Federal University of Rio de Janeiro

# Lebenslangerschicksalsschatz

Felipe Chen, Gabriel de Março e Letícia Freire

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# Contest (1)

Makefile
<span style="float:right">8 lines</span>

```
CXX = g++
CXXFLAGS = -std=c++17 -O2 -Wall -Wextra -pedantic -Wshadow -
    Wformat=2 -Wfloat-equal -Wconversion -Wlogical-op -Wshift-
    overflow=2 -Wduplicated-cond -Wcast-qual -Wcast-align -Wno
    -unused-result -Wno-sign-conversion
DEBUGFLAGS = -D_GLIBCXX_DEBUG -D_GLIBCXX_DEBUG_PEDANTIC -DLOCAL
    -fsanitize=address -fsanitize=undefined -fno-sanitize-
    recover=all -fstack-protector -D_FORTIFY_SOURCE=2

DEBUG = false
ifeq ($(DEBUG),true)
  CXXFLAGS += $(DEBUGFLAGS)
endif
```

hash.sh
<span style="float:right">3 lines</span>

```
# Hashes a file, ignoring all whitespace and comments. Use for
# verifying that code was correctly typed.
cpp -dD -P -fpreprocessed | tr -d '[:space:]'| md5sum |cut -c-6
```

hash-cpp.sh
<span style="float:right">5 lines</span>

```
# Hashes a file, ignoring all whitespace, comments and defines.
    Use for
# verifying that code was correctly typed.
# First do: chmod +x ./hash-cpp.sh
# ./hash-cpp.sh *.cpp start end
sed -n $2',',$3' p' $1 | sed '/^#w/d' | cpp -dD -P -
    fpreprocessed | tr -d '[:space:]' | md5sum |cut -c-6
```

# Mathematics (2)

## 2.1 Equations

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The extremum is given by $x = -b/2a$.

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \Rightarrow \begin{aligned} x &= \frac{ed - bf}{ad - bc} \\ y &= \frac{af - ec}{ad - bc} \end{aligned}$$

In general, given an equation $Ax = b$, the solution to a variable $x_i$ is given by

$$x_i = \frac{\det A'_i}{\det A}$$

where $A'_i$ is $A$ with the $i$'th column replaced by $b$.

## 2.2 Trigonometry

$$\sin(v + w) = \sin v \cos w + \cos v \sin w$$
$$\cos(v + w) = \cos v \cos w - \sin v \sin w$$

$$\tan(v + w) = \frac{\tan v + \tan w}{1 - \tan v \tan w}$$
$$\sin v + \sin w = 2 \sin \frac{v + w}{2} \cos \frac{v - w}{2}$$
$$\cos v + \cos w = 2 \cos \frac{v + w}{2} \cos \frac{v - w}{2}$$
$$(V + W) \tan(v - w)/2 = (V - W) \tan(v + w)/2$$

where $V, W$ are lengths of sides opposite angles $v, w$.

$$a \cos x + b \sin x = r \cos(x - \phi)$$
$$a \sin x + b \cos x = r \sin(x + \phi)$$

where $r = \sqrt{a^2 + b^2}, \phi = \operatorname{atan2}(b, a)$.

## 2.3 Geometry

### 2.3.1 Triangles

Side lengths: $a, b, c$

Semiperimeter: $p = \dfrac{a + b + c}{2}$

Area: $A = \sqrt{p(p - a)(p - b)(p - c)}$

Circumradius: $R = \dfrac{abc}{4A}$

Inradius: $r = \dfrac{A}{p}$

Length of median (divides triangle into two equal-area triangles):
$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

Length of bisector (divides angles in two):

$$s_a = \sqrt{bc \left[1 - \left(\frac{a}{b + c}\right)^2\right]}$$

Law of sines: $\dfrac{\sin \alpha}{a} = \dfrac{\sin \beta}{b} = \dfrac{\sin \gamma}{c} = \dfrac{1}{2R}$

Law of cosines: $a^2 = b^2 + c^2 - 2bc \cos \alpha$

Law of tangents: $\dfrac{a + b}{a - b} = \dfrac{\tan \dfrac{\alpha + \beta}{2}}{\tan \dfrac{\alpha - \beta}{2}}$

Pick's: A polygon on an integer grid strictly containing $i$ lattice points and having $b$ lattice points on the boundary has area $i + \frac{b}{2} - 1$. (Nothing similar in higher dimensions)
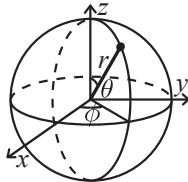
### 2.3.2 Quadrilaterals

With side lengths $a, b, c, d$, diagonals $e, f$, diagonals angle $\theta$, area $A$ and magic flux $F = b^2 + d^2 - a^2 - c^2$:

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is $180°$, $ef = ac + bd$, and $A = \sqrt{(p - a)(p - b)(p - c)(p - d)}$.

