Geometric realisation of diagrammatic sets

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Contents

1 From directed simplicial sets to d-spaces

1.1 Directed simplicial sets

We call Δ the simplex category whose objects are finite linear orders $[n] = \{0 < \cdots < n\}$ and morphisms are order-preserving functions. It is generated by the coface maps $\delta_i \colon [n-1] \to [n]$ and the codegeneracy maps $\sigma_j \colon [n+1] \to [n]$.

Definition 1 (Direction on a simplex): A *direction* on the n-simplex is a reflexive relation $\rightsquigarrow$ on the underlying set of $[n]$.

Definition 2 (Directed simplex): The *directed simplex* category $\mathrm{d}\Delta$ is the category whose

- objects are dependent pairs $([n], \rightsquigarrow)$, where $\rightsquigarrow$ is a direction on $[n]$,
- morphisms from $([n], \rightsquigarrow)$ to $([m], \looparrowright)$ are functions $f \colon [n] \to [m]$ which preserve both the order and the direction, that is, for all $i, j \in [n]$, $i \leq j$ implies $f(i) \leq f(j)$ and $i \rightsquigarrow j$ implies $f(i) \looparrowright f(j)$.

We embed Δ as a full subcategory of $\mathrm{d}\Delta$ by sending $[n]$ to $([n], [n] \times [n])$. Note that this is *not* the same as the definition used in Gylfi's note; we still need to work out a comparison.

As in Δ, we have a factorisation result.

Lemma 3 (Factorisation): Any map $f\colon ([n], \rightsquigarrow) \to ([m], \not\rightsquigarrow)$ factorizes as

$$f = d_{i_1} \dots d_{i_p} s_{j_1} \dots s_{j_q}$$

where the orderings are constructed iteratively, starting from $\rightsquigarrow$, that is:

$$s_{j_k} : ([n], s_{j_{k+1}} \dots s_{j_q}(\rightsquigarrow)) \to ([n-k], s_{j_k} s_{j_{k+1}} \dots s_{j_q}(\rightsquigarrow))$$

and similarly for d_{i_k}, except for $d_{i_1} : ([n], d_{i_2} \dots d_{i_p} s_{j_1} \dots s_{j_q}(\rightsquigarrow)) \to ([m], \not\rightsquigarrow)$ (or for s_{j_1} if $p = 0$).

Proof. (**Lemma 3**) We use the decomposition from the underlying map f in Δ, and by construction, each s_{j_k}, d_{i_k} lift to $\mathrm{d}\Delta$. □

Definition 4 (Directed simplicial set): A *directed simplicial set* is a presheaf on $\mathrm{d}\Delta$. We write **dsSet** for the category of directed simplicial sets with morphisms of presheaves.

1.2 Directed topological spaces

Definition 5 (d-spaces and d-maps): A *d-space* (X, dX) is a topological space together with a set dX of *dipaths* $\gamma\colon [0,1] \to X$ such that

1. *(closure under constant paths)* every constant path is a dipath,
2. *(closure under directed partial reparametrisations)* if $\varphi\colon [0,1] \to [0,1]$ is continuous and order-preserving, then $\gamma \circ \varphi$ is a dipath for all $\gamma \in dX$,
3. *(closure under concatenation)* the concatenation of two dipaths is a dipath.

A *d-map* (or dimap) $f\colon (X, dX) \to (Y, dY)$ is a continuous map such that for all $\gamma \in dX$, $f \circ \gamma \in dY$. With d-maps as morphisms, d-spaces form a category **dTop**.

Given a set of paths, we can consider the smallest set of dipaths that contains it.

Definition 6 (Generated dipaths): Let X be a topological space and Γ a set of paths in X. The *set of dipaths generated by* Γ is the smallest set of paths $dX(\Gamma)$ containing Γ and closed under constant paths, directed partial reparametrisation, and concatenation.

Definition 7 (Piecewise partial Γ-paths): Let X be a topological space and Γ a set of paths in X. A path $\gamma\colon [0,1] \to X$ is a *piecewise partial Γ-path* if there exist

1. a finite subdivision $(0 = a_0 < \dots < a_{n+1} = 1)$ of $[0,1]$,
2. a family of order-preserving continuous maps $(\varphi_i\colon [a_i, a_{i+1}] \to [0,1])_{i=0}^n$, and
3. a family of paths $(\gamma_i\colon [0,1] \to X)_{i=0}^n$ in Γ,

such that, for all $i \in \{0, \dots, n\}$, $\gamma|_{[a_i, a_{i+1}]} = \gamma_i \circ \varphi_i$.
We write Γ_{pp} for the set of piecewise partial Γ-paths.

