Team Contest 1
Solutions

November 15, 2010

1. The answer is $S = \{983, 984, 989, 991, 1000\}$, and it is easy to check that this satisfies the needed property. To generate S, let its elements be $a \le b \le c \le d \le e$. First, note that $a + b = 1967$ and $d + e = 1991$, since these must be the smallest and largest sums, respectively. Also, note that in computing the pairwise sums, each element of S contributes to exactly four sums, so the sum of the ten pairwise sums is equal to 4 times the sum of the elements of S. Thus, we find that $4(a + b + c + d + e) = 19788$, so $a + b + c + d + e = 4947$.

Now, we can subtract the values of $a + b$ and $d + e$ to find that $c = 989$. Next, observe that $c+e$ will be the second largest sum, since clearly $c+e \le d+e$, and all other sums are easily checked to be less than $c + e$. Similarly, $a + c$ will be the second smallest sum. Thus, $c + e = 1991$, and $e = 1000$, giving $d = 991$. Similarly, we get $a = 983, b = 984$.

2. The answer is $13\pi/4$. Let the quadrilateral be $ABCD$, with $AB = \sqrt{5}, BC = \sqrt{8}, CD = \sqrt{7}, DA = \sqrt{6}$. Then, let $AC = x$. If $x > \sqrt{13}$, then both $\angle ABC$ and $\angle ADC$ are obtuse, since $AB^2 + BC^2 = 13 < x^2$, and similarly with triangle ADC. This is a contradiction, since $\angle ABC + \angle ADC = 180°$ in a cyclic quadrilateral (because they together subtend the whole circle). On the other hand, if $x < \sqrt{13}$, the two angles are acute, again an impossibility. Thus, $x = \sqrt{13}$, and it follows that $\angle ABC = \angle ADC = 90°$ by the Pythagorean Theorem. Since these are right angles, AC is a diameter, so our circle has radius $\sqrt{13}/2$, giving us our area of $13\pi/4$.

3. First, consider the solutions (x, y) with $x = y$. Substituting into the first equation, we get $(x^2 + 6)(x - 1) = x(x^2 + 1)$, which factors as $(x - 2)(x - 3) = 0$, giving us $(x, y) = (2, 2), (3, 3)$ (and substituting $x = y$ into the second equation gives the same, so these satisfy both equations). Since we have two solutions for $x = y$, and finitely many solutions total, it suffices to show that there are an even number of solutions when $x \ne y$. But if (x, y) is a solution, then (y, x) is also seen to be a solution, since the second equation is the result of switching x and y, and vice versa. Therefore, the rest of the solutions pair up, and we have an even number of solutions in total.

4. We will compute the sum of the digits of the number, modulo 9. There are 1980 non-zero digits, and if there are k ones, the digit n occurs nk times. Thus, we find that $45k = 1980$, by the given condition, and $k = 44$.

Now, the sum of digits is $44(1 \cdot 1 + 2 \cdot 2 + \cdots + 9 \cdot 9)$. Modulo 9, this is equal to $44(1 + 4 + 0 + 7 + 7 + 0 + 4 + 1 + 0) = 44 \cdot 24 \equiv 8 \cdot 6 \equiv 3 \pmod 9$, so n is divisible by 3, but not 9. However, this means n cannot be a perfect square, because it has exactly one factor of 3, and a perfect square has to have an even number of factors of each prime number, completing the proof.

5. Let $A = \angle B_0A_0C_0$, and define B, C similarly. First, notice that $\angle A_iB_0C_0 = B/2^i$, which follows by induction and the fact that B_0A_i bisects $\angle A_{i-1}B_0C_0$. Similarly, $\angle A_iC_0B_0 = C/2^i$. It then follows that $\angle A_0B_0A_i = B(2^i-1)/2^i$, and $\angle A_0C_0A_i = C(2^i-1)/2^i$. Now, by the trigonometric form of Ceva's Theorem, we have

$$\frac{\sin \angle B_0A_0A_i}{\sin \angle C_0A_0A_i} = \frac{\sin \angle B_0C_0A_i}{\sin \angle A_0C_0A_i} \cdot \frac{\sin \angle A_0B_0A_i}{\sin \angle C_0B_0A_i} = \frac{\sin C \sin \dfrac{B(2^i-1)}{2^i}}{\sin B \sin \dfrac{C(2^i-1)}{2^i}}.$$

In a similar way, we can find expressions for the angles on either side of the cevians B_0B_i and C_0C_i, and upon multiplying these expressions, by symmetry, all terms will cancel, leaving us with 1. Therefore, by Ceva's Theorem again, the three cevians are concurrent, as desired.

