

Lattice QCD and the Proton Radius

Nesreen Hasan, Jeremy Green, Stefan Meinel, Michael Engelhardt,
Stefan Krieg, John Negele, Andrew Pochinsky, and Sergey Syritsyn*
(LHP collaboration)

[1711.11385; Phys.Rev.D97.034504]

*13th Conference on the Intersections of
Particle and Nuclear Physics
Palm Springs, CA, May 29–Jun 3, 2018*

Outline

- Nucleon form factors and radii on a lattice
- Results from the physical point
- Form factors at zero momentum

Basics of Hadron Structure in Lattice QCD

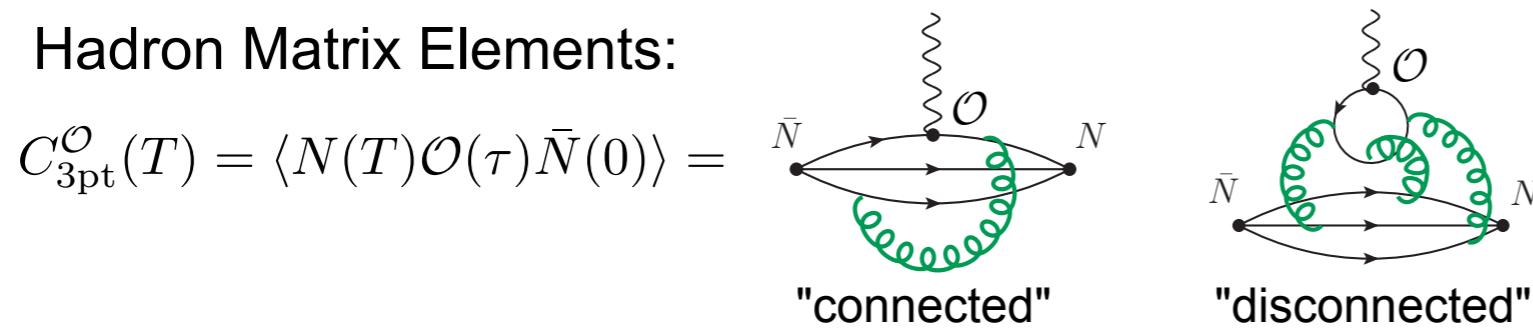
Lattice Field Theory \Leftrightarrow Numerical evaluation of the Path Integral

$$\begin{aligned} \langle q_x \bar{q}_y \dots \rangle &= \int \mathcal{D}(\text{Glue}) \int \mathcal{D}(\text{Quarks}) e^{-S_{\text{Glue}} - \bar{q}(\not{D} + m)q} [q_x \bar{q}_y \dots] \\ &= \underbrace{\int \mathcal{D}(\text{Glue}) e^{-S_{\text{Glue}}} \text{Det}(\not{D} + m)}_{\text{Hybrid Monte Carlo}} [(\not{D} + m)^{-1}_{x,y} \dots] \end{aligned}$$

Grassmann integration

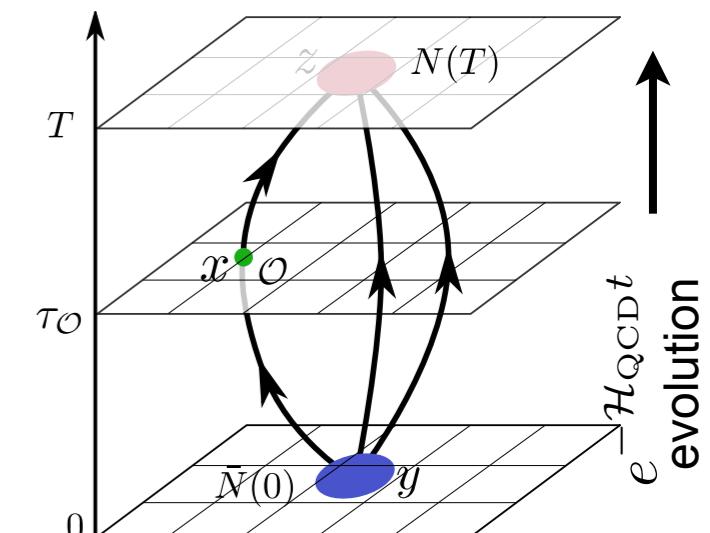
$(\not{D} + m) \cdot q = 0$
 quark motion in gluon background

Hadron Matrix Elements:



$$\begin{aligned} \langle N(T) \mathcal{O}(\tau) N(0) \rangle &= \sum_{n,m} Z_m e^{-E_n(T-\tau)} \langle n | \mathcal{O} | m \rangle e^{-E_m \tau} Z_n^* \\ &\xrightarrow[T \rightarrow \infty]{} Z_{00} e^{-M_N T} \left[\langle P' | \mathcal{O} | P \rangle + \mathcal{O} \left(\underbrace{e^{-\Delta E_{10} T}, e^{-\Delta E_{10} \tau}, e^{-\Delta E_{10}(T-\tau)}}_{\text{excited states}} \right) \right] \end{aligned}$$

Ground state
form factors

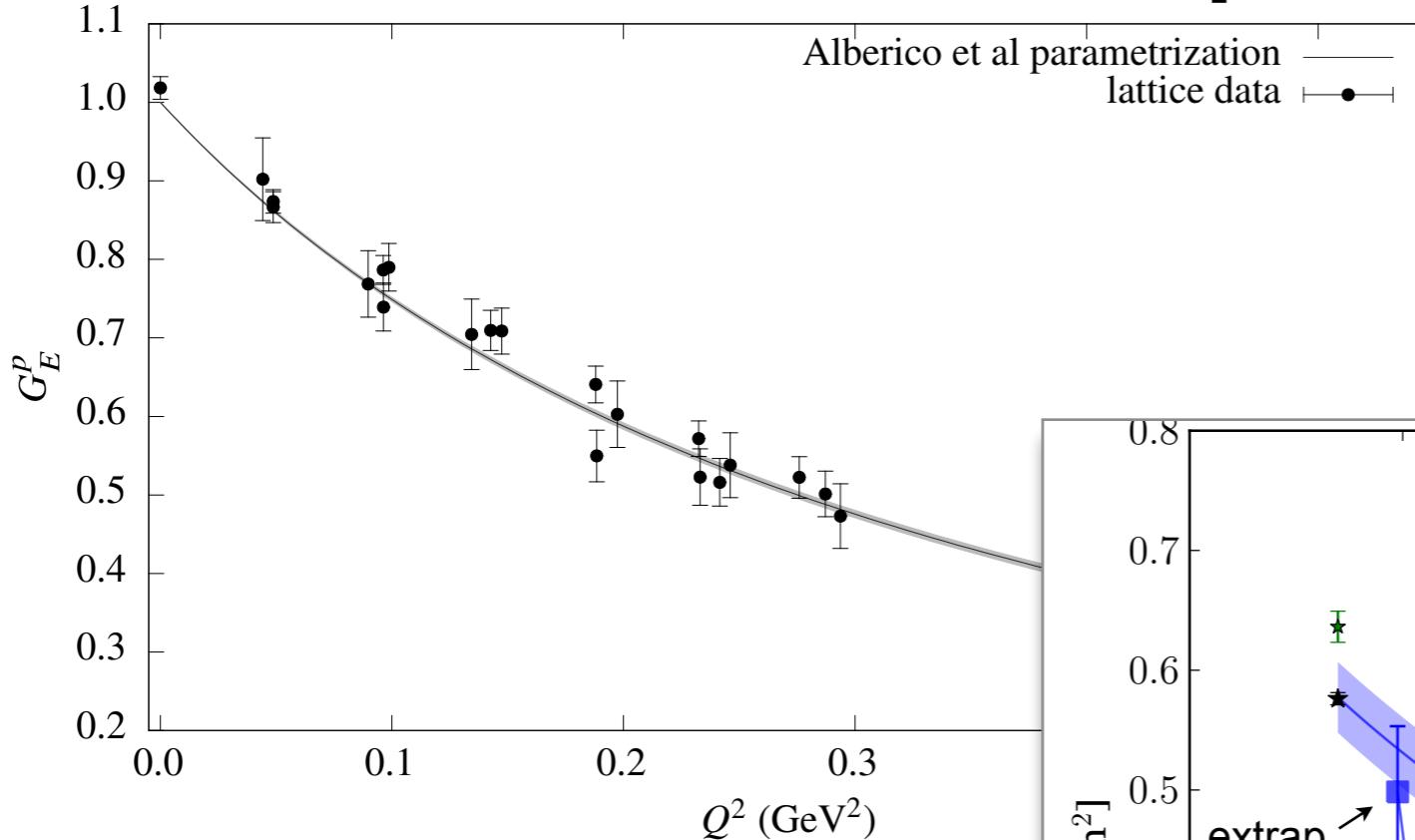


Each quark line = $(\not{D} + m)^{-1} \cdot \psi$

- Systematic effects
- excited states
 - discretization errors
 - finite volume
 - unphysical (heavy) pion mass

