Sets and Functions

Adapted from:
Chapter 2, Connecting Discrete Mathematics and Computer Science and
Chapter 1, Linear Algebra Done Right

Booleans

only takes on two values: **True** and **False**

- sometimes we use **1** and **0** instead
- can also think of it as **yes/no**, **on/off**, etc.

can be represented using 1 **bit**

- other types are represented as combinations of bits (we'll see this later)

named after George Boole, a British mathematician in the 1800s

Integers

numbers with no fractional part

we use the symbol $\mathbb{Z}$ to represent the set of all possible integers

- theres a lot of them, so unlike booleans we can't just list all integers

can be positive or negative (or zero)

- this is as opposed to **natural** numbers ($\mathbb{N}$, positive integers)

number of *bits* needed to represent an integer increases with the value

- computers like binary, so as an example the integer 5 is `101` in binary

Rational Numbers

numbers that can be represented by a ratio of two integers

- that is a number $\frac{m}{n}$ where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ and $n \neq 0$

we use the symbol $\mathbb{Q}$ to represent the set of all rational numbers

real numbers that are not rational, are called **irrational** numbers

Real Numbers

numbers that can be represented with decimals

includes all integers

- includes all number "between" integers

we use the symbol $\mathbb{R}$ to represent the set of all real numbers

typically we need more bits to store real-valued numbers

- proportional to the value and *precision*
- many representations, so we won't discuss it

Complex Numbers

Invented in order to allow for taking the root of negative numbers

Each complex number has a *real* and *imaginary* part, and is written as

- $a + bi$ where $a,b \in \mathbb{R}$

We denote the set of all complex numbers by $\mathbb{C}$

- note that $\mathbb{R}$ is a subset of $\mathbb{C}$ (where all values b are 0)

Sets

an *unordered* collection of objects

we have already been talking about these in the abstract

- set of all integers $\mathbb{Z}$
- set of all real numbers $\mathbb{R}$
- ...

we write a set using "{" and "}"

- use capital letters as names (when applicable)
- if we wanted to define the set of all algebraic operators we may use
  - $O = \{+,-,\cdot,\div\}$
- or maybe the set of all prime numbers
  - $P = \{2,3,5,7,...\}$

Set Membership

for a set *S* and an object *x*, the expression $x \in S$ is true when *x* is one of the objects in the set *S*.

we read $x \in S$ as "*x* is an element of *S*"

- or "*x* is in *S*"

there is also the notion of non-membership

- if the set *P* is the set of all primes then $\mathbf{4 \notin P}$
- that is 4 is *not* in *P*, or 4 is not prime

Set Cardinality

for a set *S*, |*S*| is the number of distinct elements in *S*

often we want to know how large a set is so we can compare sets
- for the set of operations described before $|O|=4$
- for the set of all possible bit values B, $|B|=2$

but sometimes we can't actually count how many there are
- for example $|\mathbb{Z}|=\infty$
- but (as we will see next semester) we still know that $|\mathbb{Z}|<|\mathbb{R}|$

Notation Summary

set membership	$x \in S$	x is one of the elements of S
cardinality	$\lvert S\rvert$	the number of distinct elements in the set S

Set Construction

there are two ways to define an entire set:

- **exhaustive enumeration**: simply listing all elements
- **set abstraction**: defining a set of logical rules that are true for all elements

Exhaustive Enumeration

we have already seen some of these sets, but here are some others:

- set of even prime numbers: {2}.
- set of prime numbers between 10 and 20: {11, 13, 17, 19}
- set of 2-digit perfect squares: {81, 64, 25, 16, 36, 49}
- set of bit values: {0,1}
- set of Turing Award winners in 2004, 2008, 2012, and 2016:
  - {Tim Berners-Lee, Vint Cerf, Shafi Goldwasser, Bob Kahn, Barbara Liskov, Silvio Micali}

note that not all sets need to be assigned a name

Exhaustive Enumeration

remember that sets are unordered, so the following are equivalent

- $\{2+2, 2 \cdot 2, 2 \div 2, 2-2\}$
- $\{0, 1, 4\}$
- $\{4, 0, 1\}$

note also that $|\{2+2, 2 \cdot 2, 2 \div 2, 2-2\}| = 3$

- despite there being four entries in the list of elements,
- only three are *distinct* objects

Some Quirks

the integer 2 and the set $\{2\}$ are two entirely different kinds of things

- $2 \in \{2\}$
- but $\{2\} \notin \{2\}$
- the lone element in $\{2\}$ is the number 2, not the set containing the number 2

Set Abstraction

let *U* be a set of possible elements called the **universe***.*

let *P(x)* be a condition, also called a **predicate**, that

- for every element *x*∈*U*
- *P(x)* is true or false

then we can write $\{x \in U : P(x)\}$

- which is all of the objects from *x*∈*U*
- for which *P(x)* is true

Set Abstraction

looking at some of the examples from before:

- set of even prime numbers: $\{ x \in \mathbb{Z}^{\geq 1} : x \text{ is prime and } x \text{ is even} \}$
- set of primes between 10 and 20: $\{ y \in \mathbb{Z} : y \text{ is prime and } 10 \leq y \leq 20 \}$
- set of 2-digit perfect squares: $\{ n \in \mathbb{Z} : \sqrt{n} \in \mathbb{Z} \text{ and } 10 \leq n \leq 99 \}$
- set of bits: $\{ b \in \mathbb{Z} : b^2 = b \}$

Set Abstraction

for convenience, sometimes we don't use the universe if its obvious

- this is particularly helpful when the predicate determines the universe
- in that case we could write something like $\{x : P(x)\}$

there is, of course, multiple ways to write these sets as well

- lets look at the perfect squares $\{ n \in \mathbb{Z} : \sqrt{n} \in \mathbb{Z} \text{ and } 10 \leq n \leq 99 \}$:
- $\{ n^2 : n \in \mathbb{Z} \text{ and } 10 \leq n^2 \leq 99 \}$ and
- $\{ n^2 : n \in \{4,5,6,7,8,9\} \}$ are the same set

note: many programming languages store objects in *lists*, these are ordered and distinct from sets

The Empty Set

the **empty set** is denoted by {} or simply ∅ and contains no elements

- sometimes called the null set

$|\emptyset| = 0$

- there are no elements to count

we can also define this differently

- any set with no elements can be reduced to ∅
- {x : false} (similar to an if statement with "false" in the condition)

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empty set	$\{\}$ or $\varnothing$	the set containing no elements

