EXAMPLE COMPUTATION OF "THROUGHPUT MAXIMIZATION IN MULTI-BAND OPTICAL NETWORKS WITH COLUMN GENERATION"

Cao Chen
State Key Laboratory of Advanced
Optical Communication Systems and Networks,
Shanghai Jiao Tong University,
Shanghai 200240, China
chen.cao@alumni.sjtu.edu.cn

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ABSTRACT

This report is supplementary material of a paper entitled "Throughput Maximization in Multi-Band Optical Networks with Column Generation", authored by Cao Chen, Shilin Xiao, Fen Zhou, and Massimo Tornatore. This report provides an example computation for a 4-node optical network using the Integer Linear Programming (ILP) model and Column Generation (CG) approach. Results show fewer variables required for the CG approach compared to the ILP model.

1 Problem Statement

Routing and Wavelength Assignment (RWA) is the classical resource allocation problem in optical networks[1]. This paper studies the RWA problem considering a fixed-grid optical network, where each connection occupies one wavelength. $G = (V, E)$ is used to denote an optical network with node set V and link set E. A link $l \in E$ is characterized by span number N_l. All links have an equal amount of wavelengths, denoted by W.

A lightpath p denotes a simple path from the source and destination with a particular wavelength. The assigned wavelength should be consistent across all traversed links to maintain the wavelength continuity. The carried transmission capacity C_p is mainly a function of the path length $\sum_{l \in p} N_l$ and the margin requirement on its band (the physical layer model and description of the margin requirement on optical bands are neglected in this report).

Given a normalized traffic demand matrix $\hat{D}$, i.e., $\sum_{s,d} \hat{D}_{s,d} = 1$, we need to allocate transmission capacities for each node pair using adequate lightpaths. The objective is to maximize the network throughput, providing that the traffic demand distribution follows the given traffic demand matrix $\hat{D}$. The lightpaths are subject to the following constraints,

Wavelength conflict constraint. No two lightpaths can use the same wavelength on the same link.

Wavelength limitation constraint. Available wavelengths should not exceed the spectrum resources.

Demand-capacity constraint. Traffic demand of a node pair should not exceed the total transmission capacities, so that we can evaluate the network throughput under a non-blocking state.

2 Small Example

We consider the RWA problem in a 4-node network shown in Fig. 2. There are 8 wavelengths. Each link has two fibers that support bidirectional traffic. Three demands $(1, 4)$, $(2, 3)$, and $(2, 4)$ are assumed, and the traffic demand is uniformly distributed. We discuss the number of variables required to optimize the lightpaths for these demands via two approaches, the Integer Linear Programming (ILP) model and the Column Generation (CG) approach.

Note that both approaches have assumed K different paths for each node pair, $K = 3$.

- $(1, 4) : \{p_{124}, p_{134}, p_{14}\}$ with transmission capacity $C_{p_{124}} = 100, C_{p_{134}} = 100, C_{p_{14}} = 100$.
- $(2, 3) : \{p_{243}, p_{213}, p_{2143}\}$ with transmission capacity $C_{p_{243}} = 100, C_{p_{213}} = 100, C_{p_{2143}} = 50$.
- $(2, 4) : \{p_{24}, p_{2134}, p_{214}\}$ with transmission capacity $C_{p_{24}} = 250, C_{p_{2134}} = 100, C_{p_{214}} = 50$.

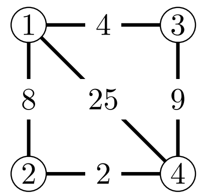

Figure 1: Illustration of a 4-node network with link number showing the length in multiples of $100\,\mathrm{km}$, $W = 8$.

2.1 Integer Linear Programming Model

This section describes the ILP model. We use $p_{s,d,k}$ to denote the k^{th} route between two nodes s and d. Let $\delta_{s,d,k,w}$ be the indicator whether the lightpath $p_{s,d,k}$ is assigned to a wavelength w, and $T_{s,d}$ represent the total transmission capacities for the lightpaths connecting (s, d). Thus, we have $T_{s,d} = \sum_k \sum_w \delta_{s,d,k,w} C_{s,d,k}$.

