Cubical Type Theory Inside a Presheaf Topos

By
Clémence Chanavat

Under the supervision of:
Pierre-Louis Curien

Master's thesis

August 2022

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Acknowledgments

First and foremost, I would like to thank Pierre-Louis Curien for his precious help and support during this internship, it was always a pleasure to meet with you and discuss proofs and ideas.

Next, I would like to thank the people in the cubical type theory community I had the chance to talk to, Nicola Gambino, Jon Sterling, and Thierry Coquand, thanks a lot, your time and explanations are what have made this thesis possible. More generally, I would also thank everyone in the community. Your work is always available free, open source, and was a vital tool for this work.

More personally, I would like to thank the people from the LMFI I met this year, and in particular Adrienne, Elaheh, and Sidney for all the interesting discussions we had. Finally, I want to thank my family for their constant love and support.

Introduction

The first models of Homotopy Type Theory, validating the univalence axiom, were constructed by Voevodsky using the standard model structure on simplicial sets [KLV12]. However, the use of classical logic prevents them to explain the univalence computationally. The most fruitful approach to finding constructive models is based on cubical sets [CMS20]. Cubical models of Homotopy Type Theory use presheaves and rely on the important feature, absent in the simplicial case, that the product of representables is again a representable. This allows us to define a representable presheaf $\mathbb{I}$, the interval object, which is a synthetic counterpart of $[0, 1]$, and to take the product with itself n times, to represent the n-dimensional cube $\mathbb{I}^n$. This interval comes with two endpoints $0, 1 : \mathbb{I}$, and has often a bunch of additional structures (distributive lattice, de Morgan algebra, etc.). In this setting, we have a natural way to implement the notion of path, if $a, b : A$, then a path $p : \texttt{Path}\ A\ a\ b$ between a and b is simply a map $p : \mathbb{I} \to A$ whose endpoints are a and b, i.e. such that $p(0) = a$ and $p(1) = b$. Thus, cubical models have built-in path types, of which we can show that they support reflexivity, path induction, and transport. However, this path type cannot directly be used as the identity type presented in [Uni13]. This is known as the regularity problem [Mö21] and implies that in particular, we do not have $\texttt{transp}^i\ A\ a = a$, whenever A is independent of i (here i is a dimension in the n-dimensional cube $\mathbb{I}^n$). Put more simply, the transport along the constant path need not be the constant function. However, there are workarounds, and one can construct Martin-Löf identity type using Path [CCHM15].

To understand the importance of the univalence axiom, it is important to notice that usually, the practice of mathematics is to work without distinction of isomorphic objects. However, when dealing with formal type theory, such identifications are by no means automatic, but can be done by the univalence axiom. Unfortunately, univalence can a priori break the constructive nature of type theory. For instance, one can lose canonicity and one can construct a closed term of natural number type that does not reduce to a numeral, or we do not necessarily have witness extraction for Σ-types [Mö21]. Thus, the introduction of cubical methods, starting with Bezem, Coquand, and Huber in [BCH19], was motivated by the goal of giving computational content to the univalence axiom.

In this thesis, we aim to study one particular model of cubical type theory under the light of tools developed for topos theory, known as Kripke-Joyal forcing.

Reading Guide

In the first section, we present the basics of the syntax of cubical type theory. We start with the underlying dependent type theory and we enrich it with some more rules and structures. Then, we briefly introduce the concept of glueing, and discuss the univalence axiom.

The second section is about topos theory. We recall and prove classical results about the subobject classifier, and we show how to internalize formulas inside a topos thanks to Kripke-Joyal semantics. Then, we move to the notion of Kripke-Joyal forcing, which allows us to check mechanically the validity of any formula easily. The material of this section could have been presented in a more general setting, but we chose to specialize our result to the particular case of presheaf toposes.

In the third part, we show how the notion of forcing can be generalized to the case of categories with families in a Hofmann-Streicher lifting of a Grothendieck universe. We present this construction and its interaction with the subobject classifier already living in the topos, and we show that the forcing in the previous section is a particular case of a more general forcing that applies to the type theory of the presheaf category.

Finally, we put the previous results into action and use forcing in a particular case. The work of [OP16] showed that it is possible for any topos to internalize cubical type theory and univalence provided that it satisfies a list of nine axioms. Thus, we pick the presheaf category from [CCHM15,] and use our previous results to prove that it satisfies this list of axioms, and thus is indeed a model of cubical type theory.

1 Cubical type theory

In this section, we present the cubical type theory developed in [CCHM15]. We first introduce a dependent type theory, on top of which we build path types and systems that we use to compose paths. Then, we briefly introduce the fundamental notion of glueing and show how it proves univalence. This section is only syntactic and the semantical counterpart of the notions presented here are to be found in the next sections.

1.1 Dependent type theory

1.1.1 Rules and interval

We use a standard Martin-Löf dependent type theory, enriched with an object $\mathbb{I}$ which is the free de Morgan algebra on a fixed infinite set of names $i, j, k, \dots$. The grammar of elements of $\mathbb{I}$ is

$$r, s ::= 0 \mid 1 \mid i \mid \neg r \mid r \wedge s \mid r \vee s$$

and the basic syntax of the type theory is

$$\begin{array}{llll} \Gamma, \Delta & ::= & () \mid \Gamma, x : A \mid \Gamma, i : \mathbb{I} & \text{Contexts} \\ t, u, A, B & ::= & x \mid \lambda x : A.t \mid t\ u \mid (x : A) \to B & \Pi\text{-types} \\ & \mid & (t, u) \mid t.1 \mid t.2 \mid (x : A) \times B & \Sigma\text{-types} \\ & \mid & \texttt{Path}\ A\ t\ u \mid \langle i \rangle\ t \mid t\ r & \text{Path types} \end{array}$$

Note that $\mathbb{I}$ has not the same status as the other types, and we cannot λ-abstract over a variable of type $i : \mathbb{I}$. Instead, we write $\langle i \rangle.t$, giving a term of type `Path` (as explained below).

We could also enrich this theory with a natural number type, and induction, but we will not use it in this thesis. We write $A \to B$ for non-dependent function and $A \times B$ for non-dependent pairs. The syntax is considered up to α-renaming and we have the usual η and β rules. We also have a substitution defined by induction with $\Delta \vdash () : ()$ and $\Delta \vdash (\sigma, x/u) : \Gamma, x : A$ if $\Delta \vdash \sigma : \Gamma$ and $\Delta \vdash u : \Gamma\sigma$. This type theory also possesses a judgmental equality for types $A = B$, and terms $a = b : A$ which are congruences and an equivalence relation on terms.

The de Morgan algebra structure on $\mathbb{I}$ means that the following equations are judgmentally true.

$\neg 0 = 1$	$\neg 1 = 0$	$\neg(r \vee s) = \neg r \wedge \neg s$	$\neg(r \wedge s) = \neg r \vee \neg s$

Note that in general, $\neg r \vee r \neq 1$ and $\neg r \wedge r \neq 0$, as we want $\mathbb{I}$ to behave like the interval $[0, 1]$ with $\vee$ representing max and $\wedge$ representing min. We extend substitution for path types, and substitution of the form $(i/0)$ and $(i/1)$ will be written $(i0)$ and $(i1)$. We have the η-rule and following inference rules for path types.

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash \texttt{Path}\ A\ a\ b} \qquad \frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash a : A}{\Gamma \vdash \langle i \rangle\ a : \texttt{Path}\ A\ a(i0)\ a(i1)}$$

$$\frac{\Gamma \vdash p : \texttt{Path}\ A\ a\ b \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash p\ r : A} \qquad \frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash a : A \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash (\langle i \rangle\ a)\ r = a(i/r) : A}$$

$$\frac{\Gamma, i : \mathbb{I} \vdash p\ i = q\ i : A}{\Gamma \vdash p = q : \texttt{Path}\ A\ p_0\ p_1} \qquad \frac{\Gamma \vdash p : \texttt{Path}\ A\ p_0\ p_1}{\Gamma \vdash p\ 0 = p_0 : A} \qquad \frac{\Gamma \vdash p : \texttt{Path}\ A\ p_0\ p_1}{\Gamma \vdash p\ 1 = p_1 : A}$$

With those rules, we can check that, for $a : A$, $1_a = \langle i \rangle\ a$ is of type $\texttt{Path}\ A\ a\ a$ and corresponds to a proof of reflexivity. Those rules also allow us to prove standard operations on identity types, for instance, if $\Gamma \vdash a : A$, $\Gamma \vdash b : A$, $\Gamma \vdash f : A \to B$ and $\Gamma \vdash p : \texttt{Path}\ A\ a\ b$, then $\Gamma \vdash \langle i \rangle\ f\ (p\ i) : \texttt{Path}\ B\ (f\ a)\ (f\ b)$. Indeed, we have that $p\ i : A$, so $f\ (p\ i) : B$ and by the second rule above, $f\ (p\ i) : \texttt{Path}\ B\ (f\ p(i0))\ (f\ p(i1))$, with $p(i0) = a$ and $p(i1) = b$. We can also prove function extensionality as:

$$\frac{\Gamma \vdash f : (x : A) \to B \qquad \Gamma \vdash g : (x : A) \to B \qquad \Gamma \vdash p : (x : A) \to \texttt{Path}\ B\ (f\ x)\ (g\ x)}{\Gamma \vdash \langle i \rangle\ \lambda x : A.\ p\ x\ i : \texttt{Path}\ ((x : A) \to B)\ f\ g}$$

Indeed, $p\ x\ i : B$ with $(p\ x\ i)(i0) = f\ x$ and $(p\ x\ i)(i1) = g\ x$, thus we have

$$\Gamma, i : \mathbb{I} \vdash \lambda x : A.\ p\ x\ i : (x : A) \to B$$

which η-reduces to f when i is 0, and to g when i is 1.

When there are n variables of dimension $i_1, \ldots, i_n : \mathbb{I}$ in the context, we are working in an n-dimensional cube. For instance

$$i : \mathbb{I}, j : \mathbb{I} \vdash A$$

corresponds to

$$\begin{tikzcd}
A(i0)(j1) \arrow[r, "A(j1)"] & A(i1)(j1) \\
A(i0)(j0) \arrow[u, "A(i0)"] \arrow[r, "A(j0)"'] & A(i1)(j0) \arrow[u, "A(i1)"']
\end{tikzcd}$$

1.1.2 Face lattice

We introduce the face lattice $\mathbb{F}$, whose elements will describe sub-polyhedra of a n-dimensional cube. Using this, we can define terms partially on the cube, and under some conditions, extend them along paths. This is composition. The face lattice $\mathbb{F}$ is the distributive lattice generated by the symbols $(i = 0)$ and $(i = 1)$ (for all dimension name i) with relation $(i = 0) \wedge (i = 1) = 0_{\mathbb{F}}$. The grammar of the elements of $\mathbb{F}$ is described by

$$\varphi, \psi ::= 0_{\mathbb{F}} \mid 1_{\mathbb{F}} \mid (i = 0) \mid (i = 1) \mid \varphi \wedge \psi \mid \varphi \vee \psi$$

We extend the context grammar with

$$\Gamma, \Delta ::= \ldots \mid \Gamma, \psi$$

and with the rule

$$\frac{\Gamma \vdash \psi : \mathbb{F}}{\Gamma, \psi \vdash}$$

For an arbitrary judgment J, $\Gamma, \psi \vdash J$ should be thought as J being valid only on the sub-polyhedron described by ψ. For instance

$$i : \mathbb{I}, j : \mathbb{I}, (i = 0) \vee (j = 1) \vdash A$$

should be illustrated as

$$\begin{tikzcd}
A(i0)(j1) \arrow[r, "A(j1)"] & A(i1)(j1) \\
A(i0)(j0) \arrow[u, "A(i0)"] &
\end{tikzcd}$$

For any context Γ, we recursively define a congruence $=_\Gamma$ on $\mathbb{F}$, that is to be thought as a partial equality. We define

$$\Gamma, \psi \vdash \varphi_0 =_{\Gamma,\psi} \varphi_1 : \mathbb{F}$$

to be equivalent to

$$\Gamma \vdash \psi \wedge \varphi_0 =_\Gamma \psi \wedge \varphi_1 : \mathbb{F}$$

and $=_\Gamma$ is the same congruence as $=_{\Gamma'}$ for any other context extension Γ' of Γ that does not involve elements $\psi : \mathbb{F}$. The congruence $=_{()}$ on the empty context is the regular equality on $\mathbb{F}$. Thus, if there is no face declared in Γ, then $=_\Gamma$ is also the equality. From now on, we drop the subscript Γ on $=_\Gamma$, and simply write $=$.

1.1.3 Systems and composition

Next, systems allow us to combine sub-polyhedra, provided that they are compatible. The syntax for systems is:

$$t, u, A, B ::= \dots \mid [\varphi_1\ a_1 \dots \varphi_n\ a_n]$$

And, provided that $\Gamma \vdash \varphi_1 \vee \cdots \vee \varphi_n = 1_\mathbb{F} : \mathbb{F}$ (the equality is the congruence defined by Γ), then we have the following rules.

$$\frac{\Gamma, \varphi_1 \vdash a_1 : A \qquad \dots \qquad \Gamma, \varphi_n \vdash a_n : A \qquad \Gamma, \varphi_i \wedge \varphi_j \vdash a_i = a_j : A\ (i \leq i, j \leq n)}{\Gamma \vdash [\varphi_1\ a_1 \dots \varphi_n\ a_n] : A}$$

$$\frac{\Gamma, \varphi_1 \vdash J \qquad \dots \qquad \Gamma, \varphi_n \vdash J}{\Gamma \vdash J} \qquad \frac{\Gamma \vdash [\varphi_1\ a_1, \dots, \varphi_n\ a_n] : A \qquad \Gamma \vdash \varphi_i = 1_\mathbb{F} : F}{\Gamma \vdash [\varphi_1\ a_1, \dots, \varphi_n\ a_n] = a_i : A}$$

We allow $n = 0$, in this case the side condition becomes $\Gamma \vdash 0_F = 1_\mathbb{F} : \mathbb{F}$, and we have that $\Gamma \vdash [] : A$, by the first rule. An element $\Gamma \vdash [\varphi_1\ a_1, \dots, \varphi_n\ a_n] : A$ is to be thought of as an element of value $a_i : A$ on the sub-polyhedron defined by φ_i. The side condition $\Gamma \vdash \varphi_1 \vee \cdots \vee \varphi_n = 1_\mathbb{F} : \mathbb{F}$ ensures that the φ_i's cover all of the cube defined by the context Γ.

