Abstract

Contents

Chapter 2

Numerical Techniques

In this chapter, I will present the numerical methods relevant to this thesis. The incompressible Navier-Stokes equations describe the time- and spatial-varying velocity field and pressure field. One of the foundations of solving partial differential equations begin with the method of weighted residuals.

2.1 Method of weighted residuals

We discuss the method of weighted residuals which provides a mathematical framework for approximating the solutions of partial differential equations. We first consider a generic linear partial differential equation as,

$$\mathbf{L}[u(x)] = 0, \quad x \in \Omega \tag{2.1}$$

where $\mathbf{L}$ refers to a spatial (linear) differential operator subjected to some boundary conditions within the domain, Ω while $u(x)$ refers to the solution. Typical examples of spatial differential operators are the Laplace, Poisson or Helmholtz operators. In seeking the solution, $u(x)$, we assume that it could be approximated by a finite number of N basis (or expansion) functions, $\Phi(x)$,

$$u(x) \approx u^{\delta}(x) = \sum_{i=0}^{N-1} \hat{u}_i \Phi_i(x), \tag{2.2}$$

where $u^{\delta}(x)$ refers to the approximate solution of $u(x)$, comprising of a linear combination of the product between the i^{th} basis coefficient, $\hat{u}_i$, and a basis function $\Phi_i(x)$, that is defined within Ω. Since $u^{\delta}(x)$ is an approximate solution of equation (2.6), we expect a non-zero difference (or 'error') between the exact solution, $u(x)$, and $u^{\delta}(x)$, known as the residual, R, given as,

$$\mathbf{L}[u^{\delta}(x)] = R[u^{\delta}(x)]. \tag{2.3}$$

The residual depends on the approximate solution $u^{\delta}(x)$, and is non-zero, varying within Ω. In other words, equation (2.6) might not be satisfied everywhere in Ω. We need to place restrictions on the residual, such that it $R \rightarrow 0$, and the approximate solution approaches the exact solution, $u^{\delta}(x) \rightarrow u(x)$. The method of residuals allow us to place a restriction on R is by using N weight (or

test) functions, $v_j(x)$, such that it is orthogonal with the residual,

$$(v_j(x), R[u^\delta(x)]) = 0, \quad j = 0, ..., N-1. \tag{2.4}$$

where $(\,\cdot\,,\,\cdot\,)$ refers to an inner-product, a measure of orthogonality between functions defined as,

$$(f, g) = \int_\Omega f(x)g(x)dx. \tag{2.5}$$

By setting the inner-product to 0, equation (2.4) becomes a system of N ordinary differential equations, where the basis coefficients, $\hat{u}_i$, could be determined as we shall see later. The choice of weight function defines the class of projection methods, and the common projection methods are shown in table 2.1. It is worth noting that the method of weighted residuals describes the projection method, and does not specify the type of basis functions employed. The choice of projection method, and basis expansions will have different solution convergence properties, i.e. how does the residual decay as the number of basis expansions increases? By considering Fourier basis expansions, one can expect exponential convergence, desirable for an efficient representation of turbulent dynamics.

Weight functions	Projection method
$v_j(x) = \delta(x - x_j)$	Collocation
$v_j(x) = \begin{cases} 1 & \text{if } x \in \Omega_j \\ 0 & \text{if } x \notin \Omega_j \end{cases}$	Finite-Volume
$v_j(x) = \phi_j$	Galerkin
$v_j(x) = \frac{\partial R}{\partial \hat{u}_j}$	Least-squares

Table 2.1: Examples of weight functions and projection methods

2.2 Galerkin Projection

The Galerkin projection is a remain as a projection method in finite elements, where the weight functions, $v(x)$, are chosen to be the same as the basis functions, $\Phi(x)$. We will elaborate on what defines being the 'same' later. To demostrate the Galerkin projection method, we consider that the spatial (linear) differential operator in equation (2.1) as a 1D Helmholtz equation,

$$\mathbf{L}[u(x)] \equiv \frac{\partial^2 u(x)}{\partial x^2} - \lambda u(x) - f(x) = 0, \quad x \in \Omega := [0, l] \tag{2.6a}$$

$$u(0) = g_D, \quad \left.\frac{\partial u}{\partial x}\right|_{x=l} = g_N. \tag{2.6b}$$

where λ is a real positive constant, $f(x)$ is the forcing function, and Ω the spatial domain bounded between 0 and l. To ensure that problem is well-posed, we impose both Dirichlet and Neumann

boundary conditions, corresponding to g_D and g_N, at $x = 0$ and $x = l$, respectively. Equation 2.6 is said to be written in the *strong* or *classical* form.

Next, we begin to construct the weak form by taking the inner product of the Helmholtz equation with a weight function, which satisfies the homogeneous Dirichlet boundary conditions by definition, and require this inner product to vanish, that is,

$$(w, \mathbf{L}[u(x)]) = \int_0^l w \left[\frac{\partial^2 u(x)}{\partial x^2} - \lambda u(x) + f(x) \right] \mathrm{d}x = 0. \tag{2.7}$$

This step is equivalent to applying the method of weighted residuals, where $u(x)$ could refer to the approximate solution, $u^\delta(x)$. We perform integration by parts next,

$$\underbrace{\int_0^l \frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\mathrm{d}x + \int_0^l \lambda v u \mathrm{d}x}_{a(v,u)} = \underbrace{\int_0^l v f \mathrm{d}x + \left[v \frac{\partial u}{\partial x} \right]_0^l}_{f(v)}. \tag{2.8}$$

This equation is typically referred to as the weak form of equation (2.6). In compact notation, we define the bilinear and linear forms as,

$$a(v, u) = f(v), \tag{2.9a}$$

where $a(v, u)$ and $f(v)$ are typically refered to as the strain energy and forcing function in structural mechanics, assumed to remain finite. To ensure this, we restrict the choice of solutions $u(x)$ to lie in the functional solution space, $\mathcal{U}$, defined as

$$\mathcal{U} := \{u | u \in H^1, u(0) = g_D\}, \tag{2.10}$$

where $u \in H^1$ contains functions of u in the Sobolev space such that the Dirichlet condition $u(0) = g_D$ is statisfied and the sum of square integral of u and its first derivatives, $\frac{\partial u}{\partial x}$ remains bounded,

$$\int_\Omega \left(u^2 + \left(\frac{\partial u}{\partial x} \right)^2 \right) \mathrm{d}\Omega < \infty. \tag{2.11}$$

We consider functions up to the first-derivatives since the it is the highest-order derivative in equation (2.7). Similarly, the functional space of weight functions, $\mathcal{V}$, is defined as,

$$\mathcal{V} := \{v | v \in H^1, v(0) = 0\}, \tag{2.12}$$

where $v \in H^1$ are functions whose values and first derivatives are square-integrable and satisfy a homogeneous Dirichlet boundary condition at $x = 0$. The generalised weak form is therefore finding $u(x) \in \mathcal{X}$, such that

$$a(v, u) = f(v), \quad \forall v \in \mathcal{W}. \tag{2.13}$$

This formulation is still infinite-dimensional, as the function space $\mathcal{U}, \mathcal{W}$ contain infinitely many

functions. To obtain an approximate solution, $u^\delta(x)$, we restrict ourselves to finite-dimensional subspaces, $\mathcal{U}^\delta \subset \mathcal{U}$, and $\mathcal{V}^\delta \subset \mathcal{V}$. The problem is then to find $u^\delta \in \mathcal{U}^\delta$, such that

$$a(v^\delta, u^\delta) = f(v^\delta), \quad v^\delta \in \mathcal{V}^\delta \tag{2.14}$$

Here, both $u^\delta \in \mathcal{U}^\delta$ and $v^\delta \in \mathcal{V}^\delta$ do not lie in same subspace, necessary for the standard Galerkin projection procedure where they should lie in the same subspace. To overcome this, we lift the solution u^δ into two parts,

$$u^\delta = u^\mathcal{H} + u^\mathcal{D}. \tag{2.15}$$

where $u^\mathcal{H} \in \mathcal{V}^\delta$ satisfies the homogeneous Dirichlet condition, lying in the same subsapce as $w^\delta \in \mathcal{V}^\delta$, and $u^\mathcal{D} \in \mathcal{U}^\delta$ satisfies the Dirichlet boundary conditions $u^\mathcal{D}(0) = g_D$. Substituting this decomposition, the standard Galerkin projection methos is to search for the solution $u^\mathcal{H} \in \mathcal{V}^\delta$ such that,

$$a(v^\delta, u^\mathcal{H}) = f(v^\delta) - a(v^\delta, u^\mathcal{D}). \tag{2.16}$$

We have briefly described the mathematical framework for approximating a solution to the linear (spatial) partial differential operator. In the following, we will expand on the type of basis and weight functions , $\Phi(x)$, specifically, using the spectral/*hp* element method, and describe essential differential and integral operations performed numerically, equation (2.16) reduces to a system of linear equations.

