A bit of topos theory in Lean

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Contents

The formalization of this paper can be found here. Note that everything was first made in Lean, and then written in LaTeX, hence the uncommon presentation. A **Fact** is a result that can be found in the mathlib library. The titles of the sections roughly correspond to the name of the file in the git repository.

1 Subobject classifier

We fiw a category $\mathcal{C}$ with a terminal object $\mathbf{1}$, we call $!_X : X \to \mathbf{1}$ the unique map from any object X to the terminal object.

1.1 Definitions and canonical constructions

Definition 1.1: Subobject classifier

We say that an object Ω and a mono truth : $\mathbf{1} \to \Omega$ form a subobject classifier if whenever we have a mono $m : S \rightarrowtail X$, there exists a map $\chi^m : X \to \Omega$ such that the maps m and $!_X$ form a limit pullback cone on $\chi^m$and truth, and if moreover χ^m is the only map having this property.

From now on, we only suppose that the category $\mathcal{C}$ has pullbacks, not necessarily a full subobject classifier.

Definition 1.2: Canonical object and inclusion

If $\sigma : X \to \Omega$ is any map, we call $\{\sigma\}$ a chosen pullback object of truth along σ. Moreover, we call c_σ the first projection of this chosen pullback. This means that the following diagram is a pullback:

$$\begin{tikzcd}
\{\sigma\} \arrow[r] \arrow[d, "\mathrm{c}_\sigma"'] & \mathbf{1} \arrow[d, "\mathrm{truth}"] \\
X \arrow[r, "\sigma"'] & \Omega
\end{tikzcd}$$

Fact 1.1: Pullback preserves monos

If w is a limit pullback cone on m and g with m a mono, then the first projection of w is a mono.

Lemma 1.1: c_σ is a mono

For every $\sigma : X \to \Omega$, c_σ is a mono.

Proof. (**Lemma 1.1**) This is **Fact 1.1**. □

Fact 1.2: Subobject

For every $c : \mathcal{C}$, we can form the category $\mathrm{sub}(c)$ whose objects are the equivalence classes of mono $m : \cdot \to c$, and maps are maps of $\mathcal{C}$ that give commutative triangles. If $S : \mathrm{sub}(c)$, we call $S^{\to}$ a chosen (via the axiom of choice) representative arrow and S^{o} its codomain. If $m : U \rightarrowtail X$ is a mono, we call $[m]$ the subobject constructed by m.

Fact 1.3: Chosen and canonical subobjects

If $m : U \rightarrowtail X$ is a mono, then we have $i : [m]^{\mathrm{o}} \simeq U$ and $m \circ i = [m]^{\to}$.

Lemma 1.2: Application to $\sigma : X \to \Omega$

For $\sigma : X \to \Omega$, we have $[\mathrm{c}_\sigma]^{\mathrm{o}} \simeq \{\sigma\}$

Proof. (**Lemma 1.2**) This is the application of **Fact 1.3** to c_σ. □

Fact 1.4: Equality of subobjects

Two subobjects $S, S' : \mathrm{sub}(c)$ are equal if we can find $i : S^\circ \simeq S'^\circ$ such that $S'^\rightarrow \circ i = S^\rightarrow$. In particular, for a mono $m : U \rightarrowtail X$, $S = [m]$ if we can find $i : S^\circ \simeq U$ such that $m \circ i = S^\rightarrow$.

Lemma 1.3: Equality on subobjects of the classifying arrow

Let $S : \mathrm{sub}(c)$. We have:

$$[\mathrm{c}_{\chi^{S^\rightarrow}}] = S$$

Proof. (**Lemma 1.3**) By **Fact 1.4**, we want an isomorphism $i : \{\chi^{S^\rightarrow}\} \simeq S^\circ$ such that $S^\rightarrow \circ i = \mathrm{c}_{\chi^{S^\rightarrow}}$. Such an isomorphism is given by the isomorphism of cones between the two limit pullback cones, the first one having first projection $\mathrm{c}_{\chi^{S^\rightarrow}}$, and the second having first projection $S^\rightarrow$. The former is a pullback by definition of the the operator c, and the latter by definition of the classifier $\chi^{S^\rightarrow}$. ☐

Definition 1.3: Subobject arrow to pullback cone

Let $\sigma : X \to \Omega$, we define a pullback cone $[[\sigma]]$ with first projection $[\mathrm{c}_\sigma]^\rightarrow$. The second projection is automatically $!_{[\mathrm{c}_\sigma]^\circ}$.

Fact 1.5: The Subobject arrow cone is limit

The pullback cone $[[\sigma]]$ is well defined, isomorphic to the canonical pullback cone given by c_σ, and thus is a limit cone.

