Team Contest 2

Solutions

January 31, 2011

1. Consider 10 'in between' points, which we get when we place one point per arc with none of the ten points in the interior, defined by each pair of adjacent points. A partition is defined by taking two different points among these ten, and we get an additional one from the partition of the empty set and the whole set, giving an answer of $\binom{10}{2} + 1 = 46$.

2. Say n has d digits. Then, appending a 1 to the right is the same as taking $10n + 1$, then appending a 2 to the left of $10n+1$ adds 10^{d+1}. Thus, $10^{d+1} + 10n + 1 = 33n$, so $n = \frac{2 \cdot 10^{d+1} + 1}{23}$; we want a d such that this is an integer (we can check that n has d digits at the end). We want $10^{d+1} \equiv 11 \pmod{23}$. $d = 2$ suffices, as $10^3 = 43 \cdot 23 + 11$. This gives $n = 2001/23 = 87$, which, indeed, has $d = 2$ (and we can see that performing the described operation gives $2871 = 33 \cdot 87$.

3. The first equation rewrites as $a^3 + (-b)^3 + (-c)^3 - 3a(-b)(-c) = 0$, and employing a well-known factorization we see that $(a - b - c)[(a + b)^2 + (b - c)^2 + (c + a)^2] = 0$, so either $a = b + c$ or, by since all squares are non-negative, $b = c = -a$. The second case fails since we require $a, b, c > 0$. In the first case, substituting into the second equation gives $a^2 = 2a$, so $a = 2$, since $a > 0$, so $b + c = 2$. As these must be positive integers, $b = c = 1$, for the lone solution $(a, b, c) = (2, 1, 1)$.

4. Let a, b, c be the lengths of the sides of ABC. By the triangle inequality, $a + b > c$, and dividing by a, $1 + \frac{b}{a} > \frac{c}{a}$. And now we use the law of sines: we have $\frac{\sin A}{\sin A} + \frac{\sin B}{\sin A} > \frac{\sin C}{\sin A}$. Multiplying through by $\sin A$, $\sin A + \sin B > \sin C$, and similar relations hold true for other permutations. Thus, $\sin A, \sin B, \sin C$ satisfy the triangle inequality, and were done.

5. For $n = 3$, the claim is obvious. We now prove the claim for $n = 4$. Let the quadrilateral be $ABCD$, and let the incenters of triangles BCD, CDA, DAB, ABC be W, X, Y, Z, respectively. We first show that $WXYZ$ is a rectangle. By symmetry, it suffices to show that $\angle WXY = 90°$. Let a, b, c, d be the measures of arcs $\widehat{AB}, \widehat{BC}, \widehat{CD}, \widehat{DA}$, respectively, in degrees. We first show that quadrilateral $WCDX$ is cyclic. From the fact that CW bisects $\angle BCD$ and that CX bisects $\angle ACD$,

$$\begin{aligned}\angle WCX &= \angle BCD - \angle BCW - \angle WCD;\\ &= \angle BCD - \frac{1}{2}\angle BCD - \frac{1}{2}\angle ACD;\\ &= \frac{1}{4}(a + d) - \frac{1}{4}a = \frac{1}{4}a.\end{aligned}$$

In a similar way, $\angle WDX = \frac{1}{4}a$, so $\angle WCX = \angle WDX$, and $WXDC$ is cyclic. Thus, $\angle WXD = 180° - \angle WCD = 180° - \frac{1}{4}(a+d)$. In a similar way, we find that $\angle YXD = 180° - \frac{1}{4}(b+d)$. Therefore, $\angle WXY = 360° - \angle WXD - \angle YXD = 360° - \frac{1}{4}(a+b+c+d) = 90°$, and $WXYZ$ is a rectangle.

The rest of the $n = 4$ case follows from the British Flag Theorem, as we get $PW^2 + PY^2 = PX^2 + PZ^2$, and our only two possible triangulations are with ABC and ADC or with BCD and BAD. The claim for $n > 4$ now follows from the $n = 4$ case. If we two triangles sharing an edge in our triangulation, eliminating the diagonal that splits the quadrilateral that they form into two and instead drawing the opposite diagonal does not change $\sum_{i=1}^{k}(PI_k)^2$, by the $n = 4$ case. It is now left to show that any two triangulations may be obtained from each other by this operation (taking a quadrilateral and switching the diagonal that splits it in two for the triangulation).

