Matrices
LHS Math Team

Many of these definitions may seem pretty arbitrary and un-motivated. The fact is, they should seem this way. When you study Linear Algebra, it will make a lot more sense where some of these things come from, but unfortunately, for now, you'll just have to be familiar with the mechanics.

A *matrix* is, essentially, a rectangular grid of numbers. For us, we're just going to use the reals. We'll also deal mostly with 2×2 matrices, but of course, larger matrices exist. Note: and $m \times n$ matrix has m rows (the things that go left to right) and n columns (the things that go up and down).

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The first thing we can do with matrices is add and subtract them. To do this, we simply add or subtract the corresponding entries. Note that we can only do this if the matrices are of the same dimensions. In the case of 2×2 matrices, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix};$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a-e & b-f \\ c-g & d-h \end{pmatrix}$$

Unfortunately, multiplication is not this easy. Again, the reason for this is far beyond the scope of high school. The multiplication of two matrices is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

Observe that the entry in the first row and first column of the product is obtained by taking the first row of the first matrix and the first column of the second matrix, and multiplying corresponding terms, then adding. That is, we take a, b and e, g, then multiply ae and bg, then add. We do a similar process for the second row and first column, second row and second column, and first row and second column.

This process generalizes to larger matrices as well. However, note that if we take two matrices, of dimensions $w \times x$ and $y \times z$, we can only multiply them if $x = y$.

Exercise: Convince yourself that the above is true: that we can only multiply two matrices of dimensions $w \times x$ and $y \times z$ if $x = y$.

To make matters even more annoying, matrix multiplication is not commutative. So in general, if we have two matrices A and B, $AB \neq BA$ (of course, this does occur for some pairs A, B). If A and B aren't square matrices of the same dimensions, one or both of the products won't even be defined. However, matrix multiplication is associative, which is at least somewhat good.

Exercise: Verify the last two assertions: that matrix multiplication is not commutative (write down two random 2×2 matrices and multiply them both ways; chances are they won't commute), and that it is associative (this will probably get ugly).

We can also multiply matrices by scalars, which does turn out nicely. That is, if we have a real number c and a matrix A, we can take cA by simply multiplying every entry in A by c. Note that

$c = -1$ gives the additive inverse of A, so that $A - A$ is the zero matrix, the matrix with all entries equal to zero.

We aren't able to talk about matrix division directly, however we can talk about multiplicative inverses. First, we need to know what the multiplicative identity is, or in other words, what we'd like to call '1'. It turns out that this, for 2×2 matrices, is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, given a 2×2 matrix A, we want to define a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. To find this A^{-1}, the general strategy is to let the entries of A^{-1} be e, f, g, h and write down explicitly the multiplication AA^{-1}. This will give us a linear system of four equations in four variables. However, it is convenient to know the following formula.

Exercise: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Verify that if we take $A^{-1} = \dfrac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, given that $ad - bc \neq 0$, we have $AA^{-1} = A^{-1}A = I$.

Note that if $ad = bc$, the inverse matrix does not exist, it exists otherwise.

Definition: Using the same A as in the previous exercise, we define $ad - bc$ to be the **determinant** of A. Sometimes, this is written as $\det(A)$. When the matrix with all of its entries is written down with vertical bars (|) instead of parentheses, it is generally understood that it refers to the determinant.

Exercise: Prove that if A, B, C are 2×2 matrices such that $A \times B = C$, then $\det(A)\det(B) = \det(C)$.