Decision Making in Heterogeneous Networks of Drift Diffusion Processes

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Abstract

We consider networks of individuals who accumulate information according to the drift diffusion process, a model of decision making found in neuroscience literature. We construct heterogeneous networks with "leaders" and "followers" and analyze the relationship between individuals' cohesion and leader characteristics, network structure, and leader placement. We begin with noise-free networks, presenting proofs and simulations of these relationships, and additionally explore these relationships in noisy networks.

Contents

1. Introduction

When groups make decisions, each individual often influences and is influenced by other individuals. These influences might rely heavily on group dynamics and structure: certain individuals can hold disproportionate amounts of influence over other individuals, and might be able to influence a limited subset of other individuals. The connections between individuals in a network, and the information available to each individual in a network, can affect decisions made both by individuals and by the group as a whole. By better understanding characteristics of these influence networks and the individual decision makers they contain, we can gain insight into the nature of decision making networks.

We can use these networks to model decisions made by humans in social networks, and to understand how the placement of heterogeneously influential and knowledgeable individuals affects the cohesion of the group.

We can apply these networks to understand a variety of different systems, such as the synchronization of coupled oscillators, the behavior of mobile sensor networks, and interactions in multivehicle systems [7].

In this project we present a qualitative and quantitative analysis of interactions between networks of decision makers. We build networks with different structures, containing both homogeneous and heterogeneous types of individuals, and simulate the effect of various interaction models.

We aim to understand how parameters of the network and population affect individuals' decision making process as well as the cohesion among the entire group.

1.1. Background

1.1.1. Drift Diffusion Model

In this project we consider the "two alternative choice task", in which decision makers must choose between two options.

Studies of this task consider two paradigms: the "interrogation" paradigm and the "free response" paradigm. Under the interrogation paradigm the individual is given a set amount of time after which they must make a decision. Under the free response paradigm the individual is given an unconstrained choice of when to make their decision.

The drift diffusion model supposes that at each time step the decision maker gathers a piece of evidence that supports one of the two alternatives, accumulating these pieces of evidence over sequential timesteps. Let $x(t)$ denote the individual's accumulated evidence at time t. Then if we consider one choice as "positive" and the other as "negative" we can numerically represent the evidence gathered at each timestep as $dx(t)$, where the magnitude indicates the strength of the evidence and the sign denotes the alternative that the evidence supports.

Under this model, the individual accumulates information according to the update rule

$$dx(t) = \beta dt + \sigma dW$$

where x represents the individual's accumulated information, β represents the correct choice (β is positive for one choice, and negative for the other), and σdW represents Gaussian noise with variance σ^2.

Each individual has a threshold value for each alternative, θ_+ and θ_-. In the interrogation paradigm, the individual chooses one of the two alternatives when $x(t) > \theta_+$ or $x(t) < \theta_-$, or when they are forced to make a decision. In the free response paradigm, the individual makes a choice once their accumulated evidence reaches the threshold θ_+ or θ_- [2].

In Figure 1 we show two trials of a decision maker accumulating evidence in a noisy situation, with $\beta = 1$, $\sigma = 0.1$, $dt = 0.001$, and $x(0) = 0$. Note that the presence of noise causes the individual to make an incorrect decision in certain situations.

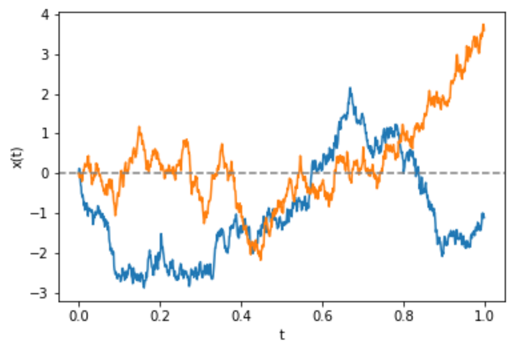


Figure 1: DDM demonstration. Two trials of a decision maker accumulating evidence in a noisy situation, with $\beta = 1$, $\sigma = 0.1$, $dt = 0.001$, and $x(0) = 0$.

We also consider a noise-free model, where $\sigma = 0$ and the individual accumulates information according to

$$dx = \beta dt$$

In behavioral experiments, researchers have observed evidence supporting this model in the brains of monkeys. Monkeys received a motion perception task in which they viewed a set of randomly moving dots, some proportion of which were set to move coherently in a given direction. The monkeys were trained to determine whether the dots were moving left or right, and the activity in certain brain regions mirrored the trajectory of evidence accumulation under the drift diffusion model [10].

1.1.2. Decision Making in Groups

In addition to individual decision makers, existing research models groups of interacting decision makers. Previous researchers proposed a process of building consensus among a group of individuals, each of whom has individual opinions and might modify their opinion states after learning of other individuals' opinions [3]. They model these interactions using a Laplacian matrix, where the decision making network is modeled as a graph with individuals as nodes and influence as edge weights, and influence between individuals depends on the graph Laplacian $L = (L_{ij})$ of the network, defined as

$$L_{ij} = \begin{cases} deg(v_i) & i = j \\ -\alpha_{ij} & i \neq j \end{cases}$$

where $deg(v_i) = \sum_j \alpha_{ij}$ and α_{ij} is the amount of influence node j exerts on node i [7].

As in Srivastava and Leonard [11], we model a network of decision makers as a system of coupled drift diffusion processes using the following update rule:

$$d\mathbf{x}(t) = (\mathbf{B} - L\mathbf{x}(t))dt + \sigma \mathbf{I}_n d\mathbf{W}_n(t)$$
$$\mathbf{x}(0) = \mathbf{x}_0$$

where $\mathbf{x} = (x_1, ..., x_n)^\top \in \mathbb{R}^n$ denotes the state of each decision maker, $x_0 \in \mathbb{R}^n$ denotes the initial state of each individual, $\mathbf{B} = (\beta_1, ..., \beta_n)^\top \in \mathbb{R}^n$ denotes the signal received by each individual, $L \in \mathbb{R}^{n\times n}$ denotes the Laplacian matrix, $\mathbf{I}_n$ is the identity matrix of size n, and $d\mathbf{W}_n$ is a vector of $N(0,1)$ Gaussians.

Each index β_i represents the information received by individual i. We allow β_i to vary across different nodes, which results in homogeneous and heterogeneous networks.

In homogeneous networks, each individual shares the same β_i. In heterogeneous networks, individuals accumulate information from different sources and therefore may have β_i with different signs and magnitudes. In leader-follower networks, only some of the individuals (leaders) have access to external information ($\beta_i \neq 0$) while other individuals (followers) only receive information from other individuals' communication ($\beta_i = 0$).

1.1.3. Related Work

Previous work has analyzed various characteristics of group decision making under various scenarios.

Ofati-Saber et al [7] define "consensus" as the asymptotic agreement between all nodes ($x^* = \alpha \mathbf{1_n}$). They consider interactions between individuals who may have different initial opinions $x_i(0)$ but who do not accumulate additional information or noise. In other words, they use a version of the coupled drift diffusion processes where $\mathbf{B} = \mathbf{0}$ and $\sigma = 0$. They find that the refusal of certain subgroups to share information or to change their state based on others' states can prevent a group from reaching consensus.

Poulakakis, Scardovi, and Leonard [8] analyze a network of interacting drift diffusion processes and analyze the effect of an individual's location on their variance, finding a measure of variance that depends on the node's communication topology. They apply this metric to specific graph structures, showing how certainty changes with node location and the number of total nodes [9]. Srivastava and Leonard look at the same model, creating a de-coupled approximation to the coupled drift diffusion model and analyzing the effect of a node's location on its error rate [11]. In both papers [8, 11] the authors analyze homogeneous coupled drift diffusion processes, ie $\mathbf{B} = \beta \mathbf{1}_n$. Srivastava and Leonard also develop a reduced model of heterogeneous networks, in which individuals may have different signal sizes [12].

Fitch and Leonard [5, 4] study the problem of selecting a set of leaders that minimize overall variance of a network's nodes. In their analysis, the leaders are a subset of nodes that receive the same external signal while the other nodes do not directly measure the external signal. In contrast to the drift diffusion update rules, they study a system in which the strength of the signal depends on the difference between a set value and the leader's opinion state

$$d\mathbf{x} = -\mathbf{k}(\mathbf{x} - \mu)dt - L\mathbf{x}dt + \sigma d\mathbf{W} \tag{1}$$

They find a metric to determine the optimal set of leaders, and observe a trade-off between information centrality and graph coverage.

We relate our work to the definition of consensus proposed by Ofati-Saber – we look for parameters that encourage agreement between nodes. We bridge the work done by Poulakakis, Scardovi, and Leonard and Srivastava and Leonard which analyzes networks composed of individual drift diffusion processes, and work by Fitch and Leonard, which considers the problem of selecting leaders and followers in a network.