We proceed by induction on n. For the case $n = 5$, the only possible triangulations arise from labeling our vertices, in order, A, B, C, D, E, and taking one of our triangles to be ACD, leaving the other two forced. If we focus on the edge, in this case, CD, which is part of a triangle in which only one of the edges of the original polygon is an edge of the triangle, note that performing the operation moves this edge over by two edges, to either AB or EA. Thus, if we perform the operation five times in a row, we may move this edge in the order CD, EA, BC, DE, AB, which covers all triangulations of $ABCDE$.

For the inductive step, note that as we have $n - 2$ triangles in the triangulation (we can prove separately that any triangulation has this many triangles) and n edges, at least two triangles must have two edges overlapping with the edges of the polygon (as no triangle has all three edges as edges of the polygon). Call such a triangle *nice*. Say we have a triangulation T_1, and we would like to turn it in to T_2. Note that it is possible to take one nice triangle from each triangulation such that the two triangles do not overlap (except perhaps in a vertex) We do this by choosing a nice triangle a_1 from T_1, and from here it is possible to choose one of T_2's (at least) two nice triangles unless both overlap with a_1, which is only possible by if we take these two be its two adjacent nice triangles. But then, the other nice triangle in T_1 can not overlap with these, since $n \geq 6$.

To finish, let a, b be nice in T_1, T_2, and operate the $(n - 1)$-gon formed by deleting a from T_1 until b is part of the triangulation (which is possible by the inductive step). Then, ignore b and operate on the resulting $(n - 1)$-gon to get T_2. This completes the proof.

6. Let $n = 2^k \cdot a$, where a is odd and k is a non-negative integer. We claim that $10^{2^k} + 1$ divides the integer, which shows that it is not prime (clearly this will be a proper factor of the integer in consideration; it is easy to check that it is much smaller as well as greater than 1). We will compute each of the summands modulo $10^{2^k} + 1$.

$$10^n \equiv 10^{a \cdot 2^k} \equiv (10^{2^k})^a \equiv (-1)^a \equiv -1 \pmod{10^{2^k} + 1},$$

since a is odd. Since $n = a \cdot 2^k$, it is easy to check that at least $k + 1$ powers of 2 divide 10^n, so we can write $10^n = b \cdot 2^{k+1}$ with b a positive integer. Then,

$$10^{10^n} \equiv (10^{2^{k+1}})^b \equiv ((10^{2^k})^2)^b \equiv 1 \pmod{10^{2^k} + 1},$$

as $k + 1 \leq n$ and thus 2^{k+1} divides 2^n, which in turn divides 10^n. In a similar way, we find that $10^{10^{10^n}} \equiv 1 \pmod{10^{2k} + 1}$. Therefore,

$$10^{10^{10^n}} + 10^{10^n} + 10^n - 1 \equiv 1 + 1 - 1 - 1 \equiv 0 \pmod{10^{2^k} + 1},$$

so we're done.

7. Multiplying both sides by a_{n+1}, we find that $a_{n+2}a_{n+1} = 1 + a_n a_{n+1}$. Now, letting $b_n = a_n a_{n+1}$, this relation simply says $b_{n+1} = 1 + b_n$. Now, we note that

$$\frac{a_{2011}}{a_{2009}} \cdot \frac{a_{2009}}{a_{2007}} \cdots \frac{a_3}{a_1} = \frac{a_{2011}}{a_1} = a_{2011},$$

but this is also equal to

$$\frac{b_{2010}}{b_{2009}} \cdot \frac{b_{2008}}{b_{2007}} \cdots \frac{b_2}{b_1}$$

Because $b_1 = a_1 a_2 = 1$, we see that $b_n = n$ for all n, so the value we want is just $\dfrac{2010 \cdot 2008 \cdots 2}{2009 \cdot 2007 \cdots 1}$.