Lemma 8 (Characterisation of generated dipaths): Let X be a topological space and Γ a set of paths in X. Let

$$\neg\Gamma := \{x! \mid x \in X \setminus \bigcup_{\gamma\in\Gamma} \gamma[0,1]\},$$

where $x!$ is the constant path at x. Then

$$dX(\Gamma) = \Gamma_{pp} + \neg\Gamma.$$

Proof. (**Lemma 8**) It suffices to show that

1. $\Gamma_{pp} + \neg\Gamma$ is generated by Γ under constant paths, directed partial reparametrisation, and concatenation, and
2. it is itself closed under these operations.

The first point is trivial for $\neg\Gamma$, since it consists only of constant paths. Let $\gamma \in \Gamma_{pp}$, and let $(a_i)_{i=0}^{n+1}, (\varphi_i)_{i=0}^{n}, (\gamma_i \in \Gamma)_{i=0}^{n}$ be given as in **Definition 7**. Then γ can be obtained by

1. precomposing each γ_i with $t \mapsto \varphi_i(a_i + (a_{i+1} - a_i)t)$;
2. concatenating all the resulting paths;
3. reparametrising piecewise linearly in such a way that the i-th factor in the concatenation is defined on $[a_i, a_{i+1}]$.

Next, we prove that every constant path $x!$ is contained in $\Gamma_{pp} + \neg\Gamma$. If x is not in the image of any $\gamma \in \Gamma$, then by definition $x! \in \neg\Gamma$. Otherwise, there exist $\gamma \in \Gamma$ and $t_x \in [0,1]$ such that $x = \gamma(t_x)$. It follows that $x!$ is equal to the partial piecewise Γ-path obtained from

$$(a_0 = 0 < 1 = a_1), \quad \varphi_0 \colon t \mapsto t_x, \quad \gamma_0 := \gamma.$$

Next, we show that both Γ_{pp} and $\neg\Gamma$ are separately closed under directed partial reparametrisation. For $\neg\Gamma$, this is immediate. Let $\gamma \in \Gamma_{pp}$ be obtained from $(a_i)_{i=0}^{n+1}, (\varphi_i)_{i=0}^{n}, (\gamma_i \in \Gamma)_{i=0}^{n}$, and let $\psi \colon [0,1] \to [0,1]$ be a directed partial reparametrisation. Let

$$b_i := \begin{cases} 0 & \text{if } a_i \le \psi(0) \ , \\ \min \psi^{-1}(a_i) & \text{if } \psi(0) < a_i < \psi(1) \ , \\ 1 & \text{if } \psi(1) \le a_i \ . \end{cases}$$

This is well-defined since $\psi^{-1}(a_i)$ is closed and bounded, so it has a minimum whenever it is non-empty, which is always true when $a_i \in [\psi(0), \psi(1)]$ because ψ is order-preserving and continuous. Moreover $0 = b_0 \le \ldots \le b_{n+1} = 1$. We define, recursively on $i \le m := |\{b_0, \ldots, b_{n+1}\}| - 1$,

$$\begin{aligned} k(0) &:= 0, \\ k(i+1) &:= \min\{j \mid b_j > b_{k(i)}\}, \end{aligned}$$

and finally

$$c_i := b_{k(i)}, \quad i \in \{0, \ldots, m+1\}$$

where by construction $0 = c_0 < \ldots < c_{m+1} = 1$. Then $\gamma \circ \psi$ is obtained from

$$(c_i)_{i=0}^{m+1}, \quad (\varphi_{k(i)} \circ \psi|_{[c_i, c_{i+1}]})_{i=0}^{m}, \quad (\gamma_{k(i)})_{i=1}^{m+1}.$$

Finally, $\neg\Gamma$ is trivially closed under concatenation, and it is straightforward to show the same for Γ_{pp}. To conclude, it suffices to observe that no endpoint of a path in Γ_{pp} can match an endpoint of a path in $\neg\Gamma$, so no other concatenation is possible. □

Lemma 9 (Maps preserving generating paths): Let $f\colon X \to Y$ be a continuous map, Γ a set of paths in X and Θ a set of paths in Y. Suppose that, for all $\gamma \in \Gamma$, we have $f \circ \gamma \in \Theta$. Then $f\colon (X, dX(\Gamma)) \to (Y, dY(\Theta))$ is a dimap.