1.2 Truth and identity

Theorem 1.1: Truth is the classifier of the identity

We have $\mathrm{c}_{\mathrm{id}_1} = \mathrm{truth}$.

Proof. (**Theorem 1.1**) This is the application of **Fact 3.10** to the mono truth. ☐

2 Image

3 Presheaves

3.1 Limits in a category of presheaves

Let $\mathcal{C}$ be a category. We call $\hat{\mathcal{C}}$ its category of presheaf, that is the category of functors $[\mathcal{C}^{\mathrm{op}}, \mathrm{Type}]$.

Fact 3.1: (co)limits in the category of functors

If $\mathcal{C}$, $\mathcal{D}$ are two categories, the category of functors $[\mathcal{C}, \mathcal{D}]$ has the same limits and colimits than $\mathcal{D}$.

Fact 3.2: The category of types

Type has a structure of category that is complete, cocomplete, and cartesian closed.

Fact 3.3: Preseahves on types

For any $\mathcal{C}$ small category, $[\mathcal{C}^{\mathrm{op}}, \mathrm{Type}]$ is complete, cocomplete, and cartesian closed.

We use the axiom of choice to chose $\mathbf{1}$, any terminal object of $\hat{\mathcal{C}}$.

3.1.1 On the terminal object

In this section, we compute a concrete terminal object $T : \hat{\mathcal{C}}$, show that it is indeed terminal, and that it is constituted of one element at each level c, then we transfer this result to "the" terminal object $\mathbf{1}$.

Fact 3.4: Constant functor

For two categories $\mathcal{C}$, $\mathcal{D}$, we have a functor $\Delta : \mathcal{D} \to [\mathcal{C}, \mathcal{D}]$, sending $d : \mathcal{D}$ to the constant functor $\Delta_c : \mathcal{C} \to \mathcal{D}$ sending everything to d (and id_d).

We call $\mathbf{1}_{\mathrm{Type}}$, a terminal object in Type.

Lemma 3.1: $\mathbf{1}_{\mathrm{Type}}$ is unique

$\mathbf{1}_{\mathrm{Type}}$ is inhabited and for every $a, b : \mathbf{1}_{\mathrm{Type}}$, a = b.

Proof. (Lemma 3.1) Consider the functions $h : \mathrm{Bool} \to \mathbf{1}_{\mathrm{Type}}$ sending every on true on a and false on b. We also have $h \circ \neg : \mathrm{Bool} \to \mathbf{1}_{\mathrm{Type}}$, hence by universal property of a terminal object, $h = h \circ \neg$, hence $h(0) = h(1)$, that is $a = b$. As a default element, we take the image of $\star$ via the unique map $\mathrm{Unit} \to \mathbf{1}_{\mathrm{Type}}$. □

Definition 3.1: T

We define the presheaves $\mathrm{T} \triangleq \Delta_{\mathbf{1}_{\mathrm{Type}}}$.

As a consequence of Lemma 3.1, for every $c : \mathcal{C}$, there is a unique object in $\mathrm{T}(c)$.

Definition 3.2: Map to T

For every $X : \hat{\mathcal{C}}$, we define $\mathrm{T\_from}^X : X \to \mathrm{T}$ such that $\mathrm{T\_from}^X_c(x)$ is the unique element of $\mathrm{T}(c)$. $\mathrm{T\_from}^X$ obviously defines a natural transformation.

Lemma 3.2: Unique map to T

For every $X : \hat{\mathcal{C}}$, there exists a unique map from X to T, given by $\mathrm{T\_from}^X$.

Proof. (Lemma 3.2) If $t : X \to \mathrm{T}$, then for all $x : X$, $t_c(x) = \mathrm{T\_from}^X_c(x)$ by uniqueness of $\mathrm{T}(c)$, thus by double extensionality, $t = \mathrm{T\_from}^X$. □

Thus T is a terminal object, hence $T \simeq \mathbf{1}$, as terminal objects are unique up to isomorphisms.

Lemma 3.3: Transfert of uniqueness via isomorphisms

If $f : X \simeq Y$, then for all $c : \mathcal{C}$, $X(c)$ is unique if and only if $Y(c)$ is unique.

Proof. (Lemma 3.3) If $f : X \simeq Y$, in particular $f_c : X(c) \simeq Y(c)$ in Type, but an isomorphism in Type is exactly a bijection, thus $X(c)$ is unique if and only if $Y(c)$ is unique. □

As a consequence, for each $c : \mathcal{C}$, there is exactly one element in $\mathbf{1}(c)$.

