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HOMEWORK 1

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Solution 1 (Time spent: 5 hour)**.** Your solution goes here.

Solution 2 (Time spent: 15 hour)**.** Your solution goes here.

(1)

$$\alpha_{k+1}(x_1) = \alpha_k(x_1)T_{x_1,x_1}M_{x_1,y_{k+1}} \tag{1}$$
$$+ \alpha_k(x_2)T_{x_2,x_1}M_{x_1,y_{k+1}} \tag{2}$$
$$\alpha_{k+1}(x_2) = \alpha_k(x_1)T_{x_1,x_2}M_{x_2,y_{k+1}} \tag{3}$$
$$+ \alpha_k(x_2)T_{x_2,x_2}M_{x_2,y_{k+1}} \tag{4}$$
$$\tag{5}$$
$$=> \begin{bmatrix} T_{x_1x_1} & T_{x_2x_1} \\ T_{x_1x_2} & T_{x_2x_2} \end{bmatrix} \otimes \begin{bmatrix} M_{x_1,y_{k+1}} \\ M_{x_2,y_{k+1}} \end{bmatrix} \begin{bmatrix} \alpha_k(X_1) \\ \alpha_k(X_2) \end{bmatrix} \tag{6}$$
$$\alpha_{k+1}(X) = T * M_{X,y_{k+1}}\alpha_k(X) \tag{7}$$
$$\tag{8}$$

$$\begin{bmatrix}
[2.50000000e-01 & 2.00000000e-01] \\
[9.00000000e-02 & 2.25000000e-02] \\
[2.25000000e-02 & 5.62500000e-03] \\
[7.03125000e-03 & 5.62500000e-03] \\
[6.32812500e-04 & 3.16406250e-03] \\
[9.49218750e-04 & 7.59375000e-04] \\
[8.54296875e-05 & 4.27148438e-04] \\
[2.56289063e-05 & 1.28144531e-04] \\
[7.68867188e-06 & 3.84433594e-05] \\
[1.15330078e-05 & 9.22640625e-06] \\
[1.03797070e-06 & 5.18985352e-06] \\
[3.11391211e-07 & 1.55695605e-06] \\
[9.34173633e-08 & 4.67086816e-07] \\
[2.80252090e-08 & 1.40126045e-07] \\
[8.40756270e-09 & 4.20378135e-08] \\
[1.26113440e-08 & 1.00890752e-08] \\
[5.67510482e-09 & 4.54008386e-09] \\
[2.04303773e-09 & 5.10759434e-10] \\
[5.10759434e-10 & 1.27689858e-10] \\
[3.19224646e-11 & 1.59612323e-10]
\end{bmatrix} \tag{9}$$

$$ \tag{10} $$

$$
\begin{aligned}
\beta_k(x_1) &= \beta_{k+1}(x_1)T_{x_1,x_1}M_{x_1,y_{k+1}} && (11)\\
&+ \beta_{k+1}(x_2)T_{x_1,x_2}M_{x_2,y_{k+1}} && (12)\\
\beta_k(x_2) &= \beta_{k+1}(x_1)T_{x_2,x_1}M_{x_1,y_{k+1}} && (13)\\
&+ \beta_{k+1}(x_2)T_{x_2,x_2}M_{x_2,y_{k+1}} && (14)
\end{aligned}
$$

$$ \tag{15} $$

$$
\begin{bmatrix} T_{x_1x_1} & T_{x_1x_2} \\ T_{x_2x_1} & T_{x_2x_2} \end{bmatrix} \Big(\begin{bmatrix} M_{x_1,y_{k+1}} \\ M_{x_2,y_{k+1}} \end{bmatrix} \otimes \begin{bmatrix} \beta_{k+1}(X_1) \\ \beta_{k+1}(X_2) \end{bmatrix} \Big) \tag{16}
$$

