Geometric realization of diagrammatic sets

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Contents

1 Directed simplex category

1.1 Combinatorics and lattice structure

We call Δ the simplex category whose objects are finite linear orders $[n] = \{0 < \cdots < n\}$ and morphisms (weakly) monotonic functions. It is generated by the face maps $\delta_i : [n-1] \to [n]$ and the coface maps $\sigma_j : [n+1] \to [n]$.

Definition 1.1 (Ordering): An *ordering* P on $[n] \in \Delta$ is a reflexive relation $P \subseteq [n] \times [n]$. We say that it is *total* if moreover, for all $0 \leq i, j \leq n$, either $(i, j) \in P$ or $(j, i) \in P$.

Definition 1.2 (Directed simplex): The *directed simplex* category $\mathrm{d}\Delta$ has object dependant pairs $([n], P \subseteq [n] \times [n])$ with P an ordering, and morphisms are the one of Δ that preserve orderings. More precisely, $f : ([n], P) \to ([m], Q)$ is a morphism $f : [n] \to [m]$ in Δ such that $f(P) \subseteq Q$, or equivalently, $P \subseteq f^{-1}(Q)$.

We consider Δ to be the subcategory of $\mathrm{d}\Delta$ by sending $[n]$ to $([n], [n] \times [n])$. As in Δ, we have the factorisation lemma of the maps.

Lemma 1.3 (Factorisation): Any map $f : ([n], P) \to ([m], Q)$ factorizes as

$$f = d_{i_1} \dots d_{i_p} s_{j_1} \dots s_{j_q}$$

where the orderings are constructed iteratively, starting from P, that is:

$$s_{j_k} : ([n], s_{j_{k+1}} \dots s_{j_q}(P)) \to ([n-k], s_{j_k} s_{j_{k+1}} \dots s_{j_q}(P))$$

and similarly for d_{i_k}, except for $d_{i_1} : ([n], d_{i_2} \dots d_{i_p} s_{j_1} \dots s_{j_q}(P)) \to ([m], Q)$ (or for s_{j_1} if $p = 0$).

Proof. (Lemma 1.3) We use the decomposition from the underlying map f in Δ, and by construction, each s_{j_k}, d_{i_k} lift to $\mathrm{d}\Delta$. $\square$

We exhibit a join semi-lattice structure on the objects of a fixed dimension, that in turn give set of sub-functors of a representable. We wall $\Delta^{[n],P}$ the presheaf represented by $([n], P)$. It all boils down to the next lemma.

Lemma 1.4 (Intersection of orderings): Let $([n], P)$ and $([n], P')$ be two objects of $\mathrm{d}\Delta$. Then for all $([m], Q)$ in $\mathrm{d}\Delta$, we have:

$$\begin{aligned} \Delta^{[n],P\cap P'}_{[m],Q} &= \Delta^{[n],P}_{[m],Q} \cap \Delta^{[n],P'}_{[m],Q} \\ \Delta^{[n],P\cup P'}_{[m],Q} &\supseteq \Delta^{[n],P}_{[m],Q} \cup \Delta^{[n],P'}_{[m],Q} \end{aligned}$$

Proof. (Lemma 1.4) Let $f : [m] \to [n]$.

$$\begin{aligned} f \in \Delta^{[n],P}_{[m],Q} \cap \Delta^{[n],P'}_{[m],Q} &\iff f(Q) \subseteq P \text{ and } f(Q) \subseteq P' \\ &\iff f(Q) \subseteq P \cap P' \\ &\iff f \in \Delta^{[n],P\cap P'}_{[m],Q} \end{aligned}$$

If $f \in \Delta^{[n],P}_{[m],Q} \cup \Delta^{[n],P'}_{[m],Q}$, then $f(Q) \subseteq P$ or $f(Q) \subseteq P'$, thus $f(Q) \subseteq P \cup P'$, meaning $f \in \Delta^{[n],P\cup P'}_{[m],Q}$. $\square$

Corollary 1.5 (Pullback of Yoneda functor): Let $([n], P)$ and $([n], P')$ be two objects of $\mathrm{d}\Delta$. Then the square:

$$\begin{tikzcd} \Delta^{[n],P\cap P'} \arrow[r, hook] \arrow[d, hook] & \Delta^{[n],P} \arrow[d, hook] \\ \Delta^{[n],P'} \arrow[r, hook] & \Delta^{[n],P\cup P'} \end{tikzcd}$$

is a pullback. The inclusions are the one induced by the map $\mathrm{id} : [n] \to [n]$, that lift to maps in $\mathrm{d}\Delta$ according to Lemma 1.4.

Proof. (Corollary 1.5) By Lemma 1.4, those squares are pointwise pullbacks. $\square$

Corollary 1.6 (Meet semilattice of representable): Let $\mathfrak{O}_n$ be all the ordering of $[n]\times[n]$. For any ordering $P \in \mathfrak{O}_n$, the bounded meet-semilattice structure $(\mathfrak{O}_n, \subseteq, \cap)$ gives a bounded meet-semilattice structure on the set of subfunctors of $\Delta^{[n],P}$, written $J_{[n],P}$, where the order is given by the lift of the identity map, and the meet by pullback. Moreover, if $P \subseteq P'$, $J_{[n],P} \subseteq J_{[n],P'}$.