8. Consider all 10-digit multiples of 7, of which there are at least $\lfloor 9 \cdot 10^{10}/7 \rfloor$, since there are $9 \cdot 10^{10}$ 10-digit integers. Now, consider the possible 10 element 'sets' of digits they contain, where we allow for duplicates in these sets. Say there are a_i occurrences of the digit i; we then have $a_0 + a_1 + \cdots + a_9 = 10$. By Balls and Urns, there are $\binom{19}{9}$ such solutions, so this many sets. We wish to show that one of these sets corresponds to at least 10^4 multiples of 7. By the Pigeonhole Principle, it is enough to show that

$$\lfloor 9 \cdot 10^{10}/7 \rfloor \geq \binom{19}{9} \cdot 10^4.$$

We can compute directly that $\binom{19}{9} < 10^5$, after which the claim becomes obvious.

9. Among the n integers less than or equal to n, $d(n)$ counts the number of these that divide n, and $\varphi(n)$ counts those that are relatively prime to n. These can only intersect in 1, because n and a divisor d of n share the common factor of d. Thus, $d(n) + \varphi(n) \geq n + 1$, as each integer other than 1 is counted at most once and 1 is counted at most twice. Therefore, we can only have $c = 0, 1$.

If $c = 1$, we have $d(n) + \varphi(n) = n + 1$, so that every integer less than or equal to n either divides n or is relatively prime to n. We claim that in this case either n is prime or $n = 4$ - we can check that these work since if n is prime, $d(n) = 2$ and $\varphi(n) = n - 1$, as every positive integer less than n is relatively prime to n ($n = 4$ can be checked directly). Now, say $n = pq$, for some $p \geq q > 1$, and $n > 4$, so that $p > 2$. Then, $(p-1)q$ cannot divide $n = pq$, since $p - 1$ does not divide p (as $p > 2$), but it shares a common factor of q with n, so $d(n) + \varphi(n) < n + 1$.

Now, say $c = 0$, so that $d(n) + \varphi(n) = n$. Then, exactly one integer less than n is neither a divisor of n nor relatively prime to n. Assume $n > 16$. Then, n is composite, or else we are back in the $c = 1$ case. Write $n = pq$ with $p \geq q > 1$, so that $p > 4$. But $(p-2)q, (p-1)q$ both are such that they do not divide pq, as $p - 2, p - 1$ cannot divide p, and they share a common factor of q with n. Thus, we must have $n \leq 16$, and we additionally know that n is composite and not equal to 4, so we only need to manually check $n = 6, 9, 10, 12, 14$. Among these, $n = 6, 9$ are the only ones that work, so we're done.

10. The answer is no. Label the squares of the large grid in the following way - start by writing across the top row: $1, 2, 3, \ldots, 12, 1, 2, 3, \ldots, 12, 1, 2$ (assuming we have 62 columns and 66 rows). Then, in the next row, directly underneath, $2, 3, 4, \ldots, 12, 1, \ldots, 1, 2, 3$. Continue in this fashion, so that reading across each row or column in one direction gives $1, 2, \ldots, 12$ in order, cycling all the way across or down. Notice that if we place a 12×1 rectangle on top of this grid, it must cover each of the numbers $1, 2, \ldots, 12$ exactly once. However, this makes it impossible to tile the large grid, if we count the number of times each number appears in the grid. We can cut off any sub-rectangle one of whose dimensions is a multiple of 12, since by symmetry these will have the same number of each number filled in. This allows us to cut off a 60×66 rectangle, then a 2×60 rectangle, leaving the 2×6 rectangle in the bottom left of our original grid. However, we see that this is filled in 123456 in the top row and 234567 in the bottom row, so we cannot have equal occurrences of each number, and thus we cannot tile the grid.

11. Consider dividing the quadrilateral into two triangles with BD. We claim that P must be on BD or AC. Assume for sake of contradiction, without loss of generality, that P is instead inside BCD, and does not lie on AC, and let AP and CP intersect BD at X and Y, respectively. Note that $[BCY][DCY] = [BPY]/[DPY] = BY/YD$. It then follows that $[BCP]/[DCP]$ is equal to this ratio as well, which we can find by writing $[BCY] = [BCP] + [BPY]$ and similarly for $[DCY]$, and cross-multiplying. We know this ratio is just 1, so Y is the midpoint of BD. However, in a similar way, X is the midpoint of BD, so we must have $X = Y$. Thus, CP and AP intersect at this point, at the midpoint of BD, that is, P is the midpoint of BD. Thus, BD bisects the area of $ABCD$, as $[BCD] = [BPC] + [DPC]$.

