Cubical Type Theory Inside a Presheaf Topos

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September 27, 2022

Context

Cubical Type Theory

- The first models of Homotopy Type Theory were non constructive.
- Thus, the need for Cubical Type Theory, based on cubical sets.
- Uses an interval object $\mathbb{I}$, which is a synthetic counterpart of $[0, 1]$.
- Leads to a notion of path $p : \texttt{Path}\ A\ a\ b$ is a map $p : \mathbb{I} \to A$.

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What we will do

- It is well known that presheaves category have an internal type theory.
- We will try to understand what is needed from such a presheaves category to be a model of Cubical Type Theory.
- For that, we will use topos theory.

Contents I

Cubical type theory

Rules and interval

1. Standard Martin-Löf dependent type theory
2. An object $\mathbb{I}$ which is the free de Morgan algebra on a fixed infinite set of names $i, j, k, \ldots$
3. Grammar of $\mathbb{I}$ is
$$r, s ::= 0 \mid 1 \mid i \mid \neg r \mid r \wedge s \mid r \vee s$$
4. Custom λ-abstraction for $i : \mathbb{I}$:
$$\langle i \rangle . t$$

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$$\begin{array}{lcll}
\Gamma, \Delta & ::= & () \mid \Gamma, x : A \mid \Gamma, i : \mathbb{I} & \text{Contexts} \\
t, u, A, B & ::= & x \mid \lambda x : A.t \mid t\ u \mid (x : A) \to B & \Pi\text{-types} \\
 & \mid & (t, u) \mid t.1 \mid t.2 \mid (x : A) \times B & \Sigma\text{-types} \\
 & \mid & \mathtt{Path}\ A\ t\ u \mid \langle i \rangle\ t \mid t\ r & \text{Path types}
\end{array}$$

Interval

Insight on $\mathbb{I}$

1. $\mathbb{I}$ is a synthetic equivalent for $[0,1]$.
2. $\vee$ represents max
3. $\wedge$ represents min
4. $\neg$ represents $1 - \cdot$
5. We write $(i0)$ and $(i1)$ for $(i/0)$ and $(i/1)$.

Jugdmental equality for $\mathbb{I}$

$\neg 0 = 1$	$\neg 1 = 0$	$\neg(r \vee s) = \neg r \wedge \neg s$	$\neg(r \wedge s) = \neg r \vee \neg s$

Path types

Rules

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash \mathtt{Path}\ A\ a\ b}
\qquad
\frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash a : A}{\Gamma \vdash \langle i \rangle\ a : \mathtt{Path}\ A\ a(i0)\ a(i1)}$$

$$\frac{\Gamma \vdash p : \mathtt{Path}\ A\ a\ b \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash p\ r : A}
\qquad
\frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash a : A \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash (\langle i \rangle\ a)\ r = a(i/r) : A}$$

$$\frac{\Gamma, i : \mathbb{I} \vdash p\ i = q\ i : A}{\Gamma \vdash p = q : \mathtt{Path}\ A\ p_0\ p_1}
\qquad
\frac{\Gamma \vdash p : \mathtt{Path}\ A\ p_0\ p_1}{\Gamma \vdash p\ 0 = p_0 : A}
\qquad
\frac{\Gamma \vdash p : \mathtt{Path}\ A\ p_0\ p_1}{\Gamma \vdash p\ 1 = p_1 : A}$$

Path types

Rules

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash \mathtt{Path}\ A\ a\ b} \qquad \frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash a : A}{\Gamma \vdash \langle i \rangle\ a : \mathtt{Path}\ A\ a(i0)\ a(i1)}$$

$$\frac{\Gamma \vdash p : \mathtt{Path}\ A\ a\ b \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash p\ r : A} \qquad \frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash a : A \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash (\langle i \rangle\ a)\ r = a(i/r) : A}$$

$$\frac{\Gamma, i : \mathbb{I} \vdash p\ i = q\ i : A}{\Gamma \vdash p = q : \mathtt{Path}\ A\ p_0\ p_1} \qquad \frac{\Gamma \vdash p : \mathtt{Path}\ A\ p_0\ p_1}{\Gamma \vdash p\ 0 = p_0 : A} \qquad \frac{\Gamma \vdash p : \mathtt{Path}\ A\ p_0\ p_1}{\Gamma \vdash p\ 1 = p_1 : A}$$

Consequences

1. Reflexivity: For $a : A$, $1_a = \langle i \rangle\ a : \mathtt{Path}\ A\ a\ a$
2. Function extensionality, from $\Gamma \vdash p : (x : A) \to \mathtt{Path}\ B\ (f\ x)\ (g\ x)$ we have

$$\Gamma \vdash \langle i \rangle\ \lambda x : A.\ p\ x\ i : \mathtt{Path}\ ((x : A) \to B)\ f\ g$$

Geometric intuition

In dimension n

n variables of dimension $i_1, \ldots, i_n : \mathbb{I}$ in the context, correspond to an n-dimensional cube.

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In dimension two

$$i : \mathbb{I}, j : \mathbb{I} \vdash A$$

$$\begin{tikzcd}
A(i0)(j1) \arrow[r, "A(j1)"] & A(i1)(j1) \\
A(i0)(j0) \arrow[u, "A(i0)"] \arrow[r, "A(j0)"'] & A(i1)(j0) \arrow[u, "A(i1)"']
\end{tikzcd}$$

Face lattice

Definition (Face lattice)

We define $\mathbb{F}$ to be the distributive lattice generated by the symbols $(i = 0)$ and $(i = 1)$ (for all dimension name i) with relation $(i = 0) \wedge (i = 1) = 0_{\mathbb{F}}$. The grammar is

$$\varphi, \psi ::= 0_{\mathbb{F}} \mid 1_{\mathbb{F}} \mid (i = 0) \mid (i = 1) \mid \varphi \wedge \psi \mid \varphi \vee \psi$$

We have the rule

$$\frac{\Gamma \vdash \psi : \mathbb{F}}{\Gamma, \psi \vdash}$$

Contexts and congruence

For context Γ, we define recursively a congruence $=_{\Gamma}$ on $\mathbb{F}$.