Proof. Let $\gamma \in dX(\Gamma)$. If γ is constant, then necessarily so is $f \circ \gamma$, and there is nothing to prove. Otherwise, by **Lemma 8**, $\gamma \in \Gamma_{pp}$ is obtained from some $(a_i)_{i=0}^{n+1}, (\varphi_i)_{i=0}^{n}, (\gamma_i \in \Gamma)_{i=0}^{n}$. Then $f \circ \gamma$ is obtained from

$$(a_i)_{i=0}^{n+1}, (\varphi_i)_{i=0}^{n}, (f \circ \gamma_i \in \Theta)_{i=0}^{n},$$

so it is a piecewise partial Θ-path. □

1.3 Geometric realisation of directed simplicial sets

We want to define a left adjoint geometric realisation functor $|-|_d\colon \mathbf{dsSet} \to \mathbf{dTop}$. Because $\mathbf{dTop}$ is cocomplete and $\mathbf{dsSet}$ is a category of presheaves on a small category, it suffices to define $|-|_d$ on $\mathrm{d}\Delta$, then take the left Kan extension along its Yoneda embedding.

Construction 10 (Elementary paths): Let $\mathbb{R}^\infty$ be the space of sequences $x = (x_i)_{i\in\mathbb{N}}$ of real numbers such that $x_i = 0$ for all but finitely many indices, with the topology that makes it a colimit of the sequence of inclusions

$$\begin{aligned} \imath_n\colon \mathbb{R}^n &\to \mathbb{R}^{n+1} \\ (x_0, \dots, x_{n-1}) &\mapsto (x_0, \dots, x_{n-1}, 0). \end{aligned}$$

Then $\mathbb{R}^\infty$ has a structure of topological real vector space, with

$$x + y := (x_i + y_i)_{i\in\mathbb{N}}, \qquad rx := (rx_i)_{i\in\mathbb{N}}$$

for all $x, y \in \mathbb{R}^\infty$ and $r \in \mathbb{R}$.
For each $i \in \mathbb{N}$, we let e_i be the point of $\mathbb{R}^\infty$ defined by

$$e_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

For each $i, j \in \mathbb{N}$, we let $\gamma_{ij}\colon [0,1] \to \mathbb{R}^\infty$ be the path defined by

$$t \mapsto (1-t)e_i + te_j.$$

Notice that $\gamma_{ij}(0) = e_i$ and $\gamma_{ij}(1) = e_j$.

Definition 11 (Geometric realisation of directed simplicial sets): The *geometric realisation* of a directed simplex $([n], \rightsquigarrow)$ is the d-space $(\Delta^n, d\Delta^n(\Gamma_\rightsquigarrow))$ where

- Δ^n is the convex hull spanned by $e_0, \dots, e_n$ in $\mathbb{R}^\infty$, that is, the closed subspace

$$\{\sum_{i=0}^{n} c_i e_i \mid c_i \geq 0, \sum_i c_i = 1\},$$

- $\Gamma_{\rightsquigarrow}$ is the set of paths

$$\{t \mapsto \sum_{i \rightsquigarrow j} c_{ij}\gamma_{ij}(t) \mid c_{ij} \geq 0, \sum_{i \rightsquigarrow j} c_{ij} = 1\}.$$

This extends to a a functor $|-|_d \colon \mathrm{d}\Delta \to \mathbf{dTop}$, sending $f \colon ([n], \rightsquigarrow) \to ([m], \looparrowright)$ to the continuous map

$$|f|_d \colon \sum_{i=0}^{n} c_i e_i \mapsto \sum_{i=0}^{n} c_i e_{f(i)} = \sum_{j=0}^{m} c'_j e_j,$$

where

$$c'_j := \sum_{j=f(i)} c_i.$$

The *geometric realisation of directed simplicial sets* is the functor $|-|_d \colon \mathbf{dsSet} \to \mathbf{dTop}$ obtained by taking the left Kan extension of $|-|_d \colon \mathrm{d}\Delta \to \mathbf{dTop}$ along the Yoneda embedding.

Observe that $|f|_d$ can be extended (non-uniquely) to a linear operator on $\mathbb{R}^\infty$.

Lemma 12 (The functor $|-|_d$ is well-defined): The functor $|-|_d$ is well-defined, that is, if $f \colon ([n], \rightsquigarrow) \to ([m], \looparrowright)$ is a map of directed simplices, then $|f|_d$ is a dimap.