Proposition 1.7 (Base change): Let $g : ([n], P) \to ([m], Q')$, and let $\Delta^{[m],Q} \overset{\text{id}}{\hookrightarrow} \Delta^{[m],Q'}$ an element of $J_{[m],Q'}$. Then

$$\begin{tikzcd}
\Delta^{[n],P\cap g^{-1}(Q)} \arrow[r, "g\circ"] \arrow[d, hook, "\text{id}"'] & \Delta^{[m],Q} \arrow[d, hook, "\text{id}"] \\
\Delta^{[n],P} \arrow[r, "g\circ"'] & \Delta^{[m],Q'}
\end{tikzcd}$$

is a pullback.

Proof. (**Proposition 1.7**) We will again check that this is a pullback pointwise. Let $([a], A) \in \mathrm{d}\Delta$.

$$\begin{tikzcd}
\Delta^{[n],P\cap g^{-1}(Q)}_{[a],A} \arrow[r, "g\circ"] \arrow[d, hook, "\text{id}"'] & \Delta^{[m],Q}_{[a],A} \arrow[d, hook, "\text{id}"] \\
\Delta^{[n],P}_{[a],A} \arrow[r, "g\circ"'] & \Delta^{[m'],Q'}_{[a],A}
\end{tikzcd}$$

We observe that $(g\circ)^{-1}(\Delta^{[m],Q}_{[a],A}) = \Delta^{[n],g^{-1}(Q)}_{[a],A}$. Indeed, $f(A) \subseteq g^{-1}(Q)$ implies $gf(A) \subseteq Q$, and if $f \in (g\circ)^{-1}(\Delta^{[m],Q}_{[a],A})$, then $gf(A) \subseteq Q$, thus $f(A) \subseteq g^{-1}gf(A) \subseteq g^{-1}(Q)$. We conclude by **Lemma 1.4** and the computation of pullbacks in **Set**. $\square$

This base change theorem indicates that the data of $\{J_{[n],P}\}_{([n],P)\in \mathrm{d}\Delta}$ can be gathered into a Grothendieck topology. We make this idea more precise:

Definition 1.8 (Cartesian coverage): A *cartesian coverage* on a category $\mathcal{C}$ consists of a function assigning to each object $U \in \mathcal{C}$ a collection of families of morphisms $\{f_i : U_i \to U\}_{i\in I}$, called covering families, such that if $\{f_i : U_i \to U\}_{i\in I}$ is a covering family and $g : V \to U$ is a morphism, then the family $\{g^*(f_i) : g^*U_i \to V\}_{i\in I}$ is a covering family of V.

We can construct a cartesian coverage on $\mathrm{d}\Delta$ by considering the covering families of $([n], P)$ to be:

$$\{\text{id}_i : ([n], P_i) \to ([n], P)\}_{i\in I}$$

such that $\bigcup_{i\in I} P_i = P$. The map id_i, that lifts $\text{id} : [n] \to [n]$, is well define by the previous condition. To check that this is a cartesian coverage is to check that, by **Proposition 1.7**, if $\bigcup_{i\in I} U_i = U$, then $\bigcup_{i\in I} Q \cap g^{-1}(P_i) = Q$ whenever $g : ([m], Q) \to ([n], P)$. We indeed have $\bigcup_{i\in I} Q \cap g^{-1}(P_i) = Q \cap g^{-1}(\bigcup_{i\in I} P_i) = Q \cap g^{-1}(Q) = Q$.

Under this coverage, we say that a presheaf $X : \mathrm{d}\Delta^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf if for all covering family $\{\text{id}_i : ([n], P_i) \to ([n], P)\}_{i\in I}$, the following diagram is an equalizer:

$$X([n],P) \longrightarrow \prod_i X([n],P_i) \rightrightarrows \prod_{i,j} X([n],P_i \cap P_j)$$

In particular, call $\equiv := ([1],\{00,11\})$, $+ := ([1],\{00,11,01\})$, $- := ([1],\{00,11,10\})$ and $\top := ([1],\{00,11,01,10\})$ the four possible orderings on $[1]$. The sheaf condition for the family $\{\mathrm{id}_\top, \mathrm{id}_+, \mathrm{id}_-, \mathrm{id}_\equiv\}$ tells us that:

$$\begin{tikzcd} X_\top \arrow[r] \arrow[d] & X_- \arrow[d] \\ X_+ \arrow[r] & X_\equiv \end{tikzcd}$$

is a pullback. We will give an interpretation of this condition in the next section, after we define the geometric realisation.

2 Geometric realisation

In this section, we describe how to realize a combinatorial computad (atoms + inclusions) into a directed topological space. The need to realize only the atoms, rather than all the category of diagrammatic sets comes from the absence of good functorial properties from the map we will describe below, in the presence of surjections between diagrams. Moreover, without the surjections, we consider the category $\mathrm{d}\Delta$ to be the directed semi-simplicial category, meaning that it is generated only by the maps δ_i with the equation $\delta_i\delta_j = \delta_{j+1}\delta_i$ whenever $i \leq j$. This is not too much of a restriction as Simon Henry's work on weak model structures shows that one can do homotopy theory just fine in this way. All the results of the previous section also apply.

