LHS Math Team

Team Contest 1 Solutions 2011-12

1. Source: 1959 IMO Problem 1

   Solution: We first note that $21n+4, 14n+3 > 1$ since n is a positive integer. Then, by the Euclidean Algorithm,

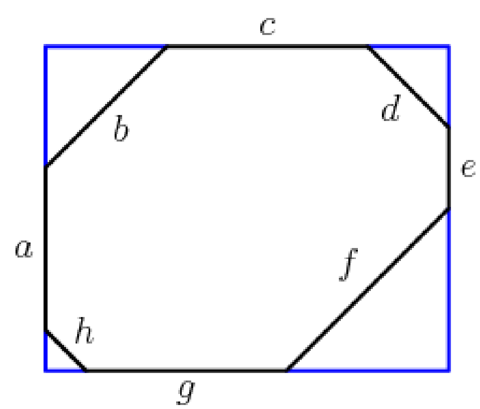
$$\begin{aligned}\gcd(21n+4, 14n+3) &= \gcd(7n+1, 14n+3)\\ &= \gcd(7n+1, 7n+2)\\ &= \gcd(7n+1, 1)\\ &= 1.\end{aligned}$$

   Thus, the fraction is irreducible since the numerator and denominator share no nontrivial common factor.

2. Answer: $\boxed{50+56\sqrt{2}}$

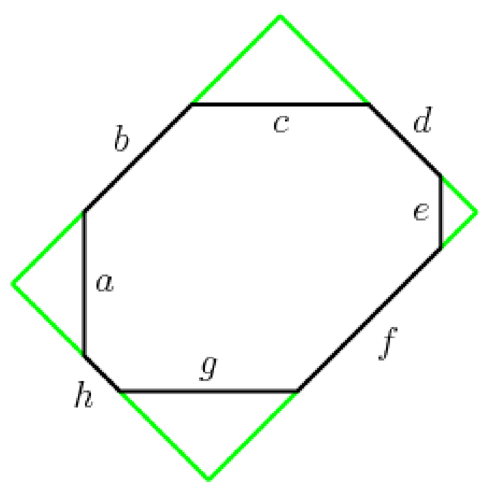
   Source: Captains

   Solution:



   We extend the sides to form a rectangle. There are 45-45-90 triangles in the corners, so since opposite sides of a rectangle are equal, we can get

$$\begin{aligned}b+c\sqrt{2}+d &= f+g\sqrt{2}+h\\ h+a\sqrt{2}+b &= d+e\sqrt{2}+f.\end{aligned}$$



We can extend the sides to form another rectangle, which gives us

$$a + b\sqrt{2} + c = e + f\sqrt{2} + g$$
$$c + d\sqrt{2} + e = g + h\sqrt{2} + a.$$

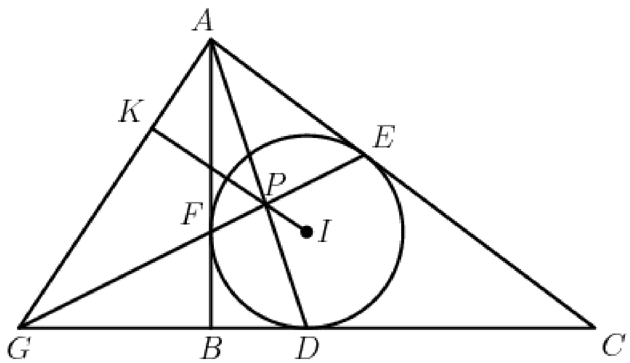
From these equations and the given information, we can find that

$$(a, b, c, d, e, f, g, h) = (4\sqrt{2}, 6, 7, 4, 2\sqrt{2}, 8, 7, 2)$$

To find the area of the octagon, we take the rectangle and subtract the four triangles. The rectangle has area $(7+5\sqrt{2})(8\sqrt{2}) = 56\sqrt{2}+80$ and the area of the triangles is $\frac{1}{2}((3\sqrt{2})^2+(2\sqrt{2})^2+(4\sqrt{2})^2+(\sqrt{2})^2) = 30$, so the area of the octagon is $56\sqrt{2} + 50$.

3. Source: Captains

   Solution:



Let the intersection of $\overline{AD}$ and $\overline{EF}$ be P. It suffices to show that $\overleftrightarrow{IP}$ is perpendicular to $\overline{AG}$.

By construction, $\overleftrightarrow{EF}$ is the polar of A, so G and P lie on the polar of A. Thus, A lies on the polar of P. By construction, D lies on the polar of G, and since G is on the polar of A, A is on the polar of G. Thus, $\overleftrightarrow{AD}$ is the polar of G and P lies on the polar of G, so G lies on the polar of P. Therefore, $\overline{AG}$ is the polar of P and $\overleftrightarrow{IP}$ is perpendicular to $\overline{AG}$, as desired.

Note: Other solutions exist, the only ones I know of hinge on the fact that if D' is the other intersection of $\overline{AD}$ with the incircle, then $m\angle GD'I = 90°$ or something very equivalent.

