# **Cubical Type Theory Inside a Presheaf Topos**

### Mathieu Chanavat

September 27, 2022

### Context

### Cubical Type Theory

- The first models of Homotopy Type Theory were non constructive.
- Thus, the need for Cubical Type Theory, based on cubical sets.
- Uses an interval object  $\mathbb{I}$ , which is a synthetic counterpart of [0,1].
- Leads to a notion of path p: Path A a b is a map p:  $\mathbb{I} \to A$ .

### Context

#### **Cubical Type Theory**

- The first models of Homotopy Type Theory were non constructive.
- Thus, the need for Cubical Type Theory, based on cubical sets.
- Uses an interval object  $\mathbb{I}$ , which is a synthetic counterpart of [0,1].
- Leads to a notion of path p: Path A a b is a map p:  $\mathbb{I} \to A$ .

#### What we will do

- It is well known that presheaves category have an internal type theory.
- We will try to understand what is needed from such a presheaves category to be a model of Cubical Type Theory.
- For that, we will use topos theory.

## Contents I

- 1. Cubical type theory Dependent type theory Glueing and univalence
- 2. Logic inside a presheaf topos Kripke-Joyal semantics Kripke-Joyal forcing
- 3. Martin-Löf Type Theory inside a presheaf category Universes and small maps Forcing for type theory
- 4. A model of cubical type theory Cubical presheaves
- 5. Conclusion

# Cubical type theory

### Rules and interval

- 1. Standard Martin-Löf dependent type theory
- 2. An object  $\mathbb I$  which is the free de Morgan algebra on a fixed infinite set of names  $i,j,k,\ldots$
- 3. Grammar of  $\mathbb{I}$  is

$$r,s ::= 0 \mid 1 \mid i \mid \neg r \mid r \wedge s \mid r \vee s$$

4. Custom  $\lambda$ -abstraction for  $i : \mathbb{I}$ :

$$\langle i \rangle.t$$

### Rules and interval

- 1. Standard Martin-Löf dependent type theory
- 2. An object  $\mathbb{I}$  which is the free de Morgan algebra on a fixed infinite set of names  $i, j, k, \ldots$
- 3. Grammar of  $\mathbb{I}$  is

$$r,s ::= 0 \mid 1 \mid i \mid \neg r \mid r \wedge s \mid r \vee s$$

4. Custom  $\lambda$ -abstraction for  $i : \mathbb{I}$ :

$$\langle i \rangle.t$$

### Interval

### Insight on ${\mathbb I}$

- 1.  $\mathbb{I}$  is a synthetic equivalent for [0,1].
- 2. ∨ represents max
- 3. ∧ represents min
- 4.  $\neg$  represents  $1 \cdot$
- 5. We write (i0) and (i1) for (i/0) and (i/1).

### Jugdmental equality for ${\mathbb I}$

# Path types

#### Rules

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash Path \ A \ a \ b} \qquad \frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash a : A}{\Gamma \vdash \langle i \rangle \ a : Path \ A \ a (i0) \ a (i1)}$$

$$\frac{\Gamma \vdash p : Path \ A \ a \ b \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash p \ r : A} \qquad \frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash a : A \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash \langle i \rangle \ a) \ r = a(i/r) : A}$$

$$\frac{\Gamma, i : \mathbb{I} \vdash p \ i = q \ i : A}{\Gamma \vdash p = q : Path \ A \ p_0 \ p_1} \qquad \frac{\Gamma \vdash p : Path \ A \ p_0 \ p_1}{\Gamma \vdash p \ 0 = p_0 : A} \qquad \frac{\Gamma \vdash p : Path \ A \ p_0 \ p_1}{\Gamma \vdash p \ 1 = p_1 : A}$$

## Path types

#### Rules

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash \text{Path } A \text{ a } b} \qquad \frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash a : A}{\Gamma \vdash \langle i \rangle \text{ a : Path } A \text{ a} (i0) \text{ a} (i1)}$$

$$\frac{\Gamma \vdash p : \text{Path } A \text{ a } b \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash p \text{ r : } A} \qquad \frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash a : A \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash \langle i \rangle \text{ a}) \text{ } r = \text{a} (i/r) : A}$$

$$\frac{\Gamma, i : \mathbb{I} \vdash p \text{ } i = \text{q } i : A}{\Gamma \vdash p \text{ } e \text{ } p \text{ } e \text{ } p \text{ } e \text{ } e$$

#### Consequences

- **1**. Reflexivity: For a:A,  $1_a=\langle i\rangle$  a: Path A a a
- 2. Function extensionality, from  $\Gamma \vdash p : (x : A) \rightarrow \text{Path } B (f x) (g x)$  we have

$$\Gamma \vdash \langle i \rangle \ \lambda x : A. \ p \times i : Path ((x : A) \rightarrow B) \ f \ g$$

### Geometric intuition

#### In dimension *n*

*n* variables of dimension  $i_1, \ldots, i_n : \mathbb{I}$  in the context, correspond to an *n*-dimensional cube.

### Geometric intuition

#### In dimension *n*

*n* variables of dimension  $i_1, \ldots, i_n : \mathbb{I}$  in the context, correspond to an *n*-dimensional cube.

#### In dimension two

$$i: \mathbb{I}, j: \mathbb{I} \vdash A$$

$$A(i0)(j1) \xrightarrow{A(j1)} A(i1)(j1)$$

$$A(i0) \uparrow \qquad \qquad \uparrow A(i1)$$

$$A(i0)(j0) \xrightarrow{A(j0)} A(i1)(j0)$$

### Face lattice

### Definition (Face lattice)

We define  $\mathbb{F}$  to be the distributive lattice generated by the symbols (i=0) and (i=1) (for all dimension name i) with relation  $(i=0) \land (i=1) = 0_{\mathbb{F}}$ . The grammar is

$$arphi, \psi ::= 0_{\mathbb{F}} \mid 1_{\mathbb{F}} \mid (i=0) \mid (i=1) \mid arphi \wedge \psi \mid arphi \vee \psi$$

We have the rule

$$\frac{\Gamma \vdash \psi : \mathbb{F}}{\Gamma, \psi \vdash}$$

#### Contexts and congruence

For context  $\Gamma$ , we define recursively a congruence  $=_{\Gamma}$  on  $\mathbb{F}$ .