2.3 Spectral/*hp* element methods

The spectral/*hp* element method is a spatial discretisation technique in which the solution domain is partitioned into a set of non-overlapping (finite) elements with size h, each consisting of a linear combination of continuous polynomial functions of up to order P. It leverages the geometric flexibility of classical finite-element methods, allowing for the representation of complex engineering geometries, and the exponential (spectral) convergence properties of classical spectral methods, where the solution error decreases exponentially. Suppose we consider $P+1$ linearly independent polynomials spanning the polynomial space of $\mathcal{P}_P$, the error of a smooth solution with element size of h and polynomial order P has the property of [Karniadakis and Sherwin, 2005],

$$||u(x) - u^\delta(x)|| \leq Ch^P ||u(x)|| \approx \mathcal{O}(h^P). \tag{2.17}$$

where C is some constant. Equation 2.17 implies that the error decreases linearly with h, and exponentially with P.

2.3.1 Domain partition

We consider a one-dimensional domain, Ω defined earlier, and seek to decompose it into a set of N_{el} elements, where Ω^e, refers to the elemental partitions with $1 \geq e \geq N_{el}$ such that they meet at their

boundaries,

$$\Omega = \bigcup_{e=1}^{N_{el}} \Omega^e, \quad \text{where} \bigcap_{e=1}^{N_{el}} \Omega^e = \emptyset \tag{2.18}$$

where e^{th} element is defined as,

$$\Omega^e = \{x | x_{e-1} \geq x \geq x_e\}, \tag{2.19}$$

Each element it then represented by a linear combination, $\phi(x)$, where x is referred to the *global* domain.

2.3.2 Standard Elements

In general, we can expect non-uniform elements so it is useful to define a *standard* element,

$$\Omega_{set} := \{\xi | -1 \geq \xi \geq 1\}, \tag{2.20}$$

where Ω_{st} refers to the standard element, defined over local coordinates, $\xi \in [-1, 1]$. By considering formulations using the standard elements, the formulations of basis expansions, differential and integration operations can be performed in local coordinates, ξ, before mapping the results back to the global domain, x. We can map the standard element into any arbitrary global coordinates based on a linear mapping $\chi^e : \Omega_{st} \to \Omega$,

$$x = \chi^e(\xi) = \frac{1-\xi}{2} x_e + \frac{1+\xi}{2} x_{e+1}, \quad \xi \in \Omega_{st} \tag{2.21}$$

which has an analytical inverse, $(\chi^e)^{-1}(x)$,

$$\xi = (\chi^e)^{-1}(x) = 2\frac{x - x_{e-1}}{x_e - x_{e-1}} - 1, \quad x \in \Omega_{st}. \tag{2.22}$$

In each standard element, we can represent the solution by using a linear combination of local expansion basis, $\phi(\xi)$,

$$\phi_0(\xi) = \frac{1-\xi}{2}, \quad \phi_1(\xi) = (1+\xi)(1-\xi), \quad \phi_2(\xi) = \frac{1+\xi}{2}, \tag{2.23}$$

where ϕ_0, ϕ_2 denotes a linear local expansion basis with $P = 1$, while ϕ_1 is a quadratic local expansion basis with $P = 2$. The approximate solution is now represented as,

$$u^\delta(x) = \sum_{i=0}^{N-1} \hat{u}_i \Phi_i(x) = \sum_{e=0}^{N_{el}-1} \sum_{i=0}^{P} \hat{u}_i^e \phi_i^e(\chi^e(\xi)). \tag{2.24}$$

where $\hat{u}_i^e$, N_{el} refers to the local expansion basis coefficients the number of elements. Consequentially, $u^\delta(x)$ now lie within $\mathcal{X}^\delta$.

$$\mathcal{X}^\delta := \left\{ u^\delta \,\middle|\, u^\delta \in H^1,\ u^\delta(\chi^e(\xi)) \in \text{span}\{\phi_0, \phi_1, \phi_2\},\ e = 1, 2, 3, 4 \right\} \tag{2.25}$$

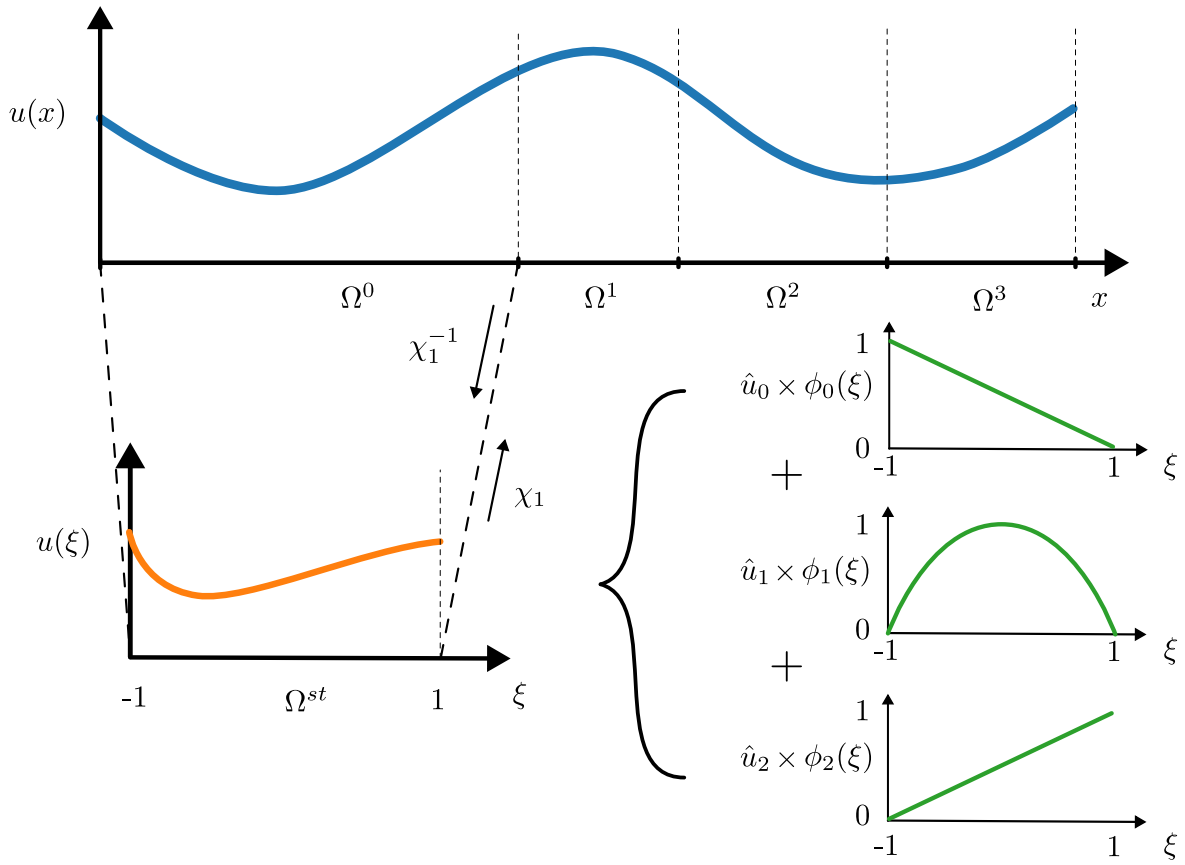

Figure 2.1: A spectral/*hp* element representation of a 1D continuous function, $u(x)$, deomposed into four non-overlapping finite elements, each containing a linear combination of three local expansion bases.

Figure 2.1 summarises the domain partition based on standard elements.