4. Source: MOP 2011 G5.2

   Solution: For an element of $\mathcal{S}$ to be divisible by p,

$$y^2 - x^3 - 1 \equiv 0 \pmod{p} \Rightarrow y^2 - 1 \equiv x^3 \pmod{p}.$$

We will show that for every $0 \le a < p$, there is exactly one $0 \le x < p$ such that $x^3 \equiv a \pmod{p}$.

Let $p = 3k + 2$. Suppose there exist distinct $0 \le x_1, x_2 < p - 1$ such that $x_1^3 \equiv x_2^3 \pmod{p}$. Then, $x_1^{3k} \equiv x_2^{3k} \pmod{p}$. If $x_1 \equiv 0 \pmod{p}$, then $x_2 \equiv 0 \pmod{p}$ since p is prime. Otherwise, by Fermat's Little Theorem, $x^{3k+1} \equiv 1 \pmod{p}$. Thus, $x^{3k} \equiv x^{-1} \pmod{p}$ and

$$x_1^{-1} \equiv x_2^{-1} \pmod{p}.$$

Since p is prime, in order for two different values to have the same inverse, they must actually be equal, so $x_1 = x_2$ and uniqueness of cube roots is established.

Since cube roots are unique modulo p, for every value of y, there is exactly one value of x such that $y^2 - x^3 - 1$ is divisible by p. There are p values that y can take, so there are at most p different values for $y^2 - x^3 - 1$ divisible by p.

5. Answer: $\boxed{\dfrac{-3\sqrt[3]{18}+\sqrt[3]{75}+2\sqrt[3]{20}}{17}}$

   Source: Captains

   Solution: Let $a=\sqrt[3]{12}$, $b=\sqrt[3]{45}$, $c=\sqrt[3]{50}$. We use the identity

   $$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca),$$

   rearranging to get

   $$\frac{1}{a+b+c}=\frac{a^2+b^2+c^2-ab-bc-ca}{a^3+b^3+c^3-3ab}.$$

   Substituting, we get

   $$\frac{1}{\sqrt[3]{12}+\sqrt[3]{45}+\sqrt[3]{50}}=\frac{-3\sqrt[3]{18}+\sqrt[3]{75}+2\sqrt[3]{20}}{17}.$$

6. Answer: $\boxed{50}$

   Source: Captains, based off an old Team Contest problem

   Solution: This line clearly goes through all points of the form $(a,2a)$, which can be seen either intuitively or by writing

   $$100=50m+b$$
   $$2=m+b.$$

   For this to be a lattice point, a must be an integer, making $2a$ an integer. Thus, a is an integer from 1 to 50 inclusive, so there are 50 possible values for a.

7. Source: 2005 Canadian MO Problem 2

   Solution: If $\gcd(a,b,c)\neq 1$, then let $\gcd(a,b,c)=d$. We write $a=a'd$, $b=b'd$, $c=c'd$, with $\gcd(a',b',c')=1$. Then, $\frac{c}{a}+\frac{c}{b}=\frac{c'}{a'}+\frac{c'}{b'}$, so it suffices to solve the problem for $\gcd(a,b,c)=1$.

   Since (a,b,c) is a triple of integers, $\frac{c}{a}+\frac{c}{b}$ is a rational number. We're looking for integer values of n, so in fact $\frac{c}{a}+\frac{c}{b}$ is an integer. We can write it as a single fraction as $\frac{c(a+b)}{ab}$, so $a|c(a+b)$.
   Since $\gcd(a,c)=1$, $a|a+b\Rightarrow a|b$, contradicting the fact that $\gcd(a,b)=1$ unless $a=1$. There are no Pythagorean triples with $a=1$, so there are no Pythagorean triples satisfying the equation and we're done.

8. Source: MOP 2011 G3.1

   Solution: Let our set S consist of the elements $\{1,2,\ldots,n\}$ and let $A_i=\{i\}$ for $1\le i\le n$. Then, for any subset $A\subseteq S$ with elements $a_1,a_2,\ldots,a_k$, from the definition of a unified function,

   $$f(A)=f(A_{a_1})\cup f(A_{a_2})\cup\cdots\cup f(A_{a_k}).$$

   In addition, all of the values of $f(A_i)$ are clearly independent from each other. Thus, the values $f(A_1),f(A_2),\ldots,f(A_n)$ completely determine the value of the function at every subset except the null set. Again from the definition, if we let $B=$,

   $$f()\cup f(A)=f(A).$$

   Thus, $f()\subseteq f(A_i)$ for $1\le i\le n$. We will use these facts to build up our unified function by selecting values for $f()$ and $f(A_i)$, counting the number of ways at each step.