6. There are a total of $\binom{25}{2} = 300$ colorings disregarding rotations, since we choose which of our two squares are yellow. Then, most colorings are counted four times, since we can rotate in exactly four ways. However, if a coloring looks the same after rotating it 180°, it is only counted twice (note that it is impossible to have a coloring be the same after rotating it 90° in either direction, since we must have a yellow square not at the center, but if we have 90° rotational symmetry we get 3 more yellow squares). There are a total of 24 colorings that have 180° symmetry, since we choose any color for the center square and the 12 pairs of squares that map to each other under a 180° rotation must have the same color. Thus, our answer is $276/4 + 24/2 = 81$.

7. Induct on n. We want to show that there exists some m with $m^3 \equiv 5 \pmod{11^n}$, for all n. First, for $n = 1$, we can take $m = 3$ (and $k = 2$). Now, assume the claim is true, that we have some m with $m^3 \equiv 5 \pmod{11^n}$. Then, we claim that for some i, $(m + i \cdot 11^n)^3 \equiv 5 \pmod{11^{n+1}}$. It suffices to show that for $i = 0, 1, \ldots, 10$, we get different values modulo 11^{n+1}, since then, one of these will have to be 0 (as they must be $0, 11^n, 2 \cdot 11^n \ldots, 10 \cdot 11^n$ in some order, by the inductive hypothesis).

By the Binomial Theorem, the quantity is equal to $m^3 + 3m^2 i \cdot 11^n + 3mi \cdot 11^{2n} + 11^{3n}$. The last two terms vanish modulo 11^{n+1}, since $3n > 2n \geq n + 1$. Then, to prove that these are all different when we make i range between 0 and 10, inclusive, it suffices to do this for $3m^2 i \cdot 11^n$, as m^3 will stay the same. Furthermore, show that the modulo 11^{n+1} residues of these are different, we can equivalently consider modulo 11 residues of $3m^2 i$. Now, note that m is relatively prime to 11, or else $m^3 \equiv 0 \pmod{11^n}$, so $3m^2$ is relatively prime to 11. This means that ranging i over the residues modulo 11 give all distinct values, so we're done.

8. For each pair of parallel faces, we have 9 lines that run perpendicular to both faces, for a total of 27 lines. Next, we have $2 \cdot 3 \cdot 3 = 18$ lines that run parallel to diagonal lines on faces, and finally four space diagonals, for a total of $27 + 18 + 4 = 49$ lines.

9. Note that $3 + 6 = 9$, and since centers of tangent circles are collinear with their point of tangency, we see that the centers of the two small circles lie on a diameter of the large circle.

Let the centers of the circles with radii $3, 6$ be A, B, and angles, and let the common external tangent meet circles A and B at points C and D, respectively. Let the chord be EF, so that C lies between E and D, and let M be the midpoint of CD. Let circles A and B be tangent at P, and say they are tangent to the large circle, centered at O, at X and Y, so that A is between X and O. Now, note that AC, OM, BD are all perpendicular to EF, so they are parallel to each other. We see that $AO = XO - XA = 6$, and $OB = OY - BY = 3$, so $CM/MD = AO/OB = 2$ (by similar triangles if we extend rays DC and BA). It follows that $MO = \dfrac{AC + 2BD}{3} = 5$, so $MF^2 = OF^2 - MO^2 = 81 - 25 = 56$. Finally, we find that $EF^2 = (2MF)^2 = 4MF^2 = 224$.

10. Let the integers be $a_1, a_2, \ldots, a_n$, and consider the n sums $a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_n$. If any of these is congruent to 0 modulo n, we are done; otherwise, some two of them are the same, since there are n sums and $n - 1$ nonzero residues modulo n. Then, noting that the set of summands of one of the sums will be a subset of the other, we can subtract the smaller (by length) from the larger, and be left with a new sum, whose sum is 0 modulo n, in which case we are also done.

11. By Fermat's Little Theorem, $2^p + 1 \equiv 2 + 1 \equiv 3 \pmod p$. If p divides $2^p + 1$, then, we need $3 \equiv 0 \pmod p$, which happens if and only if $p = 3$.

12. Let $AB = s$. Note that $\angle EBF = 360° - 60° - 60° - 90° = 150°$, and we thus have $[EFGH] = EF^2 = s^2 + s^2 - 2s^2 \cos 150° = (2 + \sqrt{3})s^2 = 3[ABCD]$ by the Law of Cosines. Thus, $[ABCD]/[EFGH] = 1/(2 + \sqrt{3}) = 2 - \sqrt{3}$.