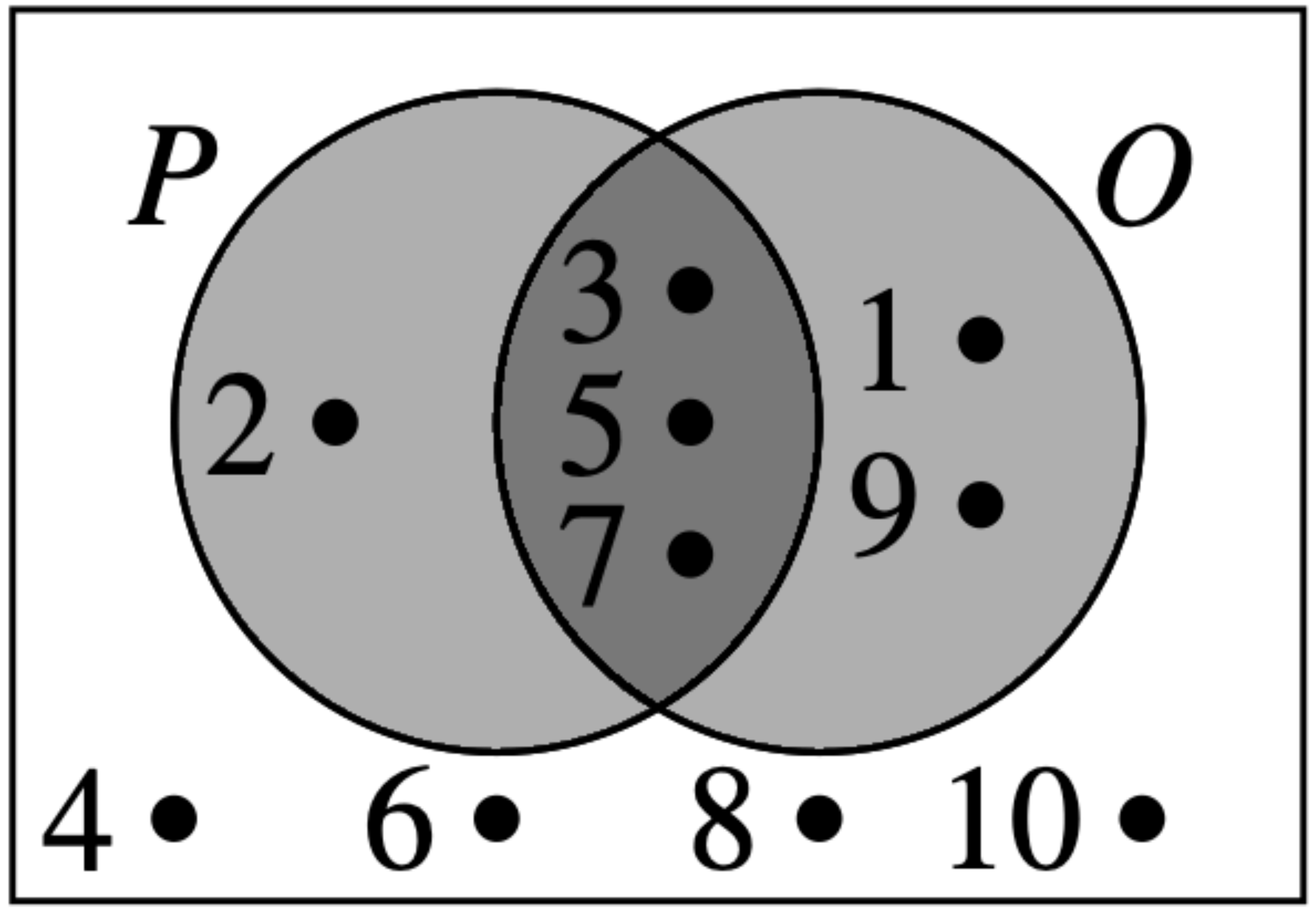
Visualizing Sets

when working with sets, sometimes its helpful to use visual representations

we will be relying on the Venn digram to see how different sets interact

lets look at this example:

- $U=\{\ x \in \mathbb{Z}^{>0} : x \leq 10\ \}$
- $P=\{\ y \in U : y$ is prime $\} = \{2,3,5,7\}$
- $O=\{\ z \in U : z \bmod 2 = 1\ \} = \{1,3,5,7,9\}$



Set Complement

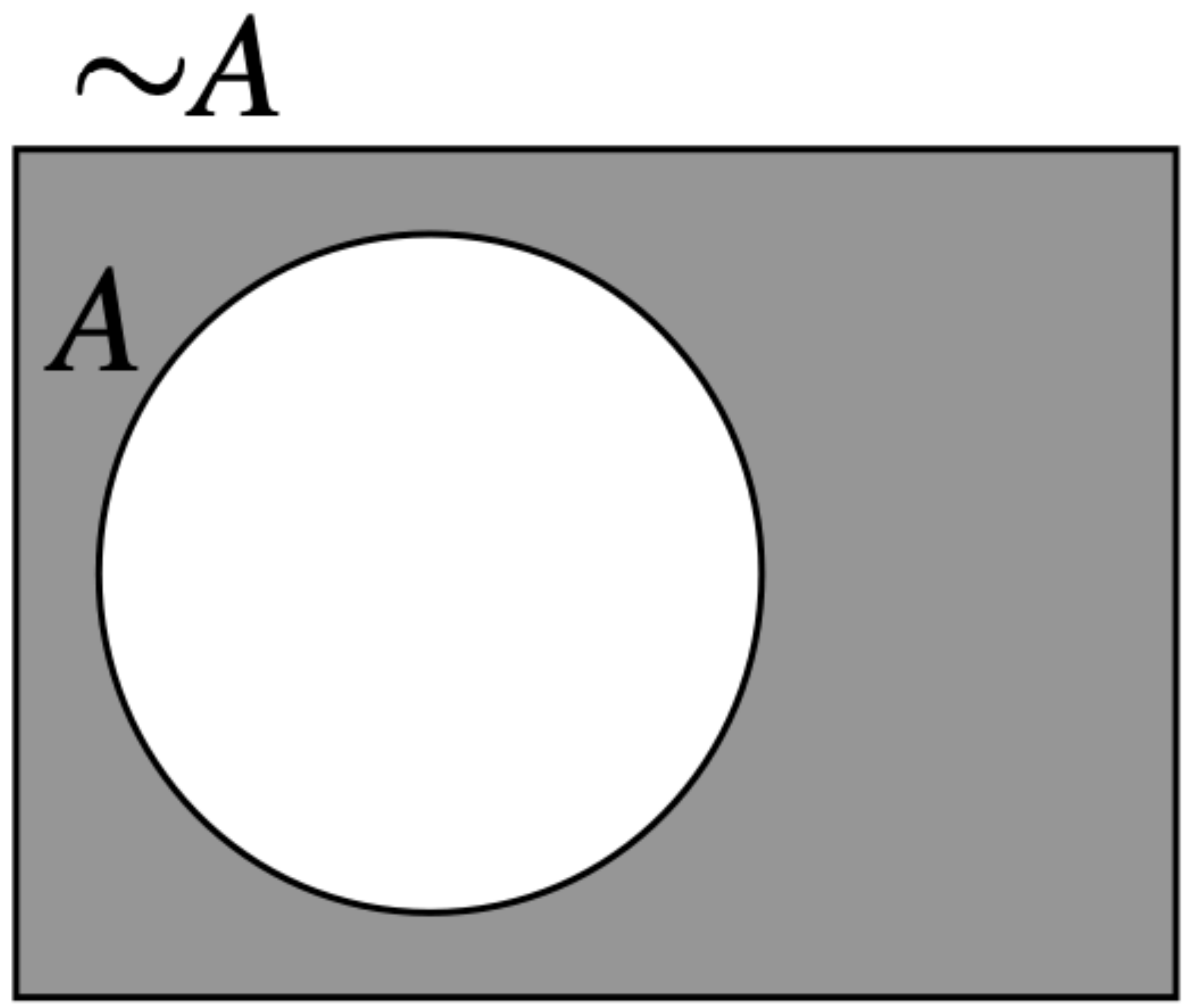
given a set S, the **complement** of the set (denoted $\sim A$ or $\bar{A}$) is

- the set of all elements from the universe not in A
- that is $\sim A = \{ x : x \notin A \}$
- since the universe is implicit from the set A we can leave it out

you can think of this as set negation

example:

- $U = \{ x \in \mathbb{Z}^{>0} : x \leq 10 \}$
- $A = \{3,4,5,6\}$
- $\sim A = \{1,2,7,8,9,10\}$



Set Union

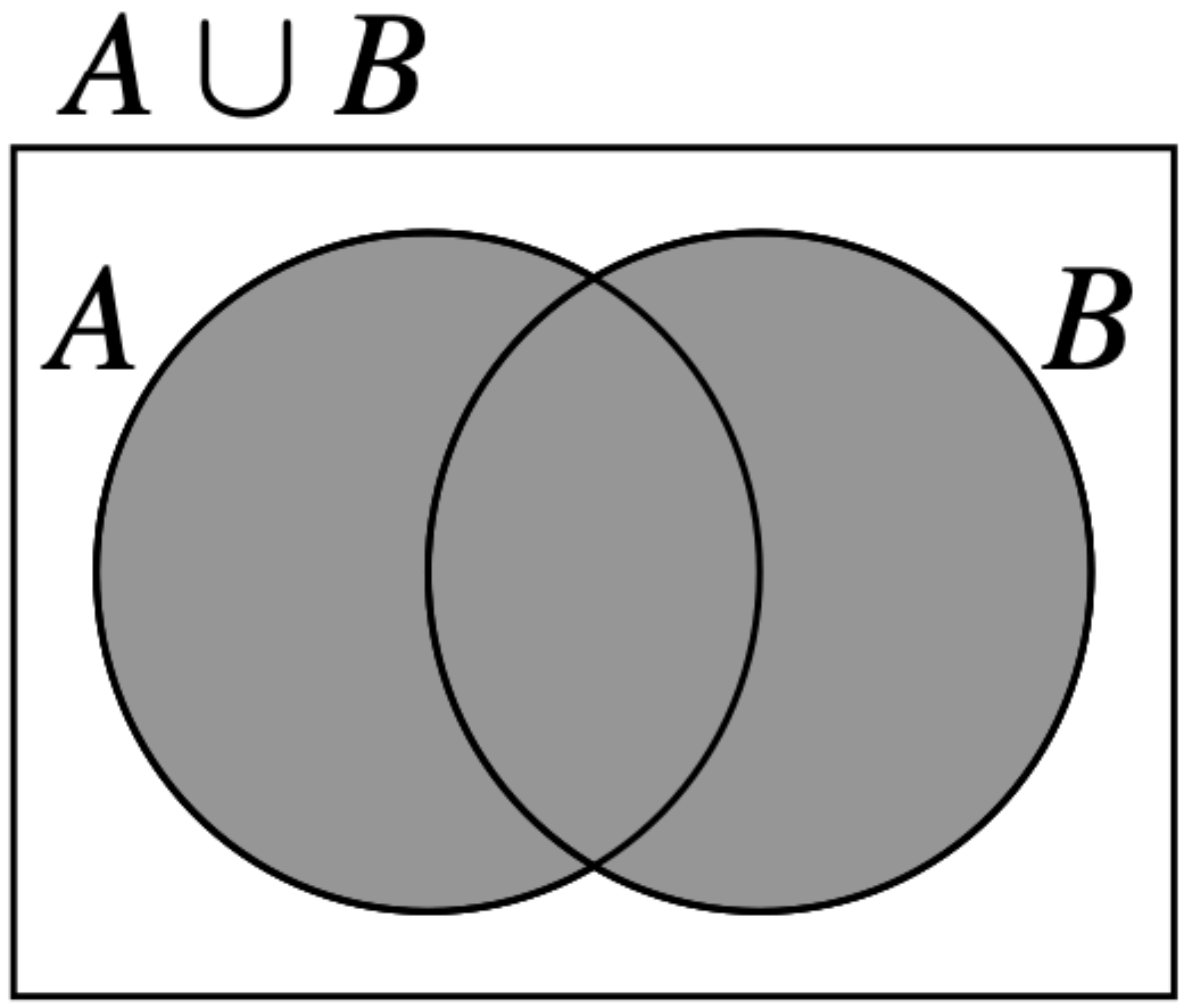
given two sets A and B, the **union** $A \cup B$