   Suppose $f()$ has k elements, $0\le k\le n$. For each k, there are $\binom{n}{k}$ ways to pick $f()$ as a $k-$element subset. For each A_i, $f()\subseteq f(A_i)$. Other than that, there are no restrictions. Each of the $n-k$ elements not in $f()$ can be in A_i or not in A_i, so this gives us 2^{n-k} choices for each value of $f(A_i)$. In total, for fixed k, there are

$$\binom{n}{k}(2^{n-k})^n = \binom{n}{k}(2^n)^{n-k}.$$

Thus, summing over all k, we have

$$\sum_{k=0}^{n}\binom{n}{k}(2^n)^{n-k}$$

unified functions. However, this is just the binomial expansion for $(2^n+1)^n$, which is a perfect power since $n \geq 2$, so we're done.

9. Answer: $\boxed{n=12}$

   Source: Tuymaada Yakut Olympiad 2005

   Solution: WLOG, suppose $a_1 < a_2 < \cdots < a_{13}$. We must show that it must be the case that $n \geq 12$, and then that $n = 12$ always suffices.

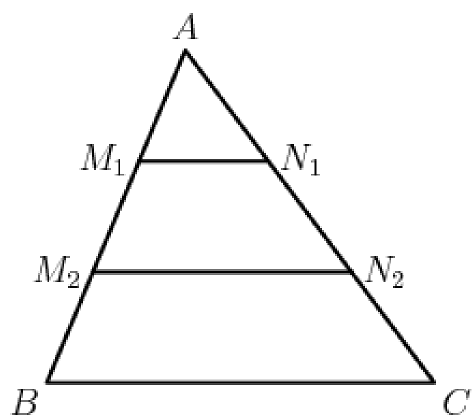
   Suppose the team leader wants the people who got scores a_2, a_4, ..., a_{12}. Then a polynomial setting the creative potentials of these people above all the others must turn around at best 11 times around $a_2, a_3, \ldots, a_{12}$. This gives us a polynomial of degree at least 12, so $n \geq 12$.

   Pick creative potentials and plot the points (a_i, c_i) such that the team leader gets what he wants. Then, there are 13 points, so we can use Lagrange Interpolation to always get a polynomial of degree at most 12, so $n = 12$ suffices.

10. Answer: $\boxed{15}$

    Source: Captains

    Solution:



    The area of triangle AM_2N_2 is $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$ of the area of $\triangle ABC$. The area of triangle AM_1N_1 is $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$ of the area of $\triangle ABC$. Thus, the area of trapezoid $M_2M_1N_1N_2$ is $\frac{4}{9} - \frac{1}{9} = \frac{1}{3}$ of the area of $\triangle ABC$. We're given that $[ABC] = 45$, so $[M_2M_1N_1N_2] = 15$.

11. Source: MOP 2011 G4.2

    Solution: Let $n = q_1^{e_1}q_2^{e_2}\cdots$ be the prime factorization of n, where the e_i's are positive and the product is finite. To show that $m^{n-1}-1$ and n share a nontrivial factor, it suffices to show that one of the q_i's divides $m^{n-1}-1$.

    First, we note that m and n must be relatively prime. Otherwise, WLOG, $q_1|m$. Then,

$$m^{p(n-1)} - 1 \equiv 0 - 1 \equiv -1 \pmod{q_1}.$$

However, $q_1|n$ and $n|m^{p(n-1)}-1$, so

$$m^{p(n-1)}-1 \equiv 0 \pmod{q_1},$$

contradiction. Thus, m and n are relatively prime.

Assume for the sake of contradiction that for each i, q_i does not divide $m^{n-1}-1$. Since $m^{p(n-1)} \equiv 1 \pmod{q_i}$, and m is relatively prime to q_i, it must be that

$$\operatorname{ord}_{q_i} m | p(n-1).$$

(To see this, note that for $a \perp p$ and $a^b \equiv a^c \equiv 1 \pmod{p}$, $a^{\gcd(b,c)} \equiv 1$ as well.)

Let $\gcd(\operatorname{ord}_{q_i} m, n-1) = g_i$. Then, if we write

$$\operatorname{ord}_{q_i} m = g_i x_i$$
$$n-1 = g_i y_i,$$

we have $\gcd(x_i, y_i) = 1$ and

$$g_i x_i | g_i y_i p \Rightarrow x_i | y_i p.$$

Thus, $x_i|p$. If $x_i = 1$, then $\operatorname{ord}_{q_i} m = g_i$ and $\operatorname{ord}_{q_i} m | n-1$. However, in this case,

$$m^{n-1} \equiv 1 \pmod{q_i},$$

contradicting the assumption there is no q_i that divides $m^{n-1}-1$. Thus, $x_i \neq 1$ and $x_i = p$ for all i. This gives us

$$\operatorname{ord}_{q_i} m = p g_i.$$

Suppose $p^k || n-1$ and $p^{l_i} || \operatorname{ord}_{q_i} m$. Then, by definition, $p^{\min\{k,l_i\}} || g_i$. From the previous result, we get $p^{\min\{k,l_i\}+1} || \operatorname{ord}_{q_i} m$. Thus,

$$\min\{k, l_i\} + 1 = l_i \Rightarrow l_i = k+1,$$

so

$$p^{k+1} || \operatorname{ord}_{q_i} m$$

for all i.