12. First, notice that without changing the circle, we may move the edges in to a different order, since an edge of a certain length in a given radius circle always takes up the same arc length in the circle. Thus, we can make our hexagon's sides alternate: $1, 2, 1, 2, 1, 2$. Now, construct an equilateral triangle of side length s by drawing the three diagonals that cut off three triangles with side lengths $1, 2, s$. Note that the major arc subtended by the obtuse angle in this triangle contains two four edges, two each of lengths 1 and 2, thus making up $2/3$ of the circle. Therefore, the measure of this obtuse angle is half of this, or $120°$. By the Law of Cosines, we may compute

$$s^2 = 1^2 + 2^2 - 2 \cdot 1 \cdot 2 \cdot \cos 120° = 7,$$

so inscribed in our circle we have an equilateral triangle of side length $\sqrt{7}$. To finish, drop a perpendicular from the center of the circle to a side of this equilateral triangle, and draw a radius to one of the endpoints of this side, giving a 30-60-90 triangle. The long leg has length $\sqrt{7}2$, so we find that the radius of the circle is $\sqrt{7}/\sqrt{3} = \sqrt{21}/3$.

13. The answer is n. We begin with the lemma that given $n+1$ equations $f(a_i) = b_i$, for $i = 1, 2, \ldots, n+1$, then at most one polynomial of degree at most n can satisfy the equations. If two different such polynomials f, g do, then $f(x) - g(x)$ has degree at most n, but it has all of the a_i as roots, which means it must be the zero polynomial (as otherwise it can have at most n roots).

Without loss of generality, say the consecutive integers to which f maps are $0, 1, \ldots, n-1$, which we can do since we can add any positive integral constant to our polynomial. Say $f(a_i) = i$ for $i = 0, 1, \ldots, n-1$. Then, using the fact that $f(a) - f(b)$ is divisible by $a - b$ for any polynomial f with integer coefficients, we see that $f(a_i) - f(a_{i-1}) = 1$ must divide $a_i - a_{i-1}$ for $i = 0, 1, \ldots, n-1$. Since the a_i are distinct, it then follows that $a_0, a_1, \ldots, a_{n-1}$ are also consecutive integers. We may assume that these are equal to $0, 1, \ldots, n-1$, since we can apply appropriate translations and reflections while keeping integer coefficients on f.

Thus, we have $f(0) = 0, f(1) = 1, \ldots, f(n-1) = n-1$. Then, the polynomial $f(x) = x(x-1)(x-2)\cdots(x-(n-1)) + x$, we see, has degree n and satisfies the needed relations, so we can get n consecutive integers. To show that we cannot map to $n+1$ consecutive integers, say that in addition we have $f(n) = n$. Then, the unique polynomial of degree at most n satisfying the $n+1$ relations is $f(x) = x$, so we have no such degree n polynomial, since $n > 1$. This completes the proof.

14. Setting $x = 0$, we get $f(y) = f(f(y))$, so we may replace the $f(f(y))$ in the original equation with $f(y)$, giving $f(x^4 + y) = x^3 f(x) + f(y)$. Next, setting $x = 1$, $y = 0$, we get $f(1) = f(1) + f(0)$, giving $f(0) = 0$. Setting $y = 0$, $f(x^4) = x^3 f(x) + f(0) = x^3 f(x)$, so substituting, $f(x^4 + y) = f(x^4) + f(y)$. Also, instead using $(-x)$, we find that $f((-x)^4) = -x^3 f(-x)$, but also $f(x^4) = x^3 f(x)$, so setting these equal, we find $f(x) = -f(-x)$, that is, f is an odd function.

We have $f(x^4 + y) = f(x^4) + f(y)$ for all x, y, and since x^4 ranges over all non-negative reals, we have $f(a+b) = f(a) + f(b)$ for all a, b with either $a \geq 0$ or $b \geq 0$. Now, if $a, b < 0$, we have $f(a+b) = -f((-a)+(-b) = f(-a)+f(-b) = f(a)+f(b)$, so this additive property holds for all a, b.