Proof. (**Lemma 12**) By **Lemma 9**, it suffices to show that, if $\gamma \in \Gamma_{\rightsquigarrow}$, then $|f|_d \circ \gamma \in \Gamma_{\looparrowright}$. Let γ be the generating path $t \mapsto \sum_{i \rightsquigarrow j} c_{ij}\gamma_{ij}(t)$. Then $|f|_d \circ \gamma$ is, by linearity,

$$\begin{aligned} t \mapsto \sum_{i \rightsquigarrow j} c_{ij} |f|_d(\gamma_{ij}(t)) &= \sum_{i \rightsquigarrow j} c_{ij}((1-t)e_{f(i)} + t e_{f(j)}) = \\ &= \sum_{i \rightsquigarrow j} c_{ij}\gamma_{f(i)f(j)}(t) = \sum_{k \looparrowright \ell} c'_{k\ell}\gamma_{k\ell}(t), \end{aligned}$$

where

$$c'_{k\ell} := \sum_{i \rightsquigarrow j, k=f(i), \ell=f(j)} c_{ij},$$

which is a path in $\Gamma_{\looparrowright}$. □

Notice that, since the relation $\rightsquigarrow$ is reflexive, $\Gamma_{\rightsquigarrow}$ already contains all constant paths, obtained as convex combinations $t \mapsto \sum_i c_i\gamma_{ii}(t) = \sum_i c_i e_i$.

2 From diagrammatic sets to directed simplicial sets

2.1 Basic definitions

We refer to [Had20] and [HK22] for the basics of oriented graded posets and regular molecules.

Definition 13 (Diagrammatic sets): A *diagrammatic set* is a presheaf on the category $\odot$ whose

- objects are *atoms* U of arbitrary dimension,
- morphisms $f \colon U \to V$ are *maps* of oriented graded posets, that is, functions satisfying

$$f(\partial_n^\alpha x) = \partial_n^\alpha f(x)$$

for all $x \in U, n \in \mathbb{N}, \alpha \in \{+,-\}$.

Diagrammatic sets and their morphisms of presheaves form a category $\odot\mathbf{Set}$.

Definition 14 (Combinatorial computads): A *combinatorial computad* is a presheaf on the wide subcategory $\odot_{in}$ of $\odot$ whose morphisms are *inclusions*, that is, injective maps. Combinatorial computads and their morphisms of presheaves form a category **CComp**.

We will define two realisation functors valued in directed simplicial sets:

1. the first from diagrammatic sets,
2. the second from combinatorial computads.

In both cases, we will do this by first defining a functor from their site of definition, namely, $\odot$ and $\odot_{in}$, respectively, then taking a left Kan extension along the Yoneda embedding. Composing with $|-|_d$ will then produce a realisation functor valued in d-spaces, as in the following picture.

$$
\begin{tikzcd}
 & \mathrm{d}\Delta \arrow[r, "|-|_d"] \arrow[d, hook, "\mathbf{y}"'] & \mathbf{dTop} \\
\odot \arrow[r, dashed, "?"] \arrow[d, hook, "\mathbf{y}"'] & \mathbf{dsSet} \arrow[ur, "|-|_d"'] & \\
\odot\mathbf{Set} \arrow[ur, "\mathrm{Lan}_{\mathbf{y}}(?)"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{d}\Delta \arrow[r, "|-|_d"] \arrow[d, hook, "\mathbf{y}"'] & \mathbf{dTop} \\
\odot_{in} \arrow[r, dashed, "?"] \arrow[d, hook, "\mathbf{y}"'] & \mathbf{dsSet} \arrow[ur, "|-|_d"'] & \\
\mathbf{CComp} \arrow[ur, "\mathrm{Lan}_{\mathbf{y}}(?)"'] & &
\end{tikzcd}
$$

Both of these will have the geometric realisation described in [Had20, Section 8.3] as underlying functor.

Definition 15 (Direction on a poset): A *direction* on a poset P is a reflexive relation $\rightsquigarrow$ on the underlying set of P. We let **dPos** be the category of dependent pairs $(P, \rightsquigarrow)$ where P is a poset and $\rightsquigarrow$ a direction on P, together with maps preserving both the order and the direction.

2.2 Realisation of diagrammatic sets

In this part, we study a different version of the geometric realisation of diagrammatic sets. We will take the same approach as in the previous section, but we will have to deal with surjective maps between molecules, and thus we will need to take more care when constructing the forgetful functor to **prePos**. We notice that we can leave unchanged the **Definition 11** on geometric realisation, as the proof of functoriality equally applies if we add degeneracies. What breaks is the construction of the preorder $\preceq$, that does not behave functorialy as the surjections do not preserve orientations in general. Thus, we will adapt it here.