- contains all of the elements from the universe
- which are in *either* A or B
- that is $A \cup B = \{ x : x \in A \text{ or } x \in B \}$

you can think of this as set addition

example:

- $\{1,2,\mathbf{3},\mathbf{4}\} \cup \{\mathbf{3},\mathbf{4},5,6\} = \{1,2,\mathbf{3},\mathbf{4},5,6\}$



Set Intersection

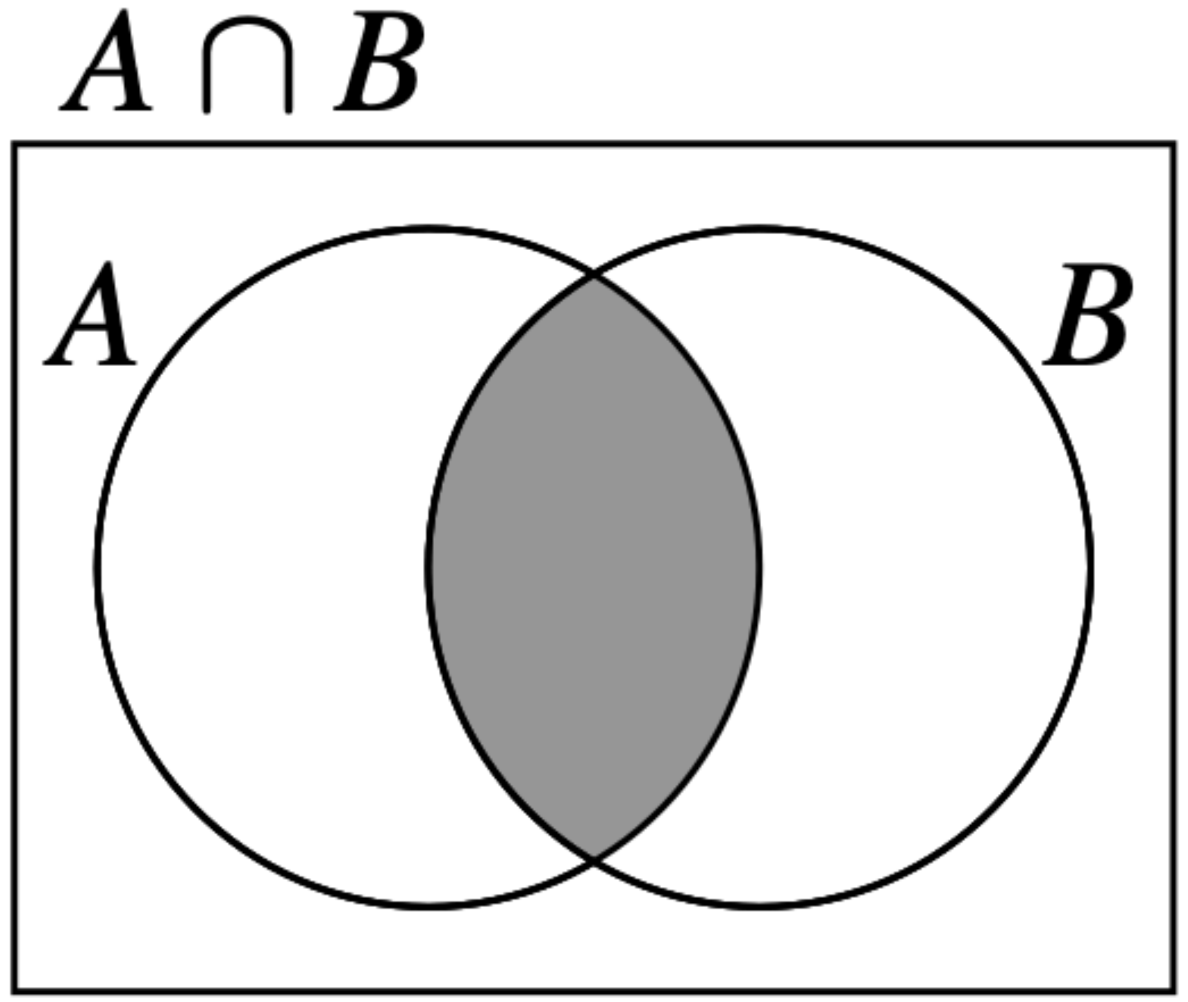
given two sets A and B, the **intersection** $A \cap B$

- contains all of the elements from the universe
- which are in *both* A and B
- that is $A \cap B = \{ x : x \in A \text{ and } x \in B \}$

this does not have a good corollary in algebra

example:

- $\{1,2,\mathbf{3},\mathbf{4}\} \cap \{\mathbf{3},\mathbf{4},5,6\} = \{\mathbf{3},\mathbf{4}\}$



Set Difference

given two sets *A* and *B*, the **difference** *A*-*B* (or *A**B*)

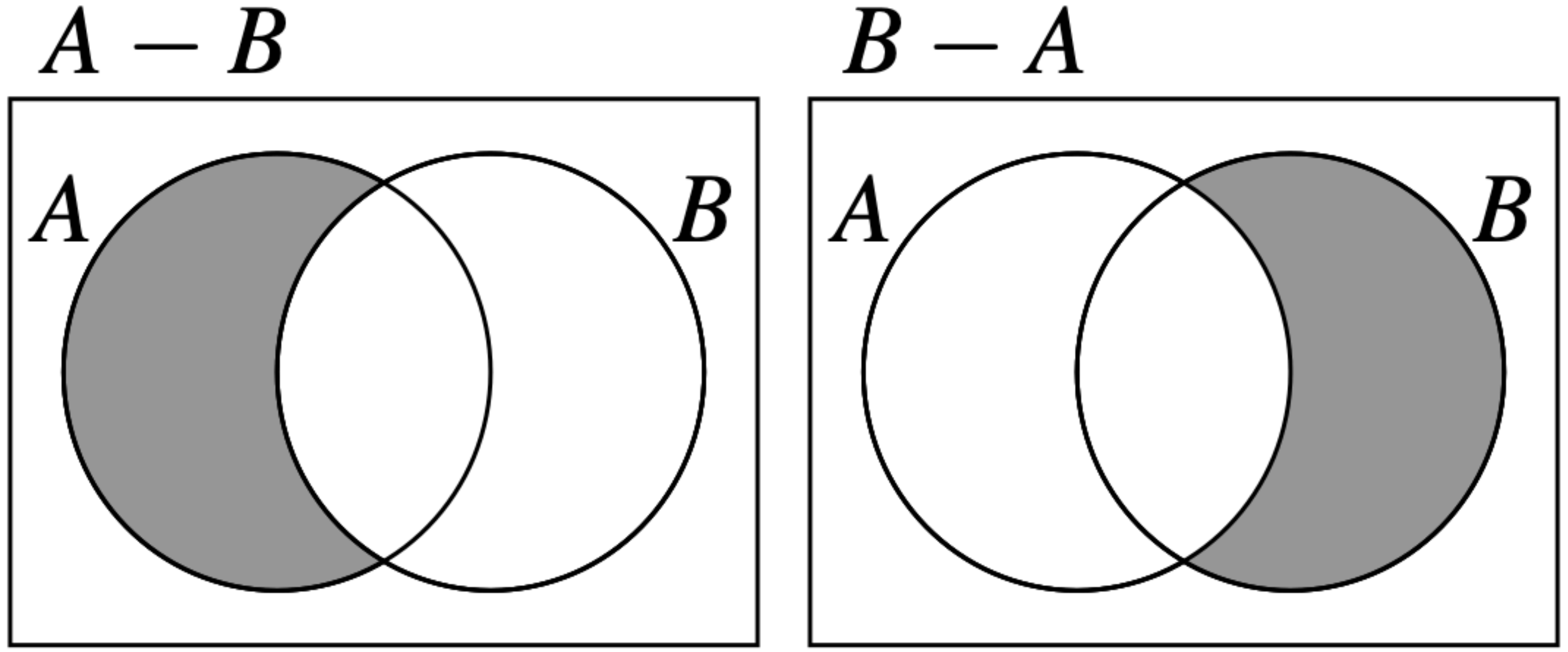
- is the set of elements from the set *A*
- that are not in the set *B*
- that is: $A\text{-}B = \{\ x : x{\in}A \text{ and } x{\notin}B\ \}$