We specifically consider the case of drift diffusion networks with leaders and followers, in which β_i is allowed to vary for each node i, and study how the placement and characteristics of leaders affects the cohesion of the system as a whole. We study both noise-free and noisy networks and compare the relationships we observe from the two types of networks.

2. Problem Statement

2.1. Approach

We construct decision making networks with different agent and interaction characteristics.

We start with noise-free networks, simulating their decision making process under the drift diffusion model and looking at resulting individual and group opinion states.

We consider the accumulated opinion states of individuals and the distance between these opinion states, and derive relationships between these quantities based on leader characteristics.

We then introduce noise to the system, simulating how this addition affects the conclusions we derive for noise-free networks.

2.2. Model

We define our model as a network of n individuals whose state at time t is represented by the $n \times 1$ vector $\mathbf{x} = (x_1, ..., x_n)^\top \in \mathbb{R}^n$. We refer to $x_i(t)$ as the individual's "opinion state" at time t. Each individual receives an external signal β_i (which may have magnitude 0), and these values are represented by the $n \times 1$ vector $\mathbf{B} = (\beta_1, ..., \beta_n)^\top \in \mathbb{R}^n$.

The individuals communicate through a network defined by the Laplacian matrix L, where for $i \neq j$ $L_{ij} = -\alpha_{ij}$ for connected nodes and 0 for unconnected nodes and $L_{ii} = deg(v_i)$ where $deg(v_i) = \sum_j \alpha_{ij}$.

In this project we consider unweighted undirected graphs, where $\alpha_{ij} = \alpha_{ji} \in \{0, 1\}$.

Individuals update their states according to the update rule

$$d\mathbf{x}(t) = (\mathbf{B} - L\mathbf{x}(t))dt + \sigma \mathbf{I}_n d\mathbf{W}_n(t) \tag{2a}$$

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{2b}$$

where $d\mathbf{W}_n$ is a vector of Weiner processes and we use $\mathbf{x}_0 = (0, ..., 0)^\top$.

2.3. Population Structures

We consider the following population types:

1. Homogeneous populations, in which all individuals share the same β_i.
2. Hetereogeneous populations, in which some individuals have different β_i
   (a) Populations with one leader (who has $\beta_i \neq 0$) and multiple followers (who have $\beta_i = 0$).

(b) Populations with two leaders (denoted by l_1 and l_2) and multiple followers (denoted by f_i, each with $\beta_{f_i} = 0$). We consider cases in which the two leaders share the same source of information ($\beta_{l_1} = \beta_{l_2}$), and when they receive different sources of information ($\beta_{l_1} \neq \beta_{l_2}$).

2.4. Network Structures

In each of our models, we consider unweighted, symmetric interactions. The strength of information flow between any two connected individuals is the same, and if individual i influences individual j then individual j influences individual i. (ie $\alpha_{ij} = \alpha_{ji} \in \{0,1\}$)

We consider the following networks:

1. A tree graph in which individuals sit in a connected network with no cycles (Figure 2a).
2. A circle graph in which individuals sit in a circle and each individual is connected to its two neighbors (Figure 2b).
3. A complete graph in which each individual is connected to every other individual (Figure 2c).
4. The graph used by Srivastava and Leonard, which we use to demonstrate networks where individuals have fixed but different centrality. We refer to this as the “Srivastava-Leonard graph" [11] (Figure 2d).
5. An Erdös-Rèyni random graph, in which each pair of individuals is connected with probability p (Figure 2e). Unless indicated otherwise, we use $p = 0.5$.

2.5. Implementation Specifics

In each of our simulations (with the exception of the graph used in [11], which contains nine nodes) we build networks containing 10 total individuals, unless indicated otherwise. We use $dt = 0.001$. We consider noise-free scenarios ($\sigma = 0$) as well as networks with noise ($\sigma > 0$).

3. Results in Noise-Free Networks

We first consider behavior in noise-free coupled DDMs. We consider the behavior of individuals as well as the cohesion of groups of individuals, in various networks with one and two leaders.

3.1. Limited Total Information Accumulation

Under our model (2.2) we find that at any given time the networks contain a fixed amount of information accumulation that must be divided among all individuals.

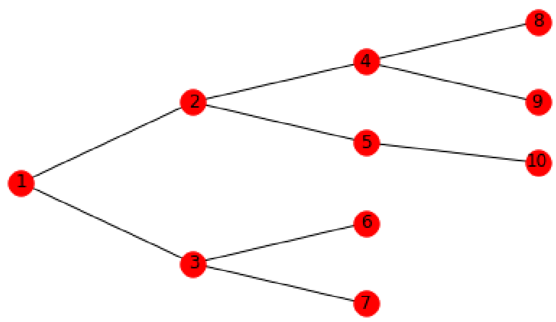
Because we use an undirected network to model interactions between individuals, in noise-free networks the population accumulates a fixed total amount of information after a specified amount of time, with the total accumulation depending only on the signal β measured by leaders in the network.

Proposition 1. *Consider a network of individuals who accumulate information according to Equation 2.2 in a noise-free network ($\sigma = 0$). At each time t, the rate of information accumulated by all individuals is a constant that depends only on* $\mathbf{B}$.

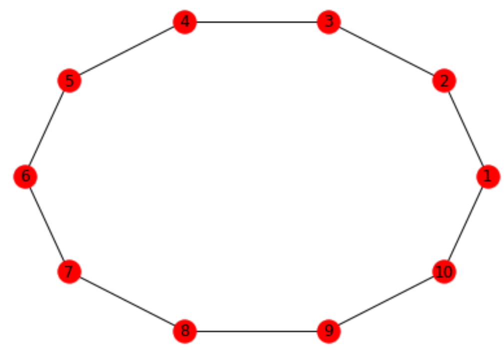
Proof. We show that $\sum \dot{x}_i = \sum \beta_i$.

For a network with n individuals, we can rewrite the update rule

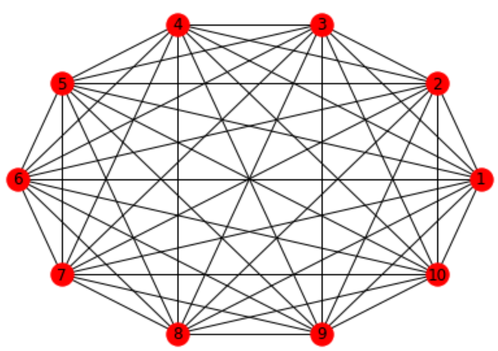
$$\dot{\mathbf{x}} = \mathbf{B} - L\mathbf{x}$$



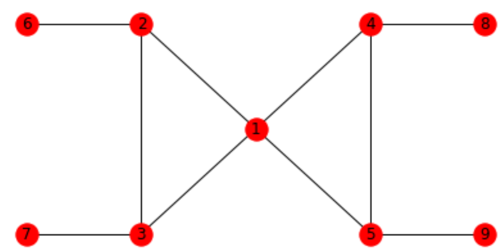
(a) Tree graph



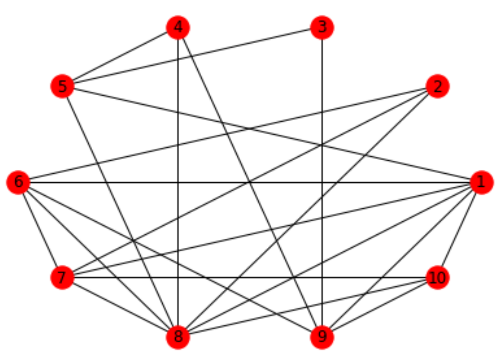
(b) Circle graph



(c) Complete graph



(d) Srivastava-Leonard graph



(e) Example of a random graph

Figure 2: We consider these graph structures in this project.

as

$$\dot{x}_i = \beta_i - \sum_{j=1}^{n} L_{ij} x_j$$

Since $L_{ii} = -\sum_{j \neq i} \alpha_{ij}$ and $L_{ij} = -\alpha_{ij}$ for $i \neq j$, we can write this as

$$\dot{x_i} = \beta_i + \sum_{j \neq i} \alpha_{ij}(x_j - x_i)$$

Then we have the total accumulation

$$\begin{aligned}
\sum_{i=1}^{n} \dot{x_i} &= \sum_{i=1}^{n} \left(\beta_i + \sum_{j \neq i} \alpha_{ij}(x_j - x_i) \right) \\
&= \sum_{i=1}^{n} \beta_i + \sum_{i=1}^{n} \sum_{j \neq i} \alpha_{ij}(x_j - x_i) \\
&= \sum_{i=1}^{n} \beta_i
\end{aligned}$$

where we have the final equality from the symmetry of influence $\alpha_{ij} = \alpha_{ji}$. □

3.1.1. Interpretation

As a consequence, there is a trade-off between how much an individual influences other nodes' accumulation of information and how quickly the individual accumulates information for their own opinion state. For instance, a more centrally located leader causes follower nodes to accumulate information more quickly but this comes at the cost of the leader's own accumulation of information.

