LHS Math Team

Team Contest 1

1. Prove that the fraction $\dfrac{21n+4}{14n+3}$ is irreducible for every positive integer n.

2. The sides of an equiangular octagon are 2, $2\sqrt{2}$, 4, $4\sqrt{2}$, 6, 7, 7, and 8, not necessarily in this order. What is the area of this octagon?

3. Let ABC be an acute triangle with $AB < AC$ and let ω be the inscribed circle of ABC. Let I be the center of circle ω and let D, E, F be the intersections of ω with $\overline{BC}$, $\overline{AC}$, $\overline{AB}$, respectively. Let G be the intersection of ray $\overrightarrow{EF}$ with ray $\overrightarrow{CB}$ and let K be the foot of the perpendicular from I to $\overline{AG}$. Prove that segments $\overline{IK}$, $\overline{AD}$, and $\overline{EF}$ are concurrent.

4. Let $p > 2$ be a prime number such that $3|(p-2)$. Let
$$\mathcal{S} = \{y^2 - x^3 - 1 \mid x \text{ and } y \text{ are integers, } 0 \le x, y \le p-1\}.$$
Prove that at most p elements of $\mathcal{S}$ are divisible by p.

5. Write
$$\frac{1}{\sqrt[3]{12} + \sqrt[3]{45} + \sqrt[3]{50}}$$
as a completely simplified fraction so that the denominator is an integer and there are no nested roots in the numerator.

6. A line segment goes from the point $P(1,2)$ to the point $Q(50,100)$. How many lattice points lie on the segment $\overline{PQ}$ (points (x,y) where x and y are both integers)? Include endpoints.

7. Define (a,b,c) to be a *Pythagorean triple* if it is a triplet of positive integers such that $a^2+b^2=c^2$. Prove that there does not exist any integer n for which we can find a Pythagorean triple (a,b,c) satisfying
$$\left(\frac{c}{a} + \frac{c}{b}\right)^2 = n.$$

8. For a finite set S with at least two elements, let $\mathcal{P}(S)$ denote the set of all subsets of S. A function $f : \mathcal{P}(S) \to \mathcal{P}(S)$ is called *unified* if $f(A) \cup f(B) = f(A \cup B)$ for every pair of subsets A and B of S. Prove that for a given finite set, the number of its unified functions is always a perfect power; that is, it can be written in the form of m^k for integers m and k with $k \ge 2$.

9. Six members of the team of Fatalia for the International Mathematical Olympiad are selected from 13 candidates. At the TST the candidates got $a_1, a_2, \ldots, a_{13}$ points with $a_i \neq a_j$ if $i \neq j$.

   The team leader (who is not a member of the team) already has in mind 6 particular candidates that he wants to be on the team. With that end in view he constructs a polynomial $P(x)$ and finds the creative potential of each candidate by the formula $c_i = P(a_i)$.

   For what minimum n can he always find a polynomial $P(x)$ of degree not exceeding n such that the creative potentials of all 6 candidates that the team leader wants are strictly more than those of the 7 others?

10. Let ABC be a triangle with area 45. Points M_1 and M_2 lie on segment $\overline{AB}$ with $AM_1 = M_1M_2 = M_2B$ and points N_1 and N_2 lie on segment $\overline{AC}$ with $AN_1 = N_1N_2 = N_2C$. Find the area of quadrilateral $M_1M_2N_2N_1$.

11. Let p be a prime number, and let m and n be integers greater than 1. Suppose that $m^{p(n-1)} - 1$ is divisible by n. Show that $m^{n-1} - 1$ and n have a common divisor greater than 1.

12. Find, with proof, all integers n such that there is a solution in nonnegative real numbers (x, y, z) to the system of equations

$$\begin{aligned} 2x^2 + 3y^2 + 6z^2 &= n \\ 3x + 4y + 5z &= 23. \end{aligned}$$

13. The unit squares of a 5×41 grid are colored in red and blue (one color to each square). Prove that there are 3 rows and 3 columns such that the 9 squares where they intersect are all the same color.

14. A quadrilateral has one vertex on each side of a square of side length 1. Show that the lengths a, b, c, d of the sides of the quadrilateral satisfy the inequalities

$$2 \leq a^2 + b^2 + c^2 + d^2 \leq 4.$$

15. Let a, b, c, d be positive real numbers. Prove that

$$\frac{1}{a+b+c+\frac{1}{abcd}+1} + \frac{1}{b+c+d+\frac{1}{abcd}+1} + \frac{1}{c+d+a+\frac{1}{abcd}+1} + \frac{1}{d+a+b+\frac{1}{abcd}+1}$$

cannot exceed $\dfrac{a+b+c+d}{a+b+c+d+1}$.

