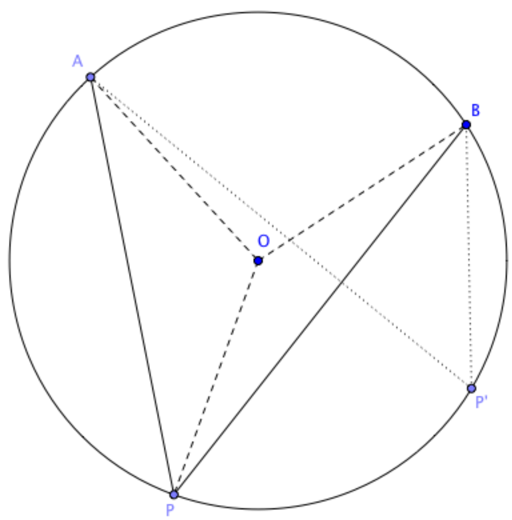
Circles
LHS Math Team

1 Angles in Circles

Given two points A and B on a circle with center O, we define the measure of arc AB, denoted $\widehat{AB}$, to be equal to the measure of $\angle AOB$. Note that we can either take the acute/obtuse angle or the reflex angle; in other words we can make $\widehat{AB}$ less than 180° or more than 180°. If $\widehat{AB} < 180^\circ$, we say that it is a *minor* arc, and if it is more, we say that it is a *major* arc.

Theorem: Given three points A, B, P on a circle with center O, and C and O on the same side of segment AB, then $\widehat{AB} = 2\angle APB$.



Proof: We'll consider the case in which O is contained in ΔABP. The other cases are similar. Draw AO, BO, PO. Note that $\Delta OAB, \Delta OAP, \Delta OBP$ are all isosceles, since $OA = OB = OP$. Let $\angle APO = a$ and $\angle BPO = b$ so that $\angle APB = a + b$. We want to show that $\angle AOB = 2(a + b)$. Since $OA = OP$, $\angle OAP = \angle OPA = a$. Now, using the fact that the angles in ΔOAP sum to 180°, $\angle AOP = 180^\circ - (\angle OAP + \angle OPA) = 180^\circ - 2a$. In a similar way, we can find that $\angle BOP = 180^\circ - 2b$. Now, $\angle AOB = 360^\circ - (\angle AOP + \angle BOP) = 2a + 2b$, so we're done (for this case).

Exercise: Finish the proof, for the cases when O is outside the angle, or one one of the sides.

It follows easily that if we keep moving P along the circle so that it stays on the same side of AB, $\angle APB$ will stay the same, since it's still equal to half of $\widehat{AB}$. In other words, given two points P, P' on the same side of AB, $\angle AP'B = \angle APB$.

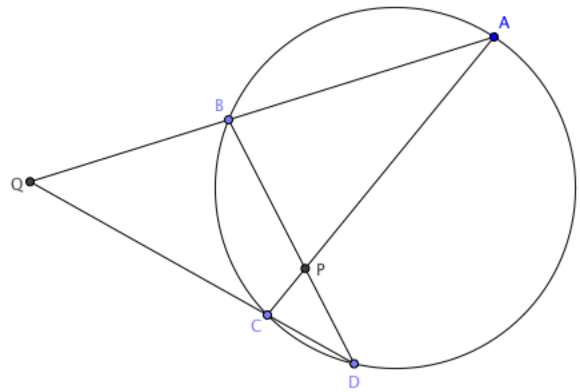
The above is essentially all you need to know about circles to do anything with them, in any problem you find. The following are useful corollaries:

1. Let A, B, C, D be 4 points on a circle, in that order. Then, $\angle ABC + \angle ADC = 180^\circ$.
2. In the same configuration as before, let AC and BD intersect at a point P inside the circle. Show that $\angle APD = \dfrac{\widehat{AD} + \widehat{BC}}{2}$, where we take the $\widehat{AD}$ so that it does not contain the other two

points, and similarly for $\widehat{BC}$.
3. With the same configuration as in Example 2, let rays AB and DC intersect outside the circle at Q. Then, $\angle APB = \dfrac{\widehat{AD} - \widehat{BC}}{2}$.

Try proving the above facts: if you get stuck, see the solutions to Fun With Proofs, October 2009.

2 Power of a Point



Theorem (Power of a Point, Part 1): Let A, B, C, D be four points on a circle, in that order, such that AC and BD meet inside the circle at P. Then, $(PA)(PC) = (PB)(PD)$.