Parameters

- $p_{s,d,k}$: path connecting (s, d) on a certain k route. It may also simply be noted as $p_{s..d}$ with the route subscript.
- $C_{s,d,k}$: transmission capacity of path $p_{s,d,k}$.

Variables

- $\delta_{s,d,k,w}$: equals 1 if path $p_{s,d,k}$ uses a wavelength w, 0 otherwise.
- TH : network throughput.

$$\max_{\delta_{s,d,k,w} \in \{0,1\}} TH$$

$$\textit{s.t.} \quad \frac{1}{3}TH + (-100)\delta_{1,4,p_{124},1} + (-100)\delta_{1,4,p_{134},1} + (-100)\delta_{1,4,p_{14},1} + (-100)\delta_{1,4,p_{124},2} + \cdots \leq 0 \tag{1a}$$

$$\frac{1}{3}TH + (-100)\delta_{2,3,p_{243},1} + (-100)\delta_{2,3,p_{213},1} + (-50)\delta_{2,3,p_{2143},1} + (-100)\delta_{2,3,p_{243},2} + \cdots \leq 0 \tag{1b}$$

$$\frac{1}{3}TH + (-250)\delta_{2,4,p_{24},1} + (-100)\delta_{2,4,p_{2134},1} + (-50)\delta_{2,4,p_{214},1} + (-250)\delta_{2,4,p_{24},2} + \cdots \leq 0 \tag{1c}$$

$$\delta_{1,4,p_{124},1} \leq 1, \qquad (\ell = \ell_{12}, w = 1) \tag{1d}$$

$$\delta_{1,4,p_{134},1} + \delta_{2,3,p_{213},1} + \delta_{2,4,p_{2134},1} \leq 1, \qquad (\ell = \ell_{13}, w = 1) \tag{1e}$$

$$\delta_{1,4,p_{14},1} + \delta_{2,3,p_{2143},1} + \delta_{2,4,p_{214},1} \leq 1, \qquad (\ell = \ell_{14}, w = 1) \tag{1f}$$

$$\cdots$$

$$\delta_{2,4,p_{214},8} \leq 1, \qquad (\ell = \ell_{21}, w = 8) \tag{1g}$$

$$\delta_{2,3,p_{243},8} + \delta_{2,3,p_{2143},8} \leq 1, \qquad (\ell = \ell_{43}, w = 8) \tag{1h}$$

The objective is to maximize the network throughput, subject to the uniformly distributed network demand profile. Note that, the constraints of this model are written following a standard linear programming form of $\max\{cx \mid Ax \leq b, x \geq 0\}$. Constraints (1a)-(1c) ensure that the traffic demand distribution of $(1, 4)$, $(2, 3)$, and $(2, 4)$ is compliant with the network demand profile. Constraints (1d)-(1h) ensure no resource conflict for each wavelength and for each link. The maximal TH of 3000 Gbps is attained by optimizing 72 boolean variables, $\delta_{s,d,k,w}$. One possible assignment is shown in Fig. 2, where we see three paths p_{14}, p_{213}, and p_{24} on the 1$^{\text{st}}$ wavelength. This means $\delta_{1,4,3,1}(\delta_{1,4,p_{14},1})$, $\delta_{2,3,2,1}(\delta_{2,3,p_{213},1})$, and $\delta_{2,4,1,1}(\delta_{2,4,p_{24},1})$ are 1, whereas the other indicators on the 1$^{\text{st}}$ wavelength are all 0.

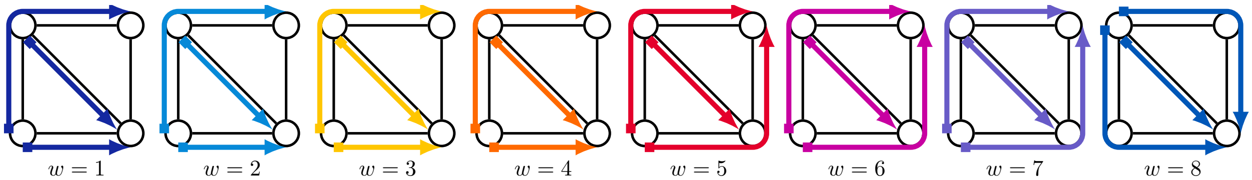

Figure 2: Illustration of a possible RWA result by the ILP model, $|\delta| = 72$.

