What is the Yoneda Lemma?

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I started studying category for almost a year now, and one of the fundamental result that still strikes me is the Yoneda Lemma. I know its statement, I use it everyday in my work, but somehow I cannot build any kind of intuition about it. I blame the notion of presheaf that is not yet intuitive to me. Here is a paper where I try to create an detailed understanding of this result and its consequences.

We assume the reader is familiar with the basics notions of category theory (categories, functors, natural transformations). If not, we encourage the reader to go through the first chapter of Emily Riehl's Category Theory in Context (from which this paper is inspired).

1 Presheaves

1.1 How to be a presheaf

Let $\mathcal{C}$ be a locally small category. A *presheaf* is a functor $X : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$. This is the data of:

- for each object $c \in \mathcal{C}$, a set $X(c)$
- for each morphism $f : c \to d$, a set-theoretic map $X(f) : X(d) \to X(c)$

This construction is functorial. It means that when we have

$$\begin{tikzcd} n \arrow[r, "g"] & d \\ b \arrow[u, "f"] \arrow[ur, "g \circ f"'] & \end{tikzcd}$$

then we have

$$\begin{tikzcd} X(c) \arrow[d, "X(f)"'] & X(d) \arrow[l, "X(g)"'] \arrow[dl, "X(g \circ f)"] \\ X(b) & \end{tikzcd}$$

In the case where the category $\mathcal{C}$ is discrete (no morphism besides identities), a presheaf is just a bunch of indexed sets, one for each $c \in \mathcal{C}$. So a good intuition for presheaves is a generalization of indexed sets. Let's strengthen this intuition to the case where the category is a preorder $\mathcal{N}$. To simplify further, suppose $\mathcal{N}$ has object the natural numbers (and so a unique morphism $n \to m$ whenever $n \leq m$). A presheaf $X : \mathcal{N}^{\mathrm{op}} \to \mathrm{Sets}$ is

- a collection of sets $X_0, X_1, X_2, \ldots$, for each natural number $n \in \mathcal{N}$
- a map $f_{n,m} : X_m \to X_n$, whenever $n \leq m$,

Is that it? No, we have the functoriality of the maps too. Let us write it in the special case $n \leq n+1 \leq n+2$, for a chosen $n \in \mathcal{N}$. We have

$$\begin{tikzcd}
n+1 \arrow[r] & n+2 \\
n \arrow[u] \arrow[ur] &
\end{tikzcd}$$

meaning that we have

$$\begin{tikzcd}
X_{n+1} \arrow[d, "f_{n,n+1}"'] & X_{n+2} \arrow[l, "f_{n+1,n+2}"'] \arrow[dl, "f_{n,n+2}"] \\
X_n &
\end{tikzcd}$$

that is

$$f_{n,n+2} = f_{n,n+1} \circ f_{n+1,n+2}$$

More generally, we can prove that for $p \geq 0$, we have

$$f_{n,n+p} = f_{n,n+1} \circ f_{n+1,n+2} \circ \cdots \circ f_{n+(p-1),n+p}$$

What does that tell us? That to specify our presheaf X, it suffices to give the maps $f_{n,n+1} : X_{n+1} \to X_n$, for all $n \geq 0$, and the rest of the maps will follow from the previous decomposition.

Back to the general category $\mathcal{C}$. It is locally small ($\mathcal{C}(c,d)$ is a set for any objects $c, d \in \mathcal{C}$), so for every $d \in \mathcal{C}$, we have the contravariant hom functor $h_d : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ defined as follow.

$$\begin{aligned}
h_d : \mathcal{C}^{\mathrm{op}} &\to \mathrm{Sets} \\
c &\mapsto \mathcal{C}(c,d) \\
f : c \to c' &\mapsto \text{-} \circ f
\end{aligned}$$

To be more explicit, if $f : c \to c'$ in $\mathcal{C}$, then we can define the post-composition function:

$$\begin{aligned}
\text{-} \circ f : \mathcal{C}(c', d) &\to \mathcal{C}(c,d) \\
u &\mapsto u \circ f
\end{aligned}$$

as in the following diagram.

$$\begin{tikzcd}
c \arrow[r, "f"] \arrow[rr, bend right, "u \circ f"'] & c' \arrow[r, "u"] & d
\end{tikzcd}$$

This construction is functorial, this is by associativity of composition.

$$(u \circ f) \circ g = u \circ (f \circ g)$$

The left hand side is $h_d(g) \circ h_d(f)(u)$ and the right hand side is $h_d(f \circ g)(u)$.