We demonstrate this in Figure 3 with the case of a Srivastava-Leonard communication network (Figure 2d) with a population that has a single leader. In the left panel the leader is placed at the most central location and the network's individuals display low discrepancies with each other. The right panel shows that as the leader moves to less central areas in the network, the leader accumulates its own information more quickly at the cost of higher discrepancies with its followers.

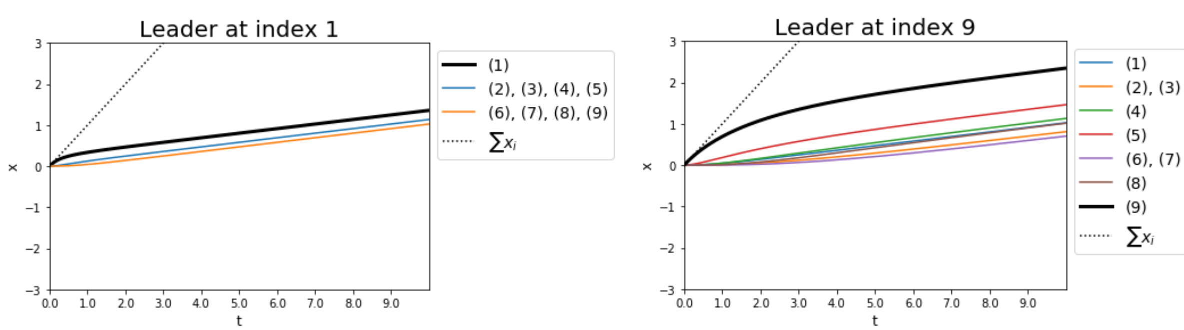


Figure 3: One leader in a Srivastava-Leonard Graph. In the left panel the leader is placed at the most central location and the network's individuals display low discrepancies with each other. The right panel shows that as the leader moves to less central areas in the network, the leader accumulates its own information more quickly at the cost of higher discrepancies with its followers.

3.2. Discrepancy Between Individuals

In addition to the opinion state of each individual, we are interested in the difference between opinion states of leaders and the individuals with whom they communicate.

Definition 3.1. *Pairwise Discrepancy* We define the "pair-wise discrepancy" d_{ij} between individuals i and j as

$$d_{ij}(t) = \begin{cases} (x_i(t) - x_j(t))^2 & if\ i,j\ connected \\ 0 & else \end{cases} \tag{3}$$

Definition 3.2. *Individual Discrepancy* We define the "discrepancy" d_i around individual i in a network of n individuals

$$d_i(t) = \sum_{j=1}^{n} d_{ij}(t) \tag{4}$$

Definition 3.3. *Total Discrepancy* We define the "total discrepancy" d_L as the sum of the discrepancies of all individuals in a network, weighted by the size of their signal and normalized by the total amount of signal

$$d_L(t) = \frac{\sum_{i=1}^{n} |\beta_i| d_i(t)}{\sum_{i=1}^{n} |\beta_i|} \tag{5}$$

In noise-free networks, we find that although individual opinion states do not necessarily converge, pair-wise discrepancies (and therefore total discrepancies) converge.

Proposition 2. *Consider a network of individuals who accumulate information according to Equation 2.2 in a noise-free network ($\sigma = 0$), with discrepancies defined according to Equations 3.2 and 3.1. As $t \to \infty$, all pairwise discrepancies d_{ij} converge to steady state values d^*_{ij} that depend on the network structure and external signals $\mathbf{B}$. We compute these steady states in Section 3.2.1.*

Proof. For any arbitrary graph, we can derive an explicit form for the pairwise discrepancies to which individuals converge.

Consider an incidence matrix E and its relationship to the Laplacian, as described in [6].

An $n \times m$ incidence matrix E of a directed graph with n nodes and m edges is defined as

$$E[v,e] = \begin{cases} -1 & v = s(e) \\ 1 & v = t(e) \\ 0 & otherwise \end{cases} \tag{6}$$

where $s(e)$ denotes the initial vertex of edge e and $t(e)$ denotes the terminal vertex of edge e. The Laplacian matrix satisfies $L = EE^\top$, where $E^\top$ denotes the transpose of E.

We define

$$y = E^\top x$$

The kth entry of y is $x_{k_1} - x_{k_2}$, where k_1 and k_2 indicate the nodes connected by the kth edge of the graph. Each entry of y therefore gives the differences between states connected by the edges of the network.

Note that this vector allows us to compute the pairwise discrepancy between every pair of nodes in the same connected component of the network by solving a system of equations involving the pairwise discrepancy between each pair of directly connected nodes in the component.

From our model of individual opinion state updates, $\dot{x} = B - Lx$, we obtain:

$$\begin{aligned}
\dot{y} = (\dot{E^\top x}) = E^\top \dot{x} &= E^\top B - E^\top Lx \\
&= E^\top B - E^\top (EE^\top)x \\
&= E^\top B - (E^\top E)(E^\top x) \\
&= E^\top B - E^\top E y
\end{aligned}$$

which has a steady state at

$$E^\top B = (E^\top E)y^*$$

The network is guaranteed to converge to the steady state. If y is not at the steady state then $E^\top B \neq (E^\top E)y$. If $(E^\top E)y > E^\top B$ then $\dot{y} < 0$, and if $(E^\top E)y < E^\top B$ then $\dot{y} > 0$. In either case, y moves toward the steady state y^*.

Note that $E^\top E$ is not necessarily non-singular. In fact, in most cases it is singular. For any matrix A, $rank(AA^\top) = rank(A^\top A)$ have the same rank. Since $L = EE^\top$, $rank(E^\top E) = rank(L)$. Furthermore, the rank of the Laplacian matrix is the number of nodes minus the number of components. Let k represent the number of components in the graph. Then $rank(E^\top E) = rank(EE^\top) = rank(L) = n - k$. Since E is an $n \times m$ matrix, $E^\top E$ is an $m \times m$ matrix. So $E^\top E$ is invertible if and only if $m = n - k$.

A graph consisting of k component trees, where a tree is a connected graph with no cycles, has $n - k$ edges since a tree with n nodes has $n - 1$ edges [13]. So for networks in which all components have tree structures, we can simply use the inverse of $E^\top E$ to compute the steady-state discrepancies.

For a complete graph with n nodes, we note that $E^\top EE^\top = nE^\top$, from which we find the steady-state discrepancies as follows:

$$\begin{aligned}
0 &= E^\top B - (E^\top E)E^\top x = E^\top B - nE^\top x = E^\top B - ny \\
y^* &= \frac{E^\top B}{n}
\end{aligned}$$

For other graph structures, we use the pseudoinverse $(E^\top E)^+$ of $E^\top E$ to calculate steady-state discrepancies.

$$(E^\top E)^+ E^\top B = (E^\top E)^+ (E^\top E)y^*$$

$(E^\top E)^+(E^\top E)y$ projects the $m \times 1$ matrix y onto the range of $(E^\top E)^*$, where $(E^\top E)^*$ is the orthogonal complement of the kernel of $E^\top E$. The linearly independent rows of the incidence matrix E correspond to the edges that induce an acyclic graph on the network [1]. If there are k such edges, then $rank(E^\top E) = k$ and $(E^\top E)^*$ has k dimensions. So $(E^\top E)^+(E^\top E)y$ projects y onto k dimensions. We only require k opinion state differences to compute discrepancies, since each individual's discrepancy depends only on its neighbors' opinion states.

For any graph we therefore can explicitly derive pairwise discrepancies to which the network converges using the following formulas.

$$y^* = (E^\top E)^{-1} E^\top B \qquad \textit{tree graph} \tag{7a}$$

$$y^* = \frac{E^\top B}{n} \qquad \textit{complete graph} \tag{7b}$$

$$(E^\top E)^+(E^\top E)y^* = (E^\top E)^+ E^\top B \qquad \textit{general graph} \tag{7c}$$

□

3.2.1. Examples

We demonstrate these three formulas using examples of a line graph, a complete graph, and a circle graph of four individuals. We specifically consider the case of one or two leaders, and use this to demonstrate the tradeoff between leaders' centrality in and coverage of the network. In this section we compute discrepancies for two leaders with the same signal; we include the general case of two leaders who do not necessarily have the same signal in Section 6.2 of the Appendix.

Our model (2.2) uses undirected graphs, so we obtain E by choosing an arbitrary orientation on the network (note that our definition of L is independent of the orientation of the graph).

Line Graph

We consider the case of four individuals in a line graph.