Lemma 1.1. *The following rules are admissible.*

$$\frac{\Gamma, 1_\mathbb{F} \vdash J}{\Gamma \vdash J}$$

$$\frac{\Gamma, \varphi, \psi \vdash J}{\Gamma, \varphi \wedge \psi \vdash J}$$

If φ is independent of i, then

$$\frac{\Gamma, i : \mathbb{I}, \varphi \vdash J}{\Gamma, \varphi, i : \mathbb{I} \vdash J}$$

We introduce some further notation. If $\Gamma, \varphi \vdash u : A$, then $\Gamma \vdash a : A[\varphi \mapsto u]$ is an abbreviation for $\Gamma \vdash a : A$ and $\Gamma, \varphi \vdash a = u : A$. That is, we have an element $a : A$ defined on all the context Γ, and when restricted to $\varphi : \mathbb{F}$, this element a is judgmentally equal to $u : A$ (which need no to be defined on all Γ, but only on Γ, φ). More generally, we write $\Gamma \vdash a : A[\varphi_1 \mapsto u_1, \dots, \varphi_n \mapsto u_n]$ for $\Gamma \vdash a : A$ and $\Gamma, \varphi_i \vdash a = u_i : A$. With these notations, we are ready to define the composition operation.

Once again, we extend the syntax with

$$t, u, A, B ::= \dots \mid \texttt{comp}^i\ A\ [\varphi \mapsto u]\ a_0$$

where u is a system define on the *extent* φ, meaning that we have $\Gamma, \varphi, \Gamma' \vdash u : A$. The rule for composition is

$$\frac{\Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash a_0 : A(i0)[\varphi \mapsto u(i0)]}{\Gamma \vdash \texttt{comp}^i\ A\ [\varphi \mapsto u]\ a_0 : A(i1)[\varphi \mapsto u(i1)]}$$

That is, if we have an element $a_0 : A(i0)$, being compatible with $u : A$ at $(i = 0)$ on a smaller cube φ, then we can transport it along the dimension i, and extend it at $(i = 1)$.

Lemma 1.2. *We have the following equality for systems:*

$$\Gamma \vdash \texttt{comp}^i\ A\ [1_{\mathbb{F}} \mapsto u]\ a = u(i1) : A(i1)$$

Proof. $\Gamma \vdash \texttt{comp}^i\ A\ [1_{\mathbb{F}} \mapsto u]\ a : A(i1)[1_{\mathbb{F}} \mapsto u(i1)]$ so in particular $\Gamma, 1_{\mathbb{F}} \vdash \texttt{comp}^i\ A\ [1_{\mathbb{F}} \mapsto u]\ a = u(i1) : A(i1)$ and hence $\Gamma \vdash \texttt{comp}^i\ A\ [1_{\mathbb{F}} \mapsto u]\ a = u(i1) : A(i1)$. □

Moreover, it is possible to extend the composition operation thanks to the operators $\wedge$ and $\vee$ in order to not only to fill the lids of open boxes with composition, but also the inside. This is known as Kan filling [CCHM15].

1.1.4 Transitivity of paths

The composition operation is enough to prove path transitivity.

Proposition 1.3. *If $\Gamma \vdash A$, $\Gamma \vdash a, b, c : A$, then the following is derivable.*

$$\frac{\Gamma \vdash p : \texttt{Path}\ A\ a\ b \qquad \Gamma \vdash q : \texttt{Path}\ A\ b\ c}{\Gamma \vdash \langle i \rangle\ \texttt{comp}^j\ A\ [(i = 0) \vee (i = 1) \mapsto [(i = 0)\ a, (i = 1)\ (q\ j)]\ (p\ i) : \texttt{Path}\ A\ a\ c}$$

Proof. Let us assume $\Gamma \vdash A$, $\Gamma \vdash a, b, c : A$, $\Gamma \vdash p : \texttt{Path}\ A\ a\ b$ and $\Gamma \vdash q : \texttt{Path}\ A\ b\ c$. Let us call $\varphi := (i = 0) \vee (i = 1)$ and $u := [(i = 0)\ a, (i = 1)\ (q\ j)]$. We seek to prove

$$\Gamma \vdash \langle i \rangle\ \texttt{comp}^j\ A\ [\varphi \mapsto u]\ (p\ i) : \texttt{Path}\ A\ a\ c$$

We have the following derivation.

$$\dfrac{\dfrac{\dfrac{\Gamma, i : \mathbb{I} \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I}, j : \mathbb{I} \vdash A \qquad \dfrac{\Pi_1}{\Gamma, i : \mathbb{I}, \varphi, j : \mathbb{I} \vdash u : A} \qquad \dfrac{\Pi_2}{\Gamma, i : \mathbb{I} \vdash p\ i : A(j0)[\varphi \mapsto u(j0)]}}{\Gamma, i : \mathbb{I} \vdash \texttt{comp}^j\ A\ [\varphi \mapsto u]\ (p\ i) : A(j1)[\varphi \mapsto u(j1)]}}{\Gamma, i : \mathbb{I} \vdash \texttt{comp}^j\ A\ [\varphi \mapsto u]\ (p\ i) : A(j1) = A}}{\Gamma \vdash \langle i \rangle\ \texttt{comp}^j\ A\ [\varphi \mapsto u]\ (p\ i) : \texttt{Path}\ A\ v(i0)\ v(i1)}$$

To complete the proof, it suffices to complete Π_1, Π_2 and show that $v(i0) = a$ and $v(i1) = c$.

Let us start with Π_1. Call $\Delta := \Gamma, i : \mathbb{I}, j : \mathbb{I}, \varphi$.

$$\dfrac{\Delta, (i = 0) \vdash a : A \qquad \dfrac{\Delta, (i = 1) \vdash q : \texttt{Path}\ A\ b\ c,\ j : \mathbb{I}}{\Delta, (i = 1) \vdash q\ j : A} \qquad \dfrac{\Delta, 0_{\mathbb{F}} \vdash q\ j = a : A}{\Delta, (i = 1) \wedge (i = 0) \vdash q\ j = a : A}}{\Gamma, i : \mathbb{I}, j : \mathbb{I}, \varphi \vdash [(i = 0)\ a, (i = 1)\ q\ j] : A}$$

We used the second inference rule for systems, so we have to check the side condition, namely

$$\Gamma, i : \mathbb{I}, j : \mathbb{I}, \varphi \vdash \varphi = 1_{\mathbb{F}} : \mathbb{F}$$

which is indeed true, since $\varphi \wedge \varphi = \varphi = \varphi \wedge 1_{\mathbb{F}}$. By Lemma 1.1, we can swap the order of φ and $j : \mathbb{I}$, hence we proved

$$\Gamma, i : \mathbb{I}, \varphi, j : \mathbb{I} \vdash u : A$$

as desired.

Let us continue with Π_2, call $\Delta := \Gamma, i : \mathbb{I}, \varphi$.

$$\dfrac{\dfrac{\Gamma \vdash p : \mathtt{Path}\ A\ a\ b \qquad \Gamma, i : \mathbb{I} \vdash i : \mathbb{I}}{\Gamma, i : \mathbb{I} \vdash p\ i : A} \qquad \dfrac{\dfrac{\Pi'(0)}{\Delta, (i = 0) \vdash p\ i = u(j0) : A} \qquad \dfrac{\Pi'(1)}{\Delta, (i = 1) \vdash p\ i = u(j0) : A}}{\Gamma, i : \mathbb{I}, \varphi \vdash p\ i = u(j0) : A}}{\Gamma, i : \mathbb{I} \vdash p\ i : A(j0)[\varphi \mapsto u(j0)]}$$

where $\Pi'(\delta)$, for $\delta \in \{0, 1\}$ is

$$\dfrac{\dfrac{\Delta, (i = \delta) \vdash [(i = 0)\ a, (i = 1)\ b] : A \qquad \Delta, (i = \delta) \vdash (i = \delta) = 1_{\mathbb{F}} : \mathbb{F}}{\Delta, (i = \delta) \vdash p\ \delta = [(i = 0)\ a, (i = 1)\ b] : A}}{\Delta, (i = \delta) \vdash p\ i = u(j0) : A}$$

The last derivation is justified by the fact that $\Delta, (i = \delta) \vdash i = \delta : \mathbb{I}$, hence

$$\Delta, (i = \delta) \vdash p\ i = p\ \delta : A$$

Finally, we want to show $v(i0) = a$ and $v(i1) = c$. In context $\Gamma, i : \mathbb{I}$, we have

$$\begin{aligned}
\big(\mathtt{comp}^j\ A\ [\varphi \mapsto u]\ (p\ i)\big)\ (i0) &= \mathtt{comp}^j\ A(i0)\ [\varphi(i0) \mapsto u(i0)]\ (p\ 0) \\
&= \mathtt{comp}^j\ A\ [1_{\mathbb{F}} \mapsto u(i0)]\ a \\
&= u(i0)(j1) \\
&= u(j1)(i0) \\
&= [(i = 0)\ a, (i = 1)\ c](i0) \\
&= [1_{\mathbb{F}}\ a, 0_{\mathbb{F}}\ c] \\
&= a
\end{aligned}$$

The last equality is justified by the fourth inference rule for systems. A similar reasoning shows $v(i1) = c$. □

The intuition is that the composition fills the following dotted arrow.

$$\begin{tikzcd}
a \arrow[r, dashed] & c \\
a \arrow[u, "a"] \arrow[r, "p\ i"'] & b \arrow[u, "q\ j"']
\end{tikzcd}$$

1.2 Towards glueing and univalence

In this section, we present informally a more advanced notion, known as glueing, next we prove the version of the univalence axiom presented in [CCHM15] and link it with a more traditional statement of the axiom.

1.2.1 Glueing

The glueing operation is analogous to the composition defined above, but it is applied to types. First, we define what it means for to type to be equivalent.

Definition 1.1 (Contractible types and equivalence)**.** *We say that a type A is contractible if the type*

$$\texttt{isContr}\ A \triangleq (x : A) \times ((y : A) \to \texttt{Path}\ A\ x\ y)$$

is inhabited. Given two types T, A and $f : T \to A$, we define

$$\texttt{isEquiv}\ T\ A\ f \triangleq (y : A) \to \texttt{isContr}\ ((x : T) \times \texttt{Path}\ A\ y\ (f\ x))$$

Finally, we define the type

$$\texttt{Equiv}\ T\ A \triangleq (f : T \to A) \times \texttt{isEquiv}\ T\ A\ f$$

Intuitively, a type A is contractible if it has a center point $x : A$, and from it, we can draw a path to any other point $y : A$. An equivalence between two types T and A is a function $f : T \to A$ whose (homotopy) fibers are all contractible. Finally, two types are equivalent if there an equivalence between them. We use this notion of equivalence to define the formation rule of glueing:

$$\frac{\Gamma \vdash A \qquad \Gamma, \varphi \vdash T \qquad \Gamma, \varphi \vdash f : \texttt{Equiv}\ T\ A}{\Gamma \vdash \texttt{Glue}\ [\varphi \mapsto (T, f)]\ A}$$

To form a glue type, we need a type A defined everywhere, and a type T, which is defined only on a *smaller* region determined by φ, we also suppose that on this region φ, the type A and T are equivalent. Then under these hypotheses, we have a new type, the glueing, that is to be thought as a copy of T inside A. Indeed, we also have the following rule for glue types.

$$\frac{\Gamma \vdash T \qquad \Gamma \vdash f : \texttt{Equiv}\ T\ A}{\Gamma \vdash \texttt{Glue}\ [1_{\mathbb{F}} \mapsto (T, f)]\ A = T}$$

Notice that here, there is no mention of the region φ, but there is no loss of generality, as it can be inside Γ, and in this case, we would have $1_{\mathbb{F}} = \varphi$. Thus, the glueing is definitionally a copy of T inside A.

Then, we have the construction of glue terms. It is done via the rule

$$\frac{\Gamma, \varphi \vdash f : \texttt{Equiv}\ T\ A \qquad \Gamma, \varphi \vdash t : T \qquad \Gamma \vdash a : A[\varphi \mapsto f\ t]}{\Gamma \vdash \texttt{glue}\ [\varphi \mapsto t]\ a : \texttt{Glue}\ [\varphi \mapsto (T, f)]\ A}$$

that transfers terms of type T into their glued version inside A, in a way that is compatible with the equivalence f. Moreover, we have an operation `unglue` that takes a term of type $\texttt{Glue}\ [\varphi \mapsto (T, f)]\ A$, and returns a term of type A. Indeed, as the glueing of T is a *copy* of T inside A, the unglueing allows us go from $b : \texttt{Glue}\ [1_{\mathbb{F}} \mapsto (T, f)]\ A$ to $\texttt{unglue}\ b : A$. More conditions that guarantee the compatibility of this operation with f on φ, and some rules state that `glue` and `unglue` are definitionally inverse of each other.

We can illustrate the type $\texttt{Glue}\ [1_{\mathbb{F}} \mapsto (T, f)]\ A$ as:

$$\begin{tikzcd}
T \arrow[rrr, dashed] \arrow[drr, "f"] \arrow[ddr] & & & \texttt{Glue}\ [1_{\mathbb{F}} \mapsto (T, f)]\ A \arrow[dr, dashed, "\texttt{unglue}"] \arrow[dd, dashed] & \\
& & A \arrow[rr] \arrow[dl] & & A \arrow[dl] \\
& \Gamma, \varphi \arrow[rr] & & \Gamma &
\end{tikzcd}$$

where the squares

$$\begin{tikzcd} T \arrow[r] \arrow[d] & \texttt{Glue}\ [1_{\mathbb{F}} \mapsto (T, f)]\ A \arrow[d] \\ \Gamma, \varphi \arrow[r] & \Gamma \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A \arrow[d] \\ \Gamma, \varphi \arrow[r] & \Gamma \end{tikzcd}$$

are pullbacks.

For instance, as we can see in [CCHM15] Example 8, the glueing allows us to construct a function

$$(A\ B : U) \to \texttt{Equiv}\ A\ B \to \texttt{Path}\ U\ A\ B$$

given by

$$\lambda(A\ B : U)(f : \texttt{Equiv}\ A\ B).\langle i \rangle\ \texttt{Glue}\ [(i = 0) \mapsto (A, f), (i = 1) \mapsto (B, \texttt{idEq}_B)]\ B$$

where $\texttt{idEq}_B : \texttt{Equiv}\ B\ B$ is the standard identity function equipped with a proof that it is an equivalence.

1.2.2 Proof of univalence

The goal here is to give of proof Corollary 11 in [CCHM15]. In their paper, it is the following.

Theorem 1.4 (Univalence axiom)**.** *For any term*

$$t : (A\ B : U) \to \texttt{Path}\ U\ A\ B \to \texttt{Equiv}\ A\ B$$

the map $t\ A\ B : \texttt{Path}\ U\ A\ B \to \texttt{Equiv}\ A\ B$ is an equivalence.