### 2.3.3 Spherical coordinates



$$\begin{aligned} x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \theta &= \operatorname{acos}(z / \sqrt{x^2 + y^2 + z^2}) \\ z &= r \cos \theta & \phi &= \operatorname{atan2}(y, x) \end{aligned}$$

## 2.4 Derivatives/Integrals

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1 - x^2}}$$
$$\frac{d}{dx} \tan x = 1 + \tan^2 x \qquad \frac{d}{dx} \arctan x = \frac{1}{1 + x^2}$$
$$\int \tan ax = -\frac{\ln|\cos ax|}{a} \qquad \int x \sin ax = \frac{\sin ax - ax \cos ax}{a^2}$$
$$\int e^{-x^2} = \frac{\sqrt{\pi}}{2} \operatorname{erf}(x) \qquad \int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax - 1)$$

Integration by parts:

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

Green's theorem:
Let $C$ be a positive, smooth, simple curve. $D$ is a region bounded by $C$.

$$\oint_C (Pdx + Qdy) = \int\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})$$

To calculate area, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, usually, picking $Q = \frac{1}{2}x$ and $P = -\frac{1}{2}y$ suffice.

Then we have

$$\frac{1}{2}\oint_C x\,dy - \frac{1}{2}\oint_C y\,dx$$

Line integral:
$C$ given by $x = x(t), y = y(t), t \in [a, b]$, then

$$\oint_C f(x, y)\,ds = \int_a^b f(x(t), y(t))\,ds$$

where, $ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$ or $\sqrt{(1 + (\frac{dy}{dx})^2}\,dx$

### 2.4.1 XOR sum

$$\bigoplus_{x=0}^{n-1} x = \{0, n-1, 1, n\}[n \bmod 4]$$

$$\bigoplus_{x=l}^{r-1} x = \bigoplus_{a=0}^{r-1} a \oplus \bigoplus_{b=0}^{l-1} b$$

## 2.5 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, (-\infty < x < \infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, (-1 < x \le 1)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \dots, (-1 \le x \le 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, (-\infty < x < \infty)$$

## 2.6 Probability theory

Let $X$ be a discrete random variable with probability $p_X(x)$ of assuming the value $x$. It will then have an expected value (mean) $\mu = \mathbb{E}(X) = \sum_x x p_X(x)$ and variance $\sigma^2 = V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$ where $\sigma$ is the standard deviation. If $X$ is instead continuous it will have a probability density function $f_X(x)$ and the sums above will instead be integrals with $p_X(x)$ replaced by $f_X(x)$.

Expectation is linear:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

For independent $X$ and $Y$,

$$V(aX + bY) = a^2 V(X) + b^2 V(Y).$$

### 2.6.1 Discrete distributions
#### Binomial distribution

The number of successes in $n$ independent yes/no experiments, each which yields success with probability $p$ is Bin$(n, p)$, $n = 1, 2, \dots, 0 \le p \le 1$.

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mu = np, \sigma^2 = np(1-p)$$

Bin$(n, p)$ is approximately Po$(np)$ for small $p$.

#### First success distribution

The number of trials needed to get the first success in independent yes/no experiments, each wich yields success with probability $p$ is Fs$(p)$, $0 \le p \le 1$.

$$p(k) = p(1-p)^{k-1}, k = 1, 2, \dots$$

$$\mu = \frac{1}{p}, \sigma^2 = \frac{1-p}{p^2}$$

#### Poisson distribution

The number of events occurring in a fixed period of time $t$ if these events occur with a known average rate $\kappa$ and independently of the time since the last event is Po$(\lambda)$, $\lambda = t\kappa$.

$$p(k) = e^{-\lambda}\frac{\lambda^k}{k!}, k = 0, 1, 2, \dots$$

$$\mu = \lambda, \sigma^2 = \lambda$$

### 2.6.2 Continuous distributions
#### Uniform distribution

If the probability density function is constant between $a$ and $b$ and 0 elsewhere it is U$(a, b)$, $a < b$.

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \frac{a+b}{2}, \sigma^2 = \frac{(b-a)^2}{12}$$

#### Exponential distribution

The time between events in a Poisson process is Exp$(\lambda)$, $\lambda > 0$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

$$\mu = \frac{1}{\lambda}, \sigma^2 = \frac{1}{\lambda^2}$$

#### Normal distribution

Most real random values with mean $\mu$ and variance $\sigma^2$ are well described by $\mathcal{N}(\mu, \sigma^2)$, $\sigma > 0$.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ then

$$aX_1 + bX_2 + c \sim \mathcal{N}(\mu_1 + \mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$$

## 2.7 Markov chains

A *Markov chain* is a discrete random process with the property that the next state depends only on the current state. Let $X_1, X_2, \dots$ be a sequence of random variables generated by the Markov process. Then there is a transition matrix $\mathbf{P} = (p_{ij})$, with $p_{ij} = \Pr(X_n = i | X_{n-1} = j)$, and $\mathbf{p}^{(n)} = \mathbf{P}^n \mathbf{p}^{(0)}$ is the probability distribution for $X_n$ (i.e., $p_i^{(n)} = \Pr(X_n = i)$), where $\mathbf{p}^{(0)}$ is the initial distribution.

$\pi$ is a stationary distribution if $\pi = \pi\mathbf{P}$. If the Markov chain is *irreducible* (it is possible to get to any state from any state), then $\pi_i = \frac{1}{\mathbb{E}(T_i)}$ where $\mathbb{E}(T_i)$ is the expected time between two visits in state $i$. $\pi_j/\pi_i$ is the expected number of visits in state $j$ between two visits in state $i$.