# Systems

#### Definition (System)

A system is a compatible union of sub-polyhedra. The grammar is:

$$t, u, A, B ::= \ldots \mid [\varphi_1 \ a_1 \ldots \varphi_n \ a_n]$$

#### Rules for systems

If  $\Gamma \vdash \varphi_1 \lor \cdots \lor \varphi_n =_{\Gamma} 1_{\mathbb{F}} : \mathbb{F}$ , then

$$\frac{\Gamma, \varphi_1 \vdash a_1 : A \qquad \dots \qquad \Gamma, \varphi_n \vdash a_n : A \qquad \Gamma, \varphi_i \land \varphi_j \vdash a_i = a_j : A \ (i \leq i, j \leq n)}{\Gamma \vdash [\varphi_1 \ a_1 \dots \varphi_n \ a_n] : A}$$

$$\frac{\Gamma, \varphi_1 \vdash J \qquad \dots \qquad \Gamma, \varphi_n \vdash J}{\Gamma \vdash J} \qquad \frac{\Gamma \vdash [\varphi_1 \ a_1, \dots, \varphi_n \ a_n] : A \qquad \Gamma \vdash \varphi_i = 1_{\mathbb{F}} : F}{\Gamma \vdash [\varphi_1 \ a_1, \dots, \varphi_n \ a_n] = a_i : A}$$

## Composition

#### Notation

We write  $\Gamma \vdash a : A[\varphi \mapsto u]$  for  $\Gamma \vdash a : A$  and  $\Gamma, \varphi \vdash a = u : A$ 

### Definition (Composition)

Composition is a way to transport an element along a variable of dimension. The grammar is:

$$t, u, A, B ::= \dots \mid \mathsf{comp}^i A [\varphi \mapsto u] a_0$$

with u a system. The rule is:

$$\frac{\Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash a_0 : A(i0)[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{comp}^i \ A \ [\varphi \mapsto u] \ a_0 : A(i1)[\varphi \mapsto u(i1)]}$$

# Transitivity

### **Transitivity**

Composition allows us to prove transitivity of paths. If  $\Gamma \vdash A$ ,  $\Gamma \vdash a$ , b, c : A, then

$$\Gamma \vdash p : \text{Path } A \text{ a } b \qquad \Gamma \vdash q : \text{Path } A \text{ b } c$$

$$comp^{j} A [(i = 0) \lor (i = 1) \mapsto [(i = 0) \land (i = 1) \land (g \mid i)] \land (g \mid i) : \text{Path } A \text{ a } c$$

$$\Gamma \vdash \langle i 
angle \hspace{0.1cm} ext{comp}^{j} \hspace{0.1cm} A \hspace{0.1cm} ext{[($i=0)$} \lor (i=1) \mapsto ext{[($i=0)$} \hspace{0.1cm} a, (i=1) \hspace{0.1cm} (q \hspace{0.1cm} j)] \hspace{0.1cm} (p \hspace{0.1cm} i) : ext{Path $A$} \hspace{0.1cm} a \hspace{0.1cm} c \hspace{$$

# Contractible types and equivalences

#### Definition (Contractible types)

A type A is contractible if

$$ext{isContr } A \stackrel{\Delta}{=} (x:A) imes ((y:A) 
ightarrow ext{Path } A imes y)$$

is inhabited.

## Contractible types and equivalences

#### Definition (Contractible types)

A type A is contractible if

$$\mathtt{isContr}\ A \stackrel{\Delta}{=} (x : A) \times ((y : A) \to \mathtt{Path}\ A \times y)$$

is inhabited.

#### Definition (Equivalence)

Given two types T, A and f:  $T \rightarrow A$ , we define

isEquiv 
$$T \land A \not \stackrel{\triangle}{=} (y : A) \rightarrow \text{isContr}((x : T) \times \text{Path } A \ y \ (f \ x))$$

We define the type

Equiv 
$$T A \stackrel{\Delta}{=} (f : T \rightarrow A) \times \text{isEquiv } T A f$$

# Glueing

### Definition (Glueing)

The glueing operation is the analogous of composition, but for types. The formation rule is:

$$\frac{\Gamma \vdash A \qquad \Gamma, \varphi \vdash T \qquad \Gamma, \varphi \vdash f : \text{Equiv } T \; A}{\Gamma \vdash \text{Glue} \; [\varphi \mapsto (T, f)] \; A}$$

# Glueing

### Definition (Glueing)

The glueing operation is the analogous of composition, but for types. The formation rule is:

$$\frac{\Gamma \vdash A \qquad \Gamma, \varphi \vdash T \qquad \Gamma, \varphi \vdash f : \text{Equiv } T \; A}{\Gamma \vdash \text{Glue} \; [\varphi \mapsto (T, f)] \; A}$$

#### Intuition for glueing

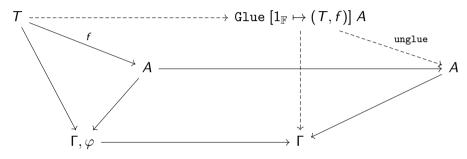
lf

- 1. Γ *⊢ A*
- 2.  $\Gamma, \varphi \vdash T$
- 3. A and T are equivalent on the region  $\varphi$

then we have the equality  $\Gamma, \varphi \vdash \texttt{Glue} [\varphi \mapsto (T, f)] A = T$ .

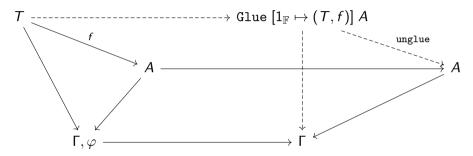
# Glueing and univalence

We can illustrate the type Glue  $[1_{\mathbb{F}} \mapsto (T, f)]$  A as:



## Glueing and univalence

We can illustrate the type Glue  $[1_{\mathbb{F}} \mapsto (T, f)]$  A as:



### Theorem (Univalence in Cubical Type Theory)

For any term

$$t:(A\ B:U) o exttt{Path}\ U\ A\ B o exttt{Equiv}\ A\ B$$

the map  $t \land B$ : Path  $U \land B \rightarrow \text{Equiv } A \land B$  is an equivalence.

