Adjunctions, monads, and monadicity

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Contents

This paper is a self-contained proof, in full details, and from (almost) scratch, of the monadicity theorem. We only assume the reader to be familiar with the basics of categories, functors, natural transformations, and the notion of coequalizer. We will give some examples and applications in further work.

1 Adjunction

Let $\mathcal{C}$, $\mathcal{D}$ be two categories and $L : \mathcal{C} \to \mathcal{D}$, $R : \mathcal{D} \to \mathcal{C}$ two functors.

1.1 Via isomorphism

Definition 1.1: Adjunction via natural isomorphism

WE say that L is left adjoint to R, and we write $L \dashv R$, whenever for all $c \in \mathcal{C}$, $d \in \mathcal{D}$, we have an isomorphism

$$\mathcal{C}(c, Rd) \overset{\phi_{c,d}}{\simeq} \mathcal{D}(Lc, d)$$

which is natural both in c and d.

The naturality of ϕ means that for $f : c \to c'$ and $g : d \to d'$, the diagram

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "u"'] & c' \arrow[d, "v"] \\ Rd \arrow[r, "Rg"'] & Rd' \end{tikzcd}$$

commutes, if and only if

$$\begin{tikzcd} Lc \arrow[r, "Lf"] \arrow[d, "\phi_{c,d}(u)"'] & Lc' \arrow[d, "\phi_{c',d'}(v)"] \\ d \arrow[r, "g"'] & d' \end{tikzcd}$$

commutes.

We call this change of square (in either direction) a *transposition*. Taking g or f to be the identity, we can transpose triangles, and this is often how we present the transposition.

Letting $c = Rd$, we obtain a morphism for free in $\mathcal{D}(LRd, d)$, namely $\varepsilon_d \overset{\Delta}{=} \phi_{Rd,d}(\mathrm{id}_{Rd})$. This defines a natural transformation, called the *counit*,

$$\varepsilon : LR \to 1_{\mathcal{D}}$$

Indeed, for the naturality of ε, we have

$$\begin{tikzcd} LRd \arrow[r, "LRg"] \arrow[d, "\phi_{Rd,d}(\mathrm{id}_{Rd})"'] & LRd' \arrow[d, "\phi_{Rd',d'}(\mathrm{id}_{Rd'})"] \\ d \arrow[r, "g"'] & d' \end{tikzcd}$$

commutes, if and only if

$$\begin{tikzcd}
Rd \arrow[r, "Rg"] \arrow[d, "\mathrm{id}_{Rd}"'] & Rd' \arrow[d, "\mathrm{id}_{Rd'}"] \\
Rd \arrow[r, "Rg"'] & Rd'
\end{tikzcd}$$

commutes, which it does.

Similarly, by letting $d = Lc$, we obtain $\eta_c \triangleq \phi^{-1}_{c,Lc}(\mathrm{id}_{Lc})$ in $\mathcal{C}(c, RLc)$ which defines a natural transformation, called the *unit*,

$$\eta : 1_{\mathcal{C}} \to RL$$

It is tempting to whisker those natural transformations, to obtain

$$\begin{array}{ll} L\eta : L \to LRL & \varepsilon L : LRL \to L \\ \eta R : R \to RLR & R\varepsilon : RLR \to R \end{array}$$

and composing what is composable, we can prove:

Lemma 1.1: Triangular identities

If $L \dashv R$, and $\eta : 1_{\mathcal{C}} \to RL$ and $\varepsilon : LR \to 1_{\mathcal{D}}$ are defined as above, then

$$\varepsilon L \circ L\eta = \mathrm{id}_L$$
$$R\varepsilon \circ \eta R = \mathrm{id}_R$$

Those are called the *triangular identities*.

Proof. (**Lemma 1.1**) It suffices to correctly transpose a square. For instance, because

$$\begin{tikzcd}
LRd \arrow[r, "L\mathrm{id}_{Rd}"] \arrow[d, "\mathrm{id}_{LRd}"'] & LRd \arrow[d, "\varepsilon_d"] \\
LRd \arrow[r, "\varepsilon_d"'] & d
\end{tikzcd}$$

commutes, so will

$$\begin{tikzcd}
Rd \arrow[r, "\mathrm{id}_{Rd}"] \arrow[d, "\phi^{-1}_{Rd,LRd}(\mathrm{id}_{LRd})"'] & Rd \arrow[d, "\phi^{-1}_{Rd,d}(\varepsilon_d)"] \\
RLRd \arrow[r, "R\varepsilon_d"'] & Rd
\end{tikzcd}$$

and we notice that by definition, $\phi^{-1}_{Rd,LRd}(\mathrm{id}_{LRd}) = \eta_{Rd}$ and $\phi^{-1}_{Rd,d}(\varepsilon_d) = \mathrm{id}_{Rd}$. Thus

$$R\varepsilon_d \circ \eta_{Rd} = \mathrm{id}_{Rd} \circ \mathrm{id}_{Rd} = \mathrm{id}_{Rd}$$

that is to say

$$(R\varepsilon \circ \eta R)_d = (\mathrm{id}_R)_d$$

$\square$

1.2 Via unit and counit

Now suppose that L and R are not yet adjoints, but instead we have two natural transformations

$$\eta : 1_{\mathcal{C}} \to RL$$
$$\varepsilon : LR \to 1_{\mathcal{D}}$$

satisfying the triangular identities

$$\varepsilon L \circ L\eta = \mathrm{id}_L$$
$$R\varepsilon \circ \eta R = \mathrm{id}_R$$