We now show that f is injective (one-to-one). Say x, y are distinct real numbers with $f(x) = f(y)$. Then, $f(x) + f(-y) = f(x-y) = 0$. But notice that if if $x \neq y$, we can repeatedly use $f(a) + f(b) = f(a+b)$ to find that $f(k(x-y)) = 0$ for all positive integers k. This contradicts the condition that only finitely many s map to zero under f, so it must be that $x = y$, so that all multiples of $x - y$ are the same. Thus, f is injective.

To finish, we have $f(f(x)) = f(x)$ for all x, so since f is injective, $f(x) = x$. This clearly satisfies both of the initial conditions, so we're done.

15. Consider nine points in a 3-by-3 array, labelled (i, j) for $0 \leq i \leq 2$ and $0 \leq j \leq 2$ in the standard way. Say $f(0,0) = a, f(1,0) = b, f(0,1) = c, f(0,2) = d, f(2,0) = e$. This forces the values of f at the other four points in the array, and we find $f(2,2) = -a + d + e$. Now, adding up f at the four corners of the array, we find $d + e = 0$. We can construct such an array given any two points such that the two points are at opposite corners; this tells us that $f(A) + f(B) = 0$ for any A, B. Then, take three points A, B, C; applying this to all three we get $f(A) = f(B) = f(C) = 0$. Since the choices of these points were arbitrary, $f(P) = 0$ for all P.

16. Without loss of generality, say $m < n$. First, notice that $(a^{2^m} + 2^{2^m})(a^{2^m} - 2^{2^m}) = a^{2^{m+1}} - 2^{2^{m+1}}$. Similarly, this, in turn, divides $a^{2^{m+2}} - 2^{2^{m+2}}$. It follows by induction that $a^{2^m} + a^{2^n}$ divides $a^{2^n} - 2^{2^n}$. Thus,

$$\gcd(a^{2^m} + 2^{2^m}, a^{2^n}, a^{2^n}) \leq \gcd(a^{2^n} - 2^{2^n}, a^{2^n} + 2^{2^n}) = \gcd(a^{2^n} - 2^{2^n}, 2 \cdot 2^{2^n}) = 1,$$

by the Euclidean Algorithm and the fact that a is odd. Thus, our two original integers are relatively prime, as desired.

17. Let the numbers be a, b, c. The first condition tells us $(a+b+c)^2 = a^2+b^2+c^2$, so $ab+bc+ca = 0$. The second tells us that $ab = c^2$ (without loss of generality we can order the variables in this way). Substituting into the previous expression, $c^2 + bc + ca = 0$, so $c(a+b+c) = 0$, and either $c = 0$ or $a+b+c = 0$. In the latter case, we find from the first equation that $a^2+b^2+c^2 = (a+b+c)^2 = 0$, and since all squares are non-negative, we must have $a = b = c = 0$. In the other case, we have $c^2 = 0$, so $ab = 0$, and either $a = 0$ or $b = 0$. We see that $(a, b, c) = (0, b, 0), (a, 0, 0)$ both satisfy the two needed conditions, so we're done.

18. Observe that for all integers k, we find by expanding directly that $a+b+c+d+e+f$ divides the sum $(a+k)(b+k)(c+k)+(d-k)(e-k)(f-k) = (abc+def)+k(ab+bc+ca-de-ef-fd)+k^2(a+b+c+d+e+f)$. Now, this is true in particular for $k = d$, in which $a+b+c+d+e+f$ divides $(a+d)(b+d)(c+d)$. If $a+b+c+d+e+f$ is prime, it either divides $a+d$, $b+d$, or $c+d$, which is impossible since it is strictly larger than each of these. Therefore, $a+b+c+d+e+f$ cannot be prime.

19. A standard computation shows that $m!$ ends in $\lfloor \frac{m}{5} \rfloor + \lfloor \frac{m}{25} \rfloor + \cdots + \lfloor \frac{m}{5^k} \rfloor + \cdots$ zeroes. Note that when when i is a positive integer, $(5i)!, (5i+1)!, (5i+2)!, (5i+3)!, (5i+4)!$ end in the same number of zeroes, since each of the summands remains the same. Furthermore, $(5i+5)!$ has strictly more zeroes than $(5i)!$. Using this information, we can determine how many integers appear as the number of zeroes at the end of $1!, 2!, \ldots, m!$, for some m.