To prove this, we rely on several lemmas.

Lemma 1.5. *For any type $A : U$, the type $(X : U) \to \texttt{Equiv}\ A\ X$ is contractible.*

The proof of this lemma is to be found in [CCHM15, Corollary 10] and relies on the fact that the map `unglue` is an equivalence.

Definition 1.2. *If $P, Q : A \to U$ are type families and $f : (x : A) \to P(x) \to Q(x)$, we define*

$$\texttt{total}(f) := \lambda w.(w.1, f(w.1, w.2)) : (x : A) \times P(x) \to (x : A) \times Q(x)$$

In [Uni13, Theorem 4.7.7], we have

Lemma 1.6. *Given $f : (x : A) \to P(x) \to Q(x)$, $\texttt{total}(f)$ is an equivalence if and only if for each $a : A$, $f(a)$ is an equivalence.*

Furthermore, in [Uni13, Lemma 3.11.3], we have

Lemma 1.7. *A type is contractible if and only if it is equivalent to the unit type* $\mathbf{1}$.

Corollary 1.8. *Any function between contractible types is an equivalence.*

Proof. Note that equivalences satisfy the 2-out-of-3 property. Let A, B be contractible types and $f : A \to B$. We have that $!_B : B \to \mathbf{1}$ is an equivalence because B is contractible, same for $!_A : A \to \mathbf{1}$. Moreover, by uniqueness, $!_A = !_B \circ f$, hence by 2-out-of-3, f is an equivalence. $\square$

From [Uni13, Lemma 3.11.8], we have

Lemma 1.9. *The type $C = (X : U) \times \texttt{Path}\ U\ A\ X$ is contractible.*

Now, the proof of Theorem 1.4 follows from the previous lemmas. Indeed, for any $A : U$, we have

$$t\ A : (B : U) \to \texttt{Path}\ U\ A\ B \to \texttt{Equiv}\ A\ B$$

thus

$$\texttt{total}(t\ A) : (B : U) \times \texttt{Path}\ U\ A\ B \to (B : U) \times \texttt{Equiv}\ A\ B$$

is an equivalence because it is a function between contractible types. Hence, by Lemma 1.6, $t\ A\ B$ is an equivalence for each B.

1.2.3 Correspondence with standard univalence

Now, we would like to show that this formulation is indeed the same as the more classical one, where we only exhibit one canonical equivalence

$$\texttt{pathToEquiv}\ A\ B : \texttt{Path}\ U\ A\ B \to \texttt{Equiv}\ A\ B$$

defined via transport. Hence, we will prove

Theorem 1.10. *For $A, B : U$, the following are equivalent.*

(i.) *Any term* $\texttt{Path}\ U\ A\ B \to \texttt{Equiv}\ A\ B$ *is an equivalence*

(ii.) $\texttt{pathToEquiv}A\ B$ *is an equivalence*

Proof. (i.) obviously implies (ii.). For the opposite direction, we show that (ii.) is equivalent to Lemma 1.5, which follows from the next lemma. $\square$

In [Uni13] Theorem 5.8.4, we have

Lemma 1.11. *For $R : A \to A \to U$ equipped with $r_0 : (a : A) \to R(a, a)$, the following are equivalent.*

(i.) *For any $a, b : A$ the map* $\texttt{transport}^{R(a)}(\text{-}, r_0(a)) : \texttt{Path}\ A\ a\ b \to R(a, b)$ *is an equivalence.*

(ii.) *For any $a : A$, the type $(b : A) \times R(a, b)$ is contractible*

Specializing with $R = \texttt{Equiv}$ and $r_0(A) = \mathrm{id}_A : \texttt{Equiv}\ A\ A$, we obtain that

$$\texttt{transport}^{\texttt{Equiv}A}(\text{-}, \mathrm{id}_A)$$

is an equivalence if and only if

$$(B : U) \to \texttt{Equiv}\ A\ B$$

is contractible. But recall that by definition,

$$\texttt{pathToEquiv}A\ B = \texttt{transport}^{\texttt{Equiv}A}(\text{-}, \mathrm{id}_A)$$

Hence

$$\begin{aligned} &(B : U) \to \texttt{Equiv}\ A\ B \text{ is contractible} \\ \iff &\texttt{pathToEquiv}A\ B \text{ is an equivalence} \end{aligned}$$

2 Logic inside a presheaf topos

In this section, we show how to internalize logic inside a presheaf topos. First, we recall some basic results about the subobject classifier in a presheaf. We then build the basic operators of logic inside the topos and show how to interpret any formula using them. This is known as Kripke-Joyal semantics, to any formula corresponds an arrow. Once this is done, we show what it means for a formula to be true inside a topos, and we introduce the notion of forcing, which allows us to decompose any formula and simplifies a lot the verification of its validity.

Finally, note that all the results below can be generalized in toposes that are not presheaves, but we chose to restrict our attention to this particular case of interest, as it simplifies the forcing theorem, and the more general theory is not needed in this paper.

2.1 Kripke-Joyal semantics

The original Kripke-Joyal semantics, as given in [SML94], was for the Mitchell-Bénabou language, it turns out that it also applies in the context of categories with families, which will be introduced in the next section. Thus, even if the internal language is not *per se* the same, the semantics of the formulas works the same.

2.1.1 The subobject classifier

Let $\mathcal{C}$ be a small category and $\mathcal{E} = [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$ the presheaf topos associated to $\mathcal{C}$. Recall that for $c \in \mathcal{C}$, the category $\mathcal{C}/c$ has objects morphisms $f : \cdot \to c$ and morphisms the commutative triangles.

Definition 2.1. *A* sieve *on $c \in \mathcal{C}$ is a subset $S \subseteq \mathrm{Obj}(\mathcal{C}/c)$ closed under post-composition. That is, if $f : d \to c$ is in S and $g : d' \to d$ in $\mathcal{C}$, then $f \circ g : d' \to c$ is in S.*

Recall that the subobject classifier Ω of $\mathcal{E}$ maps each $c \in \mathcal{C}$ to the set $\Omega(c)$ of sieves on c. For $f : c \to d$, $\Omega(f)$ maps a sieve S on d to $\{g : \cdot \to c \mid f \circ g \in S\}$. Given a sieve S on c, we can define the subfunctor $\iota_S : F_S \rightarrowtail \mathbf{y}c$ by

$$F_S(d) = \{f : d \to c \in S\}$$

and

$$F_S(d \xrightarrow{g} d')(d' \xrightarrow{f} c) = f \circ g$$

Conversely, given a subfunctor $\iota : F \rightarrowtail \mathbf{y}c$, we can define a sieve on c

$$S_F = \{g : d \to c \in C \mid \exists t : \mathbf{y}d \to F,\ \iota \circ t = \mathbf{y}g,\ d \in \mathcal{C}\}$$

Lemma 2.1. *For every sieve S we have*

$$S_{F_S} = S$$

and for every subfunctor $F \rightarrowtail \mathbf{y}c$ we have

$$F_{S_F} = F$$

Proof. If S is a sieve on c, then

$$S_{F_S} = \{g : d \to c \in C \mid \exists t : \mathbf{y}d \to F_S,\ \iota_S \circ t = \mathbf{y}g\}$$

If $g \in S_{F_S}$, then $\iota_S \circ t = \mathbf{y}g$, and so in particular in d, we have

$$\begin{tikzcd}
\mathcal{C}(d,d) \arrow[rr, "t_d"] \arrow[dr, "g\circ -"'] & & F_S(d) \arrow[dl, "\iota_d"] \\
& \mathcal{C}(d,c) &
\end{tikzcd}$$

Thus $g = \iota_d(t_d(\mathrm{id}_d)) \in S$. Conversely, if $g : d \to c \in S$, we define $t : \mathbf{y}d \to F_S$ to be the corestriction of $\mathbf{y}g : \mathbf{y}d \to \mathbf{y}c$, which is possible as $g \circ u \in S$ for all $u : d' \to d$.

If $\iota : F \rightarrowtail \mathbf{y}c$ is a subfunctor, then

$$\begin{aligned}
F_{S_F}(d) = \{g : d \to c \mid \exists t : \mathbf{y}d \to F,\ \iota \circ t = \mathbf{y}g\} &\simeq \{\iota \circ t \mid t : \mathbf{y}d \to F\} \\
&\simeq \{t : \mathbf{y}d \to F\} \\
&\simeq F(d)
\end{aligned}$$

□

With this correspondence, we can alternatively define $\Omega(c) = \mathrm{Sub}(\mathbf{y}c)$ and given $f : d \to c$, for any subfunctor $\iota : S \to \mathbf{y}c$, we can check that $\Omega(f)(S) = S \times_{\mathbf{y}c} \mathbf{y}d \in \mathrm{Sub}(\mathbf{y}d)$, that is, the following is a pullback.

$$\begin{tikzcd}
S \times_{\mathbf{y}c} \mathbf{y}d \arrow[r] \arrow[d, tail, "\Omega(f)(S)"'] & S \arrow[d, tail, "\iota"] \\
\mathbf{y}d \arrow[r, "\mathbf{y}f"] & \mathbf{y}c
\end{tikzcd}$$

Definition 2.2. *We define* $\mathrm{true} : \mathbf{1} \to \Omega$ *by picking the full subobject* $\mathbf{y}c \to \mathbf{y}c$ *at each* c.

Finally, we check that this Ω has indeed the universal property of the subobject classifier.

Lemma 2.2. *For any* $X \in \mathcal{E}$, *we have*

$$\mathrm{Sub}(X) \simeq \mathcal{E}(X, \Omega)$$

Moreover, this bijection is given by the following pullback.

$$\begin{tikzcd}
S \arrow[r] \arrow[d, tail, "s"'] & \mathbf{1} \arrow[d, tail, "\mathrm{true}"] \\
X \arrow[r, "\chi_s"'] & \Omega
\end{tikzcd}$$

Proof. Consider a subobject $s : S \rightarrowtail X$. We seek to construct $\chi_s : X \to \Omega$. For $c \in \mathcal{C}$ and $x \in X(c)$, we are looking for a subfunctor $(\chi_s)_c(x) \rightarrowtail \mathbf{y}c$. For $d \in \mathcal{C}$, we define

$$(\chi_s)_c(x)(d) = \{g : d \to c \mid X(g)(x) \in S(d)\} \subseteq \mathbf{y}c(d)$$

If $x \in S(c)$, then for all $g : d \to c$, $X(g)(x) = S(g)(x) \in S(d)$, and thus $(\chi_s)_c(x) = \mathbf{y}c = \mathrm{true}_c(\star)$, making the diagram commute. Then, if we have the following diagram

$$\begin{tikzcd}
R \arrow[drr, bend left] \arrow[dr, bend right, "t"'] & & \\
& S \arrow[r] \arrow[d, tail, "s"'] & \mathbf{1} \arrow[d, tail, "\mathrm{true}"] \\
& X \arrow[r, "\chi_s"'] & \Omega
\end{tikzcd}$$

such that true $\circ ! = \chi_s \circ t$. Then for all $c \in \mathcal{C}$ and all $x \in R(c)$, $t_c(x) \in S(c)$, as $\chi_c(t_c(x)) = \mathbf{y}c$, and thus a unique map $u_c : R(c) \to S(c)$ such that $s_c \circ u_c = t_c$. That makes our construction a pullback. Now, we need to check that χ_s is the only arrow that makes the pullback. Suppose we have $\xi : X \to \Omega$ making the diagram a pullback. It remains to show $\xi = \chi_s$. □

2.1.2 Heyting structure of the subobjects

Recall that for $X \in \mathcal{E}$, $\mathrm{Sub}(X)$, the subobjects of X, have a structure of Heyting algebra. We detail the Heyting structure on $\mathrm{Sub}(\Omega)$. First recall the isomorphism $\mathcal{E}(X, \Omega) \simeq \mathrm{Sub}(X)$.

We are looking for maps:

$$\top : \mathbf{1} \to \Omega \qquad \wedge : \Omega \times \Omega \to \Omega \qquad \neg : \Omega \to \Omega$$

$$\bot : \mathbf{1} \to \Omega \qquad \vee : \Omega \times \Omega \to \Omega \qquad \Rightarrow : \Omega \times \Omega \to \Omega$$

We take $\top :=$ true, it is the classifying map of $! : \mathbf{1} \to \mathbf{1}$, hence we have the following pullback.

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[d, tail, "!"'] & \mathbf{1} \arrow[d, tail] \\ \mathbf{1} \arrow[r, "\top"'] & \Omega \end{tikzcd}$$

Similarly, we define $\bot$ with the following pullback.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, tail, "!"'] & \mathbf{1} \arrow[d, tail, "\text{true}"] \\ \mathbf{1} \arrow[r, "\bot"'] & \Omega \end{tikzcd}$$

For $\wedge$, it is also pretty straightforward, we describe it with the following pullback.

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[d, tail, "\langle \text{true}, \text{true} \rangle"'] & \mathbf{1} \arrow[d, tail, "\text{true}"] \\ \Omega \times \Omega \arrow[r, "\wedge"'] & \Omega \end{tikzcd}$$

For $\Rightarrow$, let $\leq : \Omega_1 \rightarrowtail \Omega \times \Omega$ be the equalizer of $\pi_1 : \Omega \times \Omega \to \Omega$ and $\wedge : \Omega \times \Omega \to \Omega$. We define $\Rightarrow : \Omega \times \Omega \to \Omega$ to be the classifying arrow of $\leq$. Thus we have the following pullback.

$$\begin{tikzcd} \Omega_1 \arrow[r] \arrow[d, tail, "\leq"'] & \mathbf{1} \arrow[d, tail, "\text{true}"] \\ \Omega \times \Omega \arrow[r, "\Rightarrow"'] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\wedge"] & \Omega \\ \Omega & \end{tikzcd}$$

The operator $\vee$ is a little bit more involved and requires some work.

Definition 2.3 (Image). *A mono m is the image of an arrow f, if $f = m \circ e$ for some epi e and if whenever f factors through a mono m' then m factor through m'. We write $m = \mathrm{im}(f)$.*

Proposition 2.3. *Let $f : X \to Y$. Then f has an image $m : M \to Y$ and factors as in*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & Y \\ & M \arrow[ur, tail, "m"'] & \end{tikzcd}$$

with e epi.