Electric Form Factor

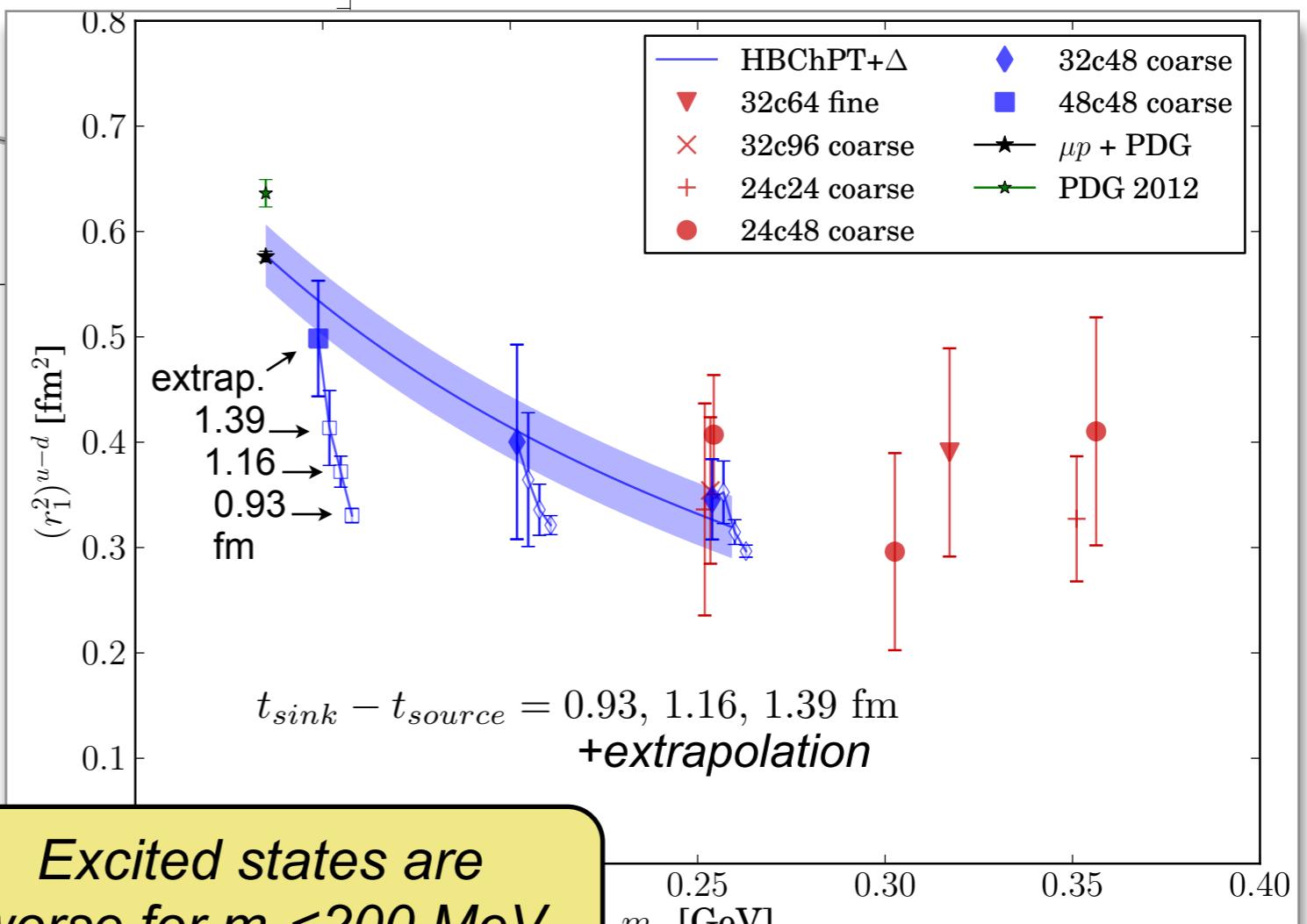
$$\langle P + q | \bar{q} \gamma^\mu q | P \rangle = \bar{U}_{P+q} \left[F_1(Q^2) \gamma^\mu + F_2(Q^2) \frac{i \sigma^{\mu\nu} q_\nu}{2M_N} \right] U_P$$



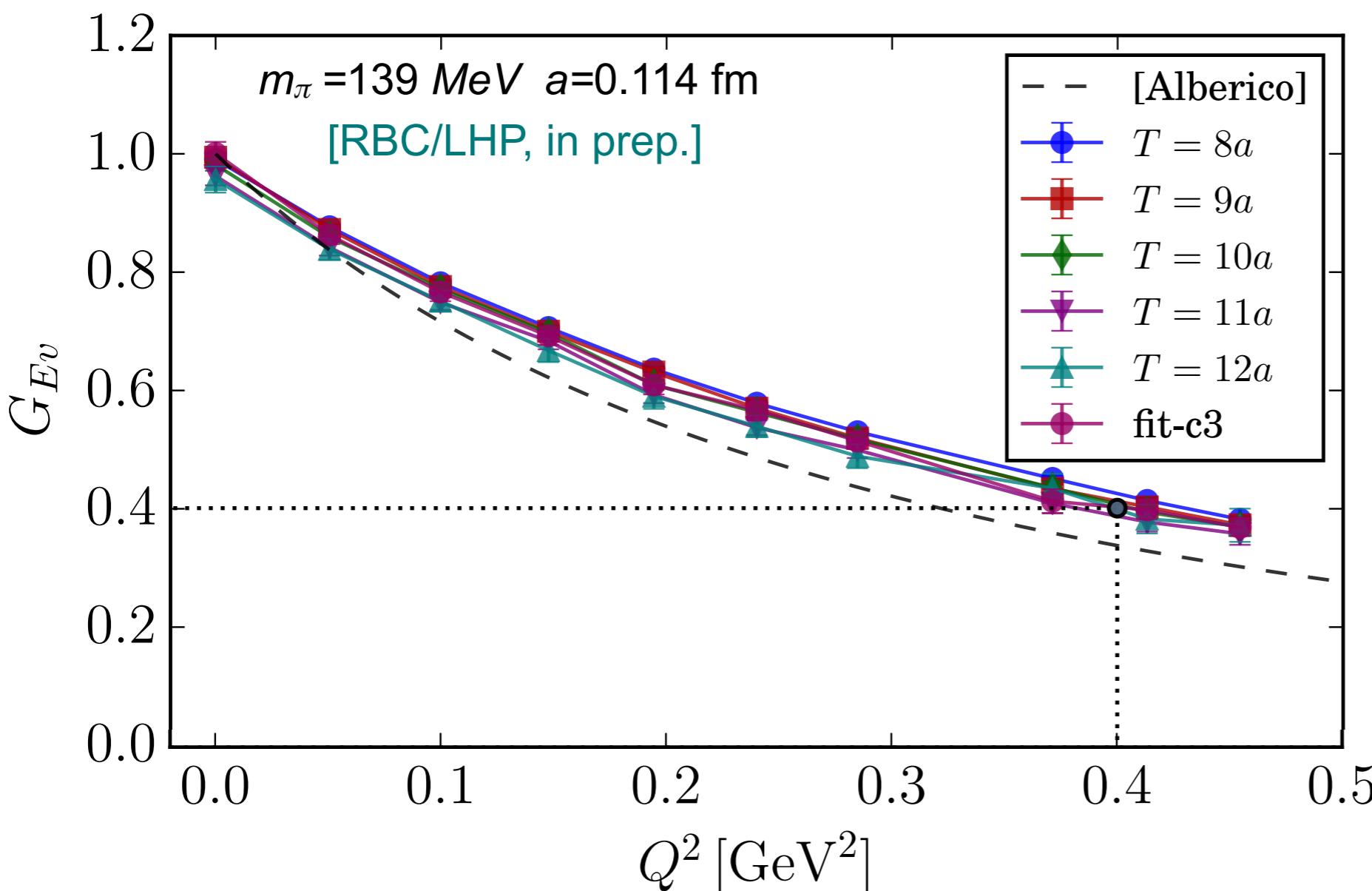
$$G_{Ep}(Q^2) \approx 1 - \frac{1}{6} Q^2 \langle r_E^2 \rangle^p + O(Q^4)$$

