LHS Math Team
Team Contest 1

1. The set S consists of five integers. If pairs of distinct elements of S are added, the following ten sums are obtained: 1967, 1972, 1973, 1974, 1975, 1980, 1983, 1984, 1989, 1991. What are the elements of S?

2. A quadrilateral inscribed in a circle has sides with lengths $\sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}$. Find the area of the circle.

3. It is known that the number of real solutions of the system

$$(y^2+6)(x-1) = y(x^2+1)$$
$$(x^2+6)(y-1) = x(y^2+1)$$

   is finite. Prove that this system has an even number of solutions.

4. The positive integer number n has 1994 digits. 14 of its digits are 0's and the number of times that the other digits: $1, 2, 3, 4, 5, 6, 7, 8, 9$ appear are in proportion $1:2:3:4:5:6:7:8:9$, respectively. Prove that n is not a perfect square.

5. Let $A_0B_0C_0$ be a triangle, and for positive integers i, let A_i be the incenter of triangle $A_{i-1}B_0C_0$. Define B_i and C_i similarly. Prove that for all positive integers i, AA_i, BB_i, and CC_i are concurrent.

6. Two of the squares of a 5×5 checkerboard are painted yellow, and the rest are painted green. Two color schemes are equivalent if one can be obtained from the other by applying a rotation in the plane of the board. How many inequivalent color schemes are possible?

7. Prove that for every positive integer n, there exists integers m and k such that $m^3 = 11^n k + 5$.

8. On a cube, 27 points are marked in the following manner: one point in each corner, one point on the middle of each edge, one point on the middle of each face, and one in the middle of the cube. How many lines contain three of these points?

9. Circles of radii 3 and 6 are externally tangent to each other and are internally tangent to a circle of radius 9. The circle of radius 9 has a chord that is a common external tangent of the other two circles. Find the square of the length of this chord.

10. Given are n integers. Prove that there is some nonempty subset of them whose sum is divisible by n.

11. Find, with proof, all prime numbers p that divide $2^p + 1$.

12. Let $ABCD$ be a square, and let E, F, G, H be points outside the square such that ABE, BCF, CDG, and DAH are all equilateral triangles. Compute the ratio of the area of $ABCD$ to the area of $EFGH$.

13. A chord ST of constant length slides around a semicricle with diameter AB. M is the midpoint of ST and P is the foot of the perpendicular from S to AB. Prove that angle SPM is constant for all positions of ST.

14. Let p be a prime. Prove that there exists a positive integer n such that p divides $2^n + 3^n + 6^n - 1$.

15. Let n be a 6-digit number, and let q and r be the quotient and remainder, respectively, when n is divided by 2010. For how many values of n is $q + r$ divisible by 7?

16. Let $t \geq 3$ be a real number, and let P be a polynomial of degree n. Prove that there exists a nonnegative integer $k \leq n+1$ with $|P(k) - t^k| \geq 1$.

17. Let n and m be given positive integers. In one move, a chess piece called an (n, m)-crocodile goes n squares horizontally or vertically and then goes m squares in a perpendicular direction. Prove that the squares of an infinite chessboard can be painted in black and white so that this chess piece always moves from a black square to a white one or vice-versa.

18. Let A be an angle with $\sin 2A = 21/25$ and $\cos A > \sin A$. Compute $\cos A - \sin A$.

19. Find all positive integers n such that $20n + 10$ divides $2011n + 2010$.

20. In triangle ABC, $AB = 14$, $BC = 16$, and $CA = 26$. Let M be the midpoint of $\overline{BC}$, and let D be a point on $\overline{BC}$ such that $\overline{AD}$ bisects $\angle BAC$. Compute PM, where P is the foot of the perpendicular from B to line AD.

21. Is $4\sqrt{4-2\sqrt{3}} - \sqrt{97-56\sqrt{3}}$ an integer?

22. Show that there exists an equiangular hexagon in the plane, whose sides measure 5, 8, 11, 14, 23, and 29 units in some order.

23. Let $a_1, ..., a_n$ be nonnegative integers, all less than or equal to m. For each $j = 1, ..., m$, let b_j be the number of values of i for which $a_i \geq j$. Prove that

$$a_1 + \cdots + a_n = b_1 + \cdots + b_m.$$

24. Let p be an odd prime and let q be a prime divisor of $M_p = 2^p - 1$. Prove that $q = 2kp + 1$ for some positive integer k.

25. Suppose that each square of a $4 \times n$ chessboard is colored either black or white. Find the smallest positive integer n for which with any such coloring, the board must contain a rectangle (formed by the horizontal and vertical lines of the board) whose four distinct unit corner squares are all of the same color.

26. In the exterior of an equilateral triangle ABC of side length α we construct an isosceles right-angled triangle ACD with $\angle CAD = 90°$. The lines DA and CB meet at point E.

    (a) Find the angle $\angle DBC$.

    (b) Express the area of triangle CDE in terms of α.

    (c) Find the length of BD.

27. At a party, no five people are all friends with each other on Facebook, and given any two groups of three people, such that within each group all three people are all Facebook friends, the two groups share a person. Prove that there are two people at the party such that upon leaving, there are no longer groups of three people, all of whom are Facebook friends with each other.

28. Determine the number of 8-tuples of nonnegative integers $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ satisfying $0 \leq a_k \leq k$, for each $k = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + a_4 + 2b_1 + 3b_2 + 4b_3 + 5b_4 = 19$.

29. Determine all strictly increasing functions f from positive integers to positive integers such that $nf(f(n)) = f(n)^2$ for all positive integers n.

30. If α is a root of $x^3 - x - 1 = 0$, compute the value of

$$\alpha^{10} + 2\alpha^8 - \alpha^7 - 3\alpha^6 - 3\alpha^5 + 4\alpha^4 + 2\alpha^3 - 4\alpha^2 - 6\alpha - 17.$$

31. A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily from the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state of the lamps there exists a sequence of changes in some of the switches at the end of which each room contains lamps which are on as well as lamps which are off.

32. Let ABC be a triangle. Point M and N lie on sides AC and BC respectively such that $MN || AB$. Points P and Q lie on sides AB and CB respectively such that $PQ || AC$. The incircle of triangle CMN touches segment AC at E. The incircle of triangle BPQ touches segment AB at F. Line EN and AB meet at R, and lines FQ and AC meet at S. Given that $AE = AF$, prove that the incenter of triangle AEF lies on the incircle of triangle ARS.

33. Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \geq 1$, where T^n denotes the n-fold application of T.

34. The equation $x^3 - 4x^2 + 5x - 1.9$ has real roots r, s, t. A box has sides of length r, s, t. Find the length of the long diagonal of the box.

35. A student of the National Technical University was reading advanced mathematics last summer for 37 days according to the following rules:

    (a) He was reading at least one hour every day.

    (b) He was reading an integer number of hours, but not more than 12, each day.

    (c) He had to read at most 60 hours in total.

    Prove that there were some successive days during which the student was reading exactly 13 hours in total.

36. Determine the last three digits of the number

$$2003^{2002^{2001}}.$$