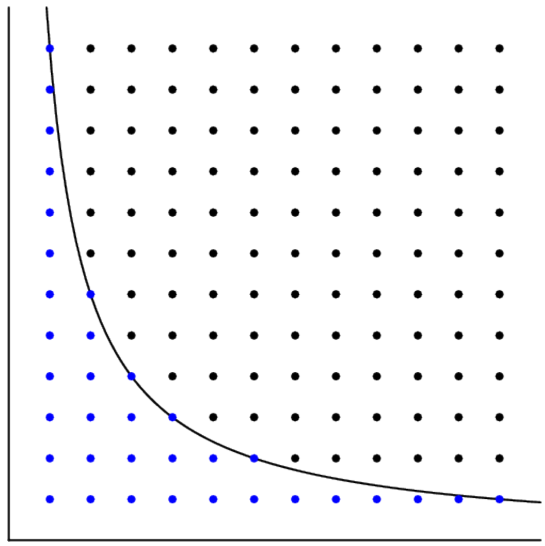
Furthermore, since $\overline{ED} \parallel \overline{AC}$,

$$h^Y_{YED} + h^Y_{YFA} = h^X_{XED} + h^X_{XCF}.$$

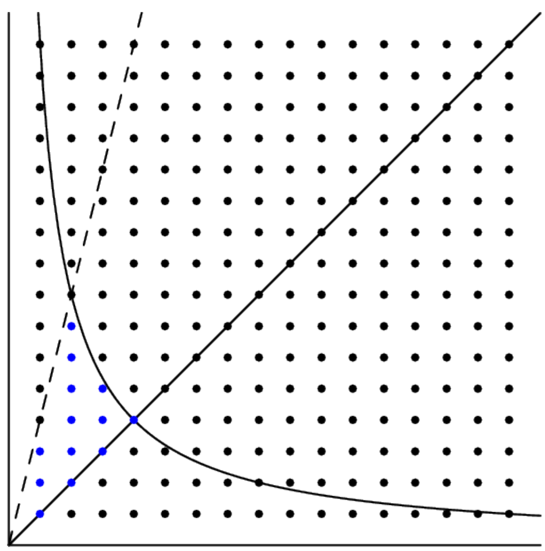
It follows that $h^Y_{YFA} = h^X_{XCF}$, i.e. $\overline{XY} \parallel \overline{AC}$.

26. Since $P(x)$ has integer coefficients, $a-b$ divides $P(a)-P(b)$. Thus, $P(x)-x|P(P(x))-P(x)|P(P(P(x)))-P(P(x))|\cdots$. Adding these terms up, we get that $P(x)-x|P(P(P(\dots(P(x))))) - x$, as desired.

27. We start by figuring out how to compute $\sum \sigma_0(n)$, where $\sigma_0(n)$ is the number of divisors of n. Let k be a divisor of n. We can match this with the lattice point $(n/k, k)$ in the first quadrant of the Cartesian plane. Furthermore, every lattice point (x, y) corresponds to a divisor x of n, with $n = xy$. Thus, $\sigma_0(1) + \sigma_0(2) + \cdots + \sigma_0(2012^2)$ is simply the number of lattice points (x, y) in the first quadrant satisfying $xy \leq 2012^2$. In other words, the point (x, y) lies nonstrictly below the positive part of the hyperbola $xy = 2012^2$.



Now we consider the bounds on the divisors. Since $k \geq \sqrt{n}$, the corresponding point (x, y) satisfies $y \geq x$. Since $k < 2\sqrt{n}$, the corresponding point (x, y) satisfies $y < 4x$. Thus, we are counting the number of lattice points nonstrictly the hyperbola $xy = 2012^2$, nonstrictly above the line $y = x$, and strictly below the line $y = 4x$.



The line $y = x$ intersects $xy = 2012^2$ at $X(2012, 2012)$. The line $y = 4x$ intersects $xy = 2012^2$ at $Y(1006, 4024)$. To do this count, we first count the number of points inside the triangle OXY, where O is the origin, and the points on the boundary that are not forbidden by the above conditions. Then, we subtract off all the points above the hyperbola and the point O.

