

Abstract

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Chapter 1

Introduction

1.1 Overview

Fluid motions driven by buoyancy and frictional forces belongs to broad class of flows known as thermoconvective shear flows. These flows exhibit rich behaviour, and are of interest in both engineering and meteorology applications spanning across a broad range of length scales. At small scales, around $L \sim 1\text{cm}$, the thermoconvection flows are relevant to the cooling of microprocessing chips. In such systems, the fluid acts medium to dissipate heat, experiences shear forces from the confining walls, and buoyancy from heating. One of the major innovation in this industry is in squeezing more transistors onto a single chip, resulting to a doubling of transistors on a single chip roughly every two years, according to Moore's law. However, one of the major limitations on further miniaturisation is the challenge of dissipating the excessive heat generated. Fluids, such as air, water or refrigerant, are often used to transport heat away from the components, thereby preventing overheating [[Kennedy and Zebib, 1983](#), [Ray and Srinivasan, 1992](#)]. At intermediate length scales, $L \sim 1\text{m}$, the interaction between buoyancy and frictional forces is important in the fabrication of uniform thin films in chemical vapour deposition (CVD) [[Evans and Greif, 1991](#), [Jensen et al., 1991](#)]. The CVD process typically involves a reactive gases carried by inert gases which flows through a channel with a heated substrate. Upon heating, the reactant gases react chemically at substrate and deposits material, forming thin films, such as silicon layers. A key challenge in the CVD process is achieving a uniform deposition and maintaining sharp interfaces between layers. The interactions between shear and buoyancy forces often gives rise to boundary layers and thermoconvective rolls, which can disrupt uniform deposition, affecting film quality. At large scales, $L \sim 1\text{km}$, the thermoconvective shear flows can be observed in the atmosphere such as the cloud streets over the Norwegian Sea. These parallel bands of cumulus clouds can stretch over hundreds of kilometres. They form when relatively warm sea surfaces heat up the colder air blowing from the North [[nor](#)]. As the colder air is heated, it rises upwards whilst carrying water vapour. As it reaches a certain altitudes, $L \sim 1 - 10\text{km}$, the water vapour condenses into visible clouds, while the cooler air falls towards the sea. This circulation is organised into parallel rotating parallel columns of air, forming distinct cloud streets.

The common thread among the examples discussed above is the interaction between shear and buoyancy forces driven fluid motion - the central focus of this thesis. By restricting our analysis to these two mechanisms, we neglect other physical mechanisms such as phase change, chemical reactions and evaporation, which may be significant in the context of cooling microprocessors, chemical vapour deposition, and atmospheric boundary layers respectively [[Vallis et al., 2019](#)]. To consider this

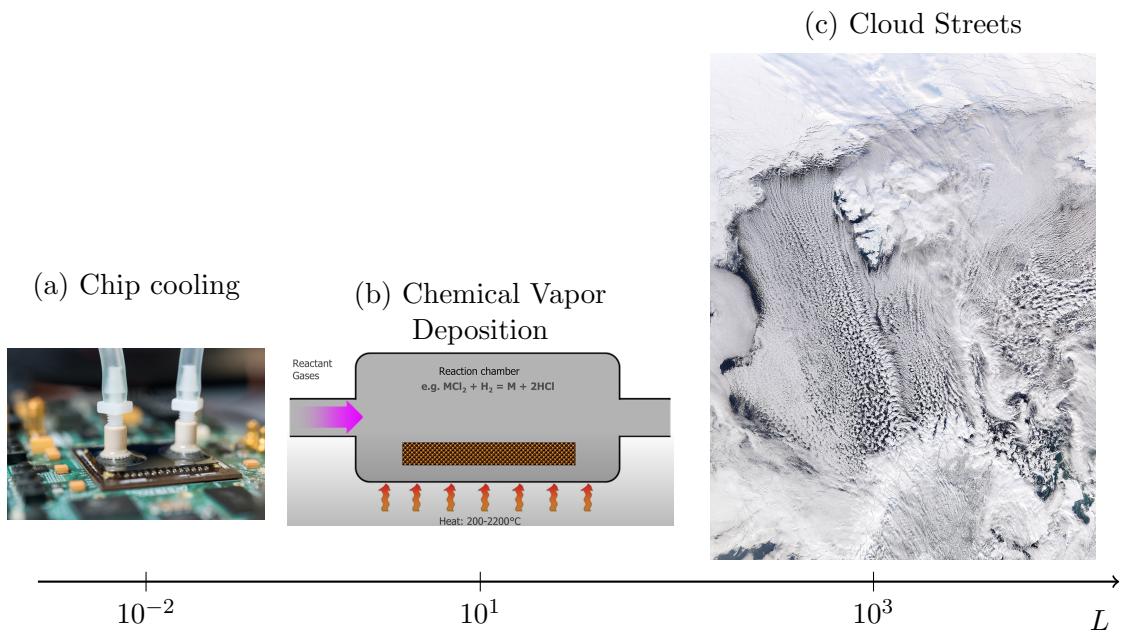


Figure 1.1: Fluid flow due to shear and buoyancy forces across length scales, $L \in [1\text{cm}, 1\text{km}]$, such as (a) chip cooling, (b) chemical vapour deposition and (c) formation of cloud streets.

interaction, we consider an idealised setup without geometric complexity, known to as the Rayleigh-Bénard-Poiseuille (RBP) flow. This system describes the fluid motion confined between two infinitely extended parallel plates, heated from below and cooled from the top, with an additional pressure gradient driving the flow. The RBP configuration combines two paradigmatic flow configurations; the classical Rayleigh-Bénard convection (RBC), driven purely by buoyancy, and plane Poiseuille flow (PPF), driven purely by shear. While the onset of convection in RBC, and the transition to subcritical shear-driven turbulence in PPF have been both extensively studied, the transitional regime in which both forces interact remains less understood. Gaining insights into this regime can have implications for various applications across a range of scales mentioned previously.

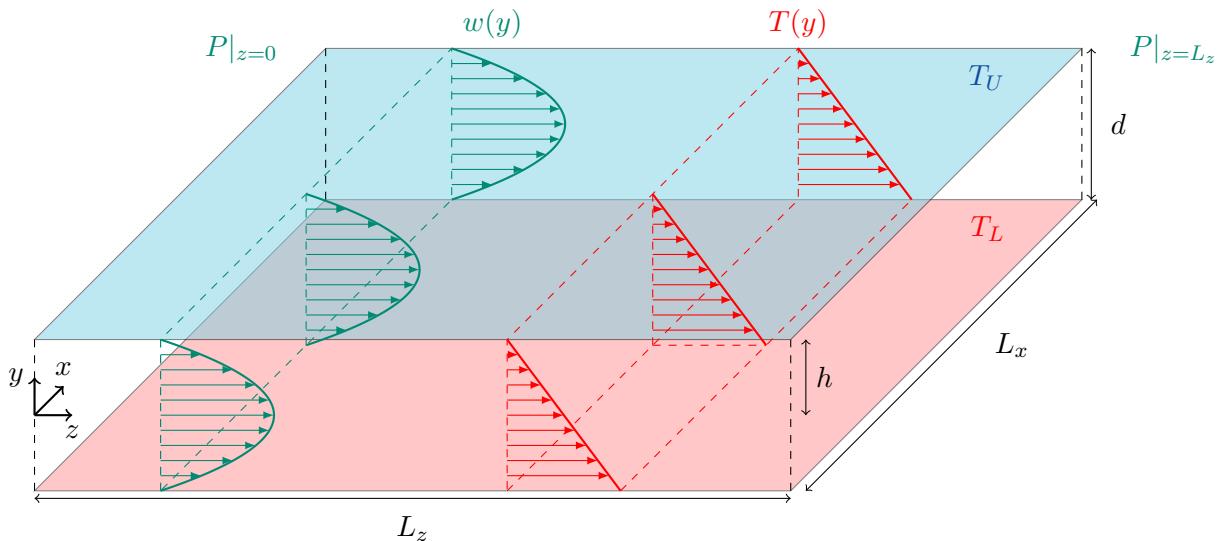


Figure 1.2: The Rayleigh-Bénard Poiseuille (RBP) flow configuration.

The RBP configuration is illustrated in figure 1.1, where $z^*, y^*, x^*, L_z, L_x, d, h$ refer to the stream-

wise, spanwise, wall-normal coordinates, length, span, depth and half-height of the domain respectively. We note that the asterisks*, refer to variables in dimensional form. The flow is driven by a pressure gradient along the streamwise z^* direction, $\Delta P^* = P^*|_{z^*=0} - P^*|_{z^*=L_z} < 0$, leading to the formation of a laminar Poiseuille flow, $w^*(y^*)$, for a sufficiently small ΔP . In this study, we will only consider fully-developed flow, where the boundary layer from the top and the bottom wall meets at the midplane, $y^* = 0$, and entrance effects are neglected. The RBP configuration is also unstably stratified, such that the temperature difference between the lower, T_L , and upper wall, T_U , is always positive, $\Delta T = T_L - T_U > 0$, leading to a stable linear conduction profile along the wall-normal direction, $T(y^*)$, if ΔT is kept sufficiently small.