To $f : c \to Rd$, we can associate

$$\phi_{c,d}(f) \triangleq \varepsilon_d \circ Lf : Lc \to d$$

and to $g : Lc \to d$, we can associate

$$\theta_{c,d}(g) \triangleq Rg \circ \eta_c : c \to Rd$$

We have:

$$\begin{aligned}
\phi_{c,d}(\theta_{c,d}(g)) &= \phi_{c,d}(Rg \circ \eta_c) && \text{by definition} \\
&= \varepsilon_d \circ L(Rg \circ \eta_c) && \text{by definition} \\
&= \varepsilon_d \circ LRg \circ L\eta_c && \text{by functoriality} \\
&= g \circ \varepsilon_{Lc} \circ L\eta_c && \text{by naturality of } \varepsilon \text{ on } g \\
&= g \circ (\varepsilon L \circ L\eta)_c && \text{by definition} \\
&= g && \text{by triangular identity}
\end{aligned}$$

with $\varepsilon_d \circ LRg = g \circ \varepsilon_{Lc}$ simply being the naturality of ε. Using the naturality of η and the other triangular identity, we also have:

$$\begin{aligned}
\theta_{c,d}(\phi_{c,d}(f)) &= \theta_{c,d}(\varepsilon_d \circ Lf) && \text{by definition} \\
&= R(\varepsilon_d \circ Lf) \circ \eta_c && \text{by definition} \\
&= R\varepsilon_d \circ RLf \circ \eta_c && \text{by functoriality} \\
&= R\varepsilon_d \circ \eta_{Rd} \circ f && \text{by naturality of } \eta \text{ on } f \\
&= (R\varepsilon \circ \eta R)_d \circ f && \text{by definition} \\
&= f && \text{by triangular identity}
\end{aligned}$$

Thus $\theta = \phi^{-1}$ and $\mathcal{C}(c, Rd) \overset{\phi_{c,d}}{\simeq} \mathcal{D}(Lc, d)$. Similar computations ensure the naturality. Thus, it is equivalent to specify an adjunction via the natural isomorphisms or via the unit and the counit. To summarize,

Theorem 1.1: Adjunction

Let $L : \mathcal{C} \to \mathcal{D}$, $R : \mathcal{D} \to \mathcal{C}$ be two functors. We say L is left adjoint to R, written $L \dashv R$, if one of the equivalent following conditions is satisfied.

1. There exists a natural isomorphism $\mathcal{C}(c, Rd) \overset{\phi_{c,d}}{\simeq} \mathcal{D}(Lc, d)$.
2. There exists two natural transformations $\eta : 1_{\mathcal{C}} \to RL$ and $\varepsilon : LR \to 1_{\mathcal{D}}$ such that $\varepsilon L \circ L\eta = \mathrm{id}_L$ and $R\varepsilon \circ \eta R = \mathrm{id}_R$.

In that case, we get from 1. to 2. by setting $\eta_c = \phi^{-1}_{c,Lc}(\mathrm{id}_{Lc})$ and $\varepsilon_d = \phi_{Rd,d}(\mathrm{id}_{Rd})$, while we get from 2. to 1. by letting $\phi_{c,d}(f) = \varepsilon_d \circ Lf$.

2 Monad

2.1 Definition

Definition 2.1: Monad

A *monad* is a functor endowed with the structure of a monoid. More precisely, let $\mathcal{C}$ be a category. We say that the data of

$$(T : \mathcal{C} \to \mathcal{C}, \eta : 1_{\mathcal{C}} \to T, \mu : T^2 \to T)$$

constitutes a monad if both

$$\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, Rightarrow, no head] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, Rightarrow, no head] \\ & T & \end{tikzcd}$$

and

$$\begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

commute.

The natural transformation η is called the *unit* of the monad while μ is called the *multiplication*. We can see that η act as the neutral element in the monoid T, as shown by the first diagram, saying

$$\mu \circ \eta T = \mathrm{id}_T = \mu \circ T\eta$$

which is to be read as

$$e \times x = x = x \times e$$

in a monoid with neutral element e, and the multiplication μ is associative thanks to the second diagram:

$$\mu \circ T\mu = \mu \circ \mu T$$

to be read as

$$a \times (b \times c) = (a \times b) \times c$$

2.2 Adjunctions create monads

Let $L \dashv R$, as previously. Recall that we have the unit and the counit

$$\eta : 1_{\mathcal{C}} \to RL$$
$$\varepsilon : LR \to 1_{\mathcal{D}}$$

satisfying the triangular identities

$$\varepsilon L \circ L\eta = \mathrm{id}_L$$
$$R\varepsilon \circ \eta R = \mathrm{id}_R$$

Let us prove that

Proposition 2.1: Monad from adjunction

$(LR, \eta, R\varepsilon L)$ defines a monad.

Proof. (**Proposition 2.1**) For the identity, we have

$$R\varepsilon L \circ RL\eta = R(\varepsilon L \circ L\eta) = R\mathrm{id}_L = \mathrm{id}_{RL}$$

and

$$R\varepsilon L \circ \eta RL = (R\varepsilon \circ \eta R)L = \mathrm{id}_R L = \mathrm{id}_{RL}$$

For the associativity,

$$R\varepsilon L \circ RLR\varepsilon L = R(\varepsilon \circ LR\varepsilon)L = R(\varepsilon \circ \varepsilon LR)L = R\varepsilon L \circ R\varepsilon LRL$$

where $\varepsilon \circ LR\varepsilon = \varepsilon \circ \varepsilon LR$ is justified by the square:

$$\begin{tikzcd} LRLRd \arrow[r, "LR(\varepsilon_d)"] \arrow[d, "\varepsilon_{LRd}"'] & LRd \arrow[d, "\varepsilon_d"] \\ LRd \arrow[r, "\varepsilon_d"'] & d \end{tikzcd}$$

using the naturality of ε on itself. $\square$

2.3 Monads creates adjunctions

Let (T, η, μ) be a monad. We will create a canonical adjunction $L^T \dashv R^T$ from it.