2.2 Column Generation

This section describes the CG approach. We begin by introducing the key concept of this approach, *wavelength configuration*. Each configuration consists of multiple link-disjoint paths in the same wavelength, denoted as c. Although we call it as wavelength configuration, they are colorless before the wavelength is assigned. The first to combine the wavelength configuration with CG can date back to 2000 [2] for ring networks and developed in 2017 [3] combined with various large-scale optimization techniques. This report develops it with capacity-efficient wavelength configurations.

The related parameters associated with a configuration are expressed as follows,

- $T_{s,d,c}$: total transmission capacities of between (s, d) on a configuration c.
- z_c : number of times a configuration c is used.
- $\delta_{s,d,k,c}$: equals 1 if the path $p_{s,d,k}$ is used by configuration c, 0 otherwise.

By using the concept of wavelength configuration, the transmission capacity between (s, d) is computed as follows, $T_{s,d} = \sum_c z_c T_{s,d,c}$. In this way, we see the summation of transmission capacities is replaced from the lightpath indicator $\delta_{s,d,k,w}$ by the configuration time z_c. Some possible wavelength configurations have been listed in Table 1. Intuitively, we can use these 14 configurations to reformulate the original problem with 14 variables, $\max \left\{ TH \mid \frac{1}{3}TH \le T_{s,d}, \forall(s,d); \sum_{1\le c\le 14} z_c \le 8; z_c \ge 0 \right\}$. However, this is unnecessary for the CG approach, as the limited columns are also able to achieve optimal performance. In the CG approach, such a problem with limited columns is called as the Restricted Master Problem (RMP).

Next, we show the process of CG approach, iteratively updating the configuration set based on the following 3 simple configurations.

- $c_1 = \{p_{124}\}$ with transmission capacity matrix $[T_{1,4,1}; T_{2,3,1}; T_{2,4,1}] = [100; 0; 0]$,
- $c_2 = \{p_{243}\}$ with transmission capacity matrix $[T_{1,4,2}; T_{2,3,2}; T_{2,4,2}] = [0; 100; 0]$,
- $c_3 = \{p_{24}\}$ with transmission capacity matrix $[T_{1,4,3}; T_{2,3,3}; T_{2,4,3}] = [0; 0; 250]$

Iteration 1 RMP.

$$\max_{z_c \ge 0} TH$$

$$\textbf{\textit{s.t.}} \quad \frac{1}{3}TH + (-100)z_1 + (-0)z_2 + (-0)z_3 \le 0 \qquad [\sigma_{1,4}] \tag{2a}$$

$$\frac{1}{3}TH + (-0)z_1 + (-100)z_2 + (-0)z_3 \le 0 \qquad [\sigma_{2,3}] \tag{2b}$$

$$\frac{1}{3}TH + (-0)z_1 + (-0)z_2 + (-250)z_3 \le 0 \qquad [\sigma_{2,4}] \tag{2c}$$

$$z_1 + z_2 + z_3 \le 8 \qquad [\sigma_W] \tag{2d}$$

The optimal solution of this RMP with three configurations is $TH^* = 1000$, z^*=$[\frac{10}{3}, \frac{10}{3}, \frac{4}{3}]$. The parameters $\sigma_{s,d}$ and σ_W are the dual variables associated with the constraints (2a)-(2c), and (2d). They can be easily retrieved by the dual function in any optimization solver. For ease of explanation, we herein write the full dual problem.

Table 1: Possible wavelength configurations.

