A directed path towards the hypermonads.

Clémence Chanavat

September 12, 2023

Contents

Abstract

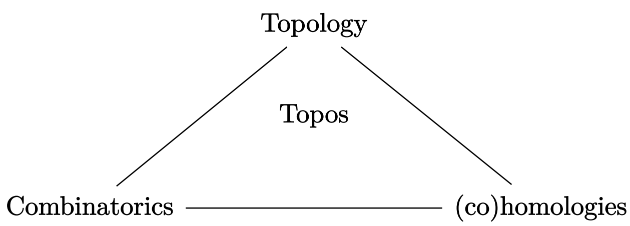
We operate a simple and fundamental split to the realm of mathematic, and give, hopefully, enough evidence that something is going on. We generalize it and create the concept of hypermonad. As a byproduct of this story, we also axiomatise point-set directed topology, compute directed (co)homologies, generalize topos theory, prove some first invariances under directed homotopies, and give hints for a modern and original definition of (∞, ∞)-category, all under the exact same idea: we keep track of the directions.

The first section explains the fundamental split that we operate systematically all along sections 2-5. In section 6, we generalize the picture by indexing the split along the shapes of $\circledast$, the category of regular molecules [HK23].

1 Introduction: why a line is not a circle

1.1 Dressing the directed picture

A portion of modern day mathematics can be found in the figure:



In this paper, we meticulously deconstruct this triangle to incorporate a notion of direction. The conceptual leap is the same for all of the four ingredients:

we systematically generalize a concept by tracking its positive part, its negative part, and how they interact.

Because we do not want our ideas to strictly collapse to the established one, the coefficients in which we store relevant data are no longer commutative groups, they are cancellative monoids, whose prototypical instance is $\mathbb{N}$. The fundamental difference we exploit can be found in the following:

Phenomenon	$\mathbb{Z}$	$\mathbb{N}$
$a+b=0$	$a=-b$	$a=0=b$

The generalization of the concepts we operate can be summarized as follow:

1. The combinatorics is now the theory of *diagrammatic sets* [Had20, HK23], the shapes have the form of *oriented* posets, whose *positive* part and *negative* part interact under the *oriented diamond property.*

2. The (co)homology theories are now given by *bichain* complexes (of monoids), where there is a *positive* differential, and a *negative* differential. The classical interaction $\delta^2 = 0$ becomes $\delta^+\delta^+ + \delta^-\delta^- = \delta^+\delta^- + \delta^-\delta^+$. The only instance of this definition the author is aware of can be found in [Pat14].

3. The theory of topological spaces now becomes the theory of directed topological spaces, which are topological spaces with a *positive* topology, and a *negative* topology. These two topologies are required to commute via the relation $\mathrm{cl}^+ \mathrm{cl}^- = \mathrm{cl}^- \mathrm{cl}^+$, where cl^α is a Kuratowski operator, a data equivalent to the one of a topological space. The example to keep in mind is the directed topology of the real line, with
$$\mathrm{cl}^+(a,b) = (a,b],$$
and
$$\mathrm{cl}^-(a,b) = [a,b).$$

4. Subsuming the directed combinatorics and the directed topologies is the notion of directed (Grothendieck) topos. A directed topos is a category with a *positive* topos structure, and a *negative* topos structure. These two structures are required to commute via $a^+a^- = a^-a^+$, where a^α is the associated sheafification operator.

Each of these notions generalize in a precise sense their undirected counterpart, the first goal of this paper is to show that they can fit in the same way into

$$\begin{tikzcd} & \text{Directed topology} \arrow[dl, dash] \arrow[dr, dash] & \\ \text{Directed combinatorics} \arrow[rr, dash] & \text{Directed topos} & \text{Directed (co)homologies} \end{tikzcd}$$

As already mentioned, the notion of directed shapes [Had20] and directed (co)homology [Pat14] can already be found in the literature. Our first contribution is to draw a connection between the two, namely by proving that directed homotopy of diagrammatic sets is an invariant of directed (co)homology. The notions of directed topological space, directed topos, and how they connect together, is original work.

The second contribution of this paper is to propose a still very conjectural generalization and abstraction of these ideas, that should lead to the concept of hypermonad. This is also original, although related ideas can be found in [Che07, HK23].

But first, to understand why this picture is possible at all, we need to answer the following question:

Why a line is not a circle?

1.2 A modern answer

A line is not a circle, because the lattice structure of their open sets behave differently. The theories of (co)homology, I argue, are not really about paths (which are a convenient proxy), but really about capturing different phenomenons on the lattice structures of the open sets. A line is not a circle because there is a particular way of reaching the top element of the lattice of open sets that is not possible with the circle. With the line, we can start from a small open $(-\varepsilon_0, \varepsilon_0)$ and continuously make it grow

$$(-\varepsilon_0, \varepsilon_0) \subseteq \cdots \subseteq (-\varepsilon_i, \varepsilon_i) \subseteq \cdots \subseteq \cdots \subseteq [-1, 1],$$

until the space is full. Now we consider a circle in $\mathbb{R}^2$:

$$\begin{tikzcd} \bullet \arrow[r, bend left, no head] \arrow[r, bend right, no head] & \bullet \end{tikzcd}$$

We try to do the same tentative. We start with a tiny open on the left, and we try to make it grow.

$$(e^{i(\pi-\varepsilon_0)}, e^{i(\pi+\varepsilon_0)}) \subseteq \cdots \subseteq (e^{i(\pi-\varepsilon_i)}, e^{i(\pi+\varepsilon_i)}) \subseteq \cdots \subset S^1,$$

the "last" inclusion is always strict: going any further would make the propagation collide with itself at the point $(0, 1)$ of $\mathbb{R}^2$. The idea of studying topological spaces with paths arises later, when we realize that making open sets grow continuously is in fact a path $[0, 1] \to X$ (or rather many paths, one per each direction where the open is growing). Hence, it is nothing but natural to have a built-in way to make the growing possible only in some particular chosen directions. This is what the input and output closures of our directed topological spaces will be for. Indeed, in traditional topology, the places where an open U is allowed to grow is given by $\partial(U)$, the boundary of U: all the points within reach of this open. Slightly generalizing, when doing directed topology, we have two different directions for the open to grow, namely, the positive direction given by a positive boundary ∂^+, and the negative direction given by a negative boundary ∂^-. Typically, the positive boundary of the open interval (a, b) will be b, and the negative boundary a. Let us see what should happen when we do directed cohomology. We consider a first directed circle:

$$\begin{tikzcd} a \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] & b \end{tikzcd}$$

It should be thought as a classical topological circle S^1, where the positive growing follows the arrows, and the negative growing opposite of the arrows. In this new directed setting, we start from a, and propagate a tiny open, trying to cover the whole circle. In this case, the propagation will have the form

$$[e^{i(\pi-\varepsilon_i)}, e^{i(\pi+\varepsilon_i)}].$$

It is close on both ends, the open interval are closed on both ends because we are growing them in the positive topology, because the arrow agree with the direction of propagation. In the positive topology, opens have form

$$\mathrm{cl}^+((a, b)) = (a, b],$$

where the right end is closed. Therefore, in the end, the collision will have shape

$$\begin{tikzcd} a \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] \\ ][\end{tikzcd}$$

which is an allowed collision: in our directed homology, it will of type $(p+q, 0) = (0, p+q)$ in $\mathbb{N} \oplus \mathbb{N}$, so p and q have to be 0, no new "degree of freedom", nothing happens. *There is no directed hole in this space*, and this is will indeed be reflected by the directed homology we develop (compare with the undirected picture, where $p + q = 0$ implies $p = -q$, so a new degree of freedom $n \mapsto (n, -n)$ appears).

Now, if we consider the directed circle

$$\begin{tikzcd} a \arrow[r, bend right, "f"'] & b \arrow[l, bend right, "g"'] \end{tikzcd}$$

then the propagation of the open sets will have shape

$$(e^{i(\pi-\varepsilon_i)}, e^{i(\pi+\varepsilon_i)}].$$

because on the f path we are still propagating with the positive topology, where opens have the shape $\mathrm{cl}^+((a,b)) = (a,b]$, but this time, on the g path, we are propagating with the negative topology, where the opens have the shape $\mathrm{cl}^-((a,b)) = [a,b)$, so we are open on the direction of growing. At the end, the collision will have shape

$$\begin{tikzcd} a \arrow[d, bend right=60, "f"'] \\ ](\arrow[u, bend right=60, "g"'] \end{tikzcd}$$

it will be a collision of type $(p,q) = (q,p)$ in $\mathbb{N} \oplus \mathbb{N}$, thus $p = q$ and a new degree of freedom of type $n \mapsto (n,n)$ appears (compare this with the undirected setting, where the exact same degree of freedom is appearing). Therefore *there is a directed hole*, again reflected by the directed homology.

2 The (co)homology of directions

We develop a (co)homology theory with value, not in abelian groups, but in commutative and cancellative (i.e. such that $a+c = b+c$ implies $a = b$) monoids, whose category we call **CCMon**. We use the notion of bichain complex. The definition already appeared in [Pat14].

2.1 Generalities about monoids

This is the first instance of the split, instead of quotienting a monoid by an equivalence relation, we quotient it by a preorder. However, it is equivalent to quotienting by the equivalence relation the preorder generates, but doing that will allow us to define, not equivalence relations on the shapes, but preorders, hence introducing a direction.

Definition 1: Let X be a set, a preorder is a reflexive and transitive relation. If $\leq$ is a preorder, we write $\uparrow x$ for the set $\{y \in X \mid x \leq y\}$, and $\downarrow x$ for the set $\{y \in X \mid y \leq x\}$.

Lemma 2: Let $(X, \leq)$ be a preorder, we have

$$\uparrow x = \uparrow x' \iff (x \leq x' \wedge x' \leq x) \iff \downarrow x = \downarrow x'.$$

Proof. (Lemma 2) We prove $\uparrow x \subseteq \uparrow x' \iff x' \leq x$, the rest of the proof is deduced by dual arguments. Suppose $\uparrow x \subseteq \uparrow x'$, then $x \in \uparrow x'$, so $x' \leq x$. Conversely, suppose $x' \leq x$. Take $y \in \uparrow x$, then $x' \leq x \leq y$, so $y \in \uparrow x'$. □

Construction 3: Let M be a monoid, and $\leq$ be congruent preorder, that is a preorder on M such that for all x, y, z, $x \leq y$ implies $x + z \leq y + z$, then we define $M/^{\leq}$ to be the monoid

$$M/^{\leq} := \{\uparrow x \mid x \in M\},$$

whose law is $\uparrow x + \uparrow y := \uparrow(x+y)$, and neutral is $0 := \uparrow 0$. Similarly, we define $M/_{\leq}$ with $\downarrow$ in place of $\uparrow$.