$$\langle r^2 \rangle = -6 \frac{dG(Q^2)}{dQ^2} \Big|_{Q^2=0}$$

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M_N^2} F_2(Q^2)$$

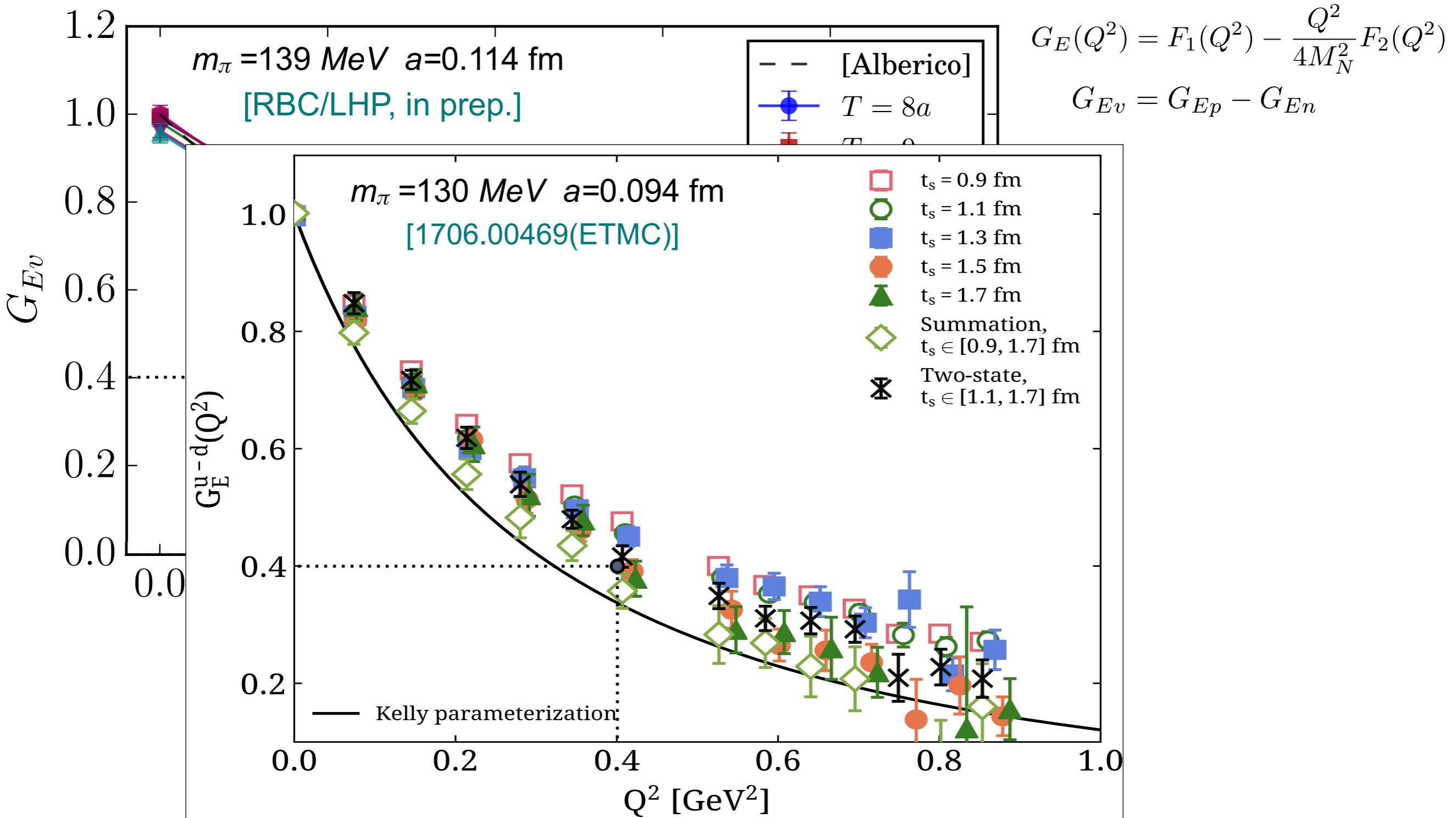


Electric Form Factor at the Physical Point

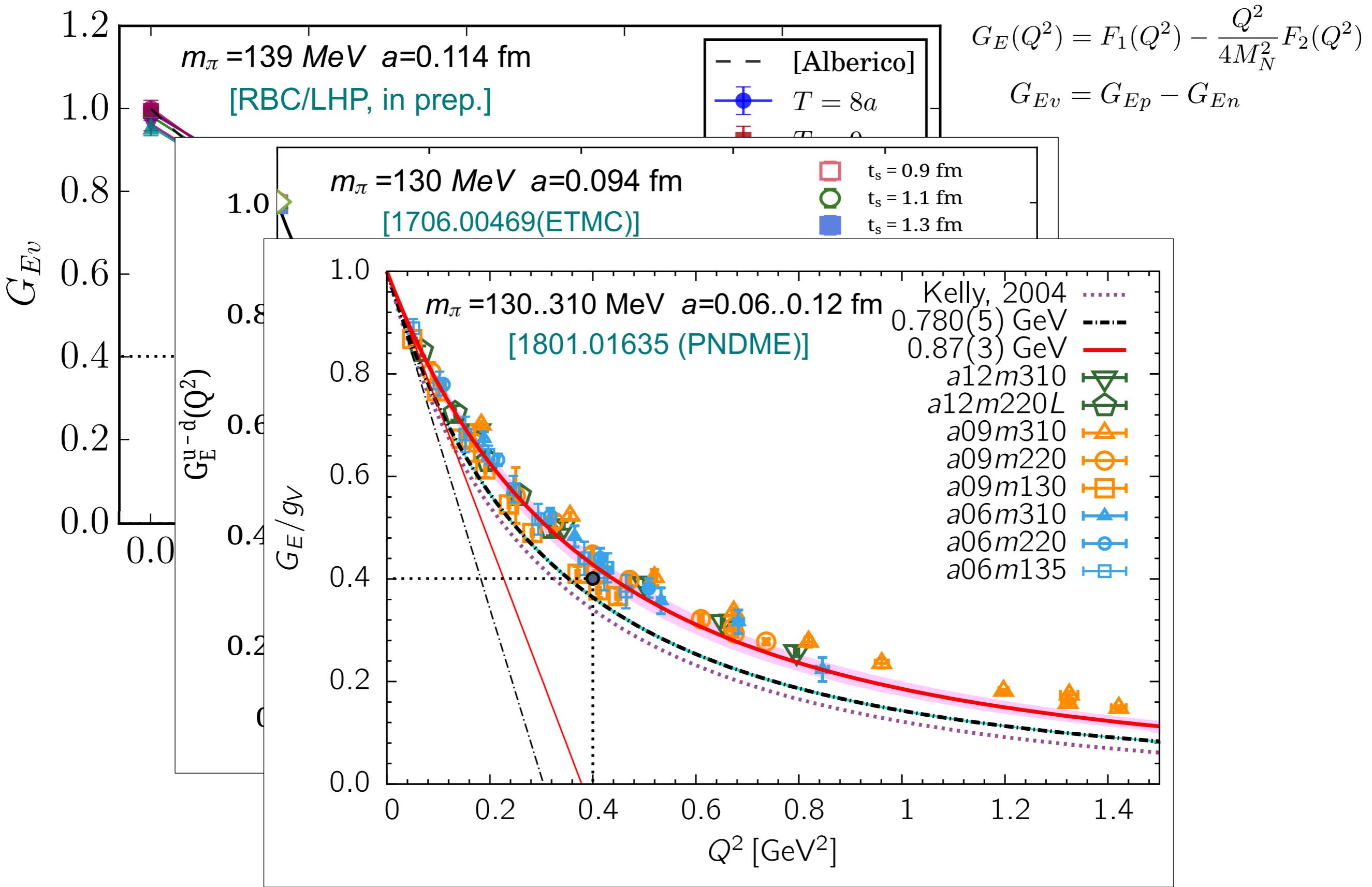


$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M_N^2} F_2(Q^2)$$
$$G_{Ev} = G_{Ep} - G_{En}$$