$$
\beta_k(X) = T^T(\beta_{k+1}(X) * M_{X,y_{k+1}}) \tag{17}
$$

$$ \tag{18} $$

$$
\begin{bmatrix}
[1.91534788e-10 & 1.91534788e-10] \\
[4.25632861e-10 & 4.25632861e-10] \\
[1.70253145e-09 & 1.70253145e-09] \\
[6.81012578e-09 & 6.81012578e-09] \\
[1.51336129e-08 & 1.51336129e-08] \\
[5.04453762e-08 & 5.04453762e-08] \\
[1.12100836e-07 & 1.12100836e-07] \\
[3.73669453e-07 & 3.73669453e-07] \\
[1.24556484e-06 & 1.24556484e-06] \\
[4.15188281e-06 & 4.15188281e-06] \\
[9.22640625e-06 & 9.22640625e-06] \\
[3.07546875e-05 & 3.07546875e-05] \\
[1.02515625e-04 & 1.02515625e-04] \\
[3.41718750e-04 & 3.41718750e-04] \\
[1.13906250e-03 & 1.13906250e-03] \\
[3.79687500e-03 & 3.79687500e-03] \\
[8.43750000e-03 & 8.43750000e-03] \\
[1.87500000e-02 & 1.87500000e-02] \\
[7.50000000e-02 & 7.50000000e-02] \\
[3.00000000e-01 & 3.00000000e-01]
\end{bmatrix} \tag{19}
$$

$$\gamma_{1:t} = \tag{20}$$

$$\begin{bmatrix} [0.55555556 & 0.44444444] \\ [0.8 & 0.2] \\ [0.8 & 0.2] \\ [0.55555556 & 0.44444444] \\ [0.16666667 & 0.83333333] \\ [0.55555556 & 0.44444444] \\ [0.16666667 & 0.83333333] \\ [0.16666667 & 0.83333333] \\ [0.16666667 & 0.83333333] \\ [0.55555556 & 0.44444444] \\ [0.16666667 & 0.83333333] \\ [0.16666667 & 0.83333333] \\ [0.16666667 & 0.83333333] \\ [0.16666667 & 0.83333333] \\ [0.16666667 & 0.83333333] \\ [0.55555556 & 0.44444444] \\ [0.55555556 & 0.44444444] \\ [0.8 & 0.2] \\ [0.8 & 0.2] \\ [0.16666667 & 0.83333333] \end{bmatrix} \tag{21}$$

$$\text{Most possible state} \tag{22}$$

$$\begin{bmatrix} [0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1] \\ [LA & LA & LA & LA & NY & LA & NY & NY & NY & LA \\ NY & NY & NY & NY & NY & LA & LA & LA & LA & NY] \end{bmatrix} \tag{23}$$

(2)

$$\xi(x, x^{'}) = P(X_k = x, X_{k+1} = x^{'} \mid Y_{1:t}) \tag{24}$$

$$= P(X_k = x \mid X_{k+1} = x^{'}, Y_{1:t}) P(X_{k+1} = x^{'} \mid Y_{1:t}) \tag{25}$$

$$= \frac{\alpha_k(x) M_{x^{'} y_{k+1}} T_{xx^{'}}}{\alpha_{k+1}(x^{'})} \alpha_{k+1}(x^{'}) \beta_{k+1}(x^{'}) \tag{26}$$

$$= \alpha_k(x) M_{x^{'} y_{k+1}} T_{xx^{'}} \beta_{k+1}(x^{'}) \tag{27}$$

(3)

$$\xi(x_1, x_1) = \alpha_k(x_1) M_{x_1 y_{k+1}} T_{x_1 x_1} \beta_{k+1}(x_1) \tag{28}$$

$$\xi(x_1, x_2) = \alpha_k(x_1) M_{x_2 y_{k+1}} T_{x_1 x_2} \beta_{k+1}(x_2) \tag{29}$$

$$\xi(x_2, x_1) = \alpha_k(x_2) M_{x_1 y_{k+1}} T_{x_2 x_1} \beta_{k+1}(x_1) \tag{30}$$

$$\xi(x_2, x_2) = \alpha_k(x_2) M_{x_2 y_{k+1}} T_{x_2 x_2} \beta_{k+1}(x_2) \tag{31}$$