# Logic inside a presheaf topos

## Topos

### Definition (A definition of a topos)

A topos is a category that

- has finite limits
- is cartesian closed
- has a subobject classifier  $(\Omega, \mathsf{true} : \mathbf{1} \rightarrowtail \Omega)$

## Topos

#### Definition (A definition of a topos)

A topos is a category that

- has finite limits
- is cartesian closed
- has a subobject classifier  $(\Omega, \text{true} : \mathbf{1} \rightarrow \Omega)$

### Definition (Subobject classifier)

For any mono  $s: S \rightarrowtail X$ , there exists a unique  $\chi_s: X \to \Omega$  such that the following is a pullback:

$$\begin{array}{ccc}
S & \longrightarrow & \mathbf{1} \\
\downarrow s & & \downarrow \text{true} \\
X & \xrightarrow{\chi_s} & \Omega
\end{array}$$

# Presheaf topos

#### Condition for the classifier

For any  $X \in \mathcal{E}$ , we have

$$\mathsf{Sub}(X) \simeq \mathcal{E}(X,\Omega)$$

## Presheaf topos

#### Condition for the classifier

For any  $X \in \mathcal{E}$ , we have

$$\mathsf{Sub}(X) \simeq \mathcal{E}(X,\Omega)$$

#### Presheaf topos

Let  $\mathcal C$  be a locally small category, then  $\mathcal E=[\mathcal C^{\operatorname{op}},\operatorname{Sets}]$  is a topos. The subobject classifier  $\Omega\in\mathcal E$  maps each  $c\in\mathcal C$  to the set of sieves on c. We define true :  $\mathbf 1\to\Omega$  by picking the maximal sieve.

## Kripke-Joyal semantics

#### Link between toposes and logic

- Each topos has its own internal logic.
- We can interpret the syntax thanks to the Kripke-Joyal semantics.
- We rely on the Heyting structure of the subobjects.

# Kripke-Joyal semantics

#### Link between toposes and logic

- Each topos has its own internal logic.
- We can interpret the syntax thanks to the Kripke-Joyal semantics.
- We rely on the Heyting structure of the subobjects.
- Then, we use the notion of Kripke-Joyal forcing to recursively unwind formulas,
- Thus transforming a formula into a concrete object of the topos.

# Heyting strucutre

### Definition (Heyting algebra)

A Heyting algebra is a bounded lattice with an operation  $\Rightarrow$  such that

$$(x \land y) \le z \iff x \land y \Rightarrow z$$

# Heyting strucutre

### Definition (Heyting algebra)

A Heyting algebra is a bounded lattice with an operation  $\Rightarrow$  such that

$$(x \wedge y) \leq z \iff x \wedge y \Rightarrow z$$

### Theorem (Heyting algebra of the subobjects)

For any  $X \in \mathcal{E}$ , Sub(X) is a Heyting algebra.

# Top and Bot

Т

We take  $\top := \text{true}$ . It is the classifying map of  $!: \mathbf{1} \to \mathbf{1}$ . We have the following pullback.



## Top and Bot

#### $\top$

We take  $\top := \text{true}$ . It is the classifying map of  $!: \mathbf{1} \to \mathbf{1}$ . We have the following pullback.





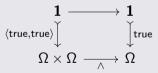
We define  $\perp$  with the following pullback.



## And

۱\

We have  $\wedge:\Omega\times\Omega\to\Omega$  with the following pullback.



## **I**mage

### Definition (Image)

A mono m is the image of an arrow f, if  $f = m \circ e$  for some epi e and if whenever f factors through a mono m' then m factor through m'. We write m = im(f).

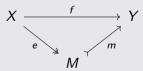
## Image

### Definition (Image)

A mono m is the image of an arrow f, if  $f = m \circ e$  for some epi e and if whenever f factors through a mono m' then m factor through m'. We write m = im(f).

### Theorem (Image in a topos)

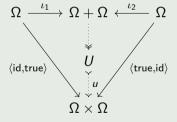
Let  $f: X \to Y$ . Then f has an image  $m: M \to Y$  and factors as in



with e epi.

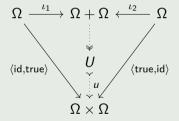
## Or

### Setup for the ∨



## Or

### Setup for the ∨



 $\vee$ 

We take  $\vee : \Omega \times \Omega \to \Omega$  to be the classifying arrow of  $u : U \to \Omega \times \Omega$  above.

# Quantifiers as pullbacks

### Adjonction to the pullback functor

We define

where  $\pi^*$  is the pullback functor along the second projection  $\pi: Y \times X \to X$  and

$$\exists_{Y} \dashv \pi^* \dashv \forall_{Y}$$

# Interpretation of formulas

### Interpretation of formulas

Suppose we have  $\sigma, \tau: X \to \Omega$ ,  $\theta: Y \times X \to \Omega$ , the we interpret

- $\sigma \wedge \tau$  by  $\wedge \circ \langle \sigma, \tau \rangle : X \to \Omega$
- $\sigma \lor \tau$  by  $\lor \circ \langle \sigma, \tau \rangle : X \to \Omega$
- $\sigma \Rightarrow \tau$  by  $\Rightarrow \circ \langle \sigma, \tau \rangle : X \to \Omega$
- $\neg \sigma$  by  $\Rightarrow \circ \langle \sigma, \bot \circ ! \rangle = \neg \circ \sigma : X \to \Omega$
- $\forall y : Y, \ \theta(y) \ \text{by} \ \forall y \theta : X \to \Omega$
- $\exists y : Y, \ \theta(y) \ \text{by} \ \exists_Y \theta : X \to \Omega$

## A very important notation

#### Notation

Every  $\sigma:X o\Omega$  is to be thought as a formula. Thus we write

$$\{x:X\mid \sigma(x)\}$$

for the subobject of X arising from the pullback of  $\sigma$  and true.

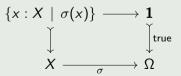
## A very important notation

#### Notation

Every  $\sigma: X \to \Omega$  is to be thought as a formula. Thus we write

$$\{x:X\mid \sigma(x)\}$$

for the subobject of X arising from the pullback of  $\sigma$  and true.



# Interpretation of equality

#### Interpretation of =

For all  $Y \in \mathcal{E}$ , let  $\Delta_Y : Y \to Y \times Y$  be the diagonal map and let  $\delta_Y$  be its classifying arrow. Let  $f, g : X \to Y$ . We interpret f = g by  $\delta_Y \circ \langle f, g \rangle$ . Thus we have the following pullback.

$$\begin{cases} x : X \mid f(x) = g(x) \end{cases} \longrightarrow \mathbf{1}$$

$$\downarrow \qquad \qquad \downarrow \text{true}$$

$$X \xrightarrow{f=g} \qquad \Omega$$

# Interpretation of equality

#### Interpretation of =

For all  $Y \in \mathcal{E}$ , let  $\Delta_Y : Y \to Y \times Y$  be the diagonal map and let  $\delta_Y$  be its classifying arrow. Let  $f, g : X \to Y$ . We interpret f = g by  $\delta_Y \circ \langle f, g \rangle$ . Thus we have the following pullback.

#### Theorem

The following diagram is an equalizer.

$$\{x: X \mid f(x) = g(x)\} \longrightarrow X \xrightarrow{f} Y$$

# Validity of a formula

### Definition (Validity)

Let  $\sigma: X \to \Omega$ . We say that  $\sigma$  is *valid* whenever  $\sigma$  factors through true :  $\mathbf{1} \to \Omega$ .



