Geometric realisation of diagrammatic sets

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1.1 Directed simplicial sets

We call Δ the simplex category whose objects are finite linear orders $[n] = \{0 < \cdots < n\}$ and morphisms are order-preserving functions. It is generated by the coface maps $\delta_i \colon [n-1] \to [n]$ and the codegeneracy maps $\sigma_j \colon [n+1] \to [n]$.

Definition 1 (Direction on a simplex): A *direction* on the n-simplex is a reflexive relation $\rightsquigarrow$ on the underlying set of $[n]$.

Definition 2 (Directed simplex): The *directed simplex* category $\mathrm{d}\Delta$ is the category whose

- objects are dependent pairs $([n], \rightsquigarrow)$, where $\rightsquigarrow$ is a direction on $[n]$,
- morphisms from $([n], \rightsquigarrow)$ to $([m], \looparrowright)$ are functions $f \colon [n] \to [m]$ which preserve both the order and the direction, that is, for all $i, j \in [n]$, $i \leq j$ implies $f(i) \leq f(j)$ and $i \rightsquigarrow j$ implies $f(i) \looparrowright f(j)$.

We embed Δ as a full subcategory of $\mathrm{d}\Delta$ by sending $[n]$ to $([n], [n] \times [n])$. Note that this is *not* the same as the definition used in Gylfi's note; we still need to work out a comparison.

As in Δ, we have a factorisation result.

Lemma 3 (Factorisation): Any map $f \colon ([n], \rightsquigarrow) \to ([m], \looparrowright)$ factorizes as

$$f = d_{i_1} \ldots d_{i_p} s_{j_1} \ldots s_{j_q}$$

where the orderings are constructed iteratively, starting from $\rightsquigarrow$, that is:

$$s_{j_k} \colon ([n], s_{j_{k+1}} \ldots s_{j_q}(\rightsquigarrow)) \to ([n-k], s_{j_k} s_{j_{k+1}} \ldots s_{j_q}(\rightsquigarrow))$$

and similarly for d_{i_k}, except for $d_{i_1}: ([n], d_{i_2} \dots d_{i_p} s_{j_1} \dots s_{j_q}(\rightsquigarrow)) \to ([m], \nrightarrow)$ (or for s_{j_1} if $p = 0$).

Proof. (**Lemma 3**) We use the decomposition from the underlying map f in Δ, and by construction, each s_{j_k}, d_{i_k} lift to $\mathrm{d}\Delta$. □

Definition 4 (Directed simplicial set): A *directed simplicial set* is a presheaf on $\mathrm{d}\Delta$. We write **dsSet** for the category of directed simplicial sets with morphisms of presheaves.

1.2 Directed topological spaces

Definition 5 (d-spaces and d-maps): A *d-space* (X, dX) is a topological space together with a set dX of *dipaths* $\gamma\colon [0,1] \to X$ such that

1. *(closure under constant paths)* every constant path is a dipath,
2. *(closure under directed partial reparametrisations)* if $\varphi\colon [0,1] \to [0,1]$ is continuous and order-preserving, then $\gamma \circ \varphi$ is a dipath for all $\gamma \in dX$,
3. *(closure under concatenation)* the concatenation of two dipaths is a dipath.

A *d-map* (or dimap) $f\colon (X, dX) \to (Y, dY)$ is a continuous map such that for all $\gamma \in dX$, $f \circ \gamma \in dY$. With d-maps as morphisms, d-spaces form a category **dTop**.

Given a set of paths, we can consider the smallest set of dipaths that contains it.

Definition 6 (Generated dipaths): Let X be a topological space and Γ a set of paths in X. The *set of dipaths generated by* Γ is the smallest set of paths $dX(\Gamma)$ containing Γ and closed under constant paths, directed partial reparametrisation, and concatenation.

Definition 7 (Piecewise partial Γ-paths): Let X be a topological space and Γ a set of paths in X. A path $\gamma\colon [0,1] \to X$ is a *piecewise partial Γ-path* if there exist

1. a finite subdivision $(0 = a_0 < \ldots < a_{n+1} = 1)$ of $[0,1]$,
2. a family of order-preserving continuous maps $(\varphi_i\colon [a_i, a_{i+1}] \to [0,1])_{i=0}^{n}$, and
3. a family of paths $(\gamma_i\colon [0,1] \to X)_{i=0}^{n}$ in Γ,

such that, for all $i \in \{0, \ldots, n\}$, $\gamma|_{[a_i, a_{i+1}]} = \gamma_i \circ \varphi_i$.
We write Γ_{pp} for the set of piecewise partial Γ-paths.

