LHS Math Team

Team Contest 2

1. Ten distinct points lie on a circle. A line is drawn, passing through none of the points, partitioning the points into two subsets. How many possible such partitions are there?

2. Find a positive integer n such that if you append the digit 2 to the beginning and the digit 1 to the end, the new number is equal to $33n$.

3. Solve the following system of equations in positive integers.

$$a^3 - b^3 - c^3 = 3abc$$
$$a^2 = 2(b + c).$$

4. Let ABC be a triangle. Show that $\sin A$, $\sin B$, $\sin C$ are the sides of some triangle.

5. Consider an $n-$gon inscribed in a circle, and let P be a point outside the circle. The $n-$gon is partitioned (triangulated) into $n-2$ triangles by diagonals, such that such diagonals only intersect at vertices of the polygon, if at all. Let $I_1, I_2, \cdots, I_{n-2}$ be the incenters of these triangles. Prove that $\sum_{k=1}^{n-2}(PI_k)^2$ does not depend on the triangulation of the $n-$gon.

6. Let n be a positive integer. Prove that $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

7. The sequence $a_1, a_2, \ldots$ satisfies the relations $a_1 = a_2 = 1$ and

$$a_{n+2} = \frac{1}{a_{n+1}} + a_n.$$

Find a_{2011}.

8. Determine, with proof, whether there exist 10,000 10-digit positive integers divisible by 7, all of which can be obtained from one another by a reordering of their digits.

9. Let $d(n)$ be the number of positive divisors of n and let $\varphi(n)$ be the number of positive integers less than n relatively prime to n. Find all non-negative integers c such that there exists n such that

$$d(n) + \varphi(n) = n + c,$$

and for such c find all values of n satisfying the above relationship.

10. Is it possible to tile a 66×62 rectangle with 12×1 rectangles, such that the 12×1 tiles are entirely within the large rectangle and no two tiles overlap?

11. Inside a convex quadrilateral $ABCD$ there is a point P such that the triangles PAB, PBC, PCD, PDA have equal areas. Prove that the area of $ABCD$ is bisected by one of the diagonals.

12. Hexagon $ABCDEF$ is inscribed in a circle, with $AB = BC = CD = 2$ and $DE = EF = FA = 1$. Find the radius of the circle.

13. Let $P(x)$ be a polynomial with integer coefficients with degree $n > 1$. Determine the largest possible number of consecutive integers that can be in the set $\{P(a) \mid a \in \mathbb{Z}\}$.

14. Let $\mathbb{R}$ denote the set of all real numbers. Find all functions f from $\mathbb{R}$ to $\mathbb{R}$ satisfying:
    (a) There are only finitely many s in $\mathbb{R}$ such that $f(s) = 0$, and
    (b) $f(x^4 + y) = x^3 f(x) + f(f(y))$ for all x, y in $\mathbb{R}$.

15. Let f be a real valued function on the plane such that if $ABCD$ is a square, $f(A)+f(B)+f(C)+f(D) = 0$. Prove that $f(P) = 0$ for all P.

16. Let a be an odd integer. Prove that $a^{2^m} + 2^{2^m}$ and $a^{2^n} + 2^{2^n}$ are relatively prime for all positive integers m and n with $m \neq n$.

17. Three real numbers satisfy the following statements:

    (a) The square of their sum equals the sum of their squares.

    (b) The product of the first two numbers is equal to the square of the third number.

    Find all ordered triples of reals that satisfy these two conditions.

18. Let a, b, c, d, e, f be positive integers and suppose that the number $a + b + c + d + e + f$ divides both $abc + def$ and $ab + bc + ca - de - ef - fd$. Prove that $a + b + c + d + e + f$ is not prime.

19. Define a positive integer n to be a factorial tail if there is some positive integer m such that the decimal representation of $m!$ ends with exactly n zeroes. How many positive integers less than 2011 are not factorial tails?

20. One thousand boxes labeled $B_1, B_2, \ldots, B_{1000}$ and $1000n$ balls are distributed among them, where n is a positive integer. A move consists of taking exactly i balls in B_i and moving them to any other box. For which values of n is it always possible to make moves until there are exactly n balls in each box?

21. In a circle with center O, chord $\overline{AB}$ equals chord $\overline{AC}$. Chord $\overline{AD}$ intersects $\overline{BC}$ at point E. If $AC = 12$ and $AE = 8$, find AD.

22. Without calculus, determine the maximum possible value of $\sin x \cos x$.

23. Let n be a positive integer. Prove that if the sum of the positive divisors of n is a power of two, then the number of divisors of n is a power of two.

24. In how many ways can a 4×4 grid be filled by nonnegative integers, such that sum of the numbers of each row and each column is 3?

25. Two circles intersect at points A and B, and tangent lines to these circles are perpendicular at each of points A and B (in other words, the circles "meet at right angles"). Let M be an arbitrary point chosen on one of the circles so that it lies inside the other circle. Denote the intersection points of lines AM and BM with the latter circle by X and Y, respectively. Prove that XY is a diameter of this circle.

26. Given a positive integer n, find the largest k such that the numbers $1, 2, \ldots, n$ can be placed in k boxes such that all of the boxes have the same sum of elements.

27. Evaluate the infinite product
$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

28. An equilateral triangle has inside it a point with distances $5, 12, 13$ from the sides. Find its side length.

29. Show that there exist 2011 consecutive integers, each of which is divisible by the cube of an integer greater than 1.

30. Let p, q, r, s be real numbers such that $p + q + r + s = 9$ and $p^2 + q^2 + r^2 + s^2 = 21$. Prove that for some permutation $(p, q, r, s) \to (a, b, c, d)$, $ab - cd \geq 2$.