Notes on sheaf cohomology in Grothendieck toposes

Clémence Chanavat

July 4, 2023

Contents

Sheaf cohomology is a big topic, and most short survey papers define sheaf cohomology on a topological space. This is a more-or-less self contained note that produces sheaf cohomology on an arbitrary Grothendieck topos. All our functors are properly enriched. An updated and corrected version will arrive soon.

1 Abelian categories

1.1 First definitions

Definition 1: An *additive category* is an **Ab** enriched category with finite coproducts. More precisely, it is a category $\mathcal{C}$ such that for all $c, d \in \mathcal{C}$, $\mathcal{C}(a, b)$ has a structure of abelian group, and the compositions:

$$\mathcal{C}(b, c) \times \mathcal{C}(a, b) \to \mathcal{C}(a, c)$$

are bilinear.

Remark 2: For all c, d in an additive category, we always have the zero morphism $0_{c,d} \in \mathcal{C}(c, d)$, which is the identity of the abelian group.

Lemma 3: In a additive category, for all $f : c \to d$, we have $0_d \circ f = 0_{c,d}$ and $f \circ 0_c = 0_{c,d}$.

Proof. (**Lemma 3**) $0 \circ f = (0 - 0) \circ f = 0 \circ f - 0 \circ f = 0$, and dually. □

Lemma 4: An additive category has a zero object.

Proof. (**Lemma 4**) We have by definition an initial object $\star$. We show that it is also terminal. First, id_0 has to be the identity of the only element in the group $\mathcal{C}(\star, \star)$. Consider the zero morphism of $\mathcal{C}(c, \star)$, and take any $f : c \to \star$. Then $f = \mathrm{id}_\star \circ f = 0 \circ f = 0$. Therefore, f is also the zero morphism of $\mathcal{C}(c, \star)$, which proves uniqueness, hence that $\star$ is terminal. □

Proposition 5: Let $\mathcal{C}$ be additive. Then any finite coproduct is also a finite product.

Proof. (**Proposition 5**) Take a binary coproduct $c \sqcup d$, and consider the following diagram:

$$\begin{tikzcd}
c \arrow[dr, "\iota_c"] \arrow[dd, "\mathrm{id}"'] & & d \arrow[dl, "\iota_d"'] \arrow[dd, "\mathrm{id}"] \\
& c \sqcup d \arrow[dl, "(\mathrm{id}_c{,}0)"] \arrow[dr, "(0{,}\mathrm{id}_d)"'] & \\
c & & d
\end{tikzcd}$$

We will show that $(c \sqcup d, (\mathrm{id}_c, 0), (0, \mathrm{id}_d))$ is a product. Consider $f : a \to c$ and $g : a \to d$ and call $\varphi : \iota_c f + \iota_d g : a \to c \sqcup d$. We compute:

$$(\mathrm{id}_c, 0) \circ (\iota_c f + \iota_d g) = (\mathrm{id}_c, 0)\iota_c f + (\mathrm{id}_c, 0)\iota_d g = \mathrm{id}_c f + 0g = f,$$

and dually

$$(0, \mathrm{id}_d) \circ (\iota_c f + \iota_d g) = g.$$

Therefore, the map φ is a morphism of cone. Observe that

$$\iota_c(\mathrm{id}_c, 0) + \iota_d(0, \mathrm{id}_d) = \mathrm{id}_{c \sqcup d},$$

as $(\iota_c(\mathrm{id}_c, 0) + \iota_d(0, \mathrm{id}_d)) \circ \iota_c = \iota_c \mathrm{id}_c + \iota_d 0 = \iota_c$, and similarly for ι_d. Thus by universal property of the coproduct, it has to be the identity. Now suppose we have $\varphi' : a \to c \sqcup d$ commuting with the the projections. Then:

$$\varphi' = \mathrm{id}_{c \sqcup d} \circ \varphi' = (\iota_c(\mathrm{id}_c, 0) + \iota_d(0, \mathrm{id}_d)) \circ \varphi' = \iota_c(\mathrm{id}_c, 0)\varphi' + \iota_d(0, \mathrm{id}_d)\varphi' = \iota_c f + \iota_d g = \varphi.$$

Finally, we conclude the proof by induction. □

Definition 6: Let $\mathcal{C}$ be an additive category, and $f : c \to d$. The *kernel* of f, if it exists, is $\ker(f)$ the pullback of f along the unique morphism $0 \to d$. Dually, the *cokernel* $\mathrm{coker}(f)$ is the pushout of f along the unique morphism $c \to 0$.