By Fermat's Little Theorem, $m^{q_i-1} \equiv 1 \pmod{q_i}$ since m and q_i are relatively prime. Thus, $\operatorname{ord}_{q_i} m | q_i - 1$ and

$$p^{k+1} | q_i - 1$$
$$q_i \equiv 1 \pmod{p^{k+1}}.$$

Since this holds for all i,

$$\prod q_i^{e_i} \equiv 1 \pmod{p^{k+1}}$$
$$n \equiv 1 \pmod{p^{k+1}}$$
$$p^{k+1} | n-1,$$

contradicting the assumption that $p^k || n-1$.

Having exhausted all cases in which $m^{n-1}-1$ and n are relatively prime, we conclude that they are, in fact, not, so we're done.

Note: the notation $\operatorname{ord}_a b$ *refers to the smallest positive integer* k *such that* $b^k \equiv 1 \pmod{a}$. *The notation* $p^k || a$ *for* p *prime means that* p^k *divides* a *but* p^{k+1} *does not.*

12. Answer: $\boxed{38 \text{ to } 126}$

Source: USAMTS 2004-05 Round 4 Problem 4

Solution: We note that $2x^2+3y^2+6z^2$ varies continuously on $3x+4y+5z=23$, so everything between the minimum and maximum is attainable.

To find the maximum, we note that the shape $2x^2+3y^2+6z^2=n$ is an ellipsoid centered at the origin, so the maximum when restricted to $3x+4y+5z=23$ will occur at one of the three intercept points of the plane with the axes. Testing $x=23/3$, $y=23/4$, and $z=23/5$, we see that the one giving the largest value is $z=23/5$, corresponding to $n=6\left(\frac{23}{5}\right)^2=\frac{3174}{25}=126\frac{24}{25}$.

To find the minimum, we use the coordinate transform $a=x\sqrt{2}, b=y\sqrt{3}, c=z\sqrt{6}$, which reduces the problem to finding the minimum of $a^2+b^2+c^2$ given

$$3\frac{a}{\sqrt{2}}+4\frac{b}{\sqrt{3}}+5\frac{c}{\sqrt{6}}=23.$$

The minimum occurs when the sphere $a^2+b^2+c^2=n$ is tangent to the transformed plane. This means that the height to the nontrivial triangular base is equal to the radius, $\sqrt{n}$. We find the volume of the tetrahedron bounded by the plane one way by finding the intercepts to be

$$a=\frac{23\sqrt{2}}{3}, b=\frac{23\sqrt{3}}{4}, c=\frac{23\sqrt{6}}{5}.$$

Thus, the volume of this tetrahedron is

$$\frac{1}{6}\frac{23\sqrt{2}}{3}\frac{23\sqrt{3}}{4}\frac{23\sqrt{6}}{5}=\frac{23^3}{60}.$$

However, the volume is equal to

$$\frac{1}{3}hB=\frac{1}{3}\sqrt{n}B,$$

where B is the area of the slanted triangular base. Through computation (cross products, sine area and law of cosines, etc.), the area of B in terms of the intercepts a, b, c is

$$\frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}.$$

Substituting the given values,

$$\begin{aligned} B &= \frac{23^2}{2}\sqrt{\frac{6}{144}+\frac{18}{400}+\frac{12}{225}} \\ &= \frac{529\sqrt{14}}{20} \end{aligned}$$

Thus, we can substitute to find that

$$\sqrt{n}=\frac{23}{\sqrt{14}} \Rightarrow n=\frac{529}{14}\approx 37.8.$$

We want integer solutions for n, so anything from 38 to 126 will do.

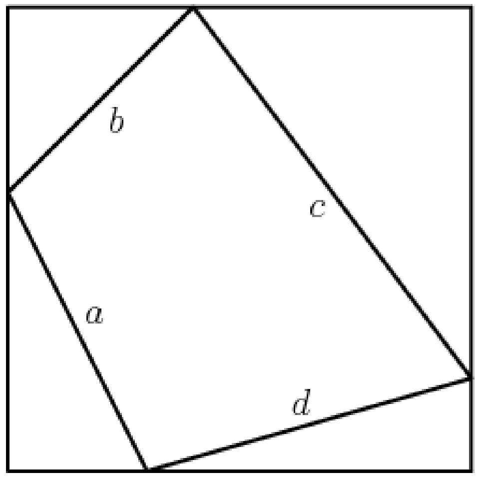
13. Source: MOP 2004

Solution: WLOG, suppose there are more columns in which the majority of the squares are red than columns in which the majority of the squares are blue. By the Pigeonhole principle, there are at least $\lceil 41/2\rceil 21$ columns with more red squares than blue. Note that in each column, there must be at least $\lceil 5/2\rceil=3$ red squares.

There are $\binom{5}{3} = 10$ ways three red squares can be placed into a column of 5 squares. Each column with a red square majority must have at least 1 of these arrangements (a column with 4 red squares will have 4, a column with 5 will have 10). Thus, by the Pigeonhole principle, some arrangement occurs at least $\lceil 21/10 \rceil = 3$ times. This gives us the desired 9 squares of the same color.