Lemma 8 (Characterisation of generated dipaths): Let X be a topological space and Γ a set of paths in X. Let

$$\neg\Gamma := \{x! \mid x \in X \setminus \bigcup_{\gamma \in \Gamma} \gamma[0,1]\},$$

where $x!$ is the constant path at x. Then

$$dX(\Gamma) = \Gamma_{pp} + \neg\Gamma.$$

Proof. (**Lemma 8**) It suffices to show that

1. $\Gamma_{pp} + \neg\Gamma$ is generated by Γ under constant paths, directed partial reparametrisation, and concatenation, and
2. it is itself closed under these operations.

The first point is trivial for $\neg\Gamma$, since it consists only of constant paths. Let $\gamma \in \Gamma_{pp}$, and let $(a_i)_{i=0}^{n+1}, (\varphi_i)_{i=0}^{n}, (\gamma_i \in \Gamma)_{i=0}^{n}$ be given as in **Definition 7**. Then γ can be obtained by

1. precomposing each γ_i with $t \mapsto \varphi_i(a_i + (a_{i+1} - a_i)t)$;
2. concatenating all the resulting paths;
3. reparametrising piecewise linearly in such a way that the i-th factor in the concatenation is defined on $[a_i, a_{i+1}]$.

Next, we prove that every constant path $x!$ is contained in $\Gamma_{pp} + \neg\Gamma$. If x is not in the image of any $\gamma \in \Gamma$, then by definition $x! \in \neg\Gamma$. Otherwise, there exist $\gamma \in \Gamma$ and $t_x \in [0,1]$ such that $x = \gamma(t_x)$. It follows that $x!$ is equal to the partial piecewise Γ-path obtained from

$$(a_0 = 0 < 1 = a_1), \quad \varphi_0\colon t \mapsto t_x, \quad \gamma_0 := \gamma.$$

Next, we show that both Γ_{pp} and $\neg\Gamma$ are separately closed under directed partial reparametrisation. For $\neg\Gamma$, this is immediate. Let $\gamma \in \Gamma_{pp}$ be obtained from $(a_i)_{i=0}^{n+1}, (\varphi_i)_{i=0}^{n}, (\gamma_i \in \Gamma)_{i=0}^{n}$, and let $\psi\colon [0,1] \to [0,1]$ be a directed partial reparametrisation. Let

$$b_i := \begin{cases} 0 & \text{if } a_i \leq \psi(0) \ , \\ \min \psi^{-1}(a_i) & \text{if } \psi(0) < a_i < \psi(1) \ , \\ 1 & \text{if } \psi(1) \leq a_i \ . \end{cases}$$

This is well-defined since $\psi^{-1}(a_i)$ is closed and bounded, so it has a minimum whenever it is non-empty, which is always true when $a_i \in [\psi(0), \psi(1)]$ because ψ is order-preserving and continuous. Moreover $0 = b_0 \leq \ldots \leq b_{n+1} = 1$. We define, recursively on $i \leq m := |\{b_0, \ldots, b_{n+1}\}| - 1$,

$$k(0) := 0,$$
$$k(i+1) := \min\{j \mid b_j > b_{k(i)}\},$$

and finally

$$c_i := b_{k(i)}, \quad i \in \{0, \ldots, m+1\}$$

where by construction $0 = c_0 < \ldots < c_{m+1} = 1$. Then $\gamma \circ \psi$ is obtained from

$$(c_i)_{i=0}^{m+1}, \quad (\varphi_{k(i)} \circ \psi|_{[c_i, c_{i+1}]})_{i=0}^{m}, \quad (\gamma_{k(i)})_{i=1}^{m+1}.$$

Finally, $\neg\Gamma$ is trivially closed under concatenation, and it is straightforward to show the same for Γ_{pp}. To conclude, it suffices to observe that no endpoint of a path in Γ_{pp} can match an endpoint of a path in $\neg\Gamma$, so no other concatenation is possible. $\square$

Lemma 9 (Maps preserving generating paths): Let $f\colon X \to Y$ be a continuous map, Γ a set of paths in X and Θ a set of paths in Y. Suppose that, for all $\gamma \in \Gamma$, we have $f \circ \gamma \in \Theta$. Then $f\colon (X, dX(\Gamma)) \to (Y, dY(\Theta))$ is a dimap.

