Team Contest

September 29, 2013

1 Geometry

1.) Two congruent circles are put inside a square, externally tangent to each other. One is tangent to two sides of the square, and the other circle is tangent to the other two sides. What is the area of one of the circles?

2.) Determine, for any 4 points, when there exists a quadrilateral with those points as the midpoints of its sides. If there does then construct it. If there doesn't then prove it.

3.) Determine, for any 5 points, when there exists a pentagon with those points as the midpoints of its sides. If there does then construct it. If there doesn't then prove it.

4.) Let ABC be a triangle, and D and E be the angle bisectors of B and C on AC and AB, prove that the circumcircles of ACE and ABD intersect on BC iff angle $A = 60°$

5.) Triangle ABC is inscribed in circle ω. Let H be the orthocenter of the triangle, and let M and N be the midpoints of segments BC and AH respectively. Point P lies on line BC such that AP is tangent to ω. Prove that $PN \perp AM$

2 Algebra

6.) Find the solutions of the equations

$$x^2 + 4xy + 4y^2 + 8x + 16y + 15 = 0$$
$$3x^2 + 6xy + 3y^2 + 5x + 5y + 2 = 0$$

7.) Find all integer polynomials $f(x)$ with all nonzero coefficients and

$$f(x^2) + 2 = f(x)^2 + 2x$$

for all x

8.) Call a set of integers lexcellent if it has the property that for any elements (not necesarily distinct) x and y of A, $x^2 + kxy + y^2$ is in A for all integers k. Find all pairs of integers (m,n) of nonzero integers such that the only lexcellent set containing m and n is the set of all integers.

9.) Prove that if $x, y, z \geq 0$ and $\frac{2ab^2}{b+c} + \frac{2bc^2}{c+a} + \frac{2ca^2}{a+c} \leq ab + bc + ac$ then

$$a^2b + b^2c + c^2a \leq a^2c + b^2a + c^2b$$

10.) Find all polynomials p(x) with integer coefficients such that for every positive integer n, the number $2^n - 1$ is divisible by p(n).

3 Number Theory

11. Is any number formed by a permutation of the digits 0-9 (for example, 7980625341) a prime?

12. How many solutions for digits a,b are there such that

$$a000000.....00000b$$

is an integer with 2015 0's and is divisible by 13.

13. For a positive integer n, consider all of its divisors (including 1 and itself). Suppose that p% of these divisors end in a 3 (for example, 117 has factors 1,3,9,13,39 and 117 $33\frac{1}{3}$% of which end in 3) Find, for a positive integer n, the maximum value of p (with proof).

14. What is $\binom{3x}{3} + \binom{3x}{6} + \binom{3x}{9} + ... + \binom{3x}{3x}$

15. Determine whether there are infinitely many quadruples (a,b,c,n) of positive integers such that

$$(a^3 - a)(b^3 - b)(c^3 - c) = n^2$$

4 Combinatorics

16. How many arrows which point parallel to the sides of the grid, can you put in an nxn grid, such that no two arrows point towards each other?

17. How many ways can you put 2013 girls and 1337 boys in a line, such that there are exactly 42 places that a boy and a girl are next to each other (YOU DO NOT NEED THE EXACT NUMBER!!! Just a simplified form!)

18. You are given the infinite plane, tiled with hexagons. Each of these hexagons may have some integral amount of drops of honey in it to start with. On a turn, you may put a bee on one of the vertices, which will put a drop of honey in each of the surrounding hexagons, or you can put an evil bee, wihch will take one drop of honey from each of the hexagons. From which starting honeycomb positions can you reach an empty honeycomb?

19. How many subsets are there of $\{1, 2, 3, ...2^n\}$ such that the sum of its elements is equal to a positive multiple of 2^n.

20. How many ways can you put n non-overlapping tetrominos to tile a $2nx2$ grid?