13. Let O be the center of the semicircle. Since the length ST is invariant, so is the angle $\angle SOT$. Now, notice that OM is perpendicular to ST, and SP is perpendicular to PO, so $\angle SPO + \angle SMO = 90° + 90° = 180°$, and $MOPS$ is cyclic. Thus, $\angle SPM = \angle SOM = \frac{1}{2}\angle SOT$, which is independent of the position of ST.

14. First, we prove the following lemma: let a, b, c, d, e, f be integers, and p be a prime, such that b, d, f are not divisible by p. Then, if $\frac{a}{b} + \frac{c}{d} = \frac{e}{f}$, then $ab^{-1} + cd^{-1} \equiv ef^{-1} \pmod p$, where $^{-1}$ denotes inverses modulo p. This is true because by multiplying through by bdf, $adf + cbf = ebd$, and multiplying by $b^{-1}d^{-1}f^{-1}$ gives what we want. Note that this generalizes to sums of any finite number of fractions.

Now, we claim that $n = p - 2$ will work, for all $p \geq 5$. Note that $2^{p-2} \equiv 2^{-1} \pmod p$, by Fermat's Little Theorem, and similarly $3^{p-2} \equiv 3^{-1} \pmod p$ and $6^{p-2} \equiv 6^{-1} \pmod p$. By the lemma, $2^{p-2} + 3^{p-2} + 6^{p-2} - 1 \equiv 2^{-1} + 3^{-1} + 6^{-1} - 1 \equiv \frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 \equiv 0 \pmod p$. Now, it suffices to check $p = 2, 3$, but any n works for $p = 2$ and any even n works for $p = 3$.

15. Note that we can write $n = 2010q + r$. But noting that 2009 is divisible by 7, we have $n = 2010q + r \equiv q + r \pmod 7$. Thus, $q + r$ is divisible by 7 if and only if n is; so we are looking for the number of 6 digit numbers divisible by 7. The smallest, as is easy to check, is $100002 = 14286 \cdot 7$, and the largest is $999999 = 142857 \cdot 7$. Thus, we have a total of 128572 values of n.

16. Assume the opposite, that is, for all $0 \leq k \leq n + 1$, we have $-1 < t^k - P(k) < 1$. Define the n-th difference operator Δ^n inductively as follows $\Delta^0(k) = P(k)$, and for all positive integers n, $\Delta^n(k) = \Delta^{n-1}(k + 1) - \Delta^{n-1}(k)$. It is not difficult to check that if P has degree n, the polynomial Δ^k has degree $n - k$, and in particular, Δ^n is constant and thus Δ^{n+1} is identically zero. However, we show that if $-1 < t^k - P(k) < 1$ for all k in the specified range, then $\Delta^{n+1}(0) > 0$, which will be a contradiction.

We prove by induction on i that $t^k(t-1)^i - 2^i < \Delta^i(k) < t^k(t-1)^i + 2^i$, for all integers $0 \leq k \leq n+1-i$ and non-negative integers i. For the base case, take $i = 0$. The claim becomes

$$t^k - 1 < \Delta^0(k) = P(k) < t^k + 1.$$

However, this is true by our assumption at the beginning. Now, say we have the claim for some i. Then,

$$\begin{gathered} t^k(t-1)^i - 2^i < \Delta^i(k) < t^k(t-1)^i + 2^i \\ t^{k+1}(t-1)^i - 2^i < \Delta^i(k+1) < t^{k+1}(t-1)^i + 2^i, \end{gathered}$$

for all $0 \leq k \leq n - i = n + 1 - (i + 1)$ (since we apply the inductive hypothesis to $k + 1$ as well as k. Now, we have that

$$\begin{aligned} \Delta^{i+1}(k) = \Delta^i(k+1) - \Delta^i(k); &> t^{k+1}(t-1)^i - 2^i - t^k(t-1)^i - 2^i \\ &= t^k(t-1)^{i+1} - 2^{i+1}, \end{aligned}$$

and

$$\begin{aligned}\Delta^{i+1}(k) &= \Delta^i(k+1) - \Delta^i(k) \\ &< t^{k+1}(t-1)^i + 2^i - t^k(t-1)^i + 2^i \\ &= t^k(t-1)^{i+1} + 2^{i+1},\end{aligned}$$

establishing the inductive step. To finish, taking $i = n+1, k = n+1-i = 0$, we see that $\Delta^{n+1}(0) > (t-1)^{k+1} - 2^{k+1} \geq 0$, since $t \geq 3$, giving the desired contradiction, so we're done.