this is like subtraction

- similarly, like subtraction *A*-*B* is not the same as *B*-*A*

example:

- {1,2,**3,4**}-{**3,4**,5,6} = {1,2}
- {**3,4**,5,6}-{1,2,**3,4**} = {5,6}



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empty set	$\{\}$ or $\varnothing$	the set containing no elements
complement	$\sim S = \{x \in U : x \notin S\}$	the set of all elements in the universe U that aren't in S; U may be left implicit if it's obvious from context
union	$S \cup T = \{x : x \in S \text{ or } x \in T\}$	the set of all elements in either S or T (or both)
intersection	$S \cap T = \{x : x \in S \text{ and } x \in T\}$	the set of all elements in both S and T
set difference	$S - T = \{x : x \in S \text{ and } x \notin T\}$	the set of all elements in S but not in T

Some extra notation

commonly we want to combine many sets together

- so similar to summation and product we can write things like:

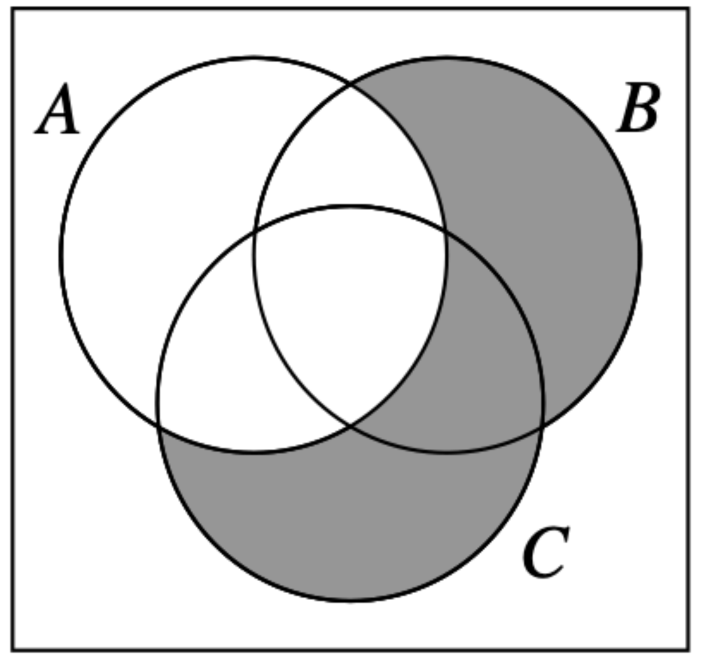
$$\bigcup_{i=1}^{n} S_i = S_1 \cup S_2 \cup \ldots \cup S_n \text{ and}$$

$$\bigcap_{i=1}^{n} S_i = S_1 \cap S_2 \cap \ldots \cap S_n$$

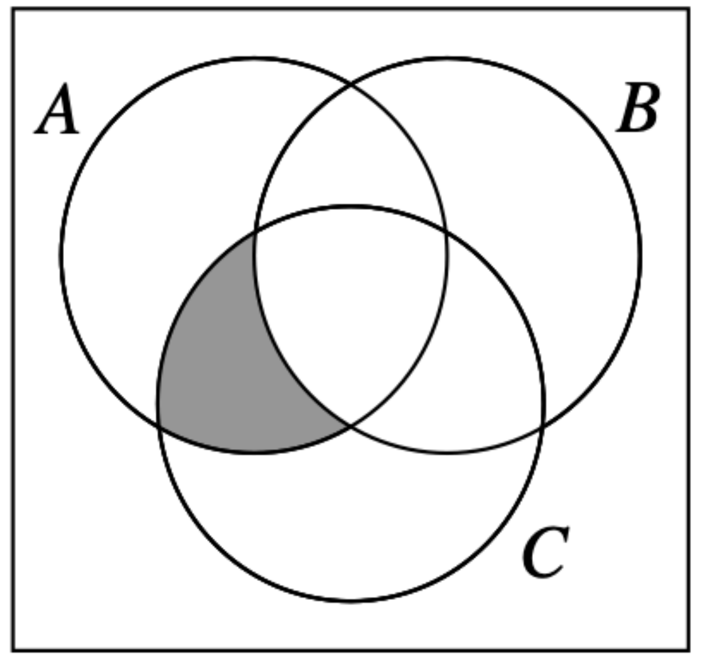
More Than Two Sets

we can combine more than just two sets

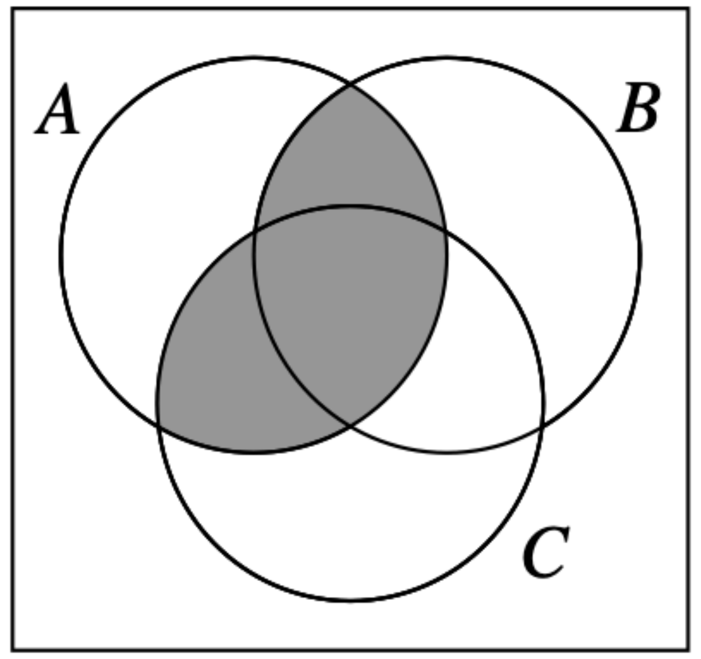
- example: $A \cup B \cup C$ is the set $\{x : x \in A \text{ or } x \in B \text{ or } x \in C\}$



$(B \cup C) - A$



$(A - B) \cap C$



$A \cap (B \cup C)$

Arithmetics and Sets

we previously saw the use of sums and products like

$$\sum_{i=3}^{5} 2^i = 8 + 16 + 32 = 56$$

but we can do the same using a set, define *S*={*3,4,5*}, then we can say

$$\sum_{x \in S} 2^x = 8 + 16 + 36 = 56$$

we can do the same with product, max, and min:

$$\prod_{x \in S} x \qquad \max_{x \in S} x \qquad \min_{x \in S} x$$

Comparing Sets

just like numbers, sometimes we need to compare whole sets

two sets A and B are **equal** if A and B have exactly the same elements

- sets A and B are *not* equal if there's an element $x \in A$ but $x \notin B$, or
- if there's an element $y \in B$ but $y \notin A$

as an example lets look at the sets {1,1,2,2} and {1,2}, are they equal?