Definition 2.2: Eilenberg-Moore category

We define $\mathcal{C}^T$, the *Eilenberg-Moore* category of T, or the category of algebras over T, ot be the category where objects are pairs

$$(c, u : Tc \to c)$$

such that

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[dr, equal] & Tc \arrow[d, "u"] \\ & c \end{tikzcd}$$

and

$$\begin{tikzcd} T^2c \arrow[r, "Tu"] \arrow[d, "\mu_c"'] & Tc \arrow[d, "u"] \\ Tc \arrow[r, "u"'] & c \end{tikzcd}$$

commutes, i.e. such that $u \circ \eta_c = \mathrm{id}_c$ and $u \circ Tu = u \circ \mu_c$. A morphism $f : (c, u) \to (c', u')$ is simply a morphism $f : c \to c'$ in $\mathcal{C}$ such that

$$\begin{tikzcd} Tc \arrow[r, "Tf"] \arrow[d, "u"'] & Tc' \arrow[d, "u'"] \\ c \arrow[r, "f"'] & c' \end{tikzcd}$$

commutes.

We have a forgetful functor $R^T : \mathcal{C}^T \to \mathcal{C}$, sending (c, u) to c and f to its underlying morphism ins $\mathcal{C}$. In the other direction, we can construct a functor $L^T : \mathcal{C} \to \mathcal{C}^T$, associating to $c \in \mathcal{C}$, its "free" algebra $(Tc, \mu_c : T^2c \to Tc)$. We indeed have $\mu_c \circ \eta_{Tc} = \mathrm{id}_{Tc}$ and $\mu_c \circ T\mu_c = \mu_c \circ \mu_{Tc}$ by the axioms of monad. L^T acts on morphisms by sending $f : c \to c'$ to $Tf : (Tc, \mu_c) \to (Tc', \mu_{c'})$, which is a morphism in $\mathcal{C}^T$ precisely because μ is a natural transformation.

Note that $R^T L^T = T$ and thus it is natural to pick $\eta : 1_{\mathcal{C}} \to T$ to be the unit of the adjunction we seek to create. For the counit, we have that $L^T R^T(c, u) = (Tc, \mu_c)$, thus we want a morphism

$$\varepsilon_{(c,u)} : (Tc, \mu_c) \to (c, u)$$

that is a morphism $\varepsilon_{(c,u)} : Tc \to c$ such that

$$\begin{tikzcd} T^2c \arrow[r, "T\varepsilon_{(c,u)}"] \arrow[d, "\mu_c"'] & Tc \arrow[d, "u"] \\ c \arrow[r, "\varepsilon_{(c,u)}"'] & c' \end{tikzcd}$$

Lemma 2.1: Counit of the adjunction

Taking $\varepsilon_{(c,u)} = u$ defines a natural transformations $\varepsilon : L^T R^T \to 1_{\mathcal{C}^T}$.

Proof. (Lemma 2.1) We have $(c, u) \in \mathcal{C}^T$, so in particular $u \circ Tu = u \circ \mu_c$, i.e. $u \circ T\varepsilon_{(c,u)} = \varepsilon_{(c,u)} \circ \mu_c$, so $\varepsilon_{(c,u)}$ is a morphism in $\mathcal{C}^T$. It remains to show that we can assemble the $\varepsilon_{(c,u)}$ into a natural transformation $\varepsilon : L^T R^T \to 1_{\mathcal{C}^T}$. For that, let $f : (c, u) \to (c', u')$, we want that

$$\begin{tikzcd}
(Tc, \mu_c) \arrow[r, "L^T R^T f"] \arrow[d, "u"'] & (Tc', \mu_{c'}) \arrow[d, "u'"] \\
(c, u) \arrow[r, "f"'] & (c', u')
\end{tikzcd}$$

which is the condition that $f : (c, u) \to (c', u')$ once we noticed that $L^T R^T f = Tf$. $\square$

Proposition 2.2: Adjunction from monad

We have $L^T \dashv R^T$.

Proof. (Proposition 2.2) It only remains to show that the triangular identities hold:

$$(\varepsilon L^T \circ L^T \eta)_c = \varepsilon_{(Tc,\mu_c)} \circ T\eta_c = \mu_c \circ T\eta_c = \mathrm{id}_{Tc} = T\mathrm{id}_c = L^T \mathrm{id}_c = \mathrm{id}_{L^T c}$$

and

$$(R^T \varepsilon \circ \eta R^T)_{(c,u)} = u \circ \eta_c = \mathrm{id}_c = id_{R^T(c,u)}$$

$\square$

Note that the subcategory $L^T \mathcal{C} \subseteq \mathcal{C}^T$ is denoted $\mathcal{C}_T$ and is called the Kleisli category. It is also possible to define this category as a category with the same objects of $\mathcal{C}$, and with a morphism between c and c' in $\mathcal{C}_T$ is a morphism $f : c \to Tc'$ in $\mathcal{C}$. The composition is then done as follow:

$$(g : c \to Td) \circ' (f : b \to Tc) = \mu_d \circ Tg \circ f$$

The composition on the left hand side is the one we define in $\mathcal{C}_T$ and on the right hand side, the one in $\mathcal{C}$.

As $L^T \dashv R^T$ is an adjunction, it creates a monad $T^T = R^T L^T$. Thankfully, we see that its unit is the unit of the adjunction, that is the unit of T, and its multiplication is $R^T \varepsilon L^T$, which evaluates in c to

$$(R^T \varepsilon L^T)_c = R^T(\varepsilon_{(Tc,\mu_c)}) = \mu_c$$

Thus, the monad created by the adjunction created by the monad is the original monad. However, the adjunction created by the monad created by the adjunction is not always the original adjunction, this is only the case when $L \dashv R$ is *monadic*, see the next part.

