

# Lecture 10: Optimization

STAT GR5206 *Statistical Computing & Introduction to Data Science*

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# Course Notes

## Exam:

- Take home final exam.
- Final exam posted on 06/28/2017 (Wednesday Morning).
- Final exam due on 06/29/2017 (Thursday 11:59pm).

## Homeworks and Labs

- Lab 7 due this Friday by 11:59pm.
- Lab 8 (~~maybe?~~) **split apply combine**
- Homework 6 (optimization) due next Tuesday by 11:59pm.
- Recommended homeworks 7 (~~MOM and MLE~~) and 8 (Split/Apply/Combine). HW 7 and HW 8 are not due. Solutions will be posted.

relevant for final

# Last Time

- **Data as Models:** The Method of Moments and the MLE. Testing model fit.

## MLE Example

Let  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \exp(\lambda)$

$$f(x | \lambda) = \lambda e^{-\lambda x}, x > 0$$

① Compute MLE of  $\lambda$

$$\begin{aligned} L(\lambda | x_1, \dots, x_n) &= \prod_{i=1}^n f(x_i | \lambda) \\ &= f(x_1 | \lambda) \cdot f(x_2 | \lambda) \cdots f(x_n | \lambda) \\ &= (\lambda e^{-\lambda x_1}) (\lambda e^{-\lambda x_2}) \cdots (\lambda e^{-\lambda x_n}) \end{aligned}$$

## Section I

$$= \lambda^n e^{-\lambda \sum x_i}$$

$$\begin{aligned} l(\lambda | x_1, \dots, x_n) &= \log(L(\lambda | x_1, \dots, x_n)) \\ &= \log(\lambda^n e^{-\lambda \sum x_i}) \\ &= n \log \lambda - \lambda \sum x_i \end{aligned}$$

$$\begin{aligned} \frac{\partial L}{\partial \lambda} &= \frac{n}{\lambda} - \sum x_i \stackrel{\text{set}}{=} 0 \\ \rightarrow \frac{n}{\lambda} &= \sum x_i \rightarrow \hat{\lambda}_{MLE} = \frac{n}{\sum x_i} \end{aligned}$$

## Functions As Objects

② Compute MOM estimator of  $\lambda$

$$E x = \int_0^{\infty} x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$\text{Equate } E x = \bar{x}, \frac{1}{\lambda} = \bar{x}, \hat{\lambda}_{MM} = \frac{1}{\bar{x}}$$

\* in this case the estimators are the same  
for MOM & MLE of  $\lambda$

# Functions as Objects

- In R, functions are objects, just like everything else!
- This means that they can be passed to functions as arguments and returned by functions as outputs as well.

# Functions as Objects

- In R, functions are objects, just like everything else!
- This means that they can be passed to functions as arguments and returned by functions as outputs as well.
- We've seen examples of this: `curve()`, `nlm()`, `apply()`, etc.

AKA: *these functions also takes other functions as args! (passed as object)*

# Functions of Functions, Computationally

- We often want to do very similar things to many different functions.
- The procedure is the same, only the function we're working with changes.
- Thus, write one function to do the job, and pass the function as an argument.
- Because R treats a function like any other object, we can do this simply: invoke the function by its argument name in the body.

# Peeking at a Function's Definition

Typing a function's name, without parentheses, gives you ~~the~~ source code. ~~for it.~~

> sample *This is the built in Sample function.*

```
function (x, size, replace = FALSE, prob = NULL)
{
  if (length(x) == 1L && is.numeric(x) && x >= 1) {
    if (missing(size))
      size <- x
    sample.int(x, size, replace, prob)
  } uses this function!
  else {
    if (missing(size))
      size <- length(x)
    x[sample.int(length(x), size, replace, prob)]
  }
}
```

<bytecode: 0x10098d6a8>

<environment: namespace:base>

You  
can  
manipulate  
this!

# Peeking at a Function's Definition

This isn't always that explicit, because some functions are defined in a lower level language (like C or Fortran).

```
> log
```

```
| function (x, base = exp(1)) .Primitive("log")
```

↳ . indicates that it is using lower level language functions.

# Peeking at a Function's Definition

```
> rowSums
```

```
function (x, na.rm = FALSE, dims = 1L)
{
  if (is.data.frame(x))
    x <- as.matrix(x)
  if (!is.array(x) || length(dn <- dim(x)) < 2L)
    stop("'x' must be an array of at least two dimensions")
  if (dims < 1L || dims > length(dn) - 1L)
    stop("invalid 'dims'")
  p <- prod(dn[-(1L:dims)])
  dn <- dn[1L:dims]
  z <- if (is.complex(x))
    .Internal(rowSums(Re(x), prod(dn), p, na.rm)) + (0+1i) *
      .Internal(rowSums(Im(x), prod(dn), p, na.rm))
  else .Internal(rowSums(x, prod(dn), p, na.rm))
  if (length(dn) > 1L) {
    dim(z) <- dn
```

# The Function Class

Functions are their own **class** in R:

```
> class(sin)
```

```
| [1] "function"
```

```
> class(sample)
```

```
| [1] "function"
```

```
> resample = function(x) {  
+   return(sample(x, size=length(x), replace=TRUE))  
+ }  
> class(resample)
```

```
| [1] "function"
```

# Facts About Functions

A call to `function()` creates and returns a function object.

- Can see the body `body()`
- Can see the arguments with `args()`
- Can see the environment with `environment()`

```
> body(resample)
```

```
{  
  return(sample(x, size = length(x), replace = TRUE))  
}
```

```
> args(resample)
```

```
function (x)  
NULL
```

# Facts About Functions

```
> environment(resample)
```

```
| <environment: R_GlobalEnv>
```

Example: `grad()`

## Gradience.

- Many problems in statistics come down to optimization. (So do lots of problems in economics, physics, computer science, etc.) *→ what we want to*
- Lots of optimization routines require the gradient of the **objective function**—this is the function that is to be minimized or maximized.
- Recall, the gradient of  $f$  at  $x = (x_1, \dots, x_n)$  is just the vector that collects all the partial derivatives: *optimize.*

*This is a  
vector  
not a  
scalar!*

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{pmatrix}$$

## Example: grad()

- Note that we do basically the same thing to get the gradient of  $f$  at  $x$  no matter what  $f$  is:
  - (1) Find partial derivative of  $f$  with respect to each component of  $x$
  - (2) Return the vector of partial derivatives

### Univariate Gradient Descent Algorithm

$$x_{t+1} = x_t - \gamma \cdot f'(x_t)$$
$$f'(x) \approx \frac{f(x+h) - f(x)}{h}, \text{ small } h$$

## Example: `grad()`

- Note that we do basically the same thing to get the gradient of  $f$  at  $x$  no matter what  $f$  is:
  - (1) Find partial derivative of  $f$  with respect to each component of  $x$
  - (2) Return the vector of partial derivatives
- It makes no sense to rewrite this every time we change  $f$ !
- Hence, write code to calculate the gradient of an arbitrary function.
- We could write our own, but there are lots of tricky issues:
  - Best way to calculate partial derivative?
  - What if  $x$  is at the edge of the domain of  $f$ ?
- Fortunately, someone has already done this for us.

## Example: grad()

From the package numDeriv:

```
> library(numDeriv)
> args(grad)
```

```
function (func, x, method = "Richardson", side = NULL, method
         ...)
```

```
NULL
```

vector  
where you  
want to  
find slope

↓  
different methods



## Example: grad()

From the package numDeriv:

```
> library(numDeriv)
> args(grad)
```

```
function (func, x, method = "Richardson", side = NULL, method
         ...)
NULL
```

- func is a function which returns a single floating-point value.
- x is a vector, at which we want to evaluate the derivative of func.
- Extra arguments in ... get passed along to func.
- Other functions in the package for, e.g., the Hessian matrix (matrix of second partial derivatives)

# Simple Example

So, does it work as advertised?

```
> simpleFun = function(x) {  
+   # x is a vector of length 2  
+   return(x[1]^2 + 1/3*x[2]^2)  
+ }  
> xpt <- runif(n = 2, min = -2, max = 2)  
> grad(simpleFun, xpt)
```

```
[1] 1.300305 -1.273476
```

```
> c(2*xpt[1], 2/3*xpt[2])
```

```
[1] 1.300305 -1.273476
```



$$f(x_1, x_2) = x_1^2 + \frac{1}{3} x_2^2$$

$$\frac{\partial f}{\partial x_1} = 2x_1$$

$$\frac{\partial f}{\partial x_2} = \frac{2}{3}x_2$$

$$\nabla f = \begin{bmatrix} 2x_1 \\ \frac{2}{3}x_2 \end{bmatrix}$$

# Basic Optimization

# Examples of Optimization Problems

Many we've already seen in this class!