For the purpose of defining the incidence matrix E, we arbitrarily choose the orientation that is shown in Figure 4.

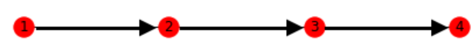


Figure 4: Graph structure with four individuals in a line.

$$E = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Using this along with Equation 7a

$$\frac{1}{4}\begin{bmatrix} -3 & 1 & 1 & 1 \\ -2 & -2 & 2 & 2 \\ -1 & -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = y$$

Substituting our definition $y = E^\top x$, this gives steady discrepancies

$$\begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -3\beta_1 + 1\beta_2 + 1\beta_3 + 1\beta_4 \\ -2\beta_1 - 2\beta_2 + 2\beta_3 + 2\beta_4 \\ -1\beta_1 - 1\beta_2 - 1\beta_3 + 3\beta_4 \end{bmatrix}$$

This gives the associated steady state total discrepancies

$$\begin{aligned} d_1^* &= \frac{(-3\beta_1 + \beta_2 + \beta_3 + \beta_4)^2}{16} \\ d_2^* &= \frac{(-3\beta_1 + \beta_2 + \beta_3 + \beta_4)^2 + (-2\beta_1 - 2\beta_2 + 2\beta_3 + 2\beta_4)^2}{16} \\ d_3^* &= \frac{(-2\beta_1 - 2\beta_2 + 2\beta_3 + 2\beta_4)^2 + (-\beta_1 - \beta_2 - \beta_3 + 3\beta_4)^2}{16} \\ d_4^* &= \frac{(-\beta_1 - \beta_2 - \beta_3 + 3\beta_4)^2}{16} \end{aligned}$$

We first consider the case of choosing the location of one leader, where $\exists!\, i : \beta_i \neq 0$. We show that choosing a leader at a more central location allows for lower leader discrepancy. If the leader is located at vertex 1 or 4 and has signal k, then the steady state total network discrepancy is $d_L^* = \frac{9k^2}{16}$. If instead the leader is located at vertex 2 or 3, then $d_L^* = \frac{5k^2}{16}$. In Table 1 we show total discrepancy for one leader receiving signal size k and for two leaders receiving signal k.

We simulate a line graph network with one and two leaders who each receive a signal size 2. Our results (in Figure 5) confirm our derivations. A single leader at index (2) or (3) gives a steady state total discrepancy of $\frac{5}{4}$, and a single leader at less central locations (1) or (4) reaches a steady state total discrepancy of $\frac{9}{4}$. Two leaders placed at either more dispersed or more central locations $(1\&4), (2\&4), (1\&4), (2\&3)$ reach a total discrepancy of 1, while two leaders placed at locations that are neither central nor dispersed reach a total discrepancy of 3.

Complete Graph

We consider the case of four individuals in a complete graph. For the purpose of defining E we fix an arbitrary orientation, which we show in Figure 6.

Leader Placements	$d_L^* = \frac{\sum_{i=1}^{n} \lvert\beta_i\rvert d_i^*}{\sum_{i=1}^{n} \lvert\beta_i\rvert}$
$(2), (3)$	$5k^2/16$
$(1), (4)$	$9k^2/16$
$(1\&3), (2\&4), (1\&4), (2\&3)$	$k^2/4$
$(1\&2), (3\&4)$	$3k^2/4$

Table 1: Total discrepancy for various leader choices in a line graph. We use (1) to denote one leader at vertex 1, and $(1\&2)$ to denote two leaders at vertices 1 and 2. To minimize discrepancy with one or two leaders receiving a fixed signal size k, we must consider both how central and how spread out the leaders are.

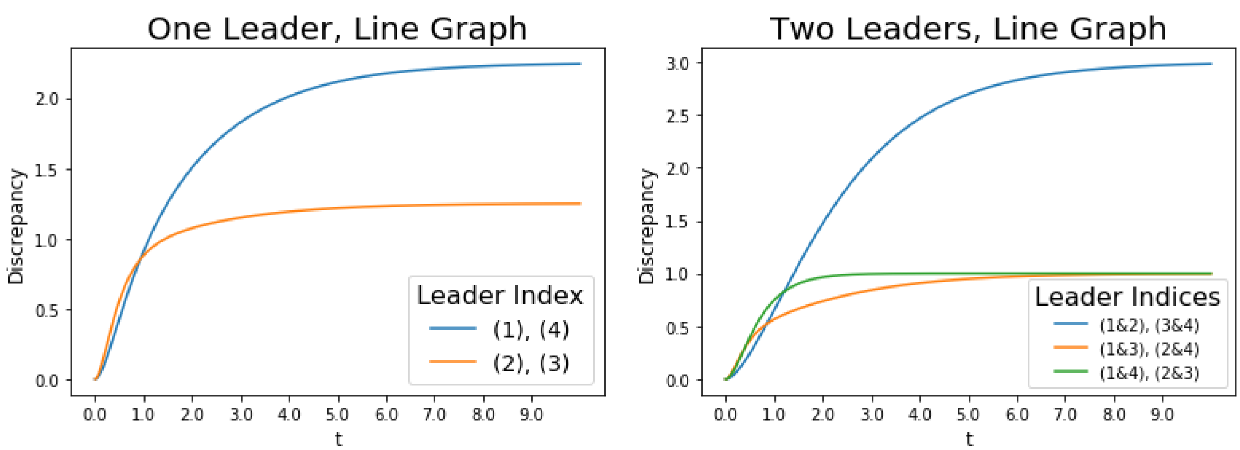


Figure 5: Simulations of one and two leaders in a line graph with four individuals, with leader signal 2.

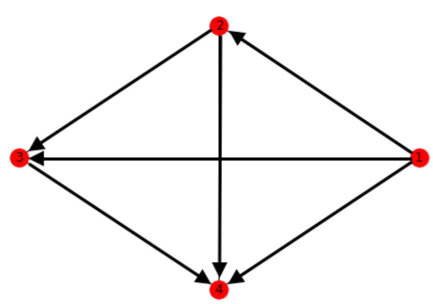


Figure 6: Graph structure with four individuals in a complete network.

For this graph, we have

$$E = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Using this along with Equation 7b gives

$$\begin{bmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_3 - x_2 \\ x_4 - x_2 \\ x_4 - x_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -\beta_1 + \beta_2 \\ -\beta_1 + \beta_3 \\ -\beta_1 + \beta_4 \\ -\beta_2 + \beta_3 \\ -\beta_2 + \beta_4 \\ -\beta_3 + \beta_4 \end{bmatrix}$$

This gives the associated steady state total discrepancies

$$d_1^* = \frac{(\beta_2 - \beta_1)^2 + (\beta_3 - \beta_1)^2 + (\beta_4 - \beta_1)^2}{16}$$
$$d_2^* = \frac{(\beta_1 - \beta_2)^2 + (\beta_3 - \beta_2)^2 + (\beta_4 - \beta_2)^2}{16}$$
$$d_3^* = \frac{(\beta_1 - \beta_3)^2 + (\beta_2 - \beta_3)^2 + (\beta_4 - \beta_3)^2}{16}$$
$$d_4^* = \frac{(\beta_1 - \beta_4)^2 + (\beta_2 - \beta_4)^2 + (\beta_3 - \beta_4)^2}{16}$$

In Table 2 we show total discrepancy for one leader with signal size k and for two leaders with signal size k. Note that the specific location of leaders is not important, since all nodes are equally connected.

Leader Placements	$d_L^* = \frac{\sum_{i=1}^{n} \lvert\beta_i\rvert d_i^*}{\sum_{i=1}^{n} \lvert\beta_i\rvert}$
one leader	$3k^2/16$
two leaders	$k^2/8$

Table 2: Total discrepancy for various leader choices in a complete graph, with leader signal size k.

We simulate these derivations to confirm our findings with leader signal size $k = 2$, and show the results in Figure 7. As predicted by our derivations, a single leader network results in a leader discrepancy of $\frac{3}{4}$ and a two-leader network results in a total discrepancy of $\frac{1}{2}$.

Circle Graph

We now consider the case of four individuals in a circle graph with an arbitrary orientation which we choose for the purpose of defining E. We show this orientation in Figure 8.