Configuration	Lightpaths	$T_{1,4}$	$T_{2,3}$	$T_{2,4}$	No. wavelengths per link*
1	p_{124}, p_{134}, p_{14}	300	0	0	[1,1,1,1,1,0,0]
2	$p_{124}, p_{134}, p_{2143}$	200	50	0	[1,1,1,1,1,1,1]
3	$p_{124}, p_{134}, p_{214}$	200	0	50	[1,1,1,1,1,1,0]
4	p_{124}, p_{14}, p_{213}	200	100	0	[1,1,1,1,0,1,0]
5	$p_{124}, p_{14}, p_{2134}$	200	0	100	[1,1,1,1,1,1,0]
6	p_{134}, p_{14}, p_{243}	200	100	0	[0,1,1,1,1,0,1]
7	p_{134}, p_{14}, p_{24}	200	0	250	[0,1,1,1,1,0,0]
8	$p_{134}, p_{243}, p_{214}$	100	100	50	[0,1,1,1,1,1,1]
9	$p_{134}, p_{2143}, p_{24}$	100	50	250	[0,1,1,1,1,1,1]
10	p_{134}, p_{24}, p_{214}	100	0	300	[0,1,1,1,1,1,0]
11	p_{14}, p_{243}, p_{213}	100	200	0	[0,1,1,1,0,1,1]
12	$p_{14}, p_{243}, p_{2134}$	100	100	100	[0,1,1,1,1,1,1]
13	p_{14}, p_{213}, p_{24}	100	100	250	[0,1,1,1,0,1,0]
14	p_{14}, p_{24}, p_{2134}	100	0	350	[0,1,1,1,1,1,0]
...	...	...	...	...	...

* The considered links include $[\ell_{12}, \ell_{13}, \ell_{14}, \ell_{24}, \ell_{34}, \ell_{21}, \ell_{43}]$.

$$
\begin{aligned}
&\min_{\sigma \geq 0, \sigma_W \geq 0} \; 8\sigma_W \\
&\textit{\textbf{s.t.}} \quad \frac{1}{3}\sigma_{1,4} + \frac{1}{3}\sigma_{2,3} + \frac{1}{3}\sigma_{2,4} \geq 1 && [TH] && \text{(3a)} \\
&\quad (-100)\sigma_{1,4} + (-0)\sigma_{2,3} + (-0)\sigma_{2,4} + \sigma_W \geq 0 && [z_1] && \text{(3b)} \\
&\quad (-0)\sigma_{1,4} + (-100)\sigma_{2,3} + (-0)\sigma_{2,4} + \sigma_W \geq 0 && [z_2] && \text{(3c)} \\
&\quad (-0)\sigma_{1,4} + (-0)\sigma_{2,3} + (-250)\sigma_{2,4} + \sigma_W \geq 0 && [z_3] && \text{(3d)}
\end{aligned}
$$

The optimal dual solution is $\sigma^* = [\sigma_{1,4}, \sigma_{2,3}, \sigma_{2,4}] = [\frac{5}{4}, \frac{5}{4}, \frac{1}{2}]$, $\sigma_W^* = 125$. Note that, by duality theory for linear programs, the current objective of the primal problem and the dual problem should be the same, i.e., $TH = 8\sigma_W$. This is true if the solution for current wavelength configurations is optimal. Consider now a case where we add a constraint to the dual problem that cuts the optimal point in the current feasible region, i.e., this constraint would make the current optimal solution (σ^*, σ_W^*) infeasible. In this case, we are expected to see the next optimal objective for the dual problem is not 1000 anymore, which could worsen to a value greater than 1000. Also by the duality theory, we can deduce that introducing such a constraint not only increases the lower bound of the dual problem (minimization problem) but also raises the upper bound of the primal problem (maximization problem). Fig. 3 illustrates the optimal primal solutions and dual solutions before and after such a constraint.

Therefore, the target is to find such a constraint that may violate the current dual problem. The violation constraint is found by adding a new configuration that minimizes the left side of the column-related constraint, like (3b), (3c), or (3d). This can be achieved by either searching the candidate wavelength configurations shown in Table 1, or solving a pricing problem as shown below.