Comparing Sets

A is a **subset** of a set B, written $A \subseteq B$, if every $x \in A$ is also an element of B

- or $A - B = \emptyset$

A is a **proper subset** of a set B, written $A \subset B$, if $A \subseteq B$ and $A \neq B$

- $A \subset B$ whenever $A \subseteq B$ but $B \not\subseteq A$, or
- $A \subseteq B$ and there is $x \in B$ thats not in A ($B - A \neq \emptyset$)

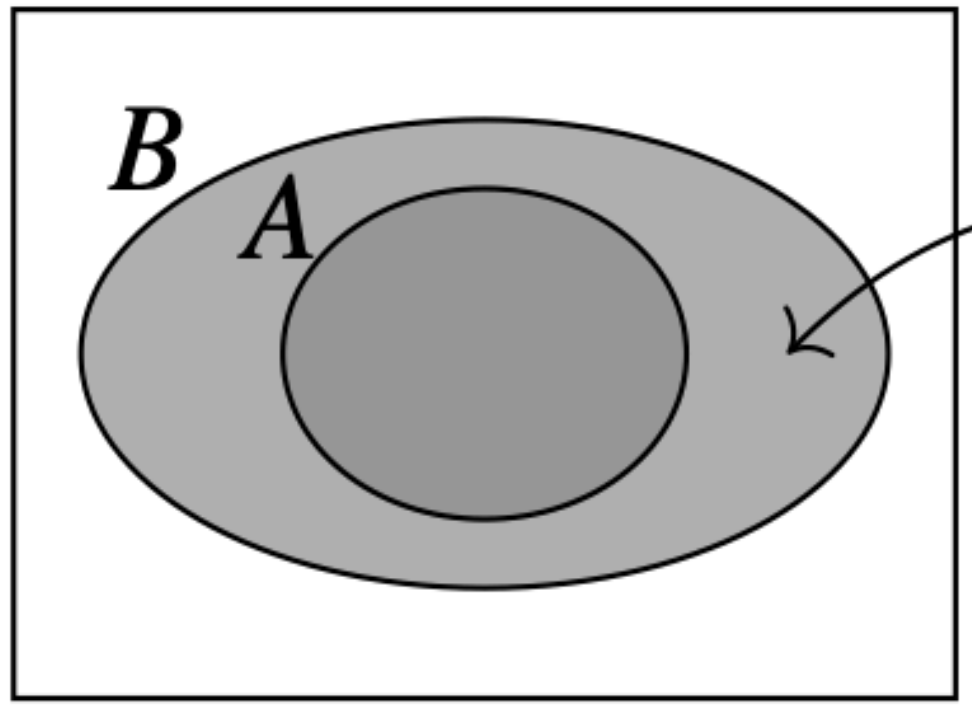
lets assume $A = \{1,2,3\}$

- $A \subseteq \{1,2,3,4\}$
- $A \subseteq \{1,2,3\}$
- $A \subset \{1,2,3,4\}$
- $A \not\subset \{1,2,3\}$

Comparing Sets

B is a **superset** of set A, written $B \supseteq A$, if $A \subseteq B$

B is a **proper superset** of A, written $B \supset A$, if $A \subset B$



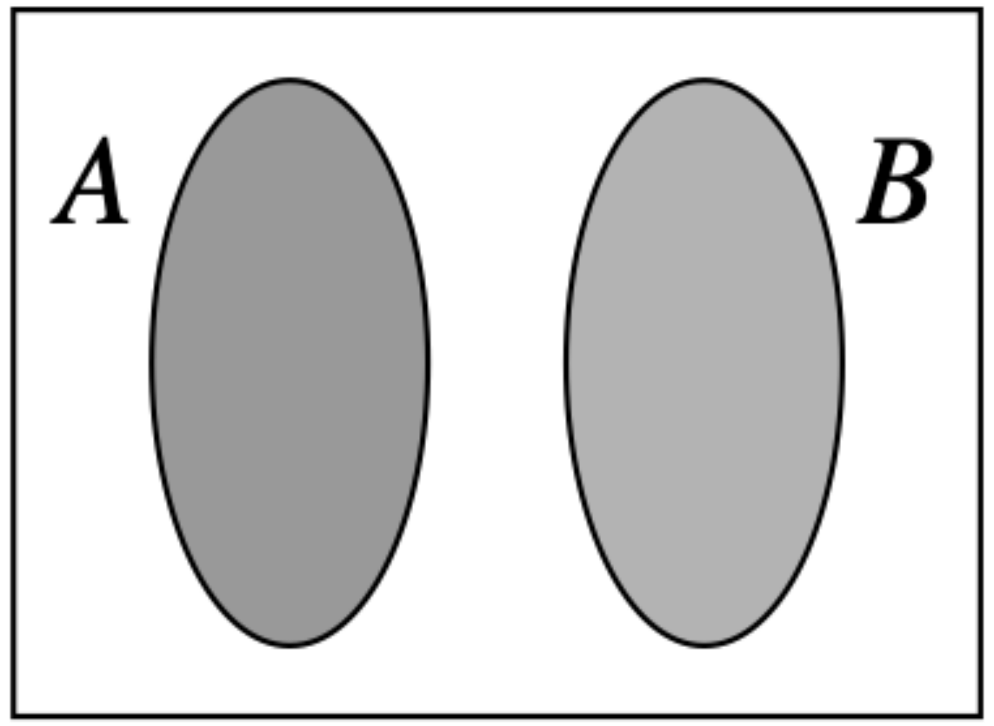
The sets satisfy $A \subset B$ (and $B \supset A$) if there's at least one element in this region, and they satisfy $A = B$ if there's no element in this region.

two sets satisfying $A \subseteq B$ and, equivalently, $B \supseteq A$

Comparing Sets

sets *A* and *B* are **disjoint** if there is no $x \in A$ where $x \in B$

- that is, when $A \cap B = \varnothing$



Exercise

let A={3, 4, 5} and B={4, 5, 6}

identify a set C satisfying the following (or state why its impossible)

- $A \subseteq C$ and $C \supseteq B$
- $A \supseteq C$ and $C \subseteq B$
- $A \supseteq C$ and $C \supseteq B$

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set difference	$S - T = \{x : x \in S \text{ and } x \notin T\}$	the set of all elements in S but not in T
set equality	$S = T$	every $x \in S$ is also in T, and every $x \in T$ is also in S
subset	$S \subseteq T$	every $x \in S$ is also in T
proper subset	$S \subset T$	$S \subseteq T$ but $S \neq T$
superset	$S \supseteq T$	every $x \in T$ is also in S
proper superset	$S \supset T$	$S \supseteq T$ but $S \neq T$

Sets of Sets

just like sums can be over other sums, we can have sets of other sets

the set $A=\{\mathbb{Z}, \mathbb{R}, \mathbb{Q}\}$ of the sets defined earlier is itself a set

- $|A| = 3$, because A has three distinct elements—namely $\mathbb{Z}$ and $\mathbb{R}$ and $\mathbb{Q}$
- though for all three elements $x \in A$, $|x|=\infty$

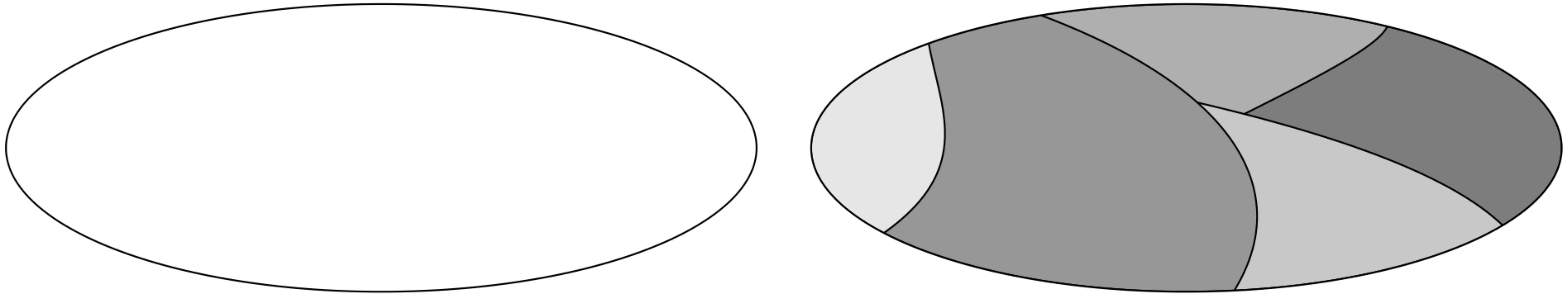
consider $B = \{\{\} , \{1, 2, 3\}\}$

- $|B|=2$; B has two elements, $\{\}$ and $\{1, 2, 3\}$
- $\{\} \in B$, but $1 \notin B$
  - although 1 *is* an element of one of the two elements of *B*
  - it is not an element of *B* itself

Partitions

a **partition** of a set S is a set $\{A_1, A_2, \dots, A_k\}$

- of nonempty sets $A_1, A_2, \dots, A_k$, for some $k \geq 1$, such that
  - (i) $A_1 \cup A_2 \cup \dots \cup A_k = S$; and
  - (ii) for any i and $j \neq i$, the sets A_i and A_j are disjoint



(a) The set S.   (b) S partitioned into 5 subsets.