Proof. Let $\gamma \in dX(\Gamma)$. If γ is constant, then necessarily so is $f \circ \gamma$, and there is nothing to prove. Otherwise, by **Lemma 8**, $\gamma \in \Gamma_{pp}$ is obtained from some $(a_i)_{i=0}^{n+1}, (\varphi_i)_{i=0}^{n}, (\gamma_i \in \Gamma)_{i=0}^{n}$. Then $f \circ \gamma$ is obtained from

$$(a_i)_{i=0}^{n+1}, (\varphi_i)_{i=0}^{n}, (f \circ \gamma_i \in \Theta)_{i=0}^{n},$$

so it is a piecewise partial Θ-path. $\square$

1.3 Realisation of directed simplicial sets

We want to define a left adjoint geometric realisation functor $|-|_{\mathrm{d}\Delta}\colon \mathbf{dsSet} \to \mathbf{dTop}$. Because $\mathbf{dTop}$ is cocomplete and $\mathbf{dsSet}$ is a category of presheaves on a small category, it suffices to define $|-|_{\mathrm{d}\Delta}$ on $\mathrm{d}\Delta$, then take the left Kan extension along its Yoneda embedding.

Construction 10 (Elementary paths): Let $\mathbb{R}^\infty$ be the space of sequences $x = (x_i)_{i\in\mathbb{N}}$ of real numbers such that $x_i = 0$ for all but finitely many indices, with the topology that makes it a colimit of the sequence of inclusions

$$\begin{aligned} \imath_n\colon \mathbb{R}^n &\to \mathbb{R}^{n+1} \\ (x_0, \ldots, x_{n-1}) &\mapsto (x_0, \ldots, x_{n-1}, 0). \end{aligned}$$

Then $\mathbb{R}^\infty$ has a structure of topological real vector space, with

$$x + y := (x_i + y_i)_{i\in\mathbb{N}}, \qquad rx := (rx_i)_{i\in\mathbb{N}}$$

for all $x, y \in \mathbb{R}^\infty$ and $r \in \mathbb{R}$.
For each $i \in \mathbb{N}$, we let e_i be the point of $\mathbb{R}^\infty$ defined by

$$e_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

For each $i, j \in \mathbb{N}$, we let $\gamma_{ij}\colon [0,1] \to \mathbb{R}^\infty$ be the path defined by

$$t \mapsto (1-t)e_i + te_j.$$

Notice that $\gamma_{ij}(0) = e_i$ and $\gamma_{ij}(1) = e_j$.

Definition 11 (Geometric realisation of directed simplicial sets): The *geometric realisation* of a directed simplex $([n], \rightsquigarrow)$ is the d-space $(\Delta^n, d\Delta^n(\Gamma_\rightsquigarrow))$ where

- Δ^n is the convex hull spanned by $e_0, \ldots, e_n$ in $\mathbb{R}^\infty$, that is, the closed subspace
$$\{\sum_{i=0}^n c_i e_i \mid c_i \geq 0, \sum_i c_i = 1\},$$
- $\Gamma_\rightsquigarrow$ is the set of paths
$$\{t \mapsto \sum_{i\rightsquigarrow j} c_{ij}\gamma_{ij}(t) \mid c_{ij} \geq 0, \sum_{i\rightsquigarrow j} c_{ij} = 1\}.$$

This extends to a a functor $|-|_{\mathrm{d}\Delta}\colon \mathrm{d}\Delta \to \mathbf{dTop}$, sending $f\colon ([n], \rightsquigarrow) \to ([m], \looparrowright)$ to the continuous map

$$|f|_{\mathrm{d}\Delta}\colon \sum_{i=0}^n c_i e_i \mapsto \sum_{i=0}^n c_i e_{f(i)} = \sum_{j=0}^m c'_j e_j,$$

where

$$c'_j := \sum_{j=f(i)} c_i.$$

The *geometric realisation of directed simplicial sets* is the functor $|-|_{\mathrm{d}\Delta}\colon \mathbf{dsSet} \to \mathbf{dTop}$ obtained by taking the left Kan extension of $|-|_{\mathrm{d}\Delta}\colon \mathrm{d}\Delta \to \mathbf{dTop}$ along the Yoneda embedding.