17. Consider the infinite graph whose vertices are squares in the chessboard, such that two vertices are connected by an edge if and only if the piece can jump directly from one to the other (this is a symmetric relation). Define an equivalence relation on the vertices, such that two vertices are in the same equivalence class if the piece can jump from one to the other in finitely many steps.

First, we show that there are no cycles of odd length in the graph. First, note that we can assume m and n are relatively prime; we can take every d-th square in each row and column, giving a new infinite chessboard, and color it as we would for an $(n/d, m/d)$-crocodile. First, say exactly one of m, n is odd. Then, $\pm m \pm n$ is always odd, no matter what the choice of sign is. Then, if we affix lattice points to each square in the grid, we always add an odd amount to the sum of the two coordinates with each jump, meaning we need an even number of jumps to get back to where we started (since we want a difference of 0 at the end). If, on the other hand, m and n are odd, consider moves in the horizontal direction, and say there are a, b, c, d moves of length $m, -m, n, -n$, respectively. Then, $am - bm + cn - dn = 0$, and reducing modulo 2 gives $a+b+c+d \equiv 0 \pmod 2$, that is, we have an even number of moves. Thus, every cycle has even length.

Now, for each equivalence class, choose one element, and color it arbitrarily; this will not disrupt the needed condition because it is impossible to jump from one equivalence class to another. Then, note that this uniquely determines the coloring of the rest of the vertices, because for each vertex, we count the number of steps in a path from it to the colored vertex in its equivalence class. This always has the same parity, because if we have two paths of different parity between two vertices, we get an odd cycle (even if they overlap - consider two vertices in which the paths overlap such that they overlap in no vertices in the paths between these two vertices). If we color vertices an odd distance away the opposite color and vertices an even distances away the same color, we're done, since the parity of path-length is independent of path, and thus any two vertices connected by an edge will have path length 1, and opposite color.

18. Let $x = \cos A$, $y = \sin A$. By the sine double angle formula, $\sin 2A = 2\sin A \cos A$, so $2xy = \dfrac{21}{25}$. Also, $x^2 + y^2 = 1$. We thus find that $(x-y)^2 = x^2 + y^2 - 2xy = 4/25$, and since $x > y$, we have $x - y = 2/5$.

19. Note that $2011n + 2010 = 2000n + 1000 + 11n + 1010 = 100(20n + 10) + 11n + 1010$, so we need $20n + 10$ to divide $11n + 1010$. Now, note that 10 divides $20n + 10$, so 10 must also divide $11n + 2010 = 10(n + 201) + n$, so n is divisible by 10. Let $n = 10k$ for some positive integer k. Now, we want $110k + 1010$ to be divisible by $200k + 10$, and dividing by 10, $11k + 101$ is divisible by $20k + 1$. This is only possible if $20k + 1 \leq 11k + 101$, so we need $k \leq 11$. We can check all these values (it makes our lives easily to consider the possible values of $(11k + 101)/(20k + 1)$, and find that the only k that works is $k = 2$, giving that 123 is divisible by 41. This corresponds to the unique solution of $n = 20$.

20. Extend BP to meet AC at E. Note that by ASA (we have right triangles, angle bisector AD, and shared side AP), we get that $BAP \cong EAP$, so BAE is isosceles and P is the midpoint of

BE. Now, $AE = 14$, so $EC = 12$. Also, P is the midpoint of BE and M is the midpoint of BC, so PM is the midline to EC in triangle BEC, and $PM = EC/2 = 6$.

21. The answer is no. First, we note that $\sqrt{4-2\sqrt{3}} = \sqrt{3}-1$, which can be seen upon squaring and noting that $\sqrt{3} > 1$. Now, $\sqrt{96-56\sqrt{3}} = 7-4\sqrt{3}$, since $7 > 4\sqrt{3}$. We now have $4(\sqrt{3}-1)-(7-4\sqrt{3}) = 8\sqrt{3}-11$, which is not an integer.

Note: We can compute these square roots by letting the square root be equal to $a - b\sqrt{3}$, squaring both sides, and equating 'like terms.'

22. Start with an equilateral triangle with side length 42, and from each corner, respectively, cut out equilateral triangle with side lengths 5, 8, and 23. This will give us the desired hexagon. (The motivation for this is to notice that given an equilangular hexagon, we can extend three of its sides to form equilateral triangles, because we have $120°$ angles at every vertex, and thus $60°$ angles upon extending every other edge.)

