Trig Identities and Inverses

LHS Math Team

For sake of brevity, on this page we'll list all of the important trig identities, and we'll outline their proofs in the later pages. You should all of the algebraic details yourself - most of the ideas of the proofs are provided. It's not recommended you try to memorize all of these identities: just have the fact that $\cos^2 x + \sin^2 x = 1$ and the addition formulas at hand (perhaps the double-angle formulas, but these are easy): everything else is a direct consequence.

1 Identities

Pythagorean Identities:
$\cos^2 x + \sin^2 x = 1$
$\tan^2 x + 1 = \sec^2 x$

Angle Addition and Subtraction:
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

Double and Triple Angle Formulas:
$\sin 2x = 2 \sin x \cos x$
$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$
$\tan 2x = \dfrac{\tan 2x}{1 - \tan^2 x}$
$\sin 3x = 3 \sin x - 4 \sin^3 x$
$\cos 3x = 4 \cos^3 x - 3 \cos x$
$\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$

Half-Angle Formulas:
Note: the sign depends on where on the unit circle the angle x is: you can either figure out for yourself when each is positive or negative, or check these things on a case-by-case basis.
$\sin \dfrac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{2}}$
$\cos \dfrac{x}{2} = \pm\sqrt{\dfrac{1 + \cos x}{2}}$
$\tan \dfrac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{1 + \cos x}} = \dfrac{1 - \cos x}{\sin x}$

Sum-to-Product Formulas:
$\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$
$\sin x \cos y = \frac{1}{2}(\sin(x + y) + \sin(x - y))$
$\cos x \sin y = \frac{1}{2}(\sin(x + y) - \sin(x - y))$
$\cos x \cos y = \frac{1}{2}(\cos(x - y) + \cos(x + y))$

Product-to-Sum Formulas:
$\sin x \pm \sin y = 2 \sin(\dfrac{x \pm y}{2}) \cos(\dfrac{x \mp y}{2})$
$\cos x + \cos y = 2 \cos(\dfrac{x + y}{2}) \cos(\dfrac{x - y}{2})$
$\cos x - \cos y = -2 \sin(\dfrac{x + y}{2}) \sin(\dfrac{x - y}{2})$

2 Proofs

Pythagorean Identities: See Unit Circle Trig for the first, and Right Triangle Trig for the second.

Angle Addition and Subtraction: Consider $e^{i(x+y)}$, and note that this is equal to $e^{ix}e^{iy}$. Write both in $\cos\theta + i\sin\theta$ form, and expand. For the tangent identity, rewrite all of the tangents in terms of sines and cosines, then work through the algebra.

Double and Triple Angle Formulas: Apply the addition formulas.

Half-Angle Formulas: Substitute $x = u/2$ into the double angle formulas. Be careful with signs!

Sum-to-Product, Product-to-Sum formulas: Note that these are direct consequences of the double angle formulas (it may be easier to take $x = 2x', y = 2y'$ instead) and the addition formulas. However, to re-derive these on the spot, we can do the following. For sum-to-product, recognize the product as one of the addends of a double angle formula, then add and subtract the appropriate ones to get what you want (you may have to play with it a little, this takes some getting used to). For product-to-sum, notice that we can write $x = (\frac{x+y}{2}) + (\frac{x-y}{2})$ and $y = (\frac{x+y}{2}) - (\frac{x-y}{2})$, and apply the addition formulas directly.

3 Inverse Trig Functions

An inverse trig function is exactly what it sounds like: its input is some real number, and the output is an angle. However, we have to be careful. If we wanted to talk about the inverse sine of say, $\frac{1}{2}$, there are many, in fact, infinitely many angles θ such that $\sin\theta = \frac{1}{2}$. Since we want this to be an inverse trig *function*, it can only take one value, meaning we have to restrict the range.

Definition: Let x be a real number such that $-1 \leq x \leq 1$. We let the inverse sine of x be the value y such that $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $\sin y = x$. We denote this $\sin^{-1} x$, or $\arcsin x$.

Note: Some people like to make a distinction between $\mathrm{Sin}^{-1}x$ and $\sin^{-1} x$, the one with the capital S being allowed to take multiple values, but for our purposes, both of these mean the same thing: the function can only take one value.

Definition: Let x be a real number such that $-1 \leq x \leq 1$. Define the inverse cosine of x to be the value of y such that $0 \leq y \leq \pi$ and $\sin y = x$. We denote this $\cos^{-1} x$, or $\arccos x$.

Definition: Let x be any real number. Define the inverse tangent of x to be the value of y such that $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and $\tan y = x$. We denote this $\tan^{-1} x$, or $\arctan x$.

Exercise: Convince yourself that these functions are, in fact, well-defined: that is, every value in the domain only has one output.