$$\left[\begin{bmatrix} \alpha_k(x_1) \\ \alpha_k(x_2) \end{bmatrix} \Big(\begin{bmatrix} M_{x_1, y_{k+1}} & M_{x_2, y_{k+1}} \end{bmatrix} \otimes \begin{bmatrix} \beta_{k+1}(x_1) & \beta_{k+1}(x_2) \end{bmatrix} \Big) \otimes \begin{bmatrix} T_{x_1 x_1} & T_{x_1 x_2} \\ T_{x_2 x_1} & T_{x_2 x_2} \end{bmatrix} \right]^T \tag{32}$$

$$\xi(X, X^{'}) = \left(\alpha_k(X) (\beta_{k+1}(X) \otimes M_{X^{'} y_{k+1}})^T T_{XX^{'}}^T \right)^T \tag{33}$$

$$\tag{34}$$

$$M^{'} = \begin{bmatrix} 0.3902439 & 0.20325203 & 0.40650407 \\ 0.06779661 & 0.70621469 & 0.2259887 \end{bmatrix} \tag{35}$$

$$T^{'} = \begin{bmatrix} 0.47023206 & 0.3526061 \\ 0.52976794 & 0.6473939 \end{bmatrix} \tag{36}$$

$$P(Y_{1:k} \mid \lambda) = 1.9153478765258798e^{-10} \leq P(Y_{1:k} \mid \lambda^{'}) = 2.2906401053990102e^{-09} \tag{37}$$

$$\tag{38}$$

Solution 3 (Time spent: 5 hour)**.** Your solution goes here.

(1) (a)

$$P(X_{k+1} = x_j \mid X_k = x_i, Y_{1:t}) = \frac{P(X_k, X_{k+1}, Y_{1:k}, Y_{k+1}, Y_{k+2:t})}{P(X_k = x_i, Y_{1:t})} \tag{39}$$

$$= P(Y_{k+2:t} \mid X_k, X_{k+1}, Y_{1:t}, Y_{t+1}) * P(X_k, X_{k+1}, Y_{1:t}, Y_{t+1}) * \eta, \tag{40}$$

$$\text{where } \eta = \Big(\alpha_k(x) * \beta_k(x)\Big)^{-1} \tag{41}$$

$$= \eta * P(Y_{k+2:t} \mid X_{k+1}) * P(Y_{k+1} \mid Y_{1:k}, X_{k+1}, X_k) * P(Y_{1:k}, X_{k+1}, X_k) \tag{42}$$

$$= \eta * P(Y_{k+2:t} \mid X_{k+1}) * P(Y_{k+1} \mid X_{k+1}) * P(X_{k+1} \mid Y_{1:k}, X_k) * P(Y_{1:k}, X_k) \tag{43}$$

$$= \eta * P(Y_{k+2:t} \mid X_{k+1}) * P(Y_{k+1} \mid X_{k+1}) * P(X_{k+1} \mid X_k) * P(Y_{1:k}, X_k) \tag{44}$$

$$= \frac{\beta_{k+1}(x) M_{x,k+1} T_{x_k, x_{k+1}}}{\beta_k(x)} \tag{45}$$

(b)

$$P(X_k = x_i \mid X_{k+1} = x_j, Y_{1:t}) = \frac{P(X_k, X_{k+1}, Y_{1:k}, Y_{k+1}, Y_{k+2:t})}{P(X_{k+1} = x_j, Y_{1:t})} \tag{46}$$

$$= P(Y_{k+2:t} \mid X_k, X_{k+1}, Y_{1:t}, Y_{t+1}) * P(X_k, X_{k+1}, Y_{1:t}, Y_{t+1}) * \eta, \tag{47}$$

$$\eta = \Big(\alpha_{k+1}(x) * \beta_{k+1}(x)\Big)^{-1} \tag{48}$$

$$= \eta * P(Y_{k+2:t} \mid X_{k+1}) * P(Y_{k+1} \mid Y_{1:k}, X_{k+1}, X_k) * P(Y_{1:k}, X_{k+1}, X_k) \tag{49}$$