3 Monadicity

3.1 Comparison functor

Let $L \dashv R$ and call $T = RL$ the associated monad. We have the following picture:

$$\begin{tikzcd} \mathcal{D} \arrow[d, shift left, "R"] \\ \mathcal{C} \arrow[u, shift left, "L"] \arrow[r, shift left, "L^T"] & \mathcal{C}^T \arrow[l, shift left, "R^T"] \end{tikzcd}$$

Definition 3.1: Comparison functor

We can define a canonical functor $K : \mathcal{D} \to \mathcal{C}^T$, called the *comparison functor*, by

$$d \mapsto (Rd, R\varepsilon_d : TRd \to Rd)$$
$$d \xrightarrow{f} d' \mapsto (Rd, R\varepsilon_d) \xrightarrow{Rf} (Rd', R\varepsilon_{d'})$$

This is well defined because $R\varepsilon_d \circ \eta_{Rd} = \mathrm{id}_{Rd}$ and by naturality on ε on ε_d and definition of μ, we have

$$R\varepsilon_d \circ TR\varepsilon_d = R(\varepsilon_d \circ LR\varepsilon_d) = R(\varepsilon_d \circ \varepsilon_{LRd}) = R\varepsilon_d \circ \mu_{Rd}$$

Thus $Kd \in \mathcal{C}^T$. The naturality of ε on Rf shows that Rf is a morphism in $\mathcal{C}^T$. The functoriality of K boils down to the one of R.

Definition 3.2: Monadicity

We say that the adjunction is *monadic* when $K : \mathcal{D} \simeq \mathcal{C}^T$ is an equivalence of category.

That means that the adjunction $L \dashv R$ comes (up to isomorphism) from a monad, the one induced by itself, by noticing that $R^T \circ K = R$, hence $L \simeq L^T$, as adjoints are unique up to isomorphisms. Let us now see what it takes for an adjunction to be monadic, we will follow and detail greatly [SML94, Theorem 4.IV.2].

3.2 Left adjoint to the comparison functor

Before diving into the full equivalence, let us just see when K has an adjoint. Morally, K does the same as the right adjoint R, so we will look for a left adjoint $J \dashv K$. Thus we will look for an isomorphism:

$$\mathcal{C}^T((c, u), (Rd, R\varepsilon_d)) \overset{\theta}{\simeq} \mathcal{D}(J(c, u), d)$$

For $f : c \to Rd$ in $\mathcal{C}$, we call $f^* = \phi_{c,d}(f)$.

Lemma 3.1: Map of algebra and coequalizer

We have a map $f : (c, u) \to (Rd, R\varepsilon_d)$ in $\mathcal{C}^T$, if and only if $f^* \circ Lu = f^* \circ \varepsilon_{Lc}$ in $\mathcal{C}$.

Proof. (**Lemma 3.1**) A map of algebra $f : (c, u) \to (Rd, R\varepsilon_d)$ is a map $f : c \to Rd$ in $\mathcal{C}$ such that

$$\begin{tikzcd} Tc \arrow[r, "Tf"] \arrow[d, "u"'] & TRd \arrow[d, "R\varepsilon_d"] \\ c \arrow[r, "f"'] & Rd \end{tikzcd}$$

commutes. By transposing this diagram, we obtain:

$$\begin{tikzcd} LTc \arrow[r, "\phi_{Tc,LRd}(Tf)"] \arrow[d, "Lu"'] & LRd \arrow[d, "\varepsilon_d"] \\ Lc \arrow[r, "\phi_{c,d}(f)"'] & d \end{tikzcd}$$

meaning that $\phi_{c,d}(f) \circ Lu = \varepsilon_d \circ \phi_{Tc,LRd}(Tf)$. But recall that by **Theorem 1.1**, $\phi_{Tc,LRd}(Tf) = \varepsilon_{LRd} \circ LRLf = Lf \circ \varepsilon_{Lc}$, hence we can rewrite

$$f^* \circ Lu = \phi_{c,d}(f) \circ Lu = \varepsilon_d \circ \phi_{Tc,LRd}(Tf) = \varepsilon_d \circ Lf \circ \varepsilon_{Lc} = \phi_{c,d}(f) \circ \varepsilon_{Lc} = f^* \circ \varepsilon_{Lc}$$

□

Suppose we have a map of algebra $f : (c, u) \to (Rd, R\varepsilon_d)$, by **Lemma 3.1**, we can translate it into the following situation:

$$\begin{tikzcd} LTc \arrow[r, shift left, "Lu"] \arrow[r, shift right, "\varepsilon_{Lc}"'] & Lc \arrow[d, "f^*"] & J(c,u) \arrow[dl, "?"] \\ & d & \end{tikzcd}$$

and we are looking to define the object $J(c, u)$ and the map "?". Such a map would arise naturally is we took $L(c, u)$ to be the coequalizer of Lu and ε_{Lc}. It need not to exist but we will care about that later.

Definition 3.3: The left adjoint to K

We the functor $J : \mathcal{C}^T \to \mathcal{C}$ that sends (c, u) to the coequalizer of Lu and ε_{Lc} and any map $f : (c, u) \to (c', u')$ to the unique dotted arrow.

$$\begin{tikzcd} LTc \arrow[r, shift left, "Lu"] \arrow[r, shift right, "\varepsilon_{Lc}"'] & Lc \arrow[r, "e"] \arrow[d, "L(f)"] & J(c,u) \arrow[d, dotted, "\exists!"] \\ LTc' \arrow[r, shift left, "Lu'"] \arrow[r, shift right, "\varepsilon_{Lc'}"'] & Lc' \arrow[r, "e'"] & J(c',u') \end{tikzcd}$$

This dotted arrow exist, indeed f is a morphism of algebra, so we have:

$$(e' \circ Lf) \circ Lu = e' \circ L(f \circ u) = e' \circ L(u' \circ Tf) = e' \circ \varepsilon_{Lc'} \circ LRLf = (e' \circ Lf) \circ \varepsilon_{Lc}$$

The universal property of the upper coequalizer ensures the existence of the arrow, and the uniqueness of such an arrow ensures the functoriality of L.