To do the first count, we apply Pick's theorem: $A = I + B/2 - 1$. By the Shoelace method, cross products, or the like, we can compute the area to be $A = 3036108$. The number of lattice points on the segment $\overline{OX}$ is $1006 - 0 + 1 = 1007$ and the number of lattice points on the segment $\overline{OY}$ is $2012 - 0 + 1 = 2013$. The segment $\overline{XY}$ has slope -2, so every x-coordinate from 1006 to 2012 inclusive corresponds to a lattice point. Thus, there are $2012 - 1006 + 1 = 1007$ lattice points on $\overline{XY}$. However, the three vertices O, X, and Y were counted twice, so $B = 1007 + 2013 + 1007 - 3 = 4024$. Substituting into Pick's theorem, we get $I = 3034097$. Adding back the boundary points on $\overline{OX}$ and $\overline{XY}$, and remembering to correct for double-counting X and for counting O and Y, we get

$$3034097 + 2013 + 1007 - 3 = 3037114.$$

Now, we must subtract the number of lattice points above the hyperbola. To do this, we take the number of lattice points on or below the segment $\overline{XY}$ and subtract the number of lattice points on or below the hyperbola from $x = 1006$ to $x = 2012$. The first count is simply

$$4024 + 4022 + \cdots + 2012 = \frac{6036 \cdot 1007}{2} = 3039126.$$

The number of lattice points nonstrictly below the hyperbola from $x = 1006$ to $x = 2012$ is

$$\sum_{n=1006}^{2012} \left\lfloor \frac{2012^2}{n} \right\rfloor = 2808487,$$

so the number of lattice points we must subtract is $3039126 - 2808487 = 230639$.

Thus, the number of lattice points in the desired region is $3037114 - 230639 = \boxed{2806475}$.

Tip: By holding off on multiplying out the large numbers, computations can be simplified greatly.

Note: The lattice point counting method can be applied to $\sum \sigma_0(n)$ *to get the result*

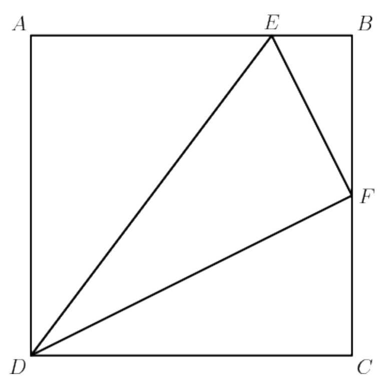
$$\sum_{n \le x} \sigma_0(n) = x \log x + O(x).$$

By drawing $y = x$ *and using symmetry, this can be improved to*

$$\sum_{n \le x} \sigma_0(n) = x \log x + (2C - 1)x + O(\sqrt{x}).$$

This error term can be improved. If the error term is given as $O(x^\theta)$*, the determination of* $\inf \theta$ *is known as the Dirichlet divisor problem.*

28.



We are given that $m\angle EDF = m\angle FDC$ and that $m\angle EFD = m\angle FCD = 90°$, so triangles EDF and FDC are similar. In addition, $m\angle EBF = 90°$ and $m\angle EFB = m\angle FDC$, so triangles EDF, FDC, and EBF are all similar.

Let $FC = a$. By the Pythagorean theorem, $DF = \sqrt{a^2+1}$. From similarities, $EF = a\sqrt{a^2+1}$, so $BF = a$. However, $BF + FC = BC = 1$, so $a = 1/2$. The area of triangle DEF is

$$\sqrt{a^2+1}(a\sqrt{a^2+1})/2 = a(a^2+1)/2 = \boxed{5/16}.$$

29. Modulo n, this expression is $1(2)\cdots(n-1) = (n-1)!$. If n is prime, then none of these factors are divisible by n, so n does not divide $(n-1)!$. Suppose there exist p and q satisfying $1 < p < q$ with $pq = n$. Then, p and q are in $(n-1)!$, so n divides the product. Such a pair (p, q) exists for all composite n except $n = p^2$, where p is a prime. For $p \ge 3$, $2p < n - 1$, so p and $2p$ are both in $(n-1)!$ and $n = p^2$ still divides $(n-1)!$. However, for $n = 2^2$, this does not hold, and indeed, 4 does not divide $3! = 6$. Thus, every composite n from 2 to 20, except 4, works, and their sum is

$$6 + 8 + 9 + 10 + 12 + 14 + 15 + 16 + 18 + 20 = \boxed{128}.$$

30. There are $\binom{15}{5} = 3003$ ways in which Carl could have gotten 10 heads and 5 tails.

Let **H** and **T** denote groups of H's and T's, respectively. The sequence must take the form **THTHTHTH**, where the first **T** and last **H** can have 0 members. The number of ways to distribute the heads is equivalent to the number of solutions in nonnegative integers to $a_1 + a_2 + a_3 + a_4 = 10$, where only a_4 may be 0, which is simply $\binom{10}{3}$ using a balls-and-urns argument. Similarly, the number of ways to distribute the tails is $\binom{5}{3}$, so the probability is

$$\frac{\binom{10}{3}\binom{5}{3}}{3003} = \boxed{\frac{400}{1001}}.$$

If you feel that a solution is wrong, an alternate solution exists, or something in a solution should be clarified, feel free to send us a message with the appropriate details.