The hom functor is a fundamental construction in category theory, and the Yoneda Lemma is all about them. One its consequences is that any presheaves is the gluing of such hom functors, in the same flavor that the rational numbers are *dense* in the real numbers, the class of hom functors

$\{h_d : \mathcal{C}^{\text{op}} \to \text{Sets} \mid d \in \mathcal{C}\}$ is also dense in the presheaves over $\mathcal{C}$. We will see how to prove such a statement later.

Let us compute the hom functor in the case of the preorder $\mathcal{N}$. We fix an $m \in \mathcal{N}$. We have

$$h_m(n) = \mathcal{N}(n,m) = \begin{cases} \{\star\} & \text{if } n \leq m \\ \emptyset & \text{if } n > m \end{cases}$$

In order to understand the maps, let us do a very quick recap about set theory.

- There is exactly one function $\emptyset_X : \emptyset \to X$ for any set X (including $X = \emptyset$), called the empty map, whose graph is $\emptyset$
- There is exactly one function $\star_X : X \to \{\star\}$ for any set X, whose graph is $\{(x, \star) \mid x \in X\}$ (hence the graph is $\emptyset$ whenever $X = \emptyset$).

As we saw previously, it suffices to understand, for every $n \in N$, where the unique map $n < n+1$ in $\mathcal{N}$ is sent by the functor h_m. Let us divide the cases, and recall that $h_m(n < n+1) : h_m(n+1) \to h_m(n)$.

- If $n < n+1 \leq m$, then $h_m(n < n+1) : \{\star\} \to \{\star\}$ is the unique map $\star_\star$ defined above
- If $m = n < n+1$, then $h_m(n < n+1) : \emptyset \to \{\star\}$ is the unique map $\emptyset_\star = \star_\emptyset$ defined above
- If $m < n < n+1$, then $h_m(n < n+1) : \emptyset \to \emptyset$ is the unique map $\emptyset_\emptyset$ defined above

1.2 Morphism of presheaves

Recall that the presheaves on a category $\mathcal{C}$ form a category where a map between $X, Y : \mathcal{C}^{\text{op}} \to \text{Sets}$ is a natural transformation $t : X \to Y$. Such a natural transformation is the data of set-theoretic maps $t_c : X(c) \to Y(c)$, for every $c \in \mathcal{C}$, such that for any arrow $f : c \to d$, the following diagram commutes.

$$\begin{tikzcd}
X(c) \arrow[d, "t_c"'] & X(d) \arrow[l, "X(f)"'] \arrow[d, "t_d"] \\
Y(c) & Y(d) \arrow[l, "Y(f)"]
\end{tikzcd}$$

That is

$$t_c \circ X(f) = Y(f) \circ t_d$$

Specializing to the example of the preorder $\mathcal{N}$, suppose we have a presheaf X which send $n \leq m$ to $f_{n,m}$ and a presheaf Y which send $n \leq m$ to $g_{n,m}$, then a natural transformation t is the data of maps $t_n : X_n \to Y_n$ such that the following infinite diagram commutes.

$$\begin{tikzcd}
X_0 \arrow[d, "t_0"'] & X_1 \arrow[l, "f_{0,1}"'] \arrow[d, "t_1"'] & \cdots & X_{n-1} \arrow[d, "t_{n-1}"'] & X_n \arrow[l, "f_{n-1,n}"'] \arrow[d, "t_n"'] & X_{n+1} \arrow[l, "f_{n,n+1}"'] \arrow[d, "t_{n+1}"'] & \cdots \\
Y_0 & Y_1 \arrow[l, "g_{0,1}"] & \cdots & Y_{n-1} & Y_n \arrow[l, "g_{n-1,n}"] & Y_{n+1} \arrow[l, "g_{n,n+1}"] & \cdots
\end{tikzcd}$$