Definition 7: An *abelian category* $\mathcal{C}$ is an additive category such that:

1. Every morphism admits a kernel and a cokernel.
2. Every mono is a kernel, and every epi is a cokernel.

Remark 8: The category **Ab** is indeed abelian. It is customary to write X/Y for the cokernel of a monomorphism $Y \hookrightarrow X$. This recovers the notion of quotient group in **Ab**.

Proposition 9: An additive category with all kernels and cokernels is abelian if and only if for all f, the canonical map

$$\mathrm{coker}(\ker(f)) \to \ker(\mathrm{coker}(f))$$

is an isomorphism.

Definition 10: If $f : c \to d$ is a morphism in an abelian category, we define its *image*, $\mathrm{im}(f)$, to be the kernel of its cokernel. According to Proposition 9, we could also have chosen the cokernel of its kernel.

Lemma 11: In an abelian category, $f : c \to d$ is mono if and only if $\ker(f) \simeq 0$, and it is epi if and only if $\mathrm{im}(f) \simeq d$.

Proof. (**Lemma 11**) Suppose f is mono, then as pullback preserves monos, we have $\iota : \ker(f) \hookrightarrow 0$, thus $\ker(f) \simeq 0$. Indeed, if we have an inclusion into a zero object, then it is an isomorphism. To prove this, consider a mono $\iota : a \hookrightarrow 0$. Next for any other a', we have the zero morphisms $0_{a',a} : a' \to a$. If we have two morphisms $h, k : a \to a'$, then $\iota h = \iota k$ as 0 is terminal, hence $h = k$, as ι is mono. Conversely, suppose $\ker(f) \simeq 0$, and take $h, k : c' \to c$ equalized by f. Then $f \circ (h - k) = 0_{c',d}$, thus via universal property of the pullback, $h - k$ factors trough $c' \to \ker(f) \simeq 0 \to c$, hence is the zero morphism, thus $h = k$.

Dually, f is epi if and only if $\mathrm{coker}(f) \simeq 0$. Thus suppose $\mathrm{coker}(f) \simeq 0$, then $\mathrm{im}(f) = \ker(\mathrm{coker}(f)) = d$, as the pullback of a map along itself is the identity. Conversely, suppose $\mathrm{im}(f) = d$, then we have the following pushout:

$$\begin{tikzcd} \ker(f) \arrow[r] \arrow[d] & 0 \arrow[d] \\ c \arrow[r, "f"'] & d \end{tikzcd}$$

which proves, via universal property, that f is epi. ☐

Remark 12: Therefore, in an abelian category, every morphism $f : c \to d$ factorises uniquely as an epi followed by a mono:

$$c \overset{p}{\twoheadrightarrow} \operatorname{im}(f) \overset{\iota}{\hookrightarrow} d.$$

Lemma 13: Suppose we have a mono $f : c \to d$, then $\ker(d \twoheadrightarrow d/c) \simeq c$.

Proof. (**Lemma 13**) We compute:

$$\ker(\operatorname{coker}(f)) \simeq \operatorname{coker}(\ker(f)) \simeq \operatorname{coker}(0_c) \simeq c.$$

where the first isomorphism uses Proposition 9, and the second Lemma 11. ☐

Definition 14: In an abelian category, a sequence

$$\cdots \to c_{n-1} \xrightarrow{f_n} c_n \xrightarrow{f_{n+1}} c_{n+1} \to \dots$$

is *long exact* if for all n, we have $\operatorname{im}(f_n) \simeq \ker(f_{n+1})$. A sequence is *short exact* if only three consecutive c_n are non zero, and we write simply:

$$0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0.$$

Proposition 15: In an abelian category, the following are equivalent.

1. $0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0$ is short exact.
2. f is mono, g is epi, and $\operatorname{im}(f) = \ker(g)$.

Proof. (**Proposition 15**) The condition $\operatorname{im}(f) = \ker(g)$ is equivalent to exactness at b, and Lemma 11 proves exactness at a and c. Indeed $\operatorname{im}(0 \to a) = 0$, which is $\ker(f)$ if and only if f is mono, and $\ker(c \to 0) = c$ which is $\operatorname{im}(g)$ if and only if g is epi. ☐

1.2 Exactness of functors

Definition 16: A functor $F : \mathcal{C} \to \mathcal{D}$ is *left exact* if it preserves finite limits, and *right exact* if it preserves finite colimits.

