“Tryout” - LHS Math Team

Monday, September 10, 2012

1. Nine points are equally spaced along a line. If a tenth distinct point is chosen uniformly at random between the 1st and 9th points on the line, what is the probability that it lies between the 4th and 7th points?

2. Hao is a TA for a programming class, and after grading final exams, Hao decides to lower the test scores of every student with last name Bhupatiraju by 50 points. If there are 25 students in the class and the class average on the test goes down by 6, how many students are there in the class with last name Bhupatiraju?

3. Mo and Bo go for a run around their neighborhood. They start off 50 meters apart and both run at 5 meters per second in the same direction. A truck comes from a corner, moving in the same direction as the runners, and passes them. The time it took for the truck to pass Bo after it has passed Mo is 5 seconds. How fast is the truck traveling in meters per second?

4. Points B, C, and D are points on a line such that D lies between B and C. Let A be a point such that $\overline{AD} \perp \overline{BC}$, $AB = 17$, $AC = 25$, and $AD = 15$. Find the perimeter of triangle ABC.

5. Zaroug and Vikas play a mathematically sophisticated game in which Zaroug counts up by 20’s starting at 3 and Vikas counts up by 12’s starting at 7. How many numbers are there less than 400 that both Zaroug and Vikas say?

6. A market lists the following exchange rates for animals.

3 Lions	$\leftrightarrow$	2 Whales
2 Elephants	$\leftrightarrow$	9 Cows
5 Turkeys	$\leftrightarrow$	1 Seal
1 Whale	$\leftrightarrow$	2 Elephants
2 Seals	$\leftrightarrow$	3 Cows

   How many turkeys would need to be exchanged for 2 lions?

7. Three of the angles in a convex polygon measure 150°, 97°, and 113°. What is the largest possible number of sides this polygon can contain, if no angle is larger than 150°?

8. Simplify $\frac{\frac{3x^2-8x-3}{x+2}}{\frac{9x^2-1}{x^2+5x+6}}$ for all x where the fraction is defined.

9. The positive integers are colored black and white such that if n is one color, then $2n$ is the other color. If all of the odd numbers are colored black, then how many numbers between 100 and 200 inclusive are colored white?

10. Let $\mathcal{S}$ be the sphere of radius 4 centered at the origin $O = (0, 0, 0)$, and let $N = (0, 0, 4)$. Point P lies on $\mathcal{S}$ such that $NP = 3$. Line $\overleftrightarrow{NP}$ intersects the plane $z = 0$ at P'. Find NP'.

11. Two solutions for x to the cubic equation $x^3 - (c^2 + 1)x + 14c = 0$, where c is a constant, are 3 and 4. Find c.

12. Determine the number of solutions for x to the equation

$$\log_{10} \sin 2x + \log_{10} \cos 2x = -1$$

   between 0 and 2π.

13. Let a, b, c, and d be unique integers from 1 to 10 inclusive selected uniformly at random. What is the probability that $a - c$ divides $ab + cd$ if $a - c$ divides $ad + bc$?

14. Five regular tetrahedra are arranged in space such that the center tetrahedron has all 4 faces touching another tetrahedron. The smallest sphere that completely contains the figure is drawn, as is the largest sphere that is completely contained within the figure. Find the ratio of the surface area of the bigger sphere to that of the smaller sphere.

15. Rohil flips a head and a tail. Afterwards, the probability he flips a tail is the ratio of the number of tails previously flipped to the total number of previous flips. What is the probability that of the next 100 flips, he will flip exactly 50 tails?

16. In triangle ABC, let A', B' and C' be the feet of the altitudes from A, B, and C, respectively. Let X, Y and Z be the midpoints of sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Given that $AB = 6$, $BC = 7$, and $AC = 8$, find $A'Y + A'Z + B'X + B'Z + C'X + C'Y$.

17. How many 3-by-3 matrices with nonnegative integer entries have the property that the numbers in each row and column sum to 6?

18. A quadrilateral $ABCD$ is inscribed in a circle with area 7π such that BD is a diameter of this circle. Given that AB has length 4 and CD has length 5, what is the distance between the centroids of triangle ABC and triangle BCD?

19. Find the largest positive integer m such that there exists a sequence of real numbers $a_0, a_1, \dots, a_m$ satisfying $a_0 = 2012$, $a_1 = 2013$, and

$$a_{n-2}^2 - a_n^2 = \frac{1}{a_{n-1}^2}$$

for $2 \le n \le m$.

20. Let $\varphi(n)$ be the number of positive integers less than or equal to n that are relatively prime to n. Given that $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$ and $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^4} = \frac{\pi^4}{90}$, compute

$$\left(\sum_{i=1}^{\infty} \frac{\varphi(i)}{i^3}\right)\left(\sum_{i=1}^{\infty} \frac{\varphi(i)}{i^4}\right).$$

Bonus: If $A = B$ and $B = C$ and C is part of D and D consists of elements of E and E is contained in A, at what time does the alligator step on the riverbank?