In that case, we write  $X \vdash \sigma$ . If  $\sigma$  is a closed formula, then we write  $\vdash \sigma$  and this amounts to say that  $\sigma = \text{true}$ .

# Characterization of validity

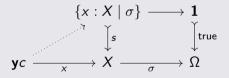
Let X be a type and  $\sigma: X \to \Omega$ . The following are equivalent.

- 1.  $X \vdash \sigma$
- 2. The subobject  $\{x: X \mid \sigma(x)\} \rightarrow X$  admits a section
- 3.  $\{x: X \mid \sigma(x)\} \simeq X$

# Kripke-Joyal forcing

### Definition (Forcing)

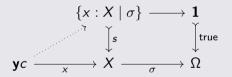
Let  $\sigma: X \to \Omega$  be a formula and  $x: yc \to X$ . We say that x forces  $\sigma$  at stage c, written  $c \Vdash \sigma(x)$ , if the following dotted arrow exists, making the left triangle commute.



# Kripke-Joyal forcing

### Definition (Forcing)

Let  $\sigma: X \to \Omega$  be a formula and  $x: yc \to X$ . We say that x forces  $\sigma$  at stage c, written  $c \Vdash \sigma(x)$ , if the following dotted arrow exists, making the left triangle commute.



#### Theorem

Let  $\sigma: X \to \Omega$ .  $X \vdash \sigma$  if and only if  $c \Vdash \sigma(x)$  for all  $x: \mathbf{y}c \to X$ .

## More theorems

### Theorem (Monotonicity)

Let  $\sigma: X \to \Omega$  be a formula and  $x: \mathbf{y}c \to X$  such that  $c \Vdash \sigma(x)$ , then for all  $f: b \to c$ ,  $b \Vdash \sigma(xf)$ .

## More theorems

## Theorem (Monotonicity)

Let  $\sigma: X \to \Omega$  be a formula and  $x: \mathbf{y}c \to X$  such that  $c \Vdash \sigma(x)$ , then for all  $f: b \to c$ ,  $b \Vdash \sigma(xf)$ .

### Theorem (Forcing with a terminal object)

If  $\mathcal C$  has a terminal object  $t\in\mathcal C$ , then a closed formula  $\sigma:\mathbf 1\to\Omega$  is valid if and only if  $t\Vdash\sigma.$ 

### The main theorem

### Theorem (Conditions for forcing)

Let  $\sigma, \tau : X \to \Omega$ ,  $\theta : Y \times X \to \Omega$  and  $x : \mathbf{y}c \to X$ , then

- $c \Vdash \bot$  never
- c ⊩ ⊤ always
- $c \Vdash \sigma(x) \land \tau(x)$  if and only if  $c \Vdash \sigma(x)$  and  $c \Vdash \tau(x)$
- $c \Vdash \sigma(x) \lor \tau(x)$  if and only if  $c \Vdash \sigma(x)$  or  $c \Vdash \tau(x)$
- $c \Vdash \sigma(x) \Rightarrow \tau(x)$  if and only if for all  $f : d \to c$ ,  $d \Vdash \sigma(xf)$  implies  $d \Vdash \tau(xf)$
- $c \Vdash \neg \sigma(x)$  if and only if for all  $f : d \to c$ , we do not have  $d \Vdash \sigma(xf)$
- $c \Vdash \exists y : Y, \ \theta(y,x)$  if and only if  $c \Vdash \theta(y,x)$  for some  $y : \mathbf{y}c \to Y$
- $c \Vdash \forall y : Y, \ \theta(y,x)$  if and only if  $d \Vdash \theta(y,xf)$  for all  $f : d \to c$  and  $y : \mathbf{y}d \to Y$

### The main theorem

### Theorem (Conditions for forcing)

Let  $\sigma, \tau : X \to \Omega$ ,  $\theta : Y \times X \to \Omega$  and  $x : \mathbf{y}c \to X$ , then

- $c \Vdash \bot$  never
- c ⊩ ⊤ always
- $c \Vdash \sigma(x) \land \tau(x)$  if and only if  $c \Vdash \sigma(x)$  and  $c \Vdash \tau(x)$
- $c \Vdash \sigma(x) \lor \tau(x)$  if and only if  $c \Vdash \sigma(x)$  or  $c \Vdash \tau(x)$
- $c \Vdash \sigma(x) \Rightarrow \tau(x)$  if and only if for all  $f : d \to c$ ,  $d \Vdash \sigma(xf)$  implies  $d \Vdash \tau(xf)$
- $c \Vdash \neg \sigma(x)$  if and only if for all  $f : d \to c$ , we do not have  $d \Vdash \sigma(xf)$
- $c \Vdash \exists y : Y, \ \theta(y,x)$  if and only if  $c \Vdash \theta(y,x)$  for some  $y : \mathbf{y}c \to Y$
- $c \Vdash \forall y : Y, \ \theta(y,x)$  if and only if  $d \Vdash \theta(y,xf)$  for all  $f : d \to c$  and  $y : \mathbf{y}d \to Y$

We can recursively unwind the condition  $c \Vdash \sigma(x)$ 

# Forcing for equality

#### Theorem (Forcing for equality)

Let  $f, g: X \to Y$  and  $x: \mathbf{y}c \to X$ , then  $c \Vdash f(x) = g(x)$  if and only if  $f \circ x = g \circ x$  as maps of  $\mathcal{E}$ .

The proof uses the equalizer characterization of equality:

$$\{x: X \mid f(x) = g(x)\} \longrightarrow X \xrightarrow{f} Y$$

$$\downarrow x \downarrow \qquad \qquad \downarrow x \downarrow \qquad \qquad$$

Martin-Löf Type Theory inside a presheaf category

### Overview

We will follow [AGH21] to define small maps, and the small map classifier  $\pi$  given by a Hofmann-Streicher lifting of a Grothendieck universe.

- We fix  $\kappa$  a (strongly) inaccessible cardinal.
- A set *small* if it has cardinality less than  $\kappa$ .
- We write  $\mathsf{Sets}_\kappa$  for the full subcategory of  $\mathsf{Sets}$  consisting of small sets (which is a Grothendieck universe).
- We fix a small (in the above sense) category C.
- ullet We call  ${\mathcal E}$  the associated presheaf topos.

# Hofmann-Streicher lifting

### Definition (Small maps)

- 1. We say that a presheaf  $A \in \mathcal{E}$  is *small* if A(c) is a small set, for all  $c \in \mathcal{C}$
- 2. We say that  $p:A\to X$  in  $\mathcal E$  is a *small map* if, for every  $x:\mathbf yc\to X$ , the presheaf  $A_x$  obtained by the pullback

$$\begin{array}{ccc}
A_{x} & \longrightarrow & A \\
\downarrow & & \downarrow p \\
\mathbf{y}c & \longrightarrow & X
\end{array}$$

is small.