Observe that $|f|_{\mathrm{d}\Delta}$ can be extended (non-uniquely) to a linear operator on $\mathbb{R}^\infty$.

Lemma 12 (The functor $|-|_{\mathrm{d}\Delta}$ is well-defined): The functor $|-|_{\mathrm{d}\Delta}$ is well-defined, that is, if $f\colon ([n], \rightsquigarrow) \to ([m], \looparrowright)$ is a map of directed simplices, then $|f|_{\mathrm{d}\Delta}$ is a dimap.

Proof. (**Lemma 12**) By **Lemma 9**, it suffices to show that, if $\gamma \in \Gamma_{\rightsquigarrow}$, then $|f|_{\mathrm{d}\Delta} \circ \gamma \in \Gamma_{\looparrowright}$. Let γ be the generating path $t \mapsto \sum_{i \rightsquigarrow j} c_{ij}\gamma_{ij}(t)$. Then $|f|_{\mathrm{d}\Delta} \circ \gamma$ is, by linearity,

$$\begin{aligned} t \mapsto \sum_{i \rightsquigarrow j} c_{ij} |f|_{\mathrm{d}\Delta}(\gamma_{ij}(t)) &= \sum_{i \rightsquigarrow j} c_{ij}((1-t)e_{f(i)} + te_{f(j)}) = \\ &= \sum_{i \rightsquigarrow j} c_{ij}\gamma_{f(i)f(j)}(t) = \sum_{k \looparrowright \ell} c'_{k\ell}\gamma_{k\ell}(t), \end{aligned}$$

where

$$c'_{k\ell} := \sum_{i \rightsquigarrow j, k = f(i), \ell = f(j)} c_{ij},$$

which is a path in $\Gamma_{\looparrowright}$. $\square$

Notice that, since the relation $\rightsquigarrow$ is reflexive, $\Gamma_{\rightsquigarrow}$ already contains all constant paths, obtained as convex combinations $t \mapsto \sum_i c_i \gamma_{ii}(t) = \sum_i c_i e_i$.

2 From diagrammatic sets to directed simplicial sets

2.1 Basic definitions

We refer to [Had20] and [HK22] for the basics of oriented graded posets and regular molecules.

Definition 13 (Diagrammatic sets): A *diagrammatic set* is a presheaf on the category $\odot$ whose

- objects are *atoms* U of arbitrary dimension,
- morphisms $f\colon U \to V$ are *maps* of oriented graded posets, that is, functions satisfying

  $$f(\partial_n^\alpha x) = \partial_n^\alpha f(x)$$

  for all $x \in U, n \in \mathbb{N}, \alpha \in \{+, -\}$.

Diagrammatic sets and their morphisms of presheaves form a category $\odot\mathbf{Set}$.

Definition 14 (Combinatorial computads): A *combinatorial computad* is a presheaf on the wide subcategory $\odot_{in}$ of $\odot$ whose morphisms are *inclusions*, that is, injective maps. Combinatorial computads and their morphisms of presheaves form a category **CComp**.

We will define two realisation functors valued in directed simplicial sets:

1. the first from diagrammatic sets,
2. the second from combinatorial computads.