- Minimize mean-squared error of regression surface (Gauss, c. 1800)
- Maximize likelihood of distribution (Fisher, c. 1918)
- Fitting general models by minimizing the training mean-squared error (GMP data)

optimizing least squares



# Formal Optimization Set-up

## Domain Reals

- Given an objective function  $f : \mathcal{D} \mapsto \mathbb{R}$ , find

minimizer of  $f$

$$\theta^* = \arg \min_{\theta} f(\theta).$$

input of  $f$  that optimizes that function

- Basics: maximizing  $f$  is minimizing  $-f$ :

$\hookleftarrow$  negative

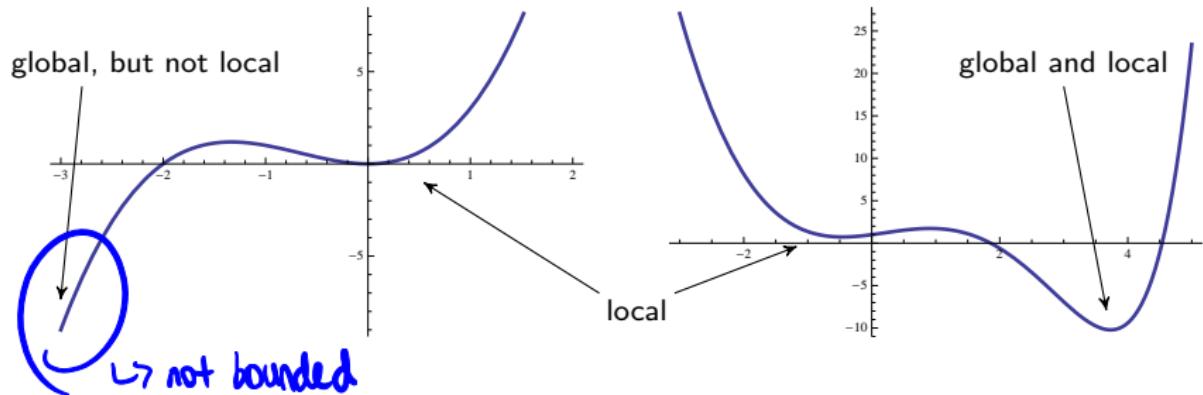
$$\arg \max_{\theta} f(\theta) = \arg \min_{\theta} (-f(\theta)).$$

- If  $h$  is strictly increasing (e.g., log), then

$\downarrow$   
going uphill

$$\arg \min_{\theta} f(\theta) = \arg \min_{\theta} h(f(\theta)).$$

# Types of Minima



## Local and global minima

A minimum of  $f$  at  $\theta^*$  is called:

- **Global** if  $f$  assumes no smaller value on its domain.
- **Local** if there is some open neighborhood  $U$  of  $\theta^*$  such that  $f(\theta^*)$  is a global minimum of  $f$  restricted to  $U$ .

↳ you need the neighbourhood

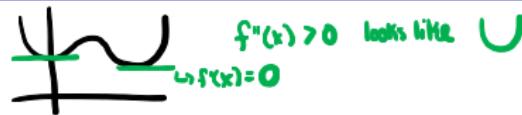
# Considerations

In general, not possible to find exact minimums (or maximums) of optimization problems, but we will learn algorithms that can achieve arbitrarily good approximations, by forming iterative guesses at the minimum (or maximum).

- **Approximation:** How close can we get to  $f(\theta^*)$ , or (better yet)  $\theta^*$ ?
- **Convergence Speed:** How many iterations does the algorithm take to get there? Varies with precision of approximation, niceness of  $f$ , size of  $\mathcal{D}$ , size of data, method...
- **Iteration Cost:** Most optimization algorithms use **successive approximation**, so must consider cost of each iteration when thinking about efficiency.

↳ every Iteration can be  
high cost.

# Optima



## Analytic criteria for local minima

Recall that one-dimensional  $x^*$  is a local minimum of smooth function  $f$  if

$$f'(x^*) = 0 \quad \text{and} \quad f''(x^*) > 0 .$$

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$$f'(x^*) = 0 \quad \text{and} \quad f''(x^*) > 0.$$

This all carries over to multiple dimensions!

For  $\theta \in \mathbb{R}^d$ ,  $f(\theta_1, \theta_2, \dots, \theta_d)$

*changed x to θ (parameters)*

$$\nabla f(\theta^*) = 0 \quad \text{and} \quad H_f(\theta^*) = \left( \frac{\partial f}{\partial \theta_i \partial \theta_j}(\theta^*) \right)_{i,j=1,\dots,d}$$

positive definite ↓

The  $d \times d$ -matrix  $H_f(\theta)$  is called the **Hessian matrix** of  $f$  at  $\theta$ . A  $d \times d$  matrix  $H$

$$\text{Ex: } \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{R}^2, I(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_1^2 + z_2^2 > 0 \quad \leftarrow$$

scalar ←

is positive definite  
if  $\underline{Z^T H Z > 0}$  for any  
 $\underline{m \times n \text{ vector } Z \in \mathbb{R}^n}$

# You Remember Multivariate Calculus, Right?

Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ ,

The **gradient** of  $f$ , denoted  $\nabla f$ , is a vector of length  $n$  with each element equal to the partial derivative of  $f$ . Meaning,

$$\nabla f = \left( \frac{\partial f}{\partial \theta_1}, \frac{\partial f}{\partial \theta_2}, \dots, \frac{\partial f}{\partial \theta_d} \right).$$

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The **Hessian matrix** of  $f$  is a square matrix of the second order partial derivatives. Meaning, for  $i, j = 1, 2, \dots, d$ ,

$$[H_f]_{i,j} = \frac{\partial^2 f}{\partial \theta_i \partial \theta_j}$$

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The Hessian matrix is **positive definite** if  $z^T H_f z > 0$  for all non-zero vectors  $z \in \mathbb{R}^n$ . (Positive semidefinite if  $z^T H_f z \geq 0$ .)  $H_f(\theta^*)$  positive definite roughly means that the function  $f$  is “curved upwards” at  $\theta^*$ .

# You Remember Multivariate Calculus, Right?

## Example

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the function  $f(\theta_1, \theta_2) = \theta_1^2 \theta_2$ ,

- Find the gradient of  $f$  and find  $\nabla f(2, 3)$ .
- Find the Hessian matrix of  $f$  and also the Hessian matrix evaluated at point  $(2, 3)$ .

Hessian Matrix :

$$f(\theta_1, \theta_2) = \theta_1^2 \theta_2$$
$$\Rightarrow \frac{\partial^2 f}{\partial \theta_1^2} = 2\theta_2 \quad \frac{\partial^2 f}{\partial \theta_1 \partial \theta_2} = 2\theta_1$$
$$\text{Hess}(f)(2, 3) = \begin{bmatrix} 2(3) & 2(2) \\ 2(2) & 0 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 4 & 0 \end{bmatrix}$$
$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial \theta_1} \\ \frac{\partial f}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} 2\theta_1\theta_2 \\ \theta_1^2 \end{bmatrix}$$
$$\nabla f(2, 3) = \begin{bmatrix} 2(2)(3) \\ 2^2 \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \end{bmatrix}$$
$$\frac{\partial f}{\partial \theta_2} = 2\theta_1$$
$$\frac{\partial f}{\partial \theta_1} = 2\theta_2$$

# You Remember Multivariate Calculus, Right?

## Example

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- Find the Hessian matrix of  $f$  and also the Hessian matrix evaluated at point  $(2, 3)$ .