# Hofmann-Streicher lifting

#### Definition (Small maps)

- 1. We say that a presheaf  $A \in \mathcal{E}$  is *small* if A(c) is a small set, for all  $c \in \mathcal{C}$
- 2. We say that  $p:A\to X$  in  $\mathcal E$  is a *small map* if, for every  $x:\mathbf yc\to X$ , the presheaf  $A_x$  obtained by the pullback

$$\begin{array}{ccc}
A_{X} & \longrightarrow & A \\
\downarrow & & \downarrow p \\
\mathbf{y}c & \xrightarrow{X} & X
\end{array}$$

is small.

#### Class of small maps

We call  $\mathcal S$  the class of small maps in  $\mathcal E$ . In the same way that we can classify the monos  $\mathcal S \rightarrowtail \mathcal X$  with the map true :  $\mathbf 1 \to \Omega$ , we define  $\pi : \mathcal E \to \mathcal U$  that classifies the maps of  $\mathcal S$ .

### Classification

Given a small map  $p: A \rightarrow X$ , there exists a pullback diagram

$$\begin{array}{ccc}
A & \longrightarrow & E \\
\downarrow p & & \downarrow \pi \\
X & \xrightarrow{c_p} & U
\end{array}$$

We say that p is classified by  $c_p$ . Conversely, we introduce a canonical pullback  $p_A$  for each  $A: X \to U$ :

$$X.A \longrightarrow E$$

$$p_A \downarrow \qquad \qquad \downarrow \pi$$

$$X \longrightarrow D$$

The map  $p_A$  is called the *display map* of A.

# Small maps and subobject classifier

#### Theorem

For all  $X \in \mathcal{E}$ , we have

$$\mathcal{E}(X,U)\simeq\{u:(\int X)^{\mathsf{op}}\to\mathcal{S}\}$$

### Similarity with $\Omega$

We call  $sf(X) = \{u : (\int X)^{op} \to S\}$  the small families over X, then we have

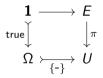
$$\mathsf{sf}(X) \simeq \mathcal{E}(X,U)$$

and with the subobject classifier, we have

$$\mathsf{Sub}(X) \simeq \mathcal{E}(X,\Omega)$$

# $\Omega$ is a type of proposition

There is an inclusion maps  $\{-\}:\Omega\to U$  which is a monomorphism and fits into a pullback of the form:



# $\Omega$ is a type of proposition

There is an inclusion maps  $\{-\}: \Omega \to U$  which is a monomorphism and fits into a pullback of the form:

$$egin{array}{ccc} \mathbf{1} & \longrightarrow & E \ \operatorname{true} & & \downarrow \pi \ \Omega & & \downarrow U \end{array}$$

Conversely, by considering the image of the universal map  $\operatorname{im}(\pi) \rightarrowtail U$ , and  $\{\operatorname{supp}(-)\}: U \to U$  is the propositional truncation.

$$\mathsf{m}(\pi) \longrightarrow \mathbf{1}$$
 $\downarrow \qquad \qquad \downarrow \mathsf{true}$ 
 $U \xrightarrow{\mathsf{supp}} \Omega$ 

# Category with families

### Definition (Category with families)

The *category with families* associated to any presheaf topos  $\mathcal{E}$ , is defined as follows:

- The contexts are the objects  $X \in \mathcal{E}$
- A type A in context X is a map  $A: X \to U$
- A term a:A in context X is a map  $a:X\to E$  such that the following diagram commutes.

$$\begin{array}{ccc}
X & \xrightarrow{a} & E \\
\parallel & & \downarrow^{\pi} \\
X & \xrightarrow{A} & U
\end{array}$$

Definitional equality on terms or type is defined via equality of maps in the topos. For instance, we have X ⊢ A = B if and only if A : X → U and B : X → U are the same maps in E. For terms, we write X ⊢ a = b : A.

## Some results

### Theorem (Closed types are presheaves)

Closed types  $A: \mathbf{1} \to U$  correspond, up to isomorphism, to small presheaves.

## Some results

#### Theorem (Closed types are presheaves)

Closed types  $A: \mathbf{1} \to U$  correspond, up to isomorphism, to small presheaves.

#### Theorem (Display maps and sections)

Let  $A : \Gamma \to U$  be a type in context  $\Gamma$ . The elements  $\Gamma \vdash a : A$  are in bijective correspondence with the sections of the display map  $p_A : \Gamma.A \to \Gamma$ .

# Type theory of a category with family

#### Substitution

If  $\Delta, \Gamma$  are two contexts, and  $A : \Gamma \to U$  a type in context  $\Gamma$ , then we define the substitution  $A[t] = A \circ t$ , which allows us to have the following substitution rule valid in our type theory.

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A \vdash b(x) : B(x)}{\Gamma \vdash b(a) : B(a)}$$

# Type theory of a category with family

#### Substitution

If  $\Delta, \Gamma$  are two contexts, and  $A : \Gamma \to U$  a type in context  $\Gamma$ , then we define the substitution  $A[t] = A \circ t$ , which allows us to have the following substitution rule valid in our type theory.

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A \vdash b(x) : B(x)}{\Gamma \vdash b(a) : B(a)}$$

#### Internal type theory

It is possible to internalize every basic construction of dependent type theory, and their formation, introduction, elimination, computation, and expansion rules [Awo18].

# Extending the forcing

#### Definition

Let  $A: X \to U$  be a type in context X, and  $x: \mathbf{y}c \to X$ . For  $a: \mathbf{y}c \to E$ , We say c forces a: A(x) written  $c \Vdash a: A(x)$  if the following diagram commutes.

$$yc \xrightarrow{a} E$$

$$x \downarrow \qquad \qquad \downarrow^{\pi}$$

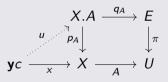
$$X \xrightarrow{A} U$$

Like in the standard forcing,  $c \Vdash a : A(x)$  is to say  $yc \vdash a : A(x)$ :

# Theorem for the forcing

#### Theorem

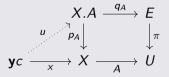
Let  $A: X \to U$  be a type in context X, and  $x: \mathbf{y}c \to X$ . An element  $a: \mathbf{y}c \to E$  is the same thing as the dotted arrow in the following diagram:



# Theorem for the forcing

#### Theorem

Let  $A: X \to U$  be a type in context X, and  $x: \mathbf{y}c \to X$ . An element  $a: \mathbf{y}c \to E$  is the same thing as the dotted arrow in the following diagram:



#### Theorem

The data of a:  $X \to E$  such that  $X \vdash a$ : A is the same as families of elements  $a_x : \mathbf{y}c \to E$  such that  $c \Vdash a_x : A(x)$ , and are uniform in the sense that  $c \Vdash a_x = a(x) : A(x)$ .