Proof. (Construction 3) If $\uparrow x = \uparrow x'$ and $\uparrow y = \uparrow y'$, then by Lemma 2, $x \leq x'$ and $y \leq y'$ so $x + y \leq x' + y'$. Dually, $x' + y' \leq x + y$, so again $\uparrow(x+y) = \uparrow(x'+y')$. □

Lemma 4: Suppose $\leq$ is a congruent preorder on M, then

$$M/^{\leq} \simeq M/_{\leq}.$$

Proof. (Lemma 4) The maps $\uparrow x \mapsto \downarrow x$ and $\downarrow x \mapsto \uparrow x$ are well-defined by Lemma 2, are morphisms of monoids by definition, and are inverse of each other. □

Lemma 5: Let $\leq$ be a congruence preorder on M, and call $\sim$ the relation on M defined by

$$x \sim y \iff \uparrow x = \uparrow y.$$

Then $\sim$ is an equivalence relation, and $M/_{\leq} \cong M/\sim$.

Proof. (**Lemma 5**) This is again Lemma 2. □

Therefore, quotienting by a congruent preorder is quotienting by the canonical equivalence relation it defines:

$$x \sim y \iff x \leq y \wedge y \leq x.$$

2.2 Bichain complexes

Definition 6: A *bichain complex* is a sequence of commutative and cancellative monoids $(C_n)_{n\in\mathbb{N}}$ togethers with maps

$$\delta^+, \delta^- : C_{n+1} \to C_n,$$

satisfying

$$\delta^+\delta^+ + \delta^-\delta^- = \delta^+\delta^- + \delta^-\delta^+.$$

We write $(C_\bullet, \delta^+, \delta^-)$. A map $f : C_\bullet \to D_\bullet$ is the data of maps $f : C_n \to D_n$ such that for $\alpha \in \{+,-\}$, $f\delta^\alpha = \delta^\alpha f$. This forms the category $\mathbf{bCh}$.

Definition 7: Let (C, δ^α) be a bichain complex, we define the submonoid

$$Z_n(C) := \{x \in C_n \mid \delta^+(x) = \delta^-(x)\},$$

and the preorder congruence $\leq$ on Z_n by

$$x \leq y \iff \exists w \in C_{n+1}, x + \delta^+(w) = y + \delta^-(w).$$

The *nth homology monoid* of (C, δ^α) is the quotient

$$H_n(C_\bullet) := Z_n/_{\leq}.$$

Remark 8: Notice that if the C_i's were abelian groups, then letting $\delta := \delta^+ - \delta^-$ would lead to a regular chain complex. Notice also that the equation

$$\delta^+\delta^+ + \delta^-\delta^- = \delta^+\delta^- + \delta^-\delta^+.$$

is the "sum" of the globular identities, and is also the oriented diamond property. This definition already lives in [Pat14], so does the preorder relation for homology, but not used because it is not a homotopical invariant (which is however good for us, we are looking for directed homotopical invariants, which is stronger).

Construction 9: Let $f : C_\bullet \to D_\bullet$ be a map of bichain complexes, then we call again f the map on homology defined by $\uparrow x \mapsto \uparrow f(x)$. This defines a functor H_n.

Proof. (**Construction 9**) The proof is the same that for regular homology, except we do not allow ourselves to use a minus sign. If $\uparrow x = \uparrow x'$, then we have w' such that $x + \delta^- w' = x' + \delta^+ w'$. Suppose $f(x) \leq y$, then there exists w such that

$$f(x) + \delta^+ w = y + \delta^- w,$$

hence

$$\begin{aligned}
f(x) + \delta^+ w &= y + \delta^- w && \Rightarrow \\
f(x) + \delta^+ w + f\delta^- w' &= y + \delta^- w + f\delta^- w' && \Rightarrow \\
f(x + \delta^- w') + \delta^+ w &= y + \delta^- w + f\delta^- w' && \Rightarrow \\
f(x' + \delta^+ w') + \delta^+ w &= y + \delta^- w + f\delta^- w' && \Rightarrow \\
f(x') + f\delta^+ w' + \delta^+ w &= y + \delta^- w + \delta^- f w' && \Rightarrow \\
f(x') + \delta^+ f w' + \delta^+ w &= y + \delta^- w + \delta^- f w' && \Rightarrow \\
f(x') + \delta^+(f w' + w) &= y + \delta^-(f w' + w).
\end{aligned}$$

Thus $f(x') \leq y$. This proves $\uparrow f(x) \subseteq \uparrow f(x')$. The other inclusion is dual. □

2.3 Directed homotopy

Definition 10: Let $C_\bullet, D_\bullet$ be bichain complexes. A *directed homotopy*, or *dihomotopy*, $\psi : f \to g$ is the data of maps $\psi : C_n \to D_{n+1}$ such that

$$g + \psi\delta^+ + \delta^+\psi = f + \psi\delta^- + \delta^-\psi.$$

A *homotopy* between f and g is the data of a dihomotopy $\psi : f \to g$, and a dihomotopy $\psi' : g \to f$. We write $\psi : f \leftrightarrow g : \psi'$.

Remark 11: The directed homotopy relation defines a preorder on the category of bichain complexes, and homotopy relation is the canonical equivalence relation associated to any preorder. If the C_i's were abelian group, then a dihomotopy gives a homotopy.

Lemma 12: If $\psi : g \to f$, then for $x \in Z_n(C)$, we have $\uparrow f(x) \subseteq \uparrow g(x)$.

Proof. (Lemma 12) Suppose $f(x) + \delta^+ w = y + \delta^- w$, then

$$\begin{aligned}
f(x) + \psi\delta^- x + \delta^-\psi x + \delta^+ w &= y + \psi\delta^- x + \delta^-\psi x\delta^- w && \Rightarrow\\
g(x) + \psi\delta^+ x + \delta^+\psi x + \delta^+ w &= y + \psi\delta^- x + \delta^-\psi x\delta^- w && \Rightarrow\\
g(x) + \psi\delta^- x + \delta^+\psi x + \delta^+ w &= y + \psi\delta^- x + \delta^-\psi x\delta^- w && \Rightarrow\\
g(x) + \delta^+\psi x + \delta^+ w &= y + \delta^-\psi x\delta^- w && \Rightarrow\\
g(x) + \delta^+(\psi x + w) &= y + \delta^-(\psi x + w).
\end{aligned}$$

Thus $g(x) \leq y$. □

Remark 13: This is the same proof as for chain complex, if we were not allowed to use the minus sign when writing $\partial = \partial^+ - \partial^-$, so we have to put it the other side of the equality.

Corollary 14: If $\psi : f \to g : \psi'$ is a homotopy, then f and g induce the same map on homology.

Proof. (Corollary 14) Using Lemma 12 both ways, we see that for all $x \in Z_n(C)$, we have $\uparrow f(x) = \uparrow g(x)$. □

Definition 15: We say that two bichain complexes are homotopic if there are two maps $f : C_\bullet \to D_\bullet$ and $g : D_\bullet \to C_\bullet$ such that $fg \leftrightarrow \mathrm{id}_D$ and $gf \leftrightarrow \mathrm{id}_C$.

Corollary 16: Two homotopical bichain complexes have the same homology.

3 The combinatorics of directions

We do not start this part from scratch, as we already have a powerful combinatorics for directions. It is a vast generalization of the theory of simplices, cubes, and globes, with care being taken in tracking a positive part and a negative part.

3.1 Diagrammatic sets

We give an very brief introduction of the combinatorics of regular molecules, which are based on [Ste93], and whose modern treatment can be found in [Had20] and [HK23].

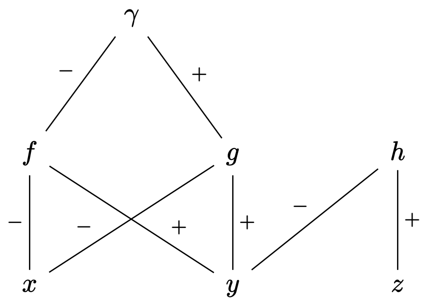
Definition 17: An *oriented graded poset* is a graded poset, that is a poset where all maximal paths starting from a given point x have the same length, which is moreover graded, that is, the edges in its covering diagram have an orientation $\alpha \in \{+, -\}$.

Remark 18: As a consequence, if x is an element of an oriented graded poset, we can consider its dimension, which is the length of any maximal path.

Consider the following pasting diagram

$$\begin{tikzcd}
x \arrow[r, bend left=50, "g"{name=G}] \arrow[r, bend right=50, "f"'{name=F}] \arrow[r, phantom, "\gamma\Uparrow"] & y \arrow[r, "h"] & z
\end{tikzcd}$$

We can naturally associate to it the following (covering diagram of the) oriented graded poset:

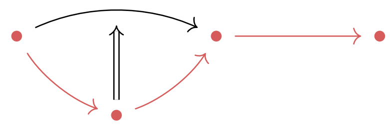


An arrow of dimension n becomes an element of dimension n in the oriented graded poset. The orientation indicates if an arrow of dimension n is in the input or the output of another arrow of dimension $n+1$.

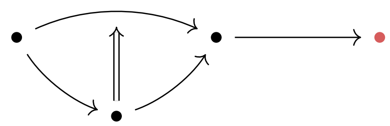
The goal is to describe a class of oriented graded poset which defines *valid* pasting diagrams. Informally, we define by examples the boundaries of an oriented graded poset.

Definition 19: Let P be an oriented graded poset. Its *input nth boundary* $\partial_n^- P$ is a closed subset representing the input of dimension n of the poset. Similarly, we define the *output nth boundary* $\partial_n^+ P$. See [HK23, 14], and the following examples.

- The input 1-boundary is in red:



- The output 0-boundary is in red:



Definition 20: We define the category of oriented graded poset, where a map $f : P \to Q$ is a function satisfying

$$\partial_n^\alpha \operatorname{cl}(f(x)) = f(\partial_n^\alpha \operatorname{cl}(x))$$

for all $x \in P$, all $n \in \mathbb{N}$, all $\alpha \in \{-,+\}$, where cl is the (downward) closure operator.

Proposition 21: Maps are order preserving, and inclusions preserve and reflect the covering relation, as well as its orientation. The inclusion of a boundary inside its poset is a map.

We now define inductively a class $\circledast$ (`\circledast`) of oriented graded poset, called the *regular molecules*.