3.1.2 On the initial object

We use the axiom of choice to chose $\mathbf{0}$, any initial object of $\hat{\mathcal{C}}$.

Fact 3.5: Initial in Type is empty

$\mathbf{0}_{\mathrm{Type}} \simeq \emptyset$, where $\emptyset : \mathrm{Type}$ is the empty type and $\mathbf{0}_{\mathrm{Type}}$ is an initial object of Type.

Fact 3.6: Strict initial

In a cartesian closed category, an initial i object is strict, meaning that any arrow $f : \cdot \to i$ is an isomorphism.

Lemma 3.4: An initial object is empty at each level

For every $c : \mathcal{C}$, $\mathbf{0}(c) \simeq \emptyset$.

Proof. (Lemma 3.4) By transitivity, it suffices to show that $\mathbf{0}(c) \simeq \mathbf{0}_{\mathrm{Type}}$, and by strict initiality, it suffices to find an arrow from $\mathbf{0}(c)$ to $\mathbf{0}_{\mathrm{Type}}$. We first take $! : \mathbf{0} \to \Delta_{\mathbf{0}_{\mathrm{Type}}}$, the (unique) arrow given by the initiality of $\mathbf{0}$, and we simply evaluate it in c. □

3.1.3 On pullbacks

We will show here that pullbacks are computed pointwise in $\hat{\mathcal{C}}$. We fix the following diagram in $\hat{\mathcal{C}}$:

$$\begin{tikzcd} & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

We denote by $s \otimes t$ any chosen pullback limit cone of s and t, we call respectively $(s \otimes t)^{\mathrm{X}}$, $(s \otimes t)^{\mathrm{fst}}$, and $(s \otimes t)^{\mathrm{snd}}$, its base point, its first projection, and its second projection. More generally, if w is a pullback cone on s and t (not necessarily a limit one), then we also call w^{X}, w^{fst} and w^{snd} its base, first projection and second projection.

Note that we say that we apply a particular thing in the presheaf world when we take it on a particular $c : \mathcal{C}$. for instance, the next Lemma is an applied equality.

Lemma 3.5: Applied commutation

If w is a pullback cone on s and t, then for every $c : \mathcal{C}$,

$$s_c \circ (w^{\mathrm{fst}})_c = t_c \circ (w^{\mathrm{snd}})_c$$

Proof. (Lemma 3.5) By the condition on the cone w, we have $s \circ w^{\mathrm{fst}} = t \circ w^{\mathrm{snd}}$, hence

$$s_c \circ (w^{\mathrm{fst}})_c = (s \circ w^{\mathrm{fst}})_c = (t \circ w^{\mathrm{snd}})_c = t_c \circ (w^{\mathrm{snd}})_c$$

□

Definition 3.3: Applied cone

If w is a pullback cone on s and t, for every $c : \mathcal{C}$, we define the pullback cone (in Type) w_c on s_c and t_c with $w_c^{\mathrm{fst}} = (w^{\mathrm{fst}})_c$, $w_c^{\mathrm{snd}} = (w^{\mathrm{snd}})_c$, and (necessarily) $w_c^{\mathrm{X}} = w^{\mathrm{X}}(c)$. The equality of Lemma 3.5 ensures that this defines a pullback cone.

For instance, with a diagram, $(s \otimes t)_c$ is just the "slice" in c of the following diagram of presheaves, and it constitutes a cone.

$$\begin{tikzcd}
(s\otimes t)^{\mathrm{X}} \arrow[r, "(s\otimes t)^{\mathrm{snd}}"] \arrow[d, "(s\otimes t)^{\mathrm{fst}}"'] & Y \arrow[d, "t"] & & (s\otimes t)^{\mathrm{X}}(c) \arrow[r, "(s\otimes t)_c^{\mathrm{snd}}"] \arrow[d, "(s\otimes t)_c^{\mathrm{fst}}"'] & Y(c) \arrow[d, "t_c"] \\
X \arrow[r, "s"'] & Z & \overset{\text{in } c}{\rightsquigarrow} & X(c) \arrow[r, "s_c"'] & Z(c)
\end{tikzcd}$$

We define walking_cospan to be the base category of a pullback cone, that is the one with three object, and two non-identity arrows, like in the following:

$$\begin{tikzcd}
 & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}$$

For f, g with same codomain in any category $\mathcal{D}$, we define cospan $f\ g$ to be the functor:

$$\text{cospan } f\ g : \text{walking\_cospan} \to \mathcal{D}$$

mapping the horizontal arrow (of the category walking_cospan) to f and the vertical arrow to g, and the objects to be the appropriate domains and codomains.