### Main theorem

### Theorem (Condition for forcing)

- 1.  $c \Vdash a : 0$  never
- 2.  $c \Vdash a : 1$  for a unique  $a = \star : \mathbf{y}c \rightarrow E$
- 3.  $c \Vdash t : (A \times B)(x)$  if and only if  $c \Vdash \pi_1(t) : A(x)$  and  $c \Vdash \pi_2(t) : B(x)$
- 4.  $c \Vdash t : (A + B)(x)$  if and only if  $c \Vdash a : A(x)$  with t = inl(a) or  $c \Vdash b : B(x)$  with t = inr(a)
- 5. We also have conditions for dependent sums and products.

### Main theorem

### Theorem (Condition for forcing)

- 1.  $c \Vdash a : 0$  never
- 2.  $c \Vdash a : 1$  for a unique  $a = \star : \mathbf{v}c \rightarrow E$
- 3.  $c \Vdash t : (A \times B)(x)$  if and only if  $c \Vdash \pi_1(t) : A(x)$  and  $c \Vdash \pi_2(t) : B(x)$
- 4.  $c \Vdash t : (A + B)(x)$  if and only if  $c \Vdash a : A(x)$  with t = inl(a) or  $c \Vdash b : B(x)$  with t = inr(a)
- 5. We also have conditions for dependent sums and products.

### Theorem (Relation the old forcing)

Let  $\sigma: X \to \Omega$  be a proposition and  $x: \mathbf{y}c \to X$ . Then the following are equivalent.

- (i)  $c \Vdash_{std} \sigma(x)$
- (ii)  $c \Vdash s : \sigma(x)$  for a (necessarily unique)  $s : \mathbf{y}c \to E$

# A model of cubical type theory

# Cubical presheaves

- We introduce the category  $\square$ . It will be the base category of a presheaf topos whose internal type theory will model cubical type theory.
- The category □ is the one from [CCHM15].
- ullet Its objects are free de Morgan algebras, whose structure will be helpful to obtain an object  $\mathbb{I}$ .
- Then, we introduce the notion of cofibration, whose behavior is important to internalize Kan filling and glueing [OP16].

# The box category

For  $n \ge 0$ , we denote by  $I_n$  the free de Morgan algebra on n generators.

#### Definition $(\Box)$

We call  $\square$  the category having as objects cardinal numbers  $[n] \ge 0$  and as morphisms in

 $\square([n],[m])$  the de Morgan homomorphisms  $f:I_m\to I_n$ .

# The box category

For  $n \ge 0$ , we denote by  $I_n$  the free de Morgan algebra on n generators.

#### Definition $(\Box)$

We call  $\square$  the category having as objects cardinal numbers  $[n] \ge 0$  and as morphisms in

- $\square([n],[m])$  the de Morgan homomorphisms  $f:I_m\to I_n$ .
  - 1. We specialize the previous sections to the case where  $\mathcal{C} = \square$ , so  $\mathcal{E}$  is the presheaf topos  $[\square^{op}, \mathsf{Sets}]$ .
  - 2.  $X_n \stackrel{\triangle}{=} X([n])$ .
  - 3.  $\mathbb{I} \stackrel{\Delta}{=} \mathbf{y}[1]$ .
  - 4.  $\mathbb{I}_n \stackrel{\Delta}{=} \square([n],[1])$  has a de Morgan structure defined pointwise.
  - 5. We call  $\sqcap_n, \sqcup_n, 0_n, 1_n$  the product, sum, zero, and one of this de Morgan algebra (that we can collect into natural transformations)

# Results about 🗌

### Theorem (Behavior of I)

- (i)  $0 \neq 1$
- (ii)  $\forall x : \mathbf{y}[n] \to \mathbb{I}$ , we have  $0 \sqcap x = 0 = x \sqcap 0$  and  $1 \sqcap x = x = x \sqcap 1$
- (iii)  $\forall x : \mathbf{y}[n] \to \mathbb{I}$ , we have  $0 \sqcup x = x = x \sqcup 0$  and  $1 \sqcup x = 1 = x \sqcup 1$

# Results about $\square$

### Theorem (Behavior of I)

- (i)  $0 \neq 1$
- (ii)  $\forall x : \mathbf{y}[n] \to \mathbb{I}$ , we have  $0 \sqcap x = 0 = x \sqcap 0$  and  $1 \sqcap x = x = x \sqcap 1$
- (iii)  $\forall x : \mathbf{y}[n] \to \mathbb{I}$ , we have  $0 \sqcup x = x = x \sqcup 0$  and  $1 \sqcup x = 1 = x \sqcup 1$

#### Theorem

 $\square$  has finite products.

# Results about

### Theorem (Behavior of I)

- (i)  $0 \neq 1$
- (ii)  $\forall x : \mathbf{y}[n] \to \mathbb{I}$ , we have  $0 \cap x = 0 = x \cap 0$  and  $1 \cap x = x = x \cap 1$
- (iii)  $\forall x : \mathbf{y}[n] \to \mathbb{I}$ , we have  $0 \sqcup x = x = x \sqcup 0$  and  $1 \sqcup x = 1 = x \sqcup 1$

#### Theorem

 $\square$  has finite products.

#### Theorem

For all  $[n] \in \square$ ,  $\mathbb{I}_n$  has decidable equality.

## Cofibrations

#### Idea

Cofibrations are a way to internalize composition and filling introduced in [CCHM15]. We assume a map cof :  $\Omega \to \Omega$  and we consider the associated subobject

$$\mathtt{Cof} = \{\varphi : \Omega \ | \ \mathsf{cof} \ \varphi\}$$

### Cofibrations

#### Idea

Cofibrations are a way to internalize composition and filling introduced in [CCHM15]. We assume a map cof :  $\Omega \to \Omega$  and we consider the associated subobject

$$\mathtt{Cof} = \{\varphi : \Omega \ | \ \mathsf{cof} \ \varphi\}$$

#### Definition (Cofibration)

A *cofibration* is a monomorphism whose classifying arrow factors trough  $\mathtt{Cof} \rightarrowtail \Omega$ .