14. Source: 1970 Canadian Math Olympiad Problem 5

    Solution:



Let $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ be the segments along the square, labeled clockwise starting from the left segment on the bottom side. We have $a_1^2 + a_2^2 = a^2$ from the Pythagorean Theorem and similar relations for the other three sides.

To show the upper bound, we write

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 &= a_1^2 + a_2^2 + b_1^2 + b_2^2 + c_1^2 + c_2^2 + d_1^2 + d_2^2 \\ &\le (a_2 + b_1)^2 + (b_2 + c_1)^2 + (c_2 + d_1)^2 + (d_2 + a_1)^2 \\ &= 1 + 1 + 1 + 1 = 4. \end{aligned}$$

To establish the lower bound, we use RMS-AM to write

$$\sqrt{\frac{a_2^2 + b_1^2}{2}} \ge \frac{a_2 + b_1}{2} \Rightarrow a_2^2 + b_1^2 \ge \frac{1}{2}.$$

Thus,

$$a_1^2 + a_2^2 + b_1^2 + b_2^2 + c_1^2 + c_2^2 + d_1^2 + d_2^2 \ge 4\left(\frac{1}{2}\right) = 2,$$

as desired.

15. Source: MOP 2011 G2.4

    Solution: First, while it is standard and more correct to write solutions forwards, this is explained going backwards because otherwise it makes even less sense.

    Let $e = \frac{1}{abcd}$, so $abcde = 1$. Substituting,

$$\frac{1}{a+b+c+e+1} + \frac{1}{a+b+d+e+1} + \frac{1}{a+c+d+e+1} + \frac{1}{b+c+d+e+1} = \frac{a+b+c+d}{a+b+c+d+1}.$$

Writing the RHS as $1 - \dfrac{1}{a+b+c+d+1}$, we can rearrange this to

$$\sum_{\text{cyc}} \frac{1}{a+b+c+d+1} \le 1.$$

We now homogenize this inequality by writing $1 = \sqrt[5]{abcde}$ on the LHS. Substituting $x_1 = \sqrt[5]{a}$, $x_2 = \sqrt[5]{b}$, and so on, we want to show

$$\sum_{\text{cyc}} \frac{x_1x_2x_3x_4x_5}{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_1x_2x_3x_4x_5} \le 1.$$

By repeated AM-GM or Muirhead's Inequality,

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 \ge x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_1x_2x_3^2x_4 + x_1x_2x_3x_4^2.$$

Substituting,

$$\begin{aligned}
\sum_{\text{cyc}} \frac{x_1x_2x_3x_4x_5}{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_1x_2x_3x_4x_5} &\le \sum_{\text{cyc}} \frac{x_1x_2x_3x_4x_5}{x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_1x_2x_3^2x_4 + x_1x_2x_3x_4^2 + x_1x_2x_3x_4x_5} \\
&= \sum_{\text{cyc}} \frac{x_5}{x_1 + x_2 + x_3 + x_4 + x_5} \\
&= 1,
\end{aligned}$$

as desired.

16. Answer: $\boxed{\text{Yes}}$

Source: IMO SL 2010 C1

Solution (construction by randomgraph): Here we give an explicit construction and show this gives 2010 possible orders. There are many degrees of freedom in this problem, so a very large number of constructions are possible. Most of these come down to subdividing the singers into two groups, forcing one group to have 30 orders, the other to have 67, and then making all members of one group come after all members of the other group.

- Singers 1,2,3 have no restrictions
- Singer 4 must come after 1,2, and 3
- Singer 5 has no restrictions
- Singer 6 comes after singers 4 and 5. This forces 6 to come after 1,2,3,4, and 5.
- Singer n comes after singer $n-1$ for $6 \le n \le 16$. This is a forced order and contributes nothing.
- Singer 17 comes after singer 13.
- Singer 18 comes <u>before</u> singer 16.
- Singer 19 comes after singers 1 through 18.
- Singer 20 comes after singers 1 through 19.

There are $3! = 6$ ways to arrange singers 1 through 4. There are then 5 places that singer 5 could go, so this gives $6 \times 5 = 30$ arrangements of the first five.

For each one of these, there are 4 possible locations for singer 17. If singer 17 is immediately after singer 13, 14, or 15, then there are 17 spots preceding singer 16 that singer 18 could go into, giving 17 possible orders in these 3 cases. If singer 17 is immediately after singer 16, then there are only 16 spots. In total, this gives $17 + 17 + 17 + 16 = 67$ ways to order the singers for any given order of the first 5. Thus, there are altogether $30 \times 67 = 2010$ orderings, as desired.