23. On the right hand side, the integer k is included once in exactly k terms, namely $b_1, b_2, \ldots, b_k$, thus contributing a total of k to the sum. This is true of all of the a_i, so the total sum on the right hand side is equal to the sum of the a_i.

24. We first prove the following lemma: that if c is the smallest positive integer such that $2^c \equiv 1 \pmod{p}$, and d is a positive integer such that $2^d \equiv 1 \pmod{p}$, then c divides d. Write $d = xc + y$, with x a non-negative integer and $0 \leq y < c$. Then, we see that $2^d \equiv (2^c)^x \cdot 2^y \equiv 1 \pmod{p}$, giving $2^y \equiv 1 \pmod{p}$, contradicting the minimality of c unless $y = 0$, that is, $c|d$.

Now, say $q|2^p - 1$. We also have $q|2^{q-1} - 1$ by Fermat's Little Theorem. Let c be the smallest positive integer with $q|2^c - 1$. Then, $c|p$, so $c = 1$ or $c = p$. However, c cannot be equal to 1, since then $q|1$, so $c = p$ and $p|q-1$. Also, note that $2^p - 1$ is odd, so as $q \equiv 1 \pmod{p}$ and $q \equiv 1 \pmod{2}$, we have $q \equiv 1 \pmod{2p}$ by the Chinese Remainder Theorem.

25. The answer is 7. Note that there is a monochromatic rectangle if and only if in two different rows, there are two corresponding pairs of squares have all four squares the same color. Then, to see that $n = 6$ has a coloring with no monochromatic rectangles, consider coloring the 6 rows in each of the 6 possible arrangements of two black and two white squares.

Clearly, if in each row we have two black and two white squares, 7 rows gives a rectangle, since by the Pigeonhole Principle there will be two identical rows. Now, assume we have a row with four of the same color, say four black squares. We compute the maximum number of additional rows without a monochromatic rectangle. First, observe that any row with at least two black squares will give us a rectangle. Also, if we add a row with four white squares, we get a rectangle in a third row, since any row has either two white or two black squares. Thus, any additional row has exactly 3 white squares, but there are only four non-identical such rows, so we will get two of the same if we add more than four, giving an upper bound of $5 < 7$ rows.

Thus, we assume we have no rows of all of the same color. Now, assume we have row with three black squares and a white square, without loss of generality colored BBBW. Then, any additional row with three black squares will have at least two of its black squares in the first three columns, which gives a black rectangle, so we don't want any of these. Now, say we have a row with three white squares. If we have WWWB, adding one more row will have some two of the same color in the first three columns, giving a monochromatic rectangle. Thus, the black cannot go on the fourth column; up to here the first three columns are equivalent, so say the second row has BWWW. Then, consider the middle two columns; in any additional row, we must have BW or WB, otherwise we get a monochromatic rectangle. However, we then need W in the first column and B in the fourth, or else we get a monochromatic rectangle, so we can add at most two additional rows.

Finally, assume BBBW is the only row without two black and two white squares. Then, the fourth column must consist only of black squares, since if we have two black squares in the first three columns, we get a black rectangle. There are, however, only 3 colorings of a row in this way, so we have at most four total rows. Thus, in all cases, we can have at most 6 rows when we avoid monochromatic rectangles, so we're done.

26. (a) Note that since ADC is isosceles, $AD = AC = AB$, so ADB is isosceles. Now, $\angle BAD = \angle BAC + \angle CAD = 150°$, so $\angle ABD = 15°$ (as it is one of the base angles).

(b) We compute $\angle EAB = 180° - \angle BAC - \angle DAC = 30°$, and $\angle ABE = 180° - \angle ABC = 120°$. Thus, $\angle AEB = 30°$, and $AB = BE$. We now have $EC = EB + BC = 2\alpha$, and $DC = \alpha\sqrt{2}$ from isosceles right triangle DAC. Finally, $\angle ECD = \angle BCA + \angle ACD = 105°$. Therefore, the area of triangle EDC is $\frac{1}{2}(EC)(CD)\sin\angle ECD = \sqrt{2}\alpha^2 \cdot \frac{\sqrt{6}+\sqrt{2}}{4} = \frac{\alpha^2(\sqrt{3}+1)}{2}$.

(c) By the Law of Cosines in BCD, $BD^2 = BC^2 + CD^2 - 2(BC)(CD)\cos\angle BCD = \alpha^2 + 2\alpha^2 + 2\sqrt{2}\alpha^2 \cdot \frac{\sqrt{6}-\sqrt{2}}{4} = \alpha^2(2+\sqrt{3})$. Thus, $BD = (\sqrt{3}+1)\alpha$.