## Cofibrations

#### Idea

Cofibrations are a way to internalize composition and filling introduced in [CCHM15]. We assume a map  $cof: \Omega \to \Omega$  and we consider the associated subobject

$$Cof = \{ \varphi : \Omega \mid cof \varphi \}$$

#### Definition (Cofibration)

A *cofibration* is a monomorphism whose classifying arrow factors trough  $Cof \rightarrow \Omega$ .

#### Theorem

Let  $\varphi: X \to \Omega$  be a proposition. For every  $x: \mathbf{y}c \to X$ , the following are equivalent.

- 1.  $c \Vdash \operatorname{cof} \varphi(x)$
- 2.  $\varphi \circ x : \mathbf{y}c \to \Omega$  factor through  $Cof \to \Omega$

## Axiom to model CTT

Consider the following nine axioms:

```
\mathbf{ax_1} \ \forall \varphi : \mathbb{I} \to \Omega, \ (\forall i : \mathbb{I}, \ \varphi \ i \lor \neg \varphi \ i) \Rightarrow (\forall i : \mathbb{I}, \ \varphi \ i) \lor (\forall i : \mathbb{I}, \ \neg \varphi \ i)
ax_2 \neg (0 = 1)
\mathbf{ax_3} \ \forall i : \mathbb{I}, \ 0 \cap i = 0 = i \cap 0 \ \land \ 1 \cap i = i = i \cap 1
\mathbf{a}\mathbf{x}_{\mathbf{A}} \ \forall i : \mathbb{I}, \ 0 \sqcup i = i = i \sqcup 0 \ \land \ 1 \sqcup i = 1 = i \sqcup 1
\mathsf{ax}_{5} \ \forall i : \mathbb{I}, \ \mathsf{cof}(i=0) \land \mathsf{cof}(i=1)
ax<sub>6</sub> \forall \varphi \ \psi : \Omega, \operatorname{cof} \varphi \Rightarrow \operatorname{cof} \psi \Rightarrow \operatorname{cof} (\varphi \lor \psi)
\mathsf{ax}_7 \ \forall \varphi \ \psi : \Omega. \ \mathsf{cof} \ \varphi \Rightarrow (\varphi \Rightarrow \mathsf{cof} \ \psi) \Rightarrow \mathsf{cof}(\varphi \land \psi)
\mathbf{ax_R} \ \forall \varphi : \mathbb{I} \to \Omega, \ (\forall i : \mathbb{I}, \ \mathsf{cof} \ \varphi \ i) \Rightarrow \mathsf{cof}(\forall i : \mathbb{I}, \ \varphi \ i)
\forall u : [\varphi], A u = B' \land s u = s' \}
```

## Axioms to model CTT

#### Theorem ([OP16])

The category with families of a presheaf topos is a model of cubical type theory if its internal logic satisfies the nine axioms above.

# The axioms, detailed

```
\mathsf{ax_1} \ \forall \varphi : \mathbb{I} \to \Omega, \ (\forall i : \mathbb{I}, \ \varphi \ i \lor \neg \varphi \ i) \Rightarrow (\forall i : \mathbb{I}, \ \varphi \ i) \lor (\forall i : \mathbb{I}, \ \neg \varphi \ i)
ax_2 \neg (0 = 1)
\mathbf{ax_3} \ \forall i : \mathbb{I}, \ 0 \cap i = 0 = i \cap 0 \ \land \ 1 \cap i = i = i \cap 1
\mathbf{ax_4} \ \forall i : \mathbb{I}, \ \mathbf{0} \sqcup i = i = i \sqcup \mathbf{0} \ \land \ \mathbf{1} \sqcup i = 1 = i \sqcup \mathbf{1}
```

# The axioms, detailed

```
axs \forall i : \mathbb{I}, cof(i = 0) \land cof(i = 1)
ax<sub>6</sub> \forall \varphi \ \psi : \Omega, \operatorname{cof} \varphi \Rightarrow \operatorname{cof} \psi \Rightarrow \operatorname{cof} (\varphi \lor \psi)
\mathsf{ax_7} \ \forall \varphi \ \psi : \Omega, \ \mathsf{cof} \ \varphi \Rightarrow (\varphi \Rightarrow \mathsf{cof} \ \psi) \Rightarrow \mathsf{cof}(\varphi \land \psi)
\mathsf{ax}_{\mathsf{R}} \ \forall \varphi : \mathbb{I} \to \Omega. \ (\forall i : \mathbb{I}, \ \mathsf{cof} \ \varphi \ i) \Rightarrow \mathsf{cof}(\forall i : \mathbb{I}, \ \varphi \ i)
```

# The axioms, detailed

```
\forall u : [\varphi], A u = B' \land s u = s' \}
```

# Kripke-Joyal forcing to prove the axioms

- 1. To prove each of these axioms, we can use Kripke-Joyal forcing and unravel each formula.
- 2. Since  $\square$  has a terminal object [0], it suffices to prove that each axiom is forced at stage [0]. That is, for  $\mathbf{k} = 1, \dots, 9$ , we have

$$\vdash ax_{k} \iff [0] \Vdash ax_{k}$$

## Axioms for the interval

## Theorem $(ax_2)$

$$[0] \Vdash \neg (0=1)$$

It suffices to show that for all [n], we do not have  $[n] \Vdash 0 = 1$ . Assume  $[n] \Vdash 0 = 1$ , then we would have  $0 = 1 : \mathbf{y}[n] \to \mathbb{I}$ , which is false as  $0_n \neq 1_n$ .

# Axioms for the interval

### Theorem $(ax_2)$

$$[0] \Vdash \neg (0=1)$$

It suffices to show that for all [n], we do not have  $[n] \Vdash 0 = 1$ . Assume  $[n] \Vdash 0 = 1$ , then we would have  $0 = 1 : \mathbf{y}[n] \to \mathbb{I}$ , which is false as  $0_n \neq 1_n$ .

### Theorem $(ax_3)$

$$[0] \Vdash (\forall i : \mathbb{I}), \ 0 \sqcap i = 0 = i \sqcap 0 \ \land \ 1 \sqcap i = i = i \sqcap 1$$

Call  $\sigma(i) = 0 \sqcap i = 0 = i \sqcap 0 \land 1 \sqcap i = i = i \sqcap 1$ . We want to show that  $[n] \Vdash \sigma(i)$  for all  $f:[n] \to [0]$  and  $i:\mathbf{y}[n] \to \mathbb{I}$ . Such a map f is unique, thus we need to show that  $[n] \Vdash \sigma(i)$ , for all  $i:\mathbf{y}[n] \to \mathbb{I}$ . Unwinding of the  $\land$  connective, it suffices to prove each equality independently. For instance with  $[n] \Vdash 0 \sqcap i = 0$ , let  $i:\mathbf{y}[n] \to \mathbb{I}$  be a generalized element.It suffices to show that  $0 \sqcap i = 0$  as maps of  $\mathcal{E}$ , and this is true.