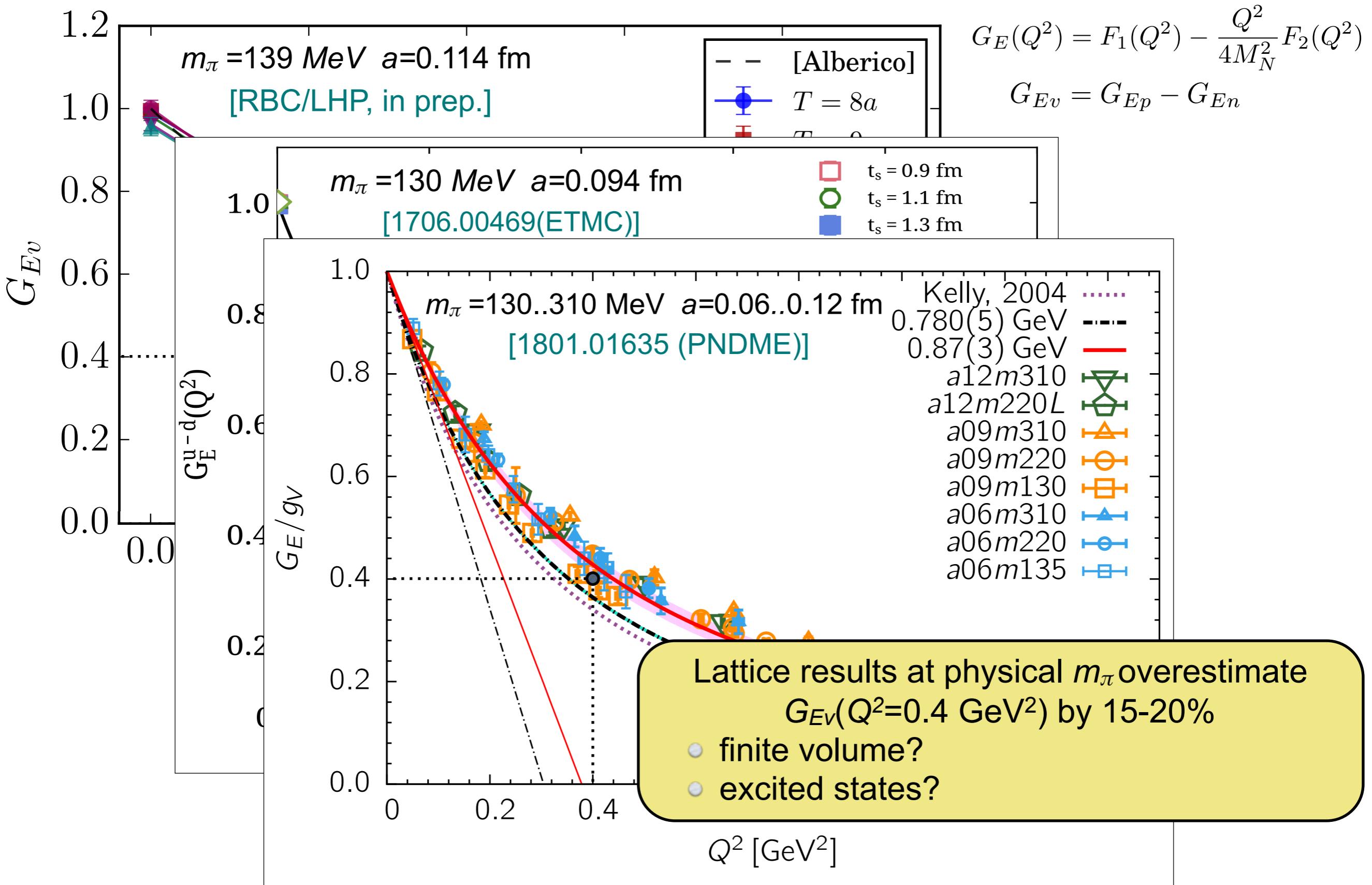
Electric Form Factor at the Physical Point



Electric Form Factor at the Physical Point



Electric Form Factor at the Physical Point



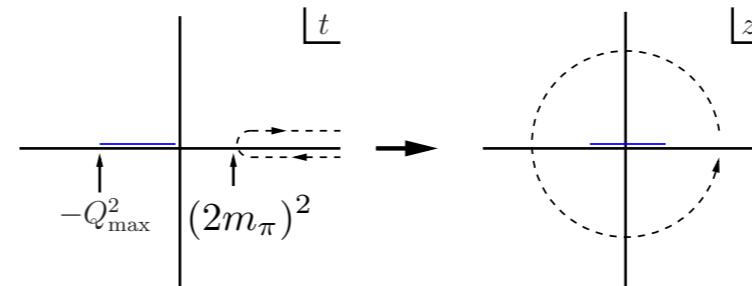
Charge Radius at/near the Physical Pion mass

- "dipole" fits

$$G(Q^2) \sim \frac{1}{(1 + Q^2/M_D^2)^2}$$

$$\langle r^2 \rangle = \frac{12}{M_D^2}$$

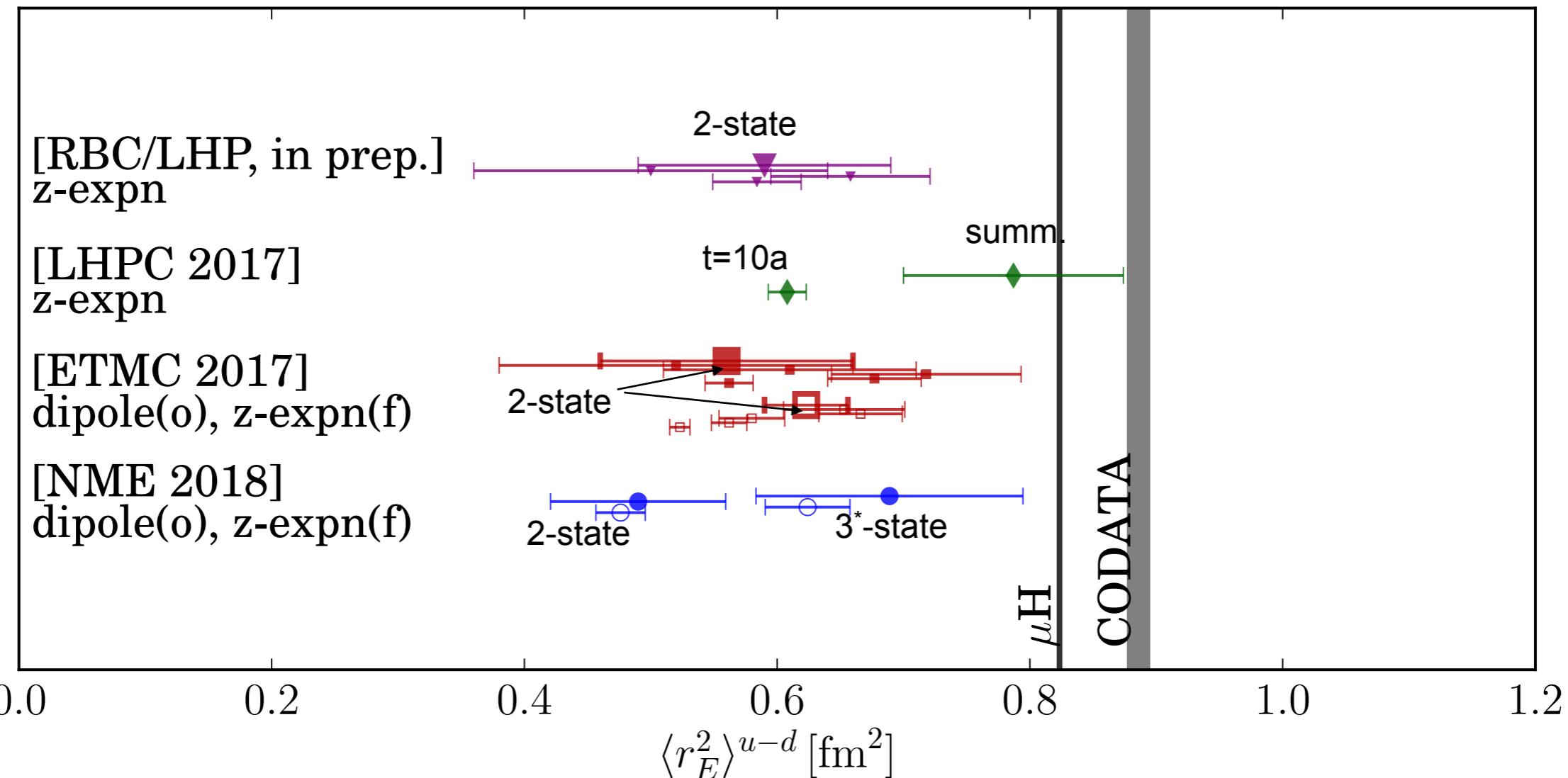
- model-independent fits (eg z-expansion)



