> $restart$

> $assume(z \geq 0)$

Computing the growth constant for the DL model, caterpillar with 2 leaves
K2 = Cat(CatCat+2Cat) where Cat is Catalan with no constant term

> $Catx := \frac{(1 - \mathrm{sqrt}(1 - 4 \cdot x))}{2}$

$$Catx := \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4x} \quad (1)$$

> $series(Catx, x, 6)$

$$x + x^2 + 2x^3 + 5x^4 + 14x^5 + O(x^6) \quad (2)$$

> $Catz := \frac{(1 - \mathrm{sqrt}(1 - 4 \cdot z))}{2}$

$$Catz := \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4z\sim} \quad (3)$$

> $Cat2z := Catz \cdot Catz + 2 \cdot Catz$

$$Cat2z := \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4z\sim}\right)^2 + 1 - \sqrt{1 - 4z\sim} \quad (4)$$

> $series(Cat2z, z)$

$$2z\sim + 3z\sim^2 + 6z\sim^3 + 15z\sim^4 + 42z\sim^5 + O(z\sim^6) \quad (5)$$

> $DL\_HIST := simplify(subs(\mathrm{x} = \mathrm{Cat2z}, \mathrm{Catx}))$

$$DL\_HIST := \frac{1}{2} - \frac{1}{2}\sqrt{-5 + 4z\sim + 6\sqrt{1 - 4z\sim}} \quad (6)$$

> $evalf(discont(DL\_HIST, z))$

$$\{-6.604101966, 0.104101966, 0.2500000000\} \quad (7)$$

> $series(DL\_HIST, z = 0, 10)$

$$2z\sim + 7z\sim^2 + 34z\sim^3 + 200z\sim^4 + 1318z\sim^5 + 9354z\sim^6 + 69864z\sim^7 + 541323z\sim^8 + 4310950z\sim^9 + O(z\sim^{10}) \quad (8)$$

> *# Growth constant*
> $evalf\left(\frac{1}{0.104101966}\right)$

$$9.605966519 \quad (9)$$

Caterpillar with 3 leaves
K3 = Ca(K2Cat+Cat+K2)

> $Cat3z := Catz \cdot DL\_HIST + Catz + DL\_HIST$

$$Cat3z := \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4z\sim}\right)\left(\frac{1}{2} - \frac{1}{2}\sqrt{-5 + 4z\sim + 6\sqrt{1 - 4z\sim}}\right) + 1 - \frac{1}{2}\sqrt{1 - 4z\sim} - \frac{1}{2}\sqrt{-5 + 4z\sim + 6\sqrt{1 - 4z\sim}} \quad (10)$$

> $series(Cat3z, z, 10)$

(11)

$$3\,z\sim+10\,z\sim^2+45\,z\sim^3+250\,z\sim^4+1590\,z\sim^5+11045\,z\sim^6+81420\,z\sim^7+625640\,z\sim^8+4954875\,z\sim^9+\mathrm{O}(z\sim^{10})\quad\textbf{(11)}$$

> $DL\_HIST\_CAT3 := simplify(subs(\mathrm{x}=\mathrm{Cat3z}, \mathrm{Catx}))$

$$DL\_HIST\_CAT3 := \frac{1}{2} - \frac{1}{2}\left(-\sqrt{1-4\,z\sim}\,\sqrt{-5+4\,z\sim+6\,\sqrt{1-4\,z\sim}} + 3\,\sqrt{-5+4\,z\sim+6\,\sqrt{1-4\,z\sim}} + 3\,\sqrt{1-4\,z\sim}-4\right)^{1/2}\quad\textbf{(12)}$$

> $evalf(discont(DL\_HIST\_CAT3, z))$

$$\{-6.604101966, 0.06360191820, 0.104101966, 0.2500000000\}\quad\textbf{(13)}$$

> $series(DL\_HIST\_CAT3, z=0, 10)$

$$3\,z\sim+19\,z\sim^2+159\,z\sim^3+1565\,z\sim^4+17022\,z\sim^5+197928\,z\sim^6+2413494\,z\sim^7+30490089\,z\sim^8+395828145\,z\sim^9+\mathrm{O}(z\sim^{10})\quad\textbf{(14)}$$

> $evalf\left(\frac{1}{0.06360191820}\right)$

$$15.72279623\quad\textbf{(15)}$$

Caterpillar with two leaves in the DLT model.

A DLT Caterpillar is a duplication tree, followed by either a single lineage (that can transfer to the other extinct lineage), and there are two choices for this,
or two parallel lineages that can transfer between them.
For a given lineage, the evolution is like a duplication tree but each internal node can be either a duplication or a HGT. So the equation for this is
CTz = z + 2 CTz^2.
Then for the whole caterpillar, we have Cat(CTz^2+2CTz)

> $solve(CTz - z - 2\cdot CTz^2, CTz)$

$$\frac{1}{4}+\frac{1}{4}\sqrt{1-8\,z\sim},\ \frac{1}{4}-\frac{1}{4}\sqrt{1-8\,z\sim}\quad\textbf{(16)}$$

> $CTz := \frac{1}{4}-\frac{1}{4}\sqrt{1-8\,z}$

$$CTz := \frac{1}{4}-\frac{1}{4}\sqrt{1-8\,z\sim}\quad\textbf{(17)}$$

> $series(CTz, z, 10)$

$$z\sim+2\,z\sim^2+8\,z\sim^3+40\,z\sim^4+224\,z\sim^5+1344\,z\sim^6+8448\,z\sim^7+54912\,z\sim^8+366080\,z\sim^9+\mathrm{O}(z\sim^{10})\quad\textbf{(18)}$$

> $Cat4z := CTz\cdot CTz + 2\cdot CTz$

$$Cat4z := \left(\frac{1}{4}-\frac{1}{4}\sqrt{1-8\,z\sim}\right)^2+\frac{1}{2}-\frac{1}{2}\sqrt{1-8\,z\sim}\quad\textbf{(19)}$$

> $DLT\_HIST := simplify(subs(\mathrm{x}=\mathrm{Cat4z}, \mathrm{Catx}))$

(20)

$$DLT\_HIST := \frac{1}{2} - \frac{1}{4}\sqrt{-6 + 10\sqrt{1 - 8\,z\sim} + 8\,z\sim} \tag{20}$$

> $series(DLT\_HIST, z = 0, 10)$

$$2\,z\sim + 9\,z\sim^2 + 56\,z\sim^3 + 405\,z\sim^4 + 3188\,z\sim^5 + 26538\,z\sim^6 + 230016\,z\sim^7 + 2056109\,z\sim^8 + 18834460\,z\sim^9 + O(z\sim^{10}) \tag{21}$$

> $evalf(discont(DLT\_HIST, z))$

$$\{-11.09016994, 0.090169942, 0.1250000000\} \tag{22}$$

> $evalf\left(\frac{1}{0.090169942}\right)$

$$11.09017016 \tag{23}$$

>