## Gradient

$$\frac{\partial f}{\partial \theta_1} = 2\theta_1\theta_2 \text{ and } \frac{\partial f}{\partial \theta_2} = \theta_1^2,$$

so

$$\nabla f = (2\theta_1\theta_2, \theta_1^2),$$

and

$$\nabla f(2, 3) = (12, 4).$$

# You Remember Multivariate Calculus, Right?

## Example

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the function  $f(\theta_1, \theta_2) = \theta_1^2\theta_2$ ,

- Find the gradient of  $f$  and find  $\nabla f(2, 3)$ .
- Find the Hessian matrix of  $f$  and also the Hessian matrix evaluated at point  $(2, 3)$ .

## Hessian

$$\frac{\partial f}{\partial \theta_1} = 2\theta_1\theta_2 \quad \text{and} \quad \frac{\partial f}{\partial \theta_2} = \theta_1^2,$$
$$\frac{\partial f}{\partial \theta_1^2} = 2\theta_2, \quad \frac{\partial f}{\partial \theta_1 \partial \theta_2} = 2\theta_1, \quad \frac{\partial f}{\partial \theta_2 \partial \theta_1} = 2\theta_1, \quad \text{and} \quad \frac{\partial f}{\partial \theta_2^2} = 0,$$

so

$$H_f = \begin{Bmatrix} 2\theta_2 & 2\theta_1 \\ 2\theta_1 & 0 \end{Bmatrix}, \quad \text{and} \quad H_f(2, 3) = \begin{Bmatrix} 6 & 4 \\ 4 & 0 \end{Bmatrix},$$

↗ dictates step size for GD.

## Numerical methods

All numerical minimization methods perform roughly the same steps:

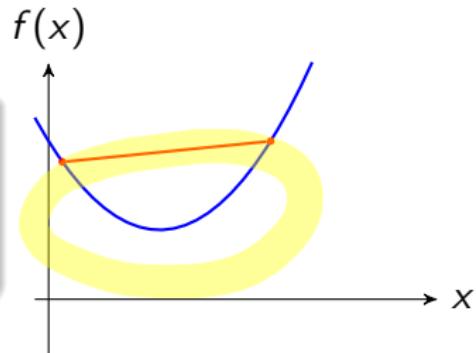
- Start with some point  $x_0$ .
- Our goal is to find a sequence  $x_0, \dots, x_m$  such that  $f(x_m)$  is a minimum.
- At a given point  $x_n$ , compute properties of  $f$  (such as  $f'(x_n)$  and  $f''(x_n)$ ).
- Based on these values, choose the next point  $x_{n+1}$ .

The information  $f'(x_n)$ ,  $f''(x_n)$  etc is always *local at  $x_n$* , and we can only decide whether a point is a local minimum, not whether it is global.

# Convex Functions

## Definition

A function  $f$  is **convex** if every line segment between function values lies above the graph of  $f$ .



concave is opposite

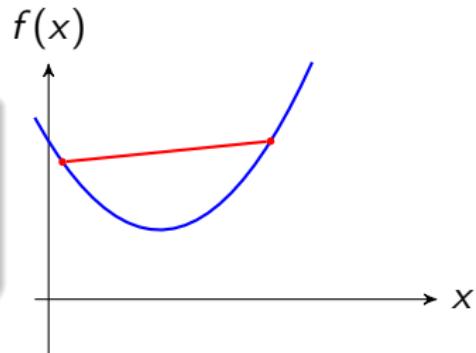


It looks like a cave!

# Convex Functions

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## Analytic criterion

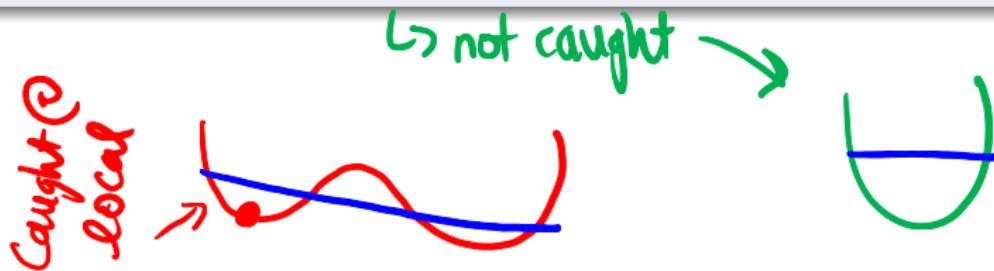
A twice differentiable function is convex if  $f''(x) \geq 0$  (or  $H_f(\theta)$  positive semidefinite) for all  $x$ .

# Convex Functions

## Implications for optimization

If  $f$  is convex, then:

- $f'(x) = 0$  is a sufficient criterion for a minimum.
- Local minima are global.
- If  $f$  is **strictly convex** ( $f'' > 0$  or  $H_f$  positive definite), there is only one minimum (which is both global and local).



# Gradients Tell Us How $f$ Changes

Recall

$$f'(x_0) = \frac{\partial f}{\partial x} \Big|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

# Gradients Tell Us How $f$ Changes

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Therefore,

$$f(x) \approx f(x_0) + (x - x_0)f'(x_0).$$

Locally, the function looks linear!

To minimize a linear function move down the slope.

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Multivariate Version

planes

$$f(\theta) \approx f(\theta_0) + (\theta - \theta_0) \cdot \nabla f(\theta_0).$$

$\nabla f(\theta_0)$  points in the direction of fastest *ascent* at  $\theta_0$ .

# Gradient Descent

## Algorithm

Gradient descent searches for a minimum of  $f$ .

1. Start with an initial guess for  $\theta$  and a step size  $\eta > 0$ .
2. While ((not too tired) and (making adequate progress))
  - Find gradient  $\nabla f(\theta)$ .
  - Set  $\theta \leftarrow \theta - \eta \nabla f(\theta)$ .
3. Return final  $\theta$  as approximation of  $\theta^*$ .

# Gradient Descent

## Algorithm

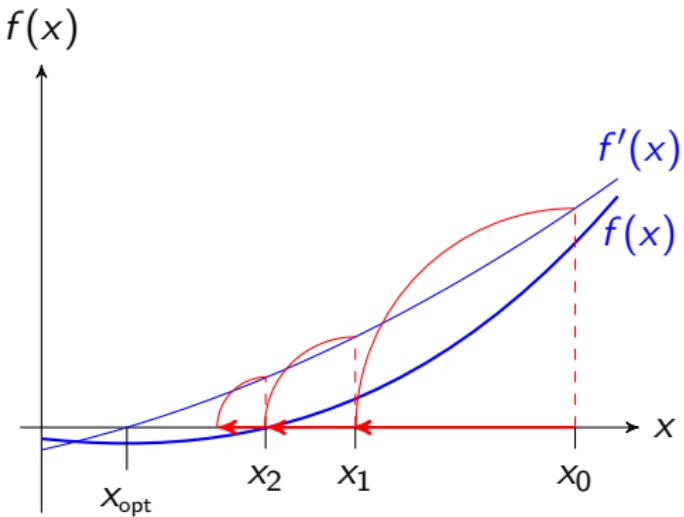
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3. Return final  $\theta$  as approximation of  $\theta^*$ .

## Variations

Adaptively adjust  $\eta$  to ensure improvement or search along the gradient direction for minimum.

# Gradient Descent



# Gradient Descent

## Pros and Cons

Pro:

- Moves in direction of greatest immediate improvement.
- If  $\eta$  is small enough, gets to a local minimum eventually, and then stops.

Con:

- ‘Small enough’  $\eta$  could be *very* small.
- Slowness or zig-zagging if components of  $\nabla f$  are very different sizes.