## The interval is connected

#### Theorem

Let  $\varphi, \psi : \mathbb{I} \to \Omega$  be two formulas. Then the following are equivalent.

- (i)  $\mathbb{I} \vdash \psi \lor \varphi$
- (ii)  $\mathbb{I} \vdash \psi$  or  $\mathbb{I} \vdash \varphi$

### The interval is connected

#### Theorem

Let  $\varphi, \psi : \mathbb{I} \to \Omega$  be two formulas. Then the following are equivalent.

- (i)  $\mathbb{I} \vdash \psi \lor \varphi$
- (ii)  $\mathbb{I} \vdash \psi$  or  $\mathbb{I} \vdash \varphi$

### Theorem $(ax_1)$

 $\vdash \forall \varphi : \mathbb{I} \to \Omega, \ (\forall i : \mathbb{I}, \ \varphi \ i \lor \neg \varphi \ i) \Rightarrow (\forall i : \mathbb{I}, \ \varphi \ i) \lor (\forall i : \mathbb{I}, \ \neg \varphi \ i)$ 

## Axioms for cofibrations

For the cofibration, we prove alternative characterization of the axioms.

#### Theorem (Characterization for cofibrations)

If the class  $Cof \in \mathcal{E}$  of cofibrant propositions is such that

- $\{i : \mathbb{I} \mid i = 0\} \rightarrow \mathbb{I}$  is a cofibration
- $\{i : \mathbb{I} \mid i = 1\} \rightarrow \mathbb{I}$  is a cofibration
- If  $f: X \rightarrowtail Y$  and  $g: Y \rightarrowtail Z$  are cofibrations, then  $g \circ f$  is a cofibration
- If  $f:X\rightarrowtail Y$  is a cofibration, then  $f^\mathbb{I}:X^\mathbb{I}\rightarrowtail Y^\mathbb{I}$  is a cofibration

then 
$$\vdash ax_k$$
, for  $k = 5, 7, 8$ 

### The axiom of strictification

#### Definition (Decidable sieves)

We define  $\Omega_{\text{dec}} \rightarrowtail \Omega$  to be the sub-presheaf consisting of decidable sieves, that is

$$\Omega_{ extsf{dec}}(c) = \{S \in \Omega(c) \mid S ext{ is a decidable subset of } \mathsf{Obj}(\mathcal{C}/c)\}$$

## The axiom of strictification

#### Definition (Decidable sieves)

We define  $\Omega_{\text{dec}} \rightarrowtail \Omega$  to be the sub-presheaf consisting of decidable sieves, that is

$$\Omega_{ exttt{dec}}(c) = \{S \in \Omega(c) \mid S ext{ is a decidable subset of } \mathsf{Obj}(\mathcal{C}/c)\}$$

#### Theorem

**ax9** is satisfied in  $\mathcal E$  if  $\mathsf{Cof} \rightarrowtail \Omega$  is contained in  $\Omega_{\mathtt{dec}} \rightarrowtail \Omega$ .

### Main theorem

#### Theorem (Model of HoTT)

If  $\mathcal{E} = [\Box^{op}, \mathsf{Sets}]$  with  $\mathbb{I} = \mathbf{y}[1]$  and  $\mathsf{Cof} = \Omega_{\mathsf{dec}}$ , then  $\vdash \mathsf{ax_k}$  for  $\mathbf{k} = \mathbf{1}, \dots, \mathbf{9}$ , thus its internal type theory is a model of cubical type theory with univalence.

### Main theorem

#### Theorem (Model of HoTT)

If  $\mathcal{E} = [\Box^{op}, \mathsf{Sets}]$  with  $\mathbb{I} = \mathbf{y}[1]$  and  $\mathsf{Cof} = \Omega_{\mathsf{dec}}$ , then  $\vdash \mathsf{ax_k}$  for  $\mathbf{k} = \mathbf{1}, \dots, \mathbf{9}$ , thus its internal type theory is a model of cubical type theory with univalence.

We did not use the same cofibrations as in [CCHM15]. They use the face lattice  $\mathbb{F}$ , seen as an object of  $\mathcal{E}$ , and define the mono  $m: \mathbb{F} \rightarrowtail \Omega$  by sending  $x \in \mathbb{F}([n])$  to the sieve

$$m_{[n]}(x) = \{\cdot \xrightarrow{f} [n] \mid \mathbb{F}(f)(x) = 1\}$$

and they take  $Cof \rightarrow \Omega$  to be the subobject given by this monomorphism m.

# Conclusion

# Summary

- Lack of computational content in e.g. simplicial models.
- In cubical settings, we can compute the univalence axiom, but the syntax of the cubical type theory tends to be technical.
- From the semantic point of view, there is not one good category of cubes [Mö21].

# Summary

- Lack of computational content in e.g. simplicial models.
- In cubical settings, we can compute the univalence axiom, but the syntax of the cubical type theory tends to be technical.
- From the semantic point of view, there is not one good category of cubes [Mö21].
- We focused on the cubes of [CCHM15] and proved that it is indeed a model of this specific cubical type theory.
- We did that thanks to [OP16], with the nine axioms.
- But we needed to answer the question of what it means for an axiom to be true in a topos.
- Thus Kripke-Joyal forcing.
- And Kripke-Joyal forcing in the context of categories with families, as in [AGH21].
- This provides a more systematic approach that could be generalized to various presheaf toposes.

# References I

- S. Awodey, N. Gambino, and S. Hazratpour, *Kripke-joyal forcing for type theory and uniform fibrations*, https://arxiv.org/abs/2110.14576, 2021.
- Steve Awodey, *Natural models of homotopy type theory*, Mathematical Structures in Computer Science **28** (2018), no. 2, 241–286.
- Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg, *Cubical Type Theory:* a constructive interpretation of the univalence axiom, 21st International Conference on Types for Proofs and Programs (Tallinn, Estonia), 21st International Conference on Types for Proofs and Programs, no. 69, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, May 2015, p. 262.
- Anders Mörtberg, Cubical methods in homotopy type theory and univalent foundations, Mathematical Structures in Computer Science (2021), 1–38.

## References II



lan Orton and Andrew M. Pitts, *Axioms for Modelling Cubical Type Theory in a Topos*, 25th EACSL Annual Conference on Computer Science Logic (CSL 2016) (Dagstuhl, Germany) (Jean-Marc Talbot and Laurent Regnier, eds.), Leibniz International Proceedings in Informatics (LIPIcs), vol. 62, Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2016, pp. 24:1–24:19.

Thanks!