2.3.3 Assembly process

As we represent our solution using local expansion basis within standard elements, their solution across the elemental boundaries may become discontinuous. In the approach of continuous Galerkin projection methods, we enforce our solution to be C^0 continuous across the elemental boundaries. In other words, the neighbouring linear interior elements must meet at the boundaries, such that the local expansion coefficients are constrained by,

$$\hat{u}_P^{e-1} = \hat{u}_0^e. \tag{2.26}$$

This constrain enforeced by consider a mapping between the (global) expansion coefficients, and local expansion coefficients,

$$\hat{\mathbf{u}}_l = \mathbf{A}\hat{\mathbf{u}}_g \tag{2.27a}$$

with,

$$\hat{\mathbf{u}}_l = \begin{bmatrix} \hat{u}_0^1 \\ \hat{u}_1^1 \\ \hat{u}_2^1 \\ \hat{u}_0^2 \\ \hat{u}_1^2 \\ \hat{u}_2^2 \\ \hat{u}_1^3 \\ \hat{u}_2^3 \\ \hat{u}_3^3 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \hat{\mathbf{u}}_g \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_3 \\ \hat{u}_4 \\ \hat{u}_5 \\ \hat{u}_6 \\ \hat{u}_7 \end{bmatrix} \tag{2.27b}$$

where $\hat{\mathbf{u}}_l, \hat{\mathbf{u}}_g, \mathbf{A} \in \mathbb{R}^{N_l, N_g}$ refers to the vector of local, global expansion and scatter matrix, and $N_l = N_{el} \times (P+1)$ refers to the total local degrees of freedom while $N_g = N_l - (N_{el} - 1)$, the global degrees of freedom. Notably, matrix $\mathbf{A}$ 'scatters' the global degrees of freedom to local degrees of freedom. In the spectral/*hp* approach, we typically define local expansion basis, and perform integration and differentiation operation in standard element. After doing so, we need to assemble the operations from the standard element to the global domain by using $\mathbf{A}^T$, is known as an assembly operation, assembling local degrees of freedom to global degrees of freedom. For instance, we wish to perform integration in the domain Ω,

$$\mathbf{I}_g[j] = (\Phi_j(x), u^\delta(x)), \tag{2.28}$$

which equivalent to performing integration using local expansion basis within standard elements, and assembling in the global space by using $\mathbf{A}^T$,

$$\mathbf{I}_g = \mathbf{A}^T \mathbf{I}_l \tag{2.29a}$$

$$\mathbf{I}_g = \begin{bmatrix} \mathbf{I}_0 \\ \vdots \\ \mathbf{I}_{N_g-1} \end{bmatrix}, \quad \mathbf{I}_l = \begin{bmatrix} \mathbf{I}^0 \\ \vdots \\ \mathbf{I}^{N_{el}-1} \end{bmatrix}, \quad \text{where} \quad \mathbf{I}^e = \begin{bmatrix} \int_{-1}^{1} \phi_0^e(\xi) u(\chi^e) \frac{\mathrm{d}\chi^e}{\mathrm{d}\xi} \, \mathrm{d}\xi \\ \vdots \\ \int_{-1}^{1} \phi_{P-1}^e(\xi) u(\chi^e) \frac{\mathrm{d}\chi^e}{\mathrm{d}\xi} \, \mathrm{d}\xi \end{bmatrix} \tag{2.29b}$$

and $\mathbf{I}_g, \mathbf{I}_l, \mathbf{I}^e$ refer to the integration operations perform in global, local and elemental space.

2.3.4 Expansion functions

Here, we discuss the expansion functions of $\phi(\xi)$, where in general could be categorised into *modal* (hierarchical) expansions or *nodal* expansions.

Modal expansions

The most common modal employed in Nektar++ are the Jacobi polynomials, denoted by $P_p^{\alpha,\beta}(x)$, which represent a family of polynomial solutions to the Sturm-Liouville problem wihin, $x \in [-1, 1]$. The Legendre polynomials are a special case of Jacobi polynomials, $L_n(\xi) = P_n^{0,0}(\xi)$ with $\alpha = \beta = 1$. Within the Nektar++ framework, it is common to use the *modified* basis based on Jacobi polynomials, $P_p^{\alpha,\beta}(\xi)$ which are modified with linear elements as

$$\phi_p(\xi) = \psi_p(\xi) = \begin{cases} \frac{1-\xi}{2} & \text{for } p = P \\ \frac{1-\xi}{2}\frac{1+\xi}{2}P_{P-1}^{1,1}(\xi) & \text{for } P \geq 2 \\ \frac{1+\xi}{2} & \text{for } p = P, \end{cases} \tag{2.30}$$

where P denotes the highest polynomial order. Figure 2.2 shows the modified Jacobi polynomials for $p \in [0, 5]$ described by equation 2.30. The boundary modes are ψ_0 and ψ_5 while the rest are boundary modes.

Nodal expansions

A popular nodal expansions are the Lagrange polynomials, common used in the spectral *element* codes such as Semtex and Nek5000. The Lagrange polynomials are given as

$$\phi_p(\xi) = h_p(\xi) = \frac{\prod_{q=0,q\neq p}^{P}(\xi - \xi_q)}{\prod_{q=0,q\neq p}^{P}(\xi_p - \xi_q)} \tag{2.31}$$

The Lagrange polynomials are particular attractive as it has a unit value at ξ_q and zero everywhere else,

$$h_p(\xi_q) = \delta_{pq}. \tag{2.32}$$

Typically, the zeros are located using the zeros of the Gauss-Lobatto polynomials where the zeros are defined using

$$\phi_p(\xi) \to h_p(\xi) = \begin{cases} 1, & \xi = \xi_p, \\ \frac{(\xi^2-1)[P_{Q-1}^{\alpha,\beta}(\xi)]'}{(Q-1)(Q+\alpha+\beta)P_{Q-1}^{\alpha,\beta}(\xi_j)(\xi-\xi_j)}, & \text{otherwise.} \end{cases} \tag{2.33}$$

Figure 2.3 presents nodal expansions based on two-dimensional and one-dimensional Lagrange polynomials, $h_p(\xi_1), h_q(\xi_2)$ respectively.

The modified basis and Lagrange exhibits stark differences.

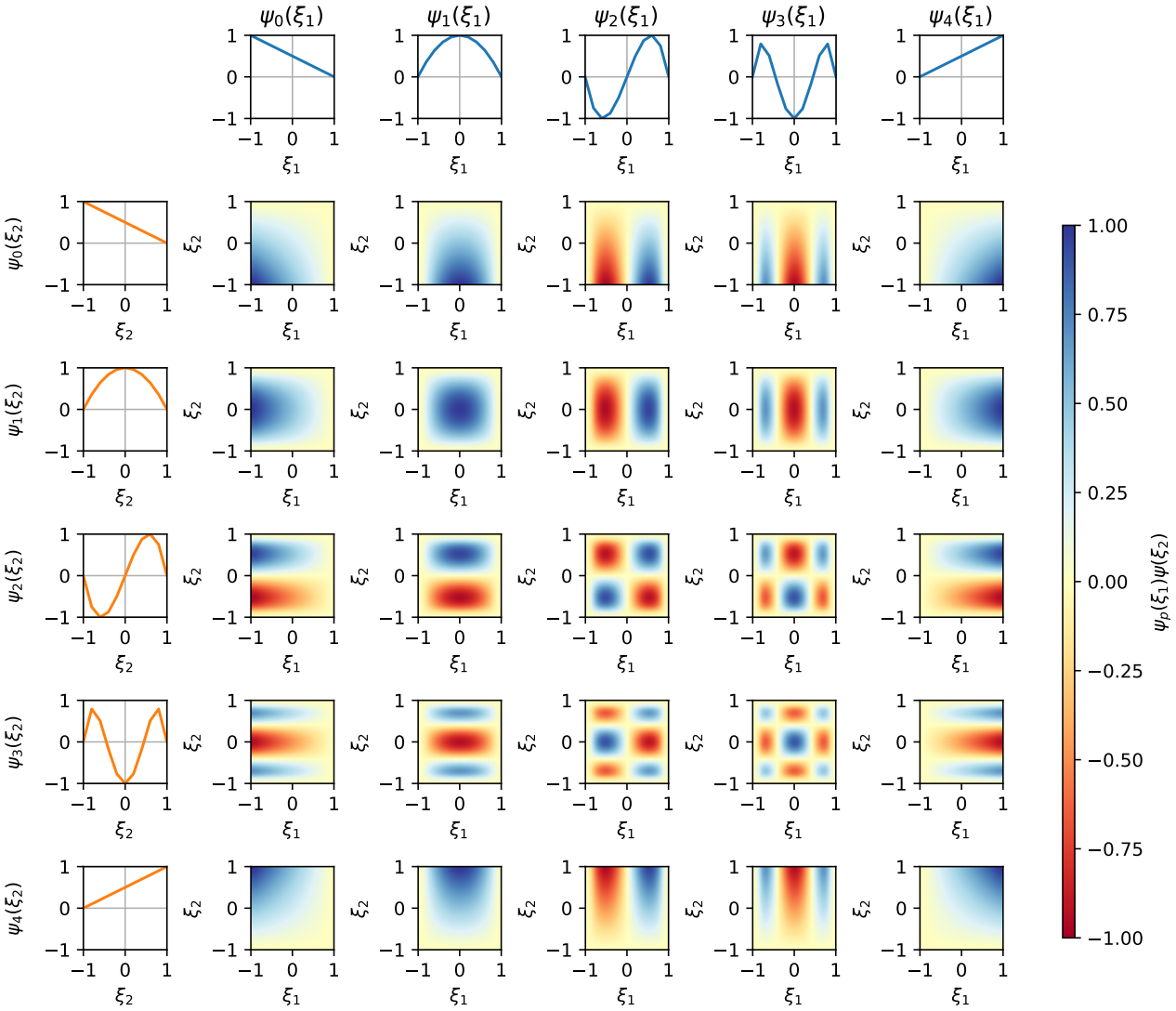

Figure 2.2: Two-dimensional and one-dimension modified basis, $\psi_p(\xi_1)$ and $\psi_q(\xi_2)$, $P = [0, 4]$.