Lemma 17: A functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories is:

1. Left exact if and only if it preserves (finite) direct sums and kernels;
2. Right exact if and only if it preserves (finite) direct sums and cokernels.

Proof. (**Lemma 17**) It is a consequence of the following facts:

1. finite (co)limits can be computed from finite (co)products and (co)equalizers;
2. direct sums of abelian groups are coproducts which are products;
3. the (co)equalizer of f and g is the (co)kernel of $f - g$.

☐

Proposition 18: Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between abelian categories. For all exact sequences

$$0 \to a \to b \to c \to 0,$$

in $\mathcal{C}$, we have

- F is left exact if and only if for all short exact sequence $0 \to a \to b \to c \to 0$ in $\mathcal{C}$, the sequence $0 \to F(a) \to F(b) \to F(c)$ is exact in $\mathcal{D}$.

- F is right exact if and only if for all short exact sequence $0 \to a \to b \to c \to 0$ in $\mathcal{C}$, the sequence $F(a) \to F(b) \to F(c) \to 0$ is exact in $\mathcal{D}$.
- F is exact if and only if for all short exact sequence $0 \to a \to b \to c \to 0$ in $\mathcal{C}$, the sequence $0 \to F(a) \to F(b) \to F(c) \to 0$ is exact in $\mathcal{D}$.

Proof. (**Proposition 18**) Suppose F is left exact. Since

$$0 \to a \xrightarrow{\iota} b \overset{p}{\twoheadrightarrow} c \to 0,$$

is short exact, we have that $0 \to a$ is the kernel of ι, and ι is the kernel of p. Therefore, by Lemma 17

$$0 \to F(a) \xrightarrow{F(\iota)} F(b) \xrightarrow{F(p)} F(c),$$

is again exact. However, to conclude that the full sequence is exact, we would need that F preserves the cokernel $c \to 0$. Conversely, for the preservation of kernel, choose $f : c \to d$, and factor it as $c \xrightarrow{f'} \mathrm{im}(f) \to d$. Then $\ker(f) = \ker(f')$ and $0 \to \ker(f) \to c \to \mathrm{im}(f) \to 0$ is exact, hence so is $0 \to F(\ker(f)) \to F(c) \to F(\mathrm{im}(f))$, and thus $F(\ker(f))$ is the kernel of $F(f)$. For the coproduct, notice that $0 \to a \to a \sqcup b \to b \to 0$ is exact in any abelian category, thus we have two exact rows in:

$$\begin{tikzcd}
0 \arrow[r] & F(a) \arrow[r] \arrow[d, "\mathrm{id}"] & F(a) \sqcup F(b) \arrow[r] \arrow[d] & F(b) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & F(a) \arrow[r] & F(a \sqcup b) \arrow[r] & F(b) &
\end{tikzcd}$$

and the snake lemma shows that $F(a) \sqcup F(b) \to F(a \sqcup b)$ is an isomorphism. $\square$

1.3 Injective objects

Definition 19: Let $\mathcal{C}$ be a category, and J a collection of morphism of $\mathcal{C}$. A *J-injective* object I of $\mathcal{C}$ is an object with the right lifting property against J, that is for all $j : c \to d \in J$, and all morphism $c \to I$, we have a dashed arrow:

$$\begin{tikzcd}
c \arrow[r, "\forall"] \arrow[d, "\forall j \in J"'] & I \\
d \arrow[ur, dotted, "\exists"'] &
\end{tikzcd}$$

If J is the class of monomorphisms of $\mathcal{C}$, we simply say that I is injective. The dual construction is called *projective*.

Proposition 20: Let $\mathcal{C}$ be an abelian category. The following are equivalent:

1. I is injective.
2. The hom functor $\mathcal{C}(-, I) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$ is exact, that is, it preserves limits and colimits.
3. For all exact sequence $0 \to c \xrightarrow{f} d$, and for all $k : c \to I$, there exists $h : d \to I$ such that $hf = k$.

Then an object I is injective if and only if $\mathcal{C}(-, I)$ is exact, that is it preserves limits and colimits.