Note that 8055! has 2011 zeroes (we can get this from some trial and error and bounding), as the sum we get is $1611 + 322 + 64 + 12 + 2 = 2010$. The same is true of $8056!, 8057!, 8058!, 8059!$, and 8055! has strictly more zeroes. Now, from 1! to 8060!, we have $8050/5 = 1611$ groups of 5 consecutive integers such that when the factorial is computed, we get the same number of zeroes at the end - furthermore, the number of zeroes is distinct for each group. Therefore, there are exactly 1611 integers between 0 and 2011, inclusive, that can be the number of zeroes at the end of $m!$, leaving 401 (none equal to zero) that cannot, so our answer is 401.

20. First, notice that we need $n \geq 500$, since if $n \leq 499$, we can start by placing $i-1$ balls in B_i, for a total of $0+1+\cdots+999 = 500 \cdot \frac{999}{2}$ balls, then remove balls from whichever boxes we want until we have the right number of total balls. Then, it is impossible to make a move from the beginning, so it is impossible to put n balls in each box.

We now show that this is possible when $n \geq 500$. The idea is that we want to move as many balls as possible into B_1 then re-distribute these one at a time until we have n balls in each box. First, remove as many balls as possible from every box and put them in B_1, remove i balls from B_i repeatedly until fewer than i balls remain in the box. This leaves at most $1+2+\cdots+999 = 500\cdot 999$ balls in boxes B_2 through B_{1000}, meaning we have at least 500 balls in box B_1.

Now, starting with B_2 and continuing one-by-one to B_{501}, add exactly enough balls from B_1 to B_i so that there are exactly i balls in B_i (unless B_i is already empty, in which we skip it and move on), then empty B_i in to B_1. This is always possible because at each step we get at least as many balls in B_1 as we had before, and we need to add at most $i-1 \leq 500$ balls at any point. After this, there are at most $501 + 502 + \cdots + 999$ balls in boxes B_2 through B_{1000}, since we have emptied $B_2, \ldots, B_{501}$, leaving much more than 1000 in B_1. Now, we can empty the rest of the boxes in a similar way to before; if the box B_i is not empty, put i balls from B_1 in it, then empty back in to B_1. This puts every ball in B_1, and from here, we can redistribute in the way we need, as desired.

21. By Power of a Point, $ED = (BE)(EC)/AE$, and we know AE, so to compute ED (which will give us AD by adding to AE), we want $(BE)(EC)$. Drop a perpendicular AH to BC, so that H is the midpoint of BC. Let $EH = x$, and $HC = y$ and without loss of generality say $BE < BH$; we wish to compute $(y-x)(y+x) = y^2-x^2$. From right triangle AEH, $AH^2 = AE^2 - EH^2 = 64 - x^2$, so by right triangle AHC, $HC^2 = AC^2 - AH^2$, giving $y^2 = 144 - (64 - x^2)$. Rearranging gives $y^2 - x^2 = 80$. Therefore, $ED = (BE)(EC)/AE = 80/8 = 10$, so $AD = AE + ED = 18$.

22. By the sine double angle formula, $\sin x \cos x = \frac{1}{2}\sin 2x \leq \frac{1}{2}$, giving a maximum of 1/2 which is achievable when $x = \frac{\pi}{4}$.

23. Use the prime factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$. The sum of the divisors of n is then $(1 + p_1 + \cdots + p_1^{e_1}) \cdots (1 + p_n + \cdots + p_n^{e_n})$, and each factor must separately be a power of 2. However, say $e_1 + 1$ has an odd factor k. Then, we see that as there are $e_1 + 1$ addends in the sum $1 + p_1 + \cdots + p_1^{e_1}$, we have that $1 + p_1 + \cdots + p_1^{k-1}$ will be a factor, and it, too, must be a power of 2. However, if $p = 2$, then this factor must be odd, and if p is odd, we have k odd addends, and since k is odd, the factor is still odd. Thus, we need $k = 1$, or else we have an odd factor of a power of 2 other than 1. This means that $e_1 + 1$ only has 1 as an odd factor, and it must be a power of 2. Similarly, $e_i + 1$ is a power of 2 for all I. To finish, the number of factors of n is $(e_1 + 1)(e_2 + 1) \cdots (e_n + 1)$, which is a power of 2, so we're done.

