Round 4: Algebra 2 – Logs, Exponents, Radicals and equations involving them

1. Given $(\log_x 5)(\log_5 64) = 1.5$, find all real solutions for x.

   Solution: From the third exercise at the bottom of the Logarithms handout, the left hand side of the equation simplifies to $\log_x 64 = 1.5$. Rewriting this in exponential notation,

   $$x^{1.5} = 64.$$

   Thus, $x = 16$. <u>This is not negative, so it is valid in the original equation.</u>

2. Given $2\sqrt[4]{x^3} - 3\sqrt{x} = 4\sqrt[4]{x} - 6$, find all real solutions for x.

   Solution: The first thing to realize when solving this problem is that we actually want to solve for $\sqrt[4]{x}$, since all variable terms in the equation are powers of this (note that $\sqrt{x} = \sqrt[4]{x^2}$. Substituting $y = \sqrt[4]{x}$,

   $$2y^3 - 3y^2 = 4y - 6 \Longrightarrow 2y^3 - 3y^2 - 4y + 6 = 0$$

   $$\Longrightarrow y^2(2y-3) - 2(2y-3) = 0$$

   $$\Longrightarrow (y^2 - 2)(2y - 3) = 0.$$

   From this factorization, $2y - 3 = 0 \Longrightarrow y = 3/2$ or $y^2 - 2 = 0 \Longrightarrow y = \pm\sqrt{2}$. In our original substitution, we can express x in terms of y as $x = y^4$. Thus, the two solutions for x are $x = 81/16$ and $x = 4$. <u>Plugging them into the original equation, they both work.</u>

3. Given $\log_x 3 + \log_9 x = \frac{17}{12}$, find all real solutions for x in simplest radical form.

   Solution: We first note that, from the second and third rules on the Logarithms handout (not the exercises), $\log_9 x = \log_3 x^{0.5} = \frac{1}{2}\log_3 x$. Now, we solve for $y = \log_3 x$. From the second exercise on the Logarithms handout, $\log_x 3 = 1/y$, so

   $$\frac{1}{y} + \frac{y}{2} = \frac{17}{12} \Longrightarrow 12 + 6y^2 = 17y$$

   $$\Longrightarrow (2y - 3)(3y - 4) = 0.$$

   This gives us the two cases $y = 3/2$ and $y = 4/3$. From the substitution we made, $y = \log_3 x \Longrightarrow x = 3^y$. This gives two solutions: $x = 3^{3/2} = 3\sqrt{3}$ and $x = 3^{4/3} = 3\sqrt[3]{3}$. <u>Both of these values are positive, so they are legitimate solutions to the original equation.</u>