Proof. (**Proposition 20**) Suppose I is injective. First in general $\mathcal{C}(-, I)$ preserves limits, that is it is left exact. Consider in $\mathcal{C}$ an exact sequence $0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0$. We want to show that:

$$\mathcal{C}(c, I) \to \mathcal{C}(b, I) \to \mathcal{C}(a, I) \to 0$$

is exact. Take any map $h : a \to I$, then as f is mono, the lifting problem:

$$\begin{tikzcd}
a \arrow[r, "h"] \arrow[d, "f"'] & I \\
b \arrow[ur, dotted] &
\end{tikzcd}$$

admits a solution. Thus, $\mathcal{C}(b, I) \to \mathcal{C}(a, I)$ is surjective. The fact that $\mathcal{C}(c, I)$ is the kernel of $\mathcal{C}(b, I) \to \mathcal{C}(a, I)$ is the direct consequence of the universal property of the kernel, and does not rely on I being injective.

Next, suppose $\mathcal{C}(-, I)$ is exact, and take $0 \to c \xrightarrow{f} d$ an exact sequence, and $k : c \to I$ a morphism. Then $\mathcal{C}(d, I) \to \mathcal{C}(c, I) \to 0$ is exact, thus $\mathcal{C}(d, I) \to \mathcal{C}(c, I)$ is epi, and we can find a preimage $h : d \to I$ to k such that $hf = k$. Finally, notice that this condition is also precisely saying that I is injective, as exact sequence $0 \to c \to d$ are exactly monomorphisms $c \to d$. $\square$

Definition 21: We say that a category $\mathcal{C}$ has *enough injectives* if every object admits a monomorphism into an injective object.

Proposition 22: The category $\mathbf{Ab}$ has enough injectives. They are given by divisible groups, that is abelian groups G such that $nG = G$, for all strictly positive integer n.

Proof. (**Proposition 22**) This proof is adapted from `https://stacks.math.columbia.edu/tag/01D6`. Take I an injective group, and suppose it is not divisible by n, that is we have some $x \in I$ such that there is no y with $ny = x$. Now consider the map $f : \mathbf{Z} \to I$ sending $m \to mx$. Then an extension along the embedding $\mathbf{Z} \hookrightarrow \frac{1}{n}\mathbf{Z}$ would be such that $nf(\frac{1}{n}) = f(1) = x$, contradiction.

Conversely, consider $A \subseteq B$ two abelian groups, I a divisible group, and $\varphi : A \to I$ any morphism. We will apply Zorn's lemma. Thus, consider the set of all morphisms $\varphi' : A' \to I$ such that $A \subseteq A' \subseteq B$ and φ' restricts to φ on A. Define the partial order $(A', \varphi') \geq (A'', \varphi'')$ if and only if $A'' \subseteq A'$, and φ' restricts to φ'' on A''. If we have an ordered collection $\{(A_k, \varphi_k)\}_{k \in K}$ of such pairs, then taking $(\cup_k A_k, \tilde{\varphi})$ where $\tilde{\varphi}(a_k) = \varphi_k(a_k)$ is well defined by the restrictions, is a maximal element. Thus, Zorn's lemma applies, and we get a maximal pair (A', φ'). It suffices then to show that in that case $A' = B$. By contradiction, take $x \in B$ and $x \notin A'$.

Suppose first that there exists no $n \in \mathbb{N}$ such that $nx \in A'$. Then $A' \oplus \mathbf{Z} \simeq A' + \mathbf{Z}x \subseteq B$, and φ' can be extended to $A' + \mathbf{Z}x$ by sending x to the identity of I, which contradicts maximality of (A', φ'). Otherwise, we take n minimal (strictly positive) such that $nx \in A'$. As I is divisible, we have some $z \in I$ such that $nz = \varphi'(nx)$. Notice that if $mx \in A'$, then $m = kn$ for an integer k, otherwise $[m \mod n] \cdot x$ would be in A', contradicting minimality. Thus, for the contradiction, we extend φ' to $A' + \mathbf{Z}x$ by sending $a + mx$ to $\varphi'(a) + mz$. This definition makes sense, as if $mx \in A'$, then $\varphi(a + mx) = \varphi(a + knx) = \varphi'(a) + k\varphi'(nx) = \varphi'(a) + mz$.

