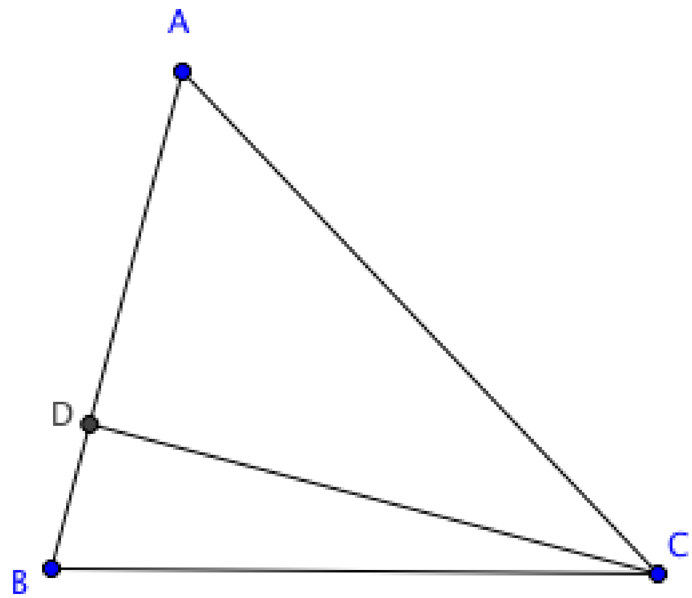
Laws of Sines and Cosines

LHS Math Team

The Laws of Sines and Cosines allow us to use trig in geometry problems. Specifically, a set of standard facts of geometry are the congruence theorems. For example, if we know the lengths of the three sides of a triangle, the rest of the triangle is *determined*, that is, there's only one set of angles that can fit in this triangle. The Laws of Sines and Cosines allow us to compute these angles directly. Here, we'll state and prove these two theorems, and leave the details of how exactly to do these computations to you.

Theorem (Sine Area Formula): Let ΔABC be a triangle. Then, its area is $\frac{1}{2}bc\sin A$ (using standard notation, see Right Triangle Trig).



Proof: Let the perpendicular from C meet AB at D. Note that $\sin A = \frac{CD}{CA} = \frac{CD}{b}$, so $CD = b\sin A$. Now, the area of the triangle is $\frac{(CD)(AB)}{2}$, since CD is a height, which is $\frac{1}{2}bc\sin A$.

Exercise: What happens when the perpendicular CD hits line AB outside the triangle?

Theorem (Law of Sines): In a triangle ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

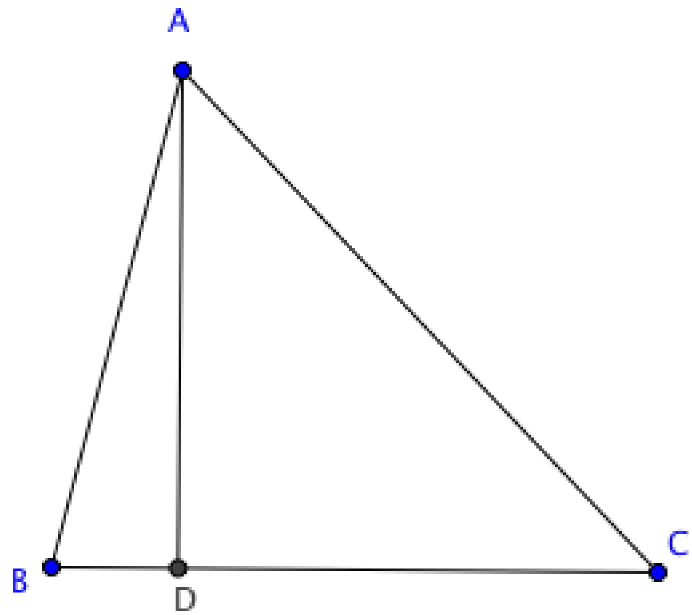
Proof: We know from the Sine Area Formula that the area is $\frac{1}{2}bc\sin A$. However, we can, in exactly the same way, get that the area is also $\frac{1}{2}ca\sin B$. Equating these gives $\frac{a}{\sin A} = \frac{b}{\sin B}$, and in the same way we can get that this is also equal to $\frac{c}{\sin C}$.

A stronger version of the Law of Sines exists, sometimes known as the Extended Law of Sines. Proving it requires some facts about circles that we'll see later, so the proof is omitted here. But it says that the three fractions that we just proved equal are also equal to $2R$, where R is the circumradius, or the radius of the unique circle passing through A, B, and C.

Theorem (Law of Cosines). Let ΔABC be any triangle. Using standard notation, we have

$$a^2 = b^2 + c^2 - 2bc\cos A;$$
$$b^2 = c^2 + a^2 - 2ca\cos B;$$
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Proof. Note that all three statements above are equivalent; we just change the roles of A, B, C to get from one to another (for example, make A into B, B into C, C into A, then apply the statement we've already established to the new triangle), so we only have to prove one of them. Draw a perpendicular from A to BC, so that it meets BC at D.



By right triangle ΔABC, $BD = AB\cos B = c\cos B$, and $AD = c\sin B$. Now, $DC = BC - BD = a - c\cos B$. Also, $AC = b$. Applying the Pythagorean Theorem to ΔADC, we have

$$AD^2 + DC^2 = AC^2;$$
$$\Leftrightarrow c^2\sin^2 B + (a - c\cos B)^2 = b^2;$$
$$\Leftrightarrow c^2(1 - \cos^2 B) + a^2 - 2ac\cos B + \cos^2 B = b^2;$$
$$\Leftrightarrow c^2 + a^2 - 2ac\cos B = b^2,$$

establishing the Law of Cosines.

Exercise: What happens if the perpendicular AD is outside the triangle?

Exercise: Why is the Law of Cosines a generalization of the Pythagorean Theorem? **Exercise:** Take the three familiar congruence theorems, SSS, SAS, and ASA, and figure out which law to use in order to solve the rest of the triangle. That is, given three sides of a triangle, how do we compute the angles? If we have two sides and an angle between them, how do we compute the third side? If we have all three angles of the triangle and one side, how do we get the other two sides?

Exercise (The ASS trap): A common misconception is that when we have an SSA (in more politically correct systems, denoted SSA) correspondence, that is, two triangles with two adjacent sides equal and an angle not between those two sides equal, the triangles are congruent. Use the appropriate law to explore this issue: convince yourself that ASS is indeed not a valid theorem, but decide what information you can determine about the rest of the triangle given two sides and an angle that is not between them.