For a connected, undirected and non-bipartite graph, where the transition probability is uniform among all neighbors, $\pi_i$ is proportional to node $i$'s degree.

A Markov chain is *ergodic* if the asymptotic distribution is independent of the initial distribution. A finite Markov chain is ergodic iff it is irreducible and *aperiodic* (i.e., the gcd of cycle lengths is 1). $\lim_{k\to\infty} \mathbf{P}^k = \mathbf{1}\pi$.

A Markov chain is an absorbing chain if
**1**. there is at least one absorbing state and
**2**. it is possible to go from any state to at least one absorbing state in a finite number of steps.
A Markov chain is an A-chain if the states can be partitioned into two sets $\mathbf{A}$ and $\mathbf{G}$, such that all states in $\mathbf{A}$ are absorbing $(p_{ii} = 1)$, and all states in $\mathbf{G}$ leads to an absorbing state in $\mathbf{A}$. The probability for absorption in state $i \in \mathbf{A}$, when the initial state is $j$, is $a_{ij} = p_{ij} + \sum_{k\in\mathbf{G}} a_{ik}p_{kj}$. The expected time until absorption, when the initial state is $i$, is $t_i = 1 + \sum_{k\in\mathbf{G}} p_{ki}t_k$.

# Data Structures (3)

## order-statistic-tree.h
**Description:** A set (not multiset!) with support for finding the n'th element, and finding the index of an element.
**Time:** $\mathcal{O}(\log N)$
<bits/extc++.h>                                                         acfa21, 19 lines
```cpp
template <typename K, typename V, typename Comp = std::less<K>>
using ordered_map = __gnu_pbds::tree<
  K, V, Comp,
  __gnu_pbds::rb_tree_tag,
  __gnu_pbds::tree_order_statistics_node_update
>;

template <typename K, typename Comp = std::less<K>>
using ordered_set = ordered_map<K, __gnu_pbds::null_type, Comp
    >;

void example() {
  ordered_set<int> t, t2; t.insert(8);
  auto it = t.insert(10).first;
  assert(it == t.lower_bound(9));
  assert(t.order_of_key(10) == 1); // num strictly smaller
  assert(t.order_of_key(11) == 2);
  assert(*t.find_by_order(0) == 8);
  t.join(t2); // assuming T < T2 or T > T2, merge t2 into t
}
```

## dsu.h
**Description:** Disjoint-set data structure.
**Time:** $\mathcal{O}(\alpha(N))$
7d5db8, 14 lines
```cpp
struct UF {
  vector<int> e;
  UF(int n) : e(n, -1) {}
  bool same_set(int a, int b) { return find(a) == find(b); }
  int size(int x) { return -e[find(x)]; }
  int find(int x) { return e[x] < 0 ? x : e[x] = find(e[x]); }
  bool unite(int a, int b) {
    a = find(a), b = find(b);
    if (a == b) return 0;
    if (e[a] > e[b]) swap(a, b);
    e[a] += e[b]; e[b] = a;
    return 1;
  }
};
```

## bipartite-dsu.h
**Description:** Disjoint-set data structure.
**Time:** $\mathcal{O}(\alpha(N))$
07774d, 31 lines
```cpp
struct DSU {
  vector<int> p, rk, color, bipartite;
  DSU(int n) : p(n), rk(n), color(n), bipartite(n, 1) {
    iota(p.begin(), p.end(), 0);
  };
  int find(int u) {
    if (u == p[u]) return u;
    int v = find(p[u]);
    color[u] ^= color[p[u]];
    return p[u] = v;
  }
  int find_color(int u) {
    find(u);
    return color[u];
  }
  // check if it doesn't create an odd cycle
  bool can(int u, int v) {
    return find(u) != find(v) || color[u] != color[v];
  }
  void unite(int u, int v) {
```

```cpp
    int pu = find(u), pv = find(v);
    if (pu == pv) {
      if (color[u] == color[v]) bipartite[pu] = false;
      return;
    }
    if (rk[pu] < rk[pv]) swap(pu, pv);
    if (color[u] == color[v]) color[pv] ^= 1;
    p[pv] = pu, rk[pu] += (rk[pu] == rk[pv]);
    if (not bipartite[pv]) bipartite[pu] = false;
  }
};
```

## dsu-rollback.h
**Description:** Disjoint-set data structure with undo.
**Usage:** int t = uf.time(); ...; uf.rollback(t);
**Time:** $\mathcal{O}(\log(N))$
7ddf1d, 21 lines
```cpp
struct RollbackUF {
  vector<int> e; vector<pair<int,int>> st;
  RollbackUF(int n) : e(n, -1) {}
  int size(int x) { return -e[find(x)]; }
  int find(int x) { return e[x] < 0 ? x : find(e[x]); }
  int time() { return st.size(); }
  void rollback(int t) {
    for (int i = time(); i --> t;)
      e[st[i].first] = st[i].second;
    st.resize(t);
  }
  bool unite(int a, int b) {
    a = find(a), b = find(b);
    if (a == b) return false;
    if (e[a] > e[b]) swap(a, b);
    st.push_back({a, e[a]});
    st.push_back({b, e[b]});
    e[a] += e[b]; e[b] = a;
    return true;
  }
};
```