Definition 22:

- (Point) The terminal oriented graded poset $\mathbf{1}$ is a regular molecule.
- (Paste) If U, V are regular molecules, such that $\partial_k^+ U \overset{\varphi}{\simeq} \partial_k^- V$ with $k < \min(\dim U, \dim V)$, then the *pasting* $U \#_k^\varphi V$, defined as the pushout,

$$\begin{tikzcd} \partial_k^+ U \overset{\varphi}{\simeq} \partial_k^- V \arrow[r, hook] \arrow[d, hook] & V \arrow[d] \\ U \arrow[r] & U \#_n^\varphi V \end{tikzcd}$$

is a regular molecule.

- (Atom) If U, V are *round* regular molecule of dimension $n+1$, such that $\partial_n U \overset{\varphi}{\simeq} \partial_n V$ restricting to $\partial_n^\alpha U \overset{\varphi}{\simeq} \partial_k^\alpha V$ for $\alpha \in \{-,+\}$. Then the *rewrite* $U \Rightarrow^\varphi V$ is a regular molecule. To construct it, we first construct $\partial(U \Rightarrow V)$ as the pushout:

$$\begin{tikzcd}
\partial_k U \overset{\varphi}{\simeq} \partial_n V \arrow[r, hook] \arrow[d, hook] & V \arrow[d] \\
U \arrow[r] & \partial(U \Rightarrow^\varphi V)
\end{tikzcd}$$

and then, we let $U \Rightarrow^\varphi V$ be the oriented graded poset obtained by adjoining a single $n+2$ element to $\partial(U \Rightarrow^\varphi V)$ (with orientation being $-$ on the U side and $+$ on the V side.

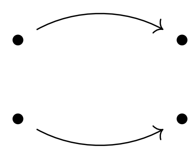
Remark 23: We asked for the (Atom) constructor that the molecules are round. Intuitively, it means that it has the shape of a globe. See [HK23, 28].

Remark 24: We can prove that isomorphisms of regular molecules are unique, and therefore, we can write (when it exists) $U \#_k V$ and $U \Rightarrow V$ without specifying the isomorphism φ.

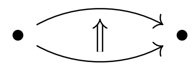
If $\mathbf{1}$ is the point, then $\mathbf{1} \Rightarrow \mathbf{1}$ is the arrow:

$$\bullet \longrightarrow \bullet$$

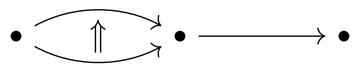
We call it I. Then, taking two copies of this arrow:



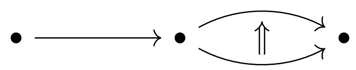
We can do the rewrite operation and obtain $I \Rightarrow I$:



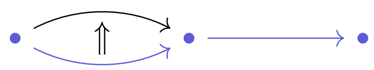
We call it O^1. Then, we can paste O^1 and I to form $O^1 \#_0 I$:



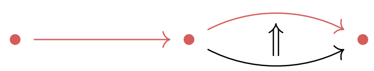
and $I \#_0 O^1$:



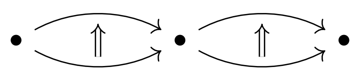
Now, we take $O^1 \#_0 I$ and $I \#_0 O^1$, and paste them along the input (blue) 1-boundary and the output (red) 1-boundary



$$\#_1$$



to obtain $(O^1 \#_0 I) \#_1 (I \#_0 O^1)$:



But notice that $(O^1 \#_0 I) \#_1 (I \#_0 O^1)$ is also $O^1 \# O^1$:

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & \bullet \end{tikzcd} \quad \#_0 \quad \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & \bullet \end{tikzcd}$$

Indeed, this is the interchange law, see [HK23, 39]. So we see that pasting decompositions are highly non-unique.

Definition 25: If $U \in \circledast$ has a top element, we say that U is an *atom*. We call $\odot$ (`\odot`) the full subcategory of $\circledast$ whose objects are atoms. We can prove that an atom is necessarily produced by (Point) or (Atom).

Definition 26: The class of *submolecule inclusions* is the smallest subclass of inclusions of regular molecules such that

1. all isomorphisms are submolecule inclusions,
2. for all regular molecules U, V and all $k \in \mathbb{N}$ such that $U \#_k V$ is defined, $U \to U \#_k V$ and $V \to U \#_k V$ are the composite of two submolecule inclusions is a submolecule inclusion.
3. the composite of two submolecule inclusions is a submolecule inclusion.

It turns out that not all inclusions are submolecule inclusions, and it is the content of [HK23] to computationally recognize them. It means that for a given pasting diagram, it may well be the case that there is a sub-pasting diagram that does not belong to any pasting decomposition.

Remark 27: If U atom, then any inclusion $U \hookrightarrow V$ is a submolecule inclusion. The inclusions $\partial_k^\alpha U \hookrightarrow U$ are always submolecule inclusions. See [HK23, 51].

3.2 Directed homotopy

We are working with the skeletal category $\odot$ of atoms and all maps.

Construction 28: Let $X : \odot^{\mathrm{op}} \to \mathbf{CCMon}$ be a presheaf of commutative and cancellative monoid. We define

$$C_n^X := \bigoplus_{\dim(U)=n} X(U),$$

and the maps $\delta^\alpha : C_{n+1}^X \to C_n^X$ for $\alpha \in \{-,+\}$, on the summand U, by

$$\delta^\alpha(x) := \sum_{a \in \Delta^\alpha U} X(\iota_a)(x).$$

It forms a bichain complex on the account of the oriented diamond property. A maps $f : X \to Y$ defines a maps $f : C_n^X \to C_n^Y$ by sending $x \in X(U)$ to $f(x) \in Y(U)$. Functoriality follows by design from naturality. Thus we have a functor

$$C_\bullet : [\odot^{\mathrm{op}}, \mathbf{CCMon}] \to \mathbf{bCh}.$$

Definition 29: Let $X \in \odot\mathbf{Set}$. The *directed homology* of X is the homology of its associated bichain complex $C_\bullet^{FX}$, where $F : \mathbf{Set} \to \mathbf{CCMon}$ is the free commutative monoid functor. We also write $C_\bullet^X$ for short.

We call I the directed cylinder, and $\iota^\alpha : \mathbf{1} \hookrightarrow I$ the two inclusions. For $U \in \odot$, we define the map

$$p_U : I \otimes U \twoheadrightarrow U$$

to be the surjection $I \otimes U \overset{! \otimes \mathrm{id}_U}{\to} \mathbf{1} \otimes U (= U)$ and

$$s_U : I \otimes U \twoheadrightarrow I$$

to be the surjection $I \otimes U \overset{\mathrm{id}_I \otimes !}{\to} I \otimes \mathbf{1}$. If $a \in \Delta^\alpha U$, we call $\iota_a : \mathrm{cl}(a) \hookrightarrow U$ the inclusion. We also call $!_U : U \twoheadrightarrow \mathbf{1}$ the terminal map.

Definition 30: Let $f, g : X \to Y$ be two natural transformation of diagrammatic sets. A *directed homotopy*, or *dihomotopy*, $\eta : f \to g$ is a natural transformation $\eta : I \otimes X \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\iota^- \otimes \mathrm{id}_X"] \arrow[dr, "f"'] & I \otimes X \arrow[d, "\eta"] & X \arrow[l, "\iota^+ \otimes \mathrm{id}_X"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

commutes. A *homotopy* between f and g is the data of $\eta : f \to g$ and $\eta' : g \to f$, and we write $\eta : f \leftrightarrow g : \eta'$.

Lemma 31: A directed homotopy $\eta : f \to g$ is equivalently the data of maps $\{\eta_p : X(U) \to Y(U)\}_{p:U\to I}$ such that $\eta_{!_U;\iota^-} = f_U$ and $\eta_{!_U;\iota^+} = g_U$, and for all $h : U \to V$ in $\odot$, and all $p : V \to I$ the diagram

$$\begin{tikzcd} X(V) \arrow[r, "Xh"] \arrow[d, "\eta_p"'] & X(U) \arrow[d, "\eta_{h;p}"] \\ Y(V) \arrow[r, "Yh"'] & Y(U) \end{tikzcd}$$

commutes.

Proof. (**Lemma 31**) We unfold the definition, and rewrite the naturality square. □

Definition 32: The diagrammatic sets X, Y are said to be *homotopic* is there are two maps $f : X \to Y$ and $g : Y \to X$ such that $gf \leftrightarrow \mathrm{id}_X$ and $fg \leftrightarrow \mathrm{id}_Y$

Proposition 33: Let $f, g : X \to Y$ be natural transformations, and let $\eta : f \to g$ be a directed homotopy. Then there is a homotopy $\eta : f \to g$ between the induced maps $f, g : C^X_\bullet \to C^Y_\bullet$.

Proof. (**Proposition 33**) For $U \in \odot$, we define $\psi : X(U) \to Y(I \otimes U)$, by

$$\psi(x) := \eta_{s_U}(X(p_U)x).$$

As $\dim(I \otimes U) = \dim(U) + 1$, this extends to a map $\psi : C^X_n \to C^Y_{n+1}$. Recall that $\Delta^\alpha I \otimes U$ has elements (α, U), with inclusion map $\iota_{(\alpha,U)} = \iota^\alpha \otimes \mathrm{id}_U$, and (I, a) for $a \in \Delta^{-\alpha} U$ with inclusion map $\iota_{(I,a)} = \mathrm{id}_I \otimes \iota_a$.