In the absence of a pressure gradient, the RBP configuration reduces to the classical Rayleigh-Bénard convection problem, bringing about buoyancy-driven convection for a sufficiently large unstable stratification. In the limiting case without unstable stratification, $\Delta T = 0$, the system reduces to the wall-bounded plane Poiseuille flow (PPF), where the transition towards subcritical shear-driven turbulence may be expected for a sufficiently large pressure gradient.

For instance, do buoyancy forces promote the transition to shear-driven turbulence and how does shear influence the convection? To describe the motion of the fluid in RBP configurations, we consider non-dimensionalised Navier-Stokes equations with Boussinesq approximations,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} + \frac{Ra}{Re^2 Pr} \theta, \quad (1.1a)$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla) \theta = \frac{1}{Re Pr} \nabla^2 \theta, \quad (1.1b)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (1.1c)$$

where $\mathbf{u}(\mathbf{x})$, $\theta(\mathbf{x})$, $p(\mathbf{x})$ refers to the nondimensionalised velocity, temperature and pressure respectively. The key control parameters for RBP flows are the Rayleigh number, Ra , Reynolds number, Re , Prandtl number Pr , which are defined as follows,

$$Ra = \eta g d^3 \Delta T / \nu \kappa, \quad Re = W_c h / \nu, \quad Pr = \kappa / \nu, \quad \Gamma = L / 2d, \quad (1.2)$$

where η , g , ΔT , ν , κ , W_c , h , d , L are the thermal expansion coefficient, acceleration due to gravity, temperature difference between the bottom and top wall, kinematic viscosity, thermal diffusivity, laminar centreline velocity, domain's half-depth, full-depth, length or span respectively.

We describe important historical of hydrodynamic stability of planar shear flows and their theoretical frameworks in §1.2. Theoretical frameworks used in the study of stability of flow such as linear stability, nonlinear dynamical systems and spatiotemporal character of transitional shear flows will be outlined. This followed the historical developments of Rayleigh-Bénard convection (RBC), where concepts of the stability of fluid flows will be utilised in §1.3. After which, we describe the historical developments of RBP flows §1.4, and the outline of the thesis will be give in §1.4.1.

1.2 Transitional wall-bounded shear flows

Wall-bounded shear flows concerns the motion of the fluid flowing in parallel to walls, typically bounded by one or more walls. The fluid closest to the wall comes to a rest, satisfying the no-slip boundary

condition in the presence of a wall. As a consequence, a velocity gradient in the direction perpendicular from the wall develops, where the fluid layer becomes *sheared* due to the presence of the wall - referred to wall-bounded sheared flows. Examples of wall-bounded shear flows include the pressure-driven plane Poiseuille flow (or channel flow), Hagen-Poiseuille (or pipe) flows, plane Couette flow and flat plate boundary layers. These geometrically simple examples enable a convenient framework amenable to the mathematical analysis of fluid motion subjected to shear. Depending on the degree of shear, the fluid motion can be either laminar, where the fluid layers move in smooth parallel 'laminates', or turbulent, characterised by chaotic eddying motions. We also note that there is a transitional regime where both states can coexist discuss later. A central question is predicting the transition from the laminar regime to the turbulence.

The earliest investigation into this transition dates back to the pipe flow experiments of [Reynolds \[1883\]](#). In his experimental setup, the flow speed through the pipe could be controlled by regulating the inlet pressure, while injecting dye to visualise the flow, as illustrated in figure 1.3(a). At low speeds, the fluid remained laminar, resulting in a single streak of steady dye in figure 1.3(b). As the speed increased, the dye began to exhibit irregular 'sinuous' motions interspersed with laminar regions shown in figure 1.3(c). This is now referred to as the transitional/intermittent regime, alternating between the laminar and turbulent states. Beyond a critical speed, the dye breaks down entirely into chaotic 'eddies', mixing with the surrounding fluid and discolouring the flow with dye downstream in figure 1.3(d). This regime is now identified as turbulence.

Reynolds conjectured that the threshold between the laminar, transitional and turbulent regimes could be characterised by the Reynolds number, $Re = UD/\nu$, where U is the centerline velocity in the pipe, D , the pipe diameter and ν , the kinematic viscosity. He observed that flow through the pipe remained 'stable' and laminar for $Re < 1900$, while it became 'unstable' and turbulent for $Re > 2000$ [[Reynolds, 1895](#)]. His remarks led to the concept of the stability of fluid flows.

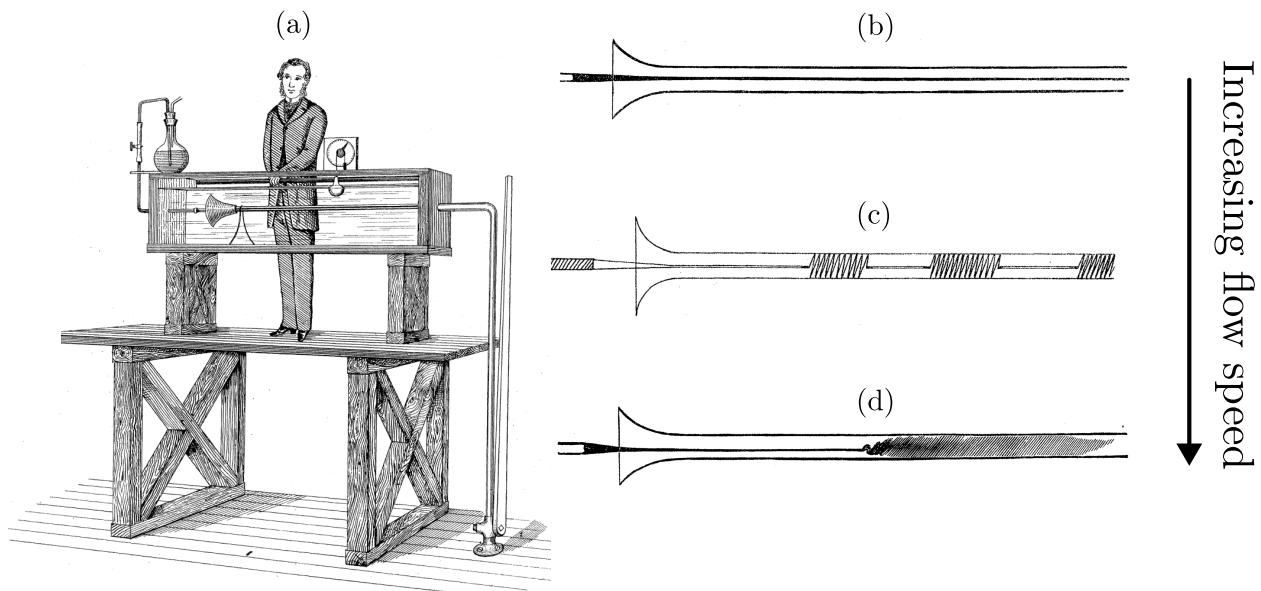


Figure 1.3: (a) Osbourne Reynolds pipe experiment with the dye injection apparatus, illustrating the (b) laminar flow, (c) intermittent regime and (d) turbulent flow as the flow speed is increased, taken from [[Reynolds, 1883](#)].

1.2.1 Linear Stability Analysis

Following Reynolds' experiment, interest towards the mathematical analysis of the stability of fluid flows grew in early 20st century. The mathematical approach typically begins by decomposing the velocity field, $\mathbf{u}(\mathbf{x}, t)$, into a laminar (base) state, $U(y)$ (assumed to depend only on the wall-normal direction here), and the velocity perturbations, $\mathbf{u}'(\mathbf{x}, t)$, with pressure similarly decomposed as,

$$\mathbf{u}(\mathbf{x}) = U(y) + \mathbf{u}'(\mathbf{x}, t), \quad \text{and} \quad p(\mathbf{x}, t) = P(x) + p'(\mathbf{x}, t). \quad (1.3)$$

Next, we substitute the formulations for the decomposed velocity and pressure into the Navier-Stokes equations of equation (1.1) and drop the nonlinear perturbations terms $(\mathbf{u}' \cdot \nabla)\mathbf{u}'$,

$$\frac{\partial \mathbf{u}'}{\partial t} + (U \cdot \nabla)\mathbf{u}' + (\mathbf{u}' \cdot \nabla)U = -\nabla p' + \frac{1}{Re}\nabla^2\mathbf{u}', \quad (1.4a)$$

$$\nabla \cdot \mathbf{u}' = 0, \quad (1.4b)$$

resulting to the linearised Navier-Stokes equations. This commonly followed by introducing a wavelike ansatz (mode) defined by streamwise and spanwise wavenumbers, α, β and complex frequency, ω . In general two ways to analyse the linearised Navier-Stokes equations by considering the behaviour of each mode independently in §1.2.1 and their coupled dynamics in §1.2.1

