02-680: Essential Mathematics and Statistics for Scientists, Fall 2023

HW1: Linear Algebra Basics - [andrew-id]

Version: 1
Due: 23:59 EST, Sept 21, 2023 on Canvas

Topics in this assignment:

1. Sets
2. Functions
3. Vectors

What to hand in.

- One write-up (**in pdf format**) providing a solution to each of the following questions.

It is required that you typeset your write-up. The editor used is not specified, but equation mode should be used when necessary to ensure the solutions are communicated correctly. The LaTeXtemplate is provided for your convenience on canvas and at https://github.com/deblasiolab/02-680_Documents, but you are free to use/create your own template. Make sure you include your Andrew ID at the top of the submission.

1. [30 points] Sets

(a) Which of the following statements is true $\forall A$, B, and C? Explain your answer. (assume A, B, and C are sets.)

i. $A - (B - C) = (A - B) - C$.

ii. $(A - B) \cap (C - B) = (A \cap C) - B$.

iii. $(A - B) \cap (C - B) = A - (B \cup C)$.

iv. if $A \cap C = B \cap C$ then $A = B$.

v. if $A \cup C = B \cup C$ then $A = B$.

(b) Let A, B, and C be subsets of some universe U. Draw a Venn diagram of each of these statements.

i. $A - B$ and $B - C$ are disjoint.

ii. $A - B$ and $C - B$ are disjoint.

iii. $A - (B \cup C)$ and $B - (A \cup C)$ are disjoint.

iv. $A - (B \cap C)$ and $B - (A \cap C)$ are disjoint.

(c) Let $P_1 = \mathscr{P}(A \cup B)$ and $P_2 = \mathscr{P}(A) \cup \mathscr{P}(B)$, answer the following and explain your responses.

i. Can P_1 and P_2 be equal?

ii. Is $P_1 \subseteq P_2$, $P_1 \supseteq P_2$, or neither?

iii. Can P_1 and P_2 have the same size (number of elements)?

2. [20 points] Functions

(a) Determine the inverse of the function $h(x) = 3x^2 + 4$

(b) Given $f(x) = x^2 + 1$ and $g(x) = 2x - 3$, find $f \circ g$ and $g \circ f$

(c) Let $\ell : X \to Y$ and $r : Y \to Z$ and $r \circ \ell : X \to Z$.

i. If $r \circ \ell$ is onto, must r be onto?

ii. If $r \circ \ell$ is one-to-one, must ℓ be one-to-one?

(d) Find the smallest codomain of the following functions assuming the domain is $\mathbb{R}$: (note: some answers will be a general set defined in class, i.e. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, others may need to be defined using predicate form.)

i. $s(x) = \sqrt{2 - x}$

ii. $t(x) = \frac{x+1}{(x-3)(x+1)}$

iii. $v(x) = x^4$

3. [20 points] Vector operation

(a) Describe geometrically (line, plane, zero, ...) all linear combinations of
   i. (3,1,2) and (12,4,8)
   ii. (2,3,0) and (0,0,1)
   iii. (4,0,0) and (0,4,4) and (4,4,6).

(b) Let $u = (-1.1, 3.7)$, $v = (2.0, 4.0)$ and $w = (0.8, 12.0)$. Calculate the dot products:
   i. $u \cdot v$
   ii. $u \cdot w$
   iii. $u \cdot (v + w)$
   iv. $w \cdot v$

(c) Find two different combinations of the vectors $u = (7, 2)$, $v = (4, 1)$ and $w = (5, 1)$ that produce the vector $a = (1, 0)$. If I take any three vectors $u, v, w \in \mathbb{R}^2$ will there always be two different combinations that produce $b = (1, 0)$?

4. [30 points] Linear independence

Determine if the following set of vectors is linearly independent.

(a) a_1,a_2,a_3: in which

- $a_1 = b_1 + b_2$,
- $a_2 = b_2 + b_3$, and
- $a_3 = b_3 + b_1$;

where b_1, b_2, b_3 are linearly independent vectors.

(b) v_1,v_2,v_3: which

- $v_1 = \left(x, -\frac{1}{2}, -\frac{1}{2}\right)$,
- $v_2 = \left(-\frac{1}{2}, x, -\frac{1}{2}\right)$, and
- $v_3 = \left(-\frac{1}{2}, -\frac{1}{2}, x\right)$;

where $x \in \mathbb{R}$.