17. Answer: $\boxed{4023^3}$ or $\boxed{65,110,360,167}$

    Source: Mandelbrot

    Solution: For the particular values of $x = 1, 2, 3, 4$, from the equations, we get

    $$a_1x^3 + a_2(x+1)^3 + a_3(x+2)^3 + a_4(x+3)^3 = (2x-1)^3.$$

    These are two single-variable cubic expressions that are equal at 4 points, so they must be equal everywhere. Letting $x = -2011$,

    $$a_1(-2011)^3 + a_2(-2010)^3 + a_3(-2009)^3 + a_4(-2008)^3 = (-4023)^3$$
    $$2011^3a_1 + 2010^3a_2 + 2009^3a_3 + 2008^3a_4 = 4023^3.$$

18. Answer: $\boxed{4}$

    Source: AHSME 1985 Problem 3

    Solution: Let M be the intersection of the circle with center A with the hypotenuse and let N be the other point. Then $AB = AM + NB - NM$. By the Pythagorean Theorem, $AB = 13$. From the definition of a circle, $AM = 12$ and $BN = 5$. Thus, $MN = 12 + 5 - 13 = 4$.

19. Answer: $\boxed{413}$

    Source: Modified, AIME 1995

    Solution: Let n be any number that is not the sum of a positive integer multiple of 60 and a positive composite number. Suppose $n = 60q + r$, $0 \le r < 60$. The condition is then equivalent to the condition that the sequence $r, r+60, r+120, \ldots, n-60$ consists of all primes. In order for n to be the largest such number, n must be composite. Our goal is then to construct the longest possible arithmetic sequence of primes with difference 60 and then the first composite number in that sequence is our answer.

    Consider the equation modulo 7. We are considering the sequence $r, r+4, r+1, \ldots$, which must be 0 modulo 7 at some point and thus composite. For each value of r modulo 7, we consider the length of the sequence before it reaches 0 modulo 7.

    - $r = 0$, $0, 4, 1, 5, 2, 6, 3, 0$, so the first composite number is at most $7(60) + r = 420 + r \le 480$.
    - $r = 1$, $1, 5, 2, 6, 3, 0$, so the first composite number is at most $5(60) + r = 300 + r \le 360$.
    - $r = 2$, $2, 6, 3, 0$, so the first composite number is at most $3(60) + r = 180 + r \le 240$.
    - $r = 3$, $3, 0$, so the first composite number is at most $1(60) + r = 60 + r \le 120$.
    - $r = 4$, $4, 1, 5, 2, 6, 3, 0$, so the first composite number is at most $6(60) + r = 360 + r \le 420$.
    - $r = 5$, $5, 2, 6, 3, 0$, so the first composite number is at most $4(60) + r = 240 + r \le 300$.
    - $r = 6$, $6, 3, 0$, so the first composite number is at most $2(60) + r = 120 + r \le 180$.

    The longest one of these came from $r = 0$ modulo 7, which means that since r is prime, $r = 7$. This sequence gives us $7, 67, 127, 187 = 11 \times 17$, so 187 is the corresponding value of n in this sequence.

    The second longest sequence was from $r = 4$ modulo 7. The largest prime less than 60 satisfying this is $r = 53$, which gives the sequence

    $$53, 113, 173, 233, 293, 353, 413 = 7 \times 59,$$

    giving $n = 413$.

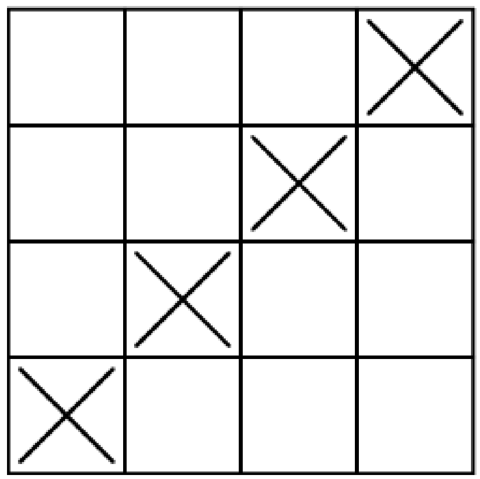
    No other sequence can possibly achieve something bigger, so 413 is the largest such number.

20. Answer: $\boxed{5184}$

    Source: Captains

    Solution: In this problem, we note the following symmetry condition: Suppose we have two sets of restraints A and B on the placement of purple and green balls. We let an *exchange* be a swapping of two rows or two columns along with all conditions on those rows or columns. Then if B can be produced through a bunch of exchanges from A, the number of arrangements under A is equal to the number of arrangements under B.

We first place the blank spots. There must be one blank spot in each row and each column, so there are $4! = 24$ ways to do this. By exchanging, every arrangement of the blank spots has the same number of arrangements for the rest of the balls, so we'll count the number of such arrangements for the following configuration.



Now, we pick two columns that will have 2 purple balls. There are $\binom{4}{2} = 6$ ways to do this, and by exchanging, these are all equivalent. Thus, we assume the first two columns have 2 purple balls. We can then do casework on all the remaining possibilities. There end up being 36 such arrangements given this arrangement of blank spots and that the first two columns are dominated by purple. Thus, the total number of arrangements is $24 \times 6 \times 36 = 5184$.