27. Rephrasing the problem in terms of graph theory, we have a finite graph with no 5-cliques, such that any two 3-cliques share a vertex, and we want to show that there are two vertices such that upon deleting them, no 3-cliques remain. Clearly, if there are 4 or fewer vertices, deleting any two vertices leaves at most 2 vertices, so we clearly have no 3-cliques. Assume from here that we have at least 5 vertices in our graph, and furthermore that each vertex has degree at least one, or else we may ignore it (it belongs to no 3-cliques).

Case 1: There is a 4-clique in the graph, $ABCD$. We claim that any other 3-clique must contain some two of A, B, C, D. Clearly, it must contain at least one, since it must share a vertex with, say, ABC. But if we have a 3-clique AEF, it shares no vertices with BCD, so we need at least two vertices of $ABCD$ for each 3-clique. Thus, say we now have a clique EAC. We now show that each clique contains either A or C. Assume this is not the case; then, a 3-clique it must intersect 4-clique $ABCD$ in at least two points, which now must be B and D. However, this 3-clique, to intersect ACE, must contain E, meaning BE and DE are both edges in our graph. However, this makes $ABCDE$ a 5-clique, a contradiction, meaning that we can delete A and C to leave no 3-cliques behind.

Case 2: There is no 4-clique in the graph, but there exist two 3-cliques ABC, BCD that share an edge. We claim that any 3-clique contains either B or C. If not, it must contain both A and D to share a vertex with ABC and BCD. But A and D are not connected by an edge, or else we have a 4-clique $ABCD$, so we can delete B and D.

Case 3: There exists a 4-cycle $ABCD$, but any two 3-clique intersect at exactly one vertex. Observe that we can assume every edge is a member of a 3-cycle, otherwise we can ignore it from the graph. Thus, let vertex E be such that ABE is a 3-clique. Then, the 3-clique containing edge CD intersects clique ABE. ACD and BCD cannot be 3-cliques, or else we are back in case 2, so ECD is a 3-clique. Now, if every 3-clique contains vertex E, we're done (we can delete E). Otherwise, take a clique that does not contain E. Note that it must intersect both EAD and EBC, so it must contain either A or D, and either B or C. However, BD and AC are not edges, so the 3-clique contains either edge AB or CD. If it contains edge CD, we need to add a 6th vertex F, but CDF and EBA will not intersect; it thus contains AB. Therefore, every 3-clique contains either E or A, so we can delete these two vertices.

Case 4: There are no 4-cycles. Start with a 3-clique ABC (if non exist, we're done), and some other 3-clique that intersects ABC at, without loss of generality, A, say ADE. If every other 3-clique contains A, we can delete A; otherwise consider a 3-clique that does not contain A. Then, it must contain either B or C, and either D or E. But including any of the four edges BD, BE, CD, CE results in a 4-cycle, which is not possible, so we're done.

This exhausts all cases, so there are always 2 vertices to delete, leaving no 3-cliques.

28. Rewrite the sum on the left hand side as $(a_1 + 2b_1) + (a_2 + 3b_2) + (a_3 + 4b_3) + (a_4 + 5b_4) = 19$. But notice that the stipulation $0 \le a_i \le i$ makes it so that $a_i + (i+1)b_i$ represents each positive integer exactly once (this is similar to writing an integer in base $i+1$). Thus, this is equivalent to finding four non-negative integers w, x, y, z with $w + x + y + z = 19$; by Balls and Urns, this is just $\binom{22}{3} = 1540$.

29. The answer is $f(n) = kn$ for all positive integers k, as well as functions f such that there exists a positive integer N so that $f(n) = n$ when $n \le N$, and $f(n) = kn$ for some positive integer $k > 1$ whenever $n > N$. These are both easily checked to satisfy the needed conditions.

First, assume that there exist positive integers a, b and distinct positive integers $k_1, k_2 > 1$ such that $f(a) = k_1 a$ and $f(b) = k_2 b$. Then, note that plugging in $n = a$ gives $af(k_1 a) = k_1^2 a^2$, and thus $f(k_1 a) = k_1^2 a$. In a similar fashion, we find that for all non-negative integers n, we have $f(k_1^n a) = k_1^{n+1} a$. Similarly, $f(k_2^n b) = k_2^{n+1} b$. Without loss of generality, say $k_2 < k_1$. Then, we can choose a large enough b' such that $a < b'$ and $f(b') = k_2 b'$, for example, by multiplying b by a large enough power of k_2.