Modal analysis

It is convenient to eliminate the pressure terms by transforming equation (1.4) using the wall-normal perturbation velocity, v' , and wall-normal vorticity, $\eta' = \partial u'/\partial z - \partial w'/\partial x$, variables. Using (v, η) , introduce an ansatz (mode) for them,

$$v'(\mathbf{x}, t) = \tilde{v}(y)e^{i(\alpha x + \beta z - \omega t)}, \quad \text{and} \quad \eta'(\mathbf{x}, t) = \tilde{\eta}(y)e^{i(\alpha x + \beta z - \omega t)}. \quad (1.5)$$

where α, β, ω denotes the streamwise and spanwise wavenumbers, and complex frequency (i.e. $\omega = \omega_r + i\omega_i$), respectively. Next, we substitute the ansatz into equation 1.4, leading to the classical Orr-Sommerfeld and Squire equations [Orr, 1907, Sommerfeld, 1909, Squire, 1933, Schmid and Henningson, 2001],

$$\begin{bmatrix} -i\omega \begin{pmatrix} k^2 - \mathcal{D}^2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \mathcal{L}_{OS} & 0 \\ i\beta U' & \mathcal{L}_{SQ} \end{pmatrix} \end{bmatrix} \begin{pmatrix} \tilde{v} \\ \tilde{\eta} \end{pmatrix} = \mathbf{0}, \quad (1.6)$$

where \mathcal{L}_{OS} and \mathcal{L}_{SQ} refers to the Orr-Sommerfeld and Squire operators given as,

$$\mathcal{L}_{OS} = i\alpha U(k^2 - \mathcal{D}^2) + i\alpha U'' + \frac{1}{Re}(k^2 - \mathcal{D}^2)^2, \quad (1.7a)$$

$$\mathcal{L}_{SQ} = i\alpha U + \frac{1}{Re}(k^2 - \mathcal{D}^2). \quad (1.7b)$$

$k^2, \mathcal{D}, U', U''$ denotes the sum of squared wavenumbers, $k^2 = \alpha^2 + \beta^2$, differential operator in y , first- and second- derivative of the laminar velocity, respectively. Equation (1.6) is simply an eigenvalue problem which could be represented as,

$$\mathbf{L}\tilde{\mathbf{q}} = i\omega\mathbf{M}\tilde{\mathbf{q}} \quad (1.8)$$

where,

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_{OS} & 0 \\ i\beta U' & \mathcal{L}_{SQ} \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} k^2 - \mathcal{D}^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{\mathbf{q}} = \begin{pmatrix} \tilde{v} \\ \tilde{\eta} \end{pmatrix}. \quad (1.9)$$

and $i\omega$ refers to the eigenvalue. The aim of linear stability analysis is to determine the critical Reynolds number, Re_c , which is defined as the lowest value of Re over α and β , such that $Im[\omega] = 0$. For $Re > Re_c$, perturbations could grow exponentially, departing from the laminar state. In other words, we consider the behaviour of each $\alpha - \beta$ mode independently, herein referred to as *modal* analysis. Squire's theorem implies that for any unstable three-dimensional perturbations, $\beta \neq 0$, there exist an unstable two-dimensional perturbation, $\beta = 0$ with a lower Re_c [Squire, 1933]. Therefore, the unstable perturbations of wall-bounded shear flows at Re_c must be two-dimensional. The theoretical calculations was first perform by Tollmien [1928] and Schlichting [1933] for a flat-plate boundary layer flow, yielding a critical Reynolds number based on streamwise distance x of $Re_{x,c} = Ux_c/\nu = 520$ [Schlichting and Gersten, 2017]. In their honour, the unstable two-dimensional perturbations of the Orr-Sommerfeld operator is referred to as Tollmien-Schlichting (T.S) waves. For plane Poiseuille flow [Orszag, 1971] with a critical wavenumber of $\alpha_c = 1.02$. However, turbulence in plane Poiseuille flows have been observed at much lower Reynolds number, $Re \sim 1000 - 2000$, contradicting the results from linear stability analysis. Likewise, the onset of turbulence appear near $Re_{x,c} \approx 5 \times 10^5$ for flat plat boundary layer. A similar result holds for the plane Couette flow [Meseguer and Trefethen, 2003]. Despite its limitation, linear analysis analysis succeeds in predicting the critical Rayleigh number in Rayleigh-Bénard convection.

Non-modal stability

One of a major limitation of linear stability analysis considered above is that it treats each eigenmode independently, referred to as *modal* analysis. However, the interaction between decaying eigenmodes may lead to a short-term amplification of perturbations, before eventually decaying. This phenonemon is referred to as *transient growth*, and the method of analysis is referred to as *non-modal* analysis, related to the normality of the linear Orr-Sommerfeld operator [Schmid, 2007]. To demonstrate an

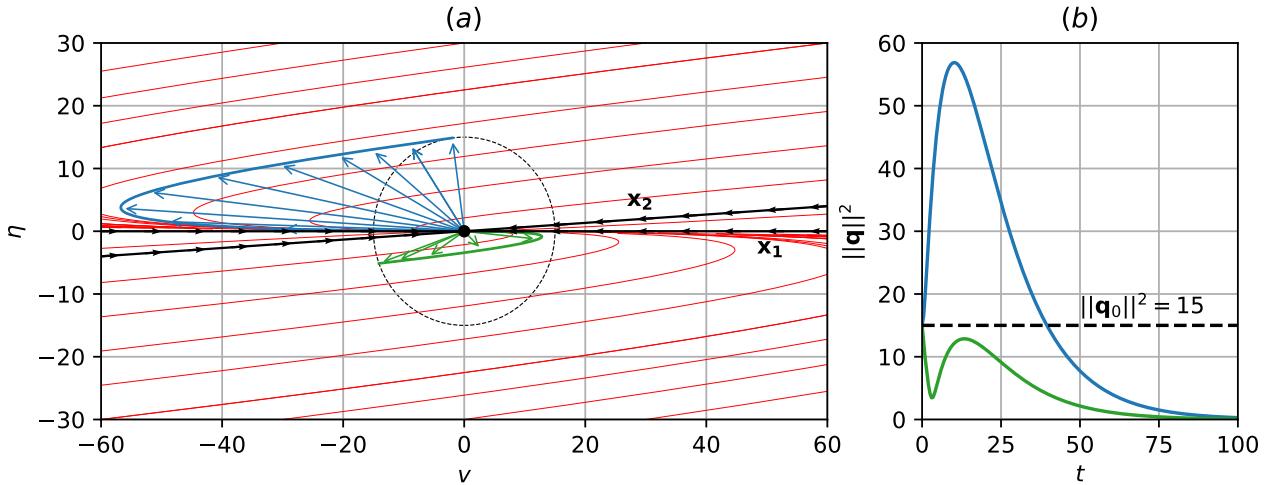


Figure 1.4: (a) The phase portrait of the toy model with $Re = 15$, (b) Transient growth.

example of transient growth, we consider a two-dimensional toy model governing the time-evolution

of \mathbf{q} ,

$$\frac{d}{dt} \begin{pmatrix} v \\ \eta \end{pmatrix} = \begin{pmatrix} -\frac{1}{Re} & -1 \\ 0 & -\frac{2}{Re} \end{pmatrix} \begin{pmatrix} v \\ \eta \end{pmatrix}, \quad (1.10)$$

where Re refers to the Reynolds number. The toy model has negative eigenvalues, $(\lambda_1, \lambda_2) = (-1/Re, -2/Re)$, and unit eigenvectors $\mathbf{x}_1 = (1, 0)$, $\mathbf{x}_2 = \frac{1}{\sqrt{Re^2+1}}(Re, 1)$. Judging from the negative eigenvalues, we conclude that $\mathbf{q}(t)$ will decay exponentially. However, as $Re \rightarrow \infty$, they become increasingly non-orthogonal approaching each other such that the angle between \mathbf{x}_1 and \mathbf{x}_2 tends towards 0. At $Re = 15$, the eigenvector pairs, \mathbf{x}_1 and \mathbf{x}_2 , are highly non-orthogonal, becoming almost linearly dependent shown in figure 1.4(a). For a randomly selected initial condition with an energy-norm of $\|\mathbf{q}_0\|_2 = 15$, where $\|\cdot\|_2$ refers to the L2-norm, the trajectory in green decays exponentially for $t \in [0, 100]$ in figure 1.4(b). In contrast, for a specifically chosen initial condition shown as the blue trajectory, $\|\mathbf{q}\|_2$ is amplified nearly four times before decaying exponentially. The toy model demonstrates the significance of transient growth for a specifically chosen initial condition.

