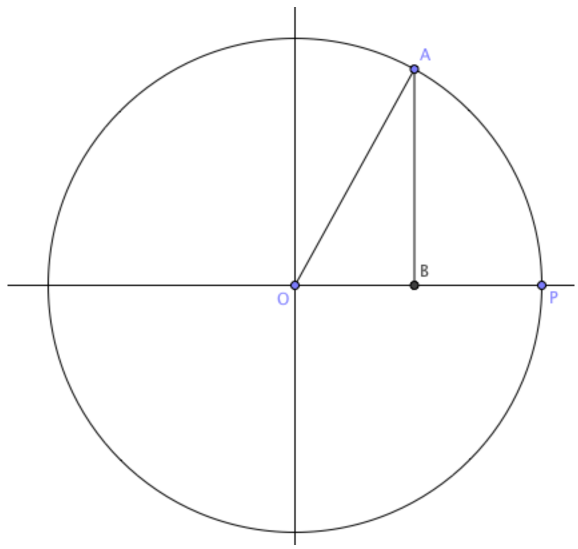
Unit Circle Trig

LHS Math Team

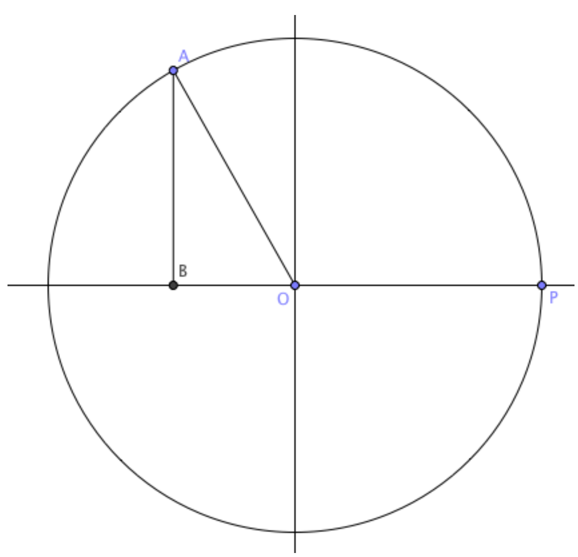
We now know how to work with trig functions of acute angles, but we want to extend this to all angles. We do this by fixing axes, and using the unit circle, the set of points that are 1 unit away from the origin in the plane.

We'll start with acute angles. Our 'base' is going to be the positive x-axis, or the segment between $O = (0, 0)$ and $P = (1, 0)$, as shown below. Then, we will take any point A on the circle in the first quadrant, and drop a perpendicular AB down to OP, giving us a right triangle ΔAOB. Let $\angle POA = \theta$.



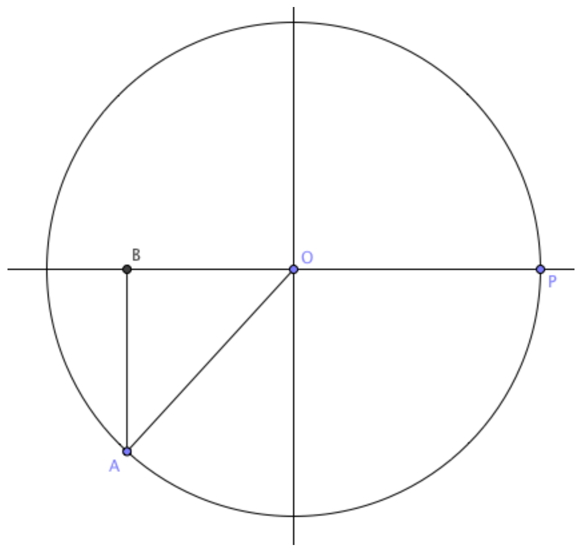
Since OA is a radius of the circle, $OA = 1$. This gives us that $AB = \sin\theta$ and $OB = \cos\theta$. However, this is just another way of saying that the coordinates of point A are $(\cos\theta, \sin\theta)$, since $\angle ABO = 90°$. Now, we have a relation between the sine and cosine of an angle and the unit circle, so we can now extend our definition to non-acute angles.

Let's say we took A in the second quadrant instead. Then, we still have $\angle POA = \theta$, except this time $\theta > 90°$. We can go so far as to redefine sine and cosine, so that $A = (\cos\theta, \sin\theta)$ (Why does this imply that $\cos^2\theta + \sin^2\theta = 1$ for all θ?) For example, let's take $\theta = 120°$.



Then, $\angle AOB = 180° - \angle AOP = 60°$. Now, $AB = \sin 60° = \frac{\sqrt{3}}{2}$, and $BO = \cos 60° = \frac{1}{2}$. Therefore, $A = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and we get $\cos 120° = -\frac{1}{2}$ and $\sin 120° = \frac{\sqrt{3}}{2}$.

Now, let's keep going, putting A in the third quadrant instead. Note that now, $\angle POA$, if we measure it counterclockwise, is a reflex angle - its measure is greater than $180°$. In this way, we can thus compute the values of trig functions of angles whose measures are greater than $180°$. As an example, we'll take $\cos 225°$ and $\sin 225°$.



We have $\angle OAB = 225° - 180° = 45°$, so $AB = \sin 45°$ and $BO = \cos 45°$. Thus, $(\cos 225°, \sin 225°) = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

Using the unit circle is the most efficient way to calculate trig functions of 'special angles' by hand. There are also many identities that relate the trig functions of angles in certain quadrants to trig functions of angles in the first quadrant (acute angles) that allow us to calculate these ratios, but memorizing these identities are, frankly, a waste of brain space. However, you should do the following:

Exercise: Convince yourself that $\sin(90° + \theta) = \cos\theta, \cos(180° - \theta) = -\cos\theta, \tan(180° + \theta) = -\tan\theta$.

Back to our original diagram: say we kept moving A counter clockwise until we got back to the first quadrant, and, for example, $\theta = 390°$. This is the exact same configuration as having $\theta = 30°$, so by convention we say that $\sin\theta = \sin(\theta + 360°)$, and similarly for all other trig functions. Also, we say that if A is such that $\angle POA = \theta$, if A' is the reflection of A over the x-axis, $\angle POA' = -\theta$.

Finally, we introduce radian measure. This may seem superficial or arbitrary now, but it turns out to be very important in calculus, and when using complex numbers and trig together (as we'll see later). It's simply a linear conversion of degree measure; π radians is equal to $180°$ degrees, and in general, k radians is equal to $\frac{180k}{\pi}$ degrees. An equivalent definition of radian measure is the length of the arc cut out in the unit circle by a sector with the given central angle. For example, a $60°$ sector in the unit circle cuts out $1/6$ of the circumference of the circle, which is $\frac{\pi}{3}$, equal to $60°$ in radians.