Systems

Definition (System)

A system is a compatible union of sub-polyhedra. The grammar is:

$$t, u, A, B ::= \ldots \mid [\varphi_1\ a_1 \ldots \varphi_n\ a_n]$$

Rules for systems

If $\Gamma \vdash \varphi_1 \vee \cdots \vee \varphi_n =_\Gamma 1_{\mathbb{F}} : \mathbb{F}$, then

$$\frac{\Gamma, \varphi_1 \vdash a_1 : A \qquad \ldots \qquad \Gamma, \varphi_n \vdash a_n : A \qquad \Gamma, \varphi_i \wedge \varphi_j \vdash a_i = a_j : A\ (i \leq i, j \leq n)}{\Gamma \vdash [\varphi_1\ a_1 \ldots \varphi_n\ a_n] : A}$$

$$\frac{\Gamma, \varphi_1 \vdash J \qquad \ldots \qquad \Gamma, \varphi_n \vdash J}{\Gamma \vdash J} \qquad \frac{\Gamma \vdash [\varphi_1\ a_1, \ldots, \varphi_n\ a_n] : A \qquad \Gamma \vdash \varphi_i = 1_{\mathbb{F}} : F}{\Gamma \vdash [\varphi_1\ a_1, \ldots, \varphi_n\ a_n] = a_i : A}$$

Composition

Notation

We write $\Gamma \vdash a : A[\varphi \mapsto u]$ for $\Gamma \vdash a : A$ and $\Gamma, \varphi \vdash a = u : A$

Definition (Composition)

Composition is a way to transport an element along a variable of dimension. The grammar is:

$$t, u, A, B ::= \dots \mid \mathtt{comp}^i\ A\ [\varphi \mapsto u]\ a_0$$

with u a system. The rule is:

$$\frac{\Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash a_0 : A(i0)[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathtt{comp}^i\ A\ [\varphi \mapsto u]\ a_0 : A(i1)[\varphi \mapsto u(i1)]}$$

Transitivity

Transitivity

Composition allows us to prove transitivity of paths. If $\Gamma \vdash A$, $\Gamma \vdash a, b, c : A$, then

$$\frac{\Gamma \vdash p : \texttt{Path}\ A\ a\ b \qquad \Gamma \vdash q : \texttt{Path}\ A\ b\ c}{\Gamma \vdash \langle i \rangle\ \texttt{comp}^j\ A\ [(i=0) \vee (i=1) \mapsto [(i=0)\ a, (i=1)\ (q\ j)]\ (p\ i) : \texttt{Path}\ A\ a\ c}$$

Contractible types and equivalences

Definition (Contractible types)

A type A is contractible if

$$\texttt{isContr}\ A \triangleq (x : A) \times ((y : A) \to \texttt{Path}\ A\ x\ y)$$

is inhabited.

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is inhabited.

Definition (Equivalence)

Given two types T, A and $f : T \to A$, we define

$$\texttt{isEquiv}\ T\ A\ f \triangleq (y : A) \to \texttt{isContr}\ ((x : T) \times \texttt{Path}\ A\ y\ (f\ x))$$

We define the type

$$\texttt{Equiv}\ T\ A \triangleq (f : T \to A) \times \texttt{isEquiv}\ T\ A\ f$$

Glueing

Definition (Glueing)

The glueing operation is the analogous of composition, but for types. The formation rule is:

$$\frac{\Gamma \vdash A \qquad \Gamma, \varphi \vdash T \qquad \Gamma, \varphi \vdash f : \mathtt{Equiv}\ T\ A}{\Gamma \vdash \mathtt{Glue}\ [\varphi \mapsto (T, f)]\ A}$$

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Intuition for glueing

If

1. $\Gamma \vdash A$
2. $\Gamma, \varphi \vdash T$
3. A and T are equivalent on the region φ

then we have the equality $\Gamma, \varphi \vdash \mathtt{Glue}\ [\varphi \mapsto (T, f)]\ A = T$.

Glueing and univalence

We can illustrate the type $\texttt{Glue}\,[1_{\mathbb{F}} \mapsto (T, f)]\; A$ as:

$$
\begin{tikzcd}
T \arrow[rrr, dashed] \arrow[drr, "f"] \arrow[ddr] & & & \texttt{Glue}\,[1_{\mathbb{F}} \mapsto (T, f)]\; A \arrow[dd, dashed] \arrow[dr, dashed, "\texttt{unglue}"] & \\
& & A \arrow[rr] \arrow[dl] & & A \arrow[dl] \\
& \Gamma, \varphi \arrow[rr] & & \Gamma &
\end{tikzcd}
$$

Glueing and univalence

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& & A \arrow[rr] \arrow[dl] & & A \arrow[dl] \\
& \Gamma, \varphi \arrow[rr] & & \Gamma &
\end{tikzcd}$$

Theorem (Univalence in Cubical Type Theory)

For any term

$$t : (A\ B : U) \to \texttt{Path}\ U\ A\ B \to \texttt{Equiv}\ A\ B$$

the map $t\ A\ B : \texttt{Path}\ U\ A\ B \to \texttt{Equiv}\ A\ B$ *is an equivalence.*

Logic inside a presheaf topos

Topos

Definition (A definition of a topos)

A *topos* is a category that

- has finite limits
- is cartesian closed
- has a subobject classifier $(\Omega, \mathrm{true} : \mathbf{1} \rightarrowtail \Omega)$