$$
\begin{aligned}
\xi^1 = &\max_{\delta_{s,d,p} \in \{0,1\}} (T_{1,4,4})\frac{5}{4} + (T_{2,3,4})\frac{5}{4} + (T_{2,4,4})\frac{1}{2} - \sigma_W \\
&\textit{\textbf{s.t.}} \quad T_{1,4,4} = 100\delta_{1,4,p_{124}} + 100\delta_{1,4,p_{134}} + 100\delta_{1,4,p_{14}} && \text{(4a)} \\
&\quad T_{2,3,4} = 100\delta_{2,3,p_{243}} + 100\delta_{2,3,p_{213}} + 50\delta_{2,3,p_{2143}} && \text{(4b)}
\end{aligned}
$$

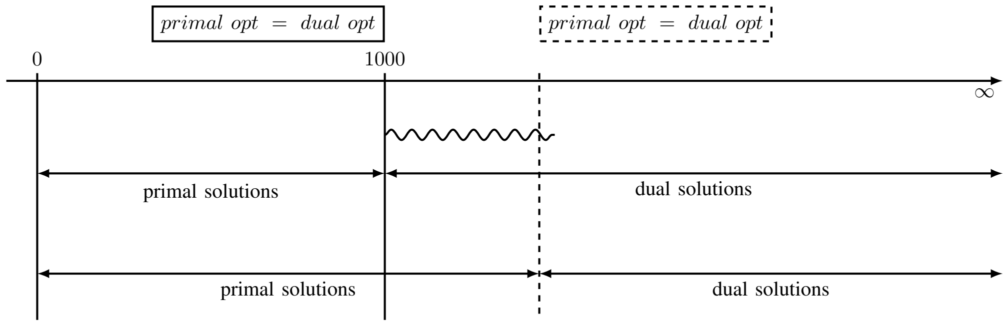

Figure 3: Solutions of the primal solution and the dual solution after a new constraint (dual problem) or a column (primal problem) is introduced.

$$T_{2,4,4} = 250\delta_{2,4,p_{24}} + 100\delta_{2,4,p_{2134}} + 50\delta_{2,4,p_{214}} \tag{4c}$$
$$\delta_{1,4,p_{124}} \leq 1 \qquad (\ell = \ell_{12}) \tag{4d}$$
$$\delta_{1,4,p_{134}} + \delta_{2,3,p_{213}} + \delta_{2,4,p_{2134}} \leq 1 \qquad (\ell = \ell_{13}) \tag{4e}$$
$$\delta_{1,4,p_{14}} + \delta_{2,3,p_{2143}} + \delta_{2,4,p_{214}} \leq 1 \qquad (\ell = \ell_{14}) \tag{4f}$$
$$\delta_{1,4,p_{124}} + \delta_{2,3,p_{243}} + \delta_{2,4,p_{24}} \leq 1 \qquad (\ell = \ell_{24}) \tag{4g}$$
$$\delta_{1,4,p_{134}} + \delta_{2,4,p_{2134}} \leq 1 \qquad (\ell = \ell_{34}) \tag{4h}$$
$$\delta_{2,3,p_{213}} + \delta_{2,3,p_{2143}} + \delta_{2,4,p_{2134}} + \delta_{2,4,p_{214}} \leq 1 \qquad (\ell = \ell_{21}) \tag{4i}$$
$$\delta_{2,3,p_{243}} + \delta_{2,3,p_{2143}} \leq 1 \qquad (\ell = \ell_{43}) \tag{4j}$$

where the constraints (4a)-(4c) compute the transmission capacities required in the objective function, and constraints (4d) - (4j) ensure the paths to be accommodated are link-disjoint. The optimal configuration, noted as $\delta_{s,d,p}$, is promising to improve the primal problem if the reduced cost ξ^1 is positive.

For the pricing problem, the optimal solution $\xi^1 = 250$, $\delta^* = [\delta_{p_{124}}, \delta_{p_{134}}, \delta_{p_{14}}, \delta_{p_{243}}, \delta_{p_{213}}, \delta_{p_{2143}}, \delta_{p_{24}}, \delta_{p_{2134}}, \delta_{p_{214}}] = [0,0,1,1,1,0,0,0,0]$. As $\xi^1 \geq 0$, continue and add the configuration $c_4 = \{p_{14}, p_{243}, p_{213}\}$ into the RMP with $[T_{1,4,4}; T_{2,3,4}; T_{2,4,4}] = [100; 200; 0]$.