2.1 Directed geometric realisation

We want to turn a directed semi-simplicial set into a directed topological space. For that, we realize the underlying semi-simplicial set the usual way, and we add a set of paths according to the ordering.

Definition 2.1 (d-spaces and d-maps): A *d-space* (X, dX) is a topological space together with a set dX of dipaths such that:

1. every constant path is a dipath,
2. If $\varphi : [0,1] \to [0,1]$ is a directed partial reparametrisation (i.e. weakly increasing), then $f \circ \varphi$ is a dipath for all $f \in dX$,
3. The concatenation of two dipaths is a dipath.

A *d-map* (or dimap) $f : (X, dX) \to (Y, dY)$ is a continuous map such that for all $h \in dX$, $f \circ h \in dY$. This gathers into the category **dTop**.

If x_i, x_j are points in $\mathbb{R}$, we call γ_{ij} the path defined by $\gamma_{ij}(t) = tx_i + (1-t)x_j$.

Definition 2.2 (Directed geometric realisation): The *directed geometric realisation* of an element $([n], P) \in \mathrm{d}\Delta$ is the directed topological space (Δ^n, dP) where Δ^n is the convex hull spanned by $n+1$ affine independent points $x_0, \ldots, x_n$ of $\mathbb{R}^{n+1}$, and dP is the closure under concatenation and reparametrisation of the set of paths:

$$\{ \sum_{(i,j)\in P} c_{ij}\gamma_{ij} \mid \sum_{(i,j)\in P} c_{ij} = 1\}$$

The directed geometric realisation has underlying topological space the standard geometric realisation. The set dP forms a set of dipaths. Indeed by definition it is closed under concatenation and reparametrisation. It also contains all the constant path, as if x is in Δ^n, then $x = \sum_{i=0}^n c_i x_i$ with $\sum_{i=0}^n c_i = 1$, thus by setting $c_{ij} = 0$ if $i \neq j$ and $c_{ij} = c_i$ if $i = j$, and by noticing that P is reflexive, we obtain the constant path at x. To make it a functor $|\text{-}|_d : \mathrm{d}\Delta \to \mathbf{dTop}$, we only need to check that the underlying continuous maps from the standard simplex lifts to maps of directed topological space. Consider $f : ([n], P) \to ([m], Q)$, and take $\gamma = \sum_{(i,j)\in P} c_{ij}\gamma_{ij}$ in dP. We want to show that $|f|\gamma$ is in dQ. By definition, $|f|$ distributes over barycentric combinations, and $|f|\gamma_{ij} = \gamma_{f(i)f(j)}$. For $(i,j) \in Q$, set

$$c'_{ij} = \sum_{(k,l)\in f^{-1}(i,j)} c_{kl}$$

where $f^{-1}(i,j)$ is shorthand for $(f^{-1}(i) \times f^{-1}(j)) \cap P$, and

$$|f| \circ \gamma = \sum_{(k,l)\in P} c_{kl}\gamma_{f(k)f(l)} = \sum_{(i,j)\in Q} c'_{ij}\gamma_{ij}$$

with $\sum_{(i,j)\in Q} c'_{ij} = \sum_{(k,l)\in P} c_{kl} = 1$, as $f(P) \subseteq Q$. Thus, $|f| \circ \gamma$ is an admissible path of dQ.

We can left-Kan extend this construction and obtain a functor $|\text{-}|_{d\Delta} : \mathbf{dsSet} \to \mathbf{dTop}$.

2.2 From combinatorial computad to directed simplicial sets

We call $\odot_i$ the skeleton of the category of combinatorial computad, that is atoms and inclusions. We have the following picture:

$$\begin{tikzcd}
 & \mathrm{d}\Delta \arrow[r, "|\text{-}|_{\mathrm{d}\Delta}"] \arrow[d, hook, "\mathbf{y}"'] & \mathbf{dTop} \\
\odot_i \arrow[r, dashed, "?"] \arrow[d, hook, "\mathbf{y}"'] & \mathbf{dsSet} \arrow[ur, "|\text{-}|_{d\Delta}"'] & \\
\odot_i\mathbf{Set} \arrow[ur, "\mathrm{Lan}_{\mathbf{y}}(?)"'] & &
\end{tikzcd}$$

We would like to find the dashed functor, thus giving a geometric realization of a diagrammatic set. For that, recall from [Had20] that we have a functor:

$$k : \odot_i \to \mathbf{sSet}$$

sending a molecule U to the simplicial nerve of its underlying poset. If U was already a simplex, it would correspond to constructing its barycentric subdivision. We would like to extend it to a functor $k_d : \odot_i \to \mathbf{dsSet}$. As $\Delta \subseteq \mathrm{d}\Delta$, we already have a functor $\mathbf{sSet} \to \mathbf{dsSet}$, however doing so would lose all the relevant information about the orientation of the diagram U in $\odot_i$ that we try to convey. Thus, we need to refine the functor k, by adding manually orientations. To that end, we introduce the intermediate category of preordered posets.