We have

$$\begin{aligned} \delta^\alpha_Y \psi x = \delta^\alpha_Y(\eta_{s_U}(X(p_U)x)) &= \sum_{a\in\Delta^\alpha I\otimes U} Y(\iota_a)(\eta_{s_U}(X(p_U)x)) = \sum_{a\in\Delta^\alpha I\otimes U} \eta_{\iota_a;s_U}(X(\iota_a)X(p_U)x) \\ &= \eta_{\iota_{(\alpha,U)};s_U}(X(\iota_{(\alpha,U)})X(p_U)x) + \sum_{a\in\Delta^{-\alpha}U} \eta_{\iota_{(I,a)};s_U}(X(\iota_{(I,a)})X(p_U)x) \\ &= \eta_{\iota_{(\alpha,U)};s_U}(X(\iota_{(\alpha,U)};p_U)x) + \sum_{a\in\Delta^{-\alpha}U} \eta_{\iota_{(I,a)};s_U}(X(\iota_{(I,a)};p_U)x) \end{aligned}$$

Now the following holds:

$$\begin{aligned} \iota_{(\alpha,U)}; s_U &= \iota^\alpha \otimes \mathrm{id}_U; s_U = !_U; \iota^\alpha \\ \iota_{(\alpha,U)}; p_U &= \iota^\alpha \otimes \mathrm{id}_U; p_U = \mathrm{id}_U \\ \iota_{(I,a)}; s_U &= \mathrm{id}_I \otimes \iota_a; s_U = s_{\mathrm{cl}(a)} \\ \iota_{(I,a)}; p_U &= \mathrm{id}_I \otimes \iota_a; s_U = p_{\mathrm{cl}(a)}; \iota_a. \end{aligned} \tag{1}$$

Therefore,

$$\delta^\alpha_Y \psi x = \eta_{!_U;\iota^\alpha}(x) + \sum_{a\in\Delta^{-\alpha}U} \eta_{s_{\mathrm{cl}(a)}}(X(p_{\mathrm{cl}(a)};\iota_a)x).$$

Next,

$$\begin{aligned} \psi\delta^\alpha_X x = \psi \sum_{a\in\Delta^\alpha U} X(\iota_a)x = \sum_{a\in\Delta^\alpha U} \psi(X(\iota_a)x) = \sum_{a\in\Delta^\alpha U} \eta_{s_{\mathrm{cl}(a)}}(X(p_{\mathrm{cl}(a)})X(\iota_a)x) \\ = \sum_{a\in\Delta^\alpha U} \eta_{s_{\mathrm{cl}(a)}}(X(p_{\mathrm{cl}(a)};\iota_a)x). \end{aligned}$$

So finally,

$$
\begin{aligned}
f(x)+\delta_Y^+\psi x+\psi\delta_X^+x &= f(x)+\eta_{!_U;\iota^+}(x)+\sum_{a\in\Delta^-U}\eta_{s_{\mathrm{cl}(a)}}(X(p_{\mathrm{cl}(a)};\iota_a)x)\\
&\quad+\sum_{a\in\Delta^+U}\eta_{s_{\mathrm{cl}(a)}}(X(p_{\mathrm{cl}(a)};\iota_a)x)\\
&= f(x)+g(x)+\sum_{a\in\Delta U}\eta_{s_{\mathrm{cl}(a)}}(X(p_{\mathrm{cl}(a)};\iota_a)x),
\end{aligned}
$$

and similarly

$$
g(x)+\delta_Y^-\psi x+\psi\delta_X^-x = f(x)+g(x)+\sum_{a\in\Delta U}\eta_{s_{\mathrm{cl}(a)}}(X(p_{\mathrm{cl}(a)};\iota_a)x),
$$

hence $f+\delta_Y^+\psi+\psi\delta_X^+ = g+\delta_Y^-\psi+\psi\delta_X^-$, which produces the desired directed homotopy. □

Remark 34: This is also somewhat how we do the proof for simplicial sets. We get $\eta : X\times\Delta[1]\to Y$ and use it to produce maps $\psi_i : X_n\to Y_{n+1}$, $0\le i\le n$. However, in that case, we are forced to use "tricks", that is, to apply the simplicial identities over and over, because the shape $[1]\times[n]$ is not representable. In this proof, we used the shape $I\otimes U$, which belongs to the category $\odot$, allowing the computations we deploy to be purely functorial and natural, modulo the existence of maps and identities of the equations (1).

Corollary 35: If $X, Y\in\odot\mathbf{Set}$ are homotopic, they have the same directed homology.

Proof. (Corollary 35) This is Proposition 33 together with Corollary 16. □

3.3 A first application

Consider the following diagrammatic set

$$
\begin{tikzcd}
a \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] & b
\end{tikzcd}
$$

Its associated bichain complex is

$$
0 \rightrightarrows \mathbb{N}[f,g] \overset{\delta^+}{\underset{\delta^-}{\rightrightarrows}} \mathbb{N}[a,b] \rightrightarrows 0
$$

with $\delta^-(f)=\delta^-(g)=a$ and $\delta^+(f)=\delta^+(g)=b$. The congruence preorder defined on $\mathbb{N}[a,b]$ is given by

$$
\begin{aligned}
(m,n)\le(m',n') &\iff \exists p,q\in\mathbb{N},(m+(p+q),n)=(m',n'+(p+q))\\
&\iff \exists k\in\mathbb{N},(m+k,n)=(m',n'+k).
\end{aligned}
$$

Thus $\uparrow(m,n)=\uparrow(m',n')$ if and only if we have k,k' such that $m+k=m'$, $m'+k'=m$, $n=n'+k$, and $n'=n+k'$. Substituting, we obtain $m'+(k+k')=m'$, therefore $k=k'=0$, hence $(m,n)=(m',n')$. The quotient is trivial, and $H_0=\mathbb{N}\oplus\mathbb{N}$. Next we compute H_1. We have $\delta^+(p,q)=(0,p+q)$ and $\delta^-(p,q)=(p+q,0)$, so $\delta^-(p,q)=\delta^-(p,q)$ implies $p+q=0$, i.e. $p=q=0$, hence $H_1=Z_1=0$.

Now consider the diagrammatic set

$$
\begin{tikzcd}
a \arrow[r, bend left, "g"] & b \arrow[l, bend left, "f"]
\end{tikzcd}
$$

Its associated bichain complex is

$$
0 \rightrightarrows \mathbb{N}[f,g] \overset{\delta^+}{\underset{\delta^-}{\rightrightarrows}} \mathbb{N}[a,b] \rightrightarrows 0
$$

with $\delta^+(f) = \delta^-(g) = a$ and $\delta^-(f) = \delta^+(g) = b$. The congruence defined on $\mathbb{N}[a, b]$ is given by

$$(m, n) \leq (m', n') \iff \exists p, q \in \mathbb{N}, (m + p, n + q) = (m' + q, n' + p)$$

The relation is symmetrical, let us write it $\sim$. Consider the monoid morphism

$$\begin{aligned} f : \mathbb{N} &\to (\mathbb{N}[a, b])/\sim \\ n &\mapsto [(n, 0)] \end{aligned}$$

It is injective as $f(n) = f(m)$ implies $(n, 0) \sim (m, 0)$ which means that we have p, q such that $(n + p, q) = (m + q, p)$, thus $p = q$, hence $n + p = m + p$, so $n = m$. Notice that letting $n, q = 0$ leads to $(m + p, 0) \sim (m, p)$, so $f(m + n) = [(m + n, 0)] = [(m, n)]$, meaning that f is surjective. The morphism f is therefore iso, so we have $H_0 = \mathbb{N}$. In fact, what we really did here was to compute $\mathbb{N} \otimes_{\mathbb{N}} \mathbb{N}$. We also compute H_1. We have $\delta^+(p, q) = (p, q)$ and $\delta^-(p, q) = (q, p)$, therefore

$$Z_1 = \{(p, q) \in \mathbb{N} \oplus \mathbb{N} \mid (p, q) = (q, p)\} = \{(n, n) \in \mathbb{N} \oplus \mathbb{N} \mid n \in \mathbb{N}\} \cong \mathbb{N},$$

and we also get $H_1 = \mathbb{N}$.

As monoids are a little bit weird, it is worth noting that $\mathbb{N} \ncong \mathbb{N} \oplus \mathbb{N}$. Indeed, suppose such iso f exists, then call $(a, b) := f(1)$. As f iso, we have k such that $(0, 1) = f(k) = kf(1) = (ka, kb)$, implying $f(1) = (0, 1)$, and k' such that $(1, 0) = k'f(1) = (k'a, k'b)$, implying $f(1) = (1, 0)$, contradiction.

To summarize, we get the following:

S^1	$a \overset{g}{\underset{f}{\rightrightarrows}} b$	$a \overset{g}{\underset{f}{\rightleftarrows}} b$
$H_0(S^1)$	$\mathbb{N} \oplus \mathbb{N}$	$\mathbb{N}$
$H_1(S^1)$	0	$\mathbb{N}$
$H_n(S^1), n \geq 2$	0	0

Thus, this homology is able to distinguish different directed circles, even though their underlying topological space are homeomorphic. Like in the traditional picture, the 0th homology monoids give the number of "biconnected" components, as on the left column, a is connected to b, but b is not connected back to a, so they define two distinct connected components. On the right column, there is only one connected component, a is connected to b which is connected to a.

4 The topology of directions

We have a way of doing directed (co)homology purely combinatorially, but we wish to include the point-set topology in the picture. In this section, we split a topological space into two parts, and give an axiomatic foundation to directed topology. This is better done not by directly looking at the open sets, but rather by looking at the equivalent data of a Kuratowski operator.

4.1 Directed topological spaces

Recall that a topological space X is equivalent to the data of Kuratowski closure

$$\mathrm{cl} : 2^X \to 2^X$$

i.e. an operator such that

- $\mathrm{cl}(\emptyset) = \emptyset$,
- for all $A \subseteq X$, $A \subseteq \mathrm{cl}(A)$,
- and for all $A, B \subseteq X$, $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

Topologies and Kuratowski closures are in bijection via

$$\mathrm{cl}(A) := \bigcap_{A \subseteq B \in \mathcal{Q}} B$$

on one way, and $A \in \mathcal{Q}$ if and only if $\mathrm{cl}(A^c) = A^c$ the other way ($\mathcal{Q}$ are the closed sets of the topology one is considering).

Definition 36: Let X be a set. A *directed topology* on X is the data of two Kuratowski closures $\mathrm{cl}^+, \mathrm{cl}^-$ such that
$$\mathrm{cl}^+ \mathrm{cl}^- = \mathrm{cl}^- \mathrm{cl}^+ .$$
If (X, cl) is a topological space, a *split* of (X, cl) is the data of a directed topology cl^α such that $\mathrm{cl} = \mathrm{cl}^+ \mathrm{cl}^- (= \mathrm{cl}^- \mathrm{cl}^+)$.

Remark 37: In term of closed sets, we would have the following, not convenient, definition. A directed topological space is a set X together with two topologies $\mathcal{O}^-, \mathcal{O}^+$ such that for all $A \subseteq X$, we have
$$\bigcap_{\left(\bigcap_{A \subseteq B \in \mathcal{O}^+} B\right) \subseteq C \in \mathcal{O}^-} C = \bigcap_{\left(\bigcap_{A \subseteq B \in \mathcal{O}^-} B\right) \subseteq C \in \mathcal{O}^+} C.$$

In term of Kuratowski closure, a function $f : X \to Y$ is continuous if for all $A \subseteq X$, we have $f(\mathrm{cl}(A)) \subseteq \mathrm{cl}(f(A))$. This motivates the definition:

Definition 38: A map $f : X \to Y$ between directed spaces is continuous if for all $A \subseteq X$, and all $\alpha \in \{-, +\}$,
$$f(\mathrm{cl}^\alpha(A)) \subseteq \mathrm{cl}^\alpha(f(A)).$$
This forms the category **dTop**.