2.3.5 Numerical integration

In the Galerkin formulation, we perform integration routinely. Suppose we want to approximate the integral of a smooth function in a standard element numerically,

$$\int_{-1}^{1} u(\xi)\, \mathrm{d}\xi = \sum_{i=0}^{Q-1} w_i u(\xi_i) + R(u), \tag{2.34}$$

where $Q, w_i, \xi_i, R(u)$ refers to the quadrature points, integration weights and zeros (or abscissae) and the integral of the error. By evaluating the integral, how are we able to minimise the integral error, $R(u)$, with the least number of quadrature points, Q, at some weights and zeros. If $u(\xi)$ is of polynomial order P, we can expect we might need at least $P + 1$ equipspaced points to accurately represent the function and evalute its integral, a rather inefficient method. Gaussain quadrature is allow us to approximate an integral of a function of order P with far lesser thatn $P + 1$ points, as we shall see later. The three generic types of Gaussian quadrature rules are known as: Gauss, Gauss-Radau and Gauss-Lobatto. The main difference between the three methods are in the treatment of the zeros,

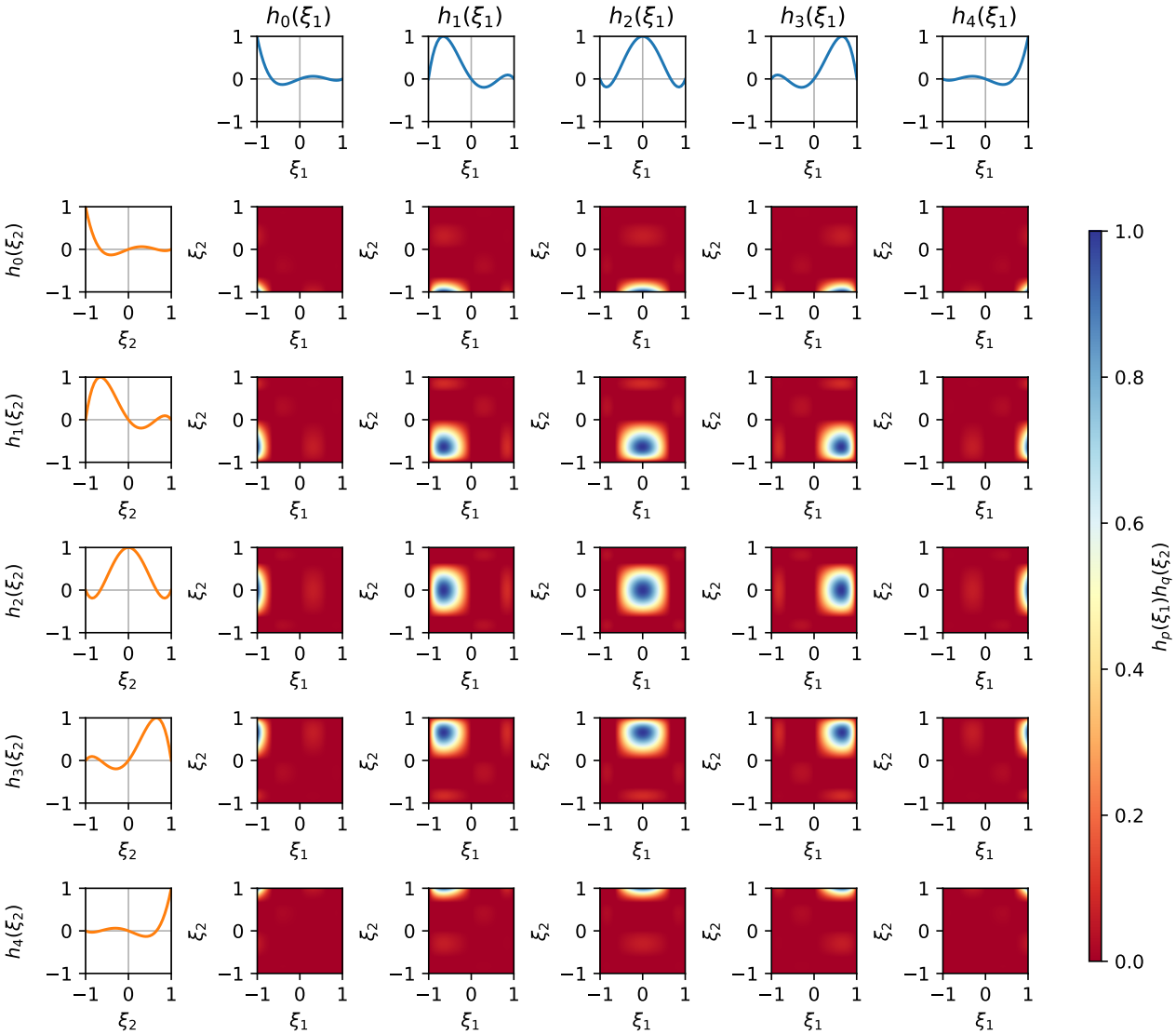

Figure 2.3: Two-dimensional and one-dimension Lagrange basis, $h_p(\xi_1)$ and $h_q(\xi_2)$, $P = [0, 4]$.

where Gauss quadrature uses zeros without the end points $\xi = \pm 1$. Gauss-Radau quadrature either select one of the end points, usually at $\xi = -1$, and Gauss-Lobatto consider the end points. We will only focus on describing the Gauss-Lobatto quadrature rules and the zeros of Jacobi polynomials known as the Gauss-Lobatto-Jacobi quadrature rules given as,

$$\xi_i^{\alpha,\beta} = \begin{cases} -1 & i = 0, \\ \xi_{i-1,Q-2}^{\alpha+1,\beta+1} & i = 1, ..., Q-2, \\ 1, & i = Q-1, \end{cases} \tag{2.35a}$$

$$w_i^{\alpha,\beta} = \begin{cases} (\beta+1)C_{0,Q-2}^{\alpha,\beta}, & i = 0, \\ C_{i,Q-2}^{\alpha,\beta}, & i = 1, ..., Q-2, \\ (\alpha+1)C_{Q-1,Q-2}^{\alpha,\beta}, & i = Q-1, \end{cases} \tag{2.35b}$$

$$C_{i,Q-2}^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+Q)\Gamma(\beta+Q)}{(Q-1)(Q-1)!\Gamma(\alpha+\beta+Q+1)[P_{Q-1}^{\alpha,\beta}(\xi_i)]^2} \tag{2.35c}$$

where $w_i^{\alpha,\beta}, \xi_i^{\alpha,\beta}$ are the zeros and weights of the Gauss-Lobatto-Jacobi quadrature rules, and Γ refers to the Gamma function. By using these conventions, we can obtain an exact integral of a continuous function, $u(\xi)$ of polynomial P, with at least $Q \geq (P+3)/2$ quadrature points.

2.3.6 Numerical differentiation

In the same fashion as Gaussian quadrature, we want to numerical differentiate efficiently, a crucial step in the weak formulation of the Helmholtz equations. Suppose that we want to differentiate in x using local coordinates given as,

$$\frac{\mathrm{d}u^\delta(\xi)}{\mathrm{d}x} = \frac{\mathrm{d}u^\delta(\xi)}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}x} = \sum_{p=0}^{P} \hat{u}_p \frac{\mathrm{d}\phi_p(\xi)}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}x}, \tag{2.36}$$

where $\mathrm{d}\xi/\mathrm{d}x$ is simply the Jacobian and the main step in differentiation is in evaluting $\mathrm{d}\phi_p(\xi)/\mathrm{d}\xi$. Now suppose that we express the solution of polynomial order P with Lagrange polynomials, the derivative of the solution commutes,

$$\frac{\mathrm{d}u(\xi)}{\mathrm{d}\xi} = \sum_{i=0}^{Q-1} u(\xi_i)\frac{\mathrm{d}}{\mathrm{d}\xi}h_i(\xi), \tag{2.37}$$

where we only require the derivative to be evaluted at the nodal points, resulting in a derivative matrix of,

$$D_{ij} = \frac{\mathrm{d}h_j(\xi)}{\mathrm{d}\xi}\Big|_{\xi=\xi_i}, \tag{2.38}$$

and the derivative of $u(\xi)$ is simply,

$$\frac{\mathrm{d}u(\xi)}{\mathrm{d}\xi}\Big|_{\xi=\xi_i} = \sum_{j=0}^{Q-1} D_{ij}\hat{u}_j. \tag{2.39}$$

A general representation of the differential operator can be presented as

$$D_{ij} = \begin{cases} \frac{p'_Q(\xi_i)}{p'_Q(\xi_j)}\frac{1}{\xi_i-\xi_j}, & i \neq j, \\ \frac{p''_Q(\xi_i)}{2p'_Q(\xi_i)}, & i = j. \end{cases} \tag{2.40}$$

where $p'_Q(\xi), p''_Q(\xi)$ are specific restricted to the quadrature used. For the Gauss-Lobatto-Jacobi quadrature rules used here, these forms could be found in Appendix C.2 in Karniadakis and Sherwin [2005].