## monotonic-queue.h
**Description:** Structure that supports all operations of a queue and get the minimum/maximum active value in the queue. Useful for sliding window 1D and 2D. For 2D problems, you will need to pre-compute another matrix, by making a row-wise traversal, and calculating the min/max value beginning in each cell. Then you just make a column-wise traverse as they were each an independent array.
**Time:** $\mathcal{O}(1)$
da8783, 41 lines
```cpp
template<typename T> struct monotonic_queue {
  vector<T> as, aas;
  vector<T> bs, bbs;
  void reserve(int N) {
    as.reserve(N); aas.reserve(N);
    bs.reserve(N); bbs.reserve(N);
  }
  void reduce() {
    while (!bs.empty()) {
      as.push_back(bs.back());
      aas.push_back(aas.empty() ? bs.back() : (bs.back() * aas.
          back()));
      bs.pop_back(); bbs.pop_back();
    }
  }
  T get() {
    if (as.empty()) reduce();
    return (bbs.empty() ? aas.back() : (aas.back() * bbs.back()
        ));
  }
  bool empty() const { return (as.empty() && bs.empty()); }
```

```cpp
  int size() const { return int(as.size()) + int(bs.size()); }
  T front() {
    if (as.empty()) reduce();
    return as.back();
  }
  void push(const T& val) {
    bs.push_back(val);
    bbs.push_back(bbs.empty() ? val : (bbs.back() * val));
  }
  void pop() {
    if (as.empty()) reduce();
    as.pop_back();
    aas.pop_back();
  }
};

struct affine_t {
  int64_t b, c;
  affine_t operator*(affine_t rhs) {
    return {(rhs.b * b) % M, (rhs.b * c + rhs.c) % M};
  }
};
```

## short-lazy-segtree.h
**Description:** Segment Tree with Lazy update (half-open interval).
**Time:** $\mathcal{O}(\lg(N) * Q)$
979def, 30 lines
```cpp
template<class T, int N> struct segtree_t {
  static_assert(__builtin_popcount(N) == 1); // N must be power
      of 2
  const T unit = 0; T f(T a, T b) { return (a + b); }
  vector<T> seg, lazy;
  segtree_t() : seg(2*N, unit), lazy(2*N) {}
  segtree_t(const vector<T>& other) : seg(2*N, unit), lazy(2*N)
      {
    for (int a = 0; a < int(other.size()); ++a) seg[a + N] =
        other[a];
    for (int a = N-1; a; --a) pull(a);
  }
  void push(int v, int L, int R) {
    seg[v] += (R - L + 1) * lazy[v]; // dependent on operation
    if (L != R) for(int i = 0; i < 2; ++i) lazy[2*v+i] += lazy[
        v];
    lazy[v] = 0;
  } // recalc values for current node
  void pull(int v) { seg[v] = f(seg[2*v], seg[2*v+1]); }
  void build() { for(int i = N-1; i > 0; --i) pull(i); }
  void upd(int mi,int ma,T delta,int v = 1,int L = 0, int R = N
      -1) {
    push(v,L,R); if (ma < L || R < mi) return;
    if (mi <= L && R <= ma) {
      lazy[v] = delta; push(v,L,R); return; }
    int M = (L+R)/2; upd(mi,ma,delta,2*v,L,M);
    upd(mi,ma,delta,2*v+1,M+1,R); pull(v);
  }
  T query(int mi, int ma, int v = 1, int L = 0, int R = N-1) {
    push(v,L,R); if (mi > R || L > ma) return unit;
    if (mi <= L && R <= ma) return seg[v];
    int M = (L+R)/2;
    return f(query(mi,ma,2*v,L,M),query(mi,ma,2*v+1,M+1,R));
  }
};
```