Moreover, by **Lemma 3.1**, $f : (c, u) \to (Rd, R\varepsilon_d)$ if and only if $f^* \circ Lu = f^* \circ \varepsilon_{Lc}$ in $\mathcal{C}$ if and only if, by coequalizer, we have a map $\theta_{(c,u),d}(f) : J(c, u) \to d$, such that $\theta_{(c,u),d}(f) \circ e = f^*$. Thus we have constructed the one to one correspondance

$$\mathcal{C}^T((c, u), (Rd, R\varepsilon_d)) \overset{\theta}{\simeq} \mathcal{D}(J(c, u), d)$$

Proposition 3.1: J is left adjoint to K

We have $J \dashv K$.

Proof. (**Proposition 3.1**) We saw that θ provides an isomorphism, the final verification is the naturality of θ. For that, let us take $f : (c, u) \to (c', u')$ and $g : d \to d'$. Then

$$\begin{tikzcd}
(c,u) \arrow[r, "f"] \arrow[d, "v"'] & (c',u') \arrow[d, "w"] \\
(Rd, R\varepsilon_d) \arrow[r, "Rg"'] & (Rd', R\varepsilon_{d'})
\end{tikzcd}$$

commutes if and only if the three following diagrams commute:

(i.)

$$\begin{tikzcd}
Lc \arrow[r, "e"] \arrow[dr, "v^*"'] & J(c,u) \arrow[d, "\theta(v)"] \\
 & d
\end{tikzcd}$$

(ii.)

$$\begin{tikzcd}
Lc \arrow[r, "Lf"] \arrow[d, "v^*"'] & Lc' \arrow[d, "w^*"] \\
d \arrow[r, "g"'] & d'
\end{tikzcd}$$

(iii.)

$$\begin{tikzcd}
Lc' \arrow[r, "e'"] \arrow[dr, "w^*"'] & J(c',u') \arrow[d, "\theta(w)"] \\
 & d'
\end{tikzcd}$$

Indeed, for the direct implication, if the upper square commutes, then the lower square commutes as the transposition of the underlying morphisms via the adjunction $L \dashv R$, and the two triangles commutes by definition, because u and v are morphism in $\mathcal{C}^T$. Conversely, if the lower diagrams commute, then the triangles indicate that the upper square is a well formed square in $\mathcal{C}^T$, by **Lemma 3.1**. The lower square indicates it commutes thanks to the transposition via $L \dashv R$. Rearranging the three shapes, we get:

$$\begin{tikzcd}
J(c,u) \arrow[rrr, bend left, "Jf"] \arrow[dr, "\theta(v)"'] & Lc \arrow[l, "e"'] \arrow[r, "Lf"] \arrow[d, "v^*"'] & c' \arrow[r, "e'"] \arrow[d, "w^*"] & J(c',u') \arrow[dl, "\theta(w)"] \\
& d \arrow[r, "g"'] & d &
\end{tikzcd}$$

and

$$\begin{aligned}
\theta(w) \circ Jf \circ e &= \theta(w) \circ e' \circ Lf && \text{by definition of } Jf \\
&= w^* \circ Lf && \text{if and only if the right triangle (iii.) commutes} \\
&= g \circ v^* && \text{if and only if the square (ii.) commutes} \\
&= g \circ \theta(v) \circ e && \text{if and only if the left triangle (i.) commutes}
\end{aligned}$$

Now, e is a coequalizer so is an epi so we can cancel it, hence

$$\theta(w) \circ Jf = g \circ \theta(v)$$

if and only if (i.), (ii.) and (iii.) commute, if and only if

$$w \circ f = Rg \circ v$$

which proves the naturality. □

In our previous construction, we used coequalizers in $\mathcal{C}$. Those are not guaranteed to exists, hence we need to add them to the hypothesis. However, we will not add all coequalizers, but only reflexive one.

Definition 3.4: Reflexive pair and coequalizer

We say that $f, g : c \to d$ in $\mathcal{C}$ is a reflexive pair if there exists a common section $s : d \to c$ of f and g, i.e. such that

$$f \circ s = \mathrm{id}_d = g \circ s$$

We say that $\mathcal{C}$ has reflexive coequalizers if it has the coequalizers of all reflexive pairs.

Theorem 3.1: Left adjoint to K

If $\mathcal{D}$ has reflexive coequalizers, then $K \dashv J$.

Proof. (**Theorem 3.1**) We constructed the left adjoint $J(c,u)$ using the coequalizer of Lu and ε_{Lc} for $(c,u) \in \mathcal{C}^T$, and $Lu \circ L\eta_c = L(u\eta_c) = \mathrm{id}_{Lc} = \varepsilon_{Lc} \circ L(\eta_c)$, thus (Lu, ε_{Lc}) is a reflexive pair, hence such a coequalizer exists by hypothesis. □