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A *topos* is a category that

- has finite limits
- is cartesian closed
- has a subobject classifier $(\Omega, \mathrm{true} : \mathbf{1} \rightarrowtail \Omega)$

Definition (Subobject classifier)

For any mono $s : S \rightarrowtail X$, there exists a unique $\chi_s : X \to \Omega$ such that the following is a pullback:

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail, "s"'] & \mathbf{1} \arrow[d, tail, "\mathrm{true}"] \\ X \arrow[r, "\chi_s"'] & \Omega \end{tikzcd}$$

Presheaf topos

Condition for the classifier

For any $X \in \mathcal{E}$, we have

$$\mathrm{Sub}(X) \simeq \mathcal{E}(X, \Omega)$$

Presheaf topos

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Presheaf topos

Let $\mathcal{C}$ be a locally small category, then $\mathcal{E} = [\mathcal{C}^{\mathsf{op}}, \mathsf{Sets}]$ is a topos. The subobject classifier $\Omega \in \mathcal{E}$ maps each $c \in \mathcal{C}$ to the set of sieves on c. We define true : $\mathbf{1} \to \Omega$ by picking the maximal sieve.

Kripke-Joyal semantics

Link between toposes and logic

- Each topos has its own internal logic.
- We can interpret the syntax thanks to the Kripke-Joyal semantics.
- We rely on the Heyting structure of the subobjects.

Kripke-Joyal semantics

Link between toposes and logic

- Each topos has its own internal logic.
- We can interpret the syntax thanks to the Kripke-Joyal semantics.
- We rely on the Heyting structure of the subobjects.
- Then, we use the notion of Kripke-Joyal forcing to recursively unwind formulas,
- Thus transforming a formula into a concrete object of the topos.

Heyting strucutre

Definition (Heyting algebra)

A *Heyting algebra* is a bounded lattice with an operation $\Rightarrow$ such that

$$(x \wedge y) \leq z \iff x \wedge y \Rightarrow z$$

Heyting strucutre

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Theorem (Heyting algebra of the subobjects)

For any $X \in \mathcal{E}$, $\mathrm{Sub}(X)$ *is a Heyting algebra.*

Top and Bot

$\top$

We take $\top :=$ true. It is the classifying map of $! : \mathbf{1} \to \mathbf{1}$. We have the following pullback.

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[d, tail, "!"'] & \mathbf{1} \arrow[d, tail] \\ \mathbf{1} \arrow[r, "\top"'] & \Omega \end{tikzcd}$$

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$\perp$

We define $\perp$ with the following pullback.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, tail, "!"'] & \mathbf{1} \arrow[d, tail, "\mathsf{true}"] \\ \mathbf{1} \arrow[r, "\perp"'] & \Omega \end{tikzcd}$$

And

$\wedge$

We have $\wedge : \Omega \times \Omega \to \Omega$ with the following pullback.

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[d, tail, "\langle \text{true}, \text{true} \rangle"'] & \mathbf{1} \arrow[d, tail, "\text{true}"] \\ \Omega \times \Omega \arrow[r, "\wedge"'] & \Omega \end{tikzcd}$$

Image

Definition (Image)

A mono m is the image of an arrow f, if $f = m \circ e$ for some epi e and if whenever f factors through a mono m' then m factor through m'. We write $m = \mathrm{im}(f)$.

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Theorem (Image in a topos)

Let $f : X \to Y$. Then f has an image $m : M \to Y$ and factors as in

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & Y \\ & M \arrow[ur, tail, "m"'] & \end{tikzcd}$$

with e epi.

Or

Setup for the $\vee$

$$
\begin{tikzcd}
\Omega \arrow[r, "\iota_1"] \arrow[ddr, "\langle \mathrm{id},\mathrm{true}\rangle"'] & \Omega + \Omega \arrow[d, dotted, two heads] & \Omega \arrow[l, "\iota_2"'] \arrow[ddl, "\langle \mathrm{true},\mathrm{id}\rangle"] \\
& U \arrow[d, dotted, tail, "u"] & \\
& \Omega \times \Omega &
\end{tikzcd}
$$

Or

Setup for the $\vee$

$$\begin{tikzcd} \Omega \arrow[r, "\iota_1"] \arrow[dr, "\langle \mathrm{id}, \mathrm{true} \rangle"'] & \Omega + \Omega \arrow[d, dotted, two heads] & \Omega \arrow[l, "\iota_2"'] \arrow[dl, "\langle \mathrm{true}, \mathrm{id} \rangle"] \\ & U \arrow[d, dotted, tail, "u"] & \\ & \Omega \times \Omega & \end{tikzcd}$$

$\vee$

We take $\vee : \Omega \times \Omega \to \Omega$ to be the classifying arrow of $u : U \to \Omega \times \Omega$ above.

Quantifiers as pullbacks

Adjonction to the pullback functor

We define

$$\begin{tikzcd}
\mathcal{E}(Y \times X, \Omega) \simeq \mathrm{Sub}(Y \times X) \arrow[d, bend right=60, "\exists_Y"'] \arrow[d, bend left=60, "\forall_Y"] \\
\mathcal{E}(X, \Omega) \simeq \mathrm{Sub}(X) \arrow[u, "\pi^*"]
\end{tikzcd}$$

where π^* is the pullback functor along the second projection $\pi : Y \times X \to X$ and

$$\exists_Y \dashv \pi^* \dashv \forall_Y$$

Interpretation of formulas

Interpretation of formulas

Suppose we have $\sigma, \tau : X \to \Omega$, $\theta : Y \times X \to \Omega$, the we interpret

- $\sigma \wedge \tau$ by $\wedge \circ \langle \sigma, \tau \rangle : X \to \Omega$
- $\sigma \vee \tau$ by $\vee \circ \langle \sigma, \tau \rangle : X \to \Omega$
- $\sigma \Rightarrow \tau$ by $\Rightarrow \circ \langle \sigma, \tau \rangle : X \to \Omega$
- $\neg \sigma$ by $\Rightarrow \circ \langle \sigma, \bot \circ ! \rangle = \neg \circ \sigma : X \to \Omega$
- $\forall y : Y,\ \theta(y)$ by $\forall_Y \theta : X \to \Omega$
- $\exists y : Y,\ \theta(y)$ by $\exists_Y \theta : X \to \Omega$

A very important notation

Notation

Every $\sigma : X \to \Omega$ is to be thought as a formula. Thus we write

$$\{x : X \mid \sigma(x)\}$$

for the subobject of X arising from the pullback of σ and true.