For this graph, we have

$$E = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

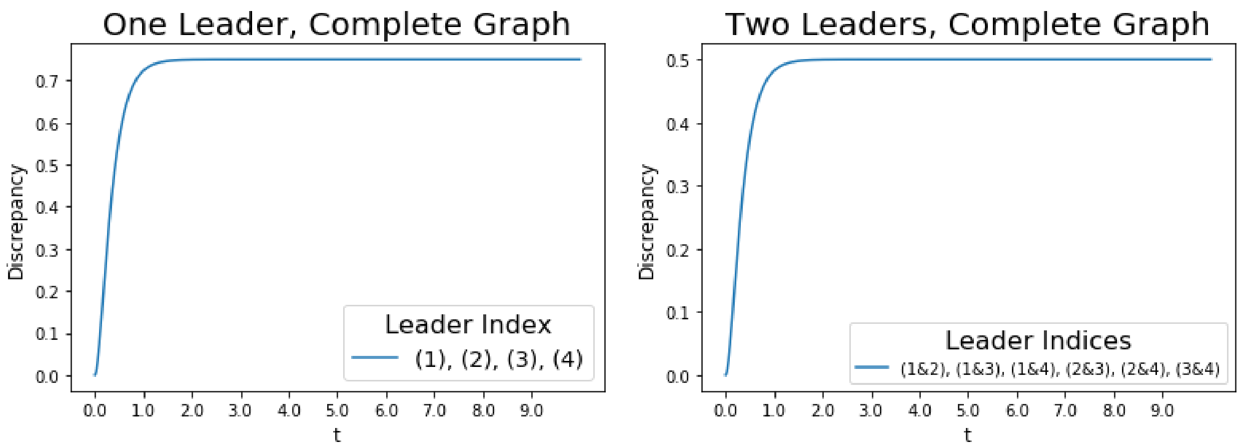


Figure 7: Simulations of one and two leaders in a complete graph with four individuals, with leader signal size $k = 2$.

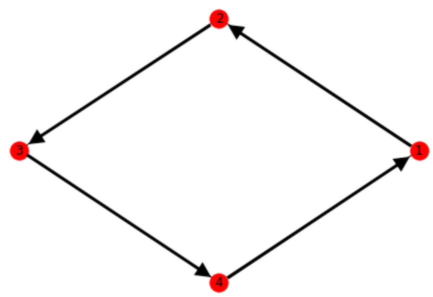


Figure 8: Graph structure with four individuals in a circle network.

Using this along with Equation 7c gives

$$\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 & -1 \\ 1 & 3 & 1 & 1 \\ -1 & 1 & 3 & -1 \\ -1 & 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_2 - x_1 \\ x_4 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -3 & 3 & 1 & -1 \\ -3 & -1 & 1 & 3 \\ -1 & -3 & 3 & 1 \\ 1 & -1 & -3 & 1 \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}$$

Which simplifies to

$$\begin{bmatrix} x_2 - x_1 \\ x_4 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -3\beta_1 + 3\beta_2 + \beta_3 - \beta_4 \\ -3\beta_1 - \beta_2 + \beta_3 + 3\beta_4 \\ -\beta_1 - 3\beta_2 + 3\beta_3 + \beta_4 \\ \beta_1 - \beta_2 - 3\beta_3 + \beta_4 \end{bmatrix}$$

This gives the associated steady state total discrepancies

$$d_1^* = \frac{(-3\beta_1+3\beta_2+\beta_3-\beta_4)^2+(-3\beta_1-\beta_2+\beta_3+3\beta_4)^2}{64}$$
$$d_2^* = \frac{(-3\beta_1+3\beta_2+\beta_3-\beta_4)^2+(-\beta_1-3\beta_2+3\beta_3+\beta_4)^2}{64}$$
$$d_3^* = \frac{(-\beta_1-3\beta_2+3\beta_3+\beta_4)^2+(\beta_1-\beta_2-3\beta_3+\beta_4)^2}{64}$$
$$d_4^* = \frac{(\beta_1-\beta_2-3\beta_3+\beta_4)^2+(-3\beta_1-\beta_2+\beta_3+3\beta_4)^2}{64}$$

In Table 3 we show discrepancies for one and two leaders, each with signal size k. As with the case of the complete network, for a single leader it does not matter which location we choose. With two leaders, it only matters how dispersed the leaders are since all nodes are equally central.

Leader Placements	$d_L^* = \frac{\sum_{i=1}^{n} \lvert\beta_i\rvert d_i^*}{\sum_{i=1}^{n} \lvert\beta_i\rvert}$
one leader	$9k^2/32$
two leaders, $(1\&2),(2\&3),(3\&4),(4\&1)$	$k^2/4$
two leaders, $(1\&3),(2\&4)$	$k^2/8$

Table 3: Total discrepancy for various leader choices in a circle graph, with leader signal size k.

Our simulations with leader signal size $k=2$ confirm our findings, and we show the results in Figure 9. As predicted by our derivations, a single leader network results in a leader discrepancy of $\frac{9}{8}$. Two adjacent leaders give a total discrepancy of 1 while choosing leaders at opposite points in the circle gives the lower discrepancy of $\frac{1}{2}$.

3.3. Reducing Discrepancy

In this section we study methods of reducing discrepancy. In particular, we analyze the effect of network connectivity, leader signal size, and leader placement on total network discrepancy.

3.3.1. Number of Leaders

Having more leaders does not necessarily reduce discrepancy. For instance, in Table 1 we saw the total discrepancy for one and two leaders in a line network of four individuals. A single leader at node (1) or node (2) results in a lower total discrepancy than having leaders at both locations (1&2). The two leaders sit in their own side of the network and their proximity to each other allows them to heavily influence each other, growing their opinion state more quickly. Because of their relative isolation from the rest of the network ((2) talks to only one follower, and (1) does not talk to any followers), the total network discrepancy increases despite the addition of more leaders.

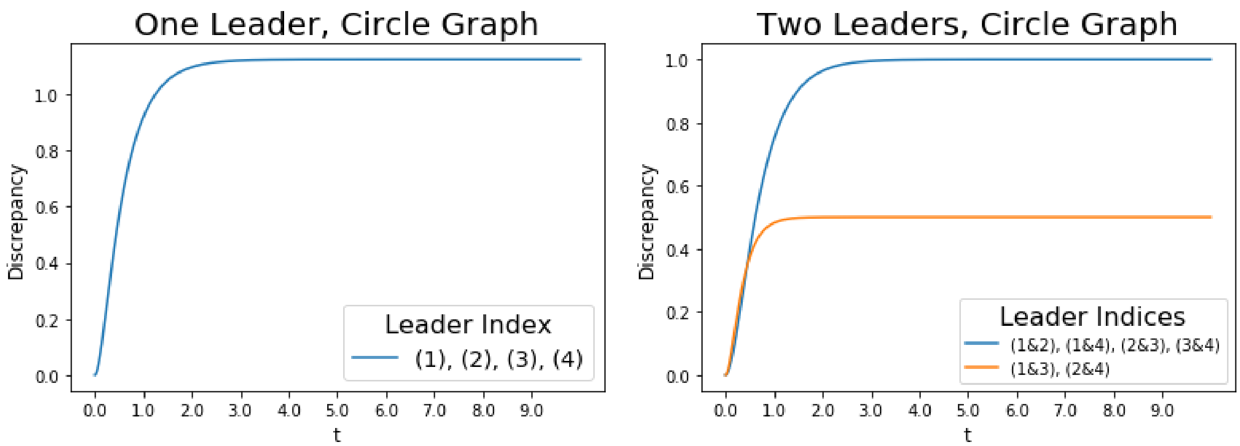


Figure 9: Simulations of one and two leaders in a circle graph with four individuals, with leader signal size $k = 2$.

3.3.2. Connectivity

As we saw from the examples in Section 3.2.1 a single leader in a more connected graph results in a lower discrepancy, and we show this in Figure 10 by contrasting a leader in graphs with increasing levels of connectivity, for a leader at index 2 with signal size 2.

As predicted by our computations in Tables 1, 2, and 3 the total discrepancies converge to $\frac{5}{4}, \frac{9}{8}$, and $\frac{3}{4}$ for a line, circle, and complete graph respectively.

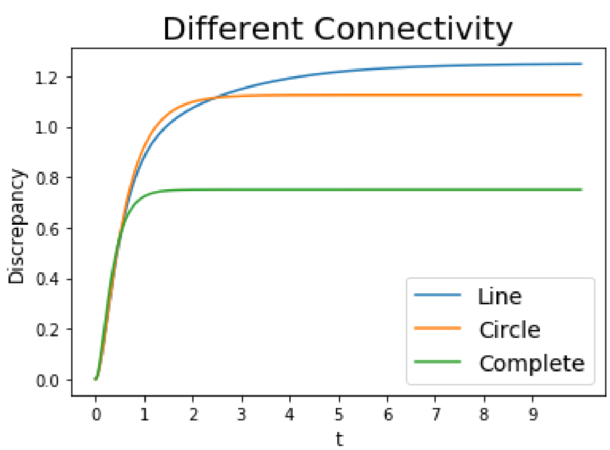


Figure 10: Simulation on various graphs with one leader. A single leader in a more connected graph results in a lower discrepancy. As predicted by our computations in Tables 1, 2, and 3 the total discrepancies converge to $\frac{5}{4}, \frac{9}{8}$, and $\frac{3}{4}$ for a line, circle, and complete graph respectively.