Definition 2.3 (Preordered poset): A *preordered poset* is a triple $(P, \leq, \preceq)$ where $(P, \leq)$ is a poset and $(P, \preceq)$ is a preorder. A map $f : (P, \leq, \preceq) \to (Q, \leq, \preceq)$ is a map respecting both the poset and the preorder structure. This form the category **prePos**.

We have a forgetful functor $\mathbf{prePos} \to \mathbf{Pos}$ and another $\odot_i \to \mathbf{Pos}$. We now describe how to factorize them via another forgetful functor from $\mathcal{U} : \odot_i \to \mathbf{prePos}$. Let V be an atom, we define $\mathcal{U}V := (\mathcal{H}U, \leq, \preceq)$, where the posetal structure is the one of its Hasse diagram, and the preorder is the transitive and reflexive closure of the relation given by:

$$x \preceq y \iff o(x \leftarrow y) = - \text{ or } o(y \leftarrow x) = +$$

As inclusions preserve orientation, this construction is functorial. We may now construct the directed nerve functor $dN : \mathbf{prePos} \to \mathbf{dsSet}$. Let $(X, \leq, \preceq)$ in **prePos**. We define $dN(X)([n], P)$ to be the set of $(n+1)$ chains $(x_0 \leq \cdots \leq x_n)$ (like in the ordinary nerve) that are moreover such that for all $(i, j) \in P$, we have $x_i \preceq x_j$. We call $k_d : \odot_i \to \mathbf{dsSet}$ the functor $dN \circ \mathcal{U}$.

We summarize our constructions with the following diagram where everything commutes. A dashed arrow indicates a forgetful functor.

$$\begin{tikzcd}
\odot_i\mathbf{Set} \arrow[drrr, bend left, "|\text{-}|_d := \mathrm{Lan}_{\mathbf{y}}(|k_d\text{-}|_{d\Delta})"] & & & \\
& \mathbf{prePos} \arrow[r, "dN"] \arrow[dd, dashed] & \mathbf{dsSet} \arrow[r, "|\text{-}|_{d\Delta}"] \arrow[dd, "\iota^*"] & \mathbf{dTop} \arrow[dd, dashed] \\
\odot_i \arrow[uu, "\mathbf{y}"] \arrow[dd, "\mathbf{y}"'] \arrow[ur, dashed] \arrow[dr, dashed] \arrow[urr, bend left, "k_d"] \arrow[drr, bend right, "k"'] & & & \\
& \mathbf{Pos} \arrow[r, "N"'] & \mathbf{sSet} \arrow[r, "|\text{-}|_{\Delta}"'] & \mathbf{Top} \\
\odot_i\mathbf{Set} \arrow[urrr, bend right, "|\text{-}| := \mathrm{Lan}_{\mathbf{y}}(|k\text{-}|_{\Delta})"'] & & &
\end{tikzcd}$$

2.3 The sheaf condition

Recall that for a presheaf $W : \mathrm{d}\Delta^{\mathrm{op}} \to \mathbf{Set}$ to be sheaf for the covering family $\{\mathrm{id}_\top, \mathrm{id}_+, \mathrm{id}_-, \mathrm{id}_\equiv\}$, the following diagram must be a pullback:

$$\begin{tikzcd}
W_\top \arrow[r] \arrow[d] & W_- \arrow[d] \\
W_+ \arrow[r] & W_\equiv
\end{tikzcd}$$

Call $\vec{I}$ the directed interval whose paths are order preserving functions, and $\overleftarrow{I}$ the directed interval whose paths are order reversing functions. Then we have $|([1], +)|_d \simeq \vec{I}$ and $|([1], -)|_d \simeq \overleftarrow{I}$. Notice also that $|([1], \equiv)|_d \simeq \dot{I}$ the directed interval with only the constant path, and thus

$\mathbf{dTop}(\dot{I}, X) = \mathbf{Top}(I, X)$. We have that $\tilde{I} := |([1], \top)|_d$, where $\tilde{I}$ is the directed interval whose paths are the one with finite change of monotonicity. Indeed, any path in $d\tilde{I}$ is the finite concatenation (and reparametrisation) of paths that are either increasing or decreasing.

Thus for a directed topological space to give a sheaf (for this family at least) via the nerve $\mathbf{dTop}(|\text{-}|_d, X)$, which is right adjoint to $|\text{-}|_d$, the diagram

$$\begin{tikzcd}
\mathbf{dTop}(\tilde{I}, X) \arrow[r] \arrow[d] & \mathbf{dTop}(\overleftarrow{I}, X) \arrow[d] \\
\mathbf{dTop}(\overrightarrow{I}, X) \arrow[r] & \mathbf{Top}(I, X)
\end{tikzcd}$$

has to be a pullback. By definition, $dX = \mathbf{dTop}(\overrightarrow{I}, X)$, and call $dX^{-1} = \mathbf{dTop}(\overleftarrow{I}, X)$, which is the set of paths γ^{-1} for $\gamma \in dX$. By the above pullback, we want $\mathbf{dTop}(\tilde{I}, X) = dX \cap dX^{-1}$.