Iteration 2
RMP. The primal solution $TH^* = \frac{12000}{7}$, $z^* = [\frac{20}{7}, 0, \frac{16}{7}, \frac{20}{7}]$, and the dual solution $\sigma = [\frac{15}{7}, 0, \frac{6}{7}]$, $\sigma_W = [\frac{1500}{7}]$.
Pricing problem. The objective function is now $\xi^2 = \max \frac{15}{7}(+T_{1,4,5}) + 0(+T_{2,3,5}) + \frac{6}{7}(+T_{2,4,5}) - \frac{1500}{7}$. Optimal solution $\xi^2 = \frac{3000}{7}$, $\delta^* = [1,1,1,0,0,0,0,0,0]$. As $\xi^2 \geq 0$, continue and add the configuration $c_5 = \{p_{124}, p_{134}, p_{14}\}$ into the RMP with $[T_{1,4,5}; T_{2,3,5}; T_{2,4,5}] = [300; 0; 0]$.

Iteration 3
RMP. The primal solution $TH^* = 2250$, $z^* = [0, 0, 3, \frac{15}{4}, \frac{5}{4}]$, and the dual solution $\sigma = [\frac{15}{16}, \frac{15}{16}, \frac{9}{8}]$, $\sigma_W = [\frac{1125}{4}]$.
Pricing problem. Optimal solution $\xi^3 = \frac{825}{4}$, $\delta^* = [0,0,1,0,0,0,1,1,0]$. As $\xi^3 \geq 0$, continue and add the configuration $c_6 = \{p_{14}, p_{24}, p_{2134}\}$ into the RMP with $[T_{1,4,6}; T_{2,3,6}; T_{2,4,6}] = [100; 0; 350]$.

Iteration 4
RMP. The primal solution $TH^* = 2800$, $z^* = [0, 0, 0, \frac{14}{3}, \frac{2}{3}, \frac{8}{3}]$, and the dual solution $\sigma = [\frac{7}{6}, \frac{7}{6}, \frac{2}{3}]$, $\sigma_W = 350$.
Pricing problem. Optimal solution $\xi^4 = 50$, $\delta^* = [0,1,1,0,0,0,1,0,0]$. As $\xi^4 \geq 0$, continue and add the configuration $c_7 = \{p_{134}, p_{14}, p_{24}\}$ into the RMP with $[T_{1,4,7}; T_{2,3,7}; T_{2,4,7}] = [200; 0; 250]$.

Iteration 5
RMP. The primal solution $TH^* = 2880$, $z^* = [0, 0, 0, \frac{24}{5}, 0, \frac{8}{5}, \frac{8}{5}]$, and the dual solution $\sigma = [\frac{4}{5}, \frac{7}{5}, \frac{4}{5}]$, $\sigma_W = 360$.
Pricing problem. Optimal solution $\xi^5 = 60$, $\delta^* = [0,0,1,0,1,0,1,0,0]$. As $\xi^5 \geq 0$, continue and add the configuration $c_8 = \{p_{14}, p_{213}, p_{24}\}$ with $[T_{1,4,8}; T_{2,3,8}; T_{2,4,8}] = [100; 100; 250]$.

Iteration 6
RMP. The primal solution $TH^* = 3000$, $z^* = [0,0,0,3,1,0,0,4]$, and the dual solution $\sigma = [\frac{5}{4}, \frac{5}{4}, \frac{1}{2}]$, $\sigma_W = 375$.
Pricing problem. Optimal solution $\xi^6 = 0$, $\delta^* = [1,1,1,0,0,0,0,0,0]$. As $\xi^6 = 0$, stop.

As the solution $z^* = [0,0,0,3,1,0,0,4]$ is already an integer, the original problem has been solved. In most common cases where z^* is not an integer, we need to pass the existing columns back to RMP and resort to a branch-and-bound algorithm [4]. This is required to guarantee each lightpath is used an integer number of times. Figure 4 illustrates the generated 8 configurations and the usages. We see that only configurations 4, 5, and 8 are adopted in the final solution. They are used 3, 1, and 4 times, respectively. The wavelength assignment for these configurations can be easily completed, as one wavelength configuration corresponds to one wavelength.

