AM-GM: Arithmetic Mean - Geometric Mean: The arithmetic mean of a set of nonnegative (or positive) numbers is greater than or equal to the geometric mean of the numbers. Formally,

If $a_1, a_2, \cdots, a_n \geq 0$, then $\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$

Equality condition: Equality holds if and only if all the numbers in the set are equal:

$\frac{a_1+a_2+\cdots+a_n}{n} = \sqrt[n]{a_1 a_2 \cdots a_n}$ if and only if $a_1 = a_2 = \cdots = a_n$

1. Given $x \geq 0$, find the minimum value of $4x + \frac{9}{x}$.

Look for hints that relate to AM-GM, such as nonnegative x. Also, many AM-GM problems involve fractions with variables in some denominators, such as in 9/x.

If you are writing a complete solution, you must also show that your value is actually achievable, in which you would use the equality condition. Even if you aren't, it's worth checking whether your value can be achieved, using the equality condition.

Other times the expression is not as straight-forward, like in the problem below, although the x > 0 hints towards AM-GM.

2. Given $x \geq 0$, find the value of x that minimizes $\frac{3x^2+8x+5}{4x}$.

AM-GM can be used in proof problems as well. For example,

3. Prove that $\frac{x}{y} + \frac{y}{x} \geq 2$ for nonnegative numbers x and y.

We could simply say, "Since x/y, y/x ≥ 0, we can apply AM-GM to x/y and y/x to get...."

Sometimes, we may need to add or multiply inequalities to get what we want. You may not figure out immediately what inequalities you should combine. Consider the following problems:

4. Show that if x, y, and z are nonnegative, then
$xy + xz + yz \geq x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}$.

Hint: Apply AM-GM to {xy, xz}, {xz, yz}, and {yz, xy}, then combine the inequalities.

5. Find the minimum value of $(x+y+z)(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})$, given that x, y, and z are nonnegative.

AM-GM can be used in geometry because all lengths must be positive. For instance,

6. A rectangular box with no top has a volume of 1000 cubic centimeters. Find its minimum possible outer surface area in square cm.

Next, we attempt to solve the following problem using AM-GM. (It can also be solved using the minimum value formula for quadratics.)

7. A family wants to build a rectangular yard beside a river. They have 60 feet of fencing for three sides of the yard. The last side is bounded by the river. Find the maximum possible area for their yard in square feet.

Let the sides of the yard perpendicular to the river have length x. Then, the other sides have length 20 - 2x. We must find the maximum area of x * (20 - x). Explain why we cannot AM-GM x and (20 - 2x). What can we do?

8. Let x, y, and z be nonnegative numbers that sum to 6. Find the maximum value of xy^2z^3.

9. Prove that $(\frac{n+1}{2})^n > n!$ for positive integers $n \geq 2$. Why can't both sides be equal?

10. Find the minimum value of $\frac{9x^2\sin^2x+4}{x\sin x}$ for $0 < x < \pi$. (AIME, 1983)

Here's a USAMO problem that showed up in 2009 that can be solved using AM-GM. Note that it can get difficult, especially if you're new at olympiad problems.

For $n \geq 2$ let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$(a_1 + a_2 + \cdots + a_n)(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}) \leq (n + \frac{1}{2})^2$.

Prove that $\max(a_1, a_2, \ldots, a_n) \leq 4\min(a_1, a_2, \ldots, a_n)$.

Hint: Prove the contrapositive of what you need, that is,

If $\max(a_1, a_2, \ldots, a_n) > 4\min(a_1, a_2, \ldots, a_n)$, then

$(a_1 + a_2 + \cdots + a_n)(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}) > (n + \frac{1}{2})^2$.