Proof. Let $x, y : Y \to Z$ be the cokernel of f, that is, the following diagram is a pushout.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "x"] \\ Y \arrow[r, "y"'] & Z \end{tikzcd}$$

We define $m : M \to Y$ to be the equalizer of x and y, it is a mono. Moreover, as $x \circ f = y \circ f$, we have by universal property an arrow $e : X \to M$ such that $f = m \circ e$. To show the universality and that e is epi, we refer the reader to [SML94] (IV.6.1). □

To define $\vee : \Omega \times \Omega \to \Omega$, we first obtain the following dotted arrow by coproduct, which we factorize with Proposition 2.3.

$$\begin{tikzcd} \Omega \arrow[r, "\iota_1"] \arrow[dr, "\langle \mathrm{id}, \mathrm{true} \rangle"'] & \Omega + \Omega \arrow[d, dotted, two heads] & \Omega \arrow[l, "\iota_2"'] \arrow[dl, "\langle \mathrm{true}, \mathrm{id} \rangle"] \\ & U \arrow[d, dotted, tail, "u"] & \\ & \Omega \times \Omega & \end{tikzcd}$$

and then we let $\vee : \Omega \times \Omega \to \Omega$ to be the classifying arrow of u, that is we have the following pullback.

$$\begin{tikzcd} U \arrow[r] \arrow[d, tail, "u"'] & \mathbf{1} \arrow[d, tail, "\mathrm{true}"] \\ \Omega \times \Omega \arrow[r, "\vee"'] & \Omega \end{tikzcd}$$

Finally, the negation is defined by implication to $\perp$, more precisely, we have

$$\langle \mathrm{id}\, \pi_1, \perp \pi_2 \rangle : \Omega \times \mathbf{1} \to \Omega \times \Omega$$

that we precompose by $\Rightarrow$ (to be fully precise, we would also have to include the isomorphism $\Omega \to \Omega \times \mathbf{1}$). Thus $\neg$ is

$$\Omega \xrightarrow{(\mathrm{id}, \perp)} \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$$

Now, suppose $\sigma, \tau : X \to \Omega$ are formula, then we interpret

- $\sigma \wedge \tau$ by $\wedge \circ \langle \sigma, \tau \rangle : X \to \Omega$
- $\sigma \vee \tau$ by $\vee \circ \langle \sigma, \tau \rangle : X \to \Omega$
- $\sigma \Rightarrow \tau$ by $\Rightarrow \circ \langle \sigma, \tau \rangle : X \to \Omega$
- $\neg \sigma$ by $\Rightarrow \circ \langle \sigma, \perp \circ ! \rangle = \neg \circ \sigma : X \to \Omega$

2.1.3 Quantifiers

Given a formula $\sigma : Y \times X \to \Omega$, we wish to define the formulas $\forall y : Y,\ \sigma(y)$ and $\exists y : Y,\ \sigma(y)$. It is a general fact in categorical logic to define existential and universal quantification respectively as left adjoint and right adjoint to a pullback functor. This is done as follows:

$$\begin{tikzcd}
\mathcal{E}(Y \times X, \Omega) \simeq \mathrm{Sub}(Y \times X) \arrow[d, bend right=60, "\exists_Y"'] \arrow[d, bend left=60, "\forall_Y"] \\
\mathcal{E}(X, \Omega) \simeq \mathrm{Sub}(X) \arrow[u, "\pi^*"]
\end{tikzcd}$$

where π^* is the pullback functor along the second projection $\pi : Y \times X \to X$ and $\exists_Y \dashv \pi^* \dashv \forall_Y$. Recall that in a locally cartesian closed category, we have a tower of adjunctions

$$\Sigma_f \dashv f^* \dashv \Pi_f$$

for every $f : A \to B$. The next lemma gives a concrete description of Σ_f.

Lemma 2.4. *Let $f : A \to B$, the left adjoint to f^*, the pullback functor along f, is given by*

$$\Sigma_f(h) = f \circ h$$

Proof. For $h : C \to A$ and $g : D \to B$, we want to show

$$\mathcal{E}/B(f \circ h, g) \simeq \mathcal{E}/A(h, f^* g)$$

Consider the following diagram.

$$\begin{tikzcd}
C \arrow[rrd, bend left, "\theta"] \arrow[rdd, bend right, "h"'] & & \\
& D' \arrow[r, "u"] \arrow[d, "f^*g"'] & D \arrow[d, "g"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

The square is a pullback. A θ such that $g \circ \theta = f \circ h$ amounts to a morphism in $\mathcal{E}/B(f \circ h, g)$. By pullback, it creates a morphism $\theta' : C \to D'$ such that $f^* g \circ \theta' = h$, that is a morphism $\theta' \in \mathcal{E}/A(h, f^* g)$. Conversely, such a $\theta' \in \mathcal{E}/A(h, f^* g)$ gives $u \circ \theta' \in \mathcal{E}/B(f \circ h, g)$. Such a correspondence is moreover natural in h and g. $\square$

As we are working with subobjects, it suffices to take the image to find the adjunction, and we can show that $\exists_Y = \mathrm{im}(\Sigma_\pi)$ and $\forall_Y = \mathrm{im}(\Pi_f)$.

Definition 2.4 (Universal and existential quantification)**.** *If $\sigma : Y \times X \to \Omega$, we interpret*

- *$\forall y : Y,\ \sigma(y)$ by $\forall_Y \sigma : X \to \Omega$*
- *$\exists y : Y,\ \sigma(y)$ by $\exists_Y \sigma : X \to \Omega$*

For instance, we have the following cartesian square.

$$\begin{tikzcd}
\{x : X \mid \forall y : Y,\ \sigma(y,x)\} \arrow[r] \arrow[d, tail] & \mathbf{1} \arrow[d, tail, "\mathrm{true}"] \\
X \arrow[r, "\forall_Y \sigma"'] & \Omega
\end{tikzcd}$$

with $\{x : X \mid \forall y : Y,\ \sigma(y,x)\} \triangleq \forall_Y \{(y,x) : Y \times X \mid \sigma(y,x)\}$.

We can unravel the definition of this existential quantification, to have something more concrete to work with.

Proposition 2.5. *We have*

$$\begin{tikzcd}
\{(y,x) : Y \times X \mid \sigma(y,x)\} \arrow[d, tail, "s"] \arrow[dr, two heads] & \\
Y \times X \arrow[d, "\pi"] & \{x : X \mid \exists y : Y,\ \sigma(y,x)\} \arrow[dl, tail] \\
X &
\end{tikzcd}$$

That is, $\{x : X \mid \exists y : Y,\ \sigma(y,x)\} \rightarrowtail X$ *is the image of* $\pi \circ s$*.*

Proof. We use Lemma 2.4 and the fact that $\exists_Y = \mathrm{im}(\Sigma_\pi)$. $\square$

2.1.4 Interpretation of equality

Finally, we describe how to interpret the symbol $=$ in the topos.

Definition 2.5 (Interpretation of equality)**.** *For all $Y \in \mathcal{E}$, let $\Delta_Y : Y \to Y \times Y$ be the diagonal map and let δ_Y be its classifying arrow, i.e. we have the following pullback.*

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, tail, "\Delta_Y"'] & \mathbf{1} \arrow[d, tail, "\mathrm{true}"] \\
Y \times Y \arrow[r, "\delta_Y"'] & \Omega
\end{tikzcd}$$

Let $f, g : X \to Y$. We define the interpretation of the formula $f = g$ by $\delta_Y \circ \langle f, g \rangle$. Thus we have the following pullback.

$$\begin{tikzcd}
\{x : X \mid f(x) = g(x)\} \arrow[r] \arrow[d, tail] & \mathbf{1} \arrow[d, tail, "\mathrm{true}"] \\
X \arrow[r, "f=g"'] & \Omega
\end{tikzcd}$$

The previous pullback can in fact be transformed into an equalizer.

Lemma 2.6. *The following diagram is an equalizer.*

$$\begin{tikzcd}
\{x : X \mid f(x) = g(x)\} \arrow[r, tail] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

Proof. In

$$\begin{tikzcd}
\{x : X \mid f(x) = g(x)\} \arrow[r, "h"] \arrow[d, tail, "e"'] & Y \arrow[r] \arrow[d, tail, "\Delta_Y"] & \mathbf{1} \arrow[d, tail, "\mathrm{true}"] \\
X \arrow[r, "\langle f,g \rangle"'] & Y \times Y \arrow[r, "\delta_Y"'] & \Omega
\end{tikzcd}$$

the right inner square and the outer square are pullbacks, hence by the pasting law,

$$\begin{tikzcd} \{x : X \mid f(x) = g(x)\} \arrow[r, "h"] \arrow[d, tail, "e"'] & Y \arrow[d, tail, "\Delta_Y"] \\ X \arrow[r, "\langle f,g \rangle"'] & Y \times Y \end{tikzcd}$$

is a pullback, which is another way of saying that e is the equalizer of f and g. Indeed

$$fe = \pi_1\langle f, g\rangle e = \pi_1\Delta h = h = \pi_2\Delta h = \pi_2\langle f, g\rangle e = ge$$

and the universal property works the same. □

2.2 Kripke-Joyal forcing

Now that we have a way to interpret formulas inside the presheaf $\mathcal{E}$, we will describe what it means for a formula to be valid inside $\mathcal{E}$ and develop a mechanical way to unravel recursively every such formula in order to check its validity. This is known as Kripke-Joyal forcing [LS86].

2.2.1 Validity of a formula

Definition 2.6 (Validity)**.** *Let $\sigma : X \to \Omega$. We say that σ is valid whenever σ factors through* true : $\mathbf{1} \to \Omega$.

$$\begin{tikzcd} \mathbf{1} \arrow[dr, tail, "\text{true}"] & \\ X \arrow[u, "!"] \arrow[r, "\sigma"'] & \Omega \end{tikzcd}$$

In that case, we write $X \vdash \sigma$. If σ is a closed formula, then we write $\vdash \sigma$ and this amounts to say that $\sigma =$ true.

Lemma 2.7. *Let X be a type and $\sigma : X \to \Omega$. The following are equivalent.*

(1) $X \vdash \sigma$

(2) The subobject $\{x : X \mid \sigma(x)\} \rightarrowtail X$ admits a section

(3) $\{x : X \mid \sigma(x)\} \simeq X$

Proof. (2) $\iff$ (3) is a general result in category theory. Suppose we have a mono $m : a \rightarrowtail b$ and $f : b \to a$ such that $m \circ f = \mathrm{id}_b$, then $m \circ f \circ m = m \circ \mathrm{id}_a$ but m is mono, hence $f \circ m = id_a$. For (1) $\iff$ (2), this is the universal property of the pullback. Indeed, consider the following diagram.

$$\begin{tikzcd} X \arrow[dr, dotted, "s"] \arrow[drr, bend left, "!"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\ & \{x : X \mid \sigma(x)\} \arrow[r] \arrow[d, tail] & \mathbf{1} \arrow[d, tail, "\text{true}"] \\ & X \arrow[r, "\sigma"'] & \Omega \end{tikzcd}$$

The square is a pullback by definition, hence we have a section of $\{x : X \mid \sigma(x)\} \rightarrowtail X$ if and only if true $\circ ! = \sigma \circ \mathrm{id}_X$. □

We can use this lemma to provide a way to introduce universal quantifiers.

Proposition 2.8. *Let $\sigma : Y \times X \to \Omega$ be a formula. Then the following are equivalent.*

(i) $X \vdash \forall y : Y,\ \sigma(y)$

(ii) $Y \times X \vdash \sigma$

Proof. $X \vdash \forall y : Y,\ \sigma(y)$ if and only if we have a section

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr, equal] & & \forall_Y\{(y,x) : Y \times X \mid \sigma(y,x)\} \arrow[dl, tail] \\ & X & \end{tikzcd}$$

thus u is a morphism in $\mathrm{Sub}(X)$ and by adjointness, this is equivalent to

$$\begin{tikzcd} Y \times X \arrow[rr, "u^*"] \arrow[dr, equal] & & \{(y,x) : Y \times X \mid \sigma(y,x)\} \arrow[dl, tail] \\ & Y \times X & \end{tikzcd}$$

meaning precisely that $Y \times X \vdash \sigma$. □

Corollary 2.9. *Let $\sigma : Y \to \Omega$ be a formula. Then the following are equivalent.*

1. $\vdash \forall y : Y,\ \sigma(y)$

2. $Y \vdash \sigma$

2.2.2 Kripke-Joyal forcing

The idea behind Kripke-Joyal forcing is to prove that a formula $\sigma : X \to \Omega$ is valid by proving that $\sigma \circ x : \mathbf{y}c \to \Omega$ is valid for all $c \in \mathcal{C}$ and $x : \mathbf{y}c \to X$. Then, the density of representable functors allows us to conclude that σ is valid. We usually write $\sigma(x)$ for $\sigma \circ x$.

Definition 2.7. *Let $\sigma : X \to \Omega$ be a formula and $x : \mathbf{y}c \to X$. We say that x forces σ at stage c, written $c \Vdash \sigma(x)$, if the following dotted arrow exists, making the left triangle commute.*

$$\begin{tikzcd} & \{x : X \mid \sigma\} \arrow[r] \arrow[d, tail, "s"] & \mathbf{1} \arrow[d, tail, "\mathrm{true}"] \\ \mathbf{y}c \arrow[ur, dotted] \arrow[r, "x"'] & X \arrow[r, "\sigma"'] & \Omega \end{tikzcd}$$

Recall that by definition, the right square in the definition above is a pullback. Note that $c \Vdash \sigma(x)$ is a way of rewriting $\mathbf{y}c \vdash \sigma(x)$. Indeed, if $\mathbf{y}c \vdash \sigma(x)$, the dotted map appears with the universal property of the pullback.