$$= \eta * P(Y_{k+2:t} \mid X_{k+1}) * P(Y_{k+1} \mid X_{k+1}) * P(X_{k+1} \mid Y_{1:k}, X_k) * P(Y_{1:k}, X_k) \tag{50}$$

$$= \eta * P(Y_{k+2:t} \mid X_{k+1}) * P(Y_{k+1} \mid X_{k+1}) * P(X_{k+1} \mid X_k) * P(Y_{1:k}, X_k) \tag{51}$$

$$= \frac{\alpha_k(x) M_{x,k+1} T_{x_k, x_{k+1}}}{\alpha_{k+1}(x)} \tag{52}$$

(c)

$$P(X_{k-1} = x_i, X_k = x_j, X_{k+1} = x_l \mid Y_{1:t}) \tag{53}$$

$$= P(X_{k-1} \mid X_k, X_{k+1}, Y_{1:t}) P(X_k \mid X_{k+1}, Y_{1:t}) P(X_{k+1} \mid Y_{1:t}) \tag{54}$$

$$= P(X_{k-1} \mid X_k, Y_{1:t}) P(X_k \mid X_{k+1}, Y_{1:t}) P(X_{k+1} \mid Y_{1:t}) \tag{55}$$

$$= \frac{\alpha_{k-1}(x) M_{x,k+1} T_{x_k, x_{k+1}}}{\alpha_k(x)} \frac{\alpha_k(x) M_{x,k+1} T_{x_k, x_{k+1}}}{\alpha_{k+1}(x)} \alpha_{k+1}(x) \beta_{k+1}(x) \tag{56}$$

$$= \alpha_{k-1}(x) M_{x,k+1}^2 T_{x_k, x_{k+1}}^2 \beta_{k+1}(x) \tag{57}$$

(2) ...

Solution 4 (Time spent: 2 hour)**.** Your solution goes here.

(1)

$$E[a_1Y_1 + a_2Y_2] = X \tag{58}$$
$$= E[a_1Y_1] + E[a_2Y_2] \tag{59}$$
$$= E[a_1h_1X] + E[\epsilon_1] + E[a_2h_2X] + E[\epsilon_2] \tag{60}$$
$$= a_1h_1E[X] + a_2h_2E[X] = X \tag{61}$$
$$\therefore a_1 + 2a_2 = 1 \tag{62}$$

(2)

$$E[(X - \hat{X})^2] = \tag{63}$$
$$= var(\hat{X}), \because E[\hat{X}] = X \tag{64}$$
$$= var(a_1Y_1 + a_2Y_2) \tag{65}$$
$$= var(a_1Y_1) + var(a_2Y_2) \tag{66}$$
$$= var(a_1h_1X + a_1\epsilon_1) + var(a_2h_2X + a_2\epsilon_2) \tag{67}$$
$$= var(a_1h_1X) + var(a_1\epsilon_1) + var(a_2h_2X) + var(a_2\epsilon_2) \tag{68}$$
$$= a_1^2h_1^2var(X) + a_2^2h_2^2var(X) + a_1^2\sigma_1^2 + a_2^2\sigma_2^2 \tag{69}$$
$$= a_1^2\sigma_1^2 + a_2^2\sigma_2^2 \tag{70}$$
$$= (1 - 2a_2)^2\sigma_1^2 + a_2^2\sigma_2^2 \tag{71}$$
$$=> \frac{dE[(X - \hat{X})^2]}{da_2} = 0 \tag{72}$$
$$=> a_2 = \frac{2 * \sigma_1^2}{4 * \sigma_1^2 + \sigma_2^2} \tag{73}$$

(3) (a)

$$\sigma_2 \gg \sigma_1 => a_2 = 0, a_1 = 1$$

(b)

$$\sigma_2 = \sigma_1 => a_2 = \frac{2}{5}, a_1 = \frac{1}{5}$$

(c)

$$\sigma_2 \ll \sigma_1 => a_2 = 1, a_1 = 0$$

(d) agreed, the coefficient shrinks when the variance of the estimator gets big since it is not reliable, and vice versa.