Proposition 2.4: For (X, dX) a directed topological space, we have

$$\mathbf{dTop}(\tilde{I}, X) = dX \cap dX^{-1}$$

Proof. (Proposition 2.4) Let $\gamma : I \to X$ be in $\mathbf{dTop}(\tilde{I}, X)$. In particular, the path $\mathrm{id} : I \to I$ and the path $1 - \mathrm{id} : I \to I$ are both in $d\tilde{I}$, thus $\gamma \circ \mathrm{id}$ and $\gamma^{-1} = \gamma \circ (1 - \mathrm{id})$ are in dX. Conversely, if γ and γ^{-1} are in dX, let $\varphi_1 \dots \varphi_n$ be a path in $d\tilde{I}$ where φ_i is weakly monotonic. Then $\gamma \circ (\varphi_1 \dots \varphi_n) = (\gamma \circ \varphi_1) \dots (\gamma \circ \varphi_n)$. If φ_i is increasing, then $\gamma \circ \varphi_i$ is a directed reparametrisation of γ so is in dX, and if φ_i is decreasing, then we can write $\gamma \circ \varphi_i = \gamma \circ (1 - \mathrm{id}_I) \circ (1 - \varphi_i) = \gamma^{-1} \circ (1 - \varphi_i)$, that is also in dX as a directed reparametrisation of γ^{-1} by $(1 - \varphi_i)$. Thus $\gamma \in \mathbf{dTop}(\tilde{I}, X)$. □

The sheaf condition is telling us nothing, it is always true for a directed topological space.

2.4 From diagrammatic sets to directed simplicial sets

In this part, we study a different version of the geometric realisation of diagrammatic sets. We will take the same approach as in the previous section, but we will have to deal with surjective maps between molecules, and thus we will need to take more care when constructing the forgetful functor to **prePos**. We notice that we can leave unchanged the Definition 2.2 on geometric realisation, as the proof of functoriality equally applies if we add degeneracies. What breaks is the construction of the preorder $\preceq$, that does not behave functorialy as the surjections do not preserve orientations in general. Thus, we will adapt it here.

We aim to construct a functor $\mathcal{U} : \odot \to \mathbf{prePos}$. For that we consider $V \in \odot$, and as usual, we associate it its underlying poset $(\mathcal{H}V, \leq)$. Now, the preorder is the reflexive and transitive closure of the relation given by:

$$x \preceq y \iff x \in \partial_0^- y \text{ or } y \in \partial_0^+ x$$

That is whenever $x \preceq y$ (and $x \neq y$), then x or y is a point (or both by transitivity). Now, a map $f : P \to Q$ of $\odot$ satisfies in particular

$$\partial_0^\alpha f(x) = f(\partial_0^\alpha x)$$

for all $x \in P$, thus if $x \in \partial_0^- y$, then $f(x) \in \partial_0^- f(y)$, making $\mathcal{U}$ a functor $\odot \to \mathbf{prePos}$. We now have the same picture as previously

$$\begin{tikzcd}
\odot\mathbf{Set} \arrow[rrrd, bend left, "|\text{-}|_d := \mathrm{Lan}_{\mathbf{y}}(|k_d\text{-}|_{d\Delta})"] & & & \\
& \mathbf{prePos} \arrow[r, "dN"] \arrow[dd, dashed] & \mathbf{dsSet} \arrow[r, "|\text{-}|_{d\Delta}"] \arrow[dd, "\iota^*"] & \mathbf{dTop} \arrow[dd, dashed] \\
\odot \arrow[uu, hook, "\mathbf{y}"] \arrow[dd, hook, "\mathbf{y}"'] \arrow[ur, dashed] \arrow[dr, dashed] \arrow[urr, bend left, "k_d"] \arrow[drr, bend right, "k"'] & & & \\
& \mathbf{Pos} \arrow[r, "N"'] & \mathbf{sSet} \arrow[r, "|\text{-}|_{\Delta}"'] & \mathbf{Top} \\
\odot\mathbf{Set} \arrow[rrru, bend right, "|\text{-}| := \mathrm{Lan}_{\mathbf{y}}(|k\text{-}|_{\Delta})"'] & & &
\end{tikzcd}$$

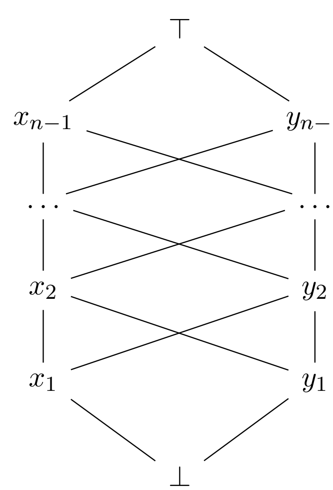
with the whole category of diagrammatic sets.

3 Examples

We may now work with the latter geometric realisation.