Theorem 2.10. *Let $\sigma : X \to \Omega$. $X \vdash \sigma$ if and only if $c \Vdash \sigma(x)$ for all $x : \mathbf{y}c \to X$.*

Proof. If $X \vdash \sigma$, then $X \simeq \{x : X \mid \sigma\}$, so the dotted arrow exists for every $x : \mathbf{y}c \to X$. Conversely, we will prove that we have a section of $s : \{x : X \mid \sigma(x)\} \rightarrowtail X$. Call $\int X$ the category of elements of X. An object of $\int X$ is the data of $c \in C$ and $x \in X(c)$, hence by the Yoneda Lemma, it is a natural transformation $x : \mathbf{y}c \to X$. Recall the density formula:

$$X = \operatorname*{colim}_{(c,x) \in \int X} \mathbf{y}c$$

and note that the injection $i_{(c,x)} : \mathbf{y}c \to \operatorname*{colim}_{(c,x)\in\int F} \mathbf{y}c$ is x itself. By hypothesis, for each $x : \mathbf{y}c \to X$, we have

$$\begin{tikzcd} & \{x : X \mid \sigma\} \arrow[d, tail, "s"] \\ \mathbf{y}c \arrow[ur, dotted, "u_{(c,x)}"] \arrow[r, "x"'] & X \end{tikzcd}$$

The maps $u_{(c,x)}$ are the maps needed to form a cone and we now check the commutativity. Let $f : (c,x) \to (c',x')$, that is $f : c \to c'$ such that $X(f)(x) = x'$, or by naturality of the isomorphism $\varphi_c : X(c) \simeq \mathbf{y}c$, a map f such that

$$\begin{tikzcd} \mathbf{y}c \arrow[rr, "f"] \arrow[dr, "x"'] & & \mathbf{y}c' \arrow[dl, "x'"] \\ & X & \end{tikzcd}$$

commutes. Such a diagram can be factorized as follows:

$$\begin{tikzcd} \mathbf{y}c \arrow[rr, "f"] \arrow[dr, "u_{c,x}"] \arrow[ddr, bend right, "x"'] & & \mathbf{y}c' \arrow[dl, "u_{(c',x')}"'] \arrow[ddl, bend left, "x'"] \\ & \{x : X \mid \sigma\} \arrow[d, tail, "s"] & \\ & X & \end{tikzcd}$$

The left and right triangles commute by hypothesis, and s is mono, hence the inner triangle commutes. We can apply the universal property of the colimit, and we obtain a map $u : X \to \{x : X \mid \sigma\}$. Rewriting that, for each injection $x : \mathbf{y}c \to X$, we have that

$$\begin{tikzcd} \mathbf{y}c \arrow[r, "x"] \arrow[ddr, "x"'] & \operatorname*{colim}_{(c,x)\in\int X} \mathbf{y}c \arrow[d, "u"] \\ & \{x : X \mid \sigma(x)\} \arrow[d, tail, "s"] \\ & X \end{tikzcd}$$

commutes. By uniqueness, we have that $s \circ u = \mathrm{id}_X$, which is the required section. □

Theorem 2.11 (Monotonicity). *Let $\sigma : X \to \Omega$ be a formula and $x : \mathbf{y}c \to X$ such that $c \Vdash \sigma(x)$, then for all $f : b \to c$, $b \Vdash \sigma(xf)$.*

Proof. If we have

$$\begin{tikzcd} & \{x : X \mid \sigma\} \arrow[d, tail, "s"] \\ \mathbf{y}c \arrow[ur, dotted] \arrow[r, "x"'] & X \end{tikzcd}$$

then post-composing by f, we obtain

$$\begin{tikzcd} & & \{x : X \mid \sigma\} \arrow[d, tail, "s"] \\ \mathbf{y}b \arrow[r, "f"'] & \mathbf{y}c \arrow[ur, dotted] \arrow[r, "x"'] & X \end{tikzcd}$$

and thus the required morphism $\mathbf{y}b$ to $\{x : X \mid \sigma\}$. □

Proposition 2.12. *If $\mathcal{C}$ has a terminal object $t \in \mathcal{C}$, then a closed formula $\sigma : \mathbf{1} \to \Omega$ is valid if and only if $t \Vdash \sigma$.*

Proof. By definition $t \Vdash \sigma$ is to say that we have a dotted map in the following diagram.

$$\begin{tikzcd} & \{\text{-} : \mathbf{1} \mid \sigma\} \arrow[d, tail, "s"] \\ \mathbf{y}t \arrow[ur, dotted] \arrow[r, "!"'] & \mathbf{1} \end{tikzcd}$$

but $\mathbf{y}t = \mathbf{1}$, so $! = \mathrm{id}_{\mathbf{1}}$, meaning that the dotted map is a section of s. □

2.2.3 Forcing to check the validity of a formula

We present a theorem that allows us to recursively unwind the condition $c \Vdash \sigma(x)$, and thus provide a mechanical way to check the validity of any formula.

Theorem 2.13 (Conditions for forcing). *Let $\sigma, \tau : X \to \Omega$, $\theta : Y \times X \to \Omega$ and $x : \mathbf{y}c \to X$, then*

- $c \Vdash \bot$ *never*
- $c \Vdash \top$ *always*
- $c \Vdash \sigma(x) \wedge \tau(x)$ *if and only if* $c \Vdash \sigma(x)$ *and* $c \Vdash \tau(x)$
- $c \Vdash \sigma(x) \vee \tau(x)$ *if and only if* $c \Vdash \sigma(x)$ *or* $c \Vdash \tau(x)$
- $c \Vdash \sigma(x) \Rightarrow \tau(x)$ *if and only if for all* $f : d \to c$, $d \Vdash \sigma(xf)$ *implies* $d \Vdash \tau(xf)$
- $c \Vdash \neg\sigma(x)$ *if and only if for all* $f : d \to c$, *we do not have* $d \Vdash \sigma(xf)$
- $c \Vdash \exists y : Y,\ \theta(y, x)$ *if and only if* $c \Vdash \theta(y, x)$ *for some* $y : \mathbf{y}c \to Y$
- $c \Vdash \forall y : Y,\ \theta(y, x)$ *if and only if* $d \Vdash \theta(y, xf)$ *for all* $f : d \to c$ *and* $y : \mathbf{y}d \to Y$

Proof. We shall not give the full proof. Note that it is very important that we are working in a presheaf topos. In more general setting, the conditions for the $\vee$ and $\exists$ operators fail and need to be adjusted.

We start with $\bot$ and $\top$. Suppose $c \Vdash \bot$, then we have a map $u : \mathbf{y}c \to \emptyset$, and by Yoneda, that means we have $u \in \emptyset(c) = \emptyset$, absurd. We have $c \Vdash \top$ by definition of $\top$.

Then we move to $\wedge$. Suppose $c \Vdash \sigma(x) \wedge \tau(x)$, then we have the following diagram:

$$\begin{tikzcd} & \{x : X \mid \sigma(x) \wedge \tau(x)\} \arrow[r] \arrow[d, tail, "s"] & \mathbf{1} \arrow[r] \arrow[d, tail, "\langle \mathrm{true}, \mathrm{true} \rangle"] & \mathbf{1} \arrow[d, tail, "\mathrm{true}"] \\ \mathbf{y}c \arrow[ur, dotted, "u"] \arrow[r, "x"'] & X \arrow[r, "\langle \sigma, \tau \rangle"'] & \Omega \times \Omega \arrow[r, "\wedge"'] & \Omega \end{tikzcd}$$

where all the squares are pullbacks. Precomposing by π_1, we obtain the following commutative square:

$$\begin{tikzcd} \{x : X \mid \sigma(x) \wedge \tau(x)\} \arrow[r] \arrow[d, tail, "s"] & \mathbf{1} \arrow[d, tail, "\mathrm{true}"] \\ X \arrow[r, "\sigma"'] & \Omega \end{tikzcd}$$

It is precisely the pullback condition for the following square.

$$\begin{tikzcd}
\{x : X \mid \sigma(x)\} \arrow[r] \arrow[d, tail] & \mathbf{1} \arrow[d, tail, "\text{true}"] \\
X \arrow[r, "\sigma"'] & \Omega
\end{tikzcd}$$

and thus we have a map $u_1 : \{x : X \mid \sigma(x) \wedge \tau(x)\} \to \{x : X \mid \sigma(x)\}$. Similarly using π_2, we obtain $u_2 : \{x : X \mid \sigma(x) \wedge \tau(x)\} \to \{x : X \mid \tau(x)\}$. Then $u_1 \circ u : \mathbf{y}c \to \{x : X \mid \sigma(x)\}$ and $u_2 \circ u : \mathbf{y}c \to \{x : X \mid \tau(x)\}$ are the desired maps.

Conversely, we have

$$\begin{tikzcd}
\mathbf{y}c \arrow[drrr, bend left, "!"] \arrow[drr, bend right, "x"'] & & & \\
& \{x : X \mid \sigma(x) \wedge \tau(x)\} \arrow[rr] \arrow[d, "s"] & & \mathbf{1} \arrow[d, "\langle \text{true}, \text{true} \rangle"] \\
& X \arrow[rr, "\langle \sigma, \tau \rangle"'] & & \Omega \times \Omega
\end{tikzcd}$$

with $\langle \sigma, \tau \rangle \circ x = \langle \text{true}, \text{true} \rangle \circ !$ (by precomposing with the projections), and thus by pullback, a map $u : \mathbf{y}c \to \{x : X \mid \sigma(x) \wedge \tau(x)\}$ such that $s \circ u = x$.

The connectives $\Rightarrow, \vee$ require more work and will not be proved here. The statement for $\neg$ is just a rephrasing of $\Rightarrow$ for $\tau(x) = \perp$. The rest of the proof can be found both in [SML94] and [LS86]. $\square$

Theorem 2.14 (Forcing for equality)**.** *Let $f, g : X \to Y$ and $x : \mathbf{y}c \to X$, then $c \Vdash f(x) = g(x)$ if and only if $f \circ x = g \circ x$ as maps of $\mathcal{E}$.*

Proof. The proof is by the universal property of the equalizer.

$$\begin{tikzcd}
\{x : X \mid f(x) = g(x)\} \arrow[r] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
& \mathbf{y}c \arrow[u, "x"] \arrow[ul, dotted] &
\end{tikzcd}$$

If $c \Vdash f(x) = g(x)$, we have the dotted arrow and then x equalizes f and g. Conversely, if x equalizes f and g, then we have the dotted arrow $\square$

Thus, we have a mechanical procedure to take any formula and check its validity inside any presheaf . If we are not working inside a presheaf category, then the theorem above doesn't apply, and minor modifications have to be made, see [SML94].

3 Martin-Löf Type Theory inside a presheaf category

We follow [AGH21] to define small maps, and the small map classifier π given by a Hofmann-Streicher lifting of a Grothendieck universe. We fix κ to be a (strongly) inaccessible cardinal, and we call a set *small* if it has cardinality less than κ. We write Sets_κ for the full subcategory of Sets consisting of small sets. Thus Sets_κ is a Grothendieck universe, that we will lift as in [HS97]. We fix a small category $\mathcal{C}$, that is $\mathrm{Obj}(\mathcal{C})$ and $\mathcal{C}(a,b)$, for every a,b, are small sets. We call $\mathcal{E}$ the associated presheaf topos.

3.1 Universes and small maps

We introduce the notion of small maps and see how they can be classified, then we show the relation between this and the subobject classifier.

3.1.1 Hofmann-Streicher lifting

Definition 3.1.

(i) We say that a presheaf $A \in \mathcal{E}$ is small *if $A(c)$ is a small set, for all $c \in \mathcal{C}$*

(ii) We say that $p : A \to X$ in $\mathcal{E}$ is a small map *if, for every $x : \mathbf{y}c \to X$, the presheaf A_x obtained by the pullback*

$$\begin{tikzcd} A_x \arrow[r] \arrow[d] & A \arrow[d, "p"] \\ \mathbf{y}c \arrow[r, "x"'] & X \end{tikzcd}$$

is small.

Let $\mathcal{S}$ be the class of small maps in $\mathcal{E}$. In the same way that we can classify the monos $S \rightarrowtail X$ with the map true : $\mathbf{1} \to \Omega$, we define $\pi : E \to U$ that classifies the maps of $\mathcal{S}$. This is given by the Hofmann-Streicher universe $U \in \mathcal{E}$, defined by letting

$$U(c) = \mathrm{ob}[(\mathcal{C}/c)^{\mathrm{op}}, \mathrm{Sets}_\kappa]$$

The presheaf $U(d \xrightarrow{f} c)(A)$ is the one sending $g : d' \to d$ to $A(f \circ g)$. Then we define $E \in \mathcal{E}$ by letting

$$E(c) = \mathrm{ob}[(\mathcal{C}/c)^{\mathrm{op}}, \mathrm{Sets}^\bullet_\kappa]$$

The forgetful functor $\mathrm{Sets}^\bullet_\kappa \to \mathrm{Sets}_\kappa$ induces a natural transformation $\pi : E \to U$ called the *small map classifier*. Given a small map $p : A \to X$, there exists a pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "p"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "c_p"'] & U \end{tikzcd}$$

We say that p is *classified* by c_p. Conversely, we introduce a canonical pullback p_A for each $A : X \to U$. First, if $A : \mathbf{y}c \to U$, we have a canonical pullback

$$\begin{tikzcd} \mathbf{y}c.A \arrow[r] \arrow[d, "p_A"'] & E \arrow[d, "\pi"] \\ \mathbf{y}c \arrow[r, "A"'] & U \end{tikzcd}$$

by letting

$$(\mathbf{y}c.A)(d) = \coprod_{f \in \mathcal{C}(d,c)} A(f)$$

Indeed, we can see $A : \mathbf{y}c \to U$ as an element of $U(c)$, that is $A : (\mathcal{C}/c)^{\mathrm{op}} \to \mathrm{Sets}_\kappa$

Lemma 3.1. *We can complete the above data into a pullback.*

Proof. The natural transformation p_A is constructed by sending each $x \in A(f)$ to f (by the universal property of the coproduct). Now we want to define a map $\mathbf{y}c.A \to E$ such that the above square commutes. Thus for each $d \in \mathcal{C}$, we want a map

$$\coprod_{f \in \mathcal{C}(d,c)} A(f) \to E(d)$$

By coproduct, it suffices to define maps

$$A(f) \to E(d)$$

for each $f : d \to c$, and we map each $x \in A(f)$, to the presheaf $A_{f,x} : (\mathcal{C}/d)^{\mathrm{op}} \to \mathrm{Sets}^\bullet_\kappa$ sending a map $g : d' \to d$ to the pointed set $(A(f \circ g), A(g)(x))$ where $A(g)$ is the action of A on the map $g : f \to g \circ f$ in $(\mathcal{C}/d)^{\mathrm{op}}$. This makes the square commutes, as the presheaf, as by the Yoneda lemma $A_d : \mathcal{C}(d,c) \to U(d)$ sends $f : d \to c$ to $U(f)(A)$ which is precisely the presheaf sending $g : d' \to d$ to $A(f \circ g)$. For the universal property, we consider a presheaf $X \in \mathcal{E}$ such that the following commutes.

$$\begin{tikzcd}
X \arrow[drr, bend left, "u"] \arrow[dr, bend right, "v"'] & & \\
& \mathbf{y}c.A \arrow[r] \arrow[d, "p_A"'] & E \arrow[d, "\pi"] \\
& \mathbf{y}c \arrow[r, "A"'] & U
\end{tikzcd}$$

and we want a map $t : X \to \mathbf{y}c.A$. This is done by letting

$$\begin{aligned} t_d : X(d) &\to \coprod_{f \in \mathcal{C}(d,c)} A(f) \\ x &\mapsto \iota_{v_d(x)}(\mathrm{pt}(u_d(x)(\mathrm{id}_d))) \end{aligned}$$

where the operator pt takes the point of a pointed set and

$$\iota_f : A(f) \to \coprod_{f \in \mathcal{C}(d,c)} A(f)$$

is the injection. This is indeed well typed because the set $u_d(x)(\mathrm{id}_d)$ is equal to the set $A(v_d(x))$ by commutation of the diagram. The map t is unique as it is defined pointwise. We do not check the commutation of the natural transformation. ☐

Finally, if $A : X \to U$, in any context X, then the density formula determines a pullback

$$\begin{tikzcd}
X.A \arrow[r] \arrow[d, "p_A"'] & E \arrow[d, "\pi"] \\
X \arrow[r, "A"'] & U
\end{tikzcd}$$

The map p_A is called the *display map* of A.

