LHS Math Team 2013 "Tryout"

1. Find the decimal expansion of $\frac{12}{35}$.

2. Zach can write a problem set with 20 questions in one hour, while Rohil can write one in 45 minutes. How long (in minutes) will it take Zach and Rohil to finish the problem set if they work together?

3. What is the tens digit of 41^{41}?

4. A circle D has three circles A, B, and C all internally tangent to it. If A, B, and C all are externally tangent to each other and have radius 1, what is the radius of D?

5. There is a complex number z with imaginary part 164 and a positive integer n such that
$$\frac{z}{z+n} = 4i.$$
Find n.

6. How many ways are there to fill a 4 by 4 grid with the numbers 1, 2, 3, 4, so that each row and each column has exactly one of each number?

7. The 'taxicab distance' between two points (x_1, y_1) and (x_2, y_2) is defined to be $|x_1 - x_2| + |y_1 - y_2|$. A cab driver starts at the origin $(0, 0)$ and drives to the point (a, b). If the driver is now 4 units away from the origin, what is the maximum taxicab distance he could have traveled?

8. You are given four distinct points A, B, C, D, not all collinear, which have the property that any circle through A and B has at least one point in common with any circle through C and D. Given that $\angle ACD = 30°$, find the value of $\angle ABD + \angle ADB + \angle ACB$.

9. Bulgarian Solitaire is a game played on a stack of 6 blocks initially split into k piles $\lambda_1, \lambda_2 \ldots \lambda_k$. Every turn, an operation is performed. In this operation, one block is taken from each pile and used to make a new pile. An initial configuration of blocks split into piles is called 'stable' if at some point during the game the configuration of piles is invariant under the operation. For example, given the configuration $(1, 2, 3)$, we find that taking one block from each pile and making a new pile gives us $(1, 2, 3)$ again. How many stable initial configurations are there?

10. Find the largest integer x such that $x^2 + 1 \mid x^3 + x - 1000$.

11. DNA sequences are long strings of A, T, C, and G, called base pairs. (e.g. AATGCA) is a DNA sequence of 6 base pairs). A DNA sequence is called *stunningly nondescript* if it contains each of A, T, C, G, in some order, in 4 consecutive base pairs somewhere in the sequence. Find the number of stunningly nondescript DNA sequences of 6 base pairs (the example above is to be included in this count).

12. You are given concave hexagon $ABCDEF$. You are given that $AD = AB + DE = CD + AF = 20$, $\angle BAF = \angle ADC = \angle ADE = 120$ degrees, and that $\angle BAD = \angle DAF$. Find the area of $ABCDEF$.

13. Notice that September 13, 2013 falls on a Friday. How many days before did the previous Friday the thirteenth fall?

14. Find $3x^2y^2$ if x and y are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$.

15. A triangle with sides a, b, c and circumradius R satisfies $R(b + c) = a\sqrt{bc}$. If $b = 10$, find the perimeter of the triangle.

16. Consider a 2014 by 2014 square grid, which is divided into 2014^2 unit squares. N unit squares on the board are marked in such a way that every square, either unmarked or marked, on the board, is adjacent to at least one marked square. Find the smallest possible value of N.

17. In parallelogram ABCD, point M is on line AB so that $\frac{AM}{AB} = \frac{17}{1000}$ and point N is on line AD so that $\frac{AN}{AD} = \frac{17}{2009}$. Let P be the intersection of lines AC and MN. Find $\frac{AC}{AP}$.

18. Given a 7 by 7 grid, the centers of k of the 49 squares are chosen. No four of the chosen points form the vertices of a rectangle whose sides are parallel to the sides of the grid. Find the largest value of k for which this is possible.

19. The system of equations

$$\log_{10}(2000xy) - (\log_{10} x)(\log_{10} y) = 4$$

$$\log_{10}(2yz) - (\log_{10} y)(\log_{10} z) = 1$$

$$\log_{10}(zx) - (\log_{10} z)(\log_{10} x) = 0$$

has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2). Find $y_1 + y_2$.

20. Sherry starts at the number 1. Whenever she's at 1, she moves one step up (to 2). Whenever she's at a number between strictly 1 and 10, she moves one step up or one step down, each with probability $\frac{1}{2}$. When she reaches 10, she stops. What is the expected number (average number) of steps that Sherry will take?