Remark 39: If (X, cl) is a topological space, then canonically $(X, \mathrm{cl}, \mathrm{cl})$ is a directed space. If $(X, \mathrm{cl}^+, \mathrm{cl}^-)$ is a directed space, then canonically $(X, \mathrm{cl}^+ \mathrm{cl}^-)$ is a topological space. This gives two functors $L : \mathbf{dTop} \to \mathbf{Top}$ and $R : \mathbf{Top} \to \mathbf{dTop}$ with $L \dashv R$, indeed, if
$$f(\mathrm{cl}^+ \mathrm{cl}^-(A)) \subseteq \mathrm{cl}(f(A)),$$
then as $\mathrm{cl}^\alpha(A) \subseteq \mathrm{cl}^\alpha \mathrm{cl}^{-\alpha}(A)$, we get
$$f(\mathrm{cl}^\alpha(A)) \subseteq f(\mathrm{cl}^\alpha(\mathrm{cl}^{-\alpha}(A))) \subseteq \mathrm{cl}(f(A)),$$
and conversely, if for all A,
$$f(\mathrm{cl}^\alpha(A)) \subseteq \mathrm{cl}(f(A)),$$
then
$$f(\mathrm{cl}^+ \mathrm{cl}^-(A)) \subseteq \mathrm{cl}(f(\mathrm{cl}^-(A))) \subseteq \mathrm{cl}\,\mathrm{cl}(f(A)) = \mathrm{cl}(f(A)).$$

We also can describe the functor $L : \mathbf{dTop} \to \mathbf{Top}$ as doing the following:

Proposition 40: If the directed topological space $(X, \mathrm{cl}^+, \mathrm{cl}^-)$ has closed sets $\mathcal{Q}^+, \mathcal{Q}^-$, then $LX = (X, \mathrm{cl}^+ \mathrm{cl}^-)$ has closed sets $\mathcal{Q}^+ \cap \mathcal{Q}^-$, and in particular, if $(X, \mathcal{Q})$ is split by $(X, \mathcal{Q}^-, \mathcal{Q}^+)$, then $\mathcal{Q}^+ \cap \mathcal{Q}^- = \mathcal{Q}$.

And we see that splitting a topological space is finding two bigger sets of closed sets whose intersection is the space we started with. However, notice that this is not an equivalence.

Informally, the prototypical motivating example is the real line. The Kuratowski closure for the Euclidean topology is the standard closure operator, i.e. the one doing:
$$\mathrm{cl}((a, b)) = [a, b].$$
We split it into two parts:
$$\mathrm{cl}^-((a, b)) = [a, b),$$
and
$$\mathrm{cl}^+((a, b)) = (a, b].$$

The point is that now:

Proposition 41: Let $f : [0, 1] \to \mathbb{R}$ be a continuous function in **dTop** with the above directed topology, then it is increasing.

Proof. (**Proposition 41**) The function f is continuous in the associated canonical topological spaces, therefore we can also treat f as a continuous function with the standard topology. Suppose

f is not increasing, then we can find a small interval $(x, y]$ on which f is strictly decreasing, then $f((x,y]) = [f(y), f(x))$, that is $f(\mathrm{cl}^+((x,y))) = [f(y), f(x)) \not\subseteq \mathrm{cl}^+ f((x,y)) = \mathrm{cl}^+(f(y), f(x)) = (f(y), f(x)]$ (a similar argument would have worked with cl^-). $\square$

The closure operators represent very naturally an input and an output for each open sets. The positive closure add the output boundary to the open set, and the negative closure add the input boundary. We also can link this definition with some established notion of directed spaces found in [FGH⁺16]. There a d-space is a topological space $(X, \mathcal{O})$ together with a set dI of continuous paths $\gamma : [0,1] \to X$. The set dI is assumed closed under concatenation, reparametrisation, and contain all constant paths. In our directed setting with (X, cl^α), if $\gamma : [0,1] \to X$ is a continuous functions where $[0,1]$ is endowed with the canonical splitting, then indeed as all reparametrisations $\varphi : [0,1] \to [0,1]$ are continuous with the this directed topology, $\gamma \circ \varphi$ is allowed path merely by the fact that continuous function composes. The same goes for concatenation. The presence of constant paths reflect the fact that constant functions are always continuous. Therefore, any directed space makes a d-space in the sense of [FGH⁺16]. We leave for future work the question of the converse, but under some topological condition, we believe that it should be possible to split the topology according to a set of path, for instance, for $(X, \mathcal{O}, dI)$ a d-space, and $U \in \mathcal{O}$, we let

$$\partial^+(U) = \{x \in \partial U \mid \exists x_0 \in U, \gamma \in dI, \gamma(0) = x_0, \gamma(1) = x\},$$

and

$$\partial^-(U) = \{x \in \partial U \mid \exists x_0 \in U, \gamma \in dI, \gamma(1) = x_0, \gamma(0) = x\}.$$

4.2 Nerve and realisation

We create an adjoint pair $|-| \dashv N$ realisation-nerve. Its definition is different from the one of, say, simplicial sets, but it is the natural thing to do once we accept that the combinatorial shapes describes movements of open sets, and that the picture

$$\begin{tikzcd}
& (a,b) \arrow[dl, dash, "\mathrm{cl}^-"'] \arrow[dr, dash, "\mathrm{cl}^+"] & \\
{[a,b)} & & (a,b]
\end{tikzcd}$$

is an oriented covering diagram of shape $\bullet \to \bullet \in \odot$ in the power-set lattice of, say, $\mathbb{R}$: there are no subset of $\mathbb{R}$ strictly between (a,b) and $(a,b]$, nor between (a,b) and $[a,b)$. The nerve-realisation pair will work directly on the open-sets, because that is where we are putting the directed structure.

Definition 42: The (undirected) realisation functor $|-| : \odot \to \mathbf{Top}$ is the functor sending any atom to its underlying poset with the Alexandrov topology.

The interesting part is that we will split this topology according to the boundary operators of the molecules. To build that, we present the consumer-producer problem, from [MP86], where the aim of the authors is to model concurrent processes using sheaf theory. The producer-consumer problem is the abstract data of three events p, t, c, respectively the production, the transfer, and the consumption. This data is organized along the following topology (of open sets):

$$\begin{tikzcd}
& \{p,t,c\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{p,t\} \arrow[dr, dash] & & \{t,c\} \arrow[dl, dash] \\
& \{t\} \arrow[d, dash] & \\
& \emptyset &
\end{tikzcd}$$

with the meaning that the transfer site is next to both the production site, and the consumption site, but that those two sites are far from each other and cannot communicate directly if not via the transfer zone t. As we our story is better told with closed sets, let us instead look at the lattice of closed sets:

$$\begin{tikzcd}[column sep=small]
& \{p,t,c\} \arrow[d, dash] & \\
& \{p,c\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{p\} \arrow[dr, dash] & & \{c\} \arrow[dl, dash] \\
& \emptyset &
\end{tikzcd}$$

The story of the producer-consumer wants that resources are sent *from* the producer *to* the consumer. This topology is symmetrical in the consumer and the producer, so it will not be able to tell the story. We can, however, split this topology to tell the story.

Proposition 43: Declaring

$$\mathcal{Q}^- := \begin{tikzcd}[column sep=small]
& \{p,t,c\} \arrow[dl, dash] \arrow[d, dash] & \\
\{p,t\} \arrow[d, dash] & \{p,c\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{p\} \arrow[dr, dash] & & \{c\} \arrow[dl, dash] \\
& \emptyset &
\end{tikzcd} \quad \text{and } \mathcal{Q}^+ := \begin{tikzcd}[column sep=small]
& \{p,t,c\} \arrow[d, dash] \arrow[dr, dash] & \\
& \{p,c\} \arrow[dl, dash] \arrow[dr, dash] & \{t,c\} \arrow[d, dash] \\
\{p\} \arrow[dr, dash] & & \{c\} \arrow[dl, dash] \\
& \emptyset &
\end{tikzcd}$$

operates a splitting of the producer-consumer topology $\mathcal{Q}$:

$$\begin{tikzcd}[column sep=small]
& \{p,t,c\} \arrow[d, dash] & \\
& \{p,c\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{p\} \arrow[dr, dash] & & \{c\} \arrow[dl, dash] \\
& \emptyset &
\end{tikzcd}$$

Proof. (**Proposition 43**) Looking at the picture, we see that $F \in \mathcal{Q}$ if and only if $F \in \mathcal{Q}^+$ and $F \in \mathcal{Q}^-$. $\square$

Remark 44: They are funny topologies that I have never seen before. We leave it to the reader to show that they are indeed stable under union and intersection.

The passage to diagrammatic sets is the following: the arrow $\bullet \to \bullet \in \odot$ is an oriented graded poset, whose underlying topological space is precisely the one of the producer-consumer problem and the splitting of this topology will be given by the boundary operators of $\bullet \to \bullet$.

Here is a first tentative of definition, that will not capture the entirety of the phenomenon we want to capture.

Definition 45: Let $U \in \odot$, with top element u. We call $\mathcal{Q}_U$ the set of all closed sets of U. We define

$$\mathcal{Q}_U^- := \mathcal{Q}_U \cup \{\{u\} \cup \partial^- U\},$$

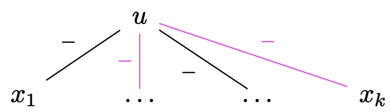
and

$$\mathcal{Q}_U^- := \mathcal{Q}_U \cup \{\{u\} \cup \partial^+ U\}.$$

This definition is ok, but it is not telling the full story. Indeed, let U be of dimension n. The splitting we just defined operates by treating the $n-1$ input layer of an atom as a unique structure. Either we take them all

$$\begin{tikzcd}
& u \arrow[dl, dash, "-"'] \arrow[d, dash, "-"'] \arrow[dr, dash, "-"] \arrow[drr, dash, "-"] & & \\
x_1 & \cdots & \cdots & x_k
\end{tikzcd}$$

or we take none of them. The thing is that this collection of input $n-1$ elements is very structured, namely, it is the top layer of a round molecules. So we would like to specify a method to take only some of them



that will tell something. This suggest that we are gonna talk about layerings. But right now, the combinatorial shapes have not enough thickness to capture, say, the continuum. They are way too finite. To remediate this problem, instead of making the combinatorics thicker, as one do with the simplicial sets $[n] \mapsto \Delta^n$, mapping a number, to a connected subset of $\mathbb{R}^n$, we are gonna make the topological spaces more combinatorial: shapes and spaces will meet in the category of oriented graded posets.

Directed topological spaces are closed under product. To define homotopy, it is natural to define a Gray product.

Definition 46: Let $(X, \mathcal{Q}^-, \mathcal{Q}^+)$ be a directed topological space. Its *dual* X^{op} is the directed space where the positive and negative topologies have been swap, that is $(\mathcal{Q}^{\mathrm{op}})^\alpha := \mathcal{Q}^{-\alpha}$.