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for the subobject of X arising from the pullback of σ and true.

$$\begin{tikzcd}
\{x : X \mid \sigma(x)\} \arrow[r] \arrow[d, tail] & \mathbf{1} \arrow[d, tail, "\text{true}"] \\
X \arrow[r, "\sigma"'] & \Omega
\end{tikzcd}$$

Interpretation of equality

Interpretation of $=$

For all $Y \in \mathcal{E}$, let $\Delta_Y : Y \to Y \times Y$ be the diagonal map and let δ_Y be its classifying arrow. Let $f, g : X \to Y$. We interpret $f = g$ by $\delta_Y \circ \langle f, g \rangle$. Thus we have the following pullback.

$$\begin{tikzcd} \{x : X \mid f(x) = g(x)\} \arrow[r] \arrow[d, tail] & \mathbf{1} \arrow[d, tail, "\text{true}"] \\ X \arrow[r, "f=g"'] & \Omega \end{tikzcd}$$

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$$\begin{tikzcd} \{x : X \mid f(x) = g(x)\} \arrow[r] \arrow[d, tail] & \mathbf{1} \arrow[d, tail, "\mathrm{true}"] \\ X \arrow[r, "f=g"'] & \Omega \end{tikzcd}$$

Theorem

The following diagram is an equalizer.

$$\{x : X \mid f(x) = g(x)\} \rightarrowtail X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

Validity of a formula

Definition (Validity)

Let $\sigma : X \to \Omega$. We say that σ is *valid* whenever σ factors through $\mathrm{true} : \mathbf{1} \to \Omega$.

$$\begin{tikzcd} \mathbf{1} \arrow[dr, tail, "\mathrm{true}"] & \\ X \arrow[u, "!"] \arrow[r, "\sigma"'] & \Omega \end{tikzcd}$$

In that case, we write $X \vdash \sigma$. If σ is a closed formula, then we write $\vdash \sigma$ and this amounts to say that $\sigma = \mathrm{true}$.

Characterization of validity

Let X be a type and $\sigma : X \to \Omega$. The following are equivalent.

1. $X \vdash \sigma$
2. The subobject $\{x : X \mid \sigma(x)\} \rightarrowtail X$ admits a section
3. $\{x : X \mid \sigma(x)\} \simeq X$

Kripke-Joyal forcing

Definition (Forcing)

Let $\sigma : X \to \Omega$ be a formula and $x : \mathbf{y}c \to X$. We say that x *forces* σ *at stage* c, written $c \Vdash \sigma(x)$, if the following dotted arrow exists, making the left triangle commute.

$$\begin{tikzcd} & \{x : X \mid \sigma\} \arrow[r] \arrow[d, tail, "s"] & \mathbf{1} \arrow[d, tail, "\mathrm{true}"] \\ \mathbf{y}c \arrow[ur, dotted] \arrow[r, "x"'] & X \arrow[r, "\sigma"'] & \Omega \end{tikzcd}$$

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\mathbf{y}c \arrow[ur, dotted] \arrow[r, "x"'] & X \arrow[r, "\sigma"'] & \Omega
\end{tikzcd}$$

Theorem

Let $\sigma : X \to \Omega$. $X \vdash \sigma$ *if and only if* $c \Vdash \sigma(x)$ *for all* $x : \mathbf{y}c \to X$.

More theorems

Theorem (Monotonicity)

Let $\sigma : X \to \Omega$ *be a formula and* $x : \mathbf{y}c \to X$ *such that* $c \Vdash \sigma(x)$, *then for all* $f : b \to c$, $b \Vdash \sigma(xf)$.

More theorems

Theorem (Monotonicity)

Let $\sigma : X \to \Omega$ *be a formula and* $x : \mathbf{y}c \to X$ *such that* $c \Vdash \sigma(x)$*, then for all* $f : b \to c$*,* $b \Vdash \sigma(xf)$*.*

Theorem (Forcing with a terminal object)

If $\mathcal{C}$ *has a terminal object* $t \in \mathcal{C}$*, then a closed formula* $\sigma : \mathbf{1} \to \Omega$ *is valid if and only if* $t \Vdash \sigma$*.*

The main theorem

Theorem (Conditions for forcing)

Let $\sigma, \tau : X \to \Omega$, $\theta : Y \times X \to \Omega$ *and* $x : \mathbf{y}c \to X$, *then*