# Gradient Descent

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**How much work does gradient descent require to find the minimum?**

# Scaling

## Big-O Notation

$$h(x) = O(g(x))$$

means

$$\lim_{x \rightarrow \infty} \frac{h(x)}{g(x)} = c,$$

for some  $c \neq 0$ .

# Scaling

## Big-O Notation

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$$\lim_{x \rightarrow \infty} \frac{h(x)}{g(x)} = c,$$

for some  $c \neq 0$ .

### Examples

- $5x^2 - 200x + 36 = O(x^2)$ .
- $\frac{e^x}{1-e^x} = O(1)$ .

*driving term*

We often talk about scaling in this way: useful to look at big picture and hide the details.

# Gradient Descent

## How Much Work is Gradient Descent?

Pro:

- **Number of Iterations:** For nice  $f$ ,  $f(\theta) \leq f(\theta^*) + \epsilon$  in  $O(\epsilon^{-2})$  iterations.
  - A little faster –  $O(\log \epsilon^{-1})$  – for *very* nice  $f$ .
- **Cost of Each Iteration:** For  $\theta \in \mathbb{R}^d$ , to get  $\nabla f(\theta)$ , must take  $d$  derivatives. Cost is  $O(d)$ .

Con:

- Taking derivative can slow down as data grows – each iteration might really be  $O(nd)$ .

# Gradient Descent

With our knowledge of `grad()`, we can now write a very useful function for performing gradient descent.

## Code Example

# Gradient Descent

Let's try it out on our simple example!

```
> x0 <- c(-1.9, -1.9)
> gd <- grad.descent(simpleFun, x0)
> gd$x
```

[1] -5.437919e-07 -1.490486e-02      *Optimal values of x*

```
> gd$k
```

[1] 144      *iterations to achieve results*

Note: the minimum here is achieved at  $(0,0)$ , so this is right

# Gradient Descent

## Example: Linear Regression

Let's set up a linear regression simulation:

```
> n <- 100
> p <- 2
> pred <- matrix(rnorm(n*p), n, p)
> beta <- c(1, 4)
> resp <- pred %*% beta + rnorm(n)
> lm.coefs <- coef(lm(resp ~ pred + 0))
> lm.coefs
```

pred1	pred2
0.8931815	3.9234320

# Gradient Descent

## Example: Linear Regression

Let's now try out gradient descent:

```
MSE <- function(beta) {  
  return(sum((resp - pred %*% beta)^2))  
}  
grad.descent(MSE, x0 = c(0,0), step.size = 0.05,  
             max.iter = 200)
```

```
Error in grad.default(f, xmat[, k - 1], ...) :  
  function returns NA at 1.20302765942042e+150  
  7.12716833990127e+148 distance from x.
```

# Gradient Descent

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```
Error in grad.default(f, xmat[, k - 1], ...) :  
  function returns NA at 1.20302765942042e+150  
  7.12716833990127e+148 distance from x.
```

## What happened??

You should practice your debugging skills to confirm this, but the step size is simply too large, and so gradient descent is not converging.

# Gradient Descent

## Example: Linear Regression

A simple fix is just to take a smaller step size

```
> out = grad.descent(MSE, x0 = c(0,0), step.size = 1e-3,  
+                      max.iter = 200)  
> out$k
```

```
| [1] 62
```

```
> out$x
```

```
| [1] 0.8931187 3.9233875
```

```
> lm.coefs
```

	pred1	pred2
0.8931815	3.9234320	

# Gradient Descent

## Up Next

We perhaps don't want to fiddle around with the step size manually.

- What's the problem with this?
- What's the problem with taking it just to be super tiny, so that we always converge?

Now we'll learn a more principled strategy

# Taylor Series

The Taylor series gives us a way to estimate the value of a function  $f$  near  $x_0$  if we know the derivatives of  $f$  at  $x_0$ .

A one-dimensional **Taylor series** around a point  $x = x_0$  is given by

$$\begin{aligned}f(x) &= f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2!}(x - x_0)^2f''(x_0) + \frac{1}{3!}(x - x_0)^3f'''(x_0) \\&\quad + \dots + \frac{1}{n!}(x - x_0)^nf^{(n)}(x_0) + \dots\end{aligned}$$

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**Taylor's theorem** states that any function satisfying certain conditions can be expressed as a Taylor series.

# Taylor Series

The degree- $n$  **Taylor polynomial** approximation to  $f$  at point  $x = x_0$  is given by

$$\begin{aligned}f(x) \approx f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2!}(x - x_0)^2f''(x_0) \\+ \dots + \frac{1}{n!}(x - x_0)^nf^{(n)}(x_0).\end{aligned}$$

# Taylor Series

Let's consider a quadratic approximation to  $f$ :

$$f(x) \approx f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0).$$

# Taylor Series

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$$f(x) \approx f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2f''(x_0).$$

Assume  $x_0$  is a minimum, then  $f'(x_0) = 0$  and the above simplifies to

$$f(x) \approx f(x_0) + \frac{1}{2}(x - x_0)^2f''(x_0).$$

**Near a minimum, a smooth function looks like a parabola!**

# Taylor Series

The same is true in the multivariate case: for a minimizing value  $\theta_0$ ,

$$\begin{aligned}f(\theta) &\approx f(\theta_0) + (\theta - \theta_0) \nabla f(\theta_0) + \frac{1}{2}(\theta - \theta_0)^T H_f(\theta_0)(\theta - \theta_0) \\&= f(\theta_0) + \frac{1}{2}(\theta - \theta_0)^T H_f(\theta_0)(\theta - \theta_0).\end{aligned}$$

**Near a minimum, a smooth function looks like a parabola!**

# Minimizing a Quadratic Function

If we know,

$$f(x) = ax^2 + bx + c,$$

we can minimize exactly:

$$f'(x) = 2ax^* + b = 0$$

$$x^* = -\frac{b}{2a}.$$

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If we know,

$$f(x) = \frac{1}{2}a(x - x_0)^2 + b(x - x_0) + c,$$

then

$$f'(x) = a(x^* - x_0) + b = 0$$

$$x^* = x_0 - a^{-1}b.$$

# Newton's Method

Let  $\theta_0$  be our current guess at a minimizing value. We find a quadratic approximation of  $f$  at  $\theta_0$ :

$$f(\theta) \approx f(\theta_0) + (\theta - \theta_0)\nabla f(\theta_0) + \frac{1}{2}(\theta - \theta_0)^T H_f(\theta_0)(\theta - \theta_0).$$

Then we update our guess by minimizing the above with

$$\nabla f(\theta) = \nabla f(\theta_0) + H_f(\theta_0)(\theta^* - \theta_0) = 0$$

$$\theta^* = \theta_0 - (H_f(\theta_0))^{-1}\nabla f(\theta_0).$$

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- If  $f$  is exactly quadratic (and  $H_f(\theta)^{-1}$  exists), this finds the minimizer exactly.
- If  $f$  isn't quadratic, keep pretending it is until we get close to  $\theta^*$ , when it's nearly true.

# Newton's Method

## Algorithm

Gradient descent searches for a minimum of  $f$ .

1. Start with an initial guess for  $\theta$ .
2. While ((not too tired) and (making adequate progress))
  - Find gradient  $\nabla f(\theta)$  and Hessian  $H_f(\theta)$ .
  - Set  $\theta \leftarrow \theta - H_f(\theta)^{-1} \nabla f(\theta)$ .
3. Return final  $\theta$  as approximation of  $\theta^*$ .

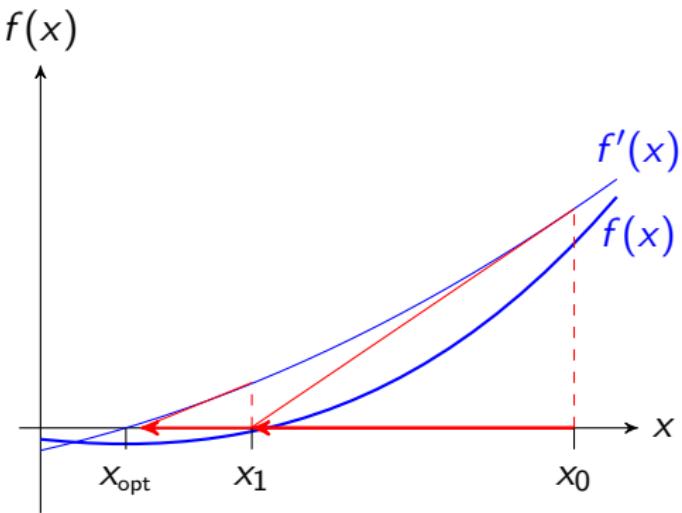
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  3. Return final  $\theta$  as approximation of  $\theta^*$ .
- 
- Like gradient descent, but with inverse Hessian as the step size.
  - Inverse Hessian tells you how far you can go in the current gradient direction.

# Newton's Method



# Newton's Method: Multiple Dimensions

## Newton's Method for Minima

$$\text{Update: } \theta_1 \leftarrow \theta_0 - H_f(\theta_0)^{-1} \nabla f(\theta_0).$$

- The inverse of  $H_f(\theta)$  exists only if the matrix is positive definite (not if it is only semidefinite), i.e.  $f$  has to be strictly convex.
- The Hessian measures the curvature of  $f$ .

# Newton's Method: Multiple Dimensions

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- The Hessian measures the curvature of  $f$ .

## Effect of the Hessian

Multiplication by  $H_f^{-1}$  in general changes the direction of  $\nabla f(\theta_0)$ . The correction takes into account how  $\nabla f(\theta)$  changes away from  $\theta_0$ , as estimated using the Hessian at  $\theta_0$ .

# Newton's Method

## Pros and Cons

Pro:

- Step-size chosen adaptively through second derivatives, much harder to get zig-zagging, over-shooting, etc.
- Always converges if  $f''(x) > 0$  (or  $H_f$  positive definite).
- Also guaranteed to need  $O(\epsilon^{-2})$  steps to get within  $\epsilon$  of the optimum.
- Only  $O(\log \log \epsilon^{-1})$  steps for very nice functions.
- Typically many fewer steps than gradient descent.

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- Typically many fewer steps than gradient descent.

Con:

- Hopeless if  $H_f$  doesn't exist or isn't invertible.
- Need to take  $O(d^2)$  second derivatives plus  $d$  first derivatives.
- Inverting  $H_f$  is  $O(d^3)$ , which becomes unwieldy in high dimensions.

# Getting Around the Hessian

Want to use Hessian to improve convergence, but don't want to keep computing it at each step.

## Approaches

- Use knowledge of the system to get approximations of the Hessian, instead of taking derivatives. ("Fisher scoring").
- Use only diagonal entries ( $d$  unmixed second derivatives).
- Use  $H_f(\theta_0)$  at initial guess  $\theta_0$  and hope  $H_f$  doesn't change that much with  $\theta$ .
- Recompute  $H_f(\theta)$  only every  $k$  steps for  $k > 1$ .
- Fast, approximate updates to the Hessian at each step (BFGS).

# Other Methods

There are lots...

For example, “Nedler-Mead” (a.k.a. the simplex method) or coordinate descent.

# Curve Fitting

# Curve Fitting by Optimizing

## Set-up

- We have data  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ .
- We also have possible curves, which we label  $r(x; \theta)$ . For example,
  - $r(x) = \theta \cdot x$
  - $r(x) = \theta_0 x^{\theta_1}$
  - $r(x) = \sum_{j=1}^q \theta_j b_j(x)$  for fixed basis functions  $b_j$ .

# Curve Fitting by Optimizing

Least squares curve fitting:

$$\hat{\theta} = \arg \min_{\theta} \frac{1}{n} \sum_{i=1}^n (y_i - r(x_i; \theta))^2.$$

'Robust' curve fitting:

$$\hat{\theta} = \arg \min_{\theta} \frac{1}{n} \sum_{i=1}^n \psi(y_i - r(x_i; \theta)),$$

where  $\psi$  is a 'robust' loss function.

# Optimization in R: optim()