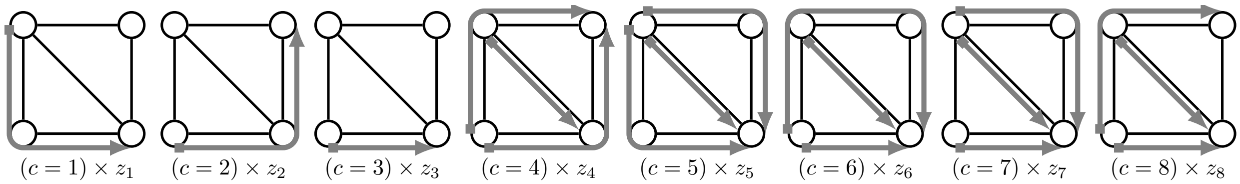

Figure 4: Illustration of the optimal wavelength configurations, $z^* = [0,0,0,3,1,0,0,4]$, $|z| = 8$.

2.3 Summary

Table 2 summarizes the performance difference between the ILP model and CG in terms of the variables and constraints. We observe only 8 variables (or 6 iterations) are required for the CG approach. Fig. 5 illustrates the variation of optimal solution during each iteration.

Table 2: Summary of the used variables and constraints for the ILP model and CG approaches

Approaches	ILP	CG
# Variables	72	8
# Constraints	3+8*7	4
Max. Throughput	3000	3000

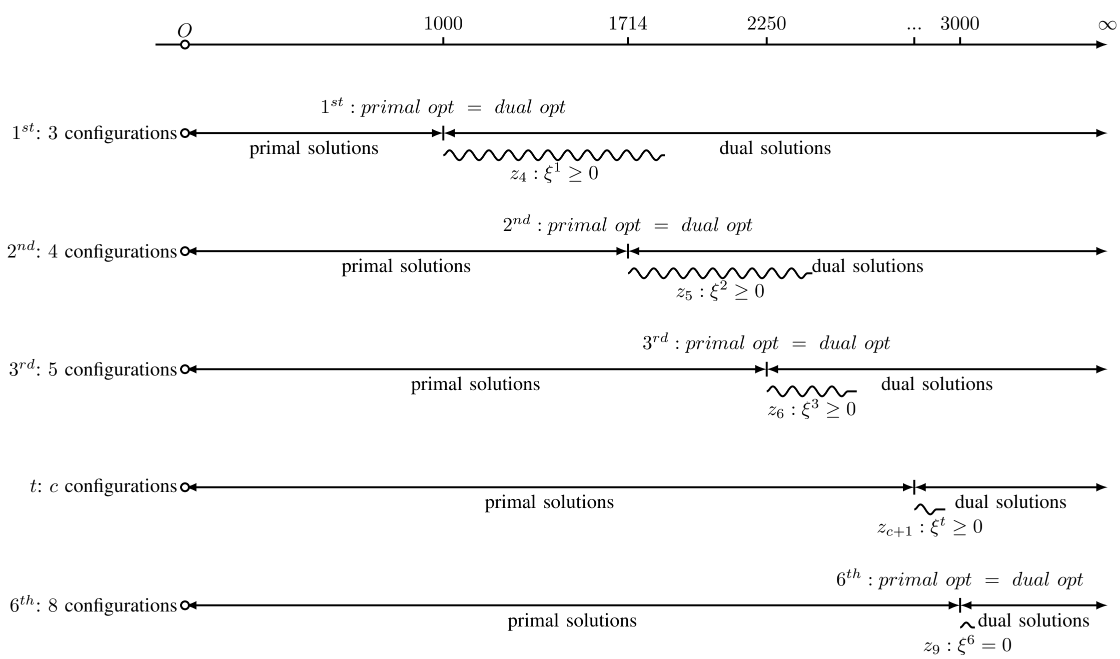

Figure 5: Status of optimal solution during iterations.

3 Conclusion

This report demonstrates the example computation of the CG approach for the RWA problem and compares the performance difference with the ILP model with respect to the variables and constraints. The CG approach shows fewer variables than the ILP model.

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