2.3.7 Example in 1D

We have outlined the basic formulation of spectral/*hp* element methods in $1D$ and we will describe its solution procedure, where we start from the weak-form of the Helmholtz equation and convert it into a system of linear equations, amenable to be solved with standard numerical linear algebra techniques. We describe the solution steps as follows,

1. Performing numerical differentiation and integration in the standard region

$$\lambda \underbrace{\int_{-1}^{1} v^{\delta} u^{\mathcal{H}} \, \mathrm{d}\xi}_{\mathbf{M}^e \hat{\mathbf{u}}^e} + \underbrace{\int_{-1}^{1} \frac{\partial v^{\delta}}{\partial \xi} \frac{\partial u^{\mathcal{H}}}{\partial \xi} \mathrm{d}\xi}_{\mathbf{L}^e \hat{\mathbf{u}}^e} = \underbrace{\int_{-1}^{1} v^{\delta} f \mathrm{d}\xi}_{\hat{\mathbf{f}}^e} \tag{2.41}$$

Elemental mass operator

Here we introduce the elemental mass operator given as $\mathbf{M}^e$,

$$\begin{aligned} \int_{-1}^{1} \sum_{i=0}^{P} \hat{v}_i^e \phi_i^e(\xi) \sum_{i=0}^{P} \hat{u}_i^e \phi_i^e(\xi) \, \mathrm{d}\xi &= \sum_{q=0}^{Q} \left[\sum_{i=0}^{P} \hat{v}_i^e \phi_i^e(\xi_q) \sum_{i=0}^{P} \hat{u}_i^e \phi_i^e(\xi_q) \right] w_q^e \\ &= (\hat{\mathbf{v}}^e)^T (\mathbf{B}^e)^T \mathbf{W}^e \mathbf{B}^e \hat{\mathbf{u}}^e \\ &= \hat{\mathbf{v}}^T \mathbf{M}^e \hat{\mathbf{u}}^e \end{aligned} \tag{2.42}$$

where $\mathbf{M}^e = (\mathbf{B}^e)^T \mathbf{W} \mathbf{B}^e$ refers to the elementral mass matrix, while $\mathbf{B}^e \in \mathbb{R}^{Q-1,P}$ and $\mathbf{W}^e \in \mathbb{R}^{Q-1,Q-1}$ refers to the elemental basis and weight matrices, a diagonal matrix consisting of integration weights, w_q^e, respectively,

$$\mathbf{B}^e = \begin{bmatrix} \phi_0(\xi_0) & \cdots & \phi_P(\xi_0) \\ \vdots & \ddots & \vdots \\ \phi_0(\xi_Q) & \cdots & \phi_P(\xi_Q) \end{bmatrix}, \quad \mathbf{W}^e = \begin{bmatrix} w_0^e & & 0 \\ & \ddots & \\ 0 & & w_Q^e \end{bmatrix} \tag{2.43}$$

Elemental laplacian matrices

Now we consider, convert the product of two first-derivatives in to matrix form,

$$\begin{aligned} \int_{-1}^{1} \sum_{i=0}^{P} \hat{v}_i^e \frac{\mathrm{d}\phi_i^e}{\mathrm{d}\xi} \sum_{i=0}^{P} \hat{u}_i^e \frac{\mathrm{d}\phi_i^e}{\mathrm{d}\xi} \, \mathrm{d}\xi &= \sum_{q=0}^{Q} \left[\sum_{i=0}^{P} \hat{v}_i^e D_{qi}^e \phi_i^e(\xi_q) \sum_{i=0}^{P} \hat{u}_i^e D_{qi}^e \phi_i^e(\xi_q) \right] w_q^e \\ &= \hat{\mathbf{v}}^T (\mathbf{B}^e)^T (\mathbf{D}^e)^T \mathbf{W}^e \mathbf{D}^e \mathbf{B}^e \hat{\mathbf{u}}^e \\ &= \hat{\mathbf{v}}^T \mathbf{L}^e \hat{\mathbf{u}}^e \end{aligned} \tag{2.44}$$

where $\mathbf{L}^e = (\mathbf{B}^e)^T (\mathbf{D}^e)^T \mathbf{W} \mathbf{D}^e \mathbf{B}^e$ refers to the elementral Laplacian matrix.

Forcing vector

Lastly, we consider the right-hand side,

$$\begin{aligned}\int_{-1}\sum_{i=0}^{P}\hat{v}_i^e\phi_i^e(\xi)f^e(\xi)\,\mathrm{d}\xi &= \sum_{q=0}^{P}\sum_{i=0}^{P}\hat{v}_i^e\phi_i^e(\xi_q)f^e(\xi_q)w_q^e, \\ &= \hat{\mathbf{v}}^T(\mathbf{B}^e)^T\mathbf{W}^e\mathbf{f}^e \\ &= \hat{\mathbf{v}}^T\hat{\mathbf{f}}^e,\end{aligned} \tag{2.45}$$

where $\hat{\mathbf{f}}^e$, is referred to the elemental forcing vector. As we consider all of the matrices, the Helmholtz equations in elemental form is simply solving for,

$$\left[\lambda\mathbf{M}^e+\mathbf{L}^e\right]\hat{\mathbf{u}}^e=\hat{\mathbf{f}}^e. \tag{2.46}$$

If we considered bolting the elements together and the boundary conditions,

$$\lambda\underbrace{\begin{bmatrix}\mathbf{M}^0+\mathbf{L}^0 & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \mathbf{M}^{N_{el}-1}+\mathbf{L}^{N_{el}-1}\end{bmatrix}}_{\mathbf{M}_l+\mathbf{L}_l}\underbrace{\begin{bmatrix}\hat{\mathbf{u}}^0\\ \vdots\\ \hat{\mathbf{u}}^{N_{el}-1}\end{bmatrix}}_{\hat{\mathbf{u}}_l}=\underbrace{\begin{bmatrix}\hat{\mathbf{f}}^0\\ \vdots\\ \hat{\mathbf{f}}^{N_{el}-1}\end{bmatrix}}_{\hat{\mathbf{f}}_l}+\underbrace{\begin{bmatrix}\mathbf{L}^0g_D\\ \vdots\\ \mathbf{0}\end{bmatrix}}_{\mathbf{g}_D}+\underbrace{\begin{bmatrix}\mathbf{0}\\ \vdots\\ g_N\end{bmatrix}}_{\mathbf{g}_N}, \tag{2.47}$$

where $\mathbf{M}_l, \mathbf{L}_l, \hat{\mathbf{u}}_l, \hat{\mathbf{f}}_l, \mathbf{g}_D, \mathbf{g}_N$ refers to the local mass matrix, local laplacian matrix, and vector of local expansion coefficients , Dirichlet and Neumann boundary conditions. Finally, we can assemble them using the assembly matrix,

$$\mathbf{A}^T(\lambda\mathbf{M}_l+\mathbf{L}_l)\mathbf{A}\hat{\mathbf{u}}_g=\mathbf{A}^T(\hat{\mathbf{f}}^l+\mathbf{g}_D+\mathbf{g}_N), \tag{2.48}$$

2.4 Numerical techniques for solving the Navier-Stokes equations

2.4.1 Velocity Correction Scheme

The spatial discretisation of the Helmholtz operator and its numerical solution procedure has been discussed using the spectral/*hp* element methods. Here, we describe the numerical methods that is used to solve the Navier-Stokes equations given as,

$$\frac{\partial u}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla p+\nu\nabla^2\mathbf{u}+\mathbf{f} \tag{2.49a}$$

$$\nabla\cdot\mathbf{u}=0, \tag{2.49b}$$

with boundary conditions,

$$\mathbf{u}=0,\quad \text{on } \partial\Omega. \tag{2.49c}$$

Here, the primitive variables are velocity and pressure $(\mathbf{u}, p)$ and we assumed unit density, $\rho = 1$, with the kinematic viscosity appearing as the control parameter. The time evolution of velocity is explicit expressed in equation (2.49a), but does not appear for the pressure, which is coupled to the velocity field, enforcing the incompressibility condition. Several strategies exist for addressing the coupled velocity-pressure fields by

1. Solving the coupled system such as the Uzawa algorithm,
2. Splitting methods,
3. Change of coordinates (vorticity-streamfunction).