## segtree.h
**Description:** Zero-indexed seg-tree. Bounds are inclusive to the left and exclusive to the right.
**Time:** $\mathcal{O}(\log N)$
db457b, 86 lines
```cpp
template<class T> struct segtree {
```

```cpp
  int H, N;
  vector<T> ts;
  segtree() {}
  explicit segtree(int N_) {
    for (H = 0, N = 1; N < N_; ++H, N *= 2) {}
    ts.resize(2*N);
    build();
  }
  template<class Q> explicit segtree(const vector<Q>& qs) {
    const int N_ = int(qs.size());
    for (H = 0, N = 1; N < N_; ++H, N *= 2) {}
    ts.resize(2*N);
    for (int i = 0; i < N_; ++i) at(i) = T(qs[i]);
    build();
  }
  T& at(int a) { return ts[a + N]; }
  void build() { for (int a = N; --a; ) merge(a); }
  inline void merge(int a) { ts[a].merge(ts[2*a], ts[2*a+1]); }
  template<class S> void update(int a, const S& v) {
    assert(0 <= a && a < N);
    ts[a += N] = T(v);
    for (; a /= 2;) merge(a);
  }
  template<class F, class... Args> void update(int a, F f, Args
      &&... args) {
    assert(0 <= a && a < N);
    (ts[a += N].*f)(args...);
    for (; a /= 2;) merge(a);
  }
  T query(int a, int b) {
    if (a == b) return T();
    a += N; b += N;
    T lhs, rhs, t;
    for (int l = a, r = b; l < r; l /= 2, r /= 2) {
      if (l & 1) { t.merge(lhs, ts[l++]); lhs = t; }
      if (r & 1) { t.merge(ts[--r], rhs); rhs = t; }
    }
    t.merge(lhs, rhs); return t;
  }
  // da1cd
  template<class Op, class E, class F, class... Args>
  auto query(int a, int b, Op op, E e, F f, Args&&... args) {
    if (a == b) return e();
    a += N; b += N;
    auto lhs = e(), rhs = e();
    for (int l = a, r = b; l < r; l /= 2, r /= 2) {
      if (l & 1) lhs = op(lhs, (ts[l++].*f)(args...));
      if (r & 1) rhs = op((ts[--r].*f)(args...), rhs);
    }
    return op(lhs, rhs);
  }
  // 1edda
  template<class F, class... Args> int find_right(int a, F f,
      Args &&... args) {
    assert(0 <= a && a <= N);
    if ((T().*f)(args...)) return a;
    if (a == N) return 1 + N;
    a += N;
    for (; ; a /= 2) if (a & 1) {
      if ((ts[a].*f)(args...)) {
        for (; a < N; ) {
          if (!(ts[a <<= 1].*f)(args...)) ++a;
        }
        return a - N + 1;
      }
      ++a;
      if (!(a & (a - 1))) return N + 1;
    }
  }
```

```cpp
  // 76b20
  template<class F, class... Args> int find_left(int a, F f,
      Args &&... args) {
    assert(0 <= a && a <= N);
    if ((T().*f)(args...)) return a;
    if (a == 0) return -1;
    a += N;
    for (; ; a /= 2) if ((a & 1) || a == 2) {
      if ((ts[a - 1].*f)(args...)) {
        for (; a <= N; ) {
          if (!(ts[(a <<= 1) - 1].*f)(args...)) --a;
        }
        return a - N - 1;
      }
      --a;
      if (!(a & (a - 1))) return -1;
    }
  }
};
```

## point-context.h

**Description:** Examples of Segment Tree

```cpp
struct seg_node { // bbfc07
  int val;
  int mi, ma;
  seg_node() : mi(INT_MAX), ma(INT_MIN), val(0) {}
  seg_node(int x) : mi(x), ma(x), val(x) {}
  void merge(const seg_node& l, const seg_node& r) {
    val = l.val + r.val;
    mi = min(l.mi, r.mi);
    ma = max(l.ma, r.ma);
  }
  void update(int x) {
    mi = ma = val = x;
  }
  bool acc_min(int& acc, int x) const {
    if (x >= mi) return true;
    if (acc > mi) acc = mi;
    return false;
  }
  bool acc_max(int& acc, int x) const {
    if (x <= ma) return true;
    if (acc < ma) acc = ma;
    return false;
  }
  bool go(int& acc, int& k) const {
    if (val <= k) {
      k -= val;
      acc += val;
      return false;
    }
    return true;
  }
};

// 1 + min of [a, N) <= x
auto find_min_right = [&](segtree<seg_node>& sg, int a, int x)
    -> int {
  int acc = INT_MAX;
  return sg.find_right(a, &seg_node::acc_min, acc, x);
};

// min of [0, a) <= x
auto find_min_left = [&](segtree<seg_node>& sg, int a, int x)
    -> int {
  int acc = INT_MAX;
  return sg.find_left(a, &seg_node::acc_min, acc, x);
};
```

```cpp
// 1 + max of [a, N) >= x
auto find_max_right = [&](segtree<seg_node>& sg, int a, int x)
    -> int {
  int acc = INT_MIN;
  return sg.find_right(a, &seg_node::acc_max, acc, x);
};

// max of [0, a) >= x
auto find_max_left = [&](segtree<seg_node>& sg, int a, int x)
    -> int {
  int acc = INT_MIN;
  return sg.find_left(a, &seg_node::acc_max, acc, x);
};

// kth one of [a, N)
auto find_kth = [&](segtree<seg_node>& sg, int a, int x) -> int
    {
  int acc = 0;
  return sg.find_right(a, &seg_node::go, acc, x);
};
```