The goal of non-modal stability analysis is to search over all initial conditions, $\tilde{\mathbf{q}}_0$, leading to the maximum amplification factor at time t , resulting in an optimisation problem,

$$G(t) = \max_{\tilde{\mathbf{q}}_0 \neq 0} \frac{\langle \tilde{\mathbf{q}}(t), \tilde{\mathbf{q}}(t) \rangle}{\langle \tilde{\mathbf{q}}_0, \tilde{\mathbf{q}}_0 \rangle}, \quad \text{s.t. } \langle \tilde{\mathbf{q}}_0, \tilde{\mathbf{q}}_0 \rangle = 1, \quad (1.11)$$

where, $\langle \cdot, \cdot \rangle$ refers to the inner-product defined as,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{\Omega} \mathbf{x}^H \mathbf{y} \, d\Omega, \quad (1.12)$$

and \mathbf{x}^H refers to the complex conjugate transpose of \mathbf{x} . By considering the linearised operator of (1.6), we can define a linear time invariant operator given as,

$$\tilde{\mathbf{q}}(t) = \mathcal{A}(t)\tilde{\mathbf{q}}_0, \quad (1.13)$$

which takes the solution from initial conditions, $\tilde{\mathbf{q}}_0$, to $\tilde{\mathbf{q}}(t)$ at time t . Substituting the expression above into equation (1.11),

$$G(t) = \max_{\tilde{\mathbf{q}}_0 \neq 0} \frac{\langle \mathcal{A}(t)\tilde{\mathbf{q}}_0, \mathcal{A}(t)\tilde{\mathbf{q}}_0 \rangle}{\langle \tilde{\mathbf{q}}_0, \tilde{\mathbf{q}}_0 \rangle} = \langle \tilde{\mathbf{q}}_0, \mathcal{A}^\dagger(t)\mathcal{A}(t)\tilde{\mathbf{q}}_0 \rangle = \lambda_{max}(\mathcal{A}^\dagger \mathcal{A}) \quad (1.14)$$

where $\mathcal{A}^\dagger(t)$ refers to the adjoint of $\mathcal{A}(t)$. The maximum amplification factor $\max G(t)$ is the largest eigenvalue, λ_{max} , of $\mathcal{A}^\dagger \mathcal{A}$, and the eigenvalue problem is given as,

$$\mathcal{A}^\dagger(t)\mathcal{A}(t)\tilde{\mathbf{q}}_0 = \lambda \tilde{\mathbf{q}}_0, \quad (1.15)$$

where $\tilde{\mathbf{q}}_0$ refers to the eigenvector denoting the optimal initial condition. For a detailed derivation of the optimal initial conditions or forcing, the reader is referred to [Butler and Farrell, 1992, Schmid, 2007]. An alternative method of computing transient growth is computing the pseudospectral of linear operators discussed in [Trefethen, 1997], outside the scope of this thesis.

Both two-dimensional, $\beta = 0$, and three-dimensional, $\beta \neq 0$, non-modal stability analysis have been studied.

In the two-dimensional form, the optimal initial conditions are in the form of near wall vortices tilted upstream, transiently energised referred to as the Orr-mechanism [Orr, 1907, Farrell, 1988, Reddy

et al., 1993]. In the three-dimension form, streamwise vortices, acting as optimal initial conditions lead to the optimal response in the form of streamwise streaks [Reddy and Henningson, 1993]. Contrary to linear stability analysis which confers two-dimensional perturbations as linearly unstable, the key result in this analysis is that three-dimensional initial conditions, $\alpha = 0$, confer the optimal initial conditions leading to large transient growth at subcritical Reynolds numbers. Figure.. shows this.

The width of the streaks happen to be robustly occur around 100 wall units, the characteristics spacing identified in many experiments [Kline, Panton, Bandybopobi]

The optimal initial conditions involve streamwise vortices which amplify streaks, related to the lift-up effect [Ellingsen and Palm, 1975, Brandt, 2014]. These modal and nonmodal mechanisms above highlight developments based on linear methods.

1.2.2 Nonlinear dynamical systems

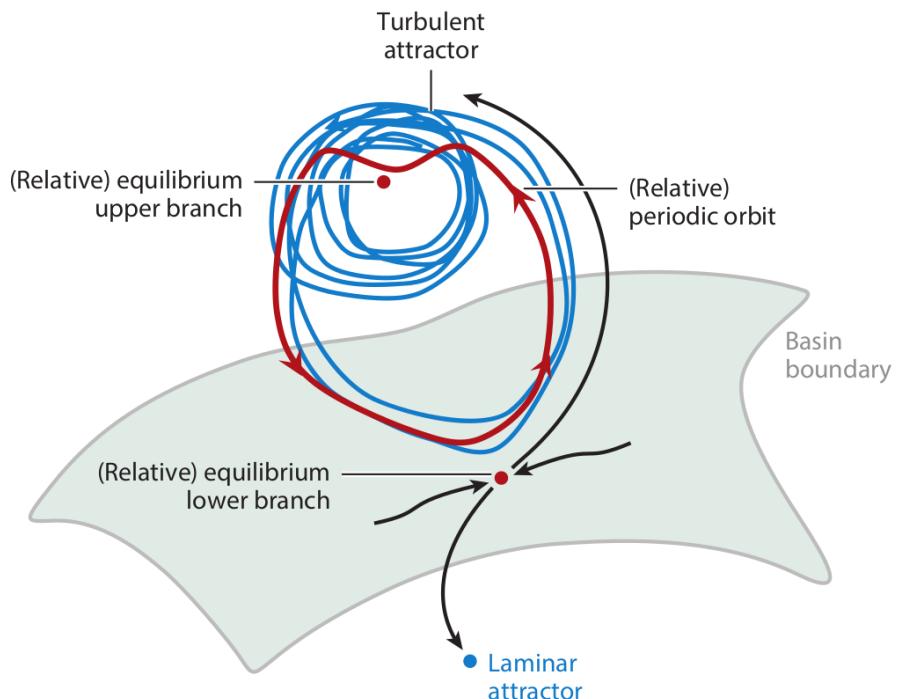


Figure 1.5: The state space organising of the upper and lower branch. Turbulence is interpreted as solution trajectories wander around the upper branch, orbiting around a network unstable invariant states. The lower branch acts as a boundary between the turbulent attractor and laminar attractor, an attractor on the edge referred to as the edge state. Taken from [Graham and Floryan, 2021].

It is well established that coherent motions defined by flow patterns that persist in space and time play an important role in the transport of momentum and heat. In parallel shear flows, these coherent structures typically appear as near-walls streaks and quasi-streamwise rollers. A persistent, quasi-periodic cycle between the regeneration of streaks and rolls, referred to as the *self-sustaining process*, appears to be a fundamental mechanism in sustaining wall-bounded turbulence *self-sustaining process* [Hamilton et al., 1995]. This mechanism is described by the generation of streaks due to quasi-streamwise rollers by redistributing the mean. These streaks become linearly unstable and breakdown, and through a nonlinear process regenerates the quasi-streamwise rollers, closing the cycle.

In the previous section, we have examined the transition process based linear mechanisms. While

such mechanisms are important, the transition to turbulence is ultimately governed fully nonlinear nature of the Navier-Stokes equations. As a result, an increasing attention has been directed towards the description of turbulence as a nonlinear dynamical systems. In this view, turbulence is interpreted as a solution trajectory evolving through a phase space composed of a network such non-trivial nonlinear solutions, commonly referred to as exact coherent states (ECS) or invariant solutions [Graham and Floryan, 2021]. These invariant solutions commonly take the form of as equilibria, travelling waves, periodic and relative periodic orbits.

In the context of parallel shear flows, Nagata was the first to discover a pair of unstable equilibrium solutions in plane Couette flow by smoothly following (homotopy) from a Taylor-Couette configuration [Nagata, 1990]. This pair consist of an unstable upper branch and lower branch emerging as a saddle-node bifurcation near $Re \approx 500$, and is disconnected from the stable laminar solution. The lower branch refers to its proximity towards the stable laminar state in phase space. A travelling-wave solution in plane Couette flow also later found by the same author [Nagata, 1997]. A family of equilibrium and travelling waves solutions was found for plane Couette and plane Poiseuille flows under various boundary conditions (i.e. stress-free, slip and no-slip) where identified by [Waleffe, 2001, 2003].

While these unstable solutions demonstrate good agreements with results from DNS such as the spanwise length scales, and mean and fluctuations, they do not capture the dynamical processes. Periodic orbits defined by time-dependent solutions that have been identified in plane Couette flow [Kawahara and Kida, 2001], describing a single regeneration cycle similar to the self-sustaining process. The chaotic trajectories of turbulence have been found to be embedded within invariant solutions and their connections between them known as heteroclinic orbits, offering a robust view of the building-blocks of of turbulence [Gibson et al., 2008, 2009, Viswanath, 2007, Halcrow et al., 2009, Graham and Floryan, 2021]

In the context of this transitional flows, the lower branch solution can be though of separating the turbulent attractor from the laminar state. Its an attractor that resides on the edge of turbulence, defined as an edge state. The graphical representation of this edge is shown in figure 1.5.

1.2.3 Spatiotemporal transitional flows

This section describes the inherent spatiotemporal intermittent description of turbulence in transitional wall-bounded shear flows commonly reported in large extended domains where the span is about fifty times the half-height of a plane Poiseuille channel, $L/h \gtrsim 50$. In this regime, turbulence is characterised by the coexistence of turbulent and laminar structures. Examples of such are found in canonical shear flow systems such as plane Couette flows [Prigent et al., 2003, Barkley and Tuckerman, 2005, 2007, Tuckerman and Barkley, 2011, Duguet et al., 2010, Reetz et al., 2019], Taylor-Couette flows [Prigent and Dauchot, 2002, Prigent et al., 2003], pipe flows [Avila et al., 2010, 2011, Song et al., 2017, Avila et al., 2023] and plane Poiseuille flows [Tsukahara et al., 2014a,c, Tuckerman et al., 2014, Tsukahara et al., 2014b, Gomé et al., 2020, Paranjape, 2019, Paranjape et al., 2020, 2023].