Fact 3.7: Evaluation functor

For any categories $\mathcal{C}$ and $\mathcal{D}$, we have a functor $\mathrm{ev}^{\mathcal{C},\mathcal{D}} : \mathcal{C} \to [[\mathcal{C}, \mathcal{D}], \mathcal{D}]$ mapping $c : \mathcal{C}$ to the functor $\mathrm{ev}_c^{\mathcal{C},\mathcal{D}} : [\mathcal{C}, \mathcal{D}] \to \mathcal{D}$ sending a functor $F : [\mathcal{C}, \mathcal{D}]$ to $F(c)$.

Lemma 3.6: Composition of evaluation and cospan

$$\mathrm{ev}^{\mathcal{C}^{\mathrm{op}},\mathrm{Type}}(c) \circ \text{cospan } s\ t = \text{cospan } s_c\ t_c$$

Proof. (Lemma 3.6) The proof is by extensionality of functors, and it suffices to look at each objects and maps in walking_cospan. ☐

If $F : \mathcal{J} \to \mathcal{D}$ is a functor with a limit, we call $\lim F$ the limit object and $\lim^j : \lim F \to F(j)$ the j^{th} projection.

Fact 3.8: Limits are computed pointwise

Let $\mathcal{J}, \mathcal{K}$ and $\mathcal{D}$ be three categories. If $F : \mathcal{J} \to [\mathcal{K}, \mathcal{D}]$ has a limit (which is then a functor in $[\mathcal{K}, \mathcal{D}]$), then for any $k : \mathcal{K}$, we have:

$$\phi_{F,k} : (\lim F)(k) \simeq \lim(\mathrm{ev}^{\mathcal{K},\mathcal{C}}(k) \circ F)$$

Fact 3.9: Limit's projections are computed pointwise

Let $\mathcal{J}, \mathcal{K}$ and $\mathcal{D}$ be three categories. If $F : \mathcal{J} \to [\mathcal{K}, \mathcal{D}]$ has a limit, then for any $k : \mathcal{K}$ and $j : \mathcal{J}$, we have:

$$(\lim^j F)_k = (\lim^j(\mathrm{ev}^{\mathcal{K},\mathcal{C}}(k) \circ F)) \circ \phi_{F,k}$$

We will now apply these facts to the case of pullbacks, to show that the canonical pullback of s_c and t_c is the isomorphic to the applied pullback $()s \otimes t)_c$.

Lemma 3.7: Base point isomorphism of applied pullback

We have:

$$\phi_{(\mathrm{cospan}\ s\ t),c} : (s \otimes t)_c^{\mathrm{X}} \simeq (s_c \otimes t_c)^{\mathrm{X}}$$

Lemma 3.8: First commutation of the applied pullback

We have:

$$(s \otimes t)_c^{\mathrm{fst}} = (s_c \otimes t_c)^{\mathrm{fst}} \circ \phi_{(\mathrm{cospan}\ s\ t),c}$$

Lemma 3.9: Second commutation of the applied pullback

We have:

$$(s \otimes t)_c^{\mathrm{snd}} = (s_c \otimes t_c)^{\mathrm{snd}} \circ \phi_{(\mathrm{cospan}\ s\ t),c}$$

Proof. (Lemma 3.7, Lemma 3.8, and Lemma 3.9) This is directly the instanciation of the previous results, but this is a mess in Lean, as there are a lot of issues concerning dependant equality, and in particular the use of Lemma 3.6. ☐

Definition 3.4: Isomorphism of applied pullback cones

We define the cone isomorphism:

$$\mathrm{iso\_app\_cone}_{s,t,c} : (s \otimes t)_c \simeq s_c \otimes t_c$$

via $\phi_{(\mathrm{cospan}\ s\ t),c}$ that can be lifted to a morphism of pullback cone thanks to Lemma 3.8 and Lemma 3.9.

Finally, we add an auxiliary lemma, to use the previous result with arbitrary pullback limits.

Lemma 3.10: Applied isomorphism from isomorphism

If w, w' are two pullback cone on s and t, and $i : w \simeq w'$, then $w_c \simeq w'_c$.

Proof. (**Lemma 3.10**) It suffices to find an isomorphism $j : w_c^{X} \simeq w_c'^{X}$ that commutes with the projections. The underlying morphism of the cone morphism i applied at c, that we call $i_c : w_c^{X} \to w_c'^{X}$ works. Indeed, the composition of cone morphism is functorial, that is the underlying morphism of the composition of two cone morphisms is the composition of the two underlying morphisms. □

Theorem 3.1: Applied cones are limits

Let w be a pullback cone on s and t. If w is a limit cone, then for every $c : \mathcal{C}$, w_c is a limit cone.