Note: Yes, this was a hastily written and lazy solution. The casework is not terribly difficult although it can be improved with more symmetry considerations.

21. Answer: $\boxed{-23}$

Source: USAMTS 2000-01 Round 3 Problem 3

Solution: Write

$$P(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)(x - r_5).$$

Then,

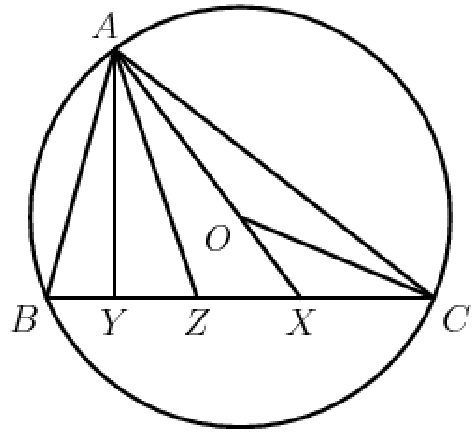
$$\begin{aligned} q(r_1)q(r_2)q(r_3)q(r_4)q(r_5) &= (r_1^2 - 2)(r_2^2 - 2)(r_3^2 - 2)(r_4^2 - 2)(r_5^2 - 2) \\ &= (-1)(\sqrt{2} - r_1)(\sqrt{2} - r_2)(\sqrt{2} - r_3)(\sqrt{2} - r_4)(\sqrt{2} - r_5) \\ &\quad (\sqrt{2} + r_1)(\sqrt{2} + r_2)(\sqrt{2} + r_3)(\sqrt{2} + r_4)(\sqrt{2} + r_5) \\ &= (-1)P(\sqrt{2})(-P(-\sqrt{2})) \\ &= P(\sqrt{2})P(-\sqrt{2}). \end{aligned}$$

From the original expression, we can evaluate $P(\sqrt{2}) = 3 + 4\sqrt{2}$ and $P(-\sqrt{2}) = 3 - 4\sqrt{2}$. Thus, $P(\sqrt{2})P(-\sqrt{2}) = -23$.

22. Answer: $\boxed{\sqrt{3}/2}$

Source: Captains

Solution:



We are given that $\overline{ZA}$ is an angle bisector, so $m\angle BAZ = m\angle ZAC = 32°$.

Since $\angle ABC$ is an inscribed angle, $m\angle AOC = 2m\angle ABC = 146°$. Triangle AOC is isosceles since $AO = OC$, so $m\angle ZAC = m\angle OAC = 17°$. Therefore, $m\angle ZAX = 15°$.

In addition, we can find that $m\angle BAY = 90° - 73° = 17°$, so $m\angle YAZ = 15°$. Thus, $\overline{AZ}$ is an angle bisector of $\angle YAX$ and

$$\frac{YZ}{ZX} = \frac{YA}{AX}.$$

Since $m\angle YAX = 30°$, $\dfrac{YA}{AX} = \dfrac{\sqrt{3}}{2}$.

23. Answer: $\boxed{1}$

    Source: Slovenia National Olympiad 2010 First Grade (note: this definitely does not correspond with first grade in American schools)

    Solution: From the first equality,

$$\begin{aligned} 2a^2 - 3abt + b^2 &= 2a^2 + abt - b^2 \\ 2b^2 &= 4abt \\ b &= 2at. \end{aligned}$$

    Substituting,

$$\begin{aligned} 2a^2 + a(2at)t - (2at)^2 &= 0 \\ 2a^2 - 2a^2t^2 &= 0 \\ t &= \pm 1. \end{aligned}$$

    Since a and b are positive reals, t is positive and thus $t = 1$.

24. Source: Brazil National Olympmiad 1998

    Solution: Suppose all the numbers are composite. We first note that $44 < \sqrt{1998} < 45$. There are 14 primes less than 44, which can be counted. There are 15 numbers, so since the numbers are relatively prime, some number can only have prime factors greater than 44. The smallest composite number with this property is $47^2 > 1998$, contradiction. Thus, some number must be prime.

25. Answer: $\boxed{5}$

    Source: Modified, Spain Math Olympiad 2010

    Solution: Let the sequence be $2n-15, 2n-13, 2n-11, \ldots, 2n+13, 2n+15$. The sum of these numbers is then $32n$, which must be a perfect cube. Thus, $n = 2k^3$ where k is a positive integer. In addition, $2n - 15 \geq 1 \Rightarrow n \geq 8$ and $2n + 15 \leq 999 \Rightarrow n \leq 492$. Substituting,

$$8 \leq 2k^3 \leq 492 \Rightarrow 2 \leq k \leq 6,$$

giving us 5 possible sequences.

26. Answer: $\boxed{134}$

    Source: MOP 2011 Handout

    Solution: Let the *blueness* of any blue point by the number of segments coming out of the point. Let the blueness of the square then be the total blueness of all the vertices. We will count this in two ways.