3.3.3. Leader Signal Size

Single Leader

We find that in graphs with a single leader, lowering the leader's β_i results in a lower discrepancy.

Intuitively, we can think about this situation as a single leader disseminating the information they receive (β_i). For a given amount of time (T) and network structure (L), a leader who receives less information is able to share a greater proportion of that information with the followers.

From Equation 7 we see why this holds. The discrepancies are determined by y, which is a function of the leader signals $\mathbf{B}$; increasing the leader signals therefore increases the corresponding steady state discrepancies.

We show this empirically by creating a random graph with 10 individuals, in which each pair of individuals is connected with probability 0.5. We randomly select one individual to serve as the leader, and experiment with $\beta_i = 1, 2, 10$. With increasing β_i, the discrepancy around the leader increases, as we see in Figure 11.

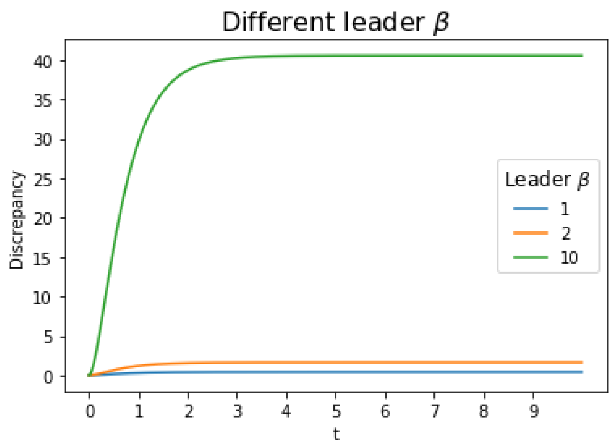


Figure 11: Simulation on random graph with one leader, changing β_i. With increasing β_i, the discrepancy around the leader increases.

Two Leaders

We further consider a system with multiple leaders, and find that when two equally connected leaders in a complete network have β_i of different magnitudes the followers tend to align more closely with the leader who has a smaller β_i.

We demonstrate this result in Figure 12. Regardless of whether the leaders have the same sign, the leader with lower magnitude β_i has lower total discrepancy, and this distinction increases with increasing difference between the β_i of the two leaders.

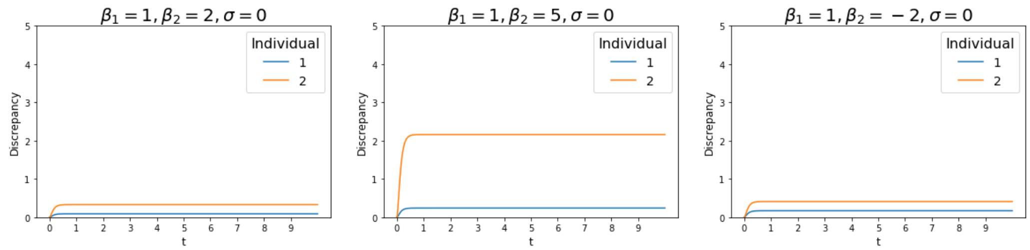
3.3.4. Centrality-Coverage Trade-Off

Consider the case in which we want to place a given number of leaders to minimize the total discrepancy around the leaders. For instance, we might have a fixed number of individuals who can receive external information ($\beta_i > 0$), and we want to place them to maximize group cohesion.

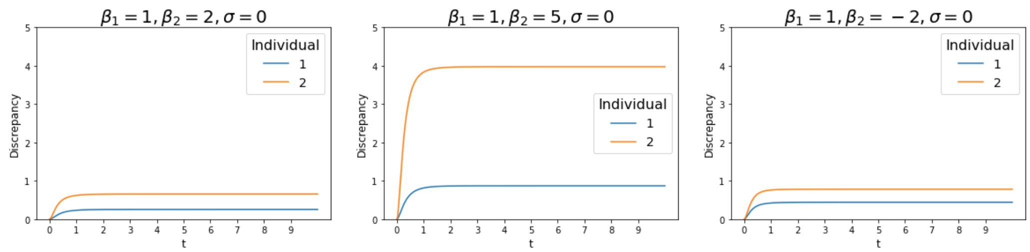
Reduction to Centrality

For networks with a single leader, we observe that this reduces to choosing the most centrally located individual, as shown for other types of networks in [5].

We demonstrate this in the case of the Srivastava-Leonard network, where a single leader can be placed at locations of differing centralities. We show the associated discrepancies in Figure 13. Note that as the centrality of the leader increases, the discrepancy around the leader decreases.



(a) Complete graph



(b) Random graph

Figure 12: Simulation with two leaders, changing β_i. Regardless of whether the leaders have the same sign, the leader with lower magnitude β_i has lower total discrepancy, and this distinction increases with increasing difference between the β_i of the two leaders.

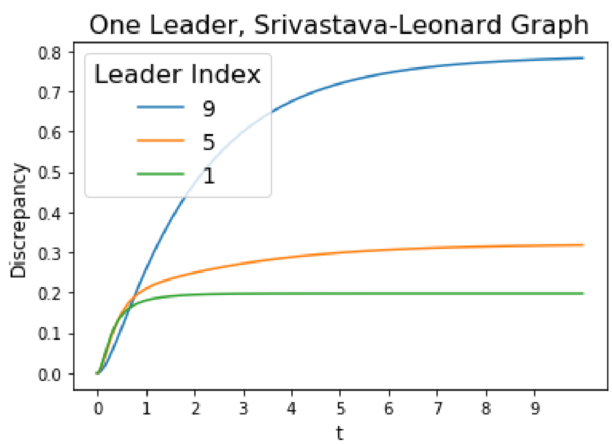


Figure 13: Simulation on Srivastava-Leonard graph with one leader. As the centrality of the leader increases, the discrepancy around the leader decreases.

Reduction to Coverage

We further observe that in systems with two leaders and fixed leader centralities, this reduces to the problem of increasing graph coverage.

We demonstrate this with the case of a circle graph, where each individual is equally centrally located. In Figure 14 we provide simulations with leaders moving farther from each other within the same network. Note that increasing the distance between the two leaders (which increases coverage of the network) results in lower discrepancies.

Trade-off

If the leader centralities are not fixed, we observe a trade-off between leaders' centrality in

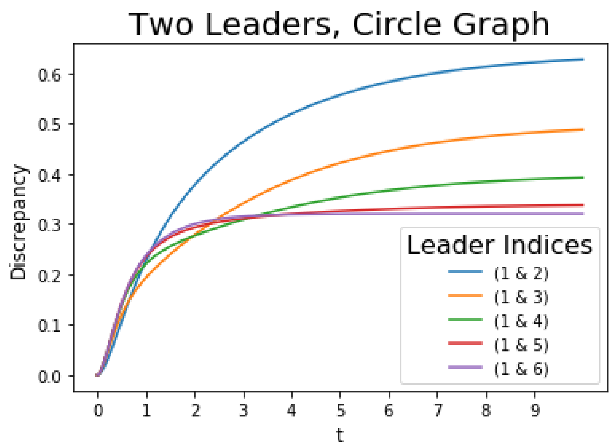


Figure 14: Simulation on circle graph with changing positions of two leaders. Increasing coverage of the network results in lower discrepancies.

and coverage of the network for decreasing overall discrepancy.

We show this using the Srivastava-Leonard Network, in which we can adjust both the centrality and coverage of our leaders.

We show the resulting discrepancies in Figure 15 and Table 4. Placing one leader at the most central location does not guarantee the lowest total discrepancy; instead, the optimal leader placement sacrifices some centrality to increase coverage of the network.

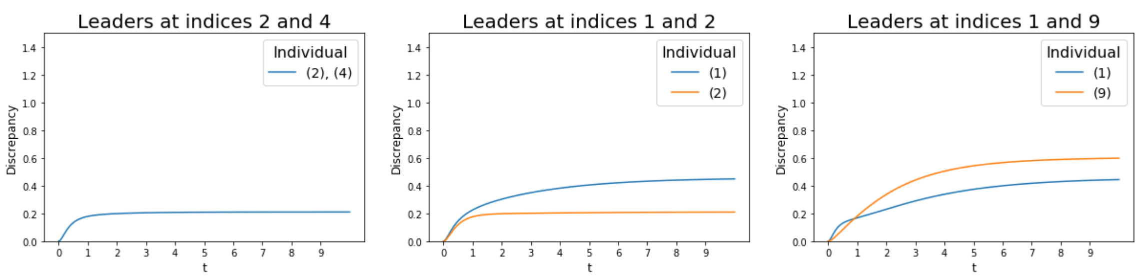


Figure 15: Discrepancies for various placements of two leaders with signal size 1 in a Srivastava-Leonard Graph. Increasing leaders' centrality or coverage does not always decrease discrepancy.