```
optim(par, fn, gr, method, control, hessian)
```

- **par**: Initial parameter guess; mandatory.
- **fn**: Function to be minimized; mandatory.
- **gr**: Gradient function; only needed for some methods.
- **method**: Defaults to a gradient-free method ('Nedler-Mead'), could be BFGS (Newton-ish).
- **control**: Optional list of control settings:
  - maximum iterations, step size, tolerance for convergence, etc.
- **hessian**: Should the final Hessian be returned? Default is FALSE.

Returns the location (`$par`) and the value (`$val`) of the optimum, diagnostics, possibly `$hessian`.

# Optimization in R: optim()

## GMP Example

```
> gmp      <- read.table("gmp.txt", header = TRUE)
> gmp$pop <- gmp$gmp/gmp$pcgmp
> # install.packages("numDeriv")
> library(numDeriv)
> mse <- function(theta) {
+   mean((gmp$pcgmp - theta[1]*gmp$pop^theta[2])^2)
+ }
> grad.mse <- function(theta) {grad(func = mse, x = theta)}
> theta0    <- c(5000, 0.15)
> fit1      <- optim(theta0, mse, grad.mse, method = "BFGS",
+                      hessian = TRUE)
```

# Optimization in R: optim()

## GMP Example

```
> fit1[1:3]
```

```
$par  
[1] 6493.2563739      0.1276921  
  
$value  
[1] 61853983  
  
$counts  
function gradient  
       63          11
```

# Optimization in R: optim()

## GMP Example

```
> fit1[4:6]
```

```
$convergence  
[1] 0
```

```
$message  
NULL
```

```
$hessian  
[,1] [,2]  
[1,] 52.5021 4422071  
[2,] 4422071.3594 375729087979
```

# Optimization in R: `nls()`

- `optim()` is a general-purpose optimizer.
- So is `nls()` – try them both if one doesn't work!
- `nls()` is for non-linear least squares.
- The default optimization method for `nls()` is a version of Newton's method.

## Optimization in R: nls()

```
nls(formula, data, start, control, [lot of others...])
```

- **formula:** Mathematical expression with response variables, predictor variable(s), and unknown parameter(s).
- **data:** Data frame with variable names matching formula.
- **start:** Initial guess at parameters (optional).
- **control:** Like with optim() (optional).

Returns an nls object with fitted values, prediction methods, etc.

# Optimization in R: nls()

## GMP Example

```
> fit2 <- nls(pcgmp ~ theta0*pop^theta1, data = gmp,  
+               start = list(theta0 = 5000, theta1 = 0.10))
```

# Optimization in R: nls()

## GMP Example

```
> summary(fit2)
```

Formula: pcgmp ~ theta0 \* pop^theta1

Parameters:

	Estimate	Std. Error	t value	Pr(> t )
theta0	6.494e+03	8.565e+02	7.582	2.87e-13 ***
theta1	1.277e-01	1.012e-02	12.612	< 2e-16 ***

---

Signif. codes: 0 `\*\*\*' 0.001 `\*\*' 0.01 `\*' 0.05 `.' 0.1 ` ' 1

Residual standard error: 7886 on 364 degrees of freedom

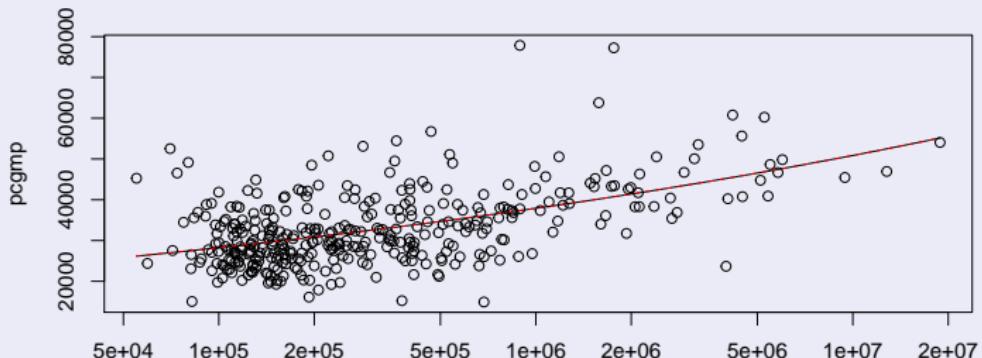
Number of iterations to convergence: 5

Achieved convergence tolerance: 1.819e-07

# Optimization in R: nls()

## GMP Example

