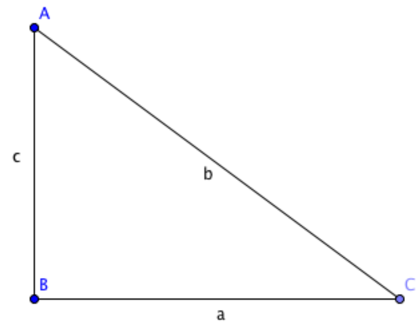
Right Triangle Trigonometry
LHS Math Team

Trigonometry is the study of triangles, and to start, we consider only right triangles, which are the simplest to work with.

We'll begin with a few definitions. Consider the following right triangle ΔABC, with $\angle ABC = 90°$.



Let $BC = a, CA = b, AB = c$, and $\angle CAB = A, \angle ABC = B, \angle BCA = C$ (note: this is standard notation that we will adopt throughout the document). We define the following:

$$\sin A = \frac{a}{b};$$
$$\cos A = \frac{c}{b};$$
$$\tan A = \frac{\sin A}{\cos A} = \frac{a}{c},$$

where $\sin, \cos, \tan$ are abbreviations of *sine, cosine, tangent*, respectively. Note that these definitions are equivalent to the following:

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}};$$
$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}};$$
$$\tan A = \frac{\text{opposite}}{\text{adjacent}},$$

where BC is the side *opposite* A, AB is the side *adjacent* to A (the hypotenuse is also adjacent, but we take the leg of the right triangle), and CA is hypotenuse. A good way to remember the above is using the pneumonic *soh-cah-toa*, using the first letters of the function, the numerator, and denominator in a row for all three functions.

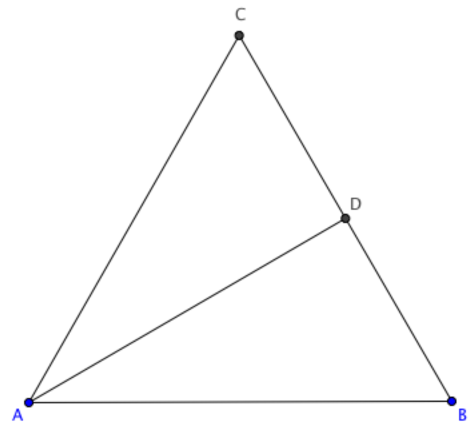
Let's say we were to repeat this, but with $C = 90° - A$. Applying the initial definition to this angle, we have

$$\sin(90° - A) = \frac{c}{b} = \cos A;$$
$$\cos(90° - A) = \frac{a}{b} = \sin A;$$
$$\tan(90° - A) = \frac{c}{a} = \frac{1}{\tan A}.$$

Now, we have our first three trig identities; $\sin(90° - A) = \cos A$, $\cos(90° - A) = \sin A$, and $\tan(90° - A) = \dfrac{1}{\tan A}$. Note that we've proved these only for acute angles A, but they turn out to be true for all angles A. However, we don't yet know exactly what it means to have the sine, cosine, or tangent of a non-acute angle A; we'll get to this later.

Example: Find $\sin 30°$, $\cos 30°$, and $\tan 30°$.
Solution: Obviously, we are dealing with a right triangle with one angle of $30°$, which is just a familiar 30-60-90 triangle. We already know the side ratios in such a triangle, but we'd like to know exactly where they come from.

We'll start with an equilateral triangle ΔABC of side length x, and draw a perpendicular from A to BC, meeting BC at D, as shown below.



Since the triangle is equilateral, all three angles are equal to $60°$. Also, $\angle BAD = \angle CAD = 30°$, and $BD = DC = \frac{x}{2}$. Now, we have that ΔABD is our desired 30-60-90 triangle. We can use our definition for sine:

$$\sin 30° = \frac{BD}{AB} = \frac{x/2}{x} = \frac{1}{2}.$$

Getting the cosine requires us to find AD. By the Pythagorean Theorem, we have

$$AD^2 = AB^2 - BD^2 = x^2 - \frac{x^2}{4} = \frac{3x^2}{4};$$
$$\Rightarrow AD = \frac{\sqrt{3}}{2}x.$$

Now, we can use our definitions for cosine and tangent:

$$\cos 30° = \frac{AD}{AB} = \frac{\sqrt{3}x/2}{x} = \frac{\sqrt{3}}{2};$$
$$\tan 30° = \frac{BD}{AD} = \frac{x/2}{\sqrt{3}x/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Exercise: Determine $\sin 45°, \cos 45°, \tan 45°$.

We'll move on to what is probably the most important trig identity.

Theorem (Pythagorean Identity). For all angles A, $(\sin A)^2 + (\cos A)^2 = 1$.
Proof. We'll only prove it for acute angles, but it turns out to be true for all angles A (again, we'll get to other angles later). Consider a right triangle ΔABC with the right angle at C. Then, applying our definitions for sine and cosine, we have

$$\begin{aligned}
&(\sin A)^2 + (\cos A)^2; \\
&= \frac{a^2}{c^2} + \frac{b^2}{c^2}; \\
&= \frac{a^2 + b^2}{c^2}.
\end{aligned}$$

Now, by the Pythagorean Theorem, $a^2 + b^2 = c^2$, so the above fraction is just equal to 1, which is what we wanted.

A quick note about notation: $(\sin A)^2$ is often denoted $\sin^2 A$, and similarly with other positive powers of trig functions. It is important to note that this is not equal to $\sin A^2$. If that wasn't confusing enough, $(\sin A)^{-1}$ is not the same as $\sin^{-1} A$, we'll see what $\sin^{-1} A$ means in section 4.

We'll conclude with three more definitions:

$$\begin{aligned}
\csc A &= \frac{1}{\sin A}; \\
\sec A &= \frac{1}{\cos A}; \\
\cot A &= \frac{1}{\tan A},
\end{aligned}$$

where $\csc, \sec, \cot$ are abbreviations of *cosecant, secant, cotangent*, respectively. In reality, these three functions are essentially useless, but you will still need to know what they are because they show up in many problem statements.

A useful corollary of the Pythagorean Identity is that $\tan^2 A + 1 = \sec^2 A$; we can see this by writing tan and sec in terms of sin and cos and multiplying both sides by $\cos^2 A$.