3.1.2 Small maps and subobject classifier

It turns out that the small map classifier works in a very similar way to the subobject classifier.

Theorem 3.2. *For all $X \in \mathcal{E}$, we have*

$$\mathcal{E}(X, U) \simeq \{u : (\textstyle\int X)^{\mathrm{op}} \to \mathcal{S}\}$$

Proof. If $\alpha : X \to U$, we define $u_\alpha : (\int X)^{\mathrm{op}} \to \mathcal{S}$ by

$$u_\alpha(c, x \in X(c)) = \alpha_c(x)(\mathrm{id}_c)$$

and if $u : (\int X)^{\mathrm{op}} \to \mathcal{S}$, we define $\chi_u : X \to U$ by

$$(\chi_u)_c(x)(d, f : d \to c) = u(d, X(f)(x))$$

We can check that these transformations are inverse of each other. □

We see the similarly with the subobject classifier Ω, if we call $\mathrm{sf}(X) = \{u : (\int X)^{\mathrm{op}} \to \mathcal{S}\}$ the small families over X, then we have

$$\mathrm{sf}(X) \simeq \mathcal{E}(X, U)$$

and with the subobject classifier, we have

$$\mathrm{Sub}(X) \simeq \mathcal{E}(X, \Omega)$$

Moreover, the subobject classifier Ω can be seen as a type of propositions. Notice that for all $c \in \mathcal{C}$, an object $S \in \Omega(c)$ is a subset of $\mathrm{Obj}(\mathcal{C}/c)$ closed under composition, thus it is a functor $S : \mathcal{C}/c \to \mathbb{2}$ where $\mathbb{2} = \{0 \leq 1\}$. Next, we have an inclusion functor $\mathbb{2} \rightharpoonup \mathrm{Sets}_\kappa$ (for instance, we can map 0 to $\emptyset$, etc. and we have an order coming from inclusion), thus this can be lifted into a map

$$\{\text{-}\} : \Omega \to \mathcal{U}$$

Proposition 3.3. *The inclusion maps $\{\text{-}\} : \Omega \to U$ is a monomorphism that fits into a pullback of the form:*

$$\begin{tikzcd}
\mathbf{1} \arrow[r] \arrow[d, tail, "\mathrm{true}"'] & E \arrow[d, "\pi"] \\
\Omega \arrow[r, tail, "\{\text{-}\}"'] & U
\end{tikzcd}$$

Proof. Call $i : \mathbb{2} \rightharpoonup \mathrm{Sets}_\kappa$. For all $c \in \mathcal{C}$, we have $\{\text{-}\}_c = i \circ \text{-}$, thus is injective in Sets_κ, that is a monomorphism. Then,

$$\begin{tikzcd}
\mathbb{1} \arrow[r] \arrow[d] & \mathrm{Sets}_\kappa^\bullet \arrow[d, "p"] \\
\mathbb{2} \arrow[r, tail, "i"'] & \mathrm{Sets}_\kappa
\end{tikzcd}$$

where $\mathbb{1}$ is the terminal category and $p : \mathrm{Sets}_\kappa^\bullet \to \mathrm{Sets}_\kappa$ is the forgetful functor, is a pullback. Indeed, the pullback category is the one that has objects pairs (s,t) with $s \in \mathbb{2}$ and $t \in \mathrm{Sets}_\kappa^\bullet$ such that $i(s) = p(t)$, thus the object of the pullback category are (in bijection with)

$$\{(s,x) \mid x \in s, s \in \mathbb{2}\}$$

but the objects of $\mathbb{2}$ were chosen to be $\emptyset$ and $\{\emptyset\}$, thus the pullback category has only one object. Similarly, we can show that there is only one morphism. By the universal property of the terminal category, we have automatically the universal arrow. Then, we obtain the desired pullback by composition, by noting that $\mathbf{1}_c \simeq \mathrm{Obj}[(\mathcal{C}/c)^{\mathrm{op}}, \mathbb{1}]$ for all $c \in \mathcal{C}$. □

Conversely, by considering the image of the universal map $\mathrm{im}(\pi) \rightarrowtail U$, we can classify it with the following pullback

$$\begin{tikzcd}
\mathrm{im}(\pi) \arrow[r] \arrow[d, tail] & \mathbf{1} \arrow[d, tail, "\mathrm{true}"] \\
U \arrow[r, "\mathrm{supp}"'] & \Omega
\end{tikzcd}$$

which allows us to define propositional truncation as the map $\{\mathrm{supp}(\text{-})\} : U \to U$.

3.2 Forcing for type theory

In any presheaf category, we can interpret a version of dependent type theory. We rely on the results of [Awo18] to present this type theory in the context of a universe $\pi : E \to U$.

3.2.1 Category with families

Definition 3.2 (Category with families)**.** *The* category with families *associated to any presheaf topos $\mathcal{E}$, is defined as follows:*

- *The contexts are the objects $X \in \mathcal{E}$*
- *A type A in context X is a map $A : X \to U$*
- *A term $a : A$ in context X is a map $a : X \to E$ such that the following diagram commutes.*

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, equal] & E \arrow[d, "\pi"] \\
X \arrow[r, "A"'] & U
\end{tikzcd}$$

- *Definitional equality on terms or type is defined via equality of maps in the topos. For instance, we have $X \vdash A = B$ if and only if $A : X \to U$ and $B : X \to U$ are the same maps in $\mathcal{E}$. For terms, we write $X \vdash a = b : A$.*

Thanks to the properties of the small map classifier, we have an isomorphism between closed types and small objects.

Lemma 3.4. *For any $A \in \mathcal{E}$, $A \to \mathbf{1}$ is a small map if and only if A is a small presheaf.*

Proof. For every $d \in \mathcal{C}$, the pullback

$$\begin{tikzcd} A_! \arrow[r] \arrow[d] & A \arrow[d, "!"] \\ \mathbf{y}d \arrow[r, "!"'] & \mathbf{1} \end{tikzcd}$$

is given by $A_! = A \times \mathbf{y}d$, and for every $c \in \mathcal{C}$, $(A \times \mathbf{y}d)(c)$ is $A(d) \times \mathcal{C}(c, d)$ which is small if and only if $A(d)$ is small, as $\mathcal{C}(c, d)$ is already a small set. □

Thus, for small presheaves A, there exists a pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "!_A"'] & E \arrow[d, "\pi"] \\ \mathbf{1} \arrow[r, "c_A"'] & U \end{tikzcd}$$

that turns the object A into a type $c_A : \mathbf{1} \to U$. Conversely, given a type $A : \mathbf{1} \to U$, we have the pullback

$$\begin{tikzcd} \mathbf{1}.A \arrow[r] \arrow[d, "p_A"'] & E \arrow[d, "\pi"] \\ \mathbf{1} \arrow[r, "A"'] & U \end{tikzcd}$$

given by the display map p_A. This proves the following proposition.

Proposition 3.5. *Closed types $A : \mathbf{1} \to U$ correspond, up to isomorphic, to small presheaves.*

Moreover, we can use the display maps to characterize elements of a given type.

Proposition 3.6. *Let $A : \Gamma \to U$ be a type in context Γ. The elements $\Gamma \vdash a : A$ are in bijective correspondence with the sections of the display map $p_A : \Gamma.A \to \Gamma$.*

Proof. Any element $\Gamma \vdash a : A$ determines a section of the display maps thanks to the following pullback.

$$\begin{tikzcd} \Gamma \arrow[dr, dashed, "(\mathrm{id}_\Gamma, a)"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "\mathrm{id}_\Gamma"'] & & \\ & \Gamma.A \arrow[r] \arrow[d, "p_A"'] & E \arrow[d, "\pi"] \\ & \Gamma \arrow[r, "A"'] & U \end{tikzcd}$$

And conversely, with the dotted arrow, we determine the element by composition with the arrow $\Gamma.A \to E$. □

It is also possible to define substitution. If Δ, Γ are two contexts, and $A : \Gamma \to U$ a type in context Γ, then we define the substitution $A[t] = A \circ t$, which allows us to have the following substitution rule valid in our type theory.

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A \vdash b(x) : B(x)}{\Gamma \vdash b(a) : B(a)}$$

For more details about substitution, see [AGH21]. Then, it is possible to internalize every basic construction of dependent type theory, and their formation, introduction, elimination, computation, and expansion rules. The proofs and constructions can be found in [Awo18].

3.2.2 Extending the forcing

In this setup, we can extend the notion of forcing presented in section 2, and obtain a similar theorem as Theorem 2.13. From now on, we will refer to the forcing presented in section 2 as *standard forcing.*

Definition 3.3. *Let $A : X \to U$ be a type in context X, and $x : \mathbf{y}c \to X$. For $a : \mathbf{y}c \to E$, We say c* forces *$a : A(x)$ written $c \Vdash a : A(x)$ if the following diagram commutes.*

$$\begin{tikzcd}
\mathbf{y}c \arrow[r, "a"] \arrow[d, "x"'] & E \arrow[d, "\pi"] \\
X \arrow[r, "A"'] & U
\end{tikzcd}$$

Like in the standard forcing, $c \Vdash a : A(x)$ is to say $\mathbf{y}c \vdash a : A(x)$, as we can see in the following diagram:

$$\begin{tikzcd}
\mathbf{y}c \arrow[rr, "a"] \arrow[d, equal] & & E \arrow[d, "\pi"] \\
\mathbf{y}c \arrow[r, "x"'] & X \arrow[r, "A"'] & U
\end{tikzcd}$$

Proposition 3.7. *Let $A : X \to U$ be a type in context X, and $x : \mathbf{y}c \to X$. An element $a : \mathbf{y}c \to E$ is the same thing as the dotted arrow in the following diagram:*

$$\begin{tikzcd}
 & X.A \arrow[r, "q_A"] \arrow[d, "p_A"'] & E \arrow[d, "\pi"] \\
\mathbf{y}c \arrow[ur, dotted, "u"] \arrow[r, "x"'] & X \arrow[r, "A"'] & U
\end{tikzcd}$$

Proof. If we have the dotted arrow u, then we can take $a = q_A \circ u$. Conversely, if we have $a : \mathbf{y}c \to E$, the universal property of the pullback gives uf the map u. ☐

In this case, the dotted map u is not unique and is determined by the element $a : A$, whereas in the standard forcing, the dotted map was necessarily unique, as it was entirely determined by x. Thus, this new forcing will have to deal with a variety of possible dotted maps, and we will have to impose uniformity conditions between them. This uniformity is expressed through the following proposition from [AGH21].

Proposition 3.8. *Let $A : X \to U$ and suppose that for each $x : \mathbf{y}c \to X$, we have an element $a_x : \mathbf{y}c \to E$ such that $c \Vdash a_x : A(x)$ and for all $f : d \to c$ in $\mathcal{C}$ $a_x \circ f = a_{xf}$. Then there exists a unique element $a : X \to E$ such that $X \vdash a : A$ and moreover for all $x : \mathbf{y}c \to X$, $a \circ x = a_x$.*

The uniformity condition can be seen in the following commutative diagram:

$$\begin{tikzcd}
\mathbf{y}d \arrow[d, "f"'] \arrow[dr, dotted, "a_{xf}"] & \\
\mathbf{y}c \arrow[r, dotted, "a_x"] \arrow[d, "x"'] & E \arrow[d, "\pi"] \\
X \arrow[r, "A"'] & U
\end{tikzcd}$$

Thus, we have the following theorem, which is analogous to Theorem 2.10

Theorem 3.9. *The data of $a : X \to E$ such that $X \vdash a : A$ is the same as uniform families of elements $a_x : \mathbf{y}c \to E$ such that $c \Vdash a_x : A(x)$, and we have that $a_x = a \circ x$.*

Finally, we can state the theorem for forcing as follow.

Theorem 3.10 (Condition for forcing)**.** *Let $A, B : X \to U$ be types and $x : \mathbf{y}c \to X$, then*

1. $c \Vdash a : 0$ *never*
2. $c \Vdash a : 1$ *for a unique* $a = \star : \mathbf{y}c \to E$
3. $c \Vdash t : (A \times B)(x)$ *if and only if* $c \Vdash \pi_1(t) : A(x)$ *and* $c \Vdash \pi_2(t) : B(x)$
4. $c \Vdash t : (A + B)(x)$ *if and only if* $c \Vdash a : A(x)$ *with* $t = \mathrm{inl}(a)$ *or* $c \Vdash b : B(x)$ *with* $t = \mathrm{inr}(a)$

Note that we also have the theorem for dependent sum and product, but we will not detail them here. All the proofs can be found in [AGH21]. Before concluding the section, we prove how this new forcing subsumes the standard one. We write $\Vdash_{\mathrm{std}}$ for the standard forcing.

Theorem 3.11. *Let $\sigma : X \to \Omega$ be a proposition and $x : \mathbf{y}c \to X$. Then the following are equivalent.*

(i) $c \Vdash_{std} \sigma(x)$

(ii) $c \Vdash s : \sigma(x)$ *for a (necessarily unique)* $s : \mathbf{y}c \to E$

Proof. By the property of Ω and by Proposition 3.3, the two following squares are pullbacks.

$$\begin{tikzcd}
\{x : X \mid \sigma(x)\} \arrow[r] \arrow[d, tail] & \mathbf{1} \arrow[r] \arrow[d, tail, "\text{true}"'] & E \arrow[d, "\pi"] \\
X \arrow[r] & \Omega \arrow[r, "\{\text{-}\}"'] & U
\end{tikzcd}$$

Thus, by the pasting law, the big square is a pullback so by the uniqueness of a pullback we have

$$\{x : X \mid \sigma(x)\} \simeq X.\{\sigma\}$$

Thus a map $u : \mathbf{y}c \to \{x : X \mid \sigma(x)\}$ amounts to a map $u : \mathbf{y}c \to X.\{\sigma\}$, which proves the theorem. $\square$

4 A model of cubical type theory

In this section, we use our previous work to show that a particular presheaf topos $[\square^{\mathrm{op}}, \mathrm{Sets}]$ is a model of cubical type theory.

