Topic 3: Tuples

02-680: Essentials of Mathematics and Statistics

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1 Tuples

Unlike sets, ***tuples*** (also called ***sequences*** or ***lists***) are an *ordered* collections of objects. Think of the position on chess board (or a 2D plane), a color in RGB, etc.

When these have small cardinality we can use terms like (ordered) pair [2], triple [3], quadruple [4], or more generically an "n-tuple".

If we're being precise, we normally use angle brackets ("$\langle$", "$\rangle$") but a lot of times we will be lazy and just use parentheses ("(",")"); But we will always differentiate from sets which use curly brackets ("{","}").

Cartesian Product. A very useful way to construct ***set of*** tuples is using the ***cartesian product*** operator, which in essence creates all possible pairs of elements from two sets.

$$S \times T = \{\langle x, y\rangle \mid x \in S \wedge y \in T\}$$

As an example, lets remember the first two sets from our examples last topic:

$$A = \{\text{“Welcome”}, \text{“to”}, \text{“02-680”}\} \text{ and}$$

$$B = \left\{x^2 \mid x = 2 \vee x = 3\right\}.$$

In this case the cartesian product is

$$A \times B = \{\langle\text{“Welcome”}, 4\rangle, \langle\text{“to”}, 4\rangle, \langle\text{“02-680”}, 4\rangle, \langle\text{“Welcome”}, 9\rangle, \langle\text{“to”}, 9\rangle, \langle\text{“02-680”}, 9\rangle\}$$

It doesn't have to be different sets in the cartesian product though, we can have the product with a set and itself. In fact this is performed so often it has its own notation:

$$B \times B = B^2 = \{\langle 4, 4\rangle, \langle 4, 9\rangle, \langle 9, 4\rangle, \langle 9, 9\rangle\}.$$

This notation also generalizes, so $S^3 = S \times S \times S$, $S^4 = S \times S \times S \times S$ and so on.

> **Quote from Liben-Newell §2.4**
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> An annoying pedantic point: we are being sloppy with notation in [the notation above]; we only defined the Cartesian product for two sets, so when we write $S \times S \times S$ we "must" mean either $S \times (S \times S)$ or $(S \times S) \times S$. We're going to ignore this issue, and simply write statements like $\langle 0, 1, 1 \rangle \in \{0,1\} \times \{0,1\} \times \{0,1\}$—even though we ought to instead be writing statements like $\langle 0, \langle 1, 1 \rangle \rangle \in \{0,1\} \times (\{0,1\} \times \{0,1\})$.

Notice that order matters in tuples (unlike sets) so $\langle 4, 9 \rangle \neq \langle 9, 4 \rangle$.

A note about this notation: Sometimes we want to have a set of tuples of different lengths (remember sets don't need to be over objects of the same type) so something like

$$B^2 \cup B^3 = \left\{ \begin{array}{c} \langle 4,4 \rangle , \langle 4,9 \rangle , \langle 9,4 \rangle , \langle 9,9 \rangle , \\ \langle 4,4,4 \rangle , \langle 4,4,9 \rangle , \langle 4,9,4 \rangle , \langle 4,9,9 \rangle , \\ \langle 9,4,4 \rangle , \langle 9,4,9 \rangle , \langle 9,9,4 \rangle , \langle 9,9,9 \rangle \end{array} \right\}$$

If we wanted to enumerate all binary numbers up to 8 digits (while omitting leading 0s):

$$\{1\} \times \bigcup_{i=1}^{7} \{0,1\}^i$$

But that leads to the notation we saw last time for strings: Σ^*. Sometimes we want the set of all tuples of any length, then we use the ***Kleene star*** (or Kleene operator); for some set S,

$$S^* = \bigcup_{i=0}^{\infty} S^i.$$

We will define $S^0 = \langle \rangle$ (the empty tuple) for any S, in the case of Σ^0 we often call it the empty string.

Useful References

Liben-Nowell, "Connecting Discrete Mathematics and Computer Science, 2e". §2.4