In both cases, we will do this by first defining a functor from their site of definition, namely, $\odot$ and $\odot_{in}$, respectively, then taking a left Kan extension along the Yoneda embedding. Composing with $|-|_{\mathrm{d}\Delta}$ will then produce a realisation functor valued in d-spaces, as in the following picture.

$$
\begin{tikzcd}
 & \mathrm{d}\Delta \arrow[r, "|-|_{\mathrm{d}\Delta}"] \arrow[d, hook, "\mathbf{y}"'] & \mathbf{dTop} \\
\odot \arrow[r, dashed, "?"] \arrow[d, hook, "\mathbf{y}"'] & \mathbf{dsSet} \arrow[ur, "|-|_{\mathrm{d}\Delta}"'] & \\
\odot\mathbf{Set} \arrow[ur, "\mathrm{Lan}_{\mathbf{y}}(?)"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{d}\Delta \arrow[r, "|-|_{\mathrm{d}\Delta}"] \arrow[d, hook, "\mathbf{y}"'] & \mathbf{dTop} \\
\odot_{in} \arrow[r, dashed, "?"] \arrow[d, hook, "\mathbf{y}"'] & \mathbf{dsSet} \arrow[ur, "|-|_{\mathrm{d}\Delta}"'] & \\
\mathbf{CComp} \arrow[ur, "\mathrm{Lan}_{\mathbf{y}}(?)"'] & &
\end{tikzcd}
$$

Both of these will have the geometric realisation described in [Had20, Section 8.3] as underlying functor. Just as the definition of that functor goes through the simplicial nerve of posets, our definition will go through a "directed" version of this nerve.

Definition 15 (Direction on a poset): A *direction on a poset* P is a reflexive relation $\rightsquigarrow$ on the underlying set of P. We let **dPos** be the category of dependent pairs $(P, \rightsquigarrow)$ where P is a poset and $\rightsquigarrow$ a direction on P, together with maps preserving both the order and the direction.

We have an obvious inclusion $\mathrm{d}\Delta \hookrightarrow \mathbf{dPos}$.

Definition 16 (Directed simplicial nerve of posets): The *directed simplicial nerve of posets* is the right adjoint $dN\colon \mathbf{dPos} \to \mathbf{dsSet}$ in the realisation-nerve pair induced by the inclusion $\mathrm{d}\Delta \hookrightarrow \mathbf{dPos}$.

Explicitly, given a poset with direction $(P, \looparrowright)$, an n-simplex with direction $\rightsquigarrow$ in $dN(P, \looparrowright)$ is given by a chain

$$(x_0 \leq \ldots \leq x_n)$$

of length n in P such that $i \rightsquigarrow j$ implies $x_i \looparrowright x_j$ for all $i, j \in [n]$.

To conclude, it will suffice to define a functor $\odot \to \mathbf{dPos}$ or a functor $\odot_{in} \to \mathbf{dPos}$.

2.2 Realisation of diagrammatic sets

The realisation functor for diagrammatic sets will be induced by the functor

$$\mathcal{U}_1\colon \odot \to \mathbf{dPos}$$

sending an atom V to its underlying poset with the direction defined by

$$x \rightsquigarrow y \iff x = y \text{ or } x \in \partial_0^- y \text{ or } y \in \partial_0^+ x.$$

This relation is reflexive by definition. If $f\colon U \to V$ is a map of atoms, it is proven in [Had20, Lemma 1.9] that it is an order-preserving map of the underlying posets. Moreover, by definition of maps, $f(\partial_0^\alpha x) = \partial_0^\alpha f(x)$ for all $x \in U, \alpha \in \{+,-\}$, so $x \rightsquigarrow y$ implies $f(x) \rightsquigarrow f(y)$. This proves that $\mathcal{U}_1$ is well-defined as a functor.

The following diagram summarises the construction of this directed geometric realisation and its relation to the "undirected" realisation.

$$\begin{tikzcd}
\odot\mathbf{Set} \arrow[rrr, bend left, "|-|_1 := \mathrm{Lan}_{\mathbf{y}}(|k_1|_{\mathrm{d}\Delta})"] & & & \\
& \mathbf{dPos} \arrow[r, "dN"] \arrow[dd, dashed] & \mathbf{dsSet} \arrow[r, "|-|_{\mathrm{d}\Delta}"] \arrow[dd, "\iota^*"] & \mathbf{dTop} \arrow[dd, dashed] \\
\odot \arrow[uu, hook, "\mathbf{y}"] \arrow[dd, hook, "\mathbf{y}"'] \arrow[ur, "\mathcal{U}_1"] \arrow[dr, dashed] \arrow[urr, bend left, "k_1"] \arrow[drr, bend right, "k"'] & & & \\
& \mathbf{Pos} \arrow[r, "N"'] & \mathbf{sSet} \arrow[r, "|\text{-}|_{\Delta}"'] & \mathbf{Top} \\
\odot\mathbf{Set} \arrow[urrr, bend right, "|\text{-}| := \mathrm{Lan}_{\mathbf{y}}(|k\text{-}|_{\Delta})"'] & & &
\end{tikzcd}$$

2.3 Realisation of combinatorial computads

The realisation defined in the previous section only cares about "direction in dimension 1": reversing n-cells for any $n > 1$ results in the same d-space up to isomorphism.