Definition 47: Let X, Y be directed topological spaces. We define $X \otimes Y$, the *Gray product* of X and Y, to be the product of X and the dual of Y:

$$X \otimes Y := X \times Y^{\mathrm{op}}.$$

Definition 48: Let $f, g : X \to Y$ be two maps between directed topological spaces. A *dihomotopy* between f and g is a continuous function

$$\varphi : [0,1] \otimes X \to Y,$$

such that $x \mapsto \varphi(0, x)$ is f and $x \mapsto \varphi(1, x)$ is g. where, as always, $[0,1]$ is the canonical splitting of its undirected version.

5 The topos of directions

We transfer the definition of directed topological spaces to toposes.

5.1 The notion of directed topos

According to the user Oscar Cunningham from this stackexchange post, the Kuratowski closure operator of topological spaces becomes the action of sheafification when passing to Grothendieck toposes. Therefore, we have a very natural generalization.

Definition 49: A *directed Grothendieck topology* on a category $\mathcal{C}$, is the data of two Grothendieck topologies J^+, J^-, such that there is a natural isomorphism

$$a^+ a^- \simeq a^- a^+,$$

where a^α is the sheafification operator associated with the topology J^α. A sheaf for $(\mathcal{C}, J^\alpha)$ will be called an α-sheaf. We also call $\mathrm{Sh}^\alpha(\mathcal{C})$ the Grothendieck toposes these topologies define.

Definition 50: Let $(\mathcal{C}, J^+, J^-)$ be a directed topology. A *directed sheaf*, or *d-sheaf*, X is a presheaf that is both a $+$-sheaf and a $-$-sheaf. A map of directed sheaf is a natural transformation between the underlying presheaves. We let $\overrightarrow{\mathrm{Sh}}(\mathcal{C})$ be the category of directed sheaves.

Lemma 51: A presheaf X is a d-sheaf if and only if $a^+ X \cong X$ and $a^- X \cong X$.

Proof. (**Lemma 51**) Follows form the fact that a presheaf is a sheaf if and only if it is isomorphic to its sheafification. ☐

Lemma 52: Let $X \in \mathrm{Sh}^\alpha(\mathcal{C})$, then $a^{-\alpha} X$ is a d-sheaf.

Proof. (Lemma 52) Recall that sheafification is idempotent. We have $a^{-\alpha}a^{-\alpha}X = a^{-\alpha}X$, and $a^{\alpha}a^{-\alpha}X = a^{-\alpha}a^{\alpha}X = a^{-\alpha}X$, because $a^{\alpha}X = X$, as X is an α-sheaf by hypothesis. $\square$

Conjecture 53: Let Ω^{α} be the subobject classifier of $\mathrm{Sh}^{\alpha}(\mathcal{C})$, then

$$a^{+}\Omega^{-} \cong a^{-}\Omega^{+},$$

and is, if not the subobject classifier of $\overrightarrow{\mathrm{Sh}}(\mathcal{C})$, something interesting.

5.2 Directed cohomology in a directed topos

Definition 54: Let $\overrightarrow{\mathrm{Sh}}(\mathcal{C})$ be a directed topos. A *bicoefficient* is the data of two sheaves $(\mathcal{F}^{-}, \mathcal{F}^{+})$ such that $\mathcal{F}^{\alpha} \in \mathrm{Sh}^{\alpha}(\mathcal{C})$ and a natural isomorphism

$$a^{+}\mathcal{F}^{-} \simeq a^{-}\mathcal{F}^{+}.$$

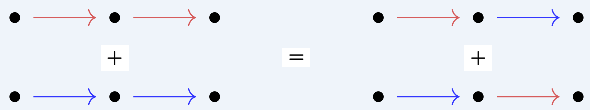
Definition 55: A *co-bichain complex* of sheaves is the data of bicoefficients $\{(\mathcal{F}_n^{-}, \mathcal{F}_n^{+})\}_{n\in\mathbb{N}}$ and maps

$$\delta^{\alpha} : \mathcal{F}_n^{\alpha} \to \mathcal{F}_{n+1}^{\alpha},$$

such that the diagram

$$\begin{tikzcd}
\dots \arrow[r] & a^{+}\mathcal{F}_{n-1}^{-} \arrow[r, "a^{+}\delta^{-}"] \arrow[d, phantom, "\cong" rotate=90] & a^{+}\mathcal{F}_{n}^{-} \arrow[r, "a^{+}\delta^{-}"] \arrow[d, phantom, "\cong" rotate=90] & a^{+}\mathcal{F}_{n+1}^{-} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \dots \\
\dots \arrow[r] & a^{-}\mathcal{F}_{n-1}^{+} \arrow[r, "a^{-}\delta^{+}"'] & a^{-}\mathcal{F}_{n}^{+} \arrow[r, "a^{-}\delta^{+}"'] & a^{-}\mathcal{F}_{n+1}^{+} \arrow[r] & \dots
\end{tikzcd}$$

follows the law of Definition 6, that we only describe pictorially here, to avoid heavy notations:



where the colors indicate which paths we are composing.

In further work, we will define the appropriate notion of injective object (that is, we will have to define directed model theory), and after fibrantly replacing a cobichain complex of monoids by its injective resolution, we will apply the global section functor, obtain bichain complex of monoid, and take the directed cohomology as in the Section 2. That is, we are gonna tell the exact same story that have been told by abelian sheaf cohomology, but this time (as it is the recurring theme) by keeping precise track of the positive part and the negative part.

5.3 The submolecule topos

We relate the combinatorial shapes to the directed topos picture, that is we exhibit a non-trivial Grothendieck topology on the category $\circledast$. Let $\circledast$ be a skeleton of the category of regular molecules and maps. A molecule $U \in \circledast$ can be thought as a (directed) space, and in this context, any decomposition of U as the pasting of other, smaller, molecules, should be regarded as an open cover. We make this idea precise by constructing a Grothendieck topology J on $\circledast$, where the open sets of U are precisely the submolecule inclusions. We show that this topos is in fact equivalent to the presheaf category $\odot$**Set** where $\odot$ is the category of maps and atoms. This will be our setting to develop sheaf cohomology.

In order to formally define the Grothendieck topology on $\circledast$, we make precise the notion of "pasting decomposition" of a regular molecule. A syntactic regular molecule is an object of the inductive type defined by the constructors (Point), (Paste), and (Atom). To every syntactic regular molecule, we associate a regular molecule in $\circledast$, which is a quotient of the inductive type. The equivalence relation is specified by the existence of an isomorphism between certain algebraic objects (the oriented graded posets) associated to each of the syntactic molecules.

Definition 56: A *pasting tree* is inductively defined:

$$T ::= U \mid (L, k, R)$$

where $U \in \circledast$, L, R are pasting trees, and k a natural number.

Definition 57: We define inductively $\mathcal{T}(U)$ the set of pasting trees associated to a syntactic regular molecule U. If U is produced by Point or Atom, then $\mathcal{T}(U) = \{U\}$. Then, inductively:

$$\mathcal{T}(U \#_k V) := \{U \#_k V\} \cup \{(L, k, R) \mid L \in \mathcal{T}(U), R \in \mathcal{T}(V)\}.$$

If $U \in \circledast$, we abuse notation and call $\mathcal{T}(U)$ the union of all sets of pasting trees $\mathcal{T}(U')$ for all syntactic molecules U' equivalent to U. Moreover, if $T \in \mathcal{T}(U)$ is a tree, we also call $T \in \circledast$ the regular molecule it defines (which is uniquely isomorphic to U).

Every pasting tree comes with associated submolecule inclusions, that cover the whole molecule. Let $U \in \circledast$, and take any tree $T \in \mathcal{T}(U)$. We define inductively a set J_T of submolecule inclusions with codomain U. If $T = V$ for a $V \in \circledast$, an inductive proof shows that $U = V$, and we let $J_T := \{\mathrm{id}_U\}$ be the associated set of submolecule inclusions. Else $T = (L, k, R)$, and $U = L \#_k R$ with submolecule inclusions ι^α, and we let $J_T := \{j; \iota^- \mid j \in J_L\} \cup \{j; \iota^+ \mid j \in J_R\}$.

Definition 58: A *presieve* on U is the collection of submolecule inclusions $J_T := \{\iota^{(k)}\}_{1 \leq k \leq n}$ associated to any pasting tree T in $\mathcal{T}(U)$.

Remark 59: By definition, presieves is just a proxy word for a pasting tree. Each presieve is specified by one and only one pasting tree. Therefore, we can talk about *the* pasting tree of a presieve. Moreover, if U is an atom, then a presieve on U necessarily contains id_U.

Lemma 60: Let S be the presieve on U and $m : W \to U$ be any map. Then there exists a presieve m^*S on W such that for all $\iota : W' \hookrightarrow W$ in m^*S, we have $\iota' : U' \hookrightarrow U \in S$, and a map $m' : W' \to S$ such that the square

$$\begin{tikzcd} W' \arrow[r, hook, "\iota"] \arrow[d, "m'"'] & W \arrow[d, "m"] \\ U' \arrow[r, hook, "\iota'"'] & U \end{tikzcd}$$

commutes.

Proof. (**Lemma 60**) We proceed by structural induction on W. If W is an atom (or the point), then m factor trough $\iota^{(k)} : U^{(k)} \hookrightarrow U$, where $U^{(k)}$ is the molecule containing the image of the top element of W. Therefore $m^*S = \{\mathrm{id}_W\}$ is a valid choice.

Next suppose $W = W^- \#_l W^+$ with inclusion ι^α. Consider two presieves $(\iota^-; m)^*S$ and $(\iota^+; m)^*S$ from the induction hypothesis, and take $m^*S = (\iota^-; m)^*S \cup (\iota^+; m)^*S$, which is a presieve on W. For any $j; \iota^\beta \in m^*S$, we have by inductive hypothesis on j and $\iota^\beta; m : W^\beta \to U$, $\iota' \in S$ and a map m' such that $m'; \iota' = j; \iota^\beta; m$, which is a desired factorisation. □

Proposition 61: The assignation $K : \circledast \to \mathbf{Set}$ that sends any molecule U to the set $K(U)$ of all presieves on U is a basis for a Grothendieck topology J on $\circledast$, called the *submolecule topology*. We call $\mathrm{Sh}(\circledast)$ its associated category of sheaves.

Proof. (**Proposition 61**) First, $\{\mathrm{id}_U : U \to U\}$ is indeed in $K(U)$. Next, the stability statement is precisely the content of Lemma 60. Finally, suppose we have $S = \{\iota^{(k)} : U^{(k)} \hookrightarrow U\}$ a presieve on $\circledast$, and for each k, a presieve $S_k = \{\kappa^{(k,l)}\}$ on $U^{(k)}$. Then as submolecule inclusions compose, the set $\{\kappa^{(k,l)}; \iota^{(k)} \mid \iota^k \in S, \kappa^{(k,l)} \in S_k\}$ is a also a presieve on U. □

Lemma 62: Let $U \in \circledast$, then the following are equivalent.