- $c \Vdash \bot$ *never*
- $c \Vdash \top$ *always*
- $c \Vdash \sigma(x) \wedge \tau(x)$ *if and only if* $c \Vdash \sigma(x)$ *and* $c \Vdash \tau(x)$
- $c \Vdash \sigma(x) \vee \tau(x)$ *if and only if* $c \Vdash \sigma(x)$ *or* $c \Vdash \tau(x)$
- $c \Vdash \sigma(x) \Rightarrow \tau(x)$ *if and only if for all* $f : d \to c$, $d \Vdash \sigma(xf)$ *implies* $d \Vdash \tau(xf)$
- $c \Vdash \neg\sigma(x)$ *if and only if for all* $f : d \to c$, *we do not have* $d \Vdash \sigma(xf)$
- $c \Vdash \exists y : Y,\ \theta(y, x)$ *if and only if* $c \Vdash \theta(y, x)$ *for some* $y : \mathbf{y}c \to Y$
- $c \Vdash \forall y : Y,\ \theta(y, x)$ *if and only if* $d \Vdash \theta(y, xf)$ *for all* $f : d \to c$ *and* $y : \mathbf{y}d \to Y$

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- $c \Vdash \sigma(x) \wedge \tau(x)$ *if and only if* $c \Vdash \sigma(x)$ *and* $c \Vdash \tau(x)$
- $c \Vdash \sigma(x) \vee \tau(x)$ *if and only if* $c \Vdash \sigma(x)$ *or* $c \Vdash \tau(x)$
- $c \Vdash \sigma(x) \Rightarrow \tau(x)$ *if and only if for all* $f : d \to c$, $d \Vdash \sigma(xf)$ *implies* $d \Vdash \tau(xf)$
- $c \Vdash \neg\sigma(x)$ *if and only if for all* $f : d \to c$, *we do not have* $d \Vdash \sigma(xf)$
- $c \Vdash \exists y : Y,\ \theta(y, x)$ *if and only if* $c \Vdash \theta(y, x)$ *for some* $y : \mathbf{y}c \to Y$
- $c \Vdash \forall y : Y,\ \theta(y, x)$ *if and only if* $d \Vdash \theta(y, xf)$ *for all* $f : d \to c$ *and* $y : \mathbf{y}d \to Y$

We can recursively unwind the condition $c \Vdash \sigma(x)$

Forcing for equality

Theorem (Forcing for equality)

Let $f, g : X \to Y$ and $x : \mathbf{y}c \to X$, then $c \Vdash f(x) = g(x)$ if and only if $f \circ x = g \circ x$ as maps of $\mathcal{E}$.

The proof uses the equalizer characterization of equality:

$$\begin{tikzcd} \{x : X \mid f(x) = g(x)\} \arrow[r] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ & \mathbf{y}c \arrow[u, "x"] \arrow[ul, dotted] & \end{tikzcd}$$

Martin-Löf Type Theory inside a presheaf category

Overview

We will follow [AGH21] to define small maps, and the small map classifier π given by a Hofmann-Streicher lifting of a Grothendieck universe.

- We fix κ a (strongly) inaccessible cardinal.
- A set *small* if it has cardinality less than κ.
- We write Sets_κ for the full subcategory of Sets consisting of small sets (which is a Grothendieck universe).
- We fix a small (in the above sense) category $\mathcal{C}$.
- We call $\mathcal{E}$ the associated presheaf topos.

Hofmann-Streicher lifting

Definition (Small maps)

1. We say that a presheaf $A \in \mathcal{E}$ is *small* if $A(c)$ is a small set, for all $c \in \mathcal{C}$
2. We say that $p : A \to X$ in $\mathcal{E}$ is a *small map* if, for every $x : \mathbf{y}c \to X$, the presheaf A_x obtained by the pullback

$$\begin{tikzcd} A_x \arrow[r] \arrow[d] & A \arrow[d, "p"] \\ \mathbf{y}c \arrow[r, "x"'] & X \end{tikzcd}$$

   is small.

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is small.

Class of small maps

We call $\mathcal{S}$ the class of small maps in $\mathcal{E}$. In the same way that we can classify the monos $S \rightarrowtail X$ with the map $\mathrm{true} : \mathbf{1} \to \Omega$, we define $\pi : E \to U$ that classifies the maps of $\mathcal{S}$.

Classification

Given a small map $p : A \to X$, there exists a pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "p"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "c_p"'] & U \end{tikzcd}$$

We say that p is *classified* by c_p. Conversely, we introduce a canonical pullback p_A for each $A : X \to U$:

$$\begin{tikzcd} X.A \arrow[r] \arrow[d, "p_A"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "A"'] & U \end{tikzcd}$$

The map p_A is called the *display map* of A.

Small maps and subobject classifier

Theorem

For all $X \in \mathcal{E}$, we have

$$\mathcal{E}(X, U) \simeq \{u : (\textstyle\int X)^{\mathrm{op}} \to \mathcal{S}\}$$

Similarity with Ω

We call $\mathrm{sf}(X) = \{u : (\int X)^{\mathrm{op}} \to \mathcal{S}\}$ the small families over X, then we have

$$\mathrm{sf}(X) \simeq \mathcal{E}(X, U)$$

and with the subobject classifier, we have

$$\mathrm{Sub}(X) \simeq \mathcal{E}(X, \Omega)$$

Ω is a type of proposition

There is an inclusion maps $\{\text{-}\} : \Omega \to U$ which is a monomorphism and fits into a pullback of the form:

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[d, tail, "\mathsf{true}"'] & E \arrow[d, "\pi"] \\ \Omega \arrow[r, tail, "\{\text{-}\}"'] & U \end{tikzcd}$$

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Conversely, by considering the image of the universal map $\mathrm{im}(\pi) \rightarrowtail U$, and $\{\mathrm{supp}(\text{-})\} : U \to U$ is the propositional truncation.

$$\begin{tikzcd} \mathrm{im}(\pi) \arrow[r] \arrow[d, tail] & \mathbf{1} \arrow[d, tail, "\mathsf{true}"] \\ U \arrow[r, "\mathsf{supp}"'] & \Omega \end{tikzcd}$$

Category with families

Definition (Category with families)

The *category with families* associated to any presheaf topos $\mathcal{E}$, is defined as follows:

- The contexts are the objects $X \in \mathcal{E}$
- A type A in context X is a map $A : X \to U$
- A term $a : A$ in context X is a map $a : X \to E$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, equal] & E \arrow[d, "\pi"] \\ X \arrow[r, "A"'] & U \end{tikzcd}$$

- Definitional equality on terms or type is defined via equality of maps in the topos. For instance, we have $X \vdash A = B$ if and only if $A : X \to U$ and $B : X \to U$ are the same maps in $\mathcal{E}$. For terms, we write $X \vdash a = b : A$.