## lazy-segtree.h

**Description:** Segment Tree with Lazy update (half-open interval).

**Time:** $\mathcal{O}(\lg(N) * Q)$

```cpp
template<class T> struct segtree_range {
  int H, N;
  vector<T> ts;
  segtree_range() {}
  explicit segtree_range(int N_) {
    for (H = 0, N = 1; N < N_; ++H, N *= 2) {}
    ts.resize(2*N);
    build();
  }
  template<class Q> explicit segtree_range(const vector<Q>& qs)
      {
    const int N_ = int(qs.size());
    for (H = 0, N = 1; N < N_; ++H, N *= 2) {}
    ts.resize(2*N);
    for (int i = 0; i < N_; ++i) at(i) = T(qs[i]);
    build();
  }
  T& at(int a) { return ts[a + N]; }
  void build() { for (int a = N; --a; ) merge(a); }
  inline void push(int a) { ts[a].push(ts[2 * a], ts[2 * a +
      1]); }
  inline void merge(int a) { ts[a].merge(ts[2*a], ts[2*a+1]); }
  // f0fcbd
  void for_parents_down(int a, int b) {
    for (int h = H; h; --h) {
      const int l = (a >> h), r = (b >> h);
      if (l == r) {
        if ((l << h) != a || (r << h) != b) push(l);
      } else {
        if ((l << h) != a) push(l);
        if ((r << h) != b) push(r);
      }
    }
  }
  // a25cb
  void for_parents_up(int a, int b) {
    for (int h = 1; h <= H; ++h) {
      const int l = (a >> h), r = (b >> h);
      if (l == r) {
        if ((l << h) != a || (r << h) != b) merge(l);
      } else {
        if ((l << h) != a) merge(l);
        if ((r << h) != b) merge(r);
```

```cpp
      }
    }
  }
  // 6014b
  template<class F, class... Args> void update(int a, int b, F
      f, Args&&... args) {
    if (a == b) return;
    a += N; b += N;
    for_parents_down(a, b);
    for (int l = a, r = b; l < r; l /= 2, r /= 2) {
      if (l & 1) (ts[l++].*f)(args...);
      if (r & 1) (ts[--r].*f)(args...);
    }
    for_parents_up(a, b);
  }
  T query(int a, int b) {
    if (a == b) return T();
    a += N; b += N;
    for_parents_down(a, b);
    T lhs, rhs, t;
    for (int l = a, r = b; l < r; l /= 2, r /= 2) {
      if (l & 1) { t.merge(lhs, ts[l++]); lhs = t; }
      if (r & 1) { t.merge(ts[--r], rhs); rhs = t; }
    }
    t.merge(lhs, rhs); return t;
  }
  // 5a862
  template<class Op, class E, class F, class... Args>
  auto query(int a, int b, Op op, E e, F f, Args&&... args) {
    if (a == b) return e();
    a += N; b += N;
    for_parents_down(a, b);
    auto lhs = e(), rhs = e();
    for (int l = a, r = b; l < r; l /= 2, r /= 2) {
      if (l & 1) { lhs = op(lhs, (ts[l++].*f)(args...)); };
      if (r & 1) { rhs = op((ts[--r].*f)(args...), rhs); };
    }
    return op(lhs, rhs);
  }
  // aab16
  // find min i s.t. T::f(args...) returns true in [a, i) from
  //     left to right
  template<class F, class... Args> int find_right(int a, F f,
      Args &&... args) {
    assert(0 <= a && a <= N);
    if ((T().*f)(args...)) return a;
    if (a == N) return 1 + N;
    a += N;
    for (int h = H; h; --h) push(a >> h);
    for (; ; a /= 2) if (a & 1) {
      if ((ts[a].*f)(args...)) {
        for (; a < N; ) {
          push(a);
          if (!(ts[a <<= 1].*f)(args...)) ++a;
        }
        return a - N + 1;
      }
      ++a;
      if (!(a & (a - 1))) return N + 1;
    }
  }
  // a033b
  // find max i s.t. T::f(args...) returns true in [i, a) from
  //     right to left
  template<class F, class... Args> int find_left(int a, F f,
      Args &&... args) {
    assert(0 <= a && a <= N);
    if ((T().*f)(args...)) return a;
    if (a == 0) return -1;
```

```cpp
    a += N;
    for (int h = H; h; --h) push((a - 1) >> h);
    for (; ; a /= 2) if ((a & 1) || a == 2) {
      if ((ts[a - 1].*f)(args...)) {
        for (; a <= N; ) {
          push(a - 1);
          if (!(ts[(a <<= 1) - 1].*f)(args...)) --a;
        }
        return a - N - 1;
      }
      --a;
      if (!(a & (a - 1))) return -1;
    }
  }
};
```