$$z = \frac{\sqrt{t_{cut} - t} - \sqrt{t_{cut} - t_0}}{\sqrt{t_{cut} - t} + \sqrt{t_{cut} - t_0}}$$

$$G(Q^2) \sim \sum a_k [z(t = -Q^2)]^k$$

- strong contributions from excited states



Smaller Momenta: Twisted Boundary Conditions

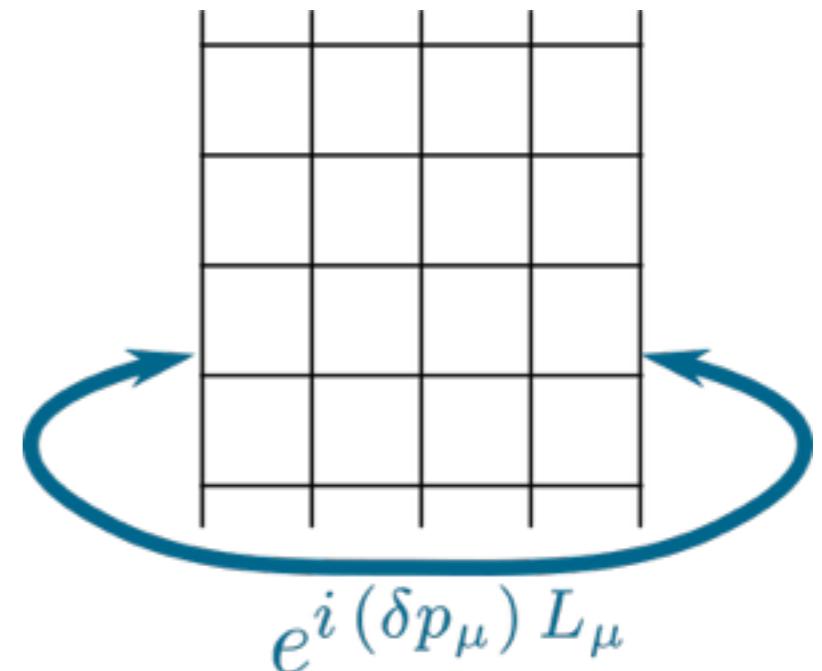
- Quantized lattice momenta for PBC

$$\psi(x + L) = \psi(x) \quad p_\mu = \frac{2\pi}{L_\mu} n_\mu$$

with minimally acceptable $m_\pi L \gtrsim 4$

$$p_{\min} \lesssim \frac{\pi}{2} m_\pi \approx 0.21 \text{ GeV}$$

$$Q_{\min}^2 \approx 0.05 \text{ GeV}^2$$



- Twisted BC $\psi(x + \hat{L}_\mu) = e^{i\theta_\mu} \psi(x)$

$$\text{arbitrary momenta } p_\mu = \frac{2\pi}{L_\mu} n_\mu + \frac{\theta_\mu}{L_\mu}$$

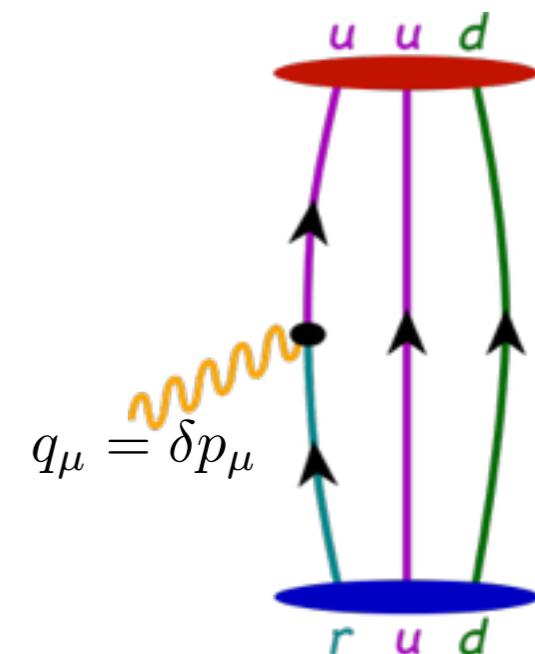
baryons with new twisted valence "flavor" r

$$\chi_{\Sigma_r} = \frac{1}{\sqrt{2}} ([rud] + [rdu]),$$

$$\chi_{\Lambda_r} = \frac{1}{\sqrt{6}} (2[udr] - [rud] - [dru]),$$

no sea twisted flavor \Rightarrow additional finite volume effects

[F.J.Jiang, B.Tiburzi (0810.1495)]



$$\langle N(\vec{p}' = 0) J_\nu(q) \bar{N}(0) \rangle$$

Expansion in Lattice Momentum: Correlators

[de Divitiis, Petronzio, Tantalo (2012)]

compute correlator expansion

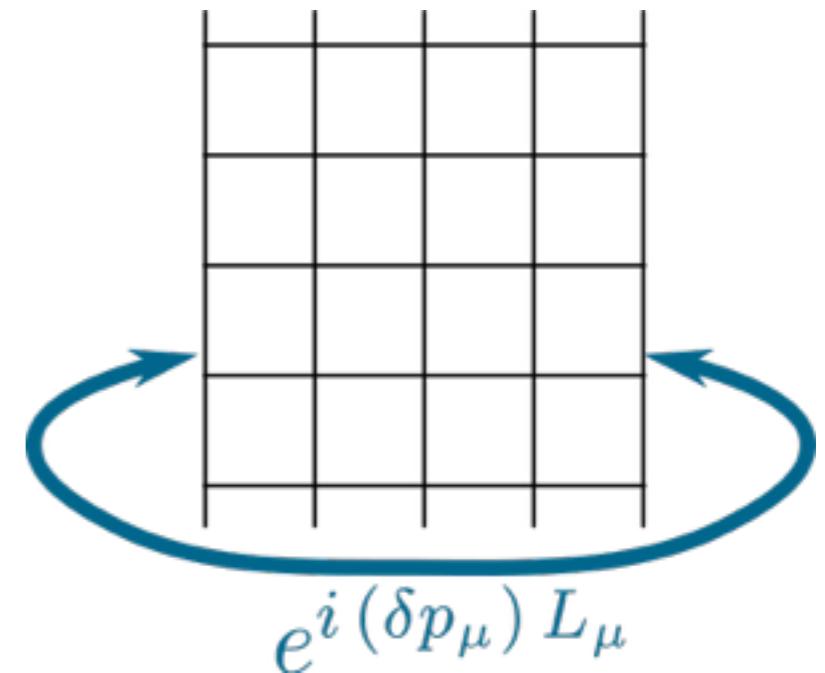
$$C(q) = C(0) + \delta p_\mu \frac{\partial C(q)}{\partial q_\mu} + \frac{1}{2} \delta p_\mu \delta p_\nu \frac{\partial^2 C(q)}{\partial q_\mu \partial q_\nu} + \dots$$

using propagator derivatives

$$\frac{\delta \not{D}^{-1}}{\delta p_\mu} = -\not{D}^{-1} \frac{\delta \not{D}}{\delta p_\mu} \not{D}^{-1}$$

$$\frac{\delta^2 \mathcal{D}^{-1}}{\delta p_\mu^2} = -\mathcal{D}^{-1} \frac{\delta^2 \mathcal{D}}{\delta p_\mu^2} \mathcal{D}^{-1} + 2\mathcal{D}^{-1} \frac{\delta \mathcal{D}}{\delta p_\mu} \mathcal{D}^{-1} \frac{\delta \mathcal{D}}{\delta p_\mu} \mathcal{D}^{-1}$$

