Topic 8: Vector Spaces

02-680: Essentials of Mathematics and Statistics

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A ***vector space*** consists of 3 elements: a set of objects V along with definitions of addition and scalar multiplication on the elements in V. To be considered a vector space, the 3 elements have to satisfy the following conditions:

(1) $\forall v_1, v_2 \in V : v_1 + v_2 \in V$

(2) $\forall v_1, v_2 \in V : v_1 + v_2 = v_2 + v_1$

(3) $\forall v_1, v_2, v_3 \in V : (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$

(4) $\exists$ unique $\mathbf{0} \in V : \forall v \in V : v + \mathbf{0} = v$

(5) $\forall v \in V : \exists$ unique $u \in V : v + u = \mathbf{0}$ (usually denoted $-v$)

(6) $\forall v \in V, \alpha \in \mathbb{R} : \alpha v \in V$

(7) $\forall v_1, v_2 \in V, \alpha \in \mathbb{R} : \alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$

(8) $\forall v_1, v_2 \in V, \alpha, \beta \in \mathbb{R} : (\alpha + \beta)v = \alpha v + \beta v$

(9) $\forall v_1, v_2 \in V, \alpha, \beta \in \mathbb{R} : \alpha(\beta v) = (\alpha\beta)v$

(10) $\exists$ unique $\mathbf{1} \in V : \forall v \in V : \mathbf{1}v = v$

Example. Lets ask if the following is a vector space:

$$V = \{\langle x_1, x_2 \rangle \mid x_1, x_2 \in \mathbb{R}\}$$

with addition as

$$\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 - b_2 \rangle$$

(note the second dimension) and scalar multiplication as

$$\alpha\langle a_1, a_2 \rangle = \langle \alpha a_1, \alpha a_2 \rangle.$$

Since multiplication is as we normally see it, we know axioms (6), (8), (9), and (10) are satisfied. We can also see that axiom (1) is satisfied. Let's check (2):

$$\begin{aligned}
\langle a_1, a_2\rangle + \langle b_1, b_2\rangle &\stackrel{?}{=} \langle b_1, b_2\rangle + \langle a_1, a_2\rangle \\
\langle a_1 + b_1, a_2 - b_2\rangle &\stackrel{?}{=} \langle b_1 + a_1, b_2 - a_2\rangle \\
a_1 + b_1 &= b_1 + a_1 \\
a_2 - b_2 &\neq b_2 - a_2
\end{aligned}$$

Therefore, what we defined (with the non-standard addition) is not a vector space.

Other Spaces. That said, with standard addition $\mathbb{R}^2$ ***is*** a vector space (I will leave it to you to verify). The most common vector spaces we will be using in this class is $\mathbb{R}^n$ (for a fixed n).

In addition to $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$ are also vector spaces (for fixed m and n) with the usual definitions of addition and scalar multiplication for matrices developed last week. (Kind of silly since $\mathbb{R}^{m\times n}$ are matrices, not vectors but thats okay.)

The set of real-valued functions F (over a fixed interval) is also a vector space, though in this case we may call it a ***function space***. We can define

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) = \alpha f(x).$$

The proof of this is left to you.

1 Subspaces

A subset S of a vector space V is called a ***subspace*** if S is also a vector space under the operations inherited from V.

Every vector space has at least two subspaces: (1) V itself, and (2) $\{\mathbf{0}\} \subseteq V$. These are both called *trivial* subspaces.

Examples. Lets look at two subsets of $\mathbb{R}^2$:

$$S = \{\langle 0, x_2\rangle \mid x_2 \in \mathbb{R}\} \quad \text{and} \quad T = \{\langle 1, x_2\rangle \mid x_2 \in \mathbb{R}\}$$

Are either of these subspaces?

We know both $S, T \subseteq \mathbb{R}^2$, so really we need to check if S and T are vector spaces.

Lets look at S first: because any real number times 0 is 0, as well as $0+0=0$, we can see that all of the axioms above hold. The first dimension always remains 0, and the second dimension inherits all of it's properties from scalar addition and multiplication.

What about T? We can start with axiom (1): for $a, b \in T$

$$\begin{aligned} a + b &\stackrel{?}{\in} T \\ \langle 1, a_2 \rangle + \langle 1, b_2 \rangle &\stackrel{?}{\in} T \\ \langle 1+1, a_2 + b_2 \rangle &\stackrel{?}{\in} T \\ \langle 2, a_2 + b_2 \rangle &\notin T \end{aligned}$$

I will leave it to you to show that several other axioms do not hold (namely axiom (6)).

1.1 Proving Subspaces

For any non-empty subset $S \subseteq V$ for vector space V. S is a subspace iff it is closed under addition and scalar multiplication.

That is, for any subset (thats not empty), if we know V is a vector space, we only need to prove (1) and (6) to show S is a subspace.

Example. Lets define

$$P_4 = \left\{ a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0 \mid a_i \in \mathbb{R}, \forall i \in [4] \right\}.$$

We mentioned (but didn't prove here) functions are a vector space, and clearly $P_4 \subseteq F$. To show P_4 is a subspace we only need to show that (1) and (6) hold.

For $a, b \in P_4$ and $\alpha \in \mathbb{R}$:

$$\begin{aligned} (a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0) + (b_4x^4 + b_3x^3 + b_2x^2 + b_1x^1 + b_0) &\stackrel{?}{\in} P_4 \\ (a_4 + b_4)x^4 + (a_3 + b_3)x^3 + (a_2 + b_2)x^2 + (a_1 + b_1)x^1 + (a_0 + b_0) &\in P_4 \end{aligned}$$

and

$$\begin{aligned} \alpha(a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0) &\stackrel{?}{\in} P_4 \\ (\alpha a_4)x^4 + (\alpha a_3)x^3 + (\alpha a_2)x^2 + (\alpha a_1)x^1 + (\alpha a_0) &\in P_4 \end{aligned}.$$

Therefore P_4 is a subspace of F.

Useful References

Isaak and Monougian, “Basic Concepts of Linear Algebra”. §2.1-2.3