Proof. Note that $\angle ABP = \angle ABD = \angle ACD = \angle PCD$, from the previous section, and $\angle APB = \angle DPC$ by vertical angles. Thus, by AA similarity, $\Delta APB \sim \Delta DPC$. Using the similar triangles, $\dfrac{PA}{PB} = \dfrac{PD}{PC}$; cross-multiplying gives the desired result.

Theorem (Power of a Point, part 2): In the same configuration as before, let rays AB and DC meet at a point Q outside the circle, Then, $(QA)(QB) = (QD)(QC)$ [note: do not confuse this with $(QB)(BA) = (QC)(CD)$, which is not true].

Proof. We have $\angle DAQ = \angle DAB = 180° - \angle BCD = \angle BCQ$. Similarly, $\angle ADQ = \angle CBQ$. Thus, $\Delta QBC \sim \Delta QDA$, and $\dfrac{QC}{QB} = \dfrac{QA}{QD}$. Cross mutiplying gives the desired result.

The use of Power of a Point is fairly straightforward; we can use it to either compute lengths directly, or solve for lengths with algebra.

3 Tangents

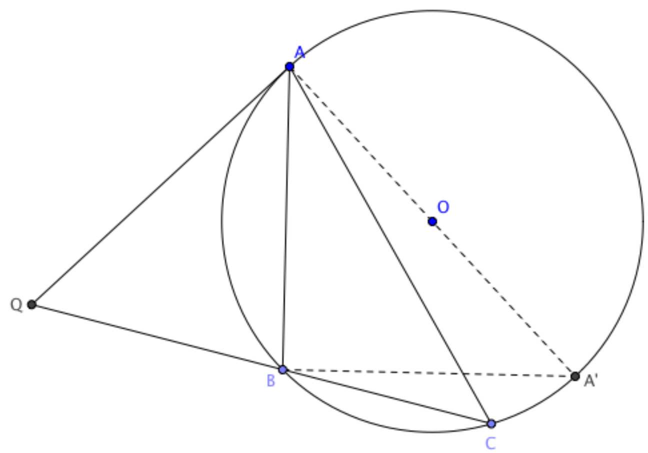
Theorem: Given a circle centered at O and a point A on the circle, let the tangent to the circle at A be l. Then, $l \perp OA$.

Proof: We use a proof by contradiction; we'll assume the opposite of what we're trying to prove, and show that this assumption leads to the something that is clearly impossible. This will show that the assumption is false, and the statement is true.

Assume that l and OA are not perpendicular. Then, drop a perpendicular from O to l; let the foot of the perpendicular be P. Then, P must lie outside the circle, because if it was on the circle, l would contain two points on the circle and not be a tangent, and if it was inside the circle, we would be able to extend AP to another point on the circle, and l would again not be a tangent. Let segment OP meet the circle at P'. Then, $OA = OP' < OP$. But note that ΔOAP is a right triangle with OA as the hypotenuse, so $OA > OP$ (because $OA^2 = OP^2 + AP^2 > OP^2$). This means that $OA < OP$ and $OA > OP$ at the same time, which is impossible. Thus, the assumption is false and l and OA must always be perpendicular.

*Note that this immediately shows that l is unique, because there can only be one line through A and perpendicular to A.

Theorem (Power of a Point, part 3): Let A, B, C be three points on a circle such that line BC and the tangent at A meet at P, with B between P and C. Then, $(PA)^2 = (PB)(PC)$.



Proof. Let the center of the circle be O. Extend AO to a second point A' on the circle (in other words, A, A' are diametrically opposite). Note that $\angle ABA' = \dfrac{\widehat{AA'}}{2} = 90°$. Thus, by $\Delta ABA'$, $\angle BAA' + \angle BA'A = 180° - \angle ABA' = 90°$. Also, from the previous theorem, $\angle PAA' = 90°$, since PA is a tangent. Now, $\angle ACP = \angle ACB = \angle AA'B = 90° - \angle BAA' = \angle PAB$. Also, $\angle APB = \angle CPA$. Thus, by AA similarity, $\Delta APB \sim \Delta CPA$, and $\dfrac{PA}{PB} = \dfrac{PC}{PA}$. Cross-multiplying gives the desired result.

Notice that this is sort of a 'limit case' of the second part of the theorem from the previous section. we can think of the tangent PA as actually intersecting the circle in two points, A and A. Of course, we need the above to establish this rigorously.

Theorem (Equal Tangents). Let A and B be two points on a circle, and let the tangents to A and B meet at P. Then, $PA = PB$.

Proof. Draw an arbitrary line through P that intersects the circle at two different points C and D. By Power of a Point, $(PA)^2 = (PC)(PD) = (PB)^2$, so $PA = PB$ (as lengths are positive).