We will focus on the plane Poiseuille flow configuration, where the spatiotemporal intermittent patterns are referred to as oblique turbulent-laminar bands illustrated in figure 1.6 at $Re = 1400$ for $L/h = 16\pi$. The the bright and dark regions highlights coexisting spatially localised turbulent and laminar regions. These turbulent-laminar bands occur over a range of Reynolds numbers, and its precise range is likely dependent on the domain's aspect ratio [Tsukahara et al., 2014b, Tuckerman et al., 2014, Paranjape et al., 2023]. Near the upper Re threshold of this regime, the domain is

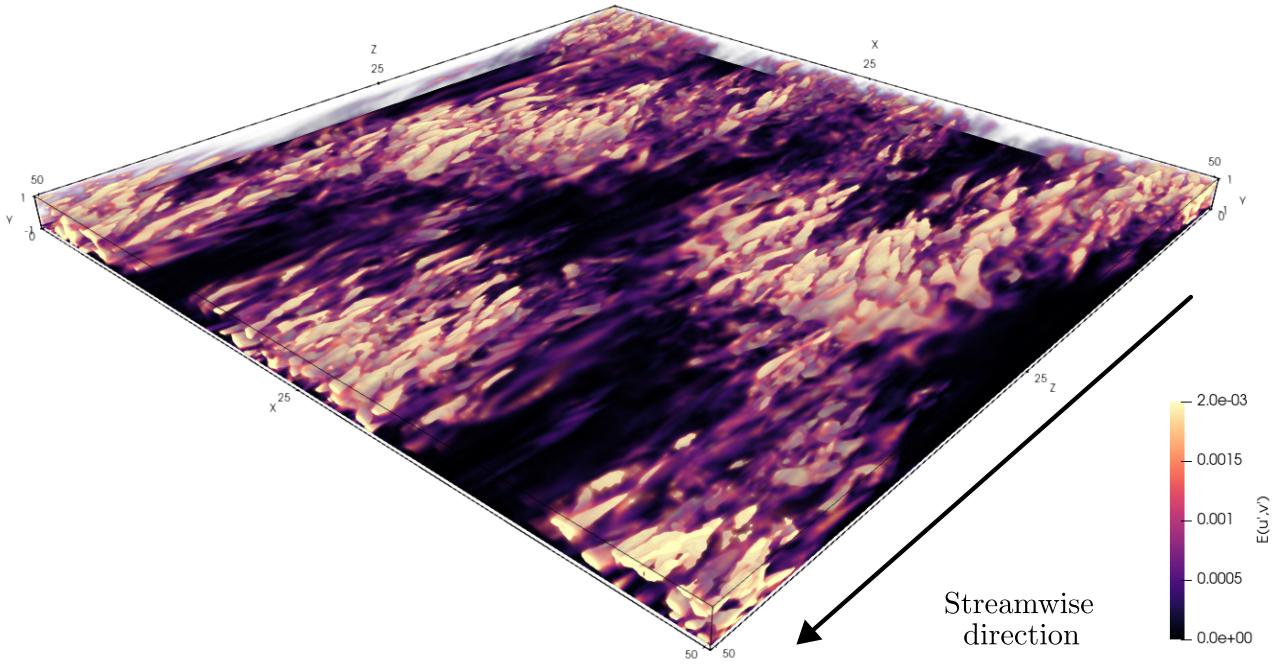


Figure 1.6: A snapshot of turbulent-laminar bands at $Re = 1400$ in a large domain $L/d = 8\pi$, depicting its spatiotemporal intermittent nature. Isovolumetric renderings are based on the spanwise, u' , and wall-normal, v' , perturbation kinetic energy, $E(u', v') = 1/2(u'^2 + v'^2)$, where the perturbation velocities are defined about the laminar state $\mathbf{u}'(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) - U_{lam}(y)$.

fully engulfed by developed turbulent regions, referred to uniform, featureless turbulence appearing at $Re = 1800$ in figure 1.6(a). As Re decreases towards $Re = 1050$, turbulent-laminar bands persist in figures 1.7(b-f). In this region, turbulent-laminar bands angles have been observed to be inclined between $20^\circ \sim 30^\circ$, with streamwise wavelengths of $\sim 60h$, and spanwise wavelengths of $\sim 20h - 30h$ [Tsukahara et al., 2014b]. To study the preference of angles, zzz performed linear stability analysis of bands and showed that preferred angle at .. degrees. Below certain Re threshold, the spatially turbulent regions spontaneously decay where the flow relaminarises asymptotically [Tuckerman et al., 2014]. This decay is shown in $Re = 1000$ near $t = 1600$ in figure 1.7(g).

Inspired from previous studies of turbulent-laminar bands in plane Couette flows [Barkley and Tuckerman, 2005, Reetz et al., 2019], narrow domains, tilted orthogonally to the band angles where considered to investigate their dynamics [Tuckerman et al., 2014, Paranjape et al., 2020, 2023]. In narrow-tilted domains inclined at 24° , the turbulent-bands convect at about $\sim 1\%$ of the bulk velocity, propagating either upstream or downstream, above or below a critical $Re \sim 1000$, independent of domain sizes for $L_z \geq 100h$ [Tuckerman et al., 2014, Gomé et al., 2020]. The characteristic spanwise wavelengths of turbulent-laminar bands are dependent on Re , appearing at $\lambda_z \sim 20h$ for $Re \geq 1400$ and $\lambda_z \sim 40h$ for $Re \leq 1100$. Indeed, between $1300 < Re < 1400$ the bands appear to alternate between two different band-widths [Tuckerman et al., 2014], merging and splitting continuously. This points towards a band splitting event in between $Re = 1100$ and $Re = 1400$, reminiscent of a puff

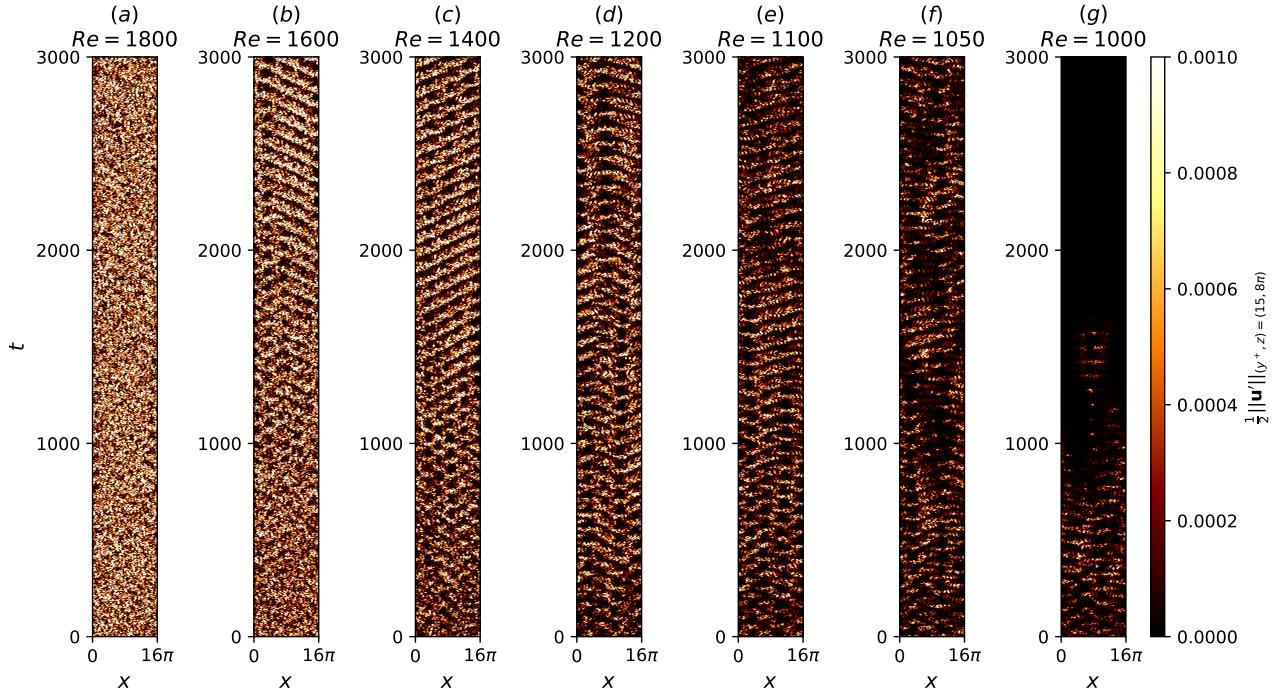


Figure 1.7: Turbulent-laminar bands for $t \in [0, 3000]$ in large domains $(L_x, L_z) = (16\pi, 16\pi)$ at (a) $Re = 1800$, (b) $Re = 1600$, (c) $Re = 1400$, (d) $Re = 1200$, (e) $Re = 1100$, (f) $Re = 1050$, (g) $Re = 1000$.

splitting in pipe flows [Avila et al., 2011].

