Topic 2: Tuples, Vectors, and Matrices

02-680: Essentials of Mathematics and Statistics

August 25, 2024

1 Tuples

Unlike sets, ***tuples*** (also called ***sequences*** or ***lists***) are an *ordered* list of objects. Think of the position on chess board (or a 2D plane), a color in RGB, etc.

When these have small cardinality we can use terms like (ordered) pair [2], triple [3], quadruple [4], or more generically an "n-tuple".

If we're being precise, we normally use angle brackets ("$\langle$", "$\rangle$") but a lot of times we will be lazy and just use parentheses ("(","")")

Cartesian Product. A very useful way to construct ***set of*** tuples is using the ***cartesian product*** operator, which in essence creates all possible pairs of elements from two sets.

$$S \times T = \{\langle x, y\rangle \mid x \in S \land y \in T\}$$

As an example, lets remember the first two sets from our examples last topic:

$$A = \{\text{"Welcome"}, \text{"to"}, \text{"02-680"}\} \text{ and}$$
$$B = \left\{x^2 \mid x = 2 \lor x = 3\right\}.$$

In this case the cartesian product is

$$A \times B = \{\langle \text{"Welcome"}, 4\rangle, \langle \text{"to"}, 4\rangle, \langle \text{"02-680"}, 4\rangle, \langle \text{"Welcome"}, 9\rangle, \langle \text{"to"}, 9\rangle, \langle \text{"02-680"}, 9\rangle\}$$

It doesn't have to be different sets in the cartesian product though, we can have the product with a set and itself. In fact this is performed so often it has its own notation:

$$B \times B = B^2 = \{\langle 4, 4\rangle, \langle 4, 9\rangle, \langle 9, 4\rangle, \langle 9, 9\rangle\}.$$

This notation also generalizes, so $S^3 = S \times S \times S$, $S^4 = S \times S \times S \times S$ and so on. Notice that order matters in tuples (unlike sets) so $\langle 4, 9\rangle \neq \langle 9, 4\rangle$.

A note about this notation: Sometimes we want to have a set of tuples of different lengths (remember sets don't need to be over objects of the same type) so something like

$$B^2 \cup B^3 = \left\{ \begin{array}{c} \langle 4,4 \rangle , \langle 4,9 \rangle , \langle 9,4 \rangle , \langle 9,9 \rangle , \\ \langle 4,4,4 \rangle , \langle 4,4,9 \rangle , \langle 4,9,4 \rangle , \langle 4,9,9 \rangle , \\ \langle 9,4,4 \rangle , \langle 9,4,9 \rangle , \langle 9,9,4 \rangle , \langle 9,9,9 \rangle \end{array} \right\}$$

If we wanted to enumerate all binary numbers up to 8 digits (while omitting leading 0s):

$$\{1\} \times \bigcup_{i=1}^{7} \{0,1\}^i$$

But that leads to the notation we saw last time for strings: Σ^*. Sometimes we want the set of all tuples of any length, then we use the ***Kleene star*** (or Kleene operator); for some set S,

$$S^* = \bigcup_{i=0}^{\infty} S^i.$$

We will define $S^0 = \langle\rangle$ (the empty tuple) for any S, in the case of Σ^0 we often call it the empty string.

2 Vectors

When the set used to define a tuple is the set of real $\mathbb{R}$ (or complex $\mathbb{C}$[1]) numbers, we call the tuple a ***vector***. Specifically an n-vector x is defined as an element in

$$x \in \mathbb{R}^n \, .$$

If we want to reference the i-th element of x we will write

$$x_i$$

(or sometimes $x[i]$, this is true of tuples as well).

2.1 Simple Operations

Graphically we can think of vectors (in $\mathbb{R}^2$) in two way, which are... somewhat equivalent: as a point on the plane, or as an arrow from the origin. The second will be useful in this section, but the latter is sometimes useful as well.

[1]It will generally be true throughout the class that the properties we're discussing also apply to complex numbers, but for simplicity we will usually only directly discuss reals.

Vector Length. If we think of the vector as an arrow, we can say the ***length*** of the vector (arrow) is the same as the hypotenuse right triangle with each leg having the same length as each one of the elements. In that case we know that for vector $x = \langle x_1, x_2 \rangle \in \mathbb{R}^2$, the length is $\sqrt{x_1^2 + x_2^2}$. To generalize this we say the length of a vector $x \in \mathbb{R}^n$

$$\|x\| = \sqrt{\sum_{i=1}^{n} (x_i)^2}.$$

Often times we call this the L_2-norm of a vector.

2.2 Dot Product

We can actually redefine the L_2-norm using the dot product:

$$\|x\| = \sqrt{x \cdot x}.$$

3 Matrices

3.1 Special Matrices

3.2 Matrix Operations