Partitions

consider the set S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

there are different ways to partition the set:

- { {1,3,5,7,9}, {2,4,6,8,10} }
- { {1, 2, 3, 4, 5, 6, 7, 8, 9} , {10} }
- { {1,4,7,10}, {2,5,8}, {3,6,9} }
- { {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9}, {10} }
- { {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} }

each of the 10 numbers from S is in one, and only one, of the listed sets

Power Sets

the **power set** of a set *S*, written $\mathscr{P}(S)$, denotes the *set of all subsets* of *S*:

- a set *A* is an element of $\mathscr{P}(S)$ precisely if *A*⊆*S*.
- $\mathscr{P}(S)$ = {*A : A*⊆*S*}

Some examples

- $\mathscr{P}\left(\{0\}\right) = \{\{\,\}, \{0\}\}$
- $\mathscr{P}\left(\{0,1\}\right) = \{\{\,\}, \{0\}, \{1\}, \{0,1\}\}$
- $\mathscr{P}\left(\{0,1,2\}\right) = \{\{\,\}, \{0\}, \{1\}, \{2\}, \{0,1\}\ \{0,2\}\ \{1,2\}, \{0,1,2\}\}$

Power Sets

another (m

$$\{\}$$

$$\{\{\}\}\quad \{\{0\}\}\quad \{\{1\}\}\quad \{\{0,1\}\}$$

$$\{\{\},\{0\}\}\quad \{\{\},\{1\}\}\quad \{\{\},\{0,1\}\}\quad \{\{0\},\{1\}\}\quad \{\{0\},\{0,1\}\}\quad \{\{1\},\{0,1\}\}$$

$$\{\{0\},\{1\},\{0,1\}\}\quad \{\{\},\{1\},\{0,1\}\}\quad \{\{\},\{0\},\{0,1\}\}\quad \{\{\},\{0\},\{1\}\}$$

$$\{\{\},\{0\},\{1\},\{0,1\}\}$$

There are a total of 16 sets; one with 0 elements, four with 1 element, six with 2 elements, four with 3 elements, and one with 4 elements

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Use of Sets

in machine learning often we **label** datapoints

we can do that using *nearest neighbor*

- given a set of labeled data and a new unlabeled point
- use the "closest" point to label the new one

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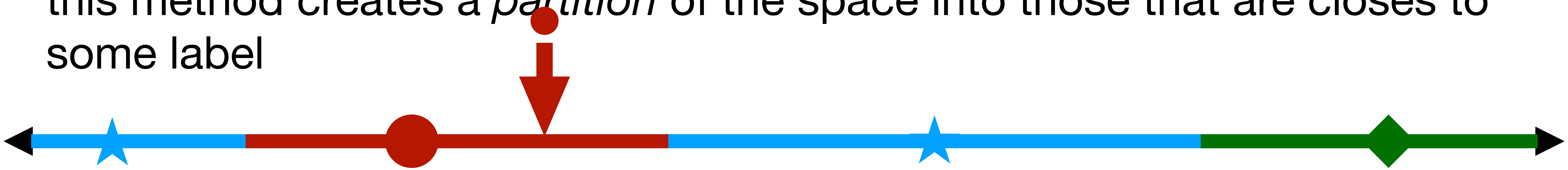
this method creates a *partition* of the space into those that are closes to some label



Use of Sets

programming languages like python include the ability to define sets

- similar to arrays, but different since they are unordered and unique

this allows you to perform functions on a whole set at once

- this is useful in machine learning
- for example if you have a set of images and you want to turn convert them into greyscale

Sets

2.118 Let S and T be two sets, with $n = |S|$ and $m = |T|$. What is the smallest cardinality that $S \cup T$ can have. Give an example of the minimum-sized sets. (You should give a *family* of examples—that is, describe a smallest-possible set for *any* values of n and m.)

2.119 Repeat for $S \cap T$. (That is, what's the smallest possible value of $|S \cap T|$ in terms of n and m? Give a family of examples.)

2.120 Repeat for $S - T$.

2.121 What's the *largest* possible value of $S \cup T$ in terms of n and m? Give a family of examples.

2.122 Repeat for the largest possible value of $S \cap T$.

2.123 Repeat for the largest possible value of $S - T$.

Sets

In a variety of CS applications, it's useful to be able to compute the similarity *of two sets A and B. (More about one of these applications, collaborative filtering, below.) There are a number of different ideas of how to measure set similarity, all based on the intuition that the larger $|A \cap B|$ is, the more similar the sets A and B are. Here are two basic measures of set similarity that are sometimes used:*

- the cardinality measure: *the similarity of A and B is* $|A \cap B|$.
- the Jaccard coefficient: *the similarity of A and B is* $\frac{|A \cap B|}{|A \cup B|}$. *(The Jaccard coefficient is named after the Swiss botanist Paul Jaccard (1868–1944), who was interested in how similar or different the distributions of various plants were in different regions [63].)*

2.124 Let $A = \{\text{chocolate}, \text{hazelnut}, \text{cheese}\}$; $B = \{\text{chocolate}, \text{cheese}, \text{cardamom}, \text{cherries}\}$; and $C = \{\text{chocolate}\}$. Compute the similarities of each pair of these sets using the cardinality measure.

2.125 Repeat the previous exercise for the Jaccard coefficient.

Functions

A **function** f from set A to set B, written $f : A \rightarrow B$, assigns to each input value $a \in A$ a unique output value $b \in B$

- the unique value b assigned to a is denoted by $f(a)$
- we sometimes say that f maps a to $f(a)$.

In algebra, we frequently encounter mathematical functions like $f(x) = x+6$

- for example we have $f(3) = 9$ and $f(4) = 10$

In programming, we often write or invoke functions that use an algorithm to transform an input into an output, like a function sort

- so that **sort**$(\langle 3, 1, 4, 1, 5, 9 \rangle) = \langle 1, 1, 3, 4, 5, 9 \rangle$, for example

Functions

for a function $f\!:\!A\rightarrow B$, the set A is called the **domain** of f, and the set B is called the **codomain** of f

the **range** or **image** of a function $f\!:\!A\rightarrow B$ is the set of all $b\in B$ such that $f(a)=b$ for some $a\in A$

- using the notation of the previous section the range of f is the set

$$\{y\in B : \text{there exists at least one } x\in A \text{ such that } f(x)=y\}$$

for function $f\!:\!A\rightarrow B$ and set $U\subseteq A$, we write $\{f(x)\!:\!x\in U\}$ as shorthand for the set