Now, choose the n such that $(k_2/k_1)^{n+1} < a/b' \le (k_2/k_1)^n$, which exists since $k_2/k_1 < 1$ and $a'/b < 1$. We then have $k_1^n a \le k_2^n b'$, but $k_1^{n+1} a > k_2^{n+1} b'$, which contradicts the strictly increasing condition since $f(k_1^n a) = k_1^{n+1} a$ and $f(k_2^n b') = k_2^{n+1} b'$.

Therefore, if there does not exist an n with $f(n) = n$, then $f(n)/n$ takes on only one value, and we get $f(n) = kn$ for some integer k (if k is not an integer, $f(n)$ will be non-integral for some n). Thus, it remains to check the case when we have some n with $f(n) = n$ and the rest with $f(n) = kn$ for some $k \ne 1$. We want to show that there exists an N such that $f(n) = n$ if and only if $n \le N$, which is equivalent to showing that if $f(n) = kn$ and $f(m) = m$, then $n > m$. Observe that $k > 1$, since it follows by an easy induction that $f(n) \ge n$ for some n (note that we don't assume k is an integer here; we could have $f(n) = n/2$ for all even n, for example). Assume instead that $n < m$, and let c be the integer such that $k^c \le m/n < k^{c+1}$. Then, we have $f(k^c n) = k^{c+1} n$, but $k^c n \le m < k^{c+1} n$, contradicting the strictly increasing requirement. This completes the proof.

30. We rewrite what we want, via long division or otherwise, as $(\alpha^3 - \alpha - 1)(\alpha^7 + \alpha^5 + 4\alpha + 2) - 15 = -15$.

31. We show that at any point in time, we can make a series of switches to reduce the number of rooms with all lamps in the same state by at least one. Say some room r_1 has all lamps in the same state. Pick a lamp, and switch its state. If its partner is in the same room, we're done, since we then have 2 in the opposite state and one in the original state, and the other rooms are unchanged. Otherwise, some lamp in a new room r_2 has changed state. If r_2 now has lamps in both states, we're done; otherwise assume they are all in the same state. If this is the case, pick a lamp different from the one we just switched, switch it, and repeat this process.

Eventually, since there are finitely many rooms, we come to a cycle (or get to a room with lamps in both states, in which case we're done); that is, we get to a room r_m such that $m > n$ and $r_m = r_n$ for some n. When we first left this room, we flipped a switch so that we had some lamps in it on and some off; without loss of generality, say 1 was on and 2 were off. When we get back to it a second time, note that we turn on one of the two that were off, since if we turn the on one back off, this means that we made a cycle earlier (that is, we came from room $r_{m-1} = r_{n-1}$). Thus, once we get back to room r_m, it starts with 1 on and 2 off, and ends with 2 on and 1 off. At each other step, we broke even, turning one room into having lamps of both states, but turning a room that originally had lamps of both states into one with lamps in only one state. Thus, we have decreased the number of rooms with lamps in only one state, so we're done.

32. We're going to re-draw the diagram, defining points backwards. Since $AE = AF$, there is a circle ω tangent to AB and AC at F and E, respectively, and that is also tangent to the incircles of BPQ and CMN (since they share a common tangency point with the two incircles). Let the tangent to this circle that is parallel to BC and lies entirely outside triangle ABC meet AB and AC at R' and S', respectively. We claim that $R = R'$ and $S = S'$.

To show this, it suffices to show that E, N, R' are collinear, and similarly, P, Q, S' are collinear. Note that triangles $AR'S'$ and MNC are homothetic, since A, M, C, B are collinear, NC is parallel to $R'S'$, and AR' is parallel to MN. The homothety that takes MNC to $AR'S'$ also takes the incircle of MNC to the incircle of $AR'S'$, which is just ω. These two circles are tangent at E, so this must be the center of homothety. Therefore, E, N, R' are collinear, as R' is the image of N under the homothety, and similarly, P, Q, S' are collinear. It follows that the incircle of triangle ARS is tangent to AB and AC at F and E, respectively.

Now, it suffices to show that the incenter of AEF lies on ω. Since AEF is isosceles, the angle bisector of $\angle EAF$ meets arc EF of ω at its midpoint X. We want to show that X is the incenter of AEF. Indeed, $\angle AEX = \dfrac{\widehat{EX}}{2} = \angle XFE = \angle XEF$, so EX bisects $\angle AEF$, and so X is the incenter of AEF.