3.3 When the unit is a natural isomorphism

We want to turn the previous adjunction into an equivalence, as we will see later, to do so it suffices that the unit and the counit are natural isomorphisms. In this part, we will see when the unit is an iso, but first what is the unit? Let us call it λ. By definition, $\lambda_{(c,u)}$ fits into

$$\begin{tikzcd} LTc \arrow[r, shift left, "Lu"] \arrow[r, shift right, "\varepsilon_{Lc}"'] & Lc \arrow[r, "e"] \arrow[d, "\lambda^*_{(c,u)}"] & J(c,u) \arrow[dl, equal, "\mathrm{id}_{J(c,u)}"] \\ & J(c,u) & \end{tikzcd}$$

where $\lambda^*_{(c,u)}$ is the transposition via the $L \dashv R$ adjunction, thus $\lambda^*_{(c,u)} = e$. We can rearrange this diagram in

$$\begin{tikzcd} LRLc \arrow[r, "Lu"] \arrow[d, "\varepsilon_{Lc}"'] & Lc \arrow[d, "\lambda^*_{(c,u)}"] \\ Lc \arrow[r, "e"'] & J(c,u) \end{tikzcd}$$

that we can $L \dashv R$ transpose to get

$$\begin{tikzcd} RLc \arrow[r, "u"] \arrow[d, "\mathrm{id}_{RLc}"'] & c \arrow[d, "\lambda_{(c,u)}"] \\ RLc \arrow[r, "Re"'] & RJ(c,u) \end{tikzcd}$$

Expanding the diagram we have that $\lambda_{(c,u)}$ fits into

$$\begin{tikzcd} RLRLc \arrow[r, shift left, "RLu"] \arrow[r, shift right, "R\varepsilon_{Lc}"'] \arrow[d, equal] & RLc \arrow[r, "u"] \arrow[d, equal] & c \arrow[d, "\lambda_{(c,u)}"] \\ RLRLc \arrow[r, shift left, "RLu"] \arrow[r, shift right, "R\varepsilon_{Lc}"'] & RLc \arrow[r, "Re"'] & RJ(c,u) \end{tikzcd}$$

Moreover, the upper part of this diagram is a coequalizer, as shown in the next proposition.

Proposition 3.2: Canonical presentation

Let (T, η, μ) be a monad and $(c, u) \in \mathcal{C}^T$. Then

$$\begin{tikzcd} (T^2c, \mu_{Tc}) \arrow[r, shift left, "Tu"] \arrow[r, shift right, "\mu_c"'] & (Tc, \mu_c) \arrow[r, "u"] & (c, u) \end{tikzcd}$$

is a coequalizer in $\mathcal{C}^T$.

Proof. (**Proposition 3.2**) First, every morphism in the diagram is indeed a morphism in $\mathcal{C}^T$, and moreover $u \circ Tu = u \circ \mu_c$, as $(c, u) \in \mathcal{C}^T$, hence it defines a cocone. If

$$\begin{tikzcd}
(T^2c, \mu_{Tc}) \arrow[r, shift left, "Tu"] \arrow[r, shift right, "\mu_c"'] & (Tc, \mu_c) \arrow[r, "u"] \arrow[d, "h"] & (c, u) \\
& (c', u') &
\end{tikzcd}$$

with $h \circ Tu = h \circ \mu_c$, we want a morphism $g : (c, u) \to (c', u')$ as our descent, such that $g \circ u = h$. Post-composing with η_c we get $g \circ u \circ \eta_c = h \circ \eta_c$, i.e. $g = h \circ \eta_c$. This proves uniqueness, and tells us what our morphism has to be. It remains to show that

$$\begin{tikzcd}
Tc \arrow[r, "T(h \circ \eta_c)"] \arrow[d, "u"'] & Tc' \arrow[d, "u'"] \\
c \arrow[r, "h \circ \eta_c"'] & c'
\end{tikzcd}$$

commutes, proving that g is a morphism in $\mathcal{C}^T$, and also that $h \circ \eta_c \circ u = h$ to ensure the commutation of the triangle in

$$\begin{tikzcd}
(T^2c, \mu_{Tc}) \arrow[r, shift left, "Tu"] \arrow[r, shift right, "\mu_c"'] & (Tc, \mu_c) \arrow[r, "u"] \arrow[d, "h"] & (c, u) \arrow[dl, "g"] \\
& (c', u') &
\end{tikzcd}$$

Both are done in one go, we have:

$$\begin{aligned}
h \circ \eta_c \circ u &= h \circ Tu \circ \eta_{Tc} && \text{by naturality of } \eta \text{ on } u \\
&= h \circ \mu_c \circ \eta_{Tc} && \text{because } h \text{ coequalizes } Tu \text{ and } \mu_c \\
&= h && \text{by definition of a monad} \\
&= h \circ \mu_c \circ T(\eta_c) && \text{by definition of a monad} \\
&= u' \circ T(h) \circ T(\eta_c) && \text{because } h \text{ is a morphism in } \mathcal{C}^T \\
&= u' \circ T(h \circ \eta_c) && \text{by functoriality}
\end{aligned}$$

$\square$

Lemma 3.2: R^T **reflects coequalizers**

Suppose

$$\begin{tikzcd} R^T(a,u) \arrow[r, shift left, "R^T f"] \arrow[r, shift right, "R^T g"'] & R^T(b,v) \arrow[r, "R^T e"] & R^T(c,w) \end{tikzcd}$$

is a coequalizer in $\mathcal{C}$, then

$$\begin{tikzcd} (a,u) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (b,v) \arrow[r, "e"] & (c,w) \end{tikzcd}$$

is a coequalizer in $\mathcal{C}^T$.