```
> plot(pcgmp ~ pop, data = gmp, log = 'x')
> pop.order <- order(gmp$pop)
> lines(gmp$pop[pop.order], fitted(fit2)[pop.order])
> curve(fit1$par[1]*x^fit1$par[2], add = TRUE,
+        lty = "dashed", col = "red")
```



# Summary

## Summary

1. Trade-offs: complexity of iteration vs. number of iterations vs. precision of approximation.
  - **Gradient descent:** less complex iterations, more guarantees, less adaptive.
  - **Newton's method:** more complex iterations, but few of them for good functions, more adaptive, less robust.
2. Start with pre-built code like `optim()` and `nls()`, implement your own as needed.

# Constrained Optimization

# Up Next: Constrained Optimization

## Set-up

A **constrained** optimization problem adds an additional requirement on  $\theta$ .

- Given an **objective function**  $f : \mathcal{D} \mapsto \mathbb{R}$ , find

$$\theta^* = \arg \min_{\theta} f(\theta),$$

subject to  $\theta \in \mathcal{G}$ ,

where  $\mathcal{G} \subset \mathcal{D}$  is called the **feasible set**.

- The set  $\mathcal{G}$  is often defined by equations, e.g.

$$\theta^* = \arg \min_{\theta} f(\theta),$$

subject to  $g(\theta) \geq 0$ .

The equation  $g$  is called a **constraint**.

# Next: Constrained Optimization

## So far

- If  $f$  is differentiable, we can search for local minima using gradient descent.
- If  $f$  is sufficiently nice (convex and twice differentiable), we know how to speed up the search process using Newton's method.

## Constrained problems

- Numerical minimizers (like those discussed) use the criterion  $\nabla f(\theta) = 0$  for the minimum.
- In a constrained problem, the minimum is *not* identified by this criterion.

# Next: Constrained Optimization

## Constrained problems

- Numerical minimizers (like those discussed) use the criterion  $\nabla f(\theta) = 0$  for the minimum.
- In a constrained problem, the minimum is *not* identified by this criterion.

## Next steps

We will figure out how the constrained minimum can be identified. We have to distinguish two cases:

- Problems involving only equalities as constraints, i.e. subject to  $g(\theta) = 0$  (sometimes easy).
- Problems also involving inequalities, i.e. subject to  $g(\theta) \geq 0$  (a bit more complex).

## Section IV

# Example: Maximum Likelihood of a Multinomial Distribution

## Example: MLE for the Multinomial

I roll a single six-sided die  $n$  times with  $n_1, n_2, n_3, n_4, n_5$ , and  $n_6$  counting the outcomes.

### Likelihood and Log-likelihood

$$L(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) = \frac{n!}{n_1!n_2!n_3!n_4!n_5!n_6!} \prod_{i=1}^6 \theta_i^{n_i}$$

$$\ell(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) = \log \frac{n!}{n_1!n_2!n_3!n_4!n_5!n_6!} + \sum_{i=1}^6 n_i \log(\theta_i).$$

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So let's find the optimizing  $\theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$  by setting the gradient (of the log-likelihood) equal to 0: for  $i = 1, 2, 3, 4, 5, 6$ ,

$$\frac{\partial \ell}{\partial \theta_i} = \frac{n_i}{\theta_i} = 0, \quad \therefore \theta_i = \infty.$$

## Example: MLE for the Multinomial

We forgot that  $\sum_{i=1}^6 \theta_i = 1!$

Use the above constraint to eliminate one of the variables:

$\theta_6 = 1 - \sum_{i=1}^5 \theta_i$ . Rewrite the log-likelihood:

$$\begin{aligned}\ell(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) &= \log \frac{n!}{n_1! n_2! n_3! n_4! n_5! n_6!} \\ &\quad + \sum_{i=1}^5 n_i \log(\theta_i) + n_6 \log(1 - \sum_{j=1}^5 \theta_j).\end{aligned}$$

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To find a minimizer, solve the following equations for  $i = 1, 2, 3, 4, 5$ :

$$\frac{\partial \ell}{\partial \theta_i} = \frac{n_i}{\theta_i} - \frac{n_6}{1 - \sum_{j=1}^5 \theta_j} = 0.$$

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Usually eliminating a variable with a constraint doesn't work out so nicely!

# Lagrange Multipliers

Constraint:  $g(\theta) = c \implies g(\theta) - c = 0.$

## Define a Lagrangian

$$\mathcal{L}(\theta, \lambda) = f(\theta) - \lambda(g(\theta) - c).$$

Note the above equals  $f$  when the constraint is satisfied.

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Note the above equals  $f$  when the constraint is satisfied.

Now do *unconstrained* optimization over  $\theta$  and  $\lambda$ :

$$\nabla_{\theta}\mathcal{L}|_{\theta^*, \lambda^*} = \nabla f(\theta^*) - \lambda^* \nabla g(\theta^*) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda}|_{\theta^*, \lambda^*} = g(\theta^*) - c = 0.$$

Optimizing the **Lagrange multiplier**  $\mathcal{L}$  enforces the constraint, but more constraints mean more multipliers.

## Example: MLE for the Multinomial

We try the dice example again.