Proof. (**Theorem 3.1**) If w is limit, then by uniqueness of the limit, $w \simeq s \otimes t$, thus by **Lemma 3.10**, $w_c \simeq (s \otimes t)_c$, and $\mathrm{iso\_app\_cone}_{s,t,c} : (s \otimes t)_c \simeq s_c \otimes t_c$, the latter being a limit cone, thus by transitivity $w_c \simeq s_c \otimes t_c$, and the limits are preserved by isomorphism, thus w_c is a limit cone. □

Fact 3.10: Mono from pullback

Let $f : c \to d$ be a morphism in a category $\mathcal{D}$. The diagram

$$\begin{tikzcd} c \arrow[r, equal, "\mathrm{id}_c"] \arrow[d, equal, "\mathrm{id}_c"'] & c \arrow[d, "f"] \\ c \arrow[r, "f"'] & d \end{tikzcd}$$

is a pullback if and only if f is a mono.

Theorem 3.2: Natural transformation and monomorphisms

Let $m : S \rightarrowtail X$ be a mono in $\hat{\mathcal{C}}$, then m_c is a mono in Type.

Proof. (**Theorem 3.2**) By **Fact 3.10**, it suffices to show that

$$\begin{tikzcd} S(c) \arrow[r, equal, "\mathrm{id}_{S(c)}"] \arrow[d, equal, "\mathrm{id}_{S(c)}"'] & S(c) \arrow[d, "m_c"] \\ S(c) \arrow[r, "m_c"'] & X(c) \end{tikzcd}$$

is a pullback, but we know, again by **Fact 3.10**, that

$$\begin{tikzcd}
S(c) \arrow[r, equal, "\mathrm{id}_S"] \arrow[d, equal, "\mathrm{id}_S"'] & S \arrow[d, "m"] \\
S(c) \arrow[r, "m"'] & X
\end{tikzcd}$$

is a pullback, hence **Theorem 3.1** telles us that its applied cone is a pullback cone. □

Note that the converse is true in any category.

3.2 Categories of presheaves are toposes

With the previous auxiliary results, we are now ready to prove that a category of presheaves is a topos. The only missing point is the construction of a subobject classifier, as we already know that a category of presheaves is cartesian closed and has all finite limits, thanks to **Fact 3.3**.

We keep the same notations as before.

3.2.1 Defining the subobject classifier

We define the functor $\Omega : \mathbf{1} \to \hat{\mathcal{C}}$ and the classifying arrow of any mono.

Recall that a sieve S on $c \in \mathcal{C}$ is a set of arrows of $\mathcal{C}$ pointing to c that are closed under composition, that is if $f : b \to c \in S$, then for every $g : a \to b$, $f \circ g \in S$.

Definition 3.5: Sieve map

For $c, d \in \mathcal{C}$, $f : c \to d$, and S a sieve on D, we define

$$\text{sieve\_map}(f)(S) \triangleq \{h : \cdot \to c \mid f \circ h \in S\}$$

Definition 3.6: Ω

We define the presheaf $\Omega \in \hat{\mathcal{C}}$, that sends c to $\Omega(c)$ the set of sieves on c and sends $f : d \to c$ in $\mathcal{C}^{\mathrm{op}}$ to $\text{sieve\_map}(f)$. This is indeed a functor as

$$\text{sieve\_map}(\mathrm{id}_c)(S) = \{h : \cdot \to c \mid \mathrm{id}_c \circ h \in S\} = S$$

and

$$\text{sieve\_map}(g \circ f)(S) = \{h : \cdot \to c \mid g \circ f \circ h \in S\} = \text{sieve\_map}(g)(\text{sieve\_map}(f)(S))$$

Fact 3.11: Lattice on $\Omega(c)$

For every $c \in \mathcal{C}$, $\Omega(c)$ has a structure of complete lattice, we call $\top_c$, or $\top$ if there is no ambiguity, its top element, consisting of all arrow pointing to c.

Definition 3.7: The truth

We define the arrow $\text{truth} : \mathbf{1} \to \Omega$ such that $\text{truth}_c(x) = \top$. This obviously defines a natural transformation.

Let $m : S \rightarrowtail X$ be a mono in $\hat{\mathcal{C}}$.