3.1 The cone

Consider the poset C_n:



and preorder $\bot \preceq x_i, y_i$ for all $1 \leq x_i, y_i \leq n-1$. Notice that this is the pre-posetal set $\mathcal{U}(O^n)$ where we remove the output 0-boundary. Abusing notations, we also call C_n the geometrical cone of $\mathbb{R}^{n+1}$ whose base is a $(n-2)$-square, whose apex $\bot$ is at the origin, and whose height is parallel to an axis that we call z. If a point x is in the cone, we call θ the angle between the z axis and the line $\bot x$. This way, each path determines another continuous path $t \mapsto (z(t), \theta(t))$. We endow C_n with a directed topological structure by taking the dipaths dC_n to be all the piecewise-linear paths whose (z, θ) value is weakly increasing.

Proposition 3.1 (Realisation of the cone): We have

$$|C_n|_d \simeq (C_n, dC_n)$$

Proof. (**Proposition 3.1**) First, it is the case that this is the realisation as (non-directed) topological spaces, indeed C_n will be the glueing along the correct faces of 2^{n-1} n-simplices $\perp u_1 \dots u_{n-1} \top$ where $u_i \in \{x_i, y_i\}$, each one determined by a path from $\perp$ to $\top$ in the Hasse diagram of the cone. Each of this n simplex share the edge $\top\perp$ and it suffice to check that in each of these simplex, the admissible path are piecewise linear, weakly increasing along the $\top\perp$ edge.

Let $(D, dD) := |\perp u_1 \dots u_{n-1} \top|_d$ be such a directed simplex. To determine its admissible paths, it suffices to determine the maximal orderings P such that for all $(i,j) \in P$, $D_i \preceq D_j$. Indeed, if there is a smaller ordering Q, then it will induce an inclusion of directed topological spaces via $\mathrm{id} : ([n], Q) \to ([n], P)$. By construction of the preorder of the cone, we have

$$P = \mathrm{Diag} \cup \{0i \mid 1 \leq i \leq n\}$$

Thus, dD is the closure under concatenation and reparametrisation of

$$\{\sum_{i=1}^{n} c_i \gamma_{0i} \mid \sum_{i=1}^{n} c_i = 1\}$$

Note that by reparametrisation, one need not to include the constant paths in the sum. These paths are piecewise linear, and indeed each γ_{0i} is weakly increasing along (z, θ).

Conversely, all linear path of increasing value along $\perp\top$ are of the form (by reparametrisation) $t \mapsto tx + (1-t)y$ with $x \leq_{(z,\theta)} y$, and such a path can be written (how?) as a sum $\sum_{i=0}^{n} c_i \gamma_{0i}$. ☐

3.2 Globe, interval, sphere

Let C'_n be the cone C_n but with preorder $x_i, y_i \preceq \perp$, i.e. $C'_n = (C_n, \leq, \preceq^{\mathrm{op}})$. If C is a preposetal set, then let C^* be the same preposetal set without its bottom element (when it exists). We have in **prePos**:

$$(C'_n)^* \simeq C_n^*$$

The globe O^n is then the pushout of inclusions in **prePos**:

$$\begin{tikzcd} C_n'^* \simeq C_n^* \arrow[r, hook] \arrow[d, hook] & C'_n \arrow[d] \\ C_n \arrow[r] & O^n \end{tikzcd}$$

We call $\perp \preceq x_i, y_i, \top \preceq \perp'$ the elements of O^n. Consider (I^n, dI^n) the directed cube $[0,1]^n$ whose dipath are the piecewise linear weakly increasing ones. In C'_n, the admissible dipaths are the ones whose (z, θ) value is weakly decreasing.

Lemma 3.2 (Admissible dipaths): A path $\gamma : [0,1] \to I^n$ is in dI^n if and only the map $t \mapsto (z(t), \theta(t))$ is weakly increasing while $z(t) \leq \frac{1}{2}\sqrt{n}$ and the map $t \mapsto (z(t), -\theta(t))$ is weakly increasing while $z(t) \geq \frac{1}{2}\sqrt{n}$.

Proof. (Lemma 3.2) Cut the n-cube $[0,1]^n$ along the bisecting hyperplane of the diagonal $0^n 1^n$ and notice that the bottom part is a directed copy of C^n and the top part is a directed copy of (a symmetric version of) C'_n. ☐

Corollary 3.3 (Realisation of the globes): We have

$$|O^n|_d \simeq (I^n, dI^n)$$

Corollary 3.4 (Realisation of the interval): The diagrammatic interval O^1 is realized as the directed interval $\overrightarrow{[0,1]}$ with weakly increasing paths.

Corollary 3.5 (Realisation of the spheres): The sphere ∂O^n is realized as $\partial \vec{I^n} = [0,1]^n \backslash (0,1)^n$ whose directed paths are the one inherited from $\vec{I^n}$

Proof. (Corollary 3.5) The same applies, except this time we only glue the boundaries of the n simplices (whose interior is then empty). ☐

3.3 Gray product

Lemma 3.6: The forgetful functor $\odot \to$ **prePos** sends Gray products to cartesian products.