$$\{b\in B : \text{there exists some } u\in U \text{ for which } f(u)=b\}$$

Functions

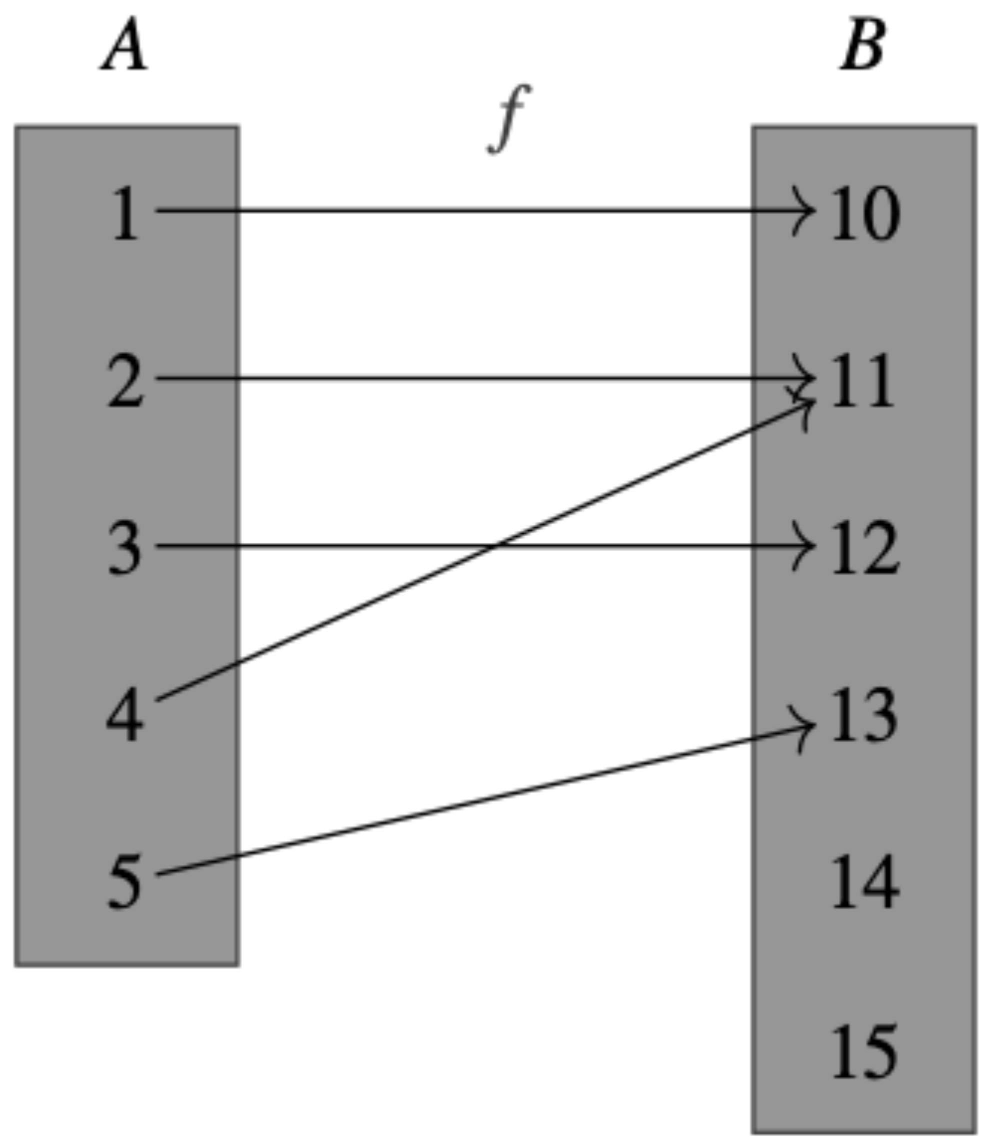


Figure 2.50 A picture of a function $f : A \to B$, where $A = \{1, \ldots, 5\}$ and $B = \{10, \ldots, 15\}$.

The function f as a table:

x	$f(x)$
1	10
2	11
3	12
4	11
5	13

What is the **domain?**

What is the **codomain?**

What is the **range?**

Function Composition

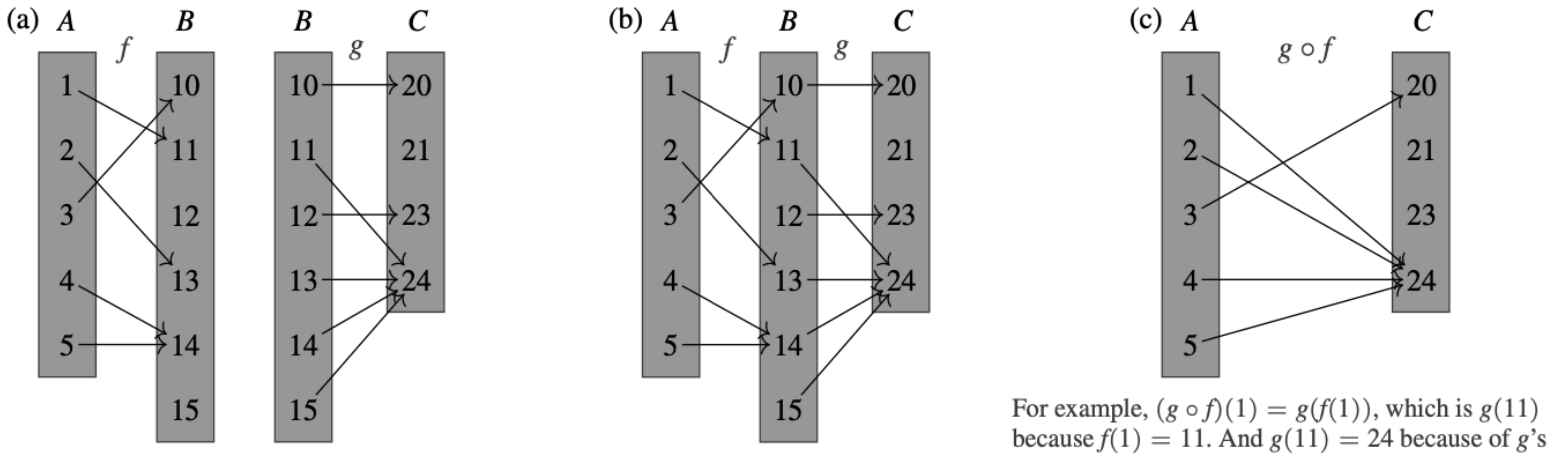
for two functions $f{:}A \to B$ and $g{:}B \to C$, the function $g \circ f{:}A \to C$ maps an element $a \in A$ to $g(f(a)) \in C$

- The function $g \circ f$ is called the **composition** of f and g

let $f{:}\mathbb{R} \to \mathbb{R}$ and $g{:}\mathbb{R} \to \mathbb{R}$ be defined by $f(x)=2x+1$ and $g(x)=x$

- the function $g \circ f$, given an input x, produces output
  $$g(f(x))=g(2x+1)=(2x+1)^2 =4x^2 +4x +1$$
- the function $f \circ g$ maps x to $f(g(x))=f(x^2)=2x^2 +1$
- the function $g \circ g$ maps x to $g(g(x))=g(x^2)=(x^2)^2 =x^4$.
- the function $f \circ f$ maps x to $f(f(x))=f(2x+1)=2(2x+1)+1=4x+3$

Composite Functions



For example, $(g \circ f)(1) = g(f(1))$, which is $g(11)$ because $f(1) = 11$. And $g(11) = 24$ because of g's arrow from 11 to 24.

Figure 2.51 Two functions $f : A \to B$ and $g : B \to C$, (a) separately and (b) pasted together. Their composition $g \circ f$ is shown in (c), based on successively following two arrows from (b).

Functions

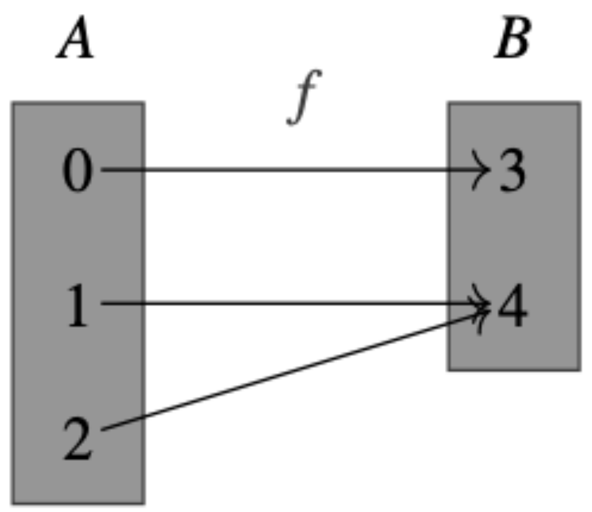
a function $f{:}A \to B$ is called **onto** if,

- for every $b \in B$, there exists at least one $a \in A$ for which $f(a)=b$
- an onto function is also sometimes called a *surjective* function

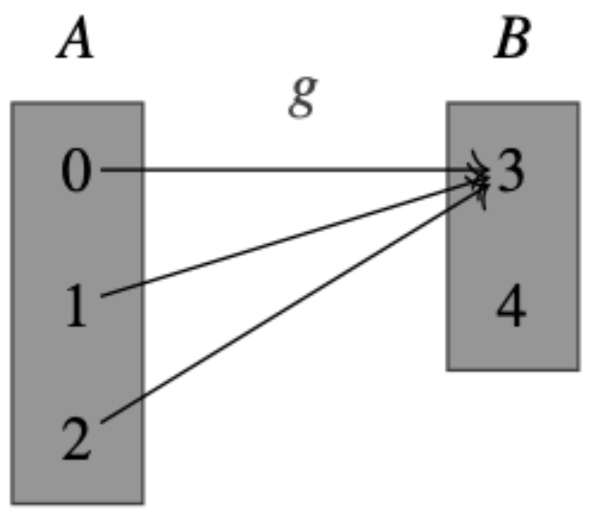
a function $f{:}A \to B$ is called **one-to-one** if,