Definition 3.8: Classifying arrow at c

For $c \in \mathcal{C}$, we define

$$\begin{aligned} \chi_c^m : X(c) &\to \Omega(c) \\ x &\mapsto \{g : d \to c \mid \exists y : S(d),\ m_d(y) = X(g)(x)\} \end{aligned}$$

If $f : d \to c \in \chi_c^m(x)$, and $g : e \to d$ in $\mathcal{C}$, then we know that there is a $w : S(d)$ such that $m_d(w) = X(f)(x)$. Moreover,

$$m_e(S(g)(w)) = (m_e \circ S(g))(w) = (X(g) \circ m_d)(w) = X(g)(X(f)(x)) = X(f \circ g)(x)$$

Thus $f \circ g \in \chi_c^m(x)$.

Definition 3.9: Classifying arrow

We define $\chi^m : X \to \Omega$ by $(\chi^m)_c = \chi_c^m$. To prove that this is a natural transformation, let $f : d \to c$ in $\mathcal{C}$, $x : X(c)$, we need to prove that for any $g : e \to d$ in $\mathcal{C}$, $g \in \chi_d^m(X(f)(x)) \iff g \in \Omega(f)(\chi_c^m(x))$, by unfolding the definition of Ω, we have that

$$\begin{aligned} g \in \Omega(f)(\chi_c^m(x)) &\iff g \in \text{sieve\_map}(f)(\chi_c^m(x)) \\ &\iff g \in \{h : \cdot \to c \mid f \circ h \in \chi_c^m(x)\} \\ &\iff f \circ g \in \chi_c^m(x) \\ &\iff \exists y : S(e),\ m_e(y) = X(f \circ g)(x) \\ &\iff \exists y : S(e),\ m_e(y) = X(g)(X(f)(x)) \\ &\iff g \in \chi_d^m(X(f)(x)) \end{aligned}$$

3.2.2 Applied classifiers

For any $d \in \mathcal{D}$, with $\mathcal{D}$ a category with a terminal object, we call $!_d : d \to \mathbf{1}_{\mathcal{D}}$ the unique arrow. We fix a mono $m : S \rightarrowtail X$ and $c \in \mathcal{C}$. We will show that m_c and $!_c$ constitutes a pullback cone on χ_c^m and truth_c.

Lemma 3.11: Maximal sieve

Let $y : X(c)$, then
$$\exists w : S(c),\ m_c(w) = y \iff \chi_c^m(y) = \top$$

Proof. (Lemma 3.11) For the direct implication, suppose we have $w : S(c)$ with $m_c(w) = y$. We want to show that for any $f : d \to c$, $f \in \chi_c^m(y)$, that is $\exists x : S(d),\ m_d(x) = X(f)(y)$, and

$$X(f)(y) = X(f)(m_c(w)) = m_d(S(d)(w))$$

which proves $f \in \chi_c^m(y)$. For the converse, we use that $\text{id}_c \in \chi_c^m(y)$, meaning that

$$\exists w : S(c),\ m_c(w) = X(\text{id}_c)(y) = \text{id}_{X(c)}(y) = y$$

as the identity function in Type is the usual identity function. ∎

Definition 3.10: A type for pullbacks

We define a type:

$$\text{pb\_obj}^{m,c} \triangleq \{y : X(c) \mid \exists x : S(c),\ m_c(x) = y\}$$

Let u be a pullback cone on χ^m and truth.

Lemma 3.12: Pullback condition as proposition

We have:

$$\forall x : u^{\mathrm{X}}(c),\ \chi^m_c \circ u^{\mathrm{fst}}_c(x) = \top$$

Proof. (Lemma 3.12) This is Lemma 3.5 evaluated at x, as $\text{truth}_c(!_{u^{\mathrm{X}}_c}(x)) = \top$. □

Definition 3.11: Lift to pb_obj

We define

$$\begin{aligned} \text{lift\_to\_pb\_obj}^{u,c} : u^{\mathrm{X}}(c) &\to \text{pb\_obj}^{m,c} \\ x &\mapsto u^{\mathrm{fst}}_c(x) \end{aligned}$$

This is well defined, indeed by Lemma 3.12, we have $\chi^m_c \circ u^{\mathrm{fst}}_c(x) = \top$, hence Lemma 3.11 indicates $\exists w : S(c),\ m_c(w) = u^{\mathrm{fst}}_c(x)$, which is the required condition to be of type $\text{pb\_obj}^{m,c}$.

Definition 3.12: Lift to $S(c)$

We define

$$\begin{aligned} \text{pb\_obj\_to\_obj}^{u,c} : \text{pb\_obj}^{m,c} &\to S(c) \\ x &\mapsto w \text{ such that } m_c(w) = u^{\mathrm{fst}}_c(x) \end{aligned}$$

by elimination of the existential quantifier on the fact that $x : \text{pb\_obj}^{m,c}$.