On the other hand, turbulent bands appear to decay at $Re = 830$ [Gomé et al., 2020], and at $re = 1100$ but surviving for $re = 1000$ [Tuckerman et al., 2014], suggesting that turbulent bands decay spontaneously. [Gomé et al., 2020] computed the probabilities distributions of turbulent band decay, $P(\Delta t^d)$, where Δt^d refers to the time it takes for decay. One of the key insight is that the probability distributions of turbulent band decay mimicks a memoryless Poisson distribution,

$$P(\Delta t^d) = \exp(-\Delta t^d/\tau^d(Re)), \quad (1.16)$$

where $\tau^d(Re)$ refers to the mean lifetime for decay as a function of Re . Similarly, the probability distribution for band splitting also follows a Poisson distribution, $P(\Delta t^s) = \exp(-\Delta t^s/\tau^s(Re))$, where $\tau^s(Re)$ refers to the mean lifetime of a splitting event dependent on Re . The mean survival lifetime of a band decaying, τ^d , and splitting, τ^s , depends superexponentially on Re , i.e. $\tau^{d,s} = \exp(\exp(Re))$. This superexponential dependence is presented in figure 1.8, with a crossover point at $Re_{cross} \approx 965$. This crossover point refers to equal mean survival lifetime of a band suggesting a critical Re for the onset of turbulent bands. While there has been substantial progress made towards understanding the behaviour of periodic turbulent-laminar bands in narrow-tilted domains, recent studies of isolated turbulent bands (ITBs) indicate different behaviour. Notably, ITBs persist at $Re \approx 700$ for $t = 10000$ (far beyond figure 1.8, characterised by streak generating head, and a diffusive upstream tail. [Xiong et al., 2015, Tao et al., 2018, Shimizu and Manneville, 2019, Xiao and Song, 2020]).

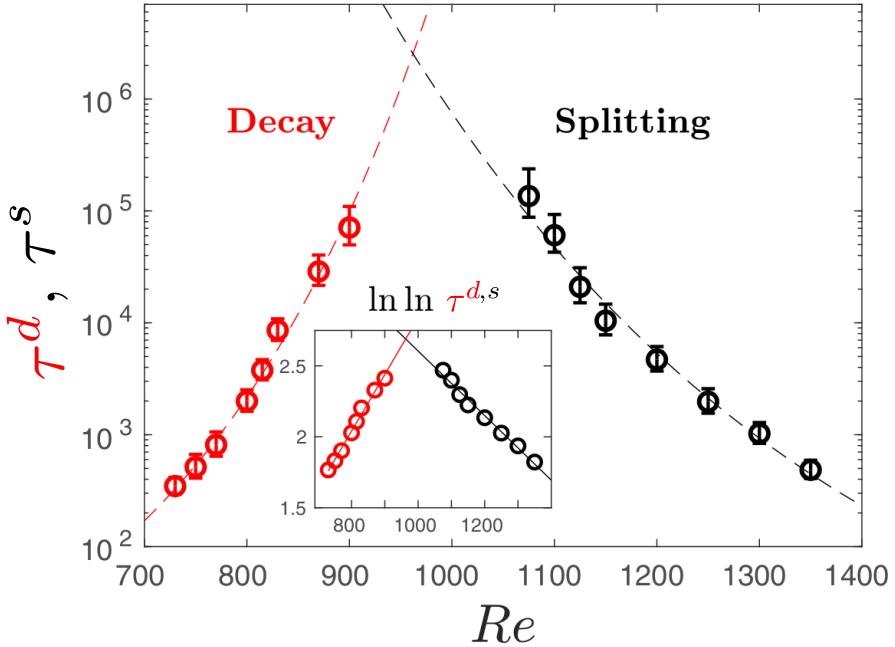


Figure 1.8: The mean decay times (red), τ^d , and mean splitting times (black), τ^s , as a function of Reynolds number, leading to a crossover point at $Re \approx 965$, adapted from [Gomé et al., 2020].

1.3 Rayleigh-Bénard convection

Rayleigh-Bénard convection (RBC) is a paradigmatic fluid configuration describing the motion of the fluid confined between two infinite-parallel plates heated from below and cooled from the top. The basic physical mechanism underpinning RBC is an increase in buoyancy due to heating, leading to fluid motion, referred to buoyancy-driven flows, offering a simple description of natural convection. As the bottom plate is heated, the bottom layer fluid becomes more buoyant and tends to rise, while the colder top fluid layer becomes relatively less buoyant and tends to sink, leading to an overturning of layers. Viscous forces between neighbouring fluid parcels act to resist the motion. As buoyancy overcomes these viscous forces, the fluid layers overturn, resulting in the initiation of convection. One of the earliest experimental work is performed by Henri Bénard [Bénard, 1901], who observed the onset of hexagonal convection cells above a certain threshold of ΔT , referred to as Bénard cells shown in figure 1.9(a) (performed by [Koschmieder and Pallas, 1974]). Utilising linear stability tools similar to §1.2.1, Rayleigh [1916] predicted the onset of convection at a critical Rayleigh number of $Ra_c = \frac{27}{4}\pi^4 = 657.5$ for stress-free boundary conditions, forming the earliest record of linear stability analysis of buoyancy convection. However, it is worth highlighting a key difference in Bénard's experiment and Rayleigh's analysis. Bénard's experiments considered thin layers of fluid heated and cooled by the ambient air [Block, 1956], where surface tension (Maragoni) effects were significant. Surface tension driven convection, now known as Bénard-Maragoni convection typically lead to the onset of hexagonal patterns [Hoard et al., 1970, Koschmieder and Pallas, 1974] shown in figure 1.9(a). In buoyancy driven convection confined to rigid plates, convection patterns usually take the form parallel elongated rollers, known as ideal straight rolls (ISRs) [Bodenschatz et al., 2000]. In the case of a circular dish, these ISRs conform to the geometry of the boundaries, taking the form of concentric rolls shown in figure 1.9(b).

It is also worth noting that Rayleigh [1916] considered a stress-free boundary conditions in his analysis, which has been extended to rigid boundary conditions, the critical Rayleigh number increased

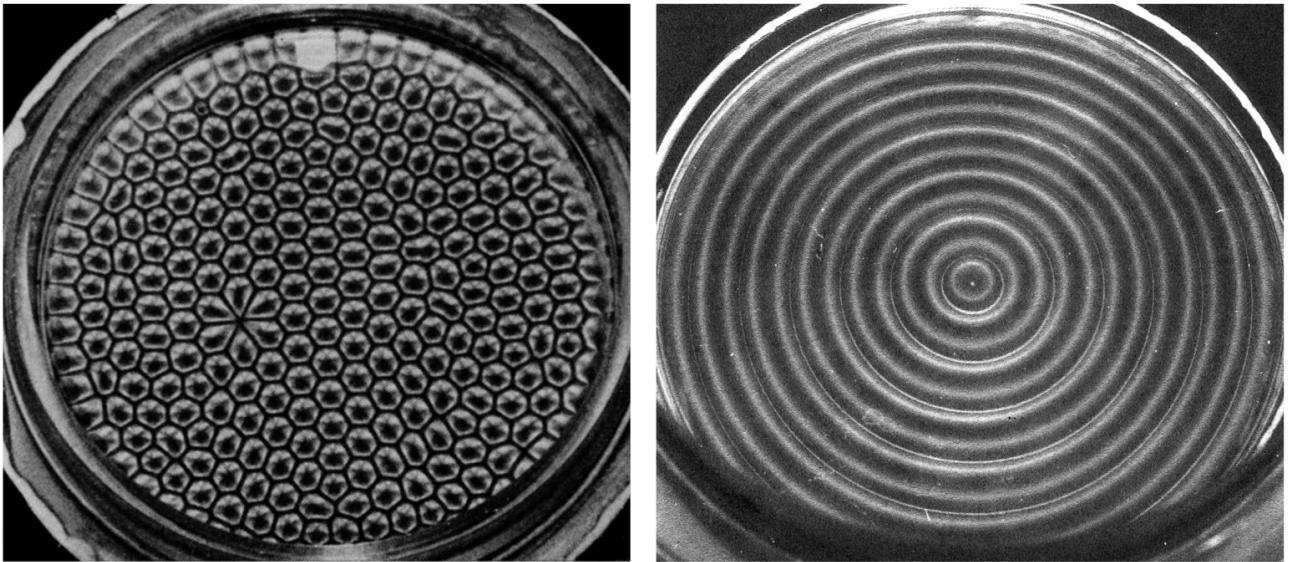


Figure 1.9: (a) Surface tension driven convection leading to the onset of hexagonal Bénard cells in a thin layer of silicone oil, heated from below and cooled by ambient air. A diamond defect appears, likely caused by plate imperfections. (b) Buoyancy driven convection in rigid plates, resulting to concentric convection rolls at 2.9 times the critical Rayleigh number. Both experiments were performed by Koschmieder and Pallas [1974], and the convection patterns were illuminated by aluminum powder, where the dark and bright regions refer to vertical and horizontal motions respectively. These higher resolution images were taken from [Van Dyke and Van Dyke, 1982].

to well known $Ra_c = 1707.8$ [Pellew and Southwell, 1940], and is independent of Pr [Bodenschatz et al., 2000]. The critical wavelength in this case is $q_c = 3.12/d$ (or $\lambda_c \approx 2d$), which suggest that distance separating the plates, d , dictates the length of a single roll, $l_{roll} = \lambda_c/2 \approx d$. In this thesis, I consider RBC with rigid-rigid boundary conditions, simply referred to RBC for the rest of this thesis.