We adopt splitting methods, which 'splits' the evolution of the Navier-Stokes equation into independent substeps. These methods, belonging to the broader family of projection methods first introduced by Teman, can be further classified into pressure-correction or velocity-correction schemes. We focus on a high-order velocity-correction scheme. We rewrite the incompressible Navier-Stokes equations in semi-discrete form with using linear and nonlinear operators as,

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{N}(\mathbf{u}) - \nabla p + \nu \mathbf{L}(\mathbf{u}), \tag{2.50a}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2.50b}$$

with boundary conditions,

$$\mathbf{u}|_{\Omega} = 0, \quad \mathbf{u}(t=0) = \mathbf{u}_0. \tag{2.50c}$$

The nonlinear, $\mathbf{N}$, linear, $\mathbf{L}$, operators are obtained from a suitable spatial-discretisation method such as the spectral/*hp* element method. The nonlinear and linear operators are defined as,

$$\mathbf{N}(\mathbf{u}) \equiv -(\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{2}\left[(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\cdot(\mathbf{u}\mathbf{u})\right], \qquad \mathbf{L}(\mathbf{u}) \equiv \nabla^2\mathbf{u}, \tag{2.51}$$

We note that the nonlinear terms are written in the skew-symmetric to minimise aliasing errors [Karniadakis et al., 1991]. To advance the velocity at time step, $\mathbf{u}^n$, to the next time step, $\mathbf{u}^{n+1}$, we integrate equation (2.50) over a time step Δt,

$$\mathbf{u}^{n+1} - \mathbf{u}^n = \underbrace{\int_{t_n}^{t_{n+1}} \mathbf{N}(\mathbf{u})\,\mathrm{d}t}_{\Delta t \sum_{q=0}^{J_e-1} \beta_q \mathbf{N}(\mathbf{u}^{n-q})} - \underbrace{\int_{t_n}^{t_{n+1}} \nabla p\,\mathrm{d}t}_{\Delta t \nabla \bar{p}^{n+1}} + \nu \underbrace{\int_{t_n}^{t_{n+1}} \mathbf{L}(\mathbf{u})\,\mathrm{d}t}_{\Delta t \sum_{q=0}^{J_i-1} \gamma_q \mathbf{L}(\mathbf{u}^{n+1-q})}. \tag{2.52}$$

The velocity correction scheme evaluates the underbraced terms in three successive independently from left to right independently, effectively 'splitting' equation (2.50) from this point onwards. The first step we perform is to extrapolate the advection velocities, by approximating the nonlinear terms using an explicit scheme such as the Adams-Bashforth family of J_e order,

$$\frac{\hat{\mathbf{u}} - \sum_{q=0}^{J_e-1} \alpha_q \mathbf{u}^{n-q}}{\Delta t} = \sum_{q=0}^{J_e-1} \beta_q \mathbf{N}(\mathbf{u}^{n-q}), \tag{2.53}$$

where $\hat{\mathbf{u}}$ is denotes the primary intermediate velocity field desired and α_e, β_e refers to the time integration coefficients for a prescribe J_e-th order, described later. After evaluting $\hat{\mathbf{u}}$, we move onto the second term in equation (2.52), which defines the pressure at time step $n+1$ as,

$$\frac{\hat{\hat{\mathbf{u}}} - \hat{\mathbf{u}}}{\Delta t} = -\nabla p^{n+1}. \tag{2.54}$$

$\hat{\hat{\mathbf{u}}}$ denotes as the secondary intermediate velocity. In this single equation, we seek to obtain two unknown solutions, $\hat{\hat{\mathbf{u}}}$ and p^{n+1}, which is ill-posed, and seek to impose certain restrictions. The splitting method assumes that the secondary intermediate velocity is divergence free, $\nabla \cdot \hat{\hat{\mathbf{u}}} = 0$, and satisfies the Dirichlet boundary conditions normal to the boundary, $\hat{\hat{\mathbf{u}}} \cdot \mathbf{n} = \mathbf{u}|_\Omega \cdot \mathbf{n}$. By considering the assumptions above and the divergence of equation (2.54), we obtain the pressure Poisson equation with the primary intermediate velocity acting as the forcing term,

$$\nabla^2 p^{n+1} = \nabla \cdot \left(\frac{\hat{\mathbf{u}}}{\Delta t} \right) \tag{2.55a}$$

and boundary conditions,

$$\frac{\partial p^{n+1}}{\partial n} = \mathbf{n} \cdot \left(\frac{\hat{\hat{\mathbf{u}}} - \hat{\mathbf{u}}}{\Delta t} \right). \tag{2.55b}$$

While the pressure boundary condition (2.55b) is straightforward to evalute, it is sensitive to large splitting errors [Karniadakis et al., 1991]. To overcome this, we consider a high-order boundary condition of pressure, obtained by taking the normal dot product of equation (2.50),

$$\frac{\partial p^{n+1}}{\partial t} = -\sum_{q=0}^{J_e - 1} \beta_q \left[\frac{1}{\Delta t} \mathbf{u}^{n-q} + \nu [\nabla \times (\nabla \times \mathbf{u}^{n-q})] + (\mathbf{u}^{n-q} \cdot \nabla) \mathbf{u}^{n-q} \right] \cdot \mathbf{n}. \tag{2.56}$$

Notably, the linear operator is expressed as $\mathbf{L}(\mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$, favouring numerical stability [**?**Karniadakis et al., 1991]. J_e is the order the explicit scheme as in equation (2.53). After solving for the pressure Poisson equation, the secondary intermediate velocity could be subsequently obtained using equation (2.54). After which, we can move onto the final substep in equation (2.52), by solving a Helmholtz equation for $\mathbf{u}^{n+1}$,

$$\frac{\gamma_0 \mathbf{u}^{n+1} - \hat{\hat{\mathbf{u}}}}{\Delta t} = \nu \sum_{q=0}^{J_i - 1} \gamma_q \mathbf{L}(\mathbf{u}^{n+1-q}), \tag{2.57}$$

where the linear terms are treated based similar to the family of Adams-Moulton implicit scheme and J_i, γ_q denotes the order of the scheme and time integration coefficients, completing the velocity correction scheme. The time integration coefficients are determined from stiffly stable schemes shown in table 2.2, an improvement from the Adams-family schemes [Karniadakis et al., 1991]. Before we do so, we have to define the test functional spaces of velocity, $\mathcal{W}$, and pressure $\mathcal{Q}$, defined as,

$$\mathcal{V} := \{ v \,|\, v \in H_0^1(\Omega),\ v|_{\partial\Omega} = 0 \} \tag{2.58a}$$

Coefficients	1^{st} order	2^{nd} order	3^{rd} order
γ_0	1	3/2	11/6
α_0	1	2	3
α_1	0	-1/2	-3/2
α_2	0	0	1/3
β_0	1	2	3
β_1	0	-1	-3
β_2	0	0	1

Table 2.2: Integration coefficient of stiffly stable schemes from Karniadakis et al. [1991].

$$\mathcal{Q} := \{q \,|\, q \in L_0^2(\Omega), \int_\Omega q \,\mathrm{d}x = 0\}. \tag{2.58b}$$

The Dirichlet boundary conditions for the test functional space, $\mathcal{V}$, is consistent with the primitive velocity, $\mathbf{u}$, while the L_0^2 denotes a zero mean instead of homogeneous Dirichlet boundary conditions. The test function space for pressure is a polynomial other lower since derivatives for pressure do not appear in the weak formulation as we shall see below. We neglect the unsteady term, leading to a steady Stokes problem, appearing as the right-hand if we consider time-advacing the solutions,

$$(\nabla \mathbf{v}, \nu \nabla \mathbf{u}) - (\nabla \cdot \mathbf{v}, p) = (\mathbf{v}, \mathbf{f}), \quad \forall \mathbf{v} \in \mathcal{V} \tag{2.59a}$$

$$(q, \nabla \cdot \mathbf{u}) = 0, \quad \forall q \in \mathcal{Q} \tag{2.59b}$$

which is a time-dependent nonlinear partial differential equation, While methods for temporal and spatial discretisation have been discussed, it is not possible to apply these techniques in a straightforward manner to the incompressible Navier-Stokes equations. This is because of the unique role of the pressure field which ensures that the time-dependent velocity field is divergence-free. However, the velocity and the pressure fields form a coupled-system through the continuity and momentum equations which requires the solution of both fields simultaneously. In general, there are 3 ways to deal with velocity-pressure coupling: (1) Coupled methods (*Uzawa* method), (2) Change of variables (streamfunction-vorticity formulation) and (3) Splitting methods which decouples velocity and pressure. The velocity correction scheme (VCS) (**?**), a type of splitting method, decouples the velocity field from the pressure field used in *nektar*++ will be discussed in this section.

