LHS Math Team
Team Contest 3

1. Show that there exists $n > 2$ such that 1991 divides $1999\dots91$ (with n 9's).

2. In a circle of radius 42, two chords of length 78 intersect at a point whose distance from the center is 18. The two chords divide the interior of the circle into four regions. Two of these regions are bordered by segments of unequal lengths. Find the area of one of these regions.

3. In triangle ABC, D is a point on $\overline{AB}$ and E is a point on $\overline{AC}$ such that $\overline{BE}$ and $\overline{CD}$ are bisectors of $\angle B$ and $\angle C$ respectively. Let Q, M and N be the feet of perpendiculars from the midpoint P of $\overline{DE}$ onto $\overline{BC}$, $\overline{AB}$ and $\overline{AC}$, respectively. Prove that $PQ = PM + PN$.

4. The sides of a convex polygon are all colored different colors. Then, every diagonal of the polygon is colored as well. Prove that there exists a triangle whose sides are sides or diagonals of the polygon whose edges are all different colors.

5. Given a positive integer n and an infinite sequence of proper fractions
$$x_0 = \frac{a_0}{n}, \dots x_i = \frac{a_i}{n+i},$$
with $a_i < n+i$ a positive integer, prove that there exist a positive integer k and integers $c_1 \dots c_k$ such that $c_1x_1 + \dots + c_kx_k = 1$.

6. Each one of 2009 distinct points in the plane is colored in blue or red, so that on every blue-centered unit circle there are exactly two red points. Find the greatest possible number of blue points.

7. Determine all pairs (a, b) of real numbers such that
$$a\lfloor bn \rfloor = b\lfloor an \rfloor$$
for all positive integers n, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

8. Finitely many circles lie in the plane, covering an area S. Prove that there exists a subset of these circles, mutually disjoint, covering an area of at least $S/9$.

9. Consider a triangle ABC with $AB = AC$, and let D be the foot of the altitude from the vertex A. The point E lies on the side AB such that $\angle ACE = \angle ECB = 18^\circ$. If $AD = 3$, find the length of the segment CE.

10. The roots of the equation $x^2 + 4x - 5 = 0$ are also the roots of the equation $2x^3 + 9x^2 - 6x - 5 = 0$. What is the third root of the second equation?

11. Find all solutions $x \in [0, 2\pi)$ to the equation $\sin^3 x(1 + \cot x) + \cos^3 x(1 + \tan x) = \cos(2x)$.

12. Determine the largest 2-digit factor of $3^{2^{2011}} - 2^{2^{2011}}$.

13. A positive integer is *troll* if at least one of its multiples begins with 2011. For example, 7 is troll because $20111 = 7 \times 2873$ is a multiple of 7 and begins with 2011. Prove that every positive integer is troll.

14. Let $c \geq 1$ be an integer, and define the sequence $a_1, a_2, a_3, \dots$ by
$$a_1 = 2$$
$$a_{n+1} = ca_n + \sqrt{(c^2-1)(a_n^2-4)}.$$
Prove that a_n is an integer for all n.

15. A rectangle $ABCD$ has side lengths $AB = m$ and $BC = n$ for m and n relatively prime odd positive integers. It is divided into unit squares, and diagonal $\overline{AC}$ intersects the boundaries of these unit squares, in order, at $A = A_1, A_2, \ldots, A_k = C$. Prove that

$$A_1A_2 - A_2A_3 + A_3A_4 - \cdots + A_{k-1}A_k = \frac{\sqrt{m^2+n^2}}{mn}.$$

16. There are n points that lie on a circle, and k segments are drawn at random between the points such that no two segments intersect on the circumference. What is the probability that no two of the segments intersect?

17. Two players A and B participate in the following game. Initially we have a pile of 2011 stones. A plays first, and he picks a divisor of 2011 and removes that number of stones from the pile. Then B picks a divisor of the number of remaining stones, and removes that number of stones from the pile, and so forth. The player who removes the last stone loses. Prove that one of the players has a winning strategy and describe it.

18. Let $a_1, a_2, \ldots, a_n$ be positive real numbers with $a_1 + a_2 + \cdots + a_n < 1$. Prove that

$$\frac{a_1a_2\cdots a_n(1-(a_1+a_2+\cdots+a_n))}{(a_1+a_2+\cdots+a_n)(1-a_1)(1-a_2)\cdots(1-a_n)} \le \frac{1}{n^{n+1}}.$$

19. Consider a cyclic quadrilateral $ABCD$, and let S be the intersection of AC and BD. Let E and F be the orthogonal projections of S on AB and CD respectively. Prove that the perpendicular bisector of segment EF meets the segments AD and BC at their midpoints.

20. Let $d(n)$ be the number of positive divisors of n. Find all positive integers n such that $d(n) = n/5$.

21. Prove that there are no positive integers x and y such that $x^5 + y^5 + 1 = (y+2)^5 + (x-3)^5$.

22. Solve in reals:

$$\begin{aligned} a^2 &= b^3 + c^3 \\ b^2 &= c^3 + d^3 \\ c^2 &= d^3 + e^3 \\ d^2 &= e^3 + a^3 \\ e^2 &= a^3 + b^3 \end{aligned}$$

23. Let ABC be a triangle with $\angle A = 45^\circ$. Let P be a point on side $\overline{BC}$ with $PB = 3$ and $PC = 5$. Let O be the circumcenter of ABC. Determine the length OP.

24. Let a, b, c, d be real numbers such that

$$\begin{gathered} a+b+c+d = -2 \\ ab+ac+ad+bc+bd+cd = 0. \end{gathered}$$

Prove that at least one of the numbers a, b, c, d is not greater than -1.

25. Let ABC be a triangle, and extend rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ to arbitrary points D and E, respectively. The bisectors of $\angle ABC$ and $\angle ACB$ meet at I, and the bisectors of $\angle DBC$ and $\angle ECB$ meet at J. Prove that the midpoint of $\overline{IJ}$ lies on the circumcircle of ABC.

26. Find all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy the following two conditions:

- $f(n)$ is a perfect square for all $n \in \mathbb{N}$
- $f(m+n) = f(m) + f(n) + 2mn$ for all $m, n \in \mathbb{N}$

27. Given 2011 people on Facebook, determine the largest possible number of pairs of people who are not Facebook friends, but share a Facebook friend in common.

28. Suppose $a_1, a_2, ..., a_{95}$ are positive reals. Show that

$$\sum_{k=1}^{95} a_k \leq 94 + \prod_{k=1}^{95} \max\{1, a_k\}$$

29. Find all prime numbers p such that $p^3 - 4p + 9$ is a perfect square.

30. George has 2011 ropes. He chooses two of the 4022 loose ends at random (possibly from the same rope), and ties them together, leaving 4020 loose ends. He again chooses two loose ends at random and joins them, and so on, until there are no loose ends. Find, with proof, the expected value of the number of loops George ends up with.