We aim to construct a functor $\mathcal{U} : \odot \to \mathbf{prePos}$. For that we consider $V \in \odot$, and as usual, we associate it its underlying poset $(\mathcal{H}V, \leq)$. Now, the preorder is the reflexive and transitive closure of the relation given by:

$$x \preceq y \iff x \in \partial_0^- y \text{ or } y \in \partial_0^+ x$$

That is whenever $x \preceq y$ (and $x \neq y$), then x or y is a point (or both by transitivity). Now, a map $f : P \to Q$ of $\odot$ satisfies in particular

$$\partial_0^\alpha f(x) = f(\partial_0^\alpha x)$$

for all $x \in P$, thus if $x \in \partial_0^- y$, then $f(x) \in \partial_0^- f(y)$, making $\mathcal{U}$ a functor $\odot \to \mathbf{prePos}$. We now have the same picture as previously

$$\begin{tikzcd}
\odot\mathbf{Set} \arrow[drrr, bend left, "|\text{-}|_d := \mathrm{Lan}_{\mathbf{y}}(|k_d\text{-}|_{d\Delta})"] & & & \\
& \mathbf{prePos} \arrow[r, "dN"] \arrow[dd, dashed] & \mathbf{dsSet} \arrow[r, "|\text{-}|_{d\Delta}"] \arrow[dd, "\iota^*"] & \mathbf{dTop} \arrow[dd, dashed] \\
\odot \arrow[uu, hook, "\mathbf{y}"] \arrow[dd, hook, "\mathbf{y}"'] \arrow[ur, dashed] \arrow[dr, dashed] \arrow[urr, bend left, "k_d"] \arrow[drr, bend right, "k"'] & & & \\
& \mathbf{Pos} \arrow[r, "N"'] & \mathbf{sSet} \arrow[r, "|\text{-}|_\Delta"'] & \mathbf{Top} \\
\odot\mathbf{Set} \arrow[urrr, bend right, "|\text{-}| := \mathrm{Lan}_{\mathbf{y}}(|k\text{-}|_\Delta)"'] & & &
\end{tikzcd}$$

with the whole category of diagrammatic sets.

2.3 Realisation of combinatorial computads

We call $\odot_i$ the skeleton of the category of combinatorial computad, that is atoms and inclusions. We have the following picture:

$$\begin{tikzcd}
& \mathrm{d}\Delta \arrow[r, "|\text{-}|_{\mathrm{d}\Delta}"] \arrow[d, hook, "\mathbf{y}"'] & \mathbf{dTop} \\
\odot_i \arrow[r, dashed, "?"] \arrow[d, hook, "\mathbf{y}"'] & \mathbf{dsSet} \arrow[ur, "|\text{-}|_{d\Delta}"'] & \\
\odot_i\mathbf{Set} \arrow[ur, "\mathrm{Lan}_{\mathbf{y}}(?)"'] & &
\end{tikzcd}$$

We would like to find the dashed functor, thus giving a geometric realization of a diagrammatic set. For that, recall from [Had20] that we have a functor:

$$k : \odot_i \to \mathbf{sSet}$$

sending a molecule U to the simplicial nerve of its underlying poset. If U was already a simplex, it would correspond to constructing its barycentric subdivision. We would like to extend it to a functor $k_d : \odot_i \to \mathbf{dsSet}$. As $\Delta \subseteq \mathrm{d}\Delta$, we already have a functor $\mathbf{sSet} \to \mathbf{dsSet}$, however doing so would lose all the relevant information about the orientation of the diagram U in $\odot_i$ that we try to convey. Thus, we need to refine the functor k, by adding manually orientations. To that end, we introduce the intermediate category of preordered posets.

Definition 16 (Preordered poset): A *preordered poset* is a triple $(P, \leq, \preceq)$ where $(P, \leq)$ is a poset and $(P, \preceq)$ is a preorder. A map $f : (P, \leq, \preceq) \to (Q, \leq, \preceq)$ is a map respecting both the poset and the preorder structure. This form the category **prePos**.