Proof. (Lemma 3.6) The forgetful functor sends Gray products to cartesian products in **Pos**. Moreover, by definition

$$x \otimes y \preceq x' \otimes y'$$

if and only if $x \otimes y \in \partial_0^- x' \otimes y'$ or $x' \otimes y' \in \partial_0^+ x \otimes y$, but the 0-boundaries commute with Gray products, that is $\partial_0^\alpha(U \otimes V) = \partial_0^\alpha U \otimes \partial_0^\alpha V$, thus $x \otimes y \preceq x' \otimes y'$ if and only if $x \preceq x'$ and $y \preceq y'$, giving indeed the cartesian product in **prePos**. ☐

Proposition 3.7 (Realisation of Gray products): Let $U, V \in \odot$, then

$$|U \otimes V|_d \simeq |U|_d \times |V|_d$$

Proof. (Proposition 3.7) First, $k_d(U \otimes V) = U \times V$ in **dsSet**, because the Gray product is sent to the cartesian products by Lemma 3.6, and the directed nerve functor is right adjoint, thus preserves products.

That means it now suffices to show that the realisation of directed simplicial sets preserves finite products. It is the case for the simplicial sets, thus is suffices to show that the admissible paths of the realisation of the product is the product of the admissible paths of the realisation. This is the case for purely formal reasons. Indeed, as the standard geometric realization preserves products, a point $x \otimes y$ is realized as the couple $(|x|, |y|)$. Thus, the path $\gamma_{ii',jj'} : (x_i, x_{i'}) \to (x_j, x_{j'})$ is the path $(\gamma_{ij}, \gamma_{i'j'})$. So, for two orderings P, Q, we can write

$$\begin{aligned}
\sum_{(ij,i'j')\in P\times Q} c_{ij,i'j'}\gamma_{ii',jj'} &= \left(\sum_{(ij,i'j')\in P\times Q} c_{ij,i'j'}\gamma_{ij}, \sum_{(ij,i'j')\in P\times Q} c_{ij,i'j'}\gamma_{i'j'} \right) \\
&= \left(\sum_{ij\in P} \left(\sum_{i'j'\in Q} c_{ij,i'j'} \right) \gamma_{ij}, \sum_{i'j'\in Q} \left(\sum_{ij\in P} c_{ij,i'j'} \right) \gamma_{i'j'} \right)
\end{aligned}$$

and both paths are admissible in their respective spaces. □

Corollary 3.8 (Realisation of the cubes): The realisation of the n-cube is $[0,1]^n$ with admissible paths the piecewise increasing ones.

Proof. (**Corollary 3.8**) The nth cube is defined to be $O^1 \otimes \cdots \otimes O^1$ n times, thus by **Proposition 3.7**, its realisation is the $\vec{I}^n$, which is the cube $[0,1]^n$ whose paths are piecewise linear increasing increasing. □

Notice that we have in **dTop**, $|O^n|_d \simeq |\square^n|_d$.

3.4 Dihomotopies and reversible paths

A diagram of shape $f : O^1 \to X$ in a diagrammatic set exhibits an admissible dipath $|f| \in dX$, as it is realized as a map $|f| : \vec{I} \to |X|_d$, and we have the isomorphism (of topological spaces) $(|X|_d)^{\vec{I}} \simeq dX$. If $x : 1 \to X$, we allow ourselves to name $x \in |X|_d$ the point determined by the map $|x|_d : 1 \to |X|_d$, thus we have that a diagram of shape $x \xrightarrow{f} y$ in X, it is realized as the concatenation of the paths $|f| = \gamma_{xf} \cdot \gamma_{fy}$.

Definition 3.9 (Dihomotopies): Let (X, dX) be a directed topological space, and let $f, g : x \to y$ be two admissible dipaths of dX. A *dihomotopy* $\varphi : f \to g$ is a continuous map $\varphi : [0,1] \to dX$ such that $\varphi(0) = f$, $\varphi(1) = g$ and for all $t \in [0,1]$, we have $\varphi(t) \in dX$. If there is a dihomotopy $\varphi : f \to g$, we write $f \sim_d g$. This is an equivalence relation.

A dihomotopy is simply a (particular) homotopy between in the space dX, thus is a morphism $\varphi : \vec{I} \times I \to X$ in **dTop**.

Lemma 3.10 (Diagrams in the realisation): Let X be a diagrammatic set.

1. A diagram of shape $x \xrightarrow{f} y \xrightarrow{g} z$ in X is realized as the concatenation of the paths in dX determined by the subdiagrams $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$.

2. A diagram of shape

$$\begin{tikzcd}
x \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Uparrow\alpha"] & y
\end{tikzcd}$$

in X proves that the paths determined by the subdiagrams $f : O^1 \to X$ and $g : O^1 \to X$ are dihomotopic.

Proof. (**Lemma 3.10**) The first point is clear as $|O^1 \#_0 O^1|_d = (|O^1|_d \coprod |O^1|_d)/(\partial^+ O^1 = \partial^- O^1)$, thus the diagram of shape $f;g : O^1 \#_0 O^1 \to X$ is indeed the concatenation of two paths and the two inclusions

$$\begin{tikzcd}
O^1 \arrow[r, hook] \arrow[dr, "f"'] & O^1 \#_0 O^1 \arrow[d, "f;g"] & O^1 \arrow[l, hook'] \arrow[dl, "g"] \\
& X &
\end{tikzcd}$$

ensure the equality of the desired paths.

