

## 02-680: Essential Mathematics and Statistics for Scientists, Fall 2023

### HW2: Linear Equations and Decomposition - [andrew-id]

*Version: 1*

*Due: 23:59 EST, October 5, 2023 on Canvas*

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**Topics** in this assignment:

1. Matrices and Systems of Linear Equations
2. Determinants
3. Eigenvalues and Eigenvectors
4. Singular Value Decomposition

**What to hand in.**

- One write-up (in pdf format) providing a solution to each of the following questions.

**It is required that you typeset your write-up.** The editor used is not specified, but equation mode should be used when necessary to ensure the solutions are communicated correctly. The L<sup>A</sup>T<sub>E</sub>X template is provided for your convenience on canvas and at [https://github.com/deblasiolab/02-680\\_Documents](https://github.com/deblasiolab/02-680_Documents), but you are free to use/create your own template. Make sure you include your Andrew ID at the top of the submission.

1. [30 points] Matrices and Systems of Linear Equations

- (a) i. Compute the product  $A\hat{x}$  where  $A = \begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix}$  and  $\hat{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ .
- ii. For the same matrix  $A$ , find a solution vector  $x$  such that  $Ax = 0$  (i.e. find the solution to the homogeneous system with zeros on the right side of all three equations).
- iii. Is there more than one solution?
- iv. Solve the following system by Gaussian Elimination (showing each step):

$$u + 4v + 2w = -2$$

$$-2u - 8v + 3w = 32$$

$$v + w = 1$$

- v. Show that  $\begin{bmatrix} 2 & 2 \\ 6 & 6 \end{bmatrix}$  has no inverse; again using Gaussian elimination (showing each step).
- vi. A. Show that for  $B \in \mathbb{R}^{m \times n}$ ,  $BB^T$  and  $B^TB$  are always symmetric, even for non-square matrices.
- B. Show by example that they may not be equal, even for square matrices.
- vii. Are all matrices with 1s down the main diagonal invertible? State true or false. Provide a short reasoning if true, and a counter-example if false.

2. [20 points] Determinants

- (a) Solve the following system of linear equations using Cramer's rule (show each step):

$$x + 2z = 9$$

$$2y + z = 8$$

$$4x - 3y = -2$$

- (b) i. If every row of a matrix  $C$  adds up to zero, and  $y$  is a column vector of ones, what is  $Cy$ ?  
ii. Does  $C$  have an inverse? Why or why not?  
iii. What can you comment about  $\det(C)$ ?

- (c) Find the inverse of  $\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$  using determinants (show your work).

- (d) What are the conditions on the entries of  $F$  for it to be invertible?

$$F = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix}$$

3. [30 points] Eigenvalues and Eigenvectors

- (a) Find the eigenvalues and eigenvector of the following (show your work):

$$\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$$

- (b) Show the steps of diagonalizing the following matrix

$$\begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$$

by supposing that we already know that  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are two eigenvectors.

$$\mathbf{v}_1 = (3, 1)^\top, \quad \mathbf{v}_2 = (2, 1)^\top$$

- (c) i. What is the characteristic equation of

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

- ii. What are the eigenvalues?

4. [20 points] Singular Value Decomposition

- (a) Find the Singular Value Decomposition of matrix  $J$ .

$$J = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

(You might use such results: the eigenvalues of  $J^\top J$  are  $\lambda_1 = 360, \lambda_2 = 90, \lambda_3 = 0$ . Corresponding unit eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

respectively.)

- (b) i. Assume you're given a singular value decomposition,  $K = U\Sigma V^\top$  for some  $K \in \mathbb{R}^{m \times n}$ . Find an SVD of  $K^\top$ .  
ii. How are the singular values of  $K$  and  $K^\top$  related?