LHS Math Team

Team Contest 2012-13

1. Triangle ABC has a right angle at A and $AB > AC$. The line connecting the midpoints of $\overline{BC}$ and $\overline{AB}$ is tangent to the incircle of the triangle. What is the area of the triangle given that the incircle has a radius of 1?

2. Let $P_n(x) = 1 + x + x^2 + \cdots + x^n$ for nonnegative integers n. Find all unique sequences of nonnegative integers $a_0, a_1, a_2, \ldots$ are there such that $a_i < 2011$ for all positive integers i and
$$a_0P_0(2011) + a_1P_1(2011) + a_2P_2(2011) + a_3P_3(2011) + \cdots + a_nP_n(2011) + \cdots = 2013_{2011},$$
where the subscript on the right hand side denotes base 2011?

3. A certain college has 20 students and offers 6 courses. Each student can enroll in any combination of the 6 courses, including none or all of them. Prove or disprove: there must exist 5 students and 2 courses such that either all 5 students are in both courses, or all 5 students are in neither course.

4. Let ABC be a triangle and let D, E, and F be the midpoints of sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let P be the midpoint of $\overline{AE}$, and suppose $\overline{AD}$ intersects $\overline{CF}$ and $\overline{FP}$ at X and Y, respectively. Prove that triangle XFY is isosceles if and only if triangle ABC is isosceles.

5. Prove that for all positive integers n,
$$\frac{1}{2n+1} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}.$$

6. Find the smallest positive integer n such that $\sqrt{n+2012} - \sqrt{n} < 2$.

7. Let ABC be an equilateral triangle with $AB = 1$. Points D and E lie on $\overline{BC}$ and $\overline{CA}$ such that $BD = CE \le EA$. Point F is constructed, not on $\overline{AB}$, such that triangle DEF is equilateral. Given that $CF = 11/31$, find CE.

8. Let z_1, z_2, and z_3 be complex numbers such that $|z_1| = |z_2| = |z_3| = 1$ and $z_1/z_2 + z_2/z_3 + z_3/z_1 = 1$. Find $|z_1 + z_2 + z_3|$.

9. Find all x such that $\lfloor x \rfloor^2 - 5\lfloor x \rfloor + 4 = 0$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

10. Triangle XYZ has area 2013 and $XY = 61$. Point P lies on $\overline{XY}$ such that $XP = 10$. Find the area of triangle XPZ.

11. At a price of \$170 per book, Amazon sells 575 copies of Principles of Economics. For each \$10 increase in the price, Amazon sells 25 fewer books. At what price should they sell books to maximize revenue?

12. Find all integers x such that $2^x + 5^{2x} = 3^{2x+1}$.

13. Consider a tournament of n people in which each pair of people plays each other exactly once and in each match, one of the two wins (i.e. no ties). Show that there must be a player who wins at least $\frac{n-1}{2}$ of their games.

14. For how many pairs of consecutive integers in the set $\{1000, 1001, 1002, \ldots, 2000\}$ is no carrying required when the two integers are added?

15. Let w_1 be a fixed circle with center O, let w_2 be a fixed circle passing through the center of w_1 and intersecting w_1 at two points, and let w_3 be a circle internally tangent to w_2 and externally tangent to w_1. Let these two tangency points be A and B. Prove there exists a point P other than O such that the circle or line through A, B, and O passes through P, independent of the choice of w_3.

16. Two parallel lines each contain 2013 points. Each point on one line is connected to one point on the other line such that every point contains is at the endpoint of exactly one line segment. One step consists of swapping two adjacent points on either line. What is the least number of steps that will guarantee there to be no intersection between any two line segments?

17. Let $a_n = 12345678910\ldots n$ be the positive integer created by concatenating all of the positive integers from 1 through n, inclusive, in order from left to right. For example, $a_{11} = 1234567891011$. What is the smallest possible n for which "20133102" is a string of consecutive digits in a_n?

18. In chess, a rook is said to attack another piece if the two pieces lie in the same row or column. On a 2013×2013 chessboard, 2013 rooks are placed such no rook attacks another rook. During each step, a rook moves to a square adjacent (but not diagonal) to the square it is on. Prove that after 2013 steps, there must exist a rook that attacks another rook.

19. Let $p(n)$ be the sum of the distinct prime factors of n. Find $p(2) + p(3) + \cdots + p(40)$.

20. Let a, b, and c be positive real numbers such that $a + b + c = 1$. Prove that

$$1 - \frac{a}{2-2a} - \frac{b}{2-2b} - \frac{c}{2-2c} < a^4 + b^4 + c^4 + 2ab^2 + 2bc^2 + 2ca^2.$$

21. For any given triangle ABC, let A' be the point on $\overline{BC}$ such that $\overline{AA'}$ bisects $\angle BAC$. Define B' and C' similarly. Let I be the incenter of triangle ABC. Find the maximum possible value of $\dfrac{IA \cdot IB \cdot IC}{AA' \cdot BB' \cdot CC'}$.

22. There are N coins, one of which is fake. The genuine coins all have the same weight while the fake coin either weighs more or less than the other coins. For which values of N can it be determined, using only two weighings on a balance, whether the fake coin weighs less or more than the real coins?

23. How many distinct planes are there that pass through at least three vertices of a regular octahedron?

24. If $P(x)$ is a degree 3 polynomial with real coefficients such that $P(0) = -2$, $P(1) = -1$, $P(2) = 1$ and $P(3) = 2$, what is the value of $P(4)$?

25. In triangle ABC, segment $\overline{AD}$ is drawn with D on side $\overline{BC}$. Point E is on side $\overline{AB}$ such that $\overline{DE}$ is parallel to side $\overline{AC}$. Point F is chosen on side $\overline{AC}$ such that $\overline{AD}$, $\overline{BF}$ and $\overline{CE}$ are concurrent. Let $\overline{FD}$ intersect $\overline{CE}$ at point X and $\overline{FE}$ intersect $\overline{AD}$ at Y. Prove that $\overline{XY}$ is parallel to $\overline{AC}$.

26. Let S be the set of functions of the form $P(P(P(\cdots P(x)\cdots))) - x$, where there are any number of P's and $P(x)$ is any nonconstant polynomial with integer coefficients besides $P(x) = x$. Prove that $P(x) - x$ divides every element in S.

27. Let a_n be the number of divisors k of n satisfying $\sqrt{n} \le k < 2\sqrt{n}$. Compute $\displaystyle\sum_{n=1}^{2012^2} a_n$, given that

$$\sum_{n=1006}^{2012} \left\lfloor \frac{2012^2}{n} \right\rfloor = 2808487.$$

28. Given square $ABCD$ with side length 1, points E and F are on sides $\overline{AB}$ and $\overline{BC}$, respectively, such that $\overline{DF}$ bisects $\angle EDC$. If $\overline{EF}$ is perpendicular to $\overline{DF}$, what is the area of triangle DEF?

29. Find the sum of all positive integers $2 < n \le 20$ for which n divides $(n+1)(n+2)\cdots(2n-1)$.

30. Carl flips 15 coins and records either heads or tails on a sheet of paper by H and T, respectively. Given that he wrote down 10 H's and 5 T's, what is the probability that there exist exactly 3 occurrences of HT?