Trig with Complex Numbers

LHS Math Team

We first introduce the notion of polar form. Normally, when we write down the coordinates of a point in the plane, we describe it in terms of how far from the axes it is. For example, $(2, 3)$ denotes the point that is 2 units to the right of the y-axis and 3 units above the x-axis. However, another way of describing a point is by its distance form the origin, and the angle which the line segment between that point and the origin makes with the x-axis. Before moving on, convince yourself that this is, in fact true, that every point can be described in this way, and that no two points are described in the same way under this system.

The distance from a point P to the origin, sometimes called the *magnitude*, is denoted r, and the angle that PO makes with the origin, called the *argument* (in the same way that we have angles on the unit circle), is denoted θ. Together, we write $P = (r, \theta)$. By convention, $r \geq 0$. Notice that if we write this in terms of the coordinates we're used to (called rectangular form), we get the point $(r \cos\theta, r \sin\theta)$.

This is where the complex numbers start coming into play. However, we're going to need to accept the following fact:

Theorem: $e^{i\theta} = \cos\theta + i \sin\theta$, where θ is measured in **radians** (this is very important!). The right hand side is sometimes denoted cisθ for short.

Unfortunately, understanding this requires quite a bit of background. e, as you may recall is the base of the natural logarithm ln, is a real constant approximately equal to 2.718.... We don't know what this constant *really* is, and we have no idea what it means to raise it to an imaginary power. Even Taylor Series are not enough for a rigorous proof - we'll just have to accept it.

Now that we have this, let's put the point (r, θ) in the complex plane instead of the real plane. Now, this corresponds to the point $r \cos\theta + ri \sin\theta$, but note that this is just the point $re^{i\theta}$, which is nice. Thus, a complex number with magnitude r and argument θ is equal to just $re^{i\theta}$.

Exercise: What is the set of points of the form $e^{i\theta}$ for some angle θ?

This gives us a very easy way of multiplying complex numbers, and in particular, raising them to very high powers. Say we have two *complex numbers* (these are not real coordinates, but points in the complex plane) $(r_1, \theta_1), (r_2, \theta_2)$ in polar form. If we want to multiply them, we can rewrite them in the form $r_1e^{i\theta_1}$ and $r_2e^{i\theta_2}$. Now, multiplying them is easy, granted we accept the fact that all of the nice properties about exponentiation that hold over the reals also hold over the complex numbers (which they do). So we have that their product is $r_1r_2e^{i\theta_1 e^{i\theta_2}} = r_1r_2e^{i(\theta_1+\theta_2)}$. $\theta_1 + \theta_2$ is, of course, an angle, so we are back in exponential form. Converting back to polar form, we get the nice formula

$$(r_1, \theta_1) \times (r_2, \theta_2) = (r_1r_2, \theta_1 + \theta_2).$$

In words, to multiply two complex numbers in polar form, we multiply the magnitudes, and add the arguments.

Raising a number in polar form to an integral power is just as easy. All we're doing when we're raising something to an integral power is multiplying it by itself lots of times, so we can use our previous formula.

Exercise: Prove that $(r, \theta)^n = (r^n, n\theta)$, first using the formula from before, then by converting both sides to exponential form. The second proof actually shows that it's true for all real numbers n, not just positive integers.

Writing this in rectangular form, and setting $r = 1$ gives a nice-looking formula, and you can verify that the result is that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, or $(\text{cis}\theta)^n = \text{cis}n\theta$.

Exercise: Compute $\sqrt{i}$.

With this, we can now solve equations like $x^3 = \sqrt{2} + \sqrt{2}i$. However, we have to be careful. Just as the equation $x^2 = 1$ cannot be solved by simply trying to take the square root of both sides, we need to do a little more work to solve this equation. The first step is to convert the number on the right hand side to polar form. Its magnitude is $\sqrt{2+2} = 2$, and we can see that the argument is $\frac{\pi}{4}$, since $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Thus, in polar form, it is $(2, \frac{\pi}{4})$.

So we're looking for (r, θ) such that $(r, \theta)^3 = (r^3, 3\theta) = (2, \frac{\pi}{4})$. This is nice, since we can just match up the values in each coordinate. We need $r^3 = 2$ and $3\theta = \frac{\pi}{4}$. r is a positive real number, so the only value of r that will work is $\sqrt[3]{2}$. For the second equation, dividing both sides by 3 gives $\theta = \frac{\pi}{12}$. But be careful! This is not the only value of θ that works. We're looking for θ in the range $[0, 2\pi)$, but by multiplying by 3, the period gets 3 times smaller. This means that we can also have $3\theta = \frac{9\pi}{4}$, which is the same as $\frac{\pi}{4}$, but θ takes a different value (and we see that we will in fact get a different corresponding complex number). Thus, we have the additional solution $\theta = \frac{3\pi}{4}$. But in fact, we can do this one more time, to get $3\theta = \frac{17\pi}{4}$, so $\theta = \frac{17\pi}{12}$. If we try to repeat this, we get $3\theta = \frac{25\pi}{4}$, so $\theta = \frac{25\pi}{12}$. However, this is the same solution as our first, so from here the solutions start to cycle.

In summary, we get $r = \sqrt[3]{2}$, and θ is $\frac{\pi}{12}$, $\frac{3\pi}{4}$, or $\frac{17\pi}{12}$. This gives us our three solutions (if you know your trig functions of 15 degrees, you can write down the solutions in rectangular form). This is good, since the original equation was a cubic, and it should have 3 solutions in the complex numbers. This will always be true: a polynomial equation of degree n will have exactly n solutions, and you shouldn't stop until you find them all.

We'll end with a particularly nice class of these types of equations, namely those of the form $x^n = 1$, where n is a positive integer. We're thus looking for the n solutions to $(r, \theta)^n = (r^n, n\theta) = (1, 0)$. $r^n = 1$, so $r = 1$, as r is a positive real. Now, $n\theta = 0$, which has the solutions $\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{2(n-1)\pi}{n}$ (why?). These are all distinct (why?), and there are n of them, so we have found all n solutions. We see that in exponential form, our solutions are

$$1, e^{2\pi i/n}, e^{4\pi i/n}, \ldots, e^{2(n-1)\pi i/n}.$$

These are called the n-th roots of unity.

Exercise: Graph the n-th roots of unity in the Complex Plane, for your favorite values of n. Connect them in a way that seems fitting. What does the picture look like? Why should it look this way? Generalize.