Now that it is established that injective objects are divisible groups, we will embed any abelian group A into a direct product of $\mathbf{Q}/\mathbf{Z}$. We state without proving that a product of injective groups is injective, and that $\mathbf{Q}/\mathbf{Z}$ is itself injective. We create by universal property a monomorphism:

$$\beta : A \to \prod_{a \in A, a \neq 0} \mathbf{Q}/\mathbf{Z},$$

which on $a \in A, a \neq 0$ is given by β_a that we define as follows. First call (a) is the subgroup generated by a, and create the map $(a) \to \mathbf{Q}/\mathbf{Z}$ sending a to an arbitrary non-zero element if the order of a is infinite, and to, say, $1/n$ if a has order n. As $\mathbf{Q}/\mathbf{Z}$ is injective, we can solve the lifting problem:

$$\begin{tikzcd} (a) \arrow[r] \arrow[d] & \mathbf{Q}/\mathbf{Z} \\ A \arrow[ur, dotted] & \end{tikzcd}$$

with β_a, which is a monomorphism. $\square$

We now wish to prove that sheaves on a Grothendieck topos have enough injectives. We consider the neat proof from [Joh14].

Theorem 23 (Barr's theorem): If $\mathcal{E}$ is a Grothendieck topos, then there is a surjective geometric morphism

$$\mathcal{F} \to \mathcal{E}$$

where $\mathcal{F}$ satisfies the axiom of choice.

For a Grothendieck topos $\mathcal{E}$, call $\mathbf{Ab}(\mathcal{E})$ the sheaves with values in $\mathbf{Ab}$.

Lemma 24 ([Joh14, 8.12]): If $f : \mathcal{E}' \to \mathcal{E}$ is a geometric morphism, then

1. The direct image $f_* : \mathbf{Ab}(\mathcal{E}') \to \mathbf{Ab}(\mathcal{E})$ preserves injectives.

2. If f is a surjection, and $\mathbf{Ab}(\mathcal{E}')$ has enough injective, so has $\mathbf{Ab}(\mathcal{E})$.

Proof. (**Lemma 24**) Suppose we have an injective object $e \in \mathcal{E}'$, then consider any diagram

$$\begin{tikzcd} a \arrow[r] \arrow[d] & f_*(e) \\ b & \end{tikzcd}$$

with $a \to b$ mono, and we transpose:

$$\begin{tikzcd} f^*(a) \arrow[r] \arrow[d] & e \\ f^*(b) & \end{tikzcd}$$

Again, $f^*(a) \to f^*(b)$ is mono, as in a topos every mono is an equalizer, and in a geometric morphism, the inverse image f^* preserve finite limits. Therefore, we can find a lift in $\mathcal{E}'$, and transpose again.

Next, suppose f is surjective. Take an abelian sheaf $a \in \mathcal{E}$, and embed it with $f^*(a) \hookrightarrow e$ with e injective in $\mathcal{E}'$. Then we consider:

$$a \xrightarrow{\eta_a} f_* f^*(a) \to f_*(e).$$

This time, $f_* f^*(a) \to f_*(e)$ is mono because f_* is a right adjoint, and the unit of a geometric morphism is by definition mono whenever it is surjective. □

Finally, we state the main result:

Theorem 25: Let $\mathcal{E}$ be a Grothendieck topos. Then $\mathbf{Ab}(\mathcal{E})$ has enough injectives.

Proof. (**Theorem 25**) Using Theorem 23, it is enough to consider the case when the Grothendieck topos satisfies the axiom of choice, but then we can exactly mimic internally the proof Proposition 22, that rely on Zorn's lemma, that is the axiom of choice, and prove that an abelian sheaf is divisible if and only if is is a divisible abelian group at each point. Then, we sheafify the constant abelian presheaf $\mathbf{Q}/\mathbf{Z}$, and prove that any abelian sheaf can be embed into its direct products, pointwise computed in a sheaf topos. □

Remark 26: We can also compute more explicitly an embedding $X \hookrightarrow I$ into an injective abelian sheaf. See `https://stacks.math.columbia.edu/tag/01DL`, which in fact rely on the same result: we first prove that $\mathbf{Ab}$ has enough injectives, and we use them pointwise to prove that abelian *presheaves* have enough injective. Then after a transfinite recursion, we conclude that we can "sheafify" this result.