4.1 Cubical presheaves

We introduce the category $\square$. It will be the base category of a presheaf topos whose internal type theory, as described in the previous chapter, will model cubical type theory. The category $\square$ is the one from [CCHM15], its objects are free de Morgan algebras, whose structure will be helpful to obtain an object $\mathbb{I}$, which we want to behave like the synthetic version of the interval $[0, 1]$. Then, we introduce the notion of cofibration, whose behavior is important to internalize Kan filling and glueing [OP16].

4.1.1 The box category

For $n \geq 0$, we denote by I_n the free de Morgan algebra on n generators.

Definition 4.1. *We call $\square$ the category having as objects cardinal numbers $[n] \geq 0$ and as morphisms in $\square([n], [m])$ the de Morgan homomorphisms $f : I_m \to I_n$.*

We specialize the previous sections to the case where $\mathcal{C} = \square$, so $\mathcal{E}$ is the presheaf topos $[\square^{\mathrm{op}}, \mathrm{Sets}]$. For a presheaf $X \in \mathcal{E}$, we will often write X_n for $X([n])$. The interval $\mathbb{I}$ is the presheaf $\mathbf{y}[1]$. Note that $\mathbb{I}_n = \square([n], [1])$ has a de Morgan structure defined pointwise. We respectively call $\sqcap_n, \sqcup_n, 0_n, 1_n$ the product, sum, zero, and one of this de Morgan algebra. We can collect this data to define natural transformations:

$$\begin{aligned} \sqcap &: \mathbb{I} \times \mathbb{I} \to \mathbb{I} \\ \sqcup &: \mathbb{I} \times \mathbb{I} \to \mathbb{I} \\ 0 &: \mathbf{1} \to \mathbb{I} \\ 1 &: \mathbf{1} \to \mathbb{I} \end{aligned}$$

defined pointwise, in the obvious way.

Proposition 4.1.

(i) $0 \neq 1$

(ii) $\forall x : \mathbf{y}[n] \to \mathbb{I}$, *we have* $0 \sqcap x = 0 = x \sqcap 0$ *and* $1 \sqcap x = x = x \sqcap 1$

(iii) $\forall x : \mathbf{y}[n] \to \mathbb{I}$, *we have* $0 \sqcup x = x = x \sqcup 0$ *and* $1 \sqcup x = 1 = x \sqcup 1$

Proof. Note that for all $n \geq 0$, $0_n \neq 1_n$ and $\forall x \in \mathbb{I}_n$, we have

$$\begin{aligned} 0_n \sqcap_n x &= 0_n = x \sqcap_n 0_n \\ 1_n \sqcap_n x &= x = x \sqcap_n 1_n \\ 0_n \sqcup_n x &= x = x \sqcup_n 0_n \\ 1_n \sqcup_n x &= 1_n = x \sqcup 1_n \end{aligned}$$

Thus, by naturality of the isomorphism $\varphi : \mathcal{E}(\mathbf{y}[n], \text{-}) \simeq (X \mapsto X_n)$, we have the results. $\square$

Proposition 4.2. *$\square$ has finite products.*

Proof. Observe that our category is equivalent to the opposite category of free de Morgan algebras, such a category has finite coproducts, given by the free product, hence $\square$ has finite products, given by $[n] \times [m] = [n+m]$. $\square$

Finally, we show a statement about the decidability of our interval object.

Lemma 4.3. *For all $[n] \in \square$, $\mathbb{I}_n$ has decidable equality.*

Proof. An element of $\mathbb{I}_n$ is a de Morgan homomorphism $f : I_1 \to I_n$, thus it is entirely determined by $f(i) \in I_n$, which is the free de Morgan algebra on n elements and has decidable equality (because it is finite in a decidable theory). $\square$

4.1.2 Cofibrations

We introduce the notion of cofibrant propositions. They are a way to internalize composition and filling introduced in [CCHM15]. The cofibrant propositions form a subobject of Ω. More precisely, we assume a map

$$\mathrm{cof} : \Omega \to \Omega$$

and we consider the associated subobject

$$\texttt{Cof} = \{\varphi : \Omega \mid \mathrm{cof}\,\varphi\}$$

We require cof to satisfy some generic properties that will allow us to model all axioms of cubical type theory and that will be verified later for a specific choice of `Cof`.

Definition 4.2 (Cofibration)**.** *A* cofibration *is a monomorphism whose classifying arrow factors trough* `Cof` $\rightarrowtail \Omega$*.*

Note that the subobject `Cof` in [OP16] is written Φ in [AGH21].

Lemma 4.4. *Let $\varphi : X \to \Omega$ be a proposition. For every $x : \mathbf{y}c \to X$, the following are equivalent.*

1. $c \Vdash \mathrm{cof}\,\varphi(x)$
2. $\varphi \circ x : \mathbf{y}c \to \Omega$ *factor through* `Cof` $\to \Omega$

Proof. Consider the following diagram:

$$\begin{tikzcd}
& \{X \mid \mathrm{cof}\,\varphi\} \arrow[r] \arrow[d, tail] & \texttt{Cof} \arrow[r] \arrow[d, tail] & \mathbf{1} \arrow[d, tail, "\mathrm{true}"] \\
\mathbf{y}c \arrow[r, "x"'] \arrow[ur, dotted] & X \arrow[r, "\varphi"'] & \Omega \arrow[r, "\mathrm{cof}"'] & \Omega
\end{tikzcd}$$

If $c \Vdash \mathrm{cof}\,\varphi(x)$, that is, if we have the dotted arrow, then indeed we have the factorization of $\varphi \circ x$ through `Cof` $\to \Omega$. Conversely, given the factorization, by pullback of the left square, we obtain the dotted arrow. $\square$

4.2 Internal validity of the axioms

This section is dedicated to proving that the cube category $\mathcal{E}$ we just defined is a model of type theory and univalence. For that, we follow [OP16] and prove that the internal logic of $\mathcal{E}$ satisfies the nine axioms presented below.

$\mathbf{ax_1}$ $\forall \varphi : \mathbb{I} \to \Omega,\ (\forall i : \mathbb{I},\ \varphi\, i \vee \neg \varphi\, i) \Rightarrow (\forall i : \mathbb{I},\ \varphi\, i) \vee (\forall i : \mathbb{I},\ \neg \varphi\, i)$

$\mathbf{ax_2}$ $\neg(0 = 1)$

$\mathbf{ax_3}$ $\forall i : \mathbb{I},\ 0 \sqcap i = 0 = i \sqcap 0\ \wedge\ 1 \sqcap i = i = i \sqcap 1$

$\mathbf{ax_4}$ $\forall i : \mathbb{I},\ 0 \sqcup i = i = i \sqcup 0\ \wedge\ 1 \sqcup i = 1 = i \sqcup 1$

$\mathbf{ax_5}$ $\forall i : \mathbb{I},\ \mathrm{cof}(i = 0) \wedge \mathrm{cof}(i = 1)$

$\mathbf{ax_6}$ $\forall \varphi\ \psi : \Omega,\ \mathrm{cof}\,\varphi \Rightarrow \mathrm{cof}\,\psi \Rightarrow \mathrm{cof}(\varphi \vee \psi)$

$\mathbf{ax_7}$ $\forall \varphi\ \psi : \Omega,\ \mathrm{cof}\,\varphi \Rightarrow (\varphi \Rightarrow \mathrm{cof}\,\psi) \Rightarrow \mathrm{cof}(\varphi \wedge \psi)$

$\mathbf{ax_8}$ $\forall \varphi : \mathbb{I} \to \Omega,\ (\forall i : \mathbb{I},\ \mathrm{cof}\,\varphi\, i) \Rightarrow \mathrm{cof}(\forall i : \mathbb{I},\ \varphi\, i)$

$\mathbf{ax_9}$ $(\varphi : \mathtt{Cof})(A : [\varphi] \to U)(B : U)(s : (u : [\varphi]) \to (A\, u \simeq B)) \to$
$(B' : U) \times \{s' : B' \simeq B \mid \forall u : [\varphi],\ A\, u = B' \wedge s\, u = s'\}$

To prove each of these axioms, we will use Kripke-Joyal forcing and unravel each formula thanks to Theorem 2.13. Since $\square$ has a terminal object $[0]$, it suffices to prove that each axiom is forced at stage $[0]$. That is, for $\mathbf{k} = \mathbf{1}, \dots, \mathbf{9}$, we have

$$\vdash \mathbf{ax_k} \iff [0] \Vdash \mathbf{ax_k}$$

4.2.1 Axioms for the interval

In this part, we will prove $\mathbf{ax_1}$ to $\mathbf{ax_4}$, relative to the structure of the interval object $\mathbb{I}$. We start with $\mathbf{ax_2}$ to $\mathbf{ax_4}$, which mean that the interval $\mathbb{I}$ has a connection structure. Notice that in the case of $\square$, the interval has in fact a full de Morgan structure, but only the connection structure will be used for the model of cubical type theory.

Theorem 4.5 ($\mathbf{ax_2}$). $[0] \Vdash \neg(0 = 1)$

Proof. By Theorem 2.10, it suffices to show that for all $[n]$, we do not have $[n] \Vdash 0 = 1$. Assume $[n] \Vdash 0 = 1$, then we would have $0 = 1 : \mathbf{y}[n] \to \mathbb{I}$, which is false as $0_n \neq 1_n$. □

Theorem 4.6 ($\mathbf{ax_3}$). $[0] \Vdash (\forall i : \mathbb{I}),\ 0 \sqcap i = 0 = i \sqcap 0\ \wedge\ 1 \sqcap i = i = i \sqcap 1$

Proof. Call $\sigma(i) = 0 \sqcap i = 0 = i \sqcap 0\ \wedge\ 1 \sqcap i = i = i \sqcap 1$. By Theorem 2.13, we want to show that $[n] \Vdash \sigma(i)$ for all $f : [n] \to [0]$ and $i : \mathbf{y}[n] \to \mathbb{I}$. Such a map f is unique, thus we need to show that $[n] \Vdash \sigma(i)$, for all $i : \mathbf{y}[n] \to \mathbb{I}$. Unwinding of the $\wedge$ connective, it suffices to prove each equality independently. For instance with $[n] \Vdash 0 \sqcap i = 0$, let $i : \mathbf{y}[n] \to \mathbb{I}$ be a generalized element. By Theorem 2.14, it suffices to show that $0 \sqcap i = 0$ as maps of $\mathcal{E}$, and this is true by Proposition 4.1. □

Theorem 4.7 ($\mathbf{ax_4}$). $[0] \Vdash (\forall i : \mathbb{I}),\ 0 \sqcup i = i = i \sqcup 0\ \wedge\ 1 \sqcup i = 1 = i \sqcup 1$

Proof. The proof is similar to the previous one. □

Now we focus on $\mathbf{ax_1}$ which says that the interval $\mathbb{I} : \mathcal{U}$ is connected. For that, we use the next auxiliary lemma.

Lemma 4.8. *Let $\varphi, \psi : \mathbb{I} \to \Omega$ be two formulas. Then the following are equivalent.*

(i) $\mathbb{I} \vdash \psi \vee \varphi$

(ii) $\mathbb{I} \vdash \psi$ *or* $\mathbb{I} \vdash \varphi$

Proof. The proof uses the fact that $\mathbb{I} = \mathbf{y}[1]$, thus $\mathbb{I} \vdash \psi \vee \varphi$ is to say that we have the dotted arrow in

$$\begin{tikzcd} & \{i : \mathbb{I} \mid \psi(i) \vee \varphi(i)\} \arrow[d, tail] \\ \mathbf{y}[1] \arrow[ur, dotted] \arrow[r, equal] & \mathbb{I} \end{tikzcd}$$

which is precisely to say that $[1] \Vdash \psi(\mathrm{id}_{\mathbb{I}}) \vee \varphi(\mathrm{id}_{\mathbb{I}})$, by Theorem 2.13, this is equivalent to $[1] \Vdash \psi(\mathrm{id}_{\mathbb{I}}) \vee [1] \Vdash \psi(\mathrm{id}_{\mathbb{I}})$ that we can rewrite $\mathbb{I} \vdash \psi \vee \mathbb{I} \vdash \varphi$. □

Theorem 4.9 ($\mathbf{ax_1}$). $\vdash \forall \varphi : \mathbb{I} \to \Omega,\ (\forall i : \mathbb{I},\ \varphi\ i \vee \neg\varphi\ i) \Rightarrow (\forall i : \mathbb{I},\ \varphi\ i) \vee (\forall i : \mathbb{I},\ \neg\varphi\ i)$

Proof. We prove something slightly different, namely that for all $\varphi : \mathbb{I} \to \Omega$,

$$\vdash (\forall i : \mathbb{I},\ \varphi\ i \vee \neg\varphi\ i) \iff (\forall i : \mathbb{I},\ \varphi\ i) \vee (\forall i : \mathbb{I},\ \neg\varphi\ i)$$

Let $\varphi : \mathbb{I} \to \Omega$, it suffices to show that $\vdash (\forall i : \mathbb{I},\ \varphi\ i \vee \neg\varphi\ i)$ if and only if $\vdash (\forall i : \mathbb{I},\ \varphi\ i) \vee (\forall i : \mathbb{I},\ \neg\varphi\ i)$. We have

$$\begin{aligned} & \vdash (\forall i : \mathbb{I},\ \varphi\ i \vee \neg\varphi\ i) & \\ \iff & \mathbb{I} \vdash \varphi\ i \vee \neg\varphi\ i & \text{by Corollary 2.9} \\ \iff & (\mathbb{I} \vdash \varphi\ i) \text{ or } (\mathbb{I} \vdash \neg\varphi\ i) & \text{by the previous lemma} \\ \iff & (\vdash \forall i : \mathbb{I},\ \varphi\ i) \text{ or } (\vdash \forall i : \mathbb{I},\ \neg\varphi\ i) & \text{by Corollary 2.9} \\ \iff & \vdash (\forall i : \mathbb{I},\ \varphi\ i) \vee (\forall i : \mathbb{I},\ \neg\varphi\ i) & \text{by Theorem 2.13} \end{aligned}$$

□

We chose to take $\mathbb{I} = \mathbf{y}[1]$, but in fact every representable has a connection structure and is connected. Indeed, the proofs of $\mathbf{ax_1} - \mathbf{ax_4}$ would have worked the same with $\mathbb{I} = \mathbf{y}[n]$, for any $n \geq 1$.

4.2.2 Axioms for cofibrations

We will not commit yet to a particular class of cofibrant propositions and will show alternative characterizations of $\mathbf{ax_5}$ to $\mathbf{ax_8}$.

Lemma 4.10. $\vdash \mathbf{ax_5}$ *if and only if the monomorphisms*

$$\{i : \mathbb{I} \mid i = 0\} \rightarrowtail \mathbb{I}$$

and

$$\{i : \mathbb{I} \mid i = 1\} \rightarrowtail \mathbb{I}$$

are cofibrations.