Definition 3.13: Applied pullback lift

e define

$$\text{lift}^{u,c} = \text{pb\_obj\_to\_obj}^{u,c} \circ \text{lift\_to\_pb\_obj}^{u,c} : u^{\mathrm{X}}(c) \to S(c)$$

Lemma 3.13: Commutation of the applied lift

We have:

$$m_c \circ \text{lift}^{u,c} = u^{\mathrm{fst}}_c$$

Proof. (Lemma 3.13) We want to show $m_c(\text{lift}^{u,c}(x)) = u^{\mathrm{fst}}_c(x)$, but by definition, $\text{lift}^{u,c}(x)$ is such that $m_c(\text{lift}^{u,c}(x)) = u^{\mathrm{fst}}_c(x)$. □

The second commutation will be automatic as the codomain is the terminal object. Now, we want to show that this lift satisfies the necessary condition to be upgraded later into a natural transformation.

Lemma 3.14: Naturality of the lift

Let $c, d \in \mathcal{C}$, and $f : d \to c$ in $\mathcal{C}$. We have

$$\mathrm{lift}^{u,d} \circ u^{\mathrm{X}}(f) = S(f) \circ \mathrm{lift}^{u,c}$$

Proof. (Lemma 3.14) We know that m_d is a mono according to **Theorem 3.2**, thus we might as well prove:

$$m_d \circ \mathrm{lift}^{u,d} \circ u^{\mathrm{X}}(f) = m_d \circ S(f) \circ \mathrm{lift}^{u,c}$$

and

$$\begin{aligned} m_d \circ S(f) \circ \mathrm{lift}^{u,c} &= X(f) \circ m_c \circ \mathrm{lift}^{u,c} && \text{by naturality of } m \\ &= X(f) \circ u_c^{\mathrm{fst}} && \text{by Lemma 3.14} \\ &= u_d^{\mathrm{fst}} \circ u^{\mathrm{X}}(f) && \text{by naturality of } u^{\mathrm{fst}} \\ &= m_d \circ \mathrm{lift}^{u,d} \circ u^{\mathrm{X}}(f) && \text{by Lemma 3.14} \end{aligned}$$

□

We basically have all the data to show that our construction is a pullback. We will now prove that χ_c^m is the only arrow that can classify m_c, that is if we have another arrow $\sigma : X \to \Omega$ such that $(m_c, !_{S(c)})$ is a limit pullback cone on σ_c and truth_c, then $\sigma_c = \chi_c^m$. To this end, let us pick $\sigma : X \to \Omega$, and suppose that m and $!_S$ defines a limit pullback cone on σ and truth.

Lemma 3.15: Maximal sieve for a pullback

For $y : X(c)$, we have

$$(\exists x : S(c),\ m_c(x) = y) \iff \sigma_c(y) = \top$$

Proof. (Lemma 3.15) The direct implication is Lemma 3.5 applied on a x such that $m_c(x) = y$. For the converse, we will use the universal property of the pullback. First we construct a cone with base $\mathbf{1}_{\mathrm{Type}}$. The first projection is $\hat{y} : \mathbf{1}_{\mathrm{Type}} \to X(c)$ that picks $y : X(c)$, and the second is the unique map from $\mathbf{1}_{\mathrm{Type}}$ to itself. The commutation is precisely ensured by the hypothesis $\sigma_c(y) = \top$. By universal property, we have a map $l : \mathbf{1}_{\mathrm{Type}} \to S(c)$, such that $m_c \circ l = \hat{y}$, evaluating this equality at $\star : \mathbf{1}_{\mathrm{Type}}$, we get $m_c(l(\star)) = y$, and thus $l(\star) : S(c)$ is the required existential witness. □

Fact 3.12: Facts on sieves

Let $f : d \to c$ in $\mathcal{C}$, then $\Omega(f)(S) = \top \iff f \in S$.

Lemma 3.16: Top sieve

For $f : d \to c$ in $\mathcal{C}$ and $y : X(c)$, we have:

$$f \in \sigma_c(y) \iff \sigma_d(X(f)(y)) = \top$$

Proof. (Lemma 3.16) We have:

$$\begin{aligned}
\sigma_d(X(f)(y)) = \top &\iff \Omega(f)(\sigma_c(y)) = \top && \text{by naturality of } \sigma \\
&\iff f \in \sigma_c(y) && \text{by Fact 3.12}
\end{aligned}$$

□

Now we give the final version of our lemma that will prove the uniqueness.

