02-680: Essential Mathematics and Statistics for Scientists, Fall 2023

HW2: Linear Equations and Decomposition - [andrew-id]

Version: 1
Due: 23:59 EST, October 5, 2023 on Canvas

Topics in this assignment:

1. Sets
2. Functions
3. Vectors

What to hand in.

- One write-up (**in pdf format**) providing a solution to each of the following questions.

It is required that you typeset your write-up. The editor used is not specified, but equation mode should be used when necessary to ensure the solutions are communicated correctly. The LaTeXtemplate is provided for your convenience on canvas and at `https://github.com/deblasiolab/02-680_Documents`, but you are free to use/create your own template. Make sure you include your Andrew ID at the top of the submission.

1. [30 points] Matrices and Systems of Linear Equations

(a) i. Compute the product $A\hat{x}$ where $A = \begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix}$ and $\hat{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

ii. For the same matrix A, find a solution vector x such that $Ax = 0$ (i.e. find the solution to the homogeneous system with with zeros on the right side of all three equations).

iii. Is there more than one solution?

iv. Solve the following system by Gaussian Elimination (showing each step):

$$u + 4v + 2w = -2$$

$$-2u - 8v + 3w = 32$$

$$v + w = 1$$

v. Show that $\begin{bmatrix} 2 & 2 \\ 6 & 6 \end{bmatrix}$ has no inverse; again using Gaussian elimination (showing each step).

vi. A. Show that for $B \in \mathbb{R}^{m \times n}$,$BB^T$ and B^TB are always symmetric, even for non-square matrices.

B. Show by example that they may not be equal, even for square matrices.

vii. Are all matrices with 1s down the main diagonal invertible? State true or false. Provide a short reasoning if true, and a counter-example if false.

2. [20 points] Determinants

(a) Solve the following system of linear equations using Cramer's rule (show each step):

$$x + 2z = 9$$

$$2y + z = 8$$

$$4x - 3y = -2$$

(b) i. If every row of a matrix C adds up to zero, and y is a column vector of ones, what is Cy?

ii. Does C have an inverse? Why or why not?

iii. What can you comment about $\det(C)$?

(c) Find the inverse of $\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$ using determinants (show your work).

(d) What are the conditions on the entries of F for it to be invertible?

$$F = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix}$$

3. [30 points] Eigenvalues and Eigenvectors

(a) Find the eigenvalues and eigenvector of the following (show your work):

$$\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$$

(b) Show the steps of diagonalizing the following matrix

$$\begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$$

by supposing that we already know that $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are two eigenvectors.

$$\boldsymbol{v_1} = (3, 1)^\top, \quad \boldsymbol{v_2} = (2, 1)^\top$$

(c) i. What is the characteristic equation of

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

ii. What are the eigenelements?

4. [20 points] Singular Value Decomposition

(a) Find the Singular Value Decomposition of matrix J.

$$G = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

(You might use such results: the eigenvalues of $J^\top J$ are $\lambda_1 = 360, \lambda_2 = 90, \lambda_3 = 0$. Corresponding unit eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

respectively.)

(b) i. Assume you're given a singular value decomposition, $K = U\Sigma V^\top$ for some $K \in \mathbb{R}^{m\times n}$. Find an SVD of $K^\top$.

ii. How are the singular values of K and $K^\top$ related?