1. U is an atom.

2. The maximal sieve on U is the only U-covering sieve.

Proof. (**Lemma 62**) Suppose U is an atom, and take S a U-covering sieve, that is S contains a presieve on U. According to Remark 59, such a presieve contains the identity on U, thus $\mathrm{id}_U \in S$, hence S is maximal. Conversely, if U is not an atom, then $U = W \#_k V$ and the presieve constituted of the two inclusions generated a covering sieve on U that does not contain the identity, hence is not maximal. □

When a presheaf X is implicit from the context, $f : U \to V$ a map in the base category, and $x \in X(V)$, we write $x \cdot f$ for $X(f)(x)$.

Remark 63: Let X be a presheaf, and S be a presieve on U, then X satisfies the sheaf condition relative to S if the following is an equalizer:

$$X(U) \longrightarrow \prod_{\iota^{(k)} \in S} X(U^{(k)}) \rightrightarrows \prod_{\substack{m:V\to U^{(k)}, p:V\to U^{(l)} \\ m;\iota^{(k)} = p;\iota^{(l)}}} X(V)$$

Proposition 64: The submolecule topology is subcanonical, that is, every representable presheaf is a sheaf.

Proof. (**Proposition 64**) Let $\mathbf{y}U$ be the presheaf representing U, take $\{V^{(k)}\}$ a presieve on V, and a family of maps $\{m_k : V^{(k)} \to U\}$ such that for any arrows h and h' such that whenever $h'; \iota^{(k')} = h; \iota^{(k)}$, then $h'; m_{k'} = h; m_k$. We wish to prove that there exists a map $m : V \to U$ such that for all k, $\iota^{(k)}; m = m_k$.

We proceed by induction on tree structure of the presieve. If the presieve is a leaf, then $V^{(1)} = V$, and we take $m = \mathrm{id}_V$. Suppose that the presieve is $(L, \#_n, R)$, and that the proof is established for L and R. Then $V = L\#_n R$. Any matching family of maps $\{m_k : V^{(k)} \to U\}_{k\in K}$ for V splits into two matching families, respectively, $\{m_k\}_{k\in K_L}$ and $\{m_k\}_{k\in K_R}$, on L and R, with $K = K_L \coprod K_R$. By inductive hypothesis, these two matching families each have a unique amalgamation $m_L : L \to U$ and $m_R : R \to U$. Notice that $\{m_L, m_R\}$ is again a matching family.

Now in the diagram

$$\begin{tikzcd}
\partial_n^+ L = \partial_n^- R \arrow[r] \arrow[d] & R \arrow[d] \arrow[ddr, bend left, "m_R"] & \\
L \arrow[r] \arrow[drr, bend right, "m_L"'] & V & \\
& & U
\end{tikzcd}$$

the square is a pushout, thus commutes, hence the restrictions of m_L and m_R to $\partial_n^+ L = \partial_n^- R$ coincide, and therefore we conclude by taking the unique map $m : V \to U$ from the universal property. □

The following result, called the comparison lemma, provides conditions to reduce a sheaf topos to a smaller one.

Lemma 65: If $\mathcal{D}$ is a small dense subcategory of a locally small site $(\mathcal{C}, J)$, then the restriction functor

$$\mathrm{Sh}(\mathcal{C}, J) \to \mathrm{Sh}(\mathcal{D}, J_{\mathcal{D}})$$

is one half of an equivalence of categories.

Proof. (**Lemma 65**) This is [Joh02, theorem C2.2.3]. □

The submolecule topology enjoys a special properties, called rigidity, from which we can deduce that $\odot$ is dense in $\circledast$.

Definition 66: Let $(\mathcal{C}, J)$ be a site. We say that $d \in \mathcal{C}$ is irreducible if the only covering sieve on d is the maximal sieve. We say that J is rigid if for all $c \in \mathcal{C}$, the family of all morphisms from irreducible objects to c generates a covering sieve.

Proposition 67: The submolecule topology is rigid, with irreducible elements given by the atoms.

Proof. (**Proposition 67**) Lemma 62 states that the irreducible elements are precisely the atoms. For $U \in \circledast$, we call A_U the set of all inclusions from an atom to U. By induction, we show that the sieve it generates covers U. If U is produced by (Point) or (Atom), then A_U contains the identity, thus generates the maximal sieve. Otherwise, $U = V\#_k W$ with inclusions ι_W, ι_V. Then by induction A_V and A_W respectively contain a presieve S_V on V and S_W on W. Then we apply the locality axiom for Grothendieck topology to the presieve $\{\iota_W, \iota_V\}$ for $\{S_V, S_W\}$ to create a presieve on U included in A_U. □

Corollary 68: We have an equivalence of categories:

$$\mathrm{Sh}(\circledast) \simeq \odot\mathbf{Set}.$$

Proof. (**Corollary 68**) We can assume that $\odot$ is small. Moreover, according to [Joh02, Definition C2.2.18], the comparison lemma for rigid topologies yields $\mathrm{Sh}(\circledast) \simeq [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$ where $\mathcal{D}$ is the full subcategory on irreducible objects, that is $\mathcal{D} \simeq \odot$. □

Remark 69: Let Γ be the set of colimit diagrams in $\circledast$ comprising the initial object, and all pushouts of inclusions. Then according to [Had20], an element of $\odot\mathbf{Set}$ is given by a Γ continuous presheaf on $\circledast$. Therefore, a presheaf on $\circledast$ is a sheaf if and only if it is Γ continuous.

We describe how to go back and forth between sheaves of molecules and diagrammatic sets. Suppose we have $X \in \mathrm{Sh}(\circledast)$, then its induced diagrammatic set is simply $U \mapsto X(U)$, it is the restriction of the sheaf to the subcategory $\odot \subseteq \circledast$. Conversely, suppose we have a diagrammatic set $X \in \odot\mathbf{Set}$, call $\bar{X}$ its associated sheaf according to Corollary 68, and take any molecule U, the set $\bar{X}(U)$ can be computed as:

$$\bar{X}(U) = \{\bar{x} \in \prod_{u \in \mathrm{Max}(U)} X(\mathrm{cl}(u)) \mid \forall x \leq u, v, X(\mathrm{cl}(x) \hookrightarrow \mathrm{cl}(u))(\bar{x}_u) = X(\mathrm{cl}(x) \hookrightarrow \mathrm{cl}(v))(\bar{x}_v)\}$$

In words, the elements of $\bar{X}(U)$ are tuples of elements of $X(\mathrm{cl}(u))$ for $u \in \mathrm{Max}(U)$, with the requirement that the elements of the tuple match when restricted to where their atoms intersect. This formula is unfolding the definition of the right Kan extension, which is the inverse functor in the comparison lemma, and then noticing that the domain of the diagram in the limit formula forms a direct system with minimal elements given by the maximal elements of the molecule. Notice that when U is already an atom, then we get $\bar{X}(U) \simeq X(U)$, which correspond to the fact that we indeed have an equivalence of category, and also that the directed system has an initial element.

Conjecture 70: The submolecule topology splits.

6 Conclusion: the hypermonads

This section is to be treated as a big conjecture. So far, the splitting we operated was a 1-splitting. We only split our theories once, along the first direction:

$$\bullet \longrightarrow \bullet$$

which is the shape $I \in \odot$. Hence, informally, letting $\odot_n$ be the subcategory of atoms of dimension less than n, and letting T be a theory, we constructed $\odot_1(T)$, by indexing a splitting along a covering diagram of shape

$$\begin{tikzcd} & \bullet \arrow[dl, dash, "-"'] \arrow[dr, dash, "+"] & \\ \bullet & & \bullet \end{tikzcd}$$

which is the shape of the only atom of dimension one. Moreover, we also have the no split at all $\odot_0(T)$, which is a theory isomorphic to T. We saw that doing that $\odot_1(T)$ for T being the topological spaces lead all naturally to the theory of directed space. There however, via the nerve-realisation, we were able to capture only direction along paths. We also want to capture directions between paths. Therefore, we should construct $\odot_2(T)$. In fact, we want to capture directed paths between directed paths between directed paths... forever. So we want to construct $\odot(T)$.

Before generalizing the splitting, we need to understand what happens when we split, and we wish to describe abstractly the kind of interaction we specified between the positive part, and the negative part. The recurring theme we observe is the one of (idempotent?) monads. A Kuratowski closure is an idempotent monad on the power-set, so is sheafification. However, the differential is not, we have $\delta^2 = 0$, not $\delta^2 = \delta$. This is because the δ operator of chain complexes is already the split of an idempotent monad given by the zero morphism. Namely:

$$\begin{tikzcd} A \arrow[rr, "0"] \arrow[dr, "\delta"'] & & C \\ & B \arrow[ur, "\delta"'] & \end{tikzcd}$$

However, it seems that it does not fit our picture so well, we have the *same* operator on the negative part and the positive part. It is that we just need to look a little bit closer, and see how this definition is instantiated, take for instance the normalized chain complex associated to a simplicial set, the differential are given by:

$$\begin{tikzcd} C_{n+1} \arrow[rr, "0"] \arrow[dr, "(-1)^{n+1}\delta_{n+1}"'] & & C_{n-1} \\ & C_n \arrow[ur, "(-1)^n\delta_n"'] & \end{tikzcd}$$

The positive part and the negative part is there, alternating with $(-1)^n$. We are simply able to hide it because we work with groups, so we can subtract and pretend there are no positive or negative part, only a part.

The formalism of this coherence is already to be found in Hadzihasanovic's [Had20, 6.25], that we copy-paste it here for the convenience of the reader.

Definition 71 ([Had20, 6.25]): Let T_1, T_2 be monads on a category $\mathcal{C}$. A *$\{T_1, T_2\}$-algebra* is a pair of a T_1-algebra $\alpha_1 : T_1X \to X$ and a T_2-algebra $\alpha_2 : T_2X \to X$ on the same object of $\mathcal{C}$. A *morphism* $f : (X, \alpha_1, \alpha_2) \to (Y, \beta_1, \beta_2)$ is a morphism of $\mathcal{C}$ compatible with both algebra structures. This forms the category $\{T_1, T_2\}$-$\mathbf{Alg}$.