Slightly above the onset of convection, described using the reduced Rayleigh number $\varepsilon = (\frac{Ra}{Ra_c} - 1)$, a set of stable convection rolls near q_c are found to exist (CITATIONS?). The theoretical foundations of performing stability analysis using expansion in powers of amplitude (*weakly nonlinear analysis*) was considered by Eckhaus [1965], in which he applied it to the problem of parallel shear flows. The important contribution was that slightly above the onset, stable stationary rolls are found within the range of $Ra > Ra_c + 3\eta(\alpha - \alpha_c)^2$, where $\eta = \frac{1}{2}\frac{\partial Ra}{\partial \alpha}|_{Ra_c}$. Busse and Whitehead [1971] then extended the same technique to Rayleigh-Bénard convection, in which η was found to depend on Pr . Whilst this technique is useful in detailing the stability regions of stationary rolls, it appears to be limited as it contradicts some experiments in which time-dependent oscillatory rolls were discovered [Rossby, 1969, Willis and Deardorff, 1970], which has been found to be a secondary instability of stationary convection rolls with complex eigenvalues. As noted by Busse [1972], the applicability of weakly nonlinear analysis is limited to small ε . This limitation is addressed by using directly computing the linear stability of saturated convection rolls computed based on a Galerkin truncation and a root-finding algorithm. The results from linear stability analysis led to the well known Busse balloon which described the stability boundary of the convection rolls in the two dimensional space of $\varepsilon - q$, known as the Busse balloon [Busse, 1978] (figure 1.10).

While the Busse balloon describes the stability boundaries of ISRs over a range of wavenumbers, predicting the wavenumber of an ISR state remains an central challenge [Bodenschatz et al., 2000]. Indeed, experimental investigations of RBC in moderate domain sizes ($\Gamma \geq 7$, where Γ refers to

the domain's aspect ratio) in rectangular (straight rolls) and cylindrical (concentric rolls) domains showed that the wavenumbers are confined within the Busse balloon. As ε was continuously modified, the ISRs with wavenumbers that are now outside of the Busse balloon, rolls dislocations and defects spontaneously nucleate, either increasing or decreasing the roll wavenumber, adhering to the stability boundaries of the Busse balloon. The hysteretic nature of the system implies that the roll wavenumber of the ISRs strongly depends on the system's history [Bodenschatz et al., 2000].

It is worth noting that the solutions in the form of ISRs appear to be an exception rather than the rule [Croquette, 1989b]. The coexistence of multiple 'non-ISR' states, in the form of squares, travelling/stationary targets, giant rotating spirals, and oscillatory convection patterns have been found over several years [Le Gal et al., 1985, Croquette, 1989a, Plapp, 1997, Hof et al., 1999, Rüdiger and Feudel, 2000, Borońska and Tuckerman, 2010a,b]. Investigation of cylindrical RBC with small aspect-ratio ($\Gamma = 2$) found eight stationary states (at the same $Ra = 142000$), and two oscillatory states ($Ra > 14200$) [Hof et al., 1999]. These findings were later supported by numerical experiments and bifurcation analyses [Ma et al., 2006, Borońska and Tuckerman, 2010a,b]. In particular, bifurcation analyses performed by Ma et al. [2006], revealed twelve stable branches in the form of symmetric and asymmetric convection rolls near onset ($Ra \leq 2500$), with the potential emergence of hundreds of branches at higher Rayleigh numbers, $Ra \leq 30000$ [Borońska and Tuckerman, 2010b]. In larger domains ($\Gamma \geq 28$), giant rotating spirals were identified and thoroughly investigated [Plapp and Bodenschatz, 1996, Plapp et al., 1998]. Experimental and numerical studies of RBC with varying side-wall boundary conditions (i.e. thermally insulating, conducting or no-slip) [Tuckerman and Barkley, 1988, Siggers, 2003, Paul et al., 2003, Bouillé et al., 2022], non-Boussinesq convection [Bodenschatz et al., 1992], and rotational effects [Hu et al., 1997] were investigated, where multiple states were also reported.

1.4 Rayleigh-Bénard Poiseuille (RBP) flows

Gage and Reid [1968] first investigated the primary instabilities of RBP flows, which can be determined by Re , Ra , Pr , and the planar x - z perturbations wavenumbers α, β respectively. For a given Ra and Pr , the neutral stability curves are limited by the development of Tollmien-Schlichting waves for $Re \geq Re_{TS} = 5772.22$ [Orszag, 1971], and convection rolls within $0 \leq Re < Re_{TS}$. Convection rolls can be categorised based on their orientation to the mean flow, namely, longitudinal ($\alpha = 0, \beta \neq 0$), transverse ($\alpha \neq 0, \beta = 0$) and oblique rolls ($\alpha \neq 0, \beta \neq 0$). The linearised system governing the onset of longitudinal rolls is analogous to the linearised RBC system, with a critical Rayleigh number, $Ra_{\parallel} = Ra_{RB} = 1708.8$ and critical wavenumber, $\alpha_{\parallel} = \alpha_{RB} = 3.13$ [Pellew and Southwell, 1940, Kelly, 1994], independent of Re and Pr . The critical Rayleigh number for oblique and transverse rolls matches that of RBC at $Re = 0$ due to horizontal isotropy, but increases as Re increases, depending on Pr , i.e., $Ra_{\perp} = f(Re, Pr)$ [Gage and Reid, 1968, Müller et al., 1992, Nicolas et al., 1997]. When spatially developing instabilities are considered, longitudinal rolls are always convectively unstable, and transverse rolls can become absolutely unstable [Müller et al., 1989, 1992, Carrière and Monkewitz, 1999]. Nonmodal stability analyses of subcritical RBP indicate that the optimal transient growth G_{max} increases gradually with Ra . The wavenumber of the optimal initial conditions, β_{max} , resembles that observed in shear flows [Reddy and Henningson, 1993], and gradually approaches the critical wavenumber of convection rolls, α_{\parallel} , as Ra increases [John Soundar Jerome et al., 2012]. For $Re > 0$, the longitudinal rolls appear as the dominant primary instability [Gage and Reid, 1968]. Secondary

stability analyses of longitudinal rolls reveal a wavy instability near $Re \sim 100$ [Clever and Busse, 1991], leading to wavy longitudinal rolls, which are convectively unstable [Pabiou et al., 2005, Nicolas et al., 2010]. The influence of finite lateral extensions in RBP flows on the stability of longitudinal and transverse rolls [Kato and Fujimura, 2000, Nicolas et al., 2000], as well as wavy rolls [Xin et al., 2006, Nicolas et al., 2010], has been reported. In finite streamwise extensions of RBP flows, the onset of convection rolls and the heat flux variations due to entrance effects have been investigated [Mahaney et al., 1988, Lee and Hwang, 1991, Nonino and Giudice, 1991]. More recently, shear-driven turbulence can enhance heat fluxes in turbulent RBP flows [Scagliarini et al., 2014, 2015, Pirozzoli et al., 2017]. RBP flows with sinusoidal heating and wavy walls have also been studied [Hossain et al., 2021]. For an in-depth discussion of RBP flows, see the reviews by Kelly [1994] and Nicolas [2002].

1.4.1 Thesis Outline

In this thesis, I am particularly focused on the transition behaviour of fluid flow driven by shear and buoyancy, addressing questions related to the onset of instabilities due to shear and buoyancy, and the (possible) competitive between shear and buoyancy driven instabilities. I would like to preface that while this thesis is dealing with onset of instabilities, it does not clearly indicate that the onset of such instabilities necessarily lead to turbulence, hence, for terminology sake, we shall be looking into transitional regimes where the fluid neither laminar nor turbulent. The main motivations are two-folds, both from an academic and applied point-of-view. Within academia, the onset and transition to turbulence in Rayleigh-Bénard Poiseuille flows remains poorly understand. Whilst there had been significant progress in our understand of transition to turbulence in independent setups, Rayleigh-Bénard convection and plane Poiseuille flows, their combined effects are not known. The thesis is structured into the follow, Chapter 1 is the introduction with literature review, chapter 2 methodology assosicated with the spectral/ hp -element method, chapter 3 with results related to the the Rayleigh and Reynolds number sweep, chapter 4 with a specific focus on the bistability between spiral defect chaos and ideal straight rolls and finally chapter 5 with concluding remarks.