24. Since all of our entries are non-negative, they must all be either 1, 2, or 3. We divide into cases for whether or not we include threes in the grid; we then divide into further subcases for the number of twos in the grid.

First, say we use no threes, and furthermore, no twos. Then, we have three ones in each row and column, or equivalently one zero in each row and column. There are $4! = 24$ ways to do this, as starting from the top row, we decide where to place the zero, which eliminates one choice for the second row, and after we choose one of the remaining three boxes, we have two choices left for the third row, etc. Next, say we have exactly one two, and we have 16 choices for where to put it. Then, its row and column have a one, so we have $3 \cdot 3 = 9$ more choices for these. By symmetry we can assume these are all in the top left corner of the grid, and we see that if we use only ones for the rest of the board, their positions are forced, giving $16 \cdot 9 = 144$ configurations. Next, if we have 2 twos, we have $16 \cdot 9/2 = 72$ choices for where to put them, as we cannot put them in the same row or column. We see that if we place a one that is in the same row or column as both of the twos, we cannot fill in the rest of the board. Without loss of generality, put $2, 0$ at the beginning of the top row and $0, 2$ at the beginning of the second. We then see that the choices of ones in the first row and column determine the rest of the board, so we have $72 \cdot 4 = 288$ configurations here.

If we have three twos, we have $16 \cdot 9 \cdot 4/6 = 96$ choices for where to put them. Without loss of generality, put them in the first three entries down the main diagonal. If we put a zero in the bottom right, the positions of the last six ones are forced. If we instead put a 1 in the bottom right, we then can decide where to put the zero in each of the fourth row and column, so we have $96 \cdot 9 = 864$ configurations here. If we have four twos, there are $4! = 24$ ways to determine where to put them. Then, to place the ones, we either pair the twos and form two 'squares' with two twos and two ones, or we cycle through the twos in some order, putting a 1 in the row of the next 2 of the cycle. There are then $3 + 3! = 9$ ways to place the ones, so $24 \cdot 9 = 216$ configurations here.

Now, consider what happens when we use threes. We either have one, two, or four of them; it is not hard to see that having exactly three is impossible. If we have four, we simply have $4! = 24$ ways to place them. With two, we have $16 \cdot 9/2 = 72$ ways to place them, then the remaining two rows and columns intersect to form a 2-by-2 'square' in which we must place a $2, 1$ row and a $1, 2$, so we have $72 \cdot 2 = 144$ configurations here. Finally, if we have exactly one three, which we can place in 16 places, the remaining 3-by-3 grid can be filled with ones, or a one and a two can be placed in each row and column. There are $3! = 6$ ways to place the twos, then only two ways to place the remaining ones, for $16(1 + 12) = 208$ configurations.

Adding everything up from all of our cases, our answer is $24 + 144 + 288 + 96 + 864 + 216 + 24 + 144 + 208 = 2008$.

25. Call the circles ω_1 and ω_2, so that M lies on ω_2 and X, Y lie on ω_1. Let l_1, l_2 be the tangents to ω_1 at A. The (directed) angle between l_1 and AY is just $\angle ABY$, from facts about angles in circles (this is the case in which we have a tangent and a secant intersecting outside the circle), and similarly, the angle between l_2 and AX is $\angle ABM$. But from ω_1, $\angle ABM = \angle ABY = \angle AXY$, so in fact the (directed) angles between l_1 and AY and between l_2 and AX are equal; thus $\angle YAX$ is equal to the angle between l_1 and l_2, which is just $90°$. Thus, $\widehat{YX}$ subtends a 180 degree arc, and it must be a diameter of ω_1.

26. The answer is $n/2$ if n is even and $(n + 1)/2$ if n is odd (that is, $\lfloor (n + 1)/2 \rfloor$). In the case of n even, we make the partition $\{1, n\}, \{2, n - 1\}, \ldots, \{n/2, n/2 + 1\}$, and in the case of n odd, we make the partition $\{1, n - 1\}, \{2, n - 2\}, \ldots, \{(n - 1)/2, (n + 1)/2\}, \{n\}$, to show that these are indeed achievable. To show that we cannot do any better, notice that at most one box may contain only one integer, or else we cannot make the sums the same. Thus, at least $k - 1$ of our k boxes must contain at least two integers, and the last must contain at least one, so that

$n \geq 2(k-1)+1 = 2k-1$. Thus, $k \leq (n+1)/2$, and since k is an integer, $k \leq \lfloor (n+1)/2 \rfloor$.

