“Tryout” - LHS Math Team

Monday, September 12, 2011

1. Ten poles lie evenly spaced apart on a straight line such that the first and last poles are 100 meters apart. What is the distance between any two adjacent poles?

2. How many prime divisors does 2011 have?

3. In the math team, there are 12 freshmen, 8 sophomores, 11 juniors, and 5 seniors. To get papers sorted, Roos selects members from the math team to be in a group to do the sorting until either there is a freshman in the group or there are 3 juniors in the group. What is the minimum number of people that Roos would need to pick to guarantee that one of these two conditions is satisfied?

4. What is the smallest possible number, greater than 1, of consecutive integers that sum to 2012?

5. Bob needs to round a positive integer n to the nearest thousand. He rounds n to the nearest ten, then rounds the result to the nearest hundred, then rounds this result to the nearest thousand. Unfortunately, Bob obtains the wrong answer. For how many values of n less than 1000 can this occur?

6. Let ABC be a triangle, and let D and E be points on $\overline{AB}$ and $\overline{AC}$ such that
$$AD : DB = AE : EC = 2 : 5.$$
What is the ratio of the area of triangle ABC to the area of triangle ADE?

7. Dale is driving his car across a hill that goes up for 80 meters and then flattens for 50 meters before dropping off a cliff. He starts at the bottom, but 3/4 of the way up, his back tires fall off, so at any given time, his speed is reduced by a factor of 2/3 from what it would normally be. After he travels 40 meters on the flat hilltop without back tires, Dale stops for 15 seconds to replace his tires and to buy some high octane jet fuel, so his speed afterwards is double the normal rate. Dale then drives so fast that he goes off the hilltop edge and 50 meters in the air (still at a level pace) before realizing that his car was not touching the ground. He then falls rapidly for 5 seconds. If this whole adventure took 140 seconds and Dale's normal uphill speed is 5 m/s less than his normal level speed, what was Dale's normal level speed?

8. A parabolic arch has a base width of 10 and a height of 8. A rectangle is inscribed in the parabolic arch with one side along the base and two vertices on the parabola, and the rectangle has height 5. What is the width of the rectangle?

9. When the number 2011^{2012} is expressed in base 7, what are the last two digits?

10. Given that
$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90},$$
find
$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots .$$

11. At a train station, a train arrives between 12:00 and 13:00, stops for 13 minutes, and then leaves. Jill goes to the station and arrives between 12:00 and 13:00, waits for 37 minutes, and then leaves. Assuming the train and Jill can each arrive at any uniformly random time in the hour interval, what is the probability that the two are both in the station at some time?

12. Find all real values of k such that $(3k - 2i)^4$ is a real number, where $i^2 = -1$.

13. A triangle has side lengths in the ratio of $4 : 6 : 7$. If θ is the smallest angle in the triangle, what is $\tan\theta$?

14. Given that
$$\log_{y/x}(x^2y^3) + \frac{\log_5(xy)}{\log_5(x/y)} = \frac{1}{3},$$
if y is multipled by $81/16$ but the equation holds true, then by what factor is x multiplied?

15. A square of side length x is cut from each corner of a larger square of side length 17. The resulting figure is folded into a box with an open top. What value of x will maximize the volume of the box?

16. Find the greatest positive integer x such that $x + 2$ divides $x^4 + 22$.

17. A particle starts from the origin and its first step is moving one unit to the east. For every subsequent step, the particle turns 45 degrees to the left and moves $\frac{1}{\sqrt{2}}$ the length of its previous step. After doing this infinitely many times, the particle is how far away from the origin?

18. Given that a, b, c, and d are positive numbers satisfying $a^2 + b^2 = 3$, $b^2 + c^2 = 7$, $c^2 + d^2 = 11$, and $ac = bd$, find $ab + bc + cd + da$.

19. Ravi needs to walk from $(0, 0)$ to $(11, 14)$ to retrieve his backpack from the ground. Each step, he goes 1 unit in either the positive x-direction or the positive y-direction. Assuming that he reached $(11, 14)$ after taking 25 steps, what is the expected value of the number of direction changes in Ravi's path?

20. Let ABC be a triangle with $m\angle A = 53^\circ$ and $m\angle B = 75^\circ$. Let D and E be points on $\overline{BC}$ and $\overline{AC}$, respectively, such that $\overline{AD} \perp \overline{BC}$ and $\overline{BE} \perp \overline{AC}$, and define H to be the intersection of $\overline{AD}$ and $\overline{BE}$. Suppose M is the midpoint of $\overline{AB}$, and let K be the intersection of ray $\overrightarrow{MH}$ with the circle passing through A, B, and C. Find $m\angle CKE$.