2 Sheaf cohomology

The outline of sheaf cohomology is that given an abelian sheaf $X \in \mathbf{Ab}(\mathcal{E})$, we can consider an injective resolution, that is an exact sequence:

$$0 \to X \to I^0 \to I^1 \to \ldots,$$

where each I^n is an injective object. Then, given a left exact functor $\Gamma : \mathbf{Ab}(\mathcal{E}) \to \mathbf{Ab}$, we have the sequence:

$$0 \to \Gamma(I^0) \to \Gamma(I^1) \to \Gamma(I^2) \to \ldots$$

obtained by applying Γ and composing $0 \to \Gamma(X) \to \Gamma(I^0)$ (which is already exact by Proposition 18). The problem is that, Γ might lack some right-exactness, and thus the sequence might not be exact. This default of exactness is measured precisely by the quotients:

$$\frac{\ker(\Gamma(I^q) \to \Gamma(I^{q+1}))}{\mathrm{im}(\Gamma(I^{q-1}) \to \Gamma(I^q))}$$

If Γ is the global section functor, that is $\Gamma = \mathcal{E}(1, -) : \mathbf{Ab}(\mathcal{E}) \to \mathbf{Ab}$, then this quotient is $H^q(\mathcal{E}, X)$, the qth cohomology group of $\mathcal{E}$ with value in X.

2.1 Injective resolution

Let $\mathbb{A}$ be an abelian category.

Definition 27: A *cochain complex* $(C^\bullet, \delta)$ in $\mathbb{A}$ is for all *non negative* integer k, an object C^k of $\mathbb{A}$, together with maps, called *differentials*,

$$\delta^k : C^k \to C^{k+1}$$

such that $\delta \circ \delta = 0$. A morphism of cochain complex $f : (C^\bullet, \delta) \to (D^\bullet, \delta')$ is the data of morphisms $f : C^k \to D^k$ in each degree commuting with the differentials. This forms the category $\mathrm{Ch}^\bullet(\mathbb{A})$.

Remark 28: It is often useful to extend a chain complex on the left by letting $C^k = 0$ for negative values of k, with trivial differentials.

Definition 29: The *qth cohomology group* of a cochain complex $(C^\bullet, \delta)$ is the abelian object:

$$H^q(C^\bullet) := \frac{\ker(\delta^q)}{\mathrm{im}(\delta^{q-1})}$$

As maps of cochain complexes commute with differentials, this define a functor $H^q : \mathrm{Ch}^\bullet(\mathbb{A}) \to \mathbb{A}$.

Definition 30: A morphism of cochain complex $f : C^\bullet \to D^\bullet$ is called a *quasi-isomorphism* if it induces an isomorphism

$$H^q(f) : H^q(C^\bullet) \simeq H^q(D^\bullet)$$

on each cohomology group.

Finally, we arrive to our definition of interest.

Definition 31: Let $X \in \mathbb{A}$. An *injective resolution* of X, is a cochain complex $I^\bullet \in \mathrm{Ch}^\bullet(\mathbb{A})$ such that for all k, I^k is an injective object, together with a quasi-isomorphism $i : X \to I^\bullet$, where X is identified with the complex:

$$X \to 0 \to 0 \to \dots$$

Remark 32: There is a model structure on $\mathrm{Ch}^\bullet(\mathbb{A})$ where cofibrations are monomorphisms (on positive degree), fibrations are epimorphisms with injective kernels, and weak equivalences are quasi-isomorphisms. We see that an injective complex $I^\bullet$ is then a fibrant object, and an injective resolution is precisely a fibrant replacement, that is, we replace a complex with another one, more suited to do homotopy theory.

Proposition 33: An injective resolution of X is equivalently the data of an exact sequence

$$0 \to X \to I^0 \to I^1 \to \dots$$

where I^k is injective.

Proof. (**Proposition 33**) Suppose we have an injective resolution

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i_0"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\ I^0 \arrow[r, "\delta^0"'] & I^1 \arrow[r, "\delta^1"'] & I^2 \arrow[r, "\delta^2"'] & \dots \end{tikzcd}$$

then $H^q(X) = X$ if $k = 0$, and $H^q(X) = 0$ for $q > 1$. Then, for $q > 0$, the induced isomorphism means that $H^{q+1}(I^\bullet) = 0$, that is $\ker(\delta^{q+1}) = \mathrm{im}(\delta^q)$, proving exactness at $q > 0$. For exactness at I^0, notice that i_0 induces an isomorphism $H^0(X) = X \simeq \ker(\delta^0) = H^0(I^\bullet)$, which also prove that i_0 is injective, as it is the kernel of δ^0. The converse uses the same argument backward. □

Remark 34: Thus, we will write $0 \to X \to I^\bullet$ for an injective resolution of X.