2.4.2 Fourier spectral/*hp* modes

Fourier-Chebyshev-Fourier type discretisation have been recognised as preferred method for performing direct numerical simulations (DNS) of transitional or turbulent channel flows [Kim et al., 1987] owing to its efficient representation of the inhomogeneous wall-normal directions and the homogeneous streamwise and spanwise directions, using Chebyshev and Fourier expansions respectively. The Fourier spectral/*hp* element method draws on this approach, where the homogeneous directions is represented by the Fourier expansions while the spectral/*hp* elements. This approach has been

commonly referred to as the Quasi-3D or (2.5D) approach, allowing for a flexible description of the inhomogeneous directions, such as riblets [doug chu]. For example, in the turbulent channel flows with riblets, the Fourier expansions are used to represent the periodic streamwise, while the spectral/*hp* elements are used to discretise the wall-normal direction. In the analysis of three-dimensional wakes of cylinders where the Fourier expansions are treated in the spanwise directions. In this thesis, we routinely use the the Quasi-3D approach, consisting of the 2D spectral/*hp* elements with 1D Fourier expansions are used to discretise the cross stream plane and streamwise flow respectively. The velocity and pressure in the spectral/*hp* plane is described by two dimensional modified bases with Fourier expansions,

$$\begin{bmatrix} \mathbf{u}^{\delta}(x,y,z,t) \\ p^{\delta}(x,y,z,t) \end{bmatrix} = \sum_{k=0}^{N_z-1} \sum_{p=0}^{P} \sum_{q=0}^{P} \psi_p(x)\psi_q(y) e^{ik\beta z} \begin{bmatrix} \hat{\mathbf{u}}_{p,q,k}(t) \\ \hat{p}_{p,q,k}(t) \end{bmatrix} = \sum_{k=0}^{N-1} e^{ik\beta z} \begin{bmatrix} \tilde{\mathbf{u}}_k(x,y,t) \\ \tilde{p}_k(x,y,t) \end{bmatrix} \tag{2.60}$$

where $\beta = \frac{2\pi}{L_z}$ is the spanwise wavenumber, L_z the spanwise length, N_z the number of Fourier expansions. Substituting equation 2.60 into the Navier-Stokes equations and taking the Fourier transform (equivalently to the Galerkin projection with respect to Fourier expansion as a test function) yields a system of N_z decoupled equations, amenable for parallel processing,

$$\frac{\partial \tilde{\mathbf{u}}_k}{\partial t} = -\tilde{\nabla}_k \tilde{p}_k + \nu(\nabla^2_{x,y} - k^2\beta^2)\tilde{\mathbf{u}}_k - [\widehat{(\mathbf{u}\cdot\nabla)\mathbf{u}}]_k \tag{2.61a}$$

$$-k\beta\tilde{\nabla}\cdot\tilde{\mathbf{u}}_k = 0, \quad k = 0, ..., N_z - 1 \tag{2.61b}$$

where, $\tilde{\nabla}_k = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, ik\beta)$, $\nabla^2_{x,y} = (\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2})$ and $[\widehat{(\mathbf{u}\cdot\nabla)\mathbf{u}}]_k$ refers to the Fourier-transformed of the k^{th} nonlinear term.

2.4.3 Enforcing constant flow rate

Due to the enforced periodicity in the streamwise z direction via Fourier expansions, a pressure drop cannot be prescribed to drive the flow for $Re > 0$ scenarios. To sustain the flow, we use a Green's function approach [**?**] to impose a constant flow rate,

$$W_b = Q(\mathbf{u}) = \frac{1}{2L_x h}\int_{x,y} \mathbf{u}\, \mathrm{d}x\mathrm{d}z, \tag{2.62}$$

where W_b and $Q(\cdot)$ refer to the desired flow rate and flow rate operator. A correction velocity, $\mathbf{u}_{corr}$, is obtained by solving the linear Stokes equation with unit forcing once and is stored for reuse. At the end of every time-step, the final velocity field, $\mathbf{u}$, is then updated by adding this correction velocity to the homogeneous velocity obtained from the velocity correction scheme,

$$\mathbf{u} = \mathbf{u}_h + \gamma\mathbf{u}_{corr}, \tag{2.63}$$

where γ defined as,

$$\gamma = \frac{W_b - Q(\mathbf{u}_h)}{Q(\mathbf{u}_{corr})}, \tag{2.64}$$

is adjusted to satisfy the desired flow rate, W_b. The flow rate, W_b, is related to the laminar centreline velocity $W_c = 3/2W_b$, which defines the Reynolds number, $Re = W_c h/\nu$. For more details on the numerical method, the reader is referred to **?**.

2.5 Stability analysis of the Navier-Stokes equations

2.5.1 Linear Stability analysis

2.5.2 Edge state computations

Consider equation 2.65 to be a solution of a 1-dimensional Poisson equation, bounded by the domain $\Omega \in [x_a, x_b]$, Next, we consider that the expansion functions, $\phi_i(x)$, belongs to an element of a Hilbert space, with a suitable inner-product.

The mathematical framework begins by first assuming that the solution, $u(x)$, is an element of a Hilbert space, $\mathcal{H}$ with a suitable inner-product $(\cdot, \cdot)$ and norm $||\cdot||$. For SEMs belong to a general class of methods known as the method of weighted residual, a generic method for approximating a solution of a differential equation. The method of weighted residual will be described with a worked example as follows. Consider that the solution of a differential equation $u(x)$ can be represented as an infinite sum of *trial* [Karniadakis and Sherwin, 2005]. functions (also known as basis functions, expansion functions, mode shapes).

(2.65)

where $\phi_i(x)$ are the *trial* functions and $\hat{u}_i$ are the trial function coefficients to be determined. with the appropriate boundary conditions, and $\mathbb{L}$ refers to a linear differential operator. Note that equation 2.65 exactly satisfies the differential equation of **??** i.e $\mathbb{L}u(x) - f(x) = 0$. The exact solution would require a computation of infinite basis coefficients $\hat{u}$ which is practically infeasible. Therefore, an approximate solution $u^\delta(x)$ is sought after by truncating an infinite number of basis expansions to a finite number,

$$u(x) \approx u^\delta(x) = \sum_{i=0}^{K} \hat{u}_i \phi_i(x), \tag{2.66}$$

where there is a finite number of K basis expansions. The approximate solution does not satisfy **??** exactly, leading to an 'error' known as a residual,

$$R(u^\delta(x)) = \mathbb{L}u^\delta(x) - f(x) \tag{2.67}$$

The method of weighted residual is a general method that allows for various types the restriction to be implemented. The method "nullifies" the residual by equating the inner product with a *test*

function, $v_j(x)$ (also known as a weight function - hence the name 'weighted residual') to zero,

$$(v_j(x), R(u^\delta(x))) = \int_{x_a}^{x_b} v_j \, R(u^\delta(x)) \, \mathrm{d}x = 0, \qquad j = 0, ..., K. \tag{2.68}$$

