Problem Set 1 - Full
Lexington Math Team

Monday, September 24, 2012

1. The digits of the number 66134889 are permuted and then each digit is colored either yellow, blue, or red. In how many ways can this be done?

2. Let $f(n)$ be a function satisfying $f(1) = 1$, $f(2n) = f(n)$, and $f(2n+1) = f(2n) + 1$ for all $n \in \mathbb{N}$. Find the maximum of $f(n)$ for $1 \leq n \leq 2012$.

3. From any point O inside a convex polygon, draw the perpendicular segments to all of the polygons sides. Prove that at least one foot of a perpendicular is inside the corresponding side (not on the extension of the side).

4. Prove that

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin(n + \frac{1}{2})\theta}{2\sin(\theta/2)}.$$

5. Let $M_a(r)$ be the number of $a \times a$ matrices with nonnegative integer entries with each row and column summing to r. Prove combinatorially that

$$M_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}.$$