In this section we will define a realisation functor which also encodes higher-dimensional directions via the existence of dipaths. However this realisation is incompatible with the presence of *collapsing* maps of atoms, as it is possible, in diagrammatic sets, to collapse a pair of cells one of which is in the input boundary of the other to a pair of lower-dimensional cells one of which is in the output boundary of the other. This produces an illegal "inversion" if directions in all dimensions are encoded via directed paths.

For this reason, this realisation only extends functorially to *inclusions* of atoms, hence determining a functor on combinatorial computads rather than diagrammatic sets.

Definition 17 (Oriented Hasse diagram): Let U be an atom. The *oriented Hasse diagram of U* is the directed graph $\mathring{\mathscr{H}}U$ whose

- set of vertices is the underlying set of U, and
- for all vertices x, y, there is an edge from y to x if and only if $y \in \Delta^- x$ or $x \in \Delta^+ y$.

The realisation functor for combinatorial computads will be induced by the functor

$$\mathcal{U}_\infty \colon \odot_{in} \to \mathbf{dPos}$$

sending an atom V to its underlying poset with the direction defined by

$$x \rightsquigarrow y \iff \text{there is an upward or downward path from } x \text{ to } y \text{ in } \mathring{\mathscr{H}}U,$$

where by upward and downward we mean "increasing or decreasing in dimension" ("zig-zag" paths are not allowed).

This relation is clearly reflexive. By [Had20, Lemma 1.11], inclusions of atoms preserve dimensions, orientations, and the covering relation, so they induce embeddings of oriented Hasse diagrams. This implies that $\mathcal{U}_\infty$ is well-defined as a functor.

The following diagram summarises this alternative directed realisation.

$$
\begin{tikzcd}
\mathbf{CComp} \arrow[drrr, bend left, "{|-|_\infty := \mathrm{Lan}_{\mathbf{y}}(|k_\infty|_{\mathrm{d}\Delta})}"] & & & \\
& \mathbf{dPos} \arrow[r, "dN"] \arrow[dd, dashed] & \mathbf{dsSet} \arrow[r, "|-|_{\mathrm{d}\Delta}"] \arrow[dd, "\iota^*"] & \mathbf{dTop} \arrow[dd, dashed] \\
\odot_{in} \arrow[uu, hook, "\mathbf{y}"] \arrow[dd, hook, "\mathbf{y}"'] \arrow[ur, "\mathcal{U}_\infty"] \arrow[urr, bend left, "k_\infty"] \arrow[dr, dashed] \arrow[drr, bend right, "k"'] & & & \\
& \mathbf{Pos} \arrow[r, "N"'] & \mathbf{sSet} \arrow[r, "|\text{-}|_\Delta"'] & \mathbf{Top} \\
\mathbf{CComp} \arrow[urrr, bend right, "{|\text{-}| := \mathrm{Lan}_{\mathbf{y}}(|k\text{-}|_\Delta)}"'] & & &
\end{tikzcd}
$$

We note that a simpler alternative to this construction would be to take the *(directed) reachability relation* on the oriented Hasse diagram of an atom (that is, also allow "zig-zag" paths). However, this has the consequence that, whenever there is a loop in the oriented Hasse diagram, which tends to happen from dimension 3 onward, every 1-simplex on the path traced by this loop is made reversible, which seems to produce a somewhat less interesting result.

References

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[HK22] Amar Hadzihasanovic and Diana Kessler. Data structures for topologically sound higher-dimensional diagram rewriting. In *Proceedings Fifth International Conference on Applied Category Theory (ACT2022)*, 2022.