- for any $b \in B$, there is *at most one* $a \in A$ such that $f(a)=b$
- we could say that $f{:}A \to B$ is one-to-one if,
  - for any $a_1 \in A$ and $a_2 \in A$ with $a_1 \neq a_2$, we have that $f(a_1) \neq f(a_2)$
- a one-to-one function is also sometimes called an *injective* function

onto



not onto



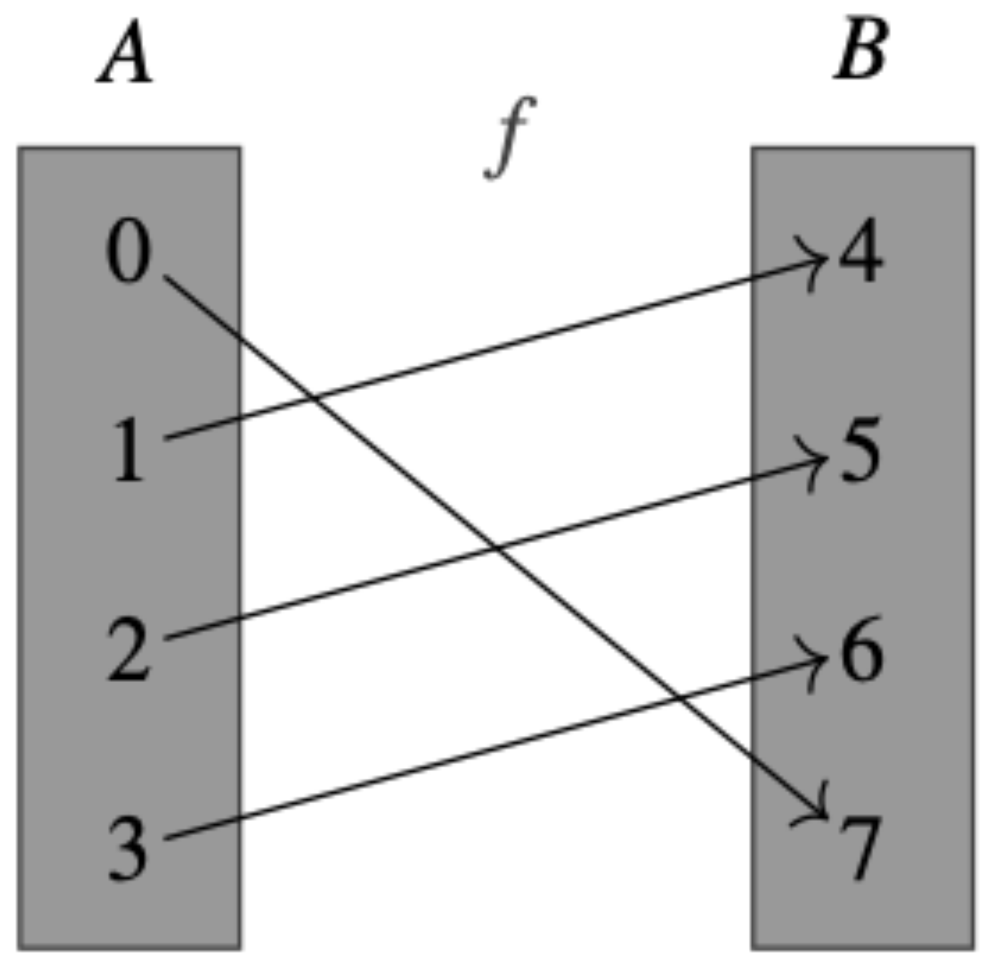
Functions

a function $f{:}A \to B$ is called a **bijection** if f is *one-to-one* and *onto*

- f is a bijection if for *every* $b \in B$, $|\{a \in A : f(a)=b\}|=1$

the function $f{:}\mathbb{R} \to \mathbb{R}$ defined by $f(x)=x-1$ is a bijection

- for every $b \in \mathbb{R}$, there is exactly one a such that $f(a)=b$
  - specifically, the value $a=b+1$



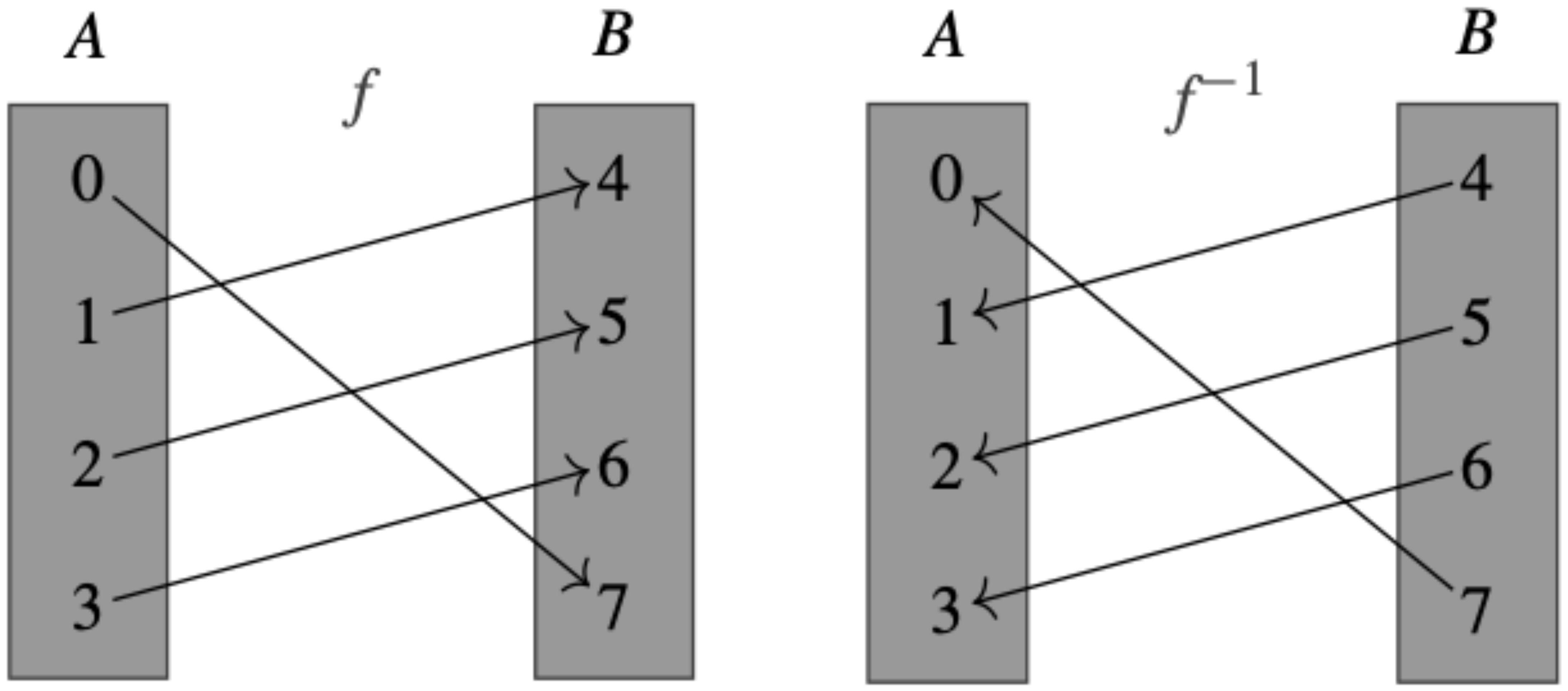
Functions

if f is a bijection, then $f^{-1}: B \rightarrow A$ is a function called the **inverse** of f, where

- $f^{-1}(b)=a$ whenever $f(a)=b$

the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x)=x-1$

- then $f^{-1}(y)=y+1$



Functions

A polynomial is a function $f:\mathbb{R}\to\mathbb{R}$ of the form $f(x)=a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$ where

- each $a_i \in \mathbb{R}$ and $a_k \neq 0$,
- for some $k \in \mathbb{Z}^{\geq 0}$
- more compactly $f(x) = \sum_{i=0}^{k} a_i x^i$

the real numbers $a_0, a_1, \ldots, a^k$ are called the **coefficients** of the polynomial

the values $a_0, a_1x, a_2x^2, \ldots, a_kx^k$ are called the **terms** of the polynomial

the **degree** of a polynomial is the maximum i such that $a_i \neq 0$

the **roots** of a polynomial $p(x)$ are the values in the set $\{x \in \mathbb{R} : p(x)=0\}$