Proof. (**Lemma 3.2**) If

$$\begin{tikzcd} (a,u) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (b,v) \arrow[r, "e"] \arrow[d, "h"] & (c,w) \\ & (d,t) & \end{tikzcd}$$

then by coequalizer in $\mathcal{C}$, we have a unique morphism $k : c \to d$ such that $k \circ e = h$. It suffices then to show that k lift to an algebra morphism, i.e. that

$$\begin{tikzcd} Tc \arrow[r, "Tk"] \arrow[d, "u"] & Td \arrow[d, "t"] \\ c \arrow[r, "k"'] & d \end{tikzcd}$$

commutes. In

$$\begin{tikzcd} Tb \arrow[r, "Te"] \arrow[d, "v"'] & Tc \arrow[r, "Tk"] \arrow[d, "u"] & Td \arrow[d, "t"] \\ b \arrow[r, "e"] & c \arrow[r, "k"'] & d \end{tikzcd}$$

the left square commutes because e is a morphism in $\mathcal{C}^t$, and the big rectangle commutes because $k \circ e = h$ is also a morphism in $\mathcal{C}^T$, hence the right square commutes. $\square$

Now by definition of J,

$$\begin{tikzcd} LTc \arrow[r, shift left, "Lu"] \arrow[r, shift right, "\varepsilon_{Lc}"'] & Lc \arrow[r, "e"] & J(c,u) \end{tikzcd}$$

is a coequalizer. Suppose it is preserved by R, then

$$
RLRLc \underset{R\varepsilon_{Lc}}{\overset{RLu}{\rightrightarrows}} RLc \xrightarrow{Re} RJ(c,u)
$$

is also a coequalizer as it is the image of

$$
(T^2c, \mu_{Tc}) \underset{R\varepsilon_{Lc}}{\overset{RLu}{\rightrightarrows}} (Tc, \mu_c) \xrightarrow{Re} (RJ(c,u), R\varepsilon_{J(c,u)})
$$

trough R^T. Indeed, the two parallel pairs are known to be morphisms in $\mathcal{C}^T$, and $Re : (Tc, \mu_c) \to (RJ(c,u), R\varepsilon_{J(c,u)})$ is also a morphism as

$$
R\varepsilon_{J(c,u)} \circ TRLe = Re \circ \varepsilon R\varepsilon_{Lc}
$$

by naturality of ε on e, and recalling that $\mu = R\varepsilon L$.

Hence, according to **Proposition 3.2** and **Lemma 3.2**, both

$$
(T^2c, \mu_{Tc}) \underset{R\varepsilon_{Lc}}{\overset{RLu}{\rightrightarrows}} (Tc, \mu_c) \xrightarrow{u} (c,u)
$$

$$
(T^2c, \mu_{Tc}) \underset{R\varepsilon_{Lc}}{\overset{RLu}{\rightrightarrows}} (Tc, \mu_c) \xrightarrow{Re} (RJ(c,u), R\varepsilon_{J(c,u)})
$$

are coequalizers. Thus we have and isomorphism in $\mathcal{C}^T$, $\lambda' : (c,u) \simeq (RJ(c,u), R\varepsilon_{J(c,u)})$ such that $\lambda' \circ u = Re$. We will now show that $\lambda'^* = \lambda^*_{(c,u)}$, i.e. that $\lambda'^* = e$. We have that λ' fits into

$$
\begin{tikzcd}
RLc \arrow[r, "u"] \arrow[d, "\mathrm{id}_{RLc}"'] & c \arrow[d, "\lambda'"] \\
RLc \arrow[r, "Re"'] & RJ(c,u)
\end{tikzcd}
$$

thus by $L \dashv R$ transposition, it also fits into

$$
\begin{tikzcd}
LRLc \arrow[r, "Lu"] \arrow[d, "\varepsilon_{Lc}"'] & Lc \arrow[d, "\lambda'^*"] \\
Lc \arrow[r, "e"'] & J(c,u)
\end{tikzcd}
$$

Thus

$$
\lambda'^* \circ Lu = e \circ \varepsilon_{Lc} = e \circ Lu
$$

and λ' is a morphism in $\mathcal{C}^T$, thus by **Lemma 3.1**,

$$
\lambda'^* \circ Lu = \lambda'^* \circ \varepsilon_{Lc}
$$

Combining these equalities, we get

$$
\lambda'^* \circ Lu = e \circ Lu
$$
$$
\lambda'^* \circ \varepsilon_{Lc} = e \circ \varepsilon_{Lc}
$$

meaning that, by the uniqueness in the universal property of the coequalizer, $\lambda'^* = e$. Hence, $\lambda'^* = \lambda^*_{(c,u)}$, so $\lambda' = \lambda_{(c,u)}$, as the transposition is a bijective operation, which proves that $\lambda_{(c,u)}$ is an isomorphism. To summarize,

Theorem 3.2: When the unit is an isomorphism

If $\mathcal{D}$ has reflexive coequalizers and R preserves such coequalizers, then $J \dashv K$, and the unit $\lambda : 1_{\mathcal{C}^T} \simeq KJ$ is an isomorphism.

3.4 When the counit is a natural isomorphism

Finally, we will see when the counit is an isomorphism. Call $\kappa : 1_{\mathcal{D}} \to JK$ the counit of the $J \dashv K$ adjunction. By definition of J and the counit, κ_d fits into

$$\begin{tikzcd} LTRd \arrow[r, shift left, "LR\varepsilon_d"] \arrow[r, shift right, "\varepsilon_{LRd}"'] & LRd \arrow[r, "e"] \arrow[d, "\mathrm{id}^*_{Rd}"] & J(Rd, R\varepsilon_d) \arrow[dl, "\kappa_d"] \\ & d & \end{tikzcd}$$

But $\mathrm{id}^*_{Rd} = \varepsilon_d$, so applying R to this diagram, we have

$$\begin{tikzcd} RLRLRd \arrow[r, shift left, "RLR\varepsilon_d"] \arrow[r, shift right, "R\varepsilon_{LRd}"'] & RLRd \arrow[r, "Re"] & J(Rd, R\varepsilon_d) \arrow[d, "R\kappa_d"] \\ RLRLRd \arrow[r, shift left, "RLR\varepsilon_d"] \arrow[r, shift right, "R\varepsilon_{LRd}"'] & RLRd \arrow[r, "R\varepsilon_d"] & Rd \end{tikzcd}$$

The following definition will prove that the lower part is a coequalizer.