Leader Indices	Leader Discrepancies	Total Discrepancies
2, 4	0.27, 0.27	0.27
1, 2	0.45, 1.0	0.73
2, 9	0.28, 4.64	2.46
1, 9	0.44, 6.87	3.66
6, 9	4.67, 4.67	4.67
2, 6	4.66, 14.06	9.36

Table 4: Centrality-coverage tradeoff. The optimal leader placement sacrifices some centrality to increase coverage of the network.

4. Results in Noisy Networks

In this section we turn to an analysis of noisy networks. We study the same model as before

$$d\mathbf{x}(t) = (\mathbf{B} - L\mathbf{x}(t))dt + \sigma \mathbf{I}_n d\mathbf{W}_n(t) \tag{8a}$$

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{8b}$$

with $\sigma \neq 0$, and look at how well our conclusions from Section 3 hold in the presence of noise.

4.1. Convergence for Opposite-Information Leaders

In Section 3.1 we showed that the network's total rate of information accumulation at each time point is determined by the total amount of leader signal, $\sum_{i=1}^{n} \dot{x_i} = \sum_{i=1}^{n} \beta_i$ and in Section 3.2 we showed that any network's discrepancy converges to stable states in our model. In networks with two leaders who have equal-magnitude and opposite-sign β_i, where $\sum_{i=1}^{n} \beta_i = 0$, the the individual opinion states (in addition to pairwise discrepancies) stabilize.

We can see this characteristic in a random graph, where the two leaders do not have equal centrality. However, we lose this convergence with the addition of noise. We show a comparison of this loss of convergence in Figure 16.

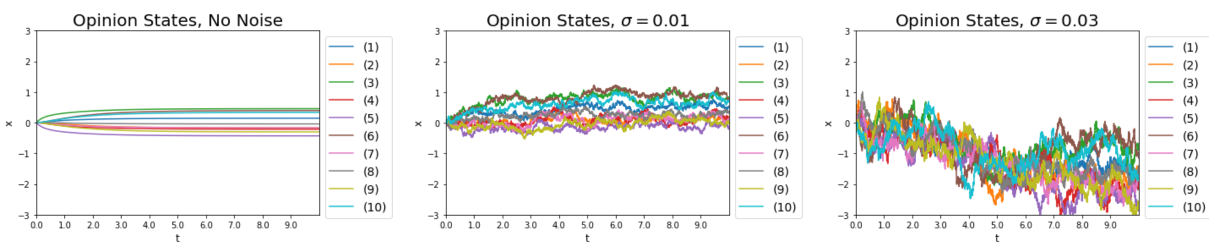


Figure 16: Simulation on random graph with two leaders at indices l_1, l_2 with $\beta_{l_1} = 1, \beta_{l_2} = -1$. Noise causes the loss of convergence observed in noise-free networks with opposite-information leaders.

4.1.1. Leader Signal Size

In noise-free networks (Figure 16) we saw that followers tend to cluster around leaders with smaller $|\beta|$. In Figure 17 we plot discrepancies with two leaders. In each column we show leaders at indices l_1, l_2 with $\beta_{l_1} = k_1, \beta_{l_2} = k_2$, and along each row we hold the amount of noise fixed. In Table 5 we show the discrepancy values in these networks. In noisy networks we continue to qualitatively and quantitatively observe this tendency for small amounts of noise. As the noise increases relative to the difference between the leaders' β_i we can no longer observe a qualitative clustering phenomenon. However, we continue to see a quantitative difference in leaders' discrepancies.

4.1.2. Connectivity

We compare discrepancies of one individual in graphs of increasing connectivity, simulating one leader at node 2 with $\beta_i = 1$ in line, circle, and complete graphs of four individuals. As in the noise-free case (Figure 10), we see a tendency toward lower discrepancy in networks with higher connectivity. In Figure 18 we see that this pattern holds in the presence of noise, although the variance introduced by increasing amounts of noise begins to overwhelm this pattern around $\sigma = 0.008$.

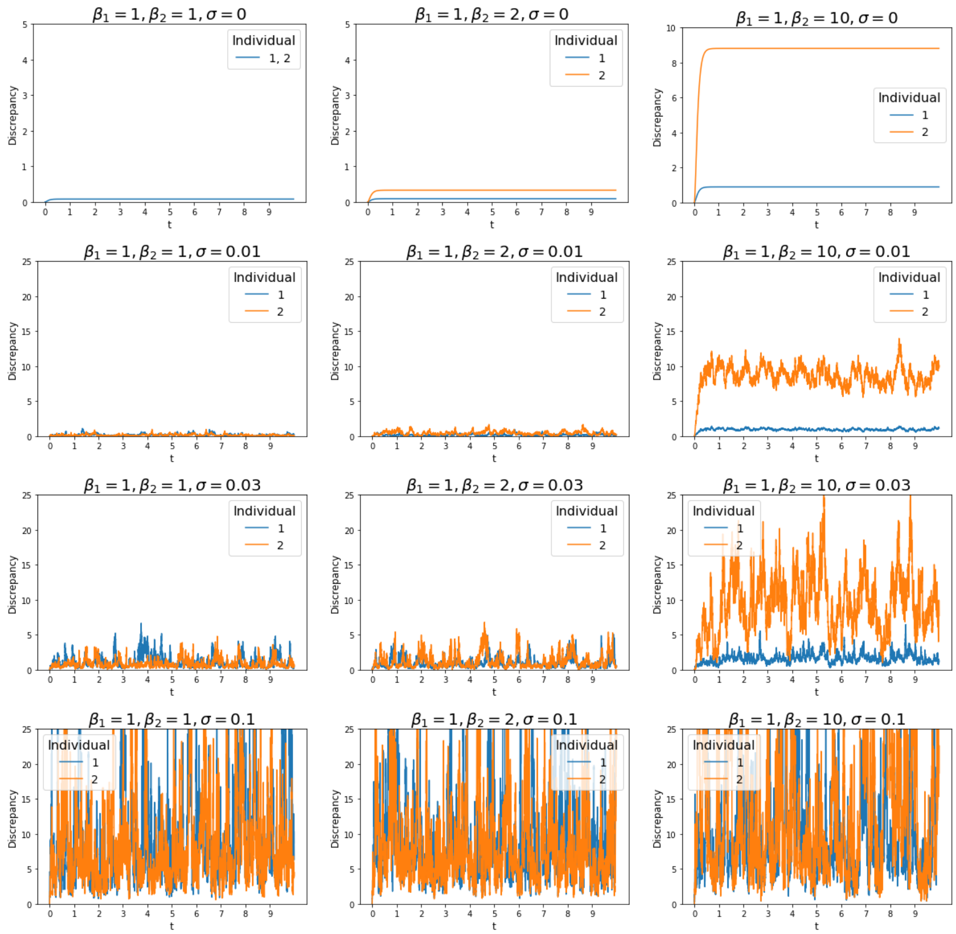


Figure 17: Noisy complete network with two leaders in a fully connected graph, qualitative comparison. As noise increases, followers no longer visibly cluster around leaders with smaller $|\beta_i|$.

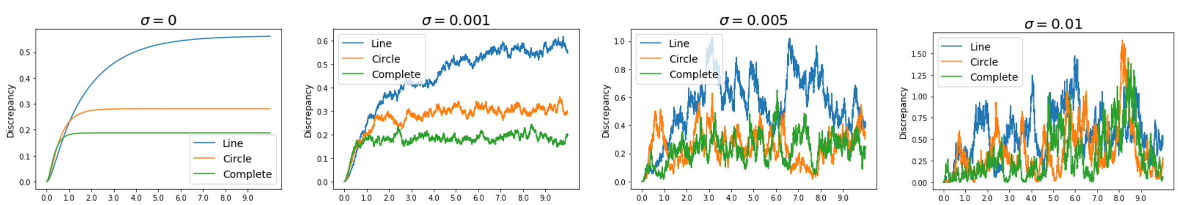


Figure 18: One leader in graphs of varying connectivity. Increasing noise begins to overwhelm the tendency towards lower discrepancies in more connected networks.

4.1.3. Centrality-Coverage Trade-Off

We consider two leaders in a noisy, fully-connected network. The average discrepancy around the more central leader is smaller, and for large centrality differences and low noise we observe a result similar to the noise-free networks we saw in Table 4. We see a similar trade-off between coverage and centrality in the Srivastsava-Leonard Network, as we show in Table 6.

