Logarithms
LHS Math Team

Definition: We say that $\log_a b = c$ if $a^c = b$, for positive numbers a and b. In English, the **logarithm** (log for short) of b, **base** a is equal to c.

In high school, if the logarithm is written without a base, that is, $\log x$, it is generally understood that this is base 10, so we have $\log_{10} x$. However, the big secret is, base 10 is completely useless outside of the fact that we're so used to doing arithmetic with it. $\ln x$ usually denotes the **natural logarithm**, which has base $e \equiv 2.718....$ The reason for this lies in calculus. This is much more useful, and most professional mathematicians just use log for the natural logarithm. However, whenever you see log on high school math competitions, it generally means base 10 (or the base is specified).

There are four basic properties of logarithms you'll have to know. All of them come from familiar rules of exponentiation.

1. $\log_a b + \log_a c = \log_a bc$.
2. $\log_a b^n = n \log_a b$.
3. $\log_{a^n} b^n = \log_a b$.
4. $\dfrac{\log_a b}{\log_a c} = \log_c b$.

Proof of 1: Let $\log_a b = x$ and $\log_a c = y$. Then, $a^x = b$ and $a^y = c$. Multiplying, $a^x a^y = bc$, but the left hand side is just a^{x+y}. Taking the log (base a) of both sides gives $x + y = \log_a bc$, or $\log_a b + \log_a c = \log_a bc$.

Proof of 2: Let $\log_a b = x$, so that $a^x = b$. Raising both sides to the n-th power, $(a^x)^n = b^n$, so $a^{nx} = b^n$. Taking the log (base a) of both sides gives $nx = \log_a b^n$, or $n \log_a b = \log_a b^n$.

Proof of 3: Let $\log_a b = x$, so that $a^x = b$. Again taking the n-th power of both sides gives $a^{nx} = b^n$, which we can rewrite as $(a^n)^x = b^n$. Now, taking the log (base a^n) of both sides, we get $x = \log_{a^n} b^n$, from which the claim follows.

Proof of 4: Let $\log_a b = x, \log_a c = y$, so that $a^x = b$ and $a^y = c$. We wish to prove that $\dfrac{x}{y} = \log_{a^y} a^x$. Using property 3, the right hand side is equal to $\log_{(a^y)^{1/y}} a^{x/y} = \log_a a^{x/y} = x/y$, which is what we wanted.

Property 4 is sometime known as the change-of-base formula. As exercises, prove the following corollaries (which follow from the above four properties):

$\log_a b - \log_a c = \log_a \dfrac{b}{c}$.

$\log_a b \cdot \log_b a = 1$.

$\log_a b \cdot \log_b c = \log_a c$.

$\log_a d \cdot \log_b c = \log_a c \cdot \log_b d$.