Some results

Theorem (Closed types are presheaves)

Closed types $A : \mathbf{1} \to U$ correspond, up to isomorphism, to small presheaves.

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Closed types $A : \mathbf{1} \to U$ correspond, up to isomorphism, to small presheaves.

Theorem (Display maps and sections)

Let $A : \Gamma \to U$ be a type in context Γ. The elements $\Gamma \vdash a : A$ are in bijective correspondence with the sections of the display map $p_A : \Gamma.A \to \Gamma$.

Type theory of a category with family

Substitution

If Δ, Γ are two contexts, and $A : \Gamma \to U$ a type in context Γ, then we define the substitution $A[t] = A \circ t$, which allows us to have the following substitution rule valid in our type theory.

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A \vdash b(x) : B(x)}{\Gamma \vdash b(a) : B(a)}$$

Type theory of a category with family

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Internal type theory

It is possible to internalize every basic construction of dependent type theory, and their formation, introduction, elimination, computation, and expansion rules [Awo18].

Extending the forcing

Definition

Let $A : X \to U$ be a type in context X, and $x : \mathbf{y}c \to X$. For $a : \mathbf{y}c \to E$, We say c *forces* $a : A(x)$ written $c \Vdash a : A(x)$ if the following diagram commutes.

$$\begin{tikzcd} \mathbf{y}c \arrow[r, "a"] \arrow[d, "x"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "A"'] & U \end{tikzcd}$$

Like in the standard forcing, $c \Vdash a : A(x)$ is to say $\mathbf{y}c \vdash a : A(x)$:

$$\begin{tikzcd} \mathbf{y}c \arrow[rr, "a"] \arrow[d, equal] & & E \arrow[d, "\pi"] \\ \mathbf{y}c \arrow[r, "x"'] & X \arrow[r, "A"'] & U \end{tikzcd}$$

Theorem for the forcing

Theorem

Let $A : X \to U$ be a type in context X, and $x : \mathbf{y}c \to X$. An element $a : \mathbf{y}c \to E$ is the same thing as the dotted arrow in the following diagram:

$$
\begin{tikzcd}
 & X.A \arrow[r, "q_A"] \arrow[d, "p_A"'] & E \arrow[d, "\pi"] \\
\mathbf{y}c \arrow[r, "x"'] \arrow[ur, dotted, "u"] & X \arrow[r, "A"'] & U
\end{tikzcd}
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$$\begin{tikzcd} & X.A \arrow[r, "q_A"] \arrow[d, "p_A"'] & E \arrow[d, "\pi"] \\ \mathbf{y}c \arrow[ur, dotted, "u"] \arrow[r, "x"'] & X \arrow[r, "A"'] & U \end{tikzcd}$$

Theorem

The data of $a : X \to E$ such that $X \vdash a : A$ is the same as families of elements $a_x : \mathbf{y}c \to E$ such that $c \Vdash a_x : A(x)$, and are uniform in the sense that $c \Vdash a_x = a(x) : A(x)$.

Main theorem

Theorem (Condition for forcing)

1. $c \Vdash a : 0$ *never*
2. $c \Vdash a : 1$ *for a unique* $a = \star : \mathbf{y}c \to E$
3. $c \Vdash t : (A \times B)(x)$ *if and only if* $c \Vdash \pi_1(t) : A(x)$ *and* $c \Vdash \pi_2(t) : B(x)$
4. $c \Vdash t : (A + B)(x)$ *if and only if* $c \Vdash a : A(x)$ *with* $t = \mathrm{inl}(a)$ *or* $c \Vdash b : B(x)$ *with* $t = \mathrm{inr}(a)$
5. *We also have conditions for dependent sums and products.*

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5. *We also have conditions for dependent sums and products.*

Theorem (Relation the old forcing)

Let $\sigma : X \to \Omega$ *be a proposition and* $x : \mathbf{y}c \to X$*. Then the following are equivalent.*

(i) $c \Vdash_{std} \sigma(x)$

(ii) $c \Vdash s : \sigma(x)$ *for a (necessarily unique)* $s : \mathbf{y}c \to E$

A model of cubical type theory

Cubical presheaves

- We introduce the category $\square$. It will be the base category of a presheaf topos whose internal type theory will model cubical type theory.
- The category $\square$ is the one from [CCHM15].
- Its objects are free de Morgan algebras, whose structure will be helpful to obtain an object $\mathbb{I}$.
- Then, we introduce the notion of cofibration, whose behavior is important to internalize Kan filling and glueing [OP16].

The box category

For $n \geq 0$, we denote by I_n the free de Morgan algebra on n generators.

Definition ($\square$)

We call $\square$ the category having as objects cardinal numbers $[n] \geq 0$ and as morphisms in $\square([n],[m])$ the de Morgan homomorphisms $f : I_m \to I_n$.