## lazy-context.h
**Description:** Examples of Segment Tree with Lazy update <sub>dc643b, 167 lines</sub>

```cpp
namespace range_flip_range_sum { //4a7f6d
// query sum a[l, r)
// update range a[i] <- !a[i]
// update range a[i] <- 1
struct seg_node {
  int sz, lz; int64_t sum;
  seg_node() : sz(1), sum(0), lz(-1) {}
  seg_node(int64_t val) : sz(1), sum(val), lz(-1) {}
  void push(const seg_node& l, const seg_node& r) {
    if (lz == 2) {
      l.flip(lz);
      r.flip(lz);
    } else if (lz != -1) {
      l.assign(lz);
      r.assign(lz);
    }
    lz = -1;
  }
  void merge(const seg_node& l, const seg_node& r) {
    sz = l.sz + r.sz;
    sum = l.sum + r.sum;
  }
  void assign(int val) {
    sum = sz * val;
    lz = val;
  }
  void flip(int val) {
    sum = sz - sum;
    if (lz == -1) lz = 2;
    else if (lz == 0) lz = 1;
    else if (lz == 1) lz = 0;
    else lz = -1;
  }
  int64_t get_sum() const { return sum; }
};
}

namespace range_add_range_sum { // d9640e
// query sum a[l, r)
// update range a[i] <- v
// update range a[i] <- a[i] + v
template<typename T = int64_t> struct seg_node {
  T val, lz_add, lz_set;
  int sz;
  bool to_set;
  seg_node(T n = 0) : val(n), lz_add(0), lz_set(0), sz(1),
      to_set(0) {}
  void push(seg_node& l, seg_node& r) {
    if (to_set) {
      l.assign(lz_set);
```

```cpp
      r.assign(lz_set);
      lz_set = 0;
      to_set = false;
    }
    if (lz_add != 0) {
      l.add(lz_add);
      r.add(lz_add);
      lz_add = 0;
    }
  }
  void merge(const seg_node& l, const seg_node& r) {
    sz = l.sz + r.sz;
    val = l.val + r.val;
  }
  void add(T v) {
    val += v * sz;
    lz_add += v;
  }
  void assign(T v) {
    val = v * sz;
    lz_add = 0;
    lz_set = v;
    to_set = true;
  }
  T get_sum() const { return val; }
};
}

namespace range_add_linear_range_sum { // a922ef
// update range a[i] <- a[i] + b * (i - s) + c
// assuming b and c are non zero, be careful
// get sum a[l, r)
template<typename T = int64_t> struct seg_node {
  T sum, lzB, lzC;
  int sz, idx;
  seg_node(int id = 0, T v = 0, int s = 0, T b = 0, T c = 0) :
      sum(v), lzB(b), lzC(c - s * b), idx(id), sz(1) {}
  void push(seg_node& l, seg_node& r) {
    l.add(lzB, lzC);
    r.add(lzB, lzC);
    lzB = lzC = 0;
  }
  void merge(const seg_node& l, const seg_node& r) {
    idx = min(l.idx, r.idx);
    sz = l.sz + r.sz;
    sum = l.sum + r.sum;
  }
  T sum_idx(T n) const { return n * (n + 1) / 2; }
  void add(T b, T c) {
    sum += b * (sum_idx(idx + sz) - sum_idx(idx)) + sz * c;
    lzB += b;
    lzC += c;
  }
  T get_sum() const { return sum; }
};
}

namespace range_affine_range_sum { // 61a09f
// update range a[i] <- b * a[i] + c
// get sum a[l, r)
struct seg_node {
  int sz; i64 sum, lzB, lzC;
  seg_node() : sz(1), sum(0), lzB(1), lzC(0) {}
  seg_node(i64 v) : sz(1), sum(v), lzB(1), lzC(0) {}
  void push(seg_node& l, seg_node& r) {
    l.add(lzB, lzC);
    r.add(lzB, lzC);
    lzB = 1, lzC = 0;
  }
```

```cpp
    void merge(const seg_node& l, const seg_node& r) {
        sz = l.sz + r.sz;
        sum = l.sum + r.sum;
    }
    void add(i64 b, i64 c) {
        sum = (b * sum + c * sz);
        lzB = (lzB * b);
        lzC = (lzC * b + c);
    }
    i64 get_sum() const { return sum; }
};
}

namespace range_chmin_chmax_point_query { // 8bab55
// update range a[i] <- min(a[i], b);
// update range a[i] <- max(a[i], b);
// get val a[i]
struct seg_node {
    int mn, mx;
    int lz0, lz1;
    seg_node() : mn(INT_MAX), mx(INT_MIN), lz0(INT_MAX), lz1(
        INT_MIN) {}
    void push(seg_node& l, seg_node& r) {
        l.minimize(lz0);
        l.maximize(lz1);
        r.minimize(lz0);
        r.maximize(lz1);
        lz0 = INT_MAX;
        lz1 = INT_MIN;
    }
    void merge(const seg_node& l, const seg_node& r) {
        mn = min(l.mn, r.mn);
        mx = max(l.mx, r.mx);
    }
    void minimize(int val) {
        mn = lz0 = min(lz0, val);
        mx = lz1 = min(lz0, lz1);
    }
    void maximize(int val) {
        mx = lz1 = max(lz1, val);
        mn = lz0 = max(lz0, lz1);
    }
    pair<int, int> get() const { return {mx, mn}; }
};
}

auto get_sum = [&](segtree_range<seg_node>& st, int a, int b) {
    return st.query(a, b, [&](auto l, auto r) -> i64 { return l +
        r; },
    [&]() -> i64 { return 0; }, &seg_node::get_sum);
};
```

## sparse-segtree.h
**Description:** Sparse Segment Tree with point update. Doesnt allocate storage for nodes with no data. Use BumpAllocator for better performance!
<span style="text-align:right">014f68, 39 lines</span>

```cpp
const int SZ = 1<<19;
template<class T> struct node_t {
    T delta = 0; node_t<T>* c[2];
    node_t() { c[0] = c[1] = nullptr; }
    void upd(int pos, T v, int L = 0, int R = SZ-1) { // add v
        if (L == pos && R == pos) { delta += v; return; }
        int M = (L + R)>>1;
        if (pos <= M) {
            if (!c[0]) c[0] = new node_t();
            c[0]->upd(pos, v, L, M);
        } else {
            if (!c[1]) c[1] = new node_t();
            c[1]->upd(pos, v, M+1, R);
```

```cpp
        }
        delta = 0;
        for (int i = 0; i < 2; ++i) if (c[i]) delta += c[i]->delta;
    }
    T query(int lx, int rx, int L = 0, int R = SZ-1) { // query
        sum of segment
        if (rx < L || R < lx) return 0;
        if (lx <= L && R <= rx) return delta;
        int M = (L + R)>>1; T res = 0;
        if (c[0]) res += c[0]->query(lx, rx, L, M);
        if (c[1]) res += c[1]->query(lx, rx, M+1, R);
        return res;
    }
    void upd(int pos, node_t *a, node_t *b, int L = 0, int R = SZ
        -1) {
        if (L != R) {
            int M = (L + R)>>1;
            if (pos <= M) {
                if (!c[0]) c[0] = new node_t();
                c[0]->upd(pos, a ? a->c[0] : nullptr, b ? b->c[0] :
                    nullptr, L, M);
            } else {
                if (!c[1]) c[1] = new node_t();
                c[1]->upd(pos,a ? a->c[1] : nullptr, b ? b->c[1] :
                    nullptr, M+1, R);
            }
        }
        delta = (a ? a->delta : 0)+(b ? b->delta : 0);
    }
};
```