    There are $16 \times 16 - 133 = 123$ blue points in the grid. Of these, $4 - 2 = 2$ are in a corner and have 2 edges coming out of them, and $4 \times 14 - 32 = 24$ are on the sides and have 3 edges coming out of them. The other $123 - 2 - 24 = 97$ are somewhere in the middle and have 4 edges. Thus, the total blueness is $2 \times 2 + 3 \times 24 + 4 \times 97 = 464$.

    Consider the edges and let there be b blue edges. Every yellow segment contributes 1 to the blueness and every blue segment contributes 2 (red contributes 0). Thus, the blueness is equal to $196 + 2b$.

    These two counts must be equal, so $196 + 2b = 464 \Rightarrow b = 134$.

27. Source: 2006 Canadian Math Olympiad Problem 5

    Solution: Let the intersection of the tangents at D and E, E and F, F and D be labeled Z, X, Y, respectively.

    It is a well-known fact that in a right triangle PQR with M the midpoint of hypotenuse PR, triangles MQR and PQM are isosceles.

    Now we do some angle-chasing:

$$\begin{aligned}
\angle EDF &= \angle EDA + \angle ADF \\
&= \angle XEA + \angle AFX \\
&= (180^\circ - \angle AEZ) + (180^\circ - \angle YFA) \\
&= 2\angle FAB + 2\angle CAE \\
&= 2(\angle FAE - 90^\circ) \\
&= 2(90^\circ - \angle EDF),
\end{aligned}$$

    whence we conclude that $\angle EDF = 60^\circ$.

    Next, we prove that triangle DYF is equilateral. To see this, note that

$$\begin{aligned}
\angle DYF &= \angle FAB + \angle BAD \\
&= \angle FDY \\
&= \angle YFD.
\end{aligned}$$

    Hence $\angle FED = 60^\circ$ as well, so triangle DEF is equilateral as desired.

28. Source: MOP Lore

    Solution: By Cauchy-Schwarz,

$$((a+b)(a+c)+(b+c)(b+a)+(c+a)(c+b))\left(\frac{a^2}{(a+b)(a+c)}+\frac{b^2}{(b+c)(b+a)}+\frac{c^2}{(c+a)(c+b)}\right) \geq (a+b+c)^2.$$

    Expanding and rewriting,

$$\begin{aligned}
\frac{a^2}{(a+b)(a+c)}+\frac{b^2}{(b+c)(b+a)}+\frac{c^2}{(c+a)(c+b)} &\geq \frac{a^2+b^2+c^2+2ab+2bc+2ca}{a^2+b^2+c^2+3ab+3bc+3ca} \\
&= 1 - \frac{ab+bc+ca}{a^2+b^2+c^2+3ab+3bc+3ca}.
\end{aligned}$$

By repeated AM-GM or Muirhead's inequality, $a^2 + b^2 + c^2 \geq ab + bc + ca$, so

$$\begin{aligned}
\frac{a^2}{(a+b)(a+c)} + \frac{b^2}{(b+c)(b+a)} + \frac{c^2}{(c+a)(c+b)} &\geq 1 - \frac{ab+bc+ca}{a^2+b^2+c^2+3ab+3bc+3ca} \\
&\geq 1 - \frac{ab+bc+ca}{4ab+4bc+4ca} \\
&= \frac{3}{4},
\end{aligned}$$

as desired.

29. Answer: $\boxed{525/29}$

Source: Captains

Solution: Point E satisfies the property that $m\angle AEB = m\angle BEC = 90°$, so E is the foot of the altitude from B to $\overline{AC}$.

Since D lies on C_1 and $\overline{AB}$ is the diameter, by the Pythagorean Theorem, $AD = 20$. We are given that $CD = 21$ and $AC = 29$, so by the converse of the Pythagorean Theorem, $m\angle ADC = 90°$ and D is the foot of the perpendicular from A to $\overline{BC}$.

We note that quadrilateral $ABDE$ is cyclic, so $m\angle CED = 180 - m\angle AED = m\angle ABD$. Thus, $\triangle ABC \sim \triangle DEC$, so if $x = ED$,

$$\frac{x}{21} = \frac{25}{29} \Rightarrow x = \frac{525}{29}.$$

30. Answer: $\boxed{86}$

Source: USAMO 1984

Solution: Let a, b, c, d be the roots and WLOG, $ab = -32$. By Vieta's formulas,

$$\begin{aligned}
a+b+c+d &= 18 \\
ab+ac+ad+bc+bd+cd &= k \\
abc+abd+acd+bcd &= -200 \\
abcd &= -1984.
\end{aligned}$$

From the last equation, we get $cd = 62$. Substituting into the third equation,

$$62(a+b) - 32(c+d) = -200.$$

Solving this along with $(a+b)+(c+d) = 18$ for $a+b$ and $c+d$, we get

$$\begin{aligned}
a+b &= 4 \\
c+d &= 14.
\end{aligned}$$

Finally, in the second equation,

$$\begin{aligned}
k &= ab+ac+ad+bc+bd+cd \\
&= (a+b)(c+d) + ab + cd \\
&= 4(14) + (-32) + 62 \\
&= 86.
\end{aligned}$$