We have a forgetful functor $\mathbf{prePos} \to \mathbf{Pos}$ and another $\odot_i \to \mathbf{Pos}$. We now describe how to factorize them via another forgetful functor from $\mathcal{U} : \odot_i \to \mathbf{prePos}$. Let V be an atom, we define $\mathcal{U}V := (\mathcal{H}U, \leq, \preceq)$, where the posetal structure is the one of its Hasse diagram, and the preorder is the transitive and reflexive closure of the relation given by:

$$x \preceq y \iff o(x \leftarrow y) = - \text{ or } o(y \leftarrow x) = +$$

As inclusions preserve orientation, this construction is functorial. We may now construct the directed nerve functor $dN : \mathbf{prePos} \to \mathbf{dsSet}$. Let $(X, \leq, \preceq)$ in **prePos**. We define $dN(X)([n], P)$ to be the set of $(n+1)$ chains $(x_0 \leq \cdots \leq x_n)$ (like in the ordinary nerve) that are moreover such that for all $(i, j) \in P$, we have $x_i \preceq x_j$. We call $k_d : \odot_i \to \mathbf{dsSet}$ the functor $dN \circ \mathcal{U}$.

We summarize our constructions with the following diagram where everything commutes. A dashed arrow indicates a forgetful functor.

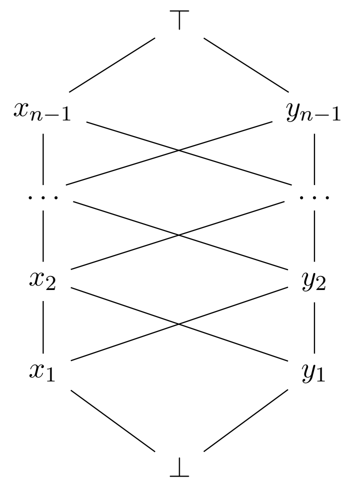
$$\begin{tikzcd}
\odot_i\mathbf{Set} \arrow[drrr, bend left, "{|\text{-}|_d := \mathrm{Lan}_{\mathbf{y}}(|k_d\text{-}|_{d\Delta})}"] & & & \\
& \mathbf{prePos} \arrow[r, "dN"] \arrow[dd, dashed] & \mathbf{dsSet} \arrow[r, "{|\text{-}|_{d\Delta}}"] \arrow[dd, "\iota^*"] & \mathbf{dTop} \arrow[dd, dashed] \\
\odot_i \arrow[uu, "\mathbf{y}"] \arrow[dd, "\mathbf{y}"'] \arrow[ur, dashed] \arrow[dr, dashed] \arrow[urr, bend left, "k_d"] \arrow[drr, bend right, "k"'] & & & \\
& \mathbf{Pos} \arrow[r, "N"'] & \mathbf{sSet} \arrow[r, "{|\text{-}|_{\Delta}}"'] & \mathbf{Top} \\
\odot_i\mathbf{Set} \arrow[urrr, bend right, "{|\text{-}| := \mathrm{Lan}_{\mathbf{y}}(|k\text{-}|_{\Delta})}"'] & & &
\end{tikzcd}$$

3 Examples

We may now work with the latter geometric realisation.

3.1 The cone

Consider the poset C_n:



and preorder $\perp \preceq x_i, y_i$ for all $1 \leq x_i, y_i \leq n-1$. Notice that this is the pre-posetal set $\mathcal{U}(O^n)$ where we remove the output 0-boundary. Abusing notations, we also call C_n the geometrical cone of $\mathbb{R}^{n+1}$ whose base is a $(n-2)$-square, whose apex $\perp$ is at the origin, and whose height is parallel to an axis that we call z. If a point x is in the cone, we call θ the angle between the z axis and the line $\perp x$. This way, each path determines another continuous path $t \mapsto (z(t), \theta(t))$. We endow C_n with a directed topological structure by taking the dipaths dC_n to be all the piecewise-linear paths whose (z, θ) value is weakly increasing.

Proposition 17 (Realisation of the cone): We have

$$|C_n|_d \simeq (C_n, dC_n)$$

Proof. (**Proposition 17**) First, it is the case that this is the realisation as (non-directed) topological spaces, indeed C_n will be the glueing along the correct faces of 2^{n-1} n-simplices $\perp u_1 \dots u_{n-1} \top$ where $u_i \in \{x_i, y_i\}$, each one determined by a path from $\perp$ to $\top$ in the Hasse diagram of the cone. Each of this n simplex share the edge $\top\perp$ and it suffice to check that in each of these simplex, the admissible path are piecewise linear, weakly increasing along the $\top\perp$ edge.

Let $(D, dD) := |\perp u_1 \dots u_{n-1} \top|_d$ be such a directed simplex. To determine its admissible paths, it suffices to determine the maximal orderings P such that for all $(i,j) \in P$, $D_i \preceq D_j$. Indeed, if there is a smaller ordering Q, then it will induce an inclusion of directed topological spaces via $\mathrm{id} : ([n], Q) \to ([n], P)$. By construction of the preorder of the cone, we have

$$P = \mathrm{Diag} \cup \{0i \mid 1 \leq i \leq n\}$$

Thus, dD is the closure under concatenation and reparametrisation of

$$\{\sum_{i=1}^{n} c_i \gamma_{0i} \mid \sum_{i=1}^{n} c_i = 1\}$$

Note that by reparametrisation, one need not to include the constant paths in the sum. These paths are piecewise linear, and indeed each γ_{0i} is weakly increasing along (z, θ).