Lemma 35: Suppose $\mathbb{A}$ has enough injectives, then any $X \in \mathbb{A}$ has an injective resolution.

Proof. (**Lemma 35**) For the sake of notation, call $I^{-1} := X$. We construct an exact sequence by induction. For the base case, take any mono $\delta^{-1} = i : X \hookrightarrow I^0$ into an injective object, which exists because $\mathbb{A}$ has enough injectives. Then $0 \to X \to I^0$ is indeed exact. Suppose we constructed $\delta^k : I^k \to I^{k+1}$ for $0 \leq k < q$. Then consider any mono $I^{q-1}/\operatorname{im}(\delta^{q-2}) \hookrightarrow I^q$ into an injective object I^q. We define the next term δ^q to be the induced map:

$$I^{q-1} \twoheadrightarrow I^{q-1}/\operatorname{im}(\delta^{q-2}) \hookrightarrow I^q.$$

Then according to Lemma 11 and Lemma 13, $\ker(\delta^q) = \ker(I^{q-1} \twoheadrightarrow I^{q-1}/\operatorname{im}(\delta^{q-2})) = \operatorname{im}(\delta^{q-2})$, so the sequence is exact at I^{q-1}, and we can continue the induction. We therefore end up with an exact sequence:

$$0 \to X \to I^0 \to I^1 \to \dots$$

made of injective objects, which is, according to Proposition 33, the same thing as an injective resolution. $\square$

2.2 Homotopy of complexes

We prove some homotopy independence result of injective resolution, that will be useful to define sheaf cohomology.

Definition 36: Two map of complexes $f, g : A^\bullet \to B^\bullet$ are said to be *homotopic* if for all $K \geq 0$, there are morphisms $\varphi^k : A^k \to B^{k-1}$ such that $f^k - g^k = \delta^{k-1} \circ \varphi^k + \varphi^{k+1} \circ \delta^k$. We write $f \simeq g$. In a diagram, we have maps:

$$\begin{tikzcd}
0 \arrow[r] & A^0 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & A^1 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, "\varphi^1"'] & A^2 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, "\varphi^2"'] & \dots \\
0 \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & \dots
\end{tikzcd}$$

where the sum of the two colored triangles equals the difference of the two parallel maps between them. We say that f is *null-homotopic* if it is homotopic to the zero map $0 : A^\bullet \to B^\bullet$. Thus, to say that f and g are homotopic is to say that $f - g$ is null-homotopic. Two complexes $A^\bullet, B^\bullet$ are *homotopic* whenever there are two maps $f : A^\bullet \to B^\bullet$ and $g : B^\bullet \to A^\bullet$ such that $g \circ f \simeq \mathrm{id}_{A^\bullet}$ and $f \circ g \simeq \mathrm{id}_{B^\bullet}$.

Lemma 37: If two maps of complex are homotopic, then they induce the same cohomology maps.

Proof. (**Lemma 37**) Suppose $f : A^\bullet \to B^\bullet$ is null homotopic, and take any $x \in H^q(A^\bullet)$, so by definition $\delta^{q+1}(x) = 0$. Then $f(x) = \delta^{q-1}\varphi(x) + \varphi\delta^{q+1}(x) = \delta^{q-1}\varphi(x) \in \operatorname{im}(\delta^{q-1})$ thus is zero in $H^q(B^\bullet)$. Hence, f is the zero map. Now, if f and g are homotopic, then $f - g$ is the zero map on the cohomology groups, thus f and g are the same map in cohomology. $\square$

Corollary 38: If two complexes are homotopic, they have the same cohomology.

Proof. (**Corollary 38**) Suppose $f : A^\bullet \to B^\bullet$ and $g : B^\bullet \to A^\bullet$ is a homotopy of complex. Then by 37, the maps $H^\bullet(f)$ and $H^\bullet(g)$ compose to the identity $H^\bullet(\mathrm{id})$ both ways, which prove the isomorphism of cohomology. $\square$

The goal is to prove that two injective resolutions are homotopic.

Theorem 39: Let $0 \to X \to I^\bullet$ and $0 \to X \to J^\bullet$ be two injective resolution of X, then $I^\bullet$ and $J^\bullet$ are homotopic.