16. Twenty singers will perform in a concert. For each singer, there is a (possibly empty) set of other singers such that he wishes to perform later than all the singers from that set. Can it happen that there are exactly 2010 orders of the singers such that all their wishes are satisfied?

17. Given that a_1, a_2, a_3, and a_4 satisfy

$$\begin{aligned} a_1 + 8a_2 + 27a_3 + 64a_4 &= 1 \\ 8a_1 + 27a_2 + 64a_3 + 125a_4 &= 27 \\ 27a_1 + 64a_2 + 125a_3 + 216a_4 &= 125 \\ 64a_1 + 125a_2 + 216a_3 + 343a_4 &= 343, \end{aligned}$$

find $2011^3 a_1 + 2010^3 a_2 + 2009^3 a_3 + 2008^3 a_4$.

18. In right triangle ABC with right angle at C with leg lengths $AC = 12$ and $BC = 5$, arcs of circles are drawn, one with center A and radius 12, the other with center B and radius 5. They intersect the hypotenuse at M and N. Find the length MN.

19. What is the largest positive integer that is not the sum of a positive integral multiple of 60 and a positive composite integer?

20. In how many ways can 6 purple balls and 6 green balls be placed into a 4×4 grid such that every row and column contains two balls of one color and one ball of the other color? Only at most one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

21. Let $P(x) = x^5 + x^2 + 1$ have roots r_1, r_2, r_3, r_4, and r_5. Let $q(x) = x^2 - 2$. Determine the product

$$q(r_1)q(r_2)q(r_3)q(r_4)q(r_5).$$

22. Let ABC be a triangle and let O be the center of the circle passing through A, B, and C. Let X be the intersection of $\overrightarrow{AO}$ with segment $\overline{BC}$ and let Y be the foot of the perpendicular from A to $\overline{BC}$. Finally, let Z be the point on $\overline{BC}$ such that $\overline{AZ}$ bisects $m\angle BAC$. Given that $m\angle A = 64°$ and $m\angle B = 73°$, find the ratio YZ/ZX.

23. For a real number t and positive real numbers a, b we have

$$2a^2 - 3abt + b^2 = 2a^2 + abt - b^2 = 0.$$

Find t.

24. 15 positive integers, all less than 1998 (and no one equal to 1), are relatively prime (no pair has a common factor > 1). Show that at least one of them must be prime.

25. A *troll* sequence is an increasing sequence of 16 consecutive odd positive integers whose sum is a perfect cube. How many troll sequences are there with all numbers less than 1000?

26. A 15×15 square is tiled with unit squares. Each vertex is colored either red or blue. There are 133 red points. Two of these red points are corners of the original square, and another 32 red points are on the sides of the original square. The sides of the unit squares are colored according to the following rule: If both endpoints are red, then it is colored red; if the points are both blue, then it is colored blue; if one point is red and the other is blue, then it is colored yellow. Suppose that there are 196 yellow sides. How many blue segments are there?

27. The vertices of a right triangle ABC inscribed in a circle divide the circumference into three arcs. The right angle is at A, so that the opposite arc BC is a semicircle while arc BA and arc AC are supplementary. To each of the three arcs, we draw a tangent such that its point of tangency is the midpoint of that portion of the tangent intercepted by the extended lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$. More precisely, the point D on arc BC is the midpoint of the segment joining the points D' and D'' where the tangent at D intersects the extended lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, respectively. Similarly, define E on arc AC and F on arc AB. Prove that triangle DEF is equilateral.

28. Show that for positive reals a, b, c,

$$\frac{a^2}{(a+b)(a+c)} + \frac{b^2}{(b+c)(b+a)} + \frac{c^2}{(c+a)(c+b)} \geq \frac{3}{4}.$$

29. Let $\triangle ABC$ be a triangle where $AB = 25$ and $AC = 29$. C_1 is a circle that has $\overline{AB}$ as a diameter and C_2 is a circle that has $\overline{BC}$ as a diameter. D is a point on C_1 so that $BD = 15$ and $CD = 21$. C_1 and C_2 clearly intersect at B; let E be the other point where C_1 and C_2 intersect. Find all possible values of ED.

30. The product of two of the four roots of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is -32. Determine the value of k.