1. Academic motivation - flow structures, statistics, transition.
2. Application motivation - shear, heat transfer. Chip cooling, thin-film fabrication and atmospheric boundary layer.

We seek to investigate the influence of unstable stratification quantified by Rayleigh number Ra , on the behaviour tuburlt-laminar bands. The onset of convection occurs at a critical Rayleigh number of $Ra_c > 1708$, in the form of a pair of convection rolls. When aligned in the streamwise direction, the convection rolls are seemingly analogous to a pair of counter-rotating vortices, an optimal initial condition for transient growth. Our investigation naturally answers a few questions related to turbulent-laminar bands. For example, does the onset of turbulent-laminar bands, Re_{cr} decrease with increasing Ra ? Do Ra -effects influence the structure of turbulent-laminar bands i.e band angle/width?

The answers to our research will have important implications Rayleigh-Bénard Poiseuille flows, ubiquitous in atmospheric, geophysical and engineering flows.

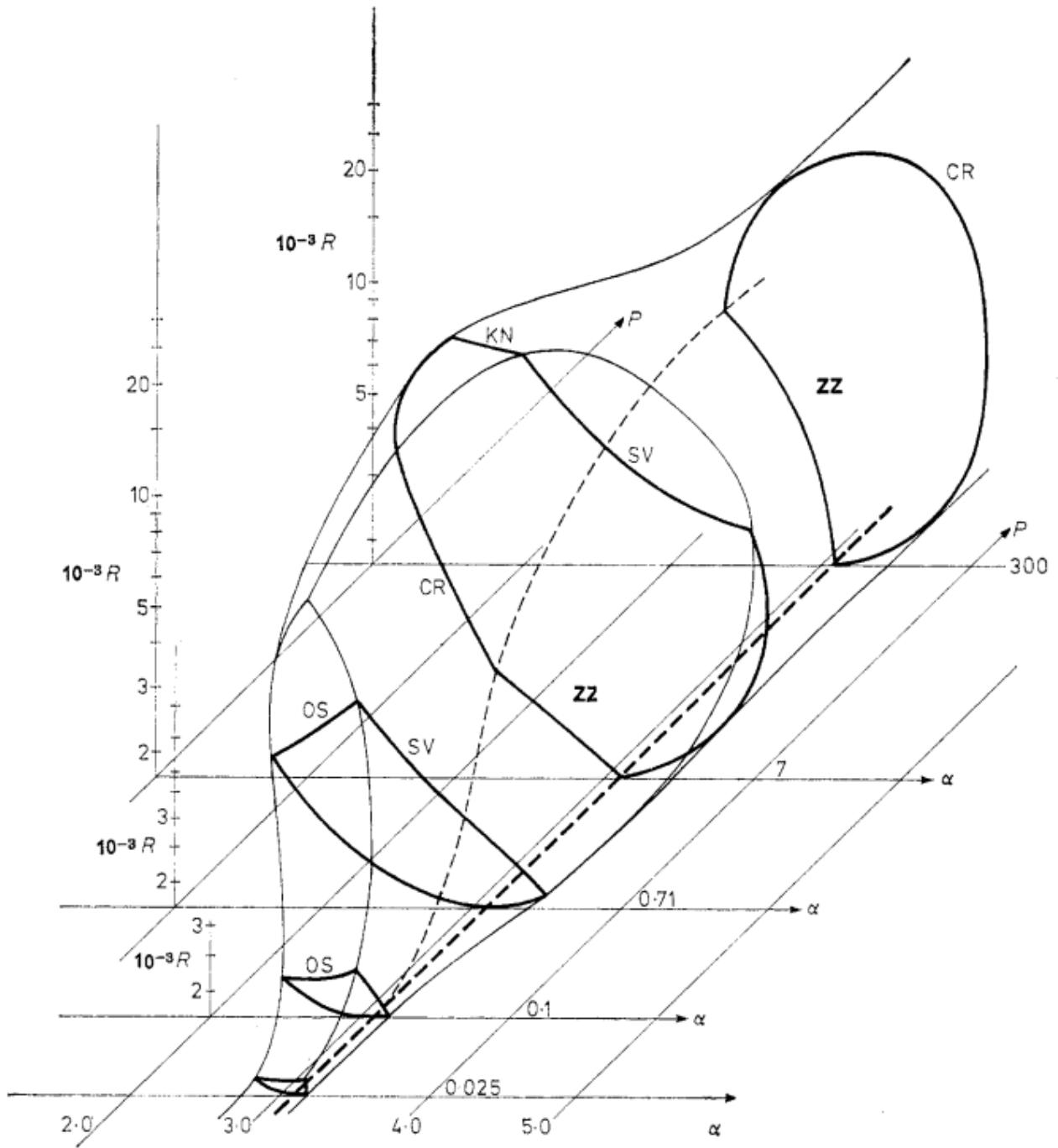


Figure 1.10: The Busse balloon describes the stability boundaries of ISRs in a $\varepsilon - q$ space. For larger wavenumbers, the instability mechanism is described by the skewed-varicose (SV) instability. For smaller wavenumbers, the instability mechanism is described by the Eckhaus instability. For large ε , the instability is described by the onset of oscillatory instability. Busse balloon digitised from [Plapp, 1997] for $Pr \approx 1$

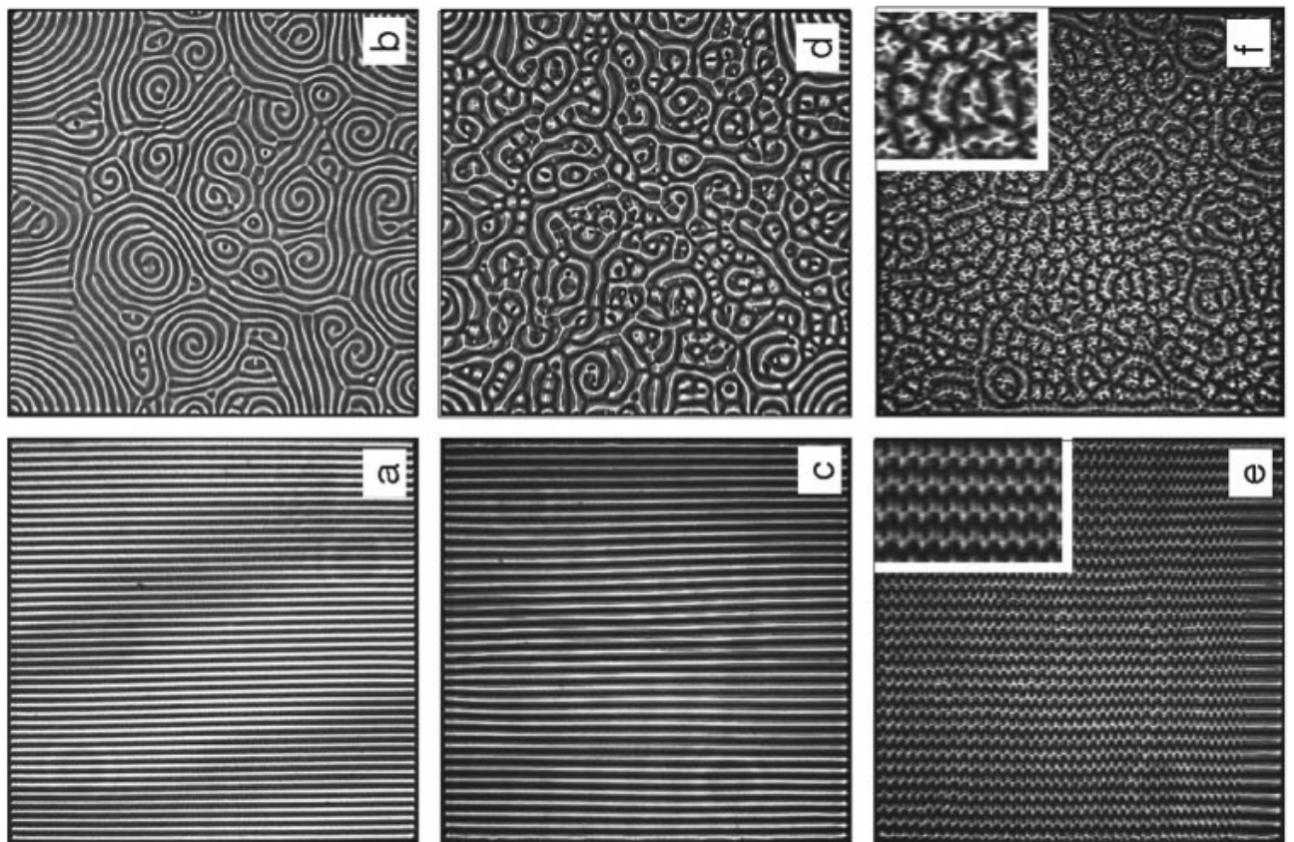


Figure 1.11: Spiral defect chaos (top row) coexisting with ISRs (bottom row) forming a bistable system.

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