2.3 Cohomology

Finally, we define sheaf cohomology inside a general Grothendieck topos, and show that it is independent of the injective resolution we chose. This last point in fact corresponds to the fact that the cohomology is really computed in the homotopy category of cochain complexes, and thus is independent of the choice of fibrant replacement, see Remark 32, but these considerations are out of scope of this note. Here, it will simply follow from the homotopic considerations of the previous section.

We fix a Grothendieck topos $\mathcal{E}$, and even though the result can be obtained more general with an abelian category $\mathbb{A}$ with enough injectives, we stick to $\mathbb{A} = \mathbf{Ab}$, and consider the category of abelian sheaves $\mathbf{Ab}(\mathcal{E})$. We use the definition from [Joh14].

Definition 40: Let $X \in \mathbf{Ab}(\mathcal{E})$. We call Γ_X the functor $\mathcal{E}(X, -) : \mathbf{Ab}(\mathcal{E}) \to \mathbf{Ab}$. If $X \in \mathcal{E}$, we may also consider Γ_X by composing with the free abelian group functor.

Lemma 41: The functor Γ_X has a left adjoint, so is left-exact.

Proof. (Lemma 41) Call $a \dashv i$ the (enriched) adjoint pair sheafification-forgetful. The proof is inspired from this post. We recall without proof that we have a tensor product of abelian groups $A \otimes B$ such that

$$A \otimes - \dashv \mathbf{Ab}(A, -),$$

and that this passes to abelian presheaves, that is, for a category $\mathcal{C}$, a natural isomorphism

$$\mathbf{Ab}(\hat{\mathcal{C}})(A \otimes X, Y) \simeq \mathbf{Ab}(A, \mathbf{Ab}(\hat{\mathcal{C}})(X, Y)),$$

where $A \otimes X$ is computed pointwise.

Call $\mathcal{C}$ a site of $\mathcal{E}$. Let A be an abelian group, and $X \in \mathbf{Ab}(\mathcal{E})$, we define the sheaf $A \cdot X$ by:

$$(A \cdot X)(c) = a(A \otimes iF(c)).$$

We show that the tensoring functor $\Delta_X : \mathbf{Ab} \to \mathbf{Ab}(\mathcal{E})$ sending A to $A \cdot X$ is the desired left adjoint. We have the natural isomorphisms:

$$\begin{aligned}
\mathbf{Ab}(\mathcal{E})(A \cdot X, Y) &\simeq \mathbf{Ab}(\mathcal{E})(a(A \otimes iF), Y) \\
&\simeq \mathbf{Ab}(\hat{\mathcal{C}})(A \otimes iX, iY) \\
&\simeq \mathbf{Ab}(A, \mathbf{Ab}(\hat{\mathcal{C}})(iX, iY)) \\
&\simeq \mathbf{Ab}(A, \mathbf{Ab}(\mathcal{E})(aiX, Y)) \\
&\simeq \mathbf{Ab}(A, \mathbf{Ab}(\mathcal{E})(X, Y)).
\end{aligned}$$

□

Construction 42: Let $U \in \mathcal{E}$. Let $F \in \mathbf{Ab}(\mathcal{E})$. The *$q$th cohomology group* of the space U with value in F is the qth cohomology group of the cochain complex $\Gamma_U(I^\bullet)$ for any injective resolution of F. More explicitly, if $0 \to U \to I^\bullet$ is an injective resolution, then define the qth cohomology group to be:

$$H^q(U, F) := \frac{\ker(\Gamma_U(I^q) \to \Gamma_U(I^{q+1}))}{\operatorname{im}(\Gamma_U(I^{q-1}) \to \Gamma_U(I^q))},$$

with convention that $I^{-1} = 0$.

Proof. (Construction 42) For this to be well defined, we need to show that it does not depend on the injective resolution we chose. We know by Theorem 39 that any two injectives resolutions of X are homotopic, and so is there image trough Γ (which is an additive functor), thus according to Corollary 38, they have isomorphic cohomology groups. □

Remark 43: If $\mathcal{E}$ is a category of sheave on a topological space X, then X as a sheaf is the terminal object, thus Γ_X is indeed the global section functor, and we recover the usual definition of cohomology.

References

[Joh14] P. T. Johnstone. *Topos theory.* Dover books on mathematics. Dover Publications, Inc, Mineola, New York, dover edition edition, 2014.