Galerkin methods are commonly found in finite/spectral element solvers, used in *nektar++*. The Galerkin method belongs to a general class of weighted residual methods that assumes the *trial* functions take on the same form as the *test* functions (Table 2.1). To describe the method, a worked example is illustrated. The Galerkin method is appplied to solve the Poisson equation **??** with the following boundary conditions,

$$B^- = g^- \quad \text{at} \quad x = x_a, \qquad B^+ = g^+ \quad \text{at} \quad x = x_b \tag{2.69}$$

where B^-, B^+ are the boundary conditions which could be either Dirichlet, Neumann or Robin conditions. Equation **??** and 2.69 together forms a boundary value problem and is said to be in the *strong* [1] form. The Galerkin method assumes that the trial functions $\phi_i(x)$ satisfies equation **??** with homogeneous boundary conditions,

$$\phi_i(x_a) = \phi_i(x_b) = 0. \tag{2.70}$$

Next, the solution $u(x)$ is decomposed into a linear combination of $\tilde{u}(x)$ and $u^H(x)$,

$$u(x) = \tilde{u}(x) + u^H(x), \tag{2.71}$$

where $\tilde{u}(x)$ is any function that satisfy the boundary conditions assosciated with equation 2.69 and $u^H(x)$ is the homogeneous solution that satisfies the homogeneous boundary conditions - $B_H^-(x_a) = B_H^+(x_b) = 0$. The resulting problem for $u^H(x)$ becomes

$$\mathbb{L}u^H(x) - h(x) = 0, \qquad x_a \leq x \leq x_b, \tag{2.72}$$

where $h = f(x) - \mathbb{L}\tilde{u}(x)$. It is worth noting that the steps thus simply mathematical, and no approximation have been made. The solutions of $u(x) = \tilde{u}(x) - u^H(x)$ represented by an infinite expansions (equation 2.65) are exact. Next, the homogeneous solution $u^H(x)$ can be approximated by a finite expansion of *trial* functions,

$$u^H(x) \approx u^{H,\delta}(x) = \sum_{i=0}^{K} \hat{u}_i^{H,\delta} \phi_i(x), \tag{2.73}$$

where $\hat{u}_i^{H,\delta}$ are the coefficients to be determined. Since $\phi_i(x)$ satisfies the homogeneous boundary conditions, $\hat{u}_i^{H,\delta}$ can take on any values and $u^{H,\delta}(x)$ will still satisfy the homogeneous boundary conditions. Substituing the approximate solution of $u^{H,\delta}(x)$ into equation 2.72, and applying the

[1] *strong* loosely mean that the trial functions are required to be both C^0 and C^1 continuous

method of weighted resiudal,

$$(R(u^{H,\delta}), v_j(x)) = \int_{x_a}^{x_b} \left(\mathbb{L}u^{H,\delta}(x) - h(x)\right) v_j(x)\, \mathrm{d}x = 0, \qquad j = 0, ..., K, \tag{2.74}$$

where $v_j(x)$ is some *test* function and there are $K+1$ finite expansions. In the Galerkin method (or Bubnov-Galerkin), the weight function $v_j(x)$ takes on the same form as the trial functions $\phi_j(x)$ (Table 2.1). In other words, the differential equation is satisfied when projected on the *test/trial* functions. Substituting equation 2.73 into the residual equation 2.74 and applying $v_j(x) = \phi_j(x)$,

$$\sum_{i=0}^{K} \hat{u}_i^{H,\delta} \int_{x_a}^{x_b} \mathbb{L}\phi_i\phi_j\, \mathrm{d}x = \int_{x_a}^{x_b} \left(f(x) - \mathbb{L}\tilde{u}(x)\right) \phi_j\, \mathrm{d}x, \qquad j = 0, .., K \tag{2.75}$$

Equation 2.75 furnishes a system of $K+1$ linear equations with $K+1$ unknowns i.e $\{\hat{u}_0^{H,\delta}, ..., \hat{u}_K^{H,\delta}\}$. Applying integration by parts to equation 2.75, the equation reduces to,

$$\sum_{i=0}^{K} \hat{u}_i^{H,\delta} \left[\int_{x_a}^{x_b} \frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial x} + \lambda\phi_j\phi_i\, \mathrm{d}x\right] = -\int_{x_a}^{x_b} \frac{\partial\tilde{u}}{\partial x}\frac{\partial\phi_j}{\partial x} + (\lambda\tilde{u} + f(x))\phi_j\, \mathrm{d}x, \tag{2.76}$$

which is known as the *weak*[2] form. The boundary conditions of the *weak* form naturally appears in the right-hand side of equation 2.77, which makes it convenient to implement. Equation 2.75 can be re-written in matrix form,

$$\begin{bmatrix} \int_{x_a}^{x_b} \frac{\partial\phi_0}{\partial x}\frac{\partial\phi_0}{\partial x} + \lambda\phi_0\phi_0\, \mathrm{d}x & \dots & \int_{x_a}^{x_b} \frac{\partial\phi_0}{\partial x}\frac{\partial\phi_K}{\partial x} + \lambda\phi_0\phi_K\, \mathrm{d}x \\ \vdots & \ddots & \vdots \\ \int_{x_a}^{x_b} \frac{\partial\phi_K}{\partial x}\frac{\partial\phi_0}{\partial x} + \lambda\phi_0\phi_0\, \mathrm{d}x & \dots & \int_{x_a}^{x_b} \frac{\partial\phi_K}{\partial x}\frac{\partial\phi_K}{\partial x} + \lambda\phi_0\phi_K\, \mathrm{d}x \end{bmatrix} \begin{bmatrix} \hat{u}_0^{H,\delta} \\ \vdots \\ \hat{u}_K^{H,\delta} \end{bmatrix} = \begin{bmatrix} -\int_{x_a}^{x_b} \frac{\partial\tilde{u}}{\partial x}\frac{\partial\phi_0}{\partial x} + (\lambda\tilde{u} + f(x))\phi_0\, \mathrm{d}x \\ \vdots \\ -\int_{x_a}^{x_b} \frac{\partial\tilde{u}}{\partial x}\frac{\partial\phi_K}{\partial x} + (\lambda\tilde{u} + f(x))\phi_K\, \mathrm{d}x \end{bmatrix} \tag{2.77}$$

where $\hat{\mathbf{u}}^{H,\delta} = [\hat{u}_0^{H,\delta}, ..., \hat{u}_K^{H,\delta}]$ is determined by solving the system of linear equations.

2.6 The Spectral/hp element methods

To represent the spatially-dependent velocity and pressure fields, spatial discretisation is performed using the spectral/*hp* element method. Other popular methods of spatial discretisation found in literature are the finite-difference methods, and finite-volume methods. The spectral/hp element method (SEM) is related to the Galerkin method in which the type of *trial* function used. The spectral/*hp* element method combines 2 traditional numerical methods, namely,

1. Finite elements:

[2] *trial* functions are only required to be C^0 continuous

The finite element method decomposes the global domain into a set of non-overlapping subdomains (finite elements), represented by linear shape functions. In a 1D domain, the size of each element is given by h and the approximate solution should covergence as h is decreased - also known as *h*-refinement. The flexibility of domain decomposition allows for complex engineering geometries to be represented.

2. Spectral method:

   The spectral method performs a global discretisation of the domain. The domain is represented by a linear combination of global continuous functions, such as the Fourier series. Spectral methods benefit from the property of *spectral convergence*, where the solution error decreases by $\mathcal{O}(c^{-N})$, where c is some constant $0 \leq c \leq 1$ and N is the number of polynomials (**?**). In other words, as the number of functions is increased, the error decreases exponentially.

The Spectral/*hp* element method leverages the advantages of both methods - geometric flexibility and spectral covergence. The spectral/*hp* method uses a series of high-order polynomials (Lagrange/Legendre) within each element. Considering each element consists of $P + 1$ linearly independent polynomials (where P refers to the highest polynomial order) spanning the polynomial space of $\mathcal{P}_P$, the error of a smooth solution with mesh-size h and polynomial order P has the property of (**?**),

$$||u(x) - u^{\delta}(x)|| \leq Ch^{P}||u(x)|| \approx \mathcal{O}(h^{P}). \tag{2.78}$$

Equation 2.78 implies that the error decreases as the h is decrease (mesh refinement) or as P is increased using higher-order polynomials.

2.7 Velocity correction scheme for incompressible Navier Stokes equations

2.8 Linear Stability Analysis

2.9 Edge Tracking

To study the dynamics of infinitetisimal perturbations about a base flow, the time evolution equation for the perturbations dynamics typically reduces to,

$$\frac{\partial}{\partial t}\mathbf{u} = \mathcal{L}\mathbf{u}, \tag{2.79}$$

where $\mathcal{L}, \mathbf{u}$ refers to the linearised operator and a vector of perturbations. Suppose the that linear operator is diagonlisable,

$$\mathcal{L} = \begin{bmatrix} | & & | \\ \mathbf{s}_1 & \cdots & \mathbf{s}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} | & & | \\ \mathbf{s}_1 & \cdots & \mathbf{s}_n \\ | & & | \end{bmatrix}^{-1} = \mathcal{S}\Lambda\mathcal{S}^{-1}. \tag{2.80}$$

Suppose we can decompose our initial conditions into a superposition of eigenmodes,

$$\mathbf{u}_0 = \alpha_{1,0}\mathbf{s}_1 + \alpha_{2,0}\mathbf{s}_2 + ... + \alpha_{N,0}\mathbf{s}_n = \sum_{i=1}^{n} \alpha_{i,0}\mathbf{s}_i, \tag{2.81}$$

and w

Edge state tracking

Computing Invariant solutions

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