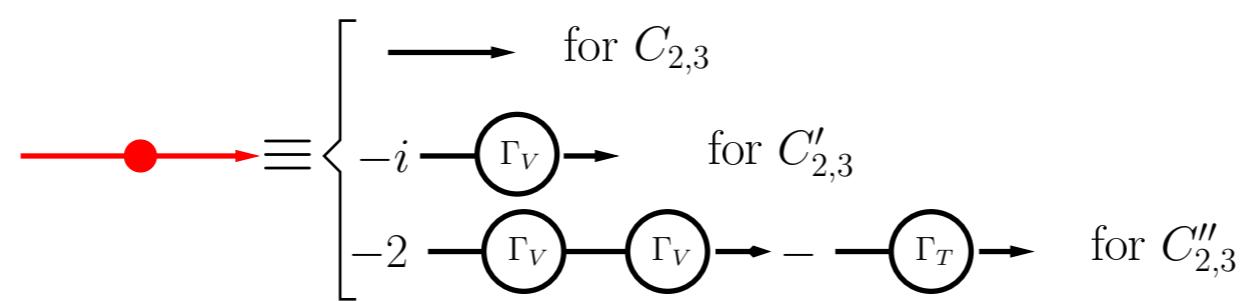
(conserved) vector current



Implementation on a lattice: compute nucleon correlators with "sequential" propagators

$$C_2^{(n)} = \begin{array}{c} \text{Diagram showing three nodes connected by curved arrows forming a loop. The top node is red, and the two bottom nodes are black. The top arrow is red, and the other two are black.} \end{array}$$

$$C_3^{(n)} = \begin{array}{c} \text{Diagram showing three nodes connected by arcs: a red circle, a black square, and a black circle. The red circle is connected to the black square by a red arc above it. The black square is connected to the black circle by a blue arc above it. A black arc connects the red circle and the black circle directly below them.} \end{array}$$



Expansion in Lattice Momentum: Matrix Elements

Estimator for matrix elements

$$R_N^X = \frac{C_3^{\mathcal{O}_X^{q,\mu}}(\vec{p}, \vec{p}', \tau, T)}{\sqrt{C_2(\vec{p}, T)C_2(\vec{p}', T)}} \quad R_S = \sqrt{\frac{C_2(\vec{p}, T - \tau)C_2(\vec{p}', \tau)}{C_2(\vec{p}', T - \tau)C_2(\vec{p}, \tau)}}$$

$$R_X^{q,\mu}(\vec{p}, \vec{p}', \tau, T) = R_N^X R_S = M_X^{q,\mu}(\vec{p}, \vec{p}') + O(e^{-\Delta E_{10}(\vec{p})\tau}) + O(e^{-\Delta E_{10}(\vec{p}')(T-\tau)}) + O(e^{-\Delta E_{min}T})$$



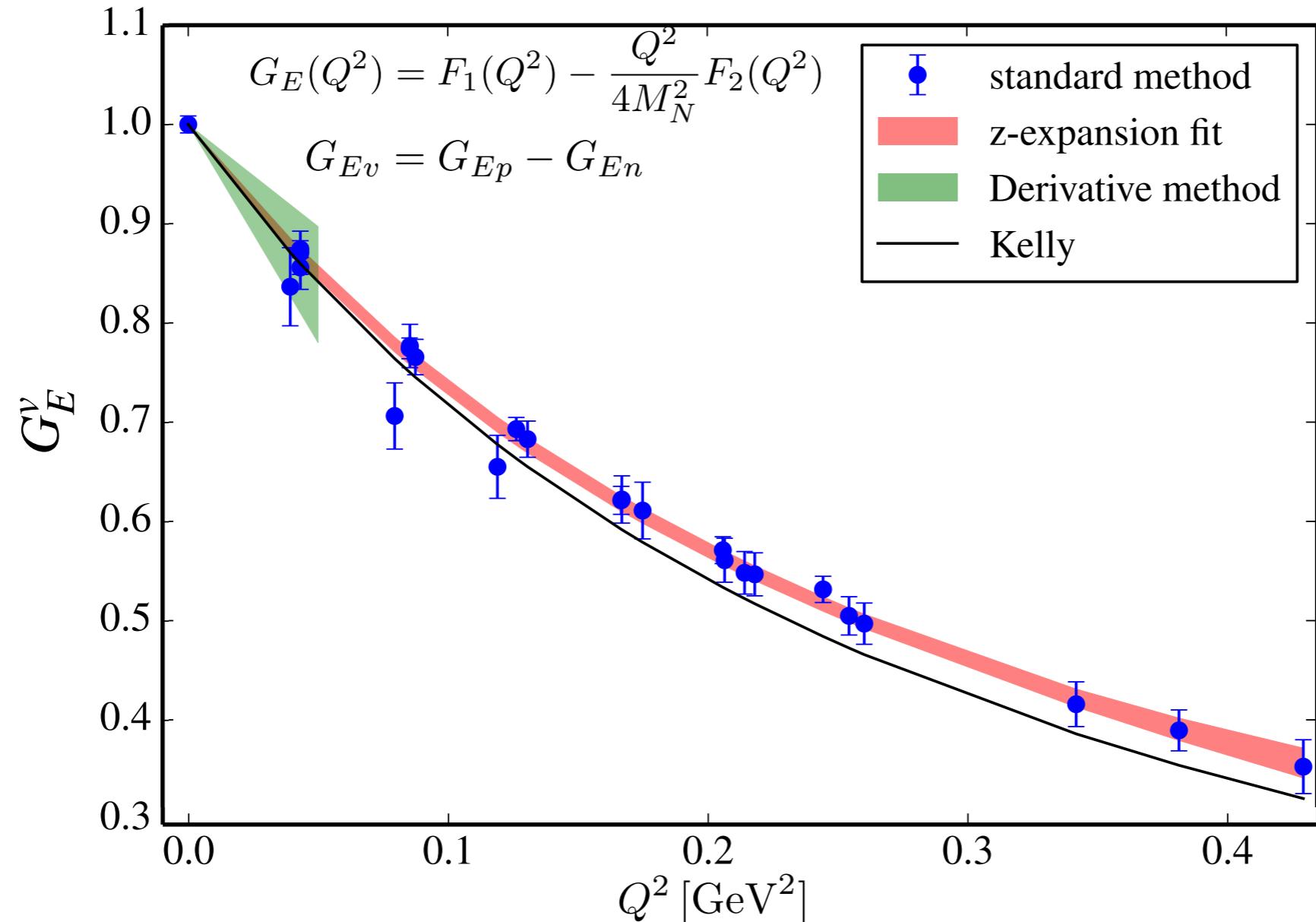
$$\frac{\sum_{\lambda, \lambda'} \bar{u}(\vec{p}, \lambda) \Gamma_{\text{pol}} u(\vec{p}', \lambda') \langle p', \lambda' | \mathcal{O}_X^{q,\mu} | p, \lambda \rangle}{4\sqrt{E(\vec{p})E(\vec{p}')(E(\vec{p})+m)(E(\vec{p}')+m)}}$$

converges to the ground state with $T \rightarrow \infty$

Vector current insertion

$$\begin{aligned} R_V^0 &= 1, & \partial_1 R_V^2 &= -\frac{i}{2m} G_M(0), \\ \partial_2 R_V^1 &= \frac{i}{2m} G_M(0) & \partial_{1,2,3}^2 R_V^0 &= -\frac{1}{4m^2} - \frac{1}{3} r_E^2, \end{aligned} \quad \rightarrow \quad \begin{aligned} \mu &= 2i m (R_V^2)', \\ r_E^2 &= -\frac{3}{4m^2} - 3 \frac{(R_V^0)''}{R_V^0} \end{aligned}$$

Isovector Electric Form Factor



comparison of G_{Ev} slope at $Q_2=0$

- z-expansion fit vs.
- twist-derivative method

$m_\pi = 135 \text{ MeV}$ $a = 0.093 \text{ fm}$ 64^4 (BMWc)
[N.Hasan et al, PRD97: 034504 (1711.11385)]