Proof. By forcing, we have $\vdash \forall i : \mathbb{I},\ \mathrm{cof}(i = 0)$ if and only if for all $i : \mathbf{y}[n] \to \mathbb{I}$, $[n] \Vdash \mathrm{cof}(i = 0)$, thus by Lemma 4.4, if and only if for all $i : \mathbf{y}[n] \to \mathbb{I}$, the map $(i = 0) : \mathbf{y}[n] \to \Omega$ factors trough $\texttt{Cof} \to \Omega$. Thus, if we have for every $i : \mathbf{y}[n] \to \mathbb{I}$ a diagram

$$\begin{tikzcd}
\mathbf{y}[n] \arrow[r, "i"] \arrow[dr, "v_i"'] & \mathbb{I} \arrow[r, "-=0"] & \Omega \\
& \mathtt{Cof} \arrow[ur, tail] &
\end{tikzcd}$$

We obtain by density a diagram

$$\begin{tikzcd}
\mathbb{I} \arrow[r, "\mathrm{id}_{\mathbb{I}}"] \arrow[dr, "\operatorname{colim} v_i"'] & \mathbb{I} \arrow[r, "-=0"] & \Omega \\
& \mathtt{Cof} \arrow[ur, tail] &
\end{tikzcd}$$

giving that $- = 0 : \mathbb{I} \to \Omega$ factors through $\mathtt{Cof} \to \Omega$, thus that $\{i : \mathbb{I} \mid i = 0\} \rightarrowtail \mathbb{I}$ is a cofibration. Conversely, given the factorization

$$\begin{tikzcd}
\mathbb{I} \arrow[r, "-=0"] \arrow[d] & \Omega \\
\mathtt{Cof} \arrow[ur, tail] &
\end{tikzcd}$$

we can compose on the left of the diagram by $i : \mathbf{y}[n] \to \mathbb{I}$ to obtain the required factorization. The proof is similar for $- = 1$. Note that we technically proved our equivalence for

$$\vdash \forall i : \mathbb{I},\ \mathrm{cof}(i = 0) \wedge \forall i : \mathbb{I},\ \mathrm{cof}(i = 1)$$

rather than

$$\vdash \forall i : \mathbb{I},\ \mathrm{cof}(i = 0) \wedge \mathrm{cof}(i = 1)$$

but the two are indeed equivalent. □

$\mathbf{ax_7}$ and $\mathbf{ax_8}$ indicate that cofibrations are closed under compositions and exponentials by $\mathbb{I}$.

Lemma 4.11.

1. *$\vdash \mathbf{ax_7}$ if and only if cofibrations are closed under compositions*

2. *$\vdash \mathbf{ax_8}$ if and only if cofibrations are closed under exponentiations by $\mathbb{I}$*

Proof. A detailed proof can be found in [OP16, Lemma 5.4]. □

Thus, to summarize, we have shown what kind of cofibration we need in order to satisfy $\mathbf{ax_5}$ to $\mathbf{ax_8}$, that is, we have the following theorem.

Theorem 4.12. *If the class $\mathtt{Cof} \in \mathcal{E}$ of cofibrant propositions is such that*

- *$\{i : \mathbb{I} \mid i = 0\} \rightarrowtail \mathbb{I}$ is a cofibration*
- *$\{i : \mathbb{I} \mid i = 1\} \rightarrowtail \mathbb{I}$ is a cofibration*
- *If $f : X \rightarrowtail Y$ and $g : Y \rightarrowtail Z$ are cofibrations, then $g \circ f$ is a cofibration*
- *If $f : X \rightarrowtail Y$ is a cofibration, then $f^{\mathbb{I}} : X^{\mathbb{I}} \rightarrowtail Y^{\mathbb{I}}$ is a cofibration*

then $\vdash \mathbf{ax_k}$, for $\mathbf{k = 5, 7, 8}$

4.2.3 The axiom of strictification

Definition 4.3. *We define $\Omega_{\mathtt{dec}} \rightarrowtail \Omega$ to be the sub-presheaf consisting of decidable sieves, that is*

$$\Omega_{\mathtt{dec}}(c) = \{S \in \Omega(c) \mid S \text{ is a decidable subset of } \mathrm{Obj}(\mathcal{C}/c)\}$$

Note that if the ambient set theory we are working with has excluded middle then $\Omega_{\mathtt{dec}} = \Omega$, but in general $\Omega_{\mathtt{dec}}$ classifies monos $s : S \rightarrowtail X$ such that for all $c \in \mathcal{C}$, $s_c : S(c) \to X(c)$ has a decidable image, indeed, when computing $(\chi_s)_c(x)(d)$, we need to decide whether or not an element belongs to $s_c(S(d))$ (that we can simply call $S(d)$).

Now, recall the axiom 9.

$$\begin{aligned}\mathbf{ax_9} : (\varphi : \mathtt{Cof})(A : [\varphi] \to U)(B : U)(s : (u : [\varphi]) \to (A\ u \simeq B)) \to \\ (B' : U) \times \{s' : B' \simeq B \mid \forall u : [\varphi]. A\ u = B' \wedge s\ u = s'\}\end{aligned}$$

It says that any cofibrant partial type A isomorphic, where it is defined, to a total type B, can be extended to a total type B' isomorphic to B. The next theorem [OP16, Theorem 8.4] gives us a sufficient condition for a cofibration to make the previous axiom true.

Theorem 4.13 (Condition for $\mathbf{ax_9}$)**.** $\mathbf{ax_9}$ *is satisfied in $\mathcal{E}$ if $\mathtt{Cof} \rightarrowtail \Omega$ is contained in $\Omega_{\mathtt{dec}} \rightarrowtail \Omega$.*

4.2.4 Main theorem

Using what we proved, or recalled, we can have an explicit model of cubical type theory inside our cubical category.

Theorem 4.14 (Model of HoTT)**.** *If $\mathcal{E} = [\square^{\mathrm{op}}, \mathrm{Sets}]$ with $\mathbb{I} = \mathbf{y}[1]$ and $\mathtt{Cof} = \Omega_{\mathtt{dec}}$, then $\vdash \mathbf{ax_k}$ for $\mathbf{k} = \mathbf{1}, \ldots, \mathbf{9}$, thus its internal type theory is a model of cubical type theory with univalence.*

Proof. The only part left is to show that this choice of $\mathtt{Cof}$ satisfies the hypotheses of Theorem 4.12. By our previous remarks, $\mathtt{Cof}$ classifies monos that have a decidable image. Hence $\{i : \mathbb{I} \mid i = 0\} \rightarrowtail \mathbb{I}$ and $\{i : \mathbb{I} \mid i = 1\} \rightarrowtail \mathbb{I}$ are cofibrations. Indeed, those monos have a decidable image whenever we can decide equality with $0_n \in \mathbb{I}_n$ for all $[n]$, and this follows from Lemma 4.3. As for $\mathbf{ax_6}$, if $\mathrm{cof}\,\varphi$ and $\mathrm{cof}\,\psi$ hold, then their associated subobject has a decidable image, and so will have the subobject associated to $\varphi \vee \psi$, and hence we have $\mathrm{cof}(\varphi \vee \psi)$. Next, the composition of $f : X \rightarrowtail Y$ and $g : Y \rightarrowtail Z$ that have both a decidable image has also a decidable image. Theorem 4.13 validates $\mathbf{ax_9}$, so the last point to check is the closure under exponentiation by $\mathbb{I}$,, which follows from the fact that $\square$ has finite products. Indeed, for all X and $[n] \in \square$, we have the following isomorphisms:

$$\begin{aligned}X^{\mathbb{I}}([n]) &\simeq \mathcal{E}(\mathbf{y}[n] \times \mathbb{I}, X) \\ &\simeq \mathcal{E}(\mathbf{y}[n] \times \mathbf{y}[1], X) \\ &\simeq \mathcal{E}(\mathbf{y}([n] \times [1]), X) \\ &\simeq X([n] \times [1])\end{aligned}$$

Thus, the functor $(\text{-})^{\mathbb{I}} : \mathcal{E} \to \mathcal{E}$ is isomorphic to the functor $\mathcal{E} \to \mathcal{E}$ induced by precomposition with $\text{-} \times [1] : \square \to \square$, which preserves the image decidability componentwise. $\square$

We want to point out that the notion of cofibration we chose here is not the same as the one in [CCHM15]. Instead, they use the face lattice $\mathbb{F}$, seen as an object of $\mathcal{E}$, and define the mono $m : \mathbb{F} \rightarrowtail \Omega$ by sending $x \in \mathbb{F}([n])$ to the sieve

$$m_{[n]}(x) = \{\cdot \xrightarrow{f} [n] \mid \mathbb{F}(f)(x) = 1\}$$

and they take $\texttt{Cof} \rightarrowtail \Omega$ to be the subobject given by this monomorphism m. One can see a proof in [OP16, Definition 8.8], that this choice of cofibration also satisfy the axioms.

4.2.5 Link with the syntax

The category with family used to model cubical type theory is not simply the one presented in section 3, rather each type comes with a composition structure. For $\varphi : \mathbf{1} \to \Omega$, we write $[\varphi]$ for the subobject $\{\text{-} : \mathbf{1} \mid \varphi\}$.

Definition 4.4. *For a type $A : X \to U$, we define the type of* cofibrant partial elements *of A to be*

$$\Box A \triangleq (\varphi : \mathbf{1} \to \texttt{Cof}) \times ([\varphi] \to A)$$

and we call an extension *of $(\varphi, f) : \Box A$ an element $a : A$ together with a proof of*

$$(\varphi, f) \uparrow a \triangleq \forall u : [\varphi],\ f\ u = a$$

Thus a cofibrant partial element is an element of type A defined partially, only on the sub-polyhedron given by φ. An extension of a cofibrant partial element is an element $a : A$ defined everywhere, such that its restriction on the sub-polyhedron coincides with the cofibrant partial element. The syntactic counterpart if the notation $a : A[\varphi \mapsto u]$ where u would be the cofibrant partial element and a the extension.

Definition 4.5. *Let $A : X \to U$ be a type, and $e : \mathbb{I}$ to be 0 or 1. A* composition structure *for A is an element of type*

$$\begin{aligned}\texttt{Comp}\ A\ e \triangleq (\varphi : \mathbf{1} \to \texttt{Cof})(f : [\varphi] \to A^{\mathbb{I}}) \to \\ \{a_0 : A\ e \mid (\varphi, \lambda u : [\varphi].f\ u\ e) \uparrow a_0\} \to \\ \{a_1 : A\ \bar{e} \mid (\varphi, \lambda u : [\varphi].f\ u\ \bar{e}) \uparrow a_1\}\end{aligned}$$

A type together with a composition structure is called fibrant.

The definition of $\texttt{Comp}$ is strictly analogous to the rule for composition given in section 1. Note that in [OP16], the composition comes in two directions, that is, for a type to be fibrant one needs an element of $\texttt{Comp}\ A\ 0$ and $\texttt{Comp}\ A\ 1$. Here, as we have a de Morgan structure on the interval which allows us to reverse the paths, we only need one or the other.

We can give a composition structure to basic type constructors and basic types. To do that, we proceed recursively. If we have $\texttt{comp}_A : \texttt{Comp}\ A$ and $\texttt{comp}_B : \texttt{Comp}\ B$, then it is possible to define elements of types $\texttt{Comp}\ (\Sigma_B A)$ or $\texttt{Comp}\ (A + B)$, etc. The details of the construction can be found in [CCHM15] as well as in [OP16]. In the latter, the approach they choose is to use an equivalent notion of *filling structure*, which is motivated by the Kan filling operation. The recursive definition of composition (or filling) structures relies heavily on the axioms.

5 Conclusion

5.1 Summary

The need for a cubical type theory, rather than, say, a simplicial one, comes from the lack of computational content the latter provides. In cubical settings, we can compute the univalence axiom. However, the drawback comes from the syntax of the cubical type theory which tends to be technical. From the semantic point of view, there is not one *good* category of cubes, and as one can see in [Mö21], there are a lot of variations, each of them with specific features.

Our main goal was here to focus on one specific cube category (the one given in [CCHM15]) and to prove that it is indeed a model of this specific cubical type theory. To prove this we relied on the work of Orton and Pitts in [OP16] who showed how the data of one topos $\mathcal{E}$, one interval object $\mathbb{I} \in \mathcal{E}$ and onebottom=2cm class of cofibrations $\texttt{Cof} \rightarrowtail \Omega$ satisfying the list of nine axioms presented in section 4.2 suffice to model cubical type theory, providing a nice and clean axiomatic framework.

However, in this new setting, new challenges appear. Indeed, we need to answer the question of what it means for an axiom to be true in a topos. Classical work from McLane and Moerdijk in [SML94] and from Lambek and Scott in [LS86] shows how it is possible to compute the internal logic of a topos using the notion of Kripke-Joyal semantics and forcing. Still, this approach does not take into account the modern needs of type theory, like proof relevance. Thus, in [AGH21], Awodey, Gambino, and Hazratpour propose a new version of Kripke-Joyal semantics. Their setting is a particular case of a topos, namely a presheaf category. There, they interpret Martin-Löf type theory thanks to the notion of category with families and extend the notion of forcing to this more general case. Thus, our work in section 2 was to give a quick presentation of the internal logic of a presheaf topos, and how one can reach the main theorem of Kripke-Joyal forcing. Then in section 3, we showed how this result interacts with the type theory of a presheaf category (which is a topos), and how the two can be combined, using the results of [AGH21]

Finally, in section 4, we could apply our results to the case of the presheaf category over the category of cubes of [CCHM15], and show that Kripke-Joyal forcing helps us to check the validity of the set of nine axioms. Thus, this work gives a new perspective on [OP16] by using the recent tools of [AGH21] to check the internal validity of the axioms. This provides a more systematic approach that could be generalized to various presheaf toposes.

5.2 Future work

The author would like to complete this work in two main directions. The first one comes from a discussion with Nicola Gambino, the author would like to extend the tools developed in [AGH21] to W-types, which are a generalization of inductive types (natural numbers, lists, trees, etc.). This work would add a line to Corollary 4.18 of [AGH21], and would give greater power to Kripke-Joyal forcing. Such work would start with a precise analysis of the semantics of W-types in the context of a Hofmann-Streicher construction on a Grothendieck universe of small maps. Preliminary investigations indicate that we will have to deal with polynomial functors, and define on them operations similar to those we can find in [Awo18]. Finally, we will have to find what kind of conditions allow us to force terms of a W-type.

The second and current direction would be the formalization in Lean a version of Theorem 2.13. Such a project is kindly hosted at CIIRC in Prague, in the laboratory of Josef Urban, where the author is developing the tools to prove such a statement, and thus to have a way to compute the internal logic of any topos.

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