Topic 5: Vectors

02-680: Essentials of Mathematics and Statistics

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We saw that we can easily build tuples using Cartesian product, and the notation is even more simplified when the sets involved are uniform (i.e. the same). When the set used to define a tuple is the set of real $\mathbb{R}$ (or complex $\mathbb{C}^*$) numbers, we call the tuple a ***vector***. Specifically an n-vector x is defined as an element in

$$x \in \mathbb{R}^n .$$

If we want to reference the i-th element of x we will write

$$x_i$$

(or sometimes $x[i]$, this is true of tuples as well).

1 Simple Operations

Graphically we can think of vectors (in $\mathbb{R}^2$) in two ways, which are somewhat equivalent: as a point on the plane, or as an arrow from the origin. The second will be useful in this section, but the latter is sometimes useful as well.

1.1 Vector Addition and Scalar Multiplication

Both of these operations are element-wise. For vectors $x, y \in \mathbb{R}^n$

$$z = x + y \quad \rightarrow \quad z_i = x_i + y_i \quad \forall 1 \leq i \leq n.$$

Thus $z \in \mathbb{R}^n$ as well.

In the context of this course, especially when talking about vectors, a ***scalar*** is a single number (i.e. $a \in \mathbb{R}$) rather than a vector. (The implication is that $\mathbb{R}$ and $\mathbb{R}^1$ are not the

*It will generally be true throughout the class that the properties we're discussing also apply to complex numbers, but for simplicity we will usually only directly discuss reals.

same thing.) When multiplying a vector $x \in \mathbb{R}^n$ by a scalar $a \in \mathbb{R}$, we once again apply this element-wise. Thus the result, $z \in \mathbb{R}^n$ can be computed as:

$$z = ax \quad \rightarrow \quad z_i = ax_i \quad \forall 1 \leq i \leq n.$$

1.2 Norms

Informally ℓ_p ***norm*** of a vector is a measure of it's size. Here $p \in \mathbb{Z}^{\geq 1}$. All norms, written as a function $\|\_\|_p : \mathbb{R}^n \mapsto \mathbb{R}$, and each has the following properties:

- $\forall x \in \mathbb{R}^n : \|x\|_p \geq 0$
- $\forall x \in \mathbb{R}^n, c \in \mathbb{R} : \|cx\|_p = |c| \cdot \|x\|_p$
- $\forall x, y \in \mathbb{R}^n : \|x + y\|_p \leq \|x\|_p + \|y\|_p$
- $\|x\|_p = 0 \iff x = 0$

The 3rd property comes up a lot, its called the ***triangle inequality***

In general the form of the ℓ_p norm is:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}.$$

Vector length. Often times called the ℓ_2- or Euclidian-norm of a vector. If we think of the vector as an arrow, we can say the ***length*** of the vector (arrow) is the same as the hypotenuse right triangle with each leg having the same length as each one of the elements. In that case we know that for vector $x = \langle x_1, x_2 \rangle \in \mathbb{R}^2$, the length is $\sqrt{x_1^2 + x_2^2}$. To generalize this we say the length of a vector $x \in \mathbb{R}^n$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} (x_i)^2}.$$

Other common norms. The ℓ_1-norm (Manhattan/Taxicab Distance):

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|.$$

The ℓ_∞-norm:

$$\|x\|_\infty = \max_{x_i \in x} \big(|x_i| \big)$$

2 Dot Product

While a scalar times a vector is a vector, it turns out a vector times a vector is a scalar! Lets define it first then we will dive in; for $x, y \in \mathbb{R}^n$

$$x \bullet y = \sum_{i=1}^{n} x_i \cdot y_i.$$

(note sometimes we will use $\bullet$ vs $\cdot$ to differentiate scalar multiplication and dot product, but generally only the latter is used since the domains of the functions are different and can be extracted from context.)

We often think of dot product as telling us how vectors go "in the same direction". Consider two cardinal vectors (going directly along an axis) on the plane; intuitively they go in totally different directions (note, this is different from *opposite* directions). WLG one vector must be $\langle a, 0\rangle$ and the other must be $\langle 0, b\rangle$ (for scalars $a, b \in \mathbb{R}$). Using the definition above $\langle a, 0\rangle \bullet \langle 0, b\rangle = 0$. Consider a third vector $\langle c, 0\rangle$ with $c \in \mathbb{R}$, but enforce that $a \geq 0$ and $c \leq 0$. In this case the result is $\langle a, 0\rangle \bullet \langle c, 0\rangle = ac$, which we know is negative; they share a lot of direction, but go opposite ways!

We can actually redefine the L_2-norm of a vector $x \in \mathbb{R}^n$ using the dot product:

$$\|x\| = \sqrt{x \cdot x}.$$

Useful References

Liben-Nowell, "Connecting Discrete Mathematics and Computer Science, 2e". §2.4