$$\mathcal{L}(\theta, \lambda) = \log \frac{n!}{\prod_{i=1}^6 n_i!} + \sum_{i=1}^6 n_i \log(\theta_i) - \lambda \left( \sum_{i=1}^6 \theta_i - 1 \right)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_i}|_{\theta^*, \lambda^*} = \frac{n_i}{\theta_i^*} - \lambda^* = 0 \implies \frac{n_i}{\lambda^*} = \theta_i^*.$$

Now we enforce the constraint:

$$1 = \sum_{i=1}^6 \theta_i^* = \sum_{i=1}^6 \frac{n_i}{\lambda^*} \implies \lambda^* = \sum_{i=1}^6 n_i.$$

Therefore,

$$\theta_i^* = \frac{n_i}{\lambda^*} = \frac{n_i}{\sum_{i=1}^6 n_i}.$$

# Inequality Constraints

Constraint:

$$h(\theta) \leq d \implies h(\theta) - d \leq 0.$$

The region where the constraint is satisfied is the **feasible set**.

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Roughly two cases:

1. Unconstrained optimum is inside the feasible set  $\implies$  constraint is **inactive**.
2. Optimum is outside the feasible set; constraint is **active** or **binding**.  
The *constrained* optimum is usually on the boundary.

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The *constrained* optimum is usually on the boundary.

Strategy

Add a Lagrange multiplier and then  $\lambda \neq 0 \implies$  constraint binds.

# Optimization Under Constraints

## Objective

Want to recover minimizing  $\theta^*$ :

$$\theta^* = \arg \min_{\theta} f(\theta),$$

subject to  $g(\theta) = 0$ .

## Idea

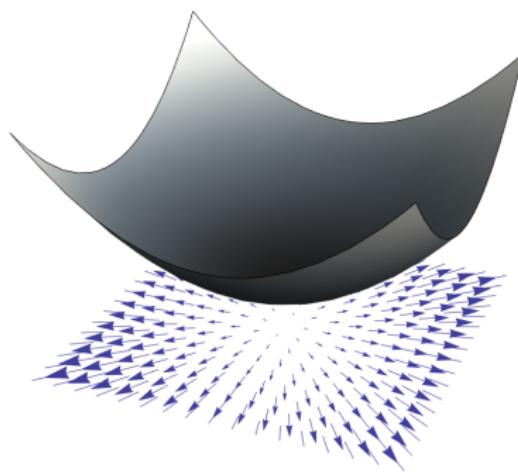
- The **feasible set** is the set of points  $\theta$  which satisfy  $g(\theta) = 0$ ,

$$G := \{\theta \mid g(\theta) = 0\}.$$

If  $g$  is reasonably smooth,  $G$  is a smooth surface in  $\mathbb{R}^d$ .

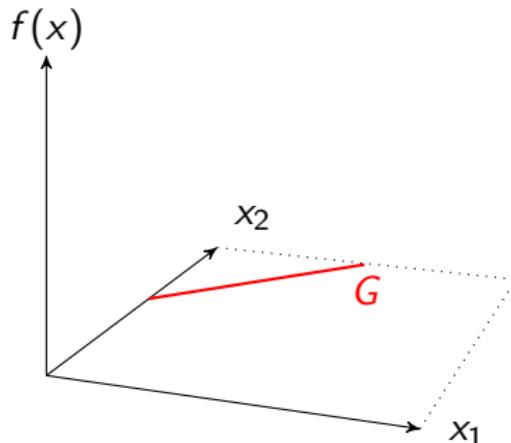
- Restrict  $f$  to this surface and call the restricted function  $f_g$ .
- Constrained optimization:** looking for the minimum of  $f_g$ .

# Optimization Under Constraints



$$f(x) = x_1^2 + x_2^2$$

The blue arrows are the gradients  $\nabla f(x)$  at various values of  $x$ .

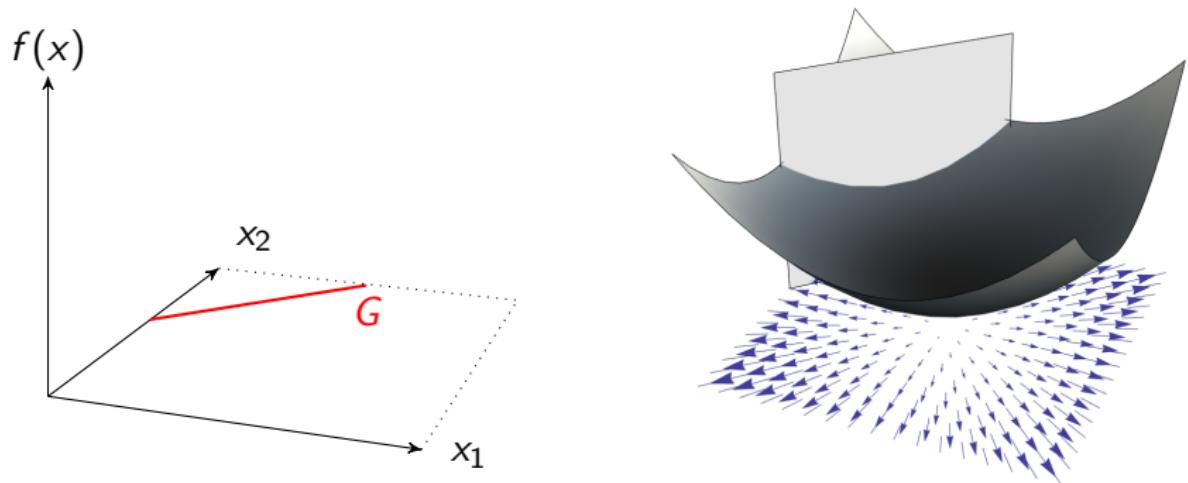


Constraint  $g$ .

Here,  $g$  is linear, so the graph of  $g$  is a (sloped) affine plane. The intersection of the plane with the  $x_1$ - $x_2$ -plane is the set  $G$  of all points  $x$  with

$$g(x) = 0.$$

# Optimization Under Constraints



- Make the function  $f_g$  given by the constraint  $g(x) = 0$  visible by placing a plane vertically through  $G$ . The graph of  $f_g$  is the intersection of the graph of  $f$  with the plane.
- Here,  $f_g$  has parabolic shape.
- The gradient of  $f$  at the minimum of  $f_g$  is *not* 0.

# Mathematical Programming

Problems of this sort have been studied for a long time.

## Linear Programming

Optimize  $f(\theta)$  subject to  $g(\theta) = c$  and  $h(\theta) \leq d$ .

- Both  $g, h$  linear in  $\theta$ .
- Here  $\theta^*$  always at a corner of the feasible set.

# An Example

## Linear Programming

A factory makes cars and trucks, using labor and steel.

- A car takes 40 hours of labor and 1 ton of steel,
- A truck takes 60 hours and 3 tons of steel,
- Total resources: 1600 hours of labor and 70 tons of steel each week.

Total revenue is 13K per car and 27K per truck.

# An Example

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Total revenue is 13K per car and 27K per truck.

## Linear Programming

Let  $c$  be the number of cars and  $t$  the number of trucks. Maximize revenue:

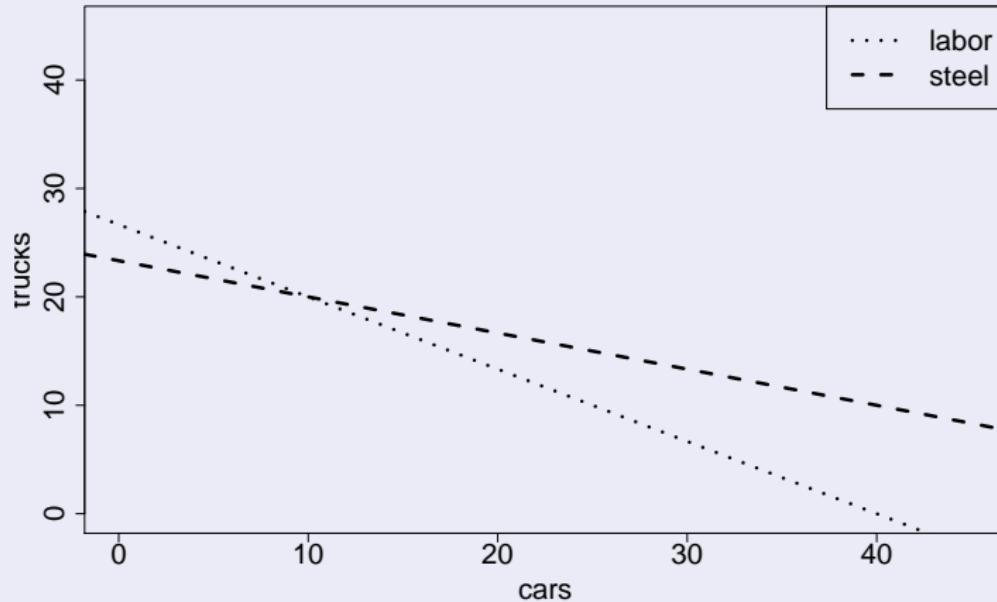
$$f(c, t) = 13c + 27t,$$

subject to

- $40c + 60t < 1600$ .
- $c + 3t < 70$ .

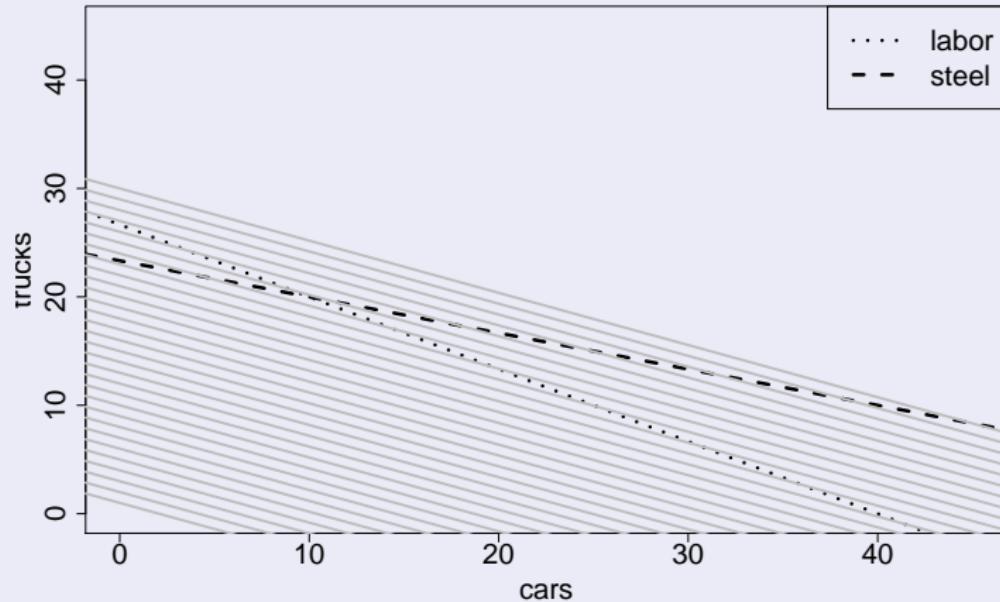
# An Example

## The feasible region



# An Example

The feasible region, plus lines of equal profit



## Maybe a More Interesting Example

- **Given:** Expected returns  $r_1, \dots, r_p$  among  $p$  financial assets, their  $p \times p$  matrix of variances and covariances  $\Sigma$ .
- **Find:** The portfolio shares  $\theta_1, \dots, \theta_p$  which maximize expected returns.
- **Such that:** total variance is below some limit, covariance with other stocks or portfolios is under some limit.

- Expected returns  $f(\theta) = r \cdot \theta$ .
- Constraints:
  - $\sum_{i=1}^p \theta_i = 1, \theta_i > 0$ .
  - Covariance constraints linear in  $\theta$ .
  - Variance constraint is quadratic, over-all variance  $\theta^T \Sigma \theta$ .

# Barrier Methods

A.K.A. “interior point”, “central path”, etc.

- Having constraints switch on and off abruptly is annoying, especially with gradient methods.
- Fix  $\mu > 0$  and try minimizing

$$f(\theta) - \mu \log(d - h(\theta))$$

- The above “pushes away” from the barrier – more and more weakly as  $\mu \rightarrow 0$

## Algorithm

- Choose initial guess of  $\theta$  in the feasible set and initial  $\mu$ .
- While ((not too tired) and (making adequate progress))
  - Minimize  $f(\theta) - \mu \log(d - h(\theta))$ ,
  - Reduce  $\mu$ .
- Return final  $\theta$  as approximation of  $\theta^*$ .

# R Implementation

`constrOptim()` implements the barrier method.