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1. We specialize the previous sections to the case where $\mathcal{C} = \square$, so $\mathcal{E}$ is the presheaf topos $[\square^{\mathrm{op}}, \mathrm{Sets}]$.
2. $X_n \triangleq X([n])$.
3. $\mathbb{I} \triangleq \mathbf{y}[1]$.
4. $\mathbb{I}_n \triangleq \square([n],[1])$ has a de Morgan structure defined pointwise.
5. We call $\sqcap_n, \sqcup_n, 0_n, 1_n$ the product, sum, zero, and one of this de Morgan algebra (that we can collect into natural transformations)

Results about $\square$

Theorem (Behavior of $\mathbb{I}$)

(i) $0 \neq 1$

(ii) $\forall x : \mathbf{y}[n] \to \mathbb{I}$, *we have* $0 \sqcap x = 0 = x \sqcap 0$ *and* $1 \sqcap x = x = x \sqcap 1$

(iii) $\forall x : \mathbf{y}[n] \to \mathbb{I}$, *we have* $0 \sqcup x = x = x \sqcup 0$ *and* $1 \sqcup x = 1 = x \sqcup 1$

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Theorem

$\square$ *has finite products.*

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Theorem

$\square$ *has finite products.*

Theorem

For all $[n] \in \square$, $\mathbb{I}_n$ *has decidable equality.*

Cofibrations

Idea

Cofibrations are a way to internalize composition and filling introduced in [CCHM15]. We assume a map $\mathsf{cof} : \Omega \to \Omega$ and we consider the associated subobject

$$\mathtt{Cof} = \{\varphi : \Omega \mid \mathsf{cof}\,\varphi\}$$

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Cofibrations are a way to internalize composition and filling introduced in [CCHM15]. We assume a map $\mathsf{cof} : \Omega \to \Omega$ and we consider the associated subobject

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Definition (Cofibration)

A *cofibration* is a monomorphism whose classifying arrow factors trough $\texttt{Cof} \rightarrowtail \Omega$.

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Definition (Cofibration)

A *cofibration* is a monomorphism whose classifying arrow factors trough $\mathtt{Cof} \rightarrowtail \Omega$.

Theorem

Let $\varphi : X \to \Omega$ be a proposition. For every $x : \mathbf{y}c \to X$, the following are equivalent.

1. $c \Vdash \mathsf{cof}\,\varphi(x)$
2. $\varphi \circ x : \mathbf{y}c \to \Omega$ *factor through* $\mathtt{Cof} \to \Omega$

Axiom to model CTT

Consider the following nine axioms:

ax$_1$ $\forall \varphi : \mathbb{I} \to \Omega,\ (\forall i : \mathbb{I},\ \varphi\ i \vee \neg\varphi\ i) \Rightarrow (\forall i : \mathbb{I},\ \varphi\ i) \vee (\forall i : \mathbb{I},\ \neg\varphi\ i)$

ax$_2$ $\neg(0 = 1)$

ax$_3$ $\forall i : \mathbb{I},\ 0 \sqcap i = 0 = i \sqcap 0\ \wedge\ 1 \sqcap i = i = i \sqcap 1$

ax$_4$ $\forall i : \mathbb{I},\ 0 \sqcup i = i = i \sqcup 0\ \wedge\ 1 \sqcup i = 1 = i \sqcup 1$

ax$_5$ $\forall i : \mathbb{I},\ \mathsf{cof}(i = 0) \wedge \mathsf{cof}(i = 1)$

ax$_6$ $\forall \varphi\ \psi : \Omega,\ \mathsf{cof}\ \varphi \Rightarrow \mathsf{cof}\ \psi \Rightarrow \mathsf{cof}(\varphi \vee \psi)$

ax$_7$ $\forall \varphi\ \psi : \Omega,\ \mathsf{cof}\ \varphi \Rightarrow (\varphi \Rightarrow \mathsf{cof}\ \psi) \Rightarrow \mathsf{cof}(\varphi \wedge \psi)$

ax$_8$ $\forall \varphi : \mathbb{I} \to \Omega,\ (\forall i : \mathbb{I},\ \mathsf{cof}\ \varphi\ i) \Rightarrow \mathsf{cof}(\forall i : \mathbb{I},\ \varphi\ i)$

ax$_9$ $(\varphi : \mathtt{Cof})(A : [\varphi] \to U)(B : U)(s : (u : [\varphi]) \to (A\ u \simeq B)) \to (B' : U) \times \{s' : B' \simeq B \mid \forall u : [\varphi],\ A\ u = B' \wedge s\ u = s'\}$

Axioms to model CTT

Theorem ([OP16])

The category with families of a presheaf topos is a model of cubical type theory if its internal logic satisfies the nine axioms above.

The axioms, detailed

ax$_1$ $\forall \varphi : \mathbb{I} \to \Omega,\ (\forall i : \mathbb{I},\ \varphi\ i \vee \neg \varphi\ i) \Rightarrow (\forall i : \mathbb{I},\ \varphi\ i) \vee (\forall i : \mathbb{I},\ \neg \varphi\ i)$

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ax₅ $\forall i : \mathbb{I},\ \mathsf{cof}(i = 0) \wedge \mathsf{cof}(i = 1)$

ax₆ $\forall \varphi\ \psi : \Omega,\ \mathsf{cof}\ \varphi \Rightarrow \mathsf{cof}\ \psi \Rightarrow \mathsf{cof}(\varphi \vee \psi)$

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ax₈ $\forall \varphi : \mathbb{I} \to \Omega,\ (\forall i : \mathbb{I},\ \mathsf{cof}\ \varphi\ i) \Rightarrow \mathsf{cof}(\forall i : \mathbb{I},\ \varphi\ i)$

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Kripke-Joyal forcing to prove the axioms

1. To prove each of these axioms, we can use Kripke-Joyal forcing and unravel each formula.
2. Since $\square$ has a terminal object $[0]$, it suffices to prove that each axiom is forced at stage $[0]$. That is, for $\mathbf{k} = \mathbf{1}, \ldots, \mathbf{9}$, we have

$$\vdash \mathbf{ax_k} \iff [0] \Vdash \mathbf{ax_k}$$

Axioms for the interval

Theorem ($\mathbf{ax_2}$)

$[0] \Vdash \neg(0 = 1)$

It suffices to show that for all $[n]$, we do not have $[n] \Vdash 0 = 1$. Assume $[n] \Vdash 0 = 1$, then we would have $0 = 1 : \mathbf{y}[n] \to \mathbb{I}$, which is false as $0_n \neq 1_n$.