The motivation for this solution is as follows: we start with the last step, noting that if the incenter of AEF lies on the incircle of ARS, then the points of tangency of the incircle of AEF must be E and F. Then, we construct this circle, and we want to show that it's the incircle of ARS, by showing that RS is tangent to it; to do this, we construct the parallel tangent and show that it is the same line as RS.

33. First, since P is a polynomial with integer coefficients, observe that if a and b are positive integers, then $a - b | P(a) - P(b)$. This follows from the fact that $a - b | a^n - b^n$; we can consider $P(a) - P(b)$ term-by-term. Thus, $p | P(pq) - P(p)$, since $p | p(q - 1)$, for primes p and q.

Define an *orbit* of length n to be a set of distinct positive integers $x_1, x_2, \ldots, x_n$ such that $T(x_i) = x_{i+1}$, where we take $x_{n+1} = x_1$. Note that all of these have $T(x_i) = n$. In particular, the number of elements enclosed in orbits of length n is divisible by n, since each element is in either one finite orbit (as there is only one 'path' back to x if we apply T to it repeatedly).

Let p and q be primes, and consider the number of elements in orbits of length pq. First, there are $P(pq)$ elements such that applying T pq times gives back the original number. However, the elements in orbits of length p and q are also counted in $P(pq)$, so we subtract these; but fixed points, elements in orbits of length 1, are double-counted in this subtraction, so we add them back. In total, we have $P(pq) - P(p) - P(q) + 1$ elements in orbits of length pq. pq divides this number, but in particular p divides this number. Since $p|P(pq) - P(p)$, we have $p|P(q) - 1$. We can repeat this for all primes p, to find that $P(q) - 1$ is divisible by infinitely many primes, so $P(q) = 1$. However, if this is also true for all primes q, our polynomial is constant (since $P(q) - 1$ has infinitely many roots), a contradiction. Thus, no such P exists.

34. By the Pythagorean Theorem, the length of the long diagonal is $\sqrt{r^2 + s^2 + t^2}$, but $r^2+s^2+t^2 = (r + s + t)^2 - 2(rs + st + tr) = 4^2 - 2(5) = 6$ by Vieta's Formulas. Thus, the length is $\sqrt{6}$.

35. Let the total number of hours read in the first n days be a_n. The sequence $a_1, \dots, a_{37}$ is an increasing sequence of positive integers with $a_{37} \leq 60$; we wish to show that some two differ by 13. Consider partitioning the possible a_i with the following sets: $\{1, 14\}, \{2, 15\}, \dots, \{13, 26\}, \{27, 40\}, \{28, 41\}, \dots, \{39, 52\}, \{53, 54, \dots, 60\}$. There are 37 a_i, and at most 8 are chosen from the last set, so at least 29 are chosen from the first 26. By the Pigeonhole Principle, some two a_i are in the same pair, giving two a_i that differ by 13. (Note: we didn't need the condition that the student never read for more than 12 hours during a single day.)

36. We want the residue of this number modulo 1000. Since $2003 \equiv 3 \pmod{1000}$, we can reduce to $3^{2002^{2001}}$. By the Fermat-Euler Theorem, $3^{400} \equiv 1 \pmod{1000}$, since $\phi(1000) = \phi(8)\phi(125) = 4 \cdot 100 = 400$. Now, we want the residue of $2002^{2001} \equiv 2^{2001}$ modulo 400. Clearly, since this is a power of 2, $2^{2001} \equiv 0 \pmod{16}$. Now, $2^{2001} \equiv 2 \cdot 2^1 \equiv 1 \pmod{25}$, since $2^{20} \equiv 1 \pmod{25}$ again by Fermat-Euler. Thus, by the Chinese Remainder Theorem, $2002^{2001} \equiv 352 \pmod{400}$, and the original number is congruent to 3^{352} modulo 1000.

To finish, we can compute this by systematic brute force. We have the following:

$$3^4 = \equiv 81 \pmod{1000}$$
$$3^8 \equiv 81^2 \equiv 561 \pmod{1000}$$
$$3^{16} \equiv 561^2 \equiv 721 \pmod{1000}$$
$$3^{32} \equiv 721^2 \equiv 841 \pmod{1000}$$
$$3^{64} \equiv 841^2 \equiv 281 \pmod{1000}$$
$$3^{128} \equiv 281^2 \equiv 961 \pmod{1000}.$$
$$3^{256} \equiv 961^2 \equiv 521 \pmod{1000}.$$

Now, $3^{352} \equiv 3^{256} \cdot 3^{64} \cdot 3^{32} \equiv 241 \pmod{1000}$.