```
> factory.n <- list(c("labor","steel"), c("car","truck"))
> factory    <- matrix(c(40, 1, 60, 3), nrow = 2,
+                         dimnames = factory.n)
> available      <- c(1600, 70)
> names(available) <- rownames(factory)
> prices          <- c(car = 13, truck = 27)
> revenue <- function(output) {return(-output %*% prices)}
> plan <- constrOptim(theta = c(5, 5), f = revenue,
+                       grad = NULL, ui = -factory,
+                       ci = -available, meth = "Nelder-Mead")
> plan$par
```

```
[1] 9.999896 20.000035
```

`constrOptim()` only works with constraints like  $\mathbf{u}\theta \geq \mathbf{c}$ , so minus signs.

# Penalties vs. Constraints

$$\arg \min_{\theta: h(\theta) \leq d} f(\theta) \leftrightarrow \arg \min_{\theta, \lambda} f(\theta) - \lambda(h(\theta) - d).$$

Note that  $d$  plays no role in minimizing  $\theta$ .

# Penalties vs. Constraints

$$\arg \min_{\theta: h(\theta) \leq d} f(\theta) \leftrightarrow \arg \min_{\theta, \lambda} f(\theta) - \lambda(h(\theta) - d).$$

Note that  $d$  plays no role in minimizing  $\theta$ .

We could just as well minimize

$$f(\theta) - \lambda h(\theta).$$

- **Constrained optimization** uses a constraint level  $d$ .
- **Penalized optimization** uses a penalty factor  $\lambda$ .

# Statistical Applications of Penalization

Mostly in this class you've seen unpenalized estimates (least squares, maximum likelihood) but lots of modern advanced methods rely on penalties.

- For when the direct estimate is too unstable.
- For handling high-dimensional cases.
- For handling non-parametrics.

# Ordinary Least Squares

No penalization; minimize MSE of the linear function  $\beta \cdot x$ :

$$\hat{\beta} = \arg \min_{\beta} \frac{1}{n} \sum_{i=1}^n (y_i - \beta \cdot x_i)^2 = \arg \min_{\beta} MSE(\beta).$$

Closed form solution if we can invert matrices:

$$\hat{\beta} = (X^T X)^{-1} X^T y,$$

where  $X$  is the  $n \times p$  design matrix of predictors and  $y$  is the  $n \times 1$  vector of responses.

# Ridge Regression

Now put a penalty on the *magnitude* of the coefficient vector:

$$\tilde{\beta} = \arg \min_{\beta} MSE(\beta) + \mu \sum_{j=1}^p \beta_j^2 = \arg \min_{\beta} MSE(\beta) + \mu \|\beta\|_2^2.$$

Penalizing  $\beta$  this way makes the estimate more *stable*; especially useful for

- Lots of noise.
- Collinear data ( $X$  not of “full rank”).
- High-dimensional,  $p > n$  data (which implies collinearity).

This is called **ridge regression**. Closed form solution:

$$\tilde{\beta} = (X^T X + \mu I)^{-1} X^T y$$

# The LASSO

Now put a penalty on the sum of the coefficient's absolute values:

$$\beta^\dagger = \arg \min_{\beta} MSE(\beta) + \mu \sum_{j=1}^p |\beta_j| = \arg \min_{\beta} MSE(\beta) + \mu \|\beta\|_1.$$

This is called **the LASSO**.

- Also stabilizes (like ridge)
- Also handles high-dimensional data (like ridge)
- Enforces **sparsity**: it likes to drive small coefficients exactly to 0.

No closed form, but very efficient interior-point algorithms (e.g., lars package)

# Spline Smoothing

Minimize the MSE of a smooth, nonlinear function, plus a penalty on curvature:

$$\hat{f} = \arg \min_f \frac{1}{n} \sum_{i=1}^n (y_i - f(x_i))^2 + \int (f''(x))^2 dx$$

This fits smooth regressions without assuming any specific functional form.

- Lets you check linear models.
- Makes you wonder why you bother with linear models.

Many different R implementations, starting with `smooth.spline`.

# How Big a Penalty?

## Remember cross-validation?

Rarely know the constraint level or the penalty factor  $\lambda$ . Lots of ways of picking, but often **cross-validation** works well:

- Divide the data into parts.
- For each value of  $\lambda$ , estimate the model on one part of the data.
- See how well the models fit the other part of the data.
- Use the  $\lambda$  which extrapolates best on average.

# Summary

- We use Lagrange multipliers to turn constrained optimization problems into unconstrained but penalized ones.
- The nature of the penalty term reflects the sort of constraint we put on the problem:
  - Shrinkage?
  - Sparsity?
  - Smoothness?

# Coordinate Descent

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Gradient descent and Newton's method adjust all coordinates of  $\theta$  at once.  
This gets harder as the number of dimensions  $d$  grows.

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Gradient descent and Newton's method adjust all coordinates of  $\theta$  at once.  
This gets harder as the number of dimensions  $d$  grows.

## Coordinate Descent

Never do more than one-dimensional optimization!

- Start with an initial guess  $\theta$ .
- While ((not too tired) and (still making adequate progress))
  - For  $i \in (1 : p)$ 
    1. Do one-dimensional optimization over the  $i^{th}$  coordinate of  $\theta$  holding all others fixed.
    2. Update  $i^{th}$  coordinate to this optimal value.
  - Return final value of  $\theta$  as approximation of  $\theta^*$ .

# Coordinate Descent

## Pros and Cons

Pro:

- Can be extremely fast and simple.

Con:

- Must have a good one-dimensional optimizer.
- Can bog down for very tricky functions, especially with lots of interactions among variables.