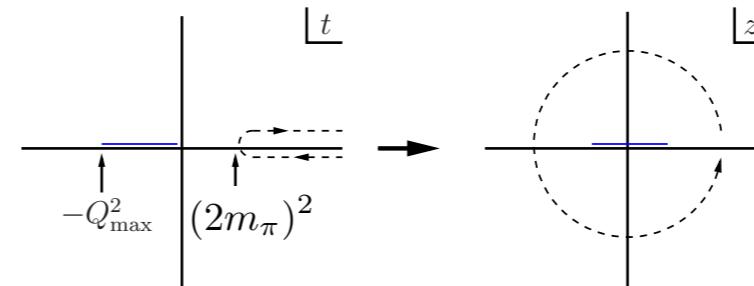
Charge Radius from FF. at Zero Momentum

- "dipole" fits

$$G(Q^2) \sim \frac{1}{(1 + Q^2/M_D^2)^2}$$

$$\langle r^2 \rangle = \frac{12}{M_D^2}$$

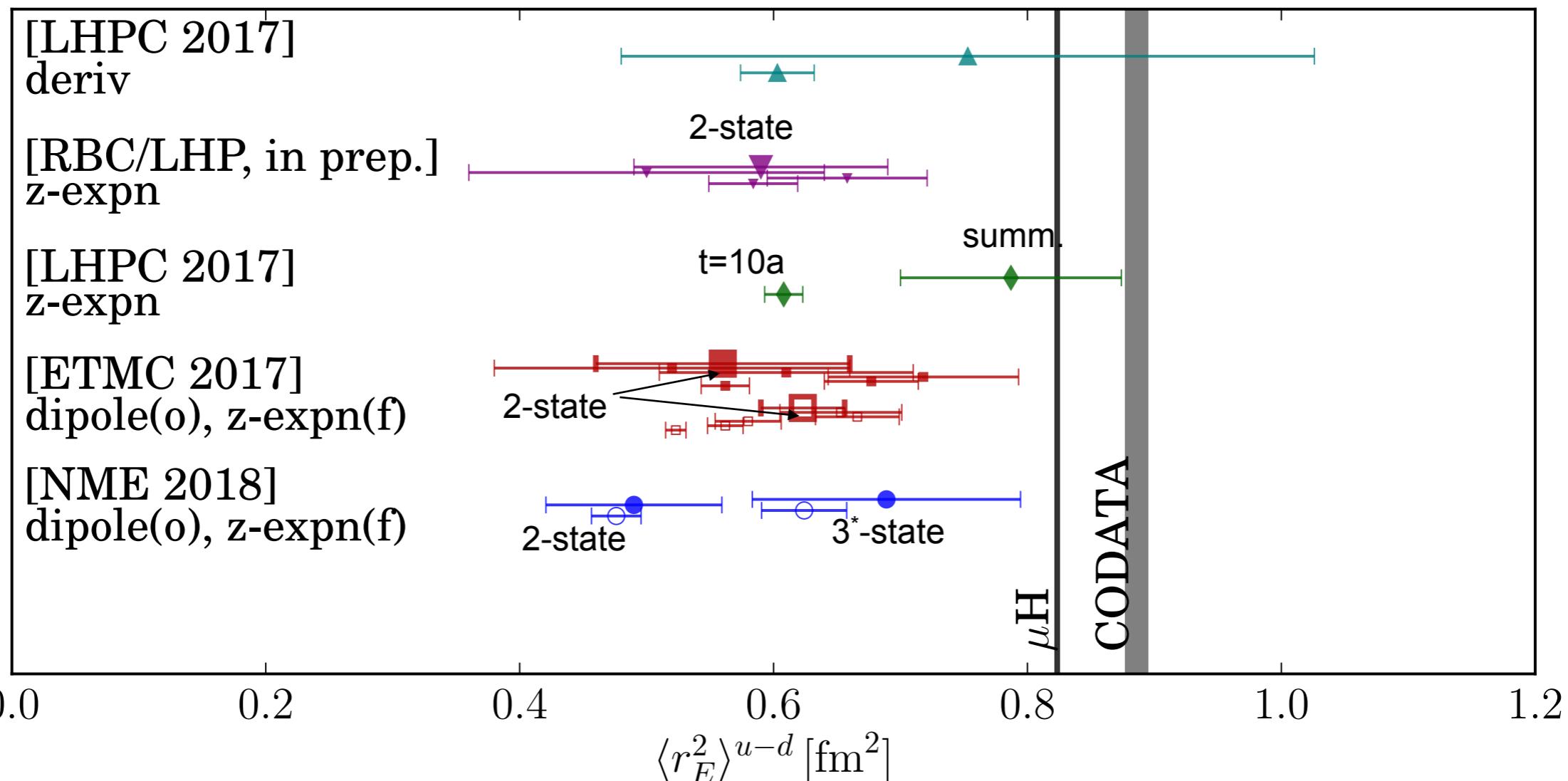
- model-independent fits (eg z-expansion)



$$z = \frac{\sqrt{t_{\text{cut}} - t} - \sqrt{t_{\text{cut}} - t_0}}{\sqrt{t_{\text{cut}} - t} + \sqrt{t_{\text{cut}} - t_0}}$$

$$G(Q^2) \sim \sum a_k [z(t = -Q^2)]^k$$

- strong contributions for excited states



Derivatives wrt. Initial and Final Momenta

Evaluate the radius from varying the initial and final momenta

[B.Tiburzi, 1407.1459]

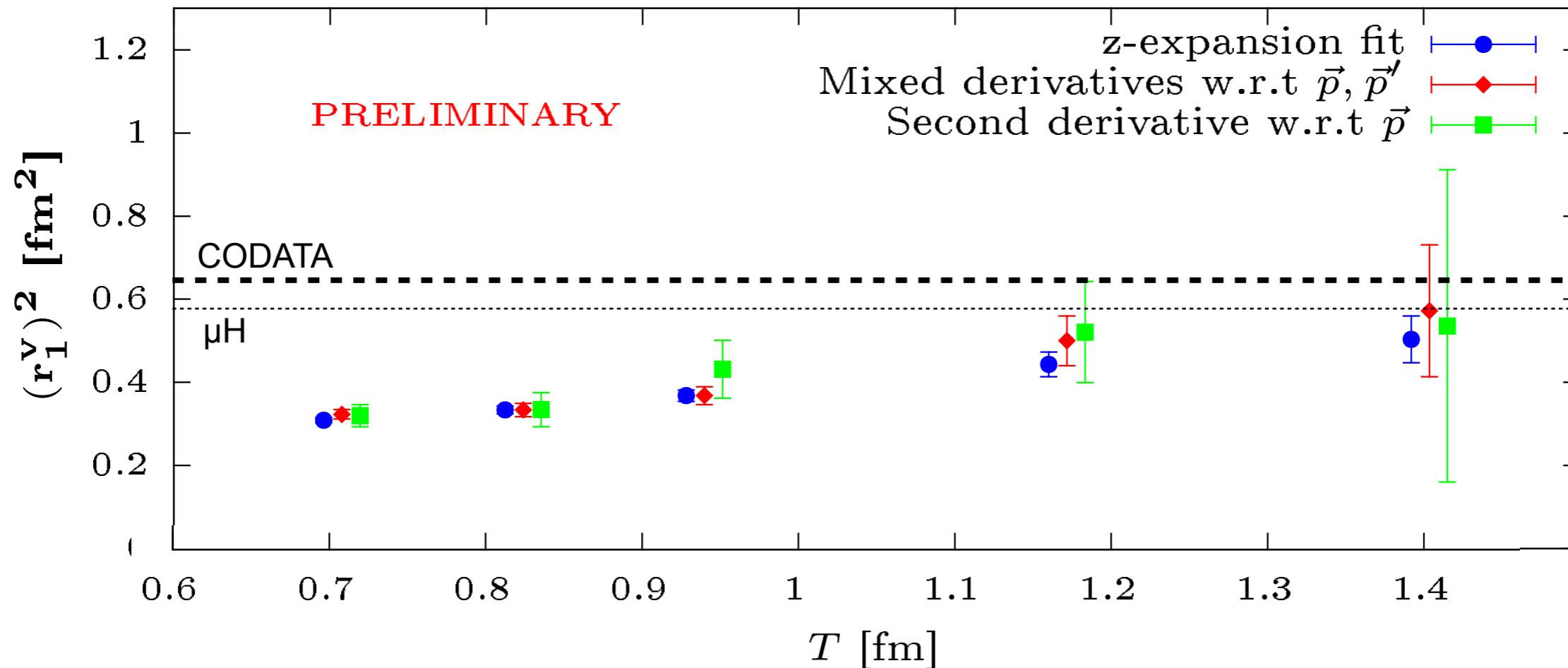
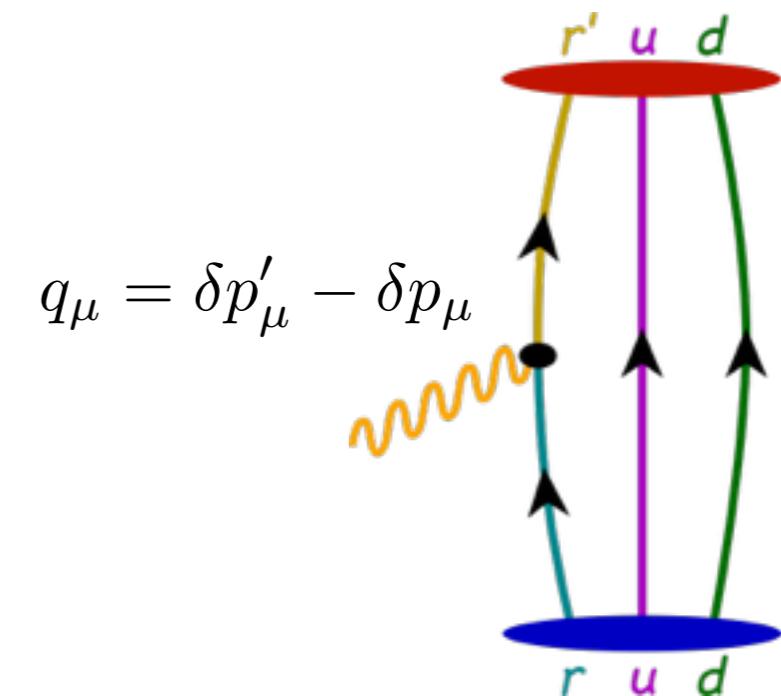
$$\frac{\partial^2}{\partial p'_i \partial p_i} \langle N(p') J^0 N(p) \rangle$$

