

Topic 3: Tuples

02-680: Essentials of Mathematics and Statistics

September 4, 2024

1 Tuples

Unlike sets, **tuples** (also called **sequences** or **lists**) are an *ordered* collections of objects. Think of the position on chess board (or a 2D plane), a color in RGB, etc.

When these have small cardinality we can use terms like (ordered) pair [2], triple [3], quadruple [4], or more generically an “ n -tuple”.

If we’re being precise, we normally use angle brackets (“ \langle ”, “ \rangle ”) but a lot of times we will be lazy and just use parentheses (“(”, “)”); But we will always differentiate from sets which use curly brackets (“{”, “}”).

Cartesian Product. A very useful way to construct **set of** tuples is using the **cartesian product** operator, which in essence creates all possible pairs of elements from two sets.

$$S \times T = \{\langle x, y \rangle \mid x \in S \wedge y \in T\}$$

As an example, lets remember the first two sets from our examples last topic:

$$A = \{\text{“Welcome”}, \text{“to”}, \text{“02-680”}\} \text{ and}$$

$$B = \{x^2 \mid x = 2 \vee x = 3\}.$$

In this case the cartesian product is

$$A \times B = \{\langle \text{“Welcome”}, 4 \rangle, \langle \text{“to”}, 4 \rangle, \langle \text{“02-680”}, 4 \rangle, \langle \text{“Welcome”}, 9 \rangle, \langle \text{“to”}, 9 \rangle, \langle \text{“02-680”}, 9 \rangle\}$$

It doesn’t have to be different sets in the cartesian product though, we can have the product with a set and itself. In fact this is performed so often it has its own notation:

$$B \times B = B^2 = \{\langle 4, 4 \rangle, \langle 4, 9 \rangle, \langle 9, 4 \rangle, \langle 9, 9 \rangle\}.$$

This notation also generalizes, so $S^3 = S \times S \times S$, $S^4 = S \times S \times S \times S$ and so on. Notice that order matters in tuples (unlike sets) so $\langle 4, 9 \rangle \neq \langle 9, 4 \rangle$.

A note about this notation: Sometimes we want to have a set of tuples of different lengths (remember sets don't need to be over objects of the same type) so something like

$$B^2 \cup B^3 = \left\{ \begin{array}{l} \langle 4, 4 \rangle, \langle 4, 9 \rangle, \langle 9, 4 \rangle, \langle 9, 9 \rangle, \\ \langle 4, 4, 4 \rangle, \langle 4, 4, 9 \rangle, \langle 4, 9, 4 \rangle, \langle 4, 9, 9 \rangle, \\ \langle 9, 4, 4 \rangle, \langle 9, 4, 9 \rangle, \langle 9, 9, 4 \rangle, \langle 9, 9, 9 \rangle \end{array} \right\}$$

If we wanted to enumerate all binary numbers up to 8 digits (while omitting leading 0s):

$$\{1\} \times \bigcup_{i=1}^7 \{0, 1\}^i$$

But that leads to the notation we saw last time for strings: Σ^* . Sometimes we want the set of all tuples of any length, then we use the ***Kleene star*** (or Kleene operator); for some set S ,

$$S^* = \bigcup_{i=0}^{\infty} S^i.$$

We will define $S^0 = \langle \rangle$ (the empty tuple) for any S , in the case of Σ^0 we often call it the empty string.

Useful References

Liben-Nowell, "Connecting Discrete Mathematics and Computer Science, 2e". §2.4