Lemma 3.17: In the sieve, existential version

For $f : d \to c$ in $\mathcal{C}$ and $y : X(c)$, we have:

$$f \in \sigma_c(y) \iff \exists x : S(d),\ m_d(x) = X(f)(y)$$

Proof. (Lemma 3.17) We have

$$\begin{aligned}
f \in \sigma_c(y) &\iff \sigma_d(X(f)(y)) = \top && \text{by Lemma 3.16} \\
&\iff \exists x : S(d),\ m_d(x) = X(f)(y) && \text{by Lemma 3.15}
\end{aligned}$$

□

Theorem 3.3: σ is the classifier

For all $y : X(c)$, we have $\sigma_c(y) = \chi^m_c(y)$.

Proof. (Theorem 3.3) We have

$$\begin{aligned}
f \in \sigma_c(y) &\iff \exists x : S(d),\ m_d(x) = X(f)(y) && \text{by Lemma 3.17} && (1) \\
&\iff \chi^m_d(X(f)(y)) = \top && \text{by Lemma 3.11} && (2) \\
&\iff f \in \chi^m_c(y) && \text{by Lemma 3.16} && (3)
\end{aligned}$$

Hence, $\sigma_c(y)$ and $\chi^m_c(y)$ have the same elements, and thus are equal. □

3.2.3 Main result

We can now gather all our previous lemma to upgrade them to the natural transformation's world, and prove that the universal property of the subobject classifier is satisfied by Ω.

Lemma 3.18: Pullback cone condition

We have:

$$\chi^m \circ m = \mathrm{truth} \circ !_S$$

Proof. (Lemma 3.18) By applying extensionality three times (on c, on $x : S(c)$, and on sieve), it suffices to show that for any $f : d \to c$, $f \in \chi^m_c(m_c(x))$, that is we want ro prove:

$$\exists y : S(d)\ m_d(y) = X(f)(m_c(x))$$

Using the naturality of m in the above proposition, we get:

$$\exists y : S(d)\ m_d(y) = m_d(S(f)(x))$$

which is clearly true using the witness $S(f)(x)$. □

Definition 3.14: Pullback cone

We define pb_cone^m to be the pullback cone on χ^m and truth with first projection m and second projection $!_S$. This indeed defines a pullback cone, thanks to the condition of **Lemma 3.18**.

Definition 3.15: Pullback lift

Let u be a pullback cone on χ^m and truth. We define $\text{lift}^u : u^{\text{X}} \to S$ to be the natural transformation whose map in $c : \mathcal{C}$ is $\text{lift}^{u,c}$. The naturality of the maps is ensured by **Lemma 3.14**.

Lemma 3.19: First commutation of the lift

Let u be a pullback cone on χ^m and truth, we have:

$$m \circ \text{lift}^u = u^{\text{fst}}$$

Proof. (**Lemma 3.19**) $(m \circ \text{lift}^u)_c = m_c \circ \text{lift}^{u,c} = u_c^{\text{fst}}$ by **Lemma 3.13**. □

Theorem 3.4: pb_cone is limit

pb_cone^m is a limit cone.

Proof. (**Theorem 3.4**) We have a cone, and for any u, a pullback cone on χ^m and truth, we have a the lift lift^u, that commutes with the first projection. It also commutes with the second projection, because $u^{\text{snd}} = \text{pb\_cone}^{\text{snd}} \circ \text{lift}^u$, as they both are maps with codomain a terminal object. The uniqueness is also easy, if $v : u^{\text{X}} \to S$ is another map that commutes with the projections, we have $m \circ v = u^{\text{fst}}$, then $m \circ v = m \circ \text{lift}^u$, hence $v = \text{lift}^u$, using that m is mono. □

Lemma 3.20: χ^m is the unique classifier

If $\sigma : X \to \Omega$ is such that m and $!_S$ defines a pullback cone on σ and truth, then $\sigma = \chi^m$.

Proof. (**Lemma 3.20**) Under these hypothesis, **Theorem 3.3** tells us that $\sigma_c(y) = \chi_c^m(y)$ for any $c : \mathcal{C}$ and $y : X(c)$, thus by extensionality $\sigma = \chi^m$. □

Theorem 3.5: $\hat{\mathcal{C}}$ is a topos

The category $\hat{\mathcal{C}}$ is a topos, with subobject classifier Ω.

Proof. (**Theorem 3.5**) The category $\hat{\mathcal{C}}$ is cartesian closed with all finite limits. The classifying arrow of a mono $m : S \rightarrowtail X$ is given by $\chi^m : X \to \Omega$. This data defines a pullback according to **Theorem 3.4**, and is the unique one that does that thanks to **Lemma 3.20**. □