27. To evaluate the infinite product, we first evaluate the product for finitely many terms, then evaluate the limit of this product as we take the number of terms to infinity. We rewrite the product as

$$\prod_{n=2}^{k} \frac{n^3-1}{n^3+1} = \prod_{n=2}^{k} \frac{n-1}{n+1} \cdot \prod_{n=2}^{k} \frac{n^2+n+1}{n^2-n+1} = \prod_{n=2}^{k} \frac{n-1}{n+1} \cdot \prod_{n=2}^{k} \frac{(n+1)^2-(n+1)+1}{n^2-n+1}.$$

In the first product, everything cancels upon expansion except the first two numerators and last two denominators, giving $\frac{2}{k(k+1)}$. In the second, everything cancels except the first denomiantor and last numerator (because the i-th numerator is the same as the $(i+1)$-st denominator), giving $\frac{k^2+k+2}{3}$. Thus, the value of the product we want is $\frac{2(k^2+k+1)}{3k(k+1)}$, which tends to $2/3$ as $k \to \infty$ (we can see this by dividing top and bottom by k^2).

28. Let the triangle be ABC, and let P be inside ABC. Letting the side length be s, draw AP, BP, CP; we can compute the area of ABC in two ways. First, since the triangle is equilateral with side length s, we have that it is equal to $s^2\sqrt{3}/4$. Then, we have split ABC into three smaller triangles, whose bases each measure s and whose altitudes measure $5, 12, 13$. Thus, the total area is $(5+12+13)s/2 = 15s = s^2\sqrt{3}/4$. Solving for s, we get $s = 60/\sqrt{3} = 20\sqrt{3}$.

29. Let $p_1, p_2, \ldots, p_{2011}$ be distinct primes. By the Chinese Remainder Theorem, there exists an integer x such that $x \equiv n \pmod{p_n^3}$, so that p_n^3 divides $x-n$. Then, simply take $x-2011, x-2010, \ldots, x-1$ to be our integers.

30. Without loss of generality, say that $p \geq q \geq r \geq s$. We first prove that $p+q \geq 5$, and so $r+s \leq 4$. Assume instead that $p+q = 5-x$ for some fixed $x > 0$, and $x \leq 1/2$ since $p+q \geq r+s$. We claim that $p^2+q^2+r^2+s^2 < 21$, which will give a contradiction; we will compute the maximum value of this expression. First, take q to be fixed. Then, to maximize $p^2+q^2+r^2+s^2$, we seek to maximize $r^2+s^2 = r^2+(4+x-r)^2$, as q is fixed, so p is also fixed (since x is fixed). This is a quadratic that opens upward, so the maximum occurs at the end-points of our domain for r, which is $(4+x)/2 \leq r \leq q$, as $q \geq r \geq s$ and $r+s = 4+x$. Noting that $(4+x)/2$ in fact gives the vertex of the quadratic, we want $r = q$, so $s = 4+x-q$. Now, we have

$$p^2+q^2+r^2+s^2 \leq (5-x-q)^2+q^2+q^2+(4+x-q)^2 = 4q^2-18q+(x^2-2x+41).$$

Note that $5-x-q \geq q \geq 4+x-q$, so $\frac{4+x}{2} \leq q \leq \frac{5-x}{2}$. This is again a quadratic in q that opens upward, with axis of symmetry at $q = -\frac{9x}{2}$, which is halfway between the two endpoints of our domain. Thus, this function is maximized at both of these endpoints, giving the value

$$4q^2+18xq+41 = 4(\frac{4+x}{2})^2 - 18(\frac{4+x}{2}) + (x^2-2x+41) = x^2-x+21 = x(x-1)+21 < 21,$$

since $0 < x \leq \frac{1}{2}$. We have reached our contradiction, so $p+q \geq 5$ and $r+s \leq 4$. To finish, $2(pq-rs) = (p+q)^2+(r-s)^2-(p^2+q^2+r^2+s^2) \geq 25+0-21 = 4$, giving $pq-rs \geq 2$.