Definition 3.5: Split coequalizer

We say that a diagram

$$\begin{tikzcd} c \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & d \arrow[r, "e"] & e \end{tikzcd}$$

with $e \circ f = e \circ g$ is a split coequalizer whenever there exist $s : c \to b$ and $t : b \to a$

$$\begin{tikzcd} c \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & d \arrow[r, "e"] \arrow[l, bend right, "t"'] & e \arrow[l, bend right, "s"'] \end{tikzcd}$$

such that $e \circ s = \mathrm{id}_c$, $f \circ t = \mathrm{id}_b$ and $g \circ t = s \circ e$. This defines a coequalizer, as whenever $h : b \to d$ is such that $h \circ f = h \circ g$, then $k = h \circ s$ is such that

$$k \circ e = h \circ s \circ e = h \circ g \circ t = h \circ f \circ t = h$$

and the uniqueness is given by $k \circ e = h$ implies $k \circ e \circ s = h \circ s$, i.e. $k = h \circ s$.

Lemma 3.3: A split coequalizer

We have

$$\begin{tikzcd} RLRLRd \arrow[r, shift left, "RLR\varepsilon_d"] \arrow[r, shift right, "R\varepsilon_{LRd}"'] & RLRd \arrow[r, "R\varepsilon_d"] & Rd \end{tikzcd}$$

is a split coequalizer.

Proof. (Lemma 3.3)

$$\begin{tikzcd} RLRLRd \arrow[r, shift left, "RLR\varepsilon_d"] \arrow[r, shift right, "R\varepsilon_{LRd}"'] & RLRd \arrow[l, bend right=60, "\eta_{RLRd}"'] \arrow[r, "R\varepsilon_d"'] & Rd \arrow[l, bend right=40, "\eta_{Rd}"'] \end{tikzcd}$$

is such that $R\varepsilon_d \circ \eta_{Rd} = \mathrm{id}_{Rd}$, $R\varepsilon_{LRd} \circ \eta_{RLRd} = \mathrm{id}_{RLRd}$ both by triangular identity, and

$$RLR\varepsilon_d \circ \eta_{RLRd} = R\varepsilon_d \circ \eta_{Rd}$$

by naturality of η on the morphism $R\varepsilon_d$. $\square$

Thus, if we suppose that

$$\begin{tikzcd} RLRLRd \arrow[r, shift left, "RLR\varepsilon_d"] \arrow[r, shift right, "R\varepsilon_{LRd}"'] & RLRd \arrow[r, "Re"] & J(Rd, R\varepsilon_d) \end{tikzcd}$$

is a coequalizer, which is true under the hypothesis of **Theorem 3.2**, then the unique map $i : J(Rd, R\varepsilon_d) \to d$ such that $i \circ Re = R\varepsilon_d$ is an isomorphism, and such a map is given by $R\kappa_d$, thus $R\kappa_d$ is an isomorphism.

Definition 3.6: Reflects isomorphism

We say that a functor $F : \mathcal{C} \to \mathcal{D}$ reflects isomorphisms if whenever $Fi : Fc \to Fd$ is an isomorphism, then $i : c \to d$ is an isomorphism.

Finally, we can state the following theorem.

Theorem 3.3: When the counit is an isomorphism

If $\mathcal{D}$ has reflexive coequalizers, R preserves such coequalizers, and R reflects isomorphisms, then $J \dashv K$, the unit $\lambda : 1_{\mathcal{C}^T} \simeq KJ$ an the counit $\kappa : JK \simeq 1_{\mathcal{D}}$ are isomorphisms.

3.5 The monadicity theorem

We have almost all the tools to state the monadicity theorem. We just need to provide a last result on adjunctions and equivalence of categories.

Definition 3.7: Equivalence of categories

We say that a functor $G : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, and we write $G : \mathcal{C} \simeq \mathcal{D}$ if G is fully faithful and essentially surjective.

Proposition 3.3: Equivalence from adjunction

If $F \dashv G$ where the unit and the counit of the adjunction are natural isomorphisms, then $G : \mathcal{C} \simeq \mathcal{D}$.

Proof. (**Proposition 3.3**) First, $\mathcal{D}(Gc, Gd) \simeq \mathcal{C}(FGc, d) \simeq \mathcal{C}(c, d)$, the first bijection is given by adjunction, and the second by counit $\varepsilon_c : FGc \simeq c$, hence G is fully faithful. Second, if $d \in \mathcal{D}$, by unit $\eta_d : d \simeq GFd$, thus G is essentially surjective. □

Finally, we can refactor **Theorem 3.3** into the monadicity theorem:

Theorem 3.4: Monadicity theorem

Let $L\mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ such that $L \dashv T$ with counit ε. If

1. $\mathcal{D}$ has reflexive coequalizers,
2. R preserves reflexive coequalizers,
3. R reflects isomorphisms.

Then $L \dashv R$ is monadic, that is the comparison functor $K : \mathcal{D} \to \mathcal{C}^T$ sending d to $(Rd, R\varepsilon_d)$ is an equivalence of categories.

Proof. (**Theorem 3.4**) Under the hypothesis, **Theorem 3.3** indicates that the unit and the counit of $K \simeq J$ are natural isomorphisms, thus according to **Proposition 3.3**, $K : \mathcal{D} \simeq \mathcal{C}^T$. □

References

[SML94] Ieke Moerdijk Saunders Mac Lane. *Sheaves in Geometry and Logic.* Springer New York, NY, 1994.