Now, let us take any $m \in \mathcal{N}$, a presheaf $X : \mathcal{N}^{\mathrm{op}} \to \mathrm{Sets}$ (sending $n \leq m$ to $f_{n,m}$), and let us see what is the data needed to define a natural transformation $t : h_m \to X$. We can rewrite the diagram above and we have

$$
\begin{tikzcd}
h_m(0) \arrow[d, "t_0"] & h_m(1) \arrow[l] \arrow[d, "t_1"] & \cdots & h_m(m-1) \arrow[d, "t_{m-1}"] & h_m(m) \arrow[l] \arrow[d, "t_m"] & h_m(m+1) \arrow[l] \arrow[d, "t_{m+1}"] & \cdots \\
X_0 & X_1 \arrow[l, "f_{0,1}"] & \cdots & X_{m-1} & X_m \arrow[l, "f_{m,m-1}"] & X_{m+1} \arrow[l, "f_{m,m+1}"] & \cdots
\end{tikzcd}
$$

and we can replace the value that we already computed, like in the following.

$$
\begin{tikzcd}
\{\star\} \arrow[d, "t_0"] & \{\star\} \arrow[l, "!_\star"'] \arrow[d, "t_1"] & \cdots & \{\star\} \arrow[d, "t_{m-1}"] & \{\star\} \arrow[l, "!_\star"'] \arrow[d, "t_m"] & \emptyset \arrow[l, "\emptyset_\star"'] \arrow[d, "t_{m+1}"] & \emptyset \arrow[l, "\emptyset_\emptyset"'] \arrow[d, "t_{m+2}"] & \cdots \\
X_0 & X_1 \arrow[l, "f_{0,1}"] & \cdots & X_{m-1} & X_m \arrow[l, "f_{m-1,m}"] & X_{m+1} \arrow[l, "f_{m,m+1}"] & X_{m+2} \arrow[l, "f_{m+1,m+2}"] & \cdots
\end{tikzcd}
$$

We can see that the data of this natural transformation is already determined for $n > m$. Indeed, $t_n : \emptyset \to X_n$ has to be $\emptyset_{X_n}$, because it is the only such map. Thus, the only data needed is $t_n : \{\star\} \to X_n$ for all $n \leq m$. Recall that a function $f : \{\star\} \to X$ is the same thing as an element $x \in X$. This is because f is completely determined by $f(\star) \in X$, and each element of X determine such a function. Thus, instead of maps $t_n : \{\star\} \to X_n$, we will pick points $t_n(\star) = x_n \in X_n$ for every $n \leq m$. Is that it? To define a natural transformation $t : h_m \to X$, do we only need to pick points $x_n \in X_n$ for $n \leq m$? No there is better. Recall that in the previous diagram, all the squares commute by naturality, hence for instance:

$$t_0 \circ !_0 = f_{0,1} \circ t_1$$

so, by applying this identity of functions to the element $\star$, we have

$$t_0 \circ !_0(\star) = f_{0,1} \circ t_1(\star)$$

that is

$$t_0(\star) = f_{0,1}(t_1(\star))$$

That means $x_0 = f_{0,1}(x_1)$. The choice of x_0 is not free, it is conditioned by the one of x_1. Similarly the choice of x_1 boils down to the choice of x_2, which reduces to the one of x_3, ..., until the choice of $x_m = t_m(\star)$. The choice of x_m reduces to nothing because the naturality condition is

$$t_m \circ \emptyset_\star = f_{m,m+1} \circ t_{m+1}$$

but the domain of these function is the empty set! Thus, we just showed that any natural transformation $t : h_m \to X$ is determined by an element $x \in X_n$. Conversely, we can apply the construction detailed above and create a natural transformation $t_x : h_m \to X$ for any $x \in X$. What we just established is the Yoneda Lemma.

Lemma 1 (Yoneda in $\mathcal{N}$)**.** *The natural transformations from h_m to X are in bijective correspondence with the set $X(m)$.*

In fact, there is more than that, such a correspondence is natural both in X and m (we will see more precisely what that means), and the element $x \in X_m$ associated to $t : h_m \to X$ is $t_m(\mathrm{id}_m) \in X_m$. This last fact was a little bit hidden, but true, in the previous example. We have $h_m(m)$ is a singleton, but this element has to be the identity map, as $\mathrm{id}_m \in h_m(m)$.

This is a pretty amazing fact. A natural transformation $t : h_m \to X$ is a priori the data of an infinite number of maps t_n for $n \geq 0$. The Yoneda lemma tells us that the data of a single point in $X(m)$ suffices to determine it completely. It is even more impressive that it is true when $h_m(n)$ are not simply $\emptyset$ or $\{\star\}$, but any sets, with why not infinite cardinals. In the next section, we will prove the Yoneda Lemma in its full generality, and see some of its impressive consequences.