4.1.4. Reduction to Centrality and Coverage

In the noise-free case, a single-leader network reduced to finding the most central leader and

Noise (σ)	Leader β_i	Mean Leader Discrepancies
0	$1, 1$	$0.08, 0.08$
0	$1, 2$	$0.09, 0.33$
0	$1, 10$	$0.88, 8.68$
0.01	$1, 1$	$0.19, 0.19$
0.01	$1, 2$	$0.48, 0.99$
0.01	$1, 10$	$0.99, 8.96$
0.03	$1, 1$	$0.90, 0.88$
0.03	$1, 2$	$1.07, 1.36$
0.03	$1, 10$	$1.76, 9.88$
0.1	$1, 1$	$9.56, 8.46$
0.1	$1, 2$	$9.34, 9.71$
0.1	$1, 10$	$8.84, 20.78$

Table 5: Noisy complete network with two leaders in a fully connected graph, quantitative comparison. As noise increases, we continue to observe a quantitative difference between leaders' discrepancies.

Leader Indices	Leader Discrepancies	Total Discrepancies
2, 4	0.21, 0.79	1.00
1, 2	0.87, 2.71	3.58
1, 9	0.43, 6.09	6.53
2, 9	0.47, 8.56	9.04
6, 9	7.60, 2.94	10.54
2, 6	4.15, 9.50	13.65

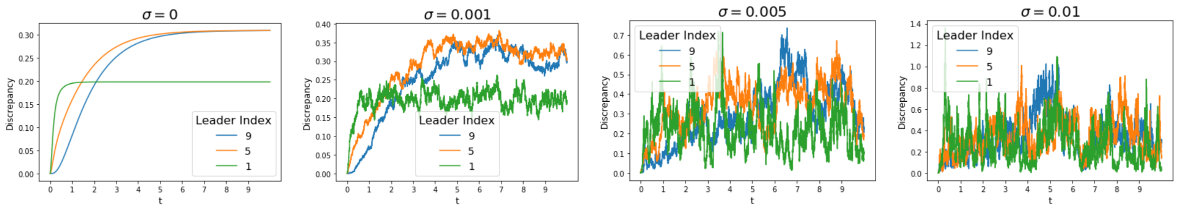
Table 6: Centrality-coverage tradeoff, with noise ($\sigma = 0.01$). In minimizing total discrepancy there is a trade-off between choosing centrally located leaders and leaders with higher joint coverage of the graph. The magnitudes differ slightly, but we see a similar pattern as in the noise-free case.

two leaders in a network of uniformly central individuals reduced to maximizing leaders' coverage of the network. In noisy networks, we see that this pattern generally holds, although higher amounts of noise introduce variability that can overwhelm this pattern. We provide examples in Figure 19.

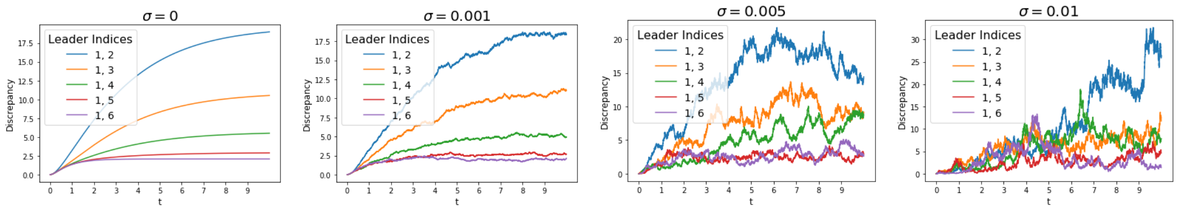
5. Conclusion

In this project we considered the behavior of networks of individual decision makers, each of whom accumulates evidence according to the drift diffusion model. We specifically looked at networks with heterogeneous populations: networks in which some individuals act as leaders and some as followers.

We observed that increasing leaders' signal size increased the discrepancy between leaders and the rest of the network, as did reducing the connectivity of a network. Furthermore, we proved that the amount of "leader information" fully determines the total amount of information accumulated among all individuals in a network and observed a number of



(a) Srivastava-Leonard graph with one leader (Centrality).



(b) Circle graph with two leaders (Coverage).

Figure 19: Reduction to coverage or centrality, both leader $\beta_i = 1$. With increasing noise, we lose the relationship between increasing coverage or centrality, and decreasing leader discrepancy.

trade-offs when trying to minimize the discrepancy between leaders: between the amount of information gathered by leaders in the network, and between leaders' centrality in and coverage of the network.

We further simulated networks with noise. We observed that for opinion state convergence noisy networks show qualitatively different behavior from noise-free networks. For relationships between discrepancy and leader or graph characteristics, small amounts of noise preserve the relationships while increasing amounts of noise eventually qualitatively overwhelm the relationships we see in noise-free networks.

5.1. Future Work

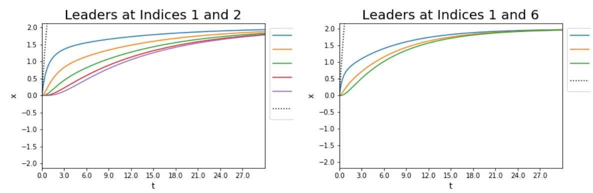
Future work could look at the model used by Fitch and Leonard [5, 4], in which leaders move towards a fixed external signal rather than accumulating information at a fixed rate (Equation 1, which we reproduce here for reference).

$$d\mathbf{x} = -\mathbf{B}(\mathbf{x} - \mu)dt - L\mathbf{x}dt + \sigma d\mathbf{W}$$

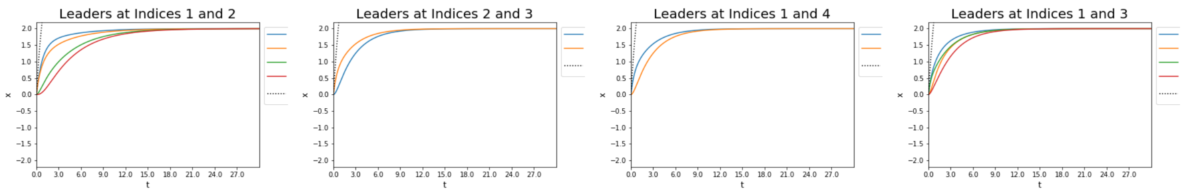
In this model all individuals converge to the same value (the measured signal μ), so we slightly modify our definition of discrepancy. Rather than measuring the difference between individuals' opinion states at the steady state, we consider discrepancies from $t = 0$ onward, using

$$\frac{\sum_{i=1}^{n} |\beta_i| \sum_{t=0}^{T} d_{ij}(t)}{\sum_{i=1}^{n} |k_i|}$$

As we show in Figure 20, preliminary simulations in noise-free networks show similar patterns as those that we find with the network of DDMs we use in this paper.



(a) Two leaders in a circle graph of ten nodes.



(b) Two leaders in a line graph of four nodes.

Figure 20: Two leader leaders in graph using a model in which leaders move towards a fixed external signal. Preliminary simulations in noise-free networks show similar patterns as those that we find with the network of DDMs we use in this paper.

Further work could investigate whether the theoretical and empirical results we found for our coupled DDM hold in the model used by Fitch and Leonard as well, in both noise-free and noisy networks.

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6. Appendix

6.1. Code

The code used to generate these simulations is available on github.

6.2. Example Graphs with Two Leaders, General Case

In this section we provide derivations of two leaders in a line, circle, and complete graph of four individuals. This is a more general case of the computations we provide in Section 3.2.1.

Graph Structure	Leader Placements	$d_L^* = \frac{\sum_{i=1}^{n} \lvert\beta_i\rvert d_i^*}{\sum_{i=1}^{n} \lvert\beta_i\rvert}$
Line	$(1\&3),(2\&4),(1\&4),(2\&3)$	$\frac{(-3k_1+k_2)^2+(-2k_1+2k_2)^2+(-k_1-k_3)^2}{16(\lvert k_1\rvert+\lvert k_2\rvert))}$
Line	$(1\&2),(3\&4)$	$\frac{(-3k_1+k_2)^2+(-3k_1+k_2)^2+(-2k_1-2k_2)^2}{16(\lvert k_1\rvert+\lvert k_2\rvert)}$
Fully Connected	any two leaders	$\frac{(k_1-k_2)^2+(k_1)^2+(k_2)^2}{8(\lvert k_1\rvert+\lvert k_2\rvert)}$
Circle	$(1\&2),(2\&3),(3\&4),(4\&1)$	$\frac{(-3k_1+3k_2)^2+(-3k_1-k_2)^2}{32(\lvert k_1\rvert+\lvert k_2\rvert)}$
Circle	$(1\&3),(2\&4)$	$\frac{(-3k_1+k_2)^2+(-k_1+3k_2)^2}{32(\lvert k_1\rvert+\lvert k_2\rvert)}$

Table 7: Total discrepancy for two leaders at indices l_1, l_2 in a line graph, $\beta_{l_1} = k_1, \beta_{l_2} = k_2$.