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Theorem (ax$_3$)

$[0] \Vdash (\forall i : \mathbb{I}),\ 0 \sqcap i = 0 = i \sqcap 0 \ \wedge\ 1 \sqcap i = i = i \sqcap 1$

Call $\sigma(i) = 0 \sqcap i = 0 = i \sqcap 0 \ \wedge\ 1 \sqcap i = i = i \sqcap 1$. We want to show that $[n] \Vdash \sigma(i)$ for all $f : [n] \to [0]$ and $i : \mathbf{y}[n] \to \mathbb{I}$. Such a map f is unique, thus we need to show that $[n] \Vdash \sigma(i)$, for all $i : \mathbf{y}[n] \to \mathbb{I}$. Unwinding of the $\wedge$ connective, it suffices to prove each equality independently. For instance with $[n] \Vdash 0 \sqcap i = 0$, let $i : \mathbf{y}[n] \to \mathbb{I}$ be a generalized element.It suffices to show that $0 \sqcap i = 0$ as maps of $\mathcal{E}$, and this is true.

The interval is connected

Theorem

Let $\varphi, \psi : \mathbb{I} \to \Omega$ be two formulas. Then the following are equivalent.

(i) $\mathbb{I} \vdash \psi \vee \varphi$

(ii) $\mathbb{I} \vdash \psi$ *or* $\mathbb{I} \vdash \varphi$

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Theorem ($\mathbf{ax_1}$)

$\vdash \forall \varphi : \mathbb{I} \to \Omega,\ (\forall i : \mathbb{I},\ \varphi\, i \vee \neg \varphi\, i) \Rightarrow (\forall i : \mathbb{I},\ \varphi\, i) \vee (\forall i : \mathbb{I},\ \neg \varphi\, i)$

Axioms for cofibrations

For the cofibration, we prove alternative characterization of the axioms.

Theorem (Characterization for cofibrations)

If the class $\mathtt{Cof} \in \mathcal{E}$ *of cofibrant propositions is such that*

- $\{i : \mathbb{I} \mid i = 0\} \rightarrowtail \mathbb{I}$ *is a cofibration*
- $\{i : \mathbb{I} \mid i = 1\} \rightarrowtail \mathbb{I}$ *is a cofibration*
- *If* $f : X \rightarrowtail Y$ *and* $g : Y \rightarrowtail Z$ *are cofibrations, then* $g \circ f$ *is a cofibration*
- *If* $f : X \rightarrowtail Y$ *is a cofibration, then* $f^{\mathbb{I}} : X^{\mathbb{I}} \rightarrowtail Y^{\mathbb{I}}$ *is a cofibration*

then $\vdash \mathbf{ax_k}$, *for* $\mathbf{k} = \mathbf{5}, \mathbf{7}, \mathbf{8}$

The axiom of strictification

Definition (Decidable sieves)

We define $\Omega_{\mathsf{dec}} \rightarrowtail \Omega$ to be the sub-presheaf consisting of decidable sieves, that is

$$\Omega_{\mathsf{dec}}(c) = \{S \in \Omega(c) \mid S \text{ is a decidable subset of } \mathrm{Obj}(\mathcal{C}/c)\}$$

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Theorem

ax$_9$ *is satisfied in* $\mathcal{E}$ *if* $\mathtt{Cof} \rightarrowtail \Omega$ *is contained in* $\Omega_{\mathtt{dec}} \rightarrowtail \Omega$.

Main theorem

Theorem (Model of HoTT)

If $\mathcal{E} = [\square^{\mathrm{op}}, \mathrm{Sets}]$ *with* $\mathbb{I} = \mathbf{y}[1]$ *and* $\mathtt{Cof} = \Omega_{\mathtt{dec}}$, *then* $\vdash \mathbf{ax_k}$ *for* $\mathbf{k} = \mathbf{1}, \ldots, \mathbf{9}$, *thus its internal type theory is a model of cubical type theory with univalence.*

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We did not use the same cofibrations as in [CCHM15]. They use the face lattice $\mathbb{F}$, seen as an object of $\mathcal{E}$, and define the mono $m : \mathbb{F} \rightarrowtail \Omega$ by sending $x \in \mathbb{F}([n])$ to the sieve

$$m_{[n]}(x) = \{\cdot \xrightarrow{f} [n] \mid \mathbb{F}(f)(x) = 1\}$$

and they take $\texttt{Cof} \rightarrowtail \Omega$ to be the subobject given by this monomorphism m.

Conclusion

Summary

- Lack of computational content in e.g. simplicial models.
- In cubical settings, we can compute the univalence axiom, but the syntax of the cubical type theory tends to be technical.
- From the semantic point of view, there is not one *good* category of cubes [Mö21].

Summary

- Lack of computational content in e.g. simplicial models.
- In cubical settings, we can compute the univalence axiom, but the syntax of the cubical type theory tends to be technical.
- From the semantic point of view, there is not one *good* category of cubes [Mö21].
- We focused on the cubes of [CCHM15] and proved that it is indeed a model of this specific cubical type theory.
- We did that thanks to [OP16], with the nine axioms.
- But we needed to answer the question of what it means for an axiom to be true in a topos.
- Thus Kripke-Joyal forcing.
- And Kripke-Joyal forcing in the context of categories with families, as in [AGH21].
- This provides a more systematic approach that could be generalized to various presheaf toposes.

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Thanks!