For the second point, we notice the path $|f|$ is the concatenation of paths $\gamma_{xf} \cdot \gamma_{fy}$, and similarly $|g| = \gamma_{xg} \cdot \gamma_{gy}$. It suffices to show that the linear interpolation of those two paths via $\varphi(t) = t|f| + (1-t)|g|$ is a dihomotopy. To show that $\varphi(t)$ is a dipath for all t, we can notice that we we can write it as the concatenation:

$$\varphi(t) = (t\gamma_{xf} + (1-t)\gamma_{xg}) \cdot (t\gamma_{fy} + (1-t)\gamma_{fy})$$

and for all t, both $t\gamma_{xf} + (1-t)\gamma_{xg}$ and $t\gamma_{fy} + (1-t)\gamma_{fy}$ constitute admissible dipaths by construction. $\square$

We could similarly define the notion of $\vec{I}$-dihomotopy as maps $\vec{I} \times \vec{I} \to X$, but the realisation we are working with will not detect the difference between

$$\begin{tikzcd}
x \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Uparrow\alpha"] & y
\end{tikzcd}$$

and

$$\begin{tikzcd}
x \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\alpha\Downarrow"] & y
\end{tikzcd}$$

and would not allow us to interpolate the paths $|f|$ and $|g|$. Indeed, it would require that the paths $t \mapsto t|f|(u) + (1-t)|g|(u)$ are directed for all $0 \le u \le 1$, but then taking for instance $u = \frac{1}{2}$, we would end up with the path $\gamma_{f\alpha} \cdot \gamma_{\alpha g}$ that does not belong to the realisation, as we do not have, for instance, $f \preceq \alpha$ or $\alpha \preceq f$.

However, this difference would be detected by the first realisation, and it would be possible to construct $\vec{I}$-dihomotopies $f \to g$ in the first diagram and $g \to f$ in the second.

Definition 3.11 (Contractible): A path $f : x \to x$ is *contractible* when there exists a dihomotopy $\varphi : f \to x$, where x is the constant path at x.

Proposition 3.12 (Contractible paths): Let X be a diagrammatic set, then a diagram of shape

$$\begin{tikzcd}
x \arrow[rr, "\varepsilon x"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Uparrow"] & x \\
& y \arrow[ur, "f^{-1}"'] &
\end{tikzcd}$$

in X proves that the path $|f| \cdot |f^{-1}| : x \to x$ is contractible, and in **Top**, we have $|f^{-1}| \sim |f|^{-1}$

Proof. (**Proposition 3.12**) A diagram of shape $\varepsilon_x =!; x : O^1 \twoheadrightarrow 1 \to X$ exhibit a path of dX, and the commutation $|\varepsilon_x| =!; |x|$ indicates it is constant. By **Lemma 3.10**, we conclude that the path $|f| \cdot |f^{-1}|$ is dihomotopic to the constant path at x. Moreover:

$$\begin{aligned}
&|f| \cdot |f^{-1}| \sim x \\
\Rightarrow &|f|^{-1} \cdot |f| \cdot |f^{-1}| \sim |f|^{-1} \cdot x \\
\Rightarrow &|f^{-1}| \sim |f|^{-1}.
\end{aligned}$$

□

Definition 3.13 (Weak dihomotopy equivalence): Two direct spaces (X, dX) and (Y, dY) are *weakly dihomotopical equivalent* if their paths spaces dX and dY are homotopical equivalent.

Proposition 3.14: Two directed spaces (X, dX) and (Y, dY) are weakly dihomotopical equivalent if and only if there exists two dimaps $f : (X, dX) \to (Y, dY)$ and $g : (Y, dY) \to (X, dX)$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$

Proof. (**Proposition 3.14**) Suppose $\Gamma : dX \to dY$ and $\Gamma' : dY \to dX$ form a homotopy equivalence. In particular, as constant paths are send to constant paths, Γ and Γ' restricts to continuous maps $\Gamma_0 : X \to Y$ and $\Gamma'_0 : Y \to X$. Take a path $\gamma \in dX$, then $t \mapsto \Gamma_0(\gamma(t)) = \Gamma \circ \gamma \in dY$, thus Γ_0 is a dimap. Similarly, Γ' is also a dimap. We have $\Gamma'\Gamma \sim \mathrm{id}_{dX}$, and this homotopy restricts to $\Gamma'_0\Gamma_0 \sim \mathrm{id}_Y$ by applying it to the constant paths. Similarly $\Gamma_0\Gamma'_0 \sim \mathrm{id}_Y$.

Conversely, suppose we have $f : (X, dX) \to (Y, dY)$ and $g : (Y, dY) \to (X, dX)$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. We let $\Gamma = f \circ -$ and $\Gamma' = g \circ -$ and conclude by $\Gamma'\Gamma = g \circ f \circ - \sim \mathrm{id}_X \circ - = \mathrm{id}_{dX}$, similarly for $\Gamma\Gamma'$. □

References

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