Complex Numbers

LHS Math Team

Note: $\mathbb{R}$ denotes the set of real numbers.

The goal of complex numbers is to be able to describe the roots of polynomials with real coefficients. For example, $x^2 - 3x + 2$ has roots 1 and 2, but to describe things like the roots of $x^3 - 1$, we can take 1 as a root, but it turns out there are two more roots in the complex roots. The Fundamental Theorem of Algebra guarantees us that when we include the complex numbers, a polynomial of degree n will have *exactly* n solutions (counting multiplicity).

Definition: Let i be a "number" such that $i^2 = -1$ (that is, $i = \sqrt{-1}$). We define the set of complex numbers, $\mathbb{C}$, to be the set of numbers of the form $z = a + bi$, where $a, b \in \mathbb{R}$. If $a = 0$, we say that z is *imaginary*, and if $b = 0$, note that z is a real number. Thus, real numbers are also complex numbers.

What can we do with complex numbers? The answer is, the same things we can do with real numbers, that is, add, subtract, multiply, and divide them. All the nice properties about the real numbers that we take for granted still hold for complex numbers, such as commutativity and associativity. We can also talk about raising numbers to complex powers, but we'll cover that at another time.

Addition and subtraction work predictably: we simply treat i as if it were an unknown variable. That is, $(a + bi) + (c + di) = (a + c) + (b + d)i$, which we see is also a complex number, since $a + c$ and $b + d$ are real. Similarly, $(a + bi) - (c + di) = (a - c) + (b - d)i$.

Multiplication also works rather predictably: we use the familiar FOIL method. So we have that $(a + bi)(c + di) = ac + (bc + ad)i + bdi^2$. However, $i^2 = -1$ by definition, so this becomes $(ac - bd) + (bc + ad)i$. When we want to raise a complex number to a (non-negative integral) power, things work the same again, for example, $(a + bi)^2 = a^2 + 2abi + b^2 i^2 = (a^2 - b^2) + 2abi$. If you're familiar with the binomial theorem, it holds over the complex numbers as well.

Division, however, is more complicated. We can't simply divide $\frac{a + bi}{c + di}$ to get $\frac{a}{c} + \frac{b}{d}i$; if you think about it, this is just as bogus as taking $\frac{10 + 50}{2 + 25}$ to get $5 + 2 = 7$. We first need the following:

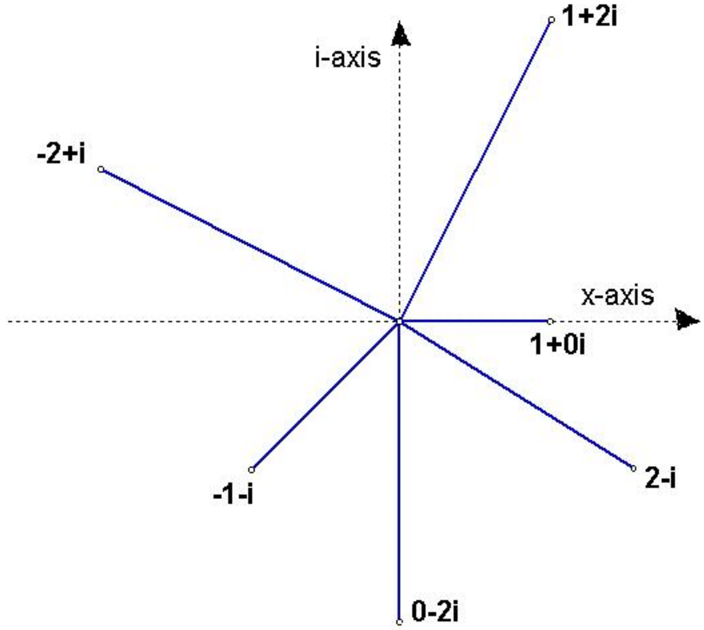
Definition: Let $z = a + bi$. We define the *conjugate* of z, denoted $\overline{z}$, to be equal to $a - bi$.

This may not seem to helpful, but $z\overline{z} = (a + bi)(a - bi) = a^2 + 0i + b^2 = a^2 + b^2$, which is a real number. It turns out that the only complex numbers w such that zw is real are the real multiples of $\overline{z}$ (which you can try to prove as an exercise).

How does this help us with division? Now, what we can do is take $\frac{a + bi}{c + di}$ and multiply top and bottom by the conjugate of the denominator, which is $c - di$. This gives us the same value, but now it is of the form $\frac{(a + bi)(c - di)}{c^2 + d^2}$. The denominator is now a single real number, so we can now multiply out the top and just divide the components by the real number in the denominator (why does this differ from the bogus division from before?). Note the similarity between this and rationalizing denominators with radicals.

Exercise: Finish the computation above, to get a formula for $\frac{a+bi}{c+di}$ in the form $x + yi$, where x and y are real numbers in terms of a, b, c, d.

We now introduce the notion of the *complex plane.* It's similar to the Cartesian plane, in which we have coordinates (x, y), defined by going x units to the right and y units up. In the complex plane, the x-axis is replaced by the real axis, and the y axis replaced by the imaginary axis. This allows us to plot all of the complex numbers in a predictable way: given $x + yi$, we go x units to the right and y units up. Several points are plotted below:



In the real numbers, the absolute value of a number is defined to be its 'distance from zero'. We can do a similar thing for the complex numbers; $|z|$ for a complex number z is defined to be the distance, in the complex plane, from the origin. That is, if $z = a+bi$, we have $|z| = \sqrt{a^2 + b^2}$ (why?).

Exercise: Prove that $|z| \cdot |w| = |zw|$.

The last thing we consider is what happens when we consider powers of i. $i^0 = 1$, by convention, $i^1 = i$, $i^2 = -1$ by definition, $i^3 = i^2 \cdot i = -i$, and $i^4 = i^2 \cdot i^2 = 1 = i^0$. Now, if we continue this, we will see that these will cycle with period 4, that is, the pattern will continue $i, -1, -i, 1, i, -1, -i, 1, \ldots$.