No tadpole insertions in propagator derivatives

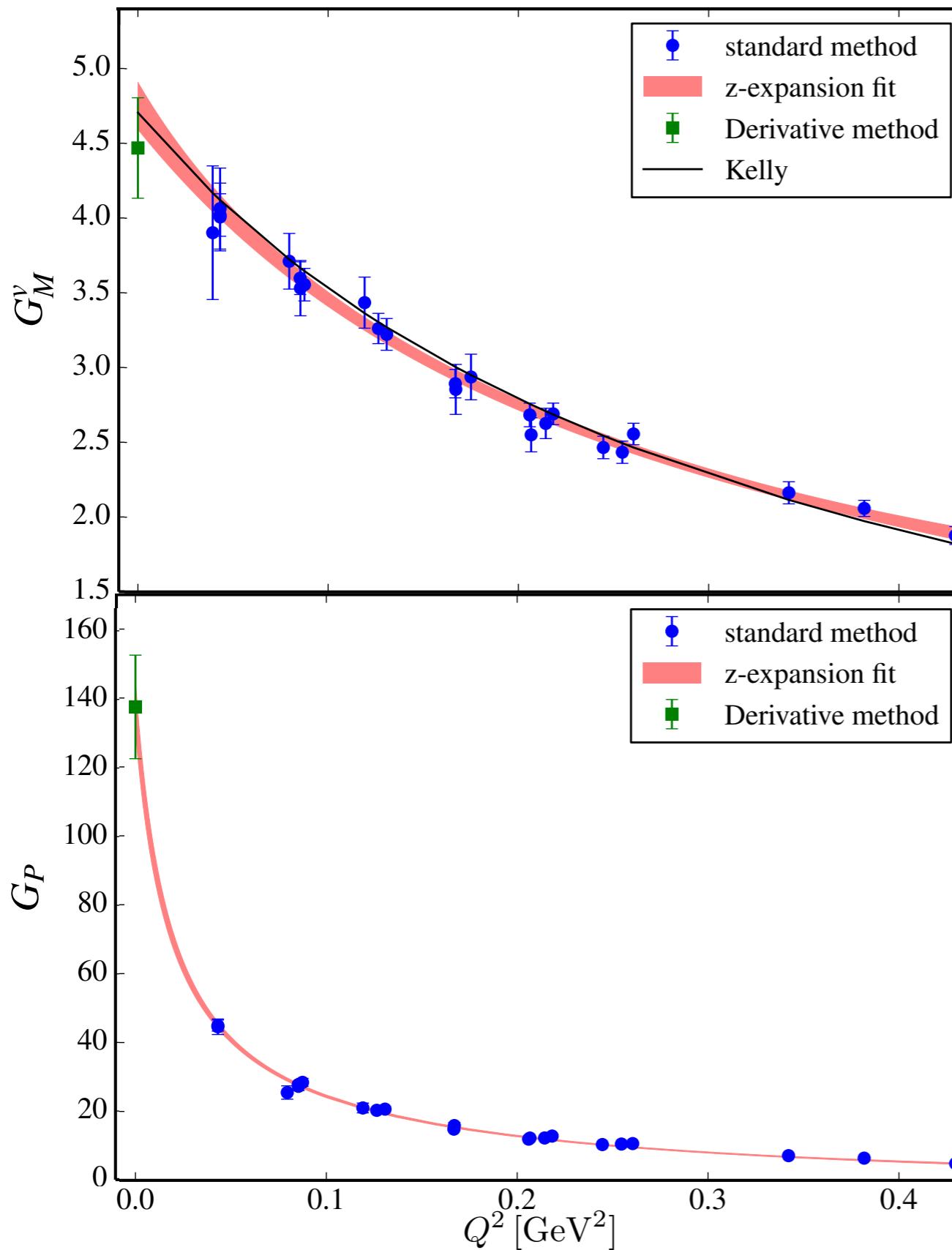
$m_\pi = 135$ MeV $a = 0.116$ fm 48^4 (BMWc)

[N.Hasan, LATTICE2017]

$$\frac{\frac{\partial}{\partial p'^j} \frac{\partial}{\partial p^j} C_3^0(\vec{p}', \vec{p}, T, \tau)|_{\vec{p}=\vec{p}'=0}}{C_2(0, \tau)} = \frac{1}{4m^2} [F_1 + 2F_2] + \frac{1}{3} F_1 [r_1]^2$$



Form Factors Vanishing in Forward Nucleon M.E.



- Sachs magnetic form factor

$$\langle p' | V_\mu | p \rangle = \bar{u}' \left[F_1 \gamma_\mu + F_2 \frac{\sigma_{\mu\nu} q^\mu}{2m_N} \right] u$$

$$G_M = F_1(Q^2) + F_2(Q^2)$$

$$\partial_1 R_V^2 = -\frac{i}{2m} G_M(0)$$

- Induced pseudoscalar form factors

$$\langle p' | A_\mu | p \rangle = \bar{u}' \left[G_A \gamma_\mu \gamma_5 + G_P \frac{q_\mu \gamma_5}{m_N} \right] u$$

$$G_P(0) = m^2 (\partial_1^2 R_A^3 + \partial_2^2 R_A^3 - 2\partial_3^2 R_A^3)$$

$m_\pi = 135$ MeV $a = 0.093$ fm 64^4 (BMWc)

[N.Hasan et al, PRD97: 034504 (1711.11385)]

Summary

- Multiple lattice results for nucleon form factors at the physical point
Chiral extrapolation in m_π no longer required
- Large systematic bias seen by all lattice groups
Overestimate $G_{E\nu}(Q^2=0.4 \text{ GeV}^2)$ by 15-20%
Underestimate isovector radius by 20-25%
- Precision for charge radius is insufficient for any conclusions
Both statistical and systematic uncertainty
- Multiple potential sources of systematic uncertainty to explore
Excited state effects
Finite volume effects
Zero-momentum extrapolation
- New promising methods for nucleon structure at zero momentum
Charge radius
Form factors vanishing from forward matrix elements