Conversely, all linear path of increasing value along $\perp\top$ are of the form (by reparametrisation) $t \mapsto tx + (1-t)y$ with $x \leq_{(z,\theta)} y$, and such a path can be written (how?) as a sum $\sum_{i=0}^{n} c_i \gamma_{0i}$. □

3.2 Globe, interval, sphere

Let C'_n be the cone C_n but with preorder $x_i, y_i \preceq \perp$, i.e. $C'_n = (C_n, \leq, \preceq^{\mathrm{op}})$. If C is a preposetal set, then let C^* be the same preposetal set without its bottom element (when it exists). We have in **prePos**:

$$(C'_n)^* \simeq C^*_n$$

The globe O^n is then the pushout of inclusions in **prePos**:

$$\begin{tikzcd} C'^*_n \simeq C^*_n \arrow[r, hook] \arrow[d, hook] & C'_n \arrow[d] \\ C_n \arrow[r] & O^n \end{tikzcd}$$

We call $\perp \preceq x_i, y_i, \top \preceq \perp'$ the elements of O^n. Consider (I^n, dI^n) the directed cube $[0,1]^n$ whose dipath are the piecewise linear weakly increasing ones. In C'_n, the admissible dipaths are the ones whose (z,θ) value is weakly decreasing.

Lemma 18 (Admissible dipaths): A path $\gamma : [0,1] \to I^n$ is in dI^n if and only the map $t \mapsto (z(t), \theta(t))$ is weakly increasing while $z(t) \leq \frac{1}{2}\sqrt{n}$ and the map $t \mapsto (z(t), -\theta(t))$ is weakly increasing while $z(t) \geq \frac{1}{2}\sqrt{n}$.

Proof. (**Lemma 18**) Cut the n-cube $[0,1]^n$ along the bisecting hyperplane of the diagonal $0^n 1^n$ and notice that the bottom part is a directed copy of C^n and the top part is a directed copy of (a symmetric version of) C'_n. □

Corollary 19 (Realisation of the globes): We have

$$|O^n|_d \simeq (I^n, dI^n)$$

Corollary 20 (Realisation of the interval): The diagrammatic interval O^1 is realized as the directed interval $\vec{[0,1]}$ with weakly increasing paths.

Corollary 21 (Realisation of the spheres): The sphere ∂O^n is realized as $\partial \vec{I^n} = [0,1]^n \backslash (0,1)^n$ whose directed paths are the one inherited from $\vec{I^n}$

Proof. (**Corollary 21**) The same applies, except this time we only glue the boundaries of the n simplices (whose interior is then empty). □

3.3 Dihomotopies and reversible paths

A diagram of shape $f : O^1 \to X$ in a diagrammatic set exhibits an admissible dipath $|f| \in dX$, as it is realized as a map $|f| : \vec{I} \to |X|_d$, and we have the isomorphism (of topological spaces) $(|X|_d)^{\vec{I}} \simeq dX$. If $x : 1 \to X$, we allow ourselves to name $x \in |X|_d$ the point determined by the map $|x|_d : 1 \to |X|_d$, thus we have that a diagram of shape $x \xrightarrow{f} y$ in X, it is realized as the concatenation of the paths $|f| = \gamma_{xf} \cdot \gamma_{fy}$.

Definition 22 (Dihomotopies): Let (X, dX) be a directed topological space, and let $f, g : x \to y$ be two admissible dipaths of dX. A *dihomotopy* $\varphi : f \to g$ is a continuous map $\varphi : [0,1] \to dX$ such that $\varphi(0) = f$, $\varphi(1) = g$ and for all $t \in [0,1]$, we have $\varphi(t) \in dX$. If there is a dihomotopy $\varphi : f \to g$, we write $f \sim_d g$. This is an equivalence relation.

A dihomotopy is simply a (particular) homotopy between in the space dX, thus is a morphism $\varphi : \vec{I} \times I \to X$ in **dTop**.

Lemma 23 (Diagrams in the realisation): Let X be a diagrammatic set.