Notice that the condition on the morphism is exactly the condition we imposed on our objects. We give an example: Let $(X, \mathrm{cl}^+_X, \mathrm{cl}^-_X)$, $(Y, \mathrm{cl}^+_Y, \mathrm{cl}^-_Y)$ be two directed topological spaces. A morphism between the two is $f : X \to Y$ that respect both the positive and negative closed sets. Moreover, a Kuratowski closure is some monad on the functor category $[X, 2]$, whose algebras are closed sets, indeed $A \subseteq \mathrm{cl}(A)$ always, and having an algebra is having a morphism $\mathrm{cl}(A) \to A$, which, is the inclusion $\mathrm{cl}(A) \subseteq A$. Therefore A is closed. A functor

$$f^{-1} : \{\mathrm{cl}^+_Y, \mathrm{cl}^-_Y\}\text{-}\mathbf{Alg} \to \{\mathrm{cl}^+_X, \mathrm{cl}^-_X\}\text{-}\mathbf{Alg}$$

is a function $f^{-1} : 2^Y \to 2^X$ that respect the α-algebra structure, that is continuous for this topology: every α-closed set of Y has to be an α-closed set of X, that is, we have a bijection between $\mathbf{dTop}(X, Y)$, and the functor category $[\{\mathrm{cl}^+_Y, \mathrm{cl}^-_Y\}\text{-}\mathbf{Alg}, \{\mathrm{cl}^+_X, \mathrm{cl}^-_X\}\text{-}\mathbf{Alg}]$, all of this is realized by a contravariant functor of type space-quantity.

Importantly, there is a pullback square of forgetful functors:

$$\begin{tikzcd} \{T_1, T_2\}\mathbf{Alg} \arrow[r] \arrow[d] & T_1\mathbf{Alg} \arrow[d] \\ T_2\mathbf{Alg} \arrow[r] & \mathcal{C} \end{tikzcd}$$

Recall now from the submolecule topology Remark 69 that a presheaf $T : \circledast^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf if and only if it transforms all pushouts of shape

$$\begin{tikzcd} \partial^+_k U = \partial^-_k V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & U \#_k V \end{tikzcd}$$

into pullback squares

$$\begin{tikzcd} X(U \#_k V) \arrow[r] \arrow[d] & X(V) \arrow[d] \\ X(U) \arrow[r] & X(\partial^+_k U = \partial^-_k V) \end{tikzcd}$$

This is the same pullback square if we accept that X is a functor $X : \circledast^{\mathrm{op}} \to \mathbf{Alg}$, where $\mathbf{Alg}$ is the category of algebras over the category of monads.

In an hypermonad, there will be monads, and we will ask *their algebras* to combine along a sheaf condition. This is the polygraphic part. But combining monads along a sheaf condition keep them on the same dimension, there is no higher picture appearing. To go higher, we notice that every time we split a monad, we asked for a compatibility condition, some kind of commutation

$$TS \cong ST.$$

Well, it is a distributive law $TS \to ST$ that happens to be an isomorphism, because we were working on an (∞, n)-picture, so the last distributive law we looked at was an isomorphism. We do not want to stop ever, so the splitting process will happen along a general distributive law

$$l : TS \to ST.$$

In fact, building higher categories with distributive laws is not new, and can be found in [Che07, Theorem 2.1]. For the convenience of the reader, we copy-paste (modulo the big diagram) here the main theorem of her paper.

Theorem 72 ([Che07, 2.1]): Let $n \geq 3$. Let $T_1, \dots, T_n$ be monads on a category $\mathcal{C}$, equipped with

- for all $i > j$ a distributive law $\lambda_{ij} : T_i T_j \Rightarrow T_j T_i$, satisfying
- for all $i > j > k$ the "Yang-Baxter" equation given by the commutativity of the following diagram

[...] Then for all $1 \leq i < n$ we have canonical monads

$$T_1 T_2 \dots T_i \text{ and } T_{i+1} T_{i+2} \dots T_n$$

together with a distributive law of $T_{i+1} T_{i+2} \dots T_n$ over $T_1 T_2 \dots T_i$ i.e.

$$(T_{i+1} T_{i+2} \dots T_n)(T_1 T_2 \dots T_i) \Rightarrow (T_1 T_2 \dots T_i)(T_{i+1} T_{i+2} \dots T_n)$$

given by the [...] composites of the λ_{ij}. Moreover, all the induced monad structures on $T_1 T_2 \dots T_n$ are the same.

Cheng then uses it to construct the "free strict n-category monad on n-dimensional globular sets". In summary, Hadzihasanovic composed the algebras and Cheng iterated the monads, ensuring coherence coherence via Yang-Baxter equations. In the next main (but prototypical) definition of this paper, we do both and ensure coherence via a sheaf condition.

Definition 73: An *hypermonad* T is a presheaf of monads

$$T : \circledast^{\mathrm{op}} \to \mathbf{Mon}$$

such that

$$a(T\text{-}\mathbf{Alg}) \cong T\text{-}\mathbf{Alg}$$

where $-\text{-}\mathbf{Alg} : \mathbf{Mon} \to \mathbf{Alg}$ is the functor sending a monad to its category of algebras, and a is the sheafification operator of the submolecule topos $a : [\circledast^{\mathrm{op}}, \mathbf{Mon}] \to \mathrm{Sh}(\circledast)$. But we know,

$$\overrightarrow{\mathrm{Sh}}(\circledast) \cong [\odot^{\mathrm{op}}, \mathbf{Mon}],$$

therefore, an hypermonad is equivalently a presheaf

$$T \in \circledast\mathbf{Mon},$$

such that

$$T\text{-}\mathbf{Alg} \in \odot\mathbf{Alg}.$$

Everything written here is not fully precise *yet*, but a distributive law of S over T is indeed equivalent to a lifting of the monad T' on S-**Alg** [BW02], so everything lives uniformly in **Mon**, the 2-category of monads. The same way there is a free forgetful adjunction

$$\mathcal{C} \rightleftarrows T\text{-}\mathbf{Alg}$$

for a monad T, the functor -**Alg** : **Mon** $\to$ **Alg**, from the category of monads, to the category of their algebras, most likely belongs to a free-forgetful adjunction

$$\mathbf{Mon} \rightleftarrows \mathbf{Alg}.$$

I think there is some kind of deep stability property of the monoidal structures in general, they are insensible to the split operation, so we exploit this invariance. It is the phenomenon we talked about earlier, $p + q = 0$ implies $p = 0 = q$, or the fact that the upward-quotient of a monoid is the same as its downward quotient, it is like the concept of mon(oid—ad—etc...) is not direction sensitive. If one says to a monoid, here is the plus and here is the minus, it will make them equal, closing the loop.

The Isbell duality also belongs to the same picture. I call this a theorem, it feels like one, but the proof sketch is far from being rigorous or too serious (yet).

Theorem 74: Hypermonads are precisely the Isbell self-duals.

Proof. (**Theorem 74**) Copy-pasting the nlab on Isbell duality if T is a syntactic category of a $\mathcal{V}$-enriched Lawvere theory, if we call T-**Alg** the category of T algebras, we have $T \subset T\text{-}\mathbf{Alg}^{\mathrm{op}}$ as a *Yoneda embedding*, and if $T \subset C \subset T$-**Alg**, then there is an Isbell adjunction

$$\begin{tikzcd} T\text{-}\mathbf{Alg}^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}"'] \arrow[r, phantom, "\perp"] & {[C^{\mathrm{op}}, \mathcal{V}]} \arrow[l, shift right, "\mathcal{O}"'] \end{tikzcd}$$

Now instantiate with $T := \odot$, $C := \circledast$, $\mathcal{V} := \mathbf{Mon}$, and we get

$$\begin{tikzcd} \odot\text{-}\mathbf{Alg}^{\mathrm{op}} \arrow[r, shift right, "\mathrm{Spec}"'] \arrow[r, phantom, "\perp"] & \circledast\mathbf{Mon} \arrow[l, shift right, "\mathcal{O}"'] \end{tikzcd}$$

but then, the condition for being an hypermonad is to be in $\odot$-**Alg**. □

I am also not sure *where* is the homotopy hypothesis, precisely. However, all of this is still very conjectural, presheaves of monads are not presheaves of sets, this is the very very beginning, and there is a lot of work to do to in proving that this theory does not collapse. If it does not though, and if it does what it pretends, then...

Definition 75: The same way a category is a monad, an *(∞, ∞)-category* is gonna be some kind of hypermonad (or maybe they *are* the hypermonads).

Definition 76: The same way a topological space is a monad, an *(∞, ∞)-topological space* is gonna be some kind of hypermonad (or maybe an hyperspace, i.e. an hypermonad with copresheaf instead of presheaf in the definition).

Definition 77: The same way a topos is a monad, an *(∞, ∞)-topos* is gonna be some kind of hypermonad.

Definition 78: The same way a chain complex is a monad, an *(∞, ∞)-chain complex* is gonna be some kind of hypermonad.

And we are on our way to study (∞, ∞)-spaces with an *(∞, ∞)-theory of cohomology*, and also on our way to an *(∞, ∞)-theory of computation*, because conceptually a distributive law does:

$$a \cdot (b + c) \mapsto a \cdot b + a \cdot c.$$

It is a single unit of computation, it is something whose abstract shape is an atom of $\odot$. A molecule of $\circledast$ is a program, it assembles many atoms together. Our example of computation is an equality here, but we want to systematically keep track the direction in which this equality is being computed. Also, programs are proofs, and somehow, I feel like this

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

should be the space of proofs that $P \wedge Q \Rightarrow P \vee Q$.

References

[BW02] Michael Barr and Charles Wells. Toposes, triples and theories, 2002.

[Che07] Eugenia Cheng. Iterated distributive laws, 2007.

[FGH+16] Lisbeth Fajstrup, Eric Goubault, Emmanuel Haucourt, Samuel Mimram, and Martin Raussen. *Directed Algebraic Topology and Concurrency*. Springer International Publishing, Cham, 2016.

[Had20] Amar Hadzihasanovic. Diagrammatic sets and rewriting in weak higher categories, July 2020. arXiv:2007.14505 [math].

[HK23] Amar Hadzihasanovic and Diana Kessler. Higher-dimensional subdiagram matching, April 2023. arXiv:2304.09216 [cs, math].

[Joh02] P. T. Johnstone. *Sketches of an elephant: a topos theory compendium.* Number 43-<44 > in Oxford logic guides. Oxford University Press, Oxford ; New York, 2002.

[MP86] Luís Monteiro and Fernando C. N. Pereira. A sheaf-theoretic model of concurrency. In Albert Meyer, editor, *Proceedings of the First Annual IEEE Symp. on Logic in Computer Science, LICS 1986*, pages 66–76. IEEE Computer Society Press, June 1986.

[Pat14] Alex Patchkoria. Cohomology monoids of monoids with coefficients in semimodules I. *Journal of Homotopy and Related Structures*, 9(1):239–255, April 2014.

[Ste93] Richard Steiner. The algebra of directed complexes. *Applied Categorical Structures*, 1(3):247–284, 1993.