1. A diagram of shape $x \xrightarrow{f} y \xrightarrow{g} z$ in X is realized as the concatenation of the paths in dX determined by the subdiagrams $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$.
2. A diagram of shape

$$\begin{tikzcd} x \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Uparrow\alpha"] & y \end{tikzcd}$$

   in X proves that the paths determined by the subdiagrams $f : O^1 \to X$ and $g : O^1 \to X$ are dihomotopic.

Proof. (**Lemma 23**) The first point is clear as $|O^1 \#_0 O^1|_d = (|O^1|_d \coprod |O^1|_d)/(\partial^+ O^1 = \partial^- O^1)$, thus the diagram of shape $f;g : O^1 \#_0 O^1 \to X$ is indeed the concatenation of two paths and the two inclusions

$$\begin{tikzcd} O^1 \arrow[r, hook] \arrow[dr, "f"'] & O^1 \#_0 O^1 \arrow[d, "f;g"] & O^1 \arrow[l, hook'] \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

ensure the equality of the desired paths.

For the second point, we notice the path $|f|$ is the concatenation of paths $\gamma_{xf} \cdot \gamma_{fy}$, and similarly $|g| = \gamma_{xg} \cdot \gamma_{gy}$. It suffices to show that the linear interpolation of those two paths via $\varphi(t) = t|f| + (1-t)|g|$ is a dihomotopy. To show that $\varphi(t)$ is a dipath for all t, we can notice that we we can write it as the concatenation:

$$\varphi(t) = (t\gamma_{xf} + (1-t)\gamma_{xg}) \cdot (t\gamma_{fy} + (1-t)\gamma_{fy})$$

and for all t, both $t\gamma_{xf} + (1-t)\gamma_{xg}$ and $t\gamma_{fy} + (1-t)\gamma_{fy}$ constitute admissible dipaths by construction. $\square$

We could similarly define the notion of $\vec{I}$-dihomotopy as maps $\vec{I} \times \vec{I} \to X$, but the realisation we are working with will not detect the difference between

$$\begin{tikzcd} x \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Uparrow\alpha"] & y \end{tikzcd}$$

and

$$\begin{tikzcd} x \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\alpha\Downarrow"] & y \end{tikzcd}$$

and would not allow us to interpolate the paths $|f|$ and $|g|$. Indeed, it would require that the paths $t \mapsto t|f|(u) + (1-t)|g|(u)$ are directed for all $0 \leq u \leq 1$, but then taking for instance $u = \frac{1}{2}$, we would end up with the path $\gamma_{f\alpha} \cdot \gamma_{\alpha g}$ that does not belong to the realisation, as we do not have, for instance, $f \preceq \alpha$ or $\alpha \preceq f$.

However, this difference would be detected by the first realisation, and it would be possible to construct $\vec{I}$-dihomotopies $f \to g$ in the first diagram and $g \to f$ in the second.

Definition 24 (Contractible): A path $f : x \to x$ is *contractible* when there exists a dihomotopy $\varphi : f \to x$, where x is the constant path at x.

Proposition 25 (Contractible paths): Let X be a diagrammatic set, then a diagram of shape

$$\begin{tikzcd} x \arrow[rr, "\varepsilon x"] \arrow[dr, "f"'] & {} & x \\ & y \arrow[u, phantom, "\Uparrow"] \arrow[ur, "f^{-1}"'] & \end{tikzcd}$$

in X proves that the path $|f| \cdot |f^{-1}| : x \to x$ is contractible, and in **Top**, we have $|f^{-1}| \sim |f|^{-1}$

Proof. (**Proposition 25**) A diagram of shape $\varepsilon_x = !; x : O^1 \twoheadrightarrow 1 \to X$ exhibit a path of dX, and the commutation $|\varepsilon_x| = !; |x|$ indicates it is constant. By **Lemma 23**, we conclude that the path $|f| \cdot |f^{-1}|$ is dihomotopic to the constant path at x. Moreover:

$$\begin{aligned} & |f| \cdot |f^{-1}| \sim x \\ \Rightarrow & |f|^{-1} \cdot |f| \cdot |f^{-1}| \sim |f|^{-1} \cdot x \\ \Rightarrow & |f^{-1}| \sim |f|^{-1}. \end{aligned}$$

$\square$

References

[Had20] Amar Hadzihasanovic. Diagrammatic sets and rewriting in weak higher categories, July 2020. arXiv:2007.14505 [math].

[HK22] Amar Hadzihasanovic and Diana Kessler. Data structures for topologically sound higher-dimensional diagram rewriting. In *Proceedings Fifth International Conference on Applied Category Theory (ACT2022)*, 2022.