## segtree-2d.h
**Description:** 2D Segment Tree.
**Time:** $\mathcal{O}\left(N \log^2 N\right)$ of memory, $\mathcal{O}\left(\log^2 N\right)$ per query
<span>"sparse_seg_tree.h"</span> <span style="text-align:right">09098e, 25 lines</span>

```cpp
template<class T> struct Node {
    node_t<T> seg; Node* c[2];
    Node() { c[0] = c[1] = nullptr; }
    void upd(int x, int y, T v, int L = 0, int R = SZ-1) { // add
        v
        if (L == x && R == x) { seg.upd(y,v); return; }
        int M = (L+R)>>1;
        if (x <= M) {
            if (!c[0]) c[0] = new Node();
            c[0]->upd(x,y,v,L,M);
        } else {
            if (!c[1]) c[1] = new Node();
            c[1]->upd(x,y,v,M+1,R);
        }
        seg.upd(y,v); // only for addition
            // seg.upd(y,c[0]?&c[0]->seg:nullptr,c[1]?&c[1]->
            //     seg:nullptr);
    }
    T query(int x1, int x2, int y1, int y2, int L = 0, int R = SZ
        -1) { // query sum of rectangle
        if (x1 <= L && R <= x2) return seg.query(y1,y2);
        if (x2 < L || R < x1) return 0;
        int M = (L+R)>>1; T res = 0;
        if (c[0]) res += c[0]->query(x1, x2, y1, y2, L, M);
        if (c[1]) res += c[1]->query(x1, x2, y1, y2, M+1, R);
        return res;
    }
};
```

## persistent-segtree.h
**Description:** Persistent implementation of a segment tree. This one compute the kth smallest element in a subarray $[a, b]$.
<span style="text-align:right">d277eb, 31 lines</span>

```cpp
struct segtree_t {
```

```cpp
    struct snapshot {
        int cnt, linkl, linkr;
        snapshot() : cnt(0), linkl(0), linkr(0) {}
        snapshot(int _cnt, int l, int r) : cnt(_cnt), linkl(l),
            linkr(r) {}
    };
    int id;
    vector<snapshot> tree;
    segtree_t() {}
    segtree_t(int n) : id(1), tree(20*n) {}
    int update(int v, int l, int r, int x) {
        if (x < l || x > r) return v;
        if (l == r) {
            tree[id] = snapshot(1, 0, 0);
            return id++;
        }
        int m = (l + r) >> 1;
        int lx = update(tree[v].linkl, l, m, x);
        int rx = update(tree[v].linkr, m+1, r, x);
        tree[id] = snapshot(tree[lx].cnt + tree[rx].cnt, lx, rx);
        return id++;
    }
    int query(int a, int b, int l, int r, int k) { // kth
        if (l == r) return l;
        int m = (l + r) >> 1;
        int cnt = tree[tree[b].linkl].cnt - tree[tree[a].linkl].cnt
        if (k <= cnt)
            return query(tree[a].linkl, tree[b].linkl, l, m, k);
        return query(tree[a].linkr, tree[b].linkr, m+1, r, k-cnt);
    }
};
```

## rmq.h
**Description:** Range Minimum/Maximum Queries on an array. Returns min(V[a], V[a + 1], ... V[b]) in constant time. Returns a pair that holds the answer, first element is the value and the second is the index.
**Usage:** rmq_t<pair<int, int>> rmq(values);
// values is a vector of pairs {val(i), index(i)}
rmq.query(inclusive, exclusive)
rmq_t<pair<int, int>, greater<pair<int, int>>> rmq(values)
//max query
**Time:** $\mathcal{O}\left(|V| \log |V| + Q\right)$
<span style="text-align:right">8c53c5, 19 lines</span>

```cpp
template<typename T, typename Cmp=less<T>>
struct rmq_t : private Cmp {
    int N = 0;
    vector<vector<T>> table;
    const T& min(const T& a, const T& b) const { return Cmp::
        operator()(a, b) ? a : b; }
    rmq_t() {}
    rmq_t(const vector<T>& values) : N(int(values.size())), table
        (__lg(N) + 1) {
        table[0] = values;
        for (int a = 1; a < int(table.size()); ++a) {
            table[a].resize(N - (1 << a) + 1);
            for (int b = 0; b + (1 << a) <= N; ++b)
                table[a][b] = min(table[a-1][b], table[a-1][b + (1 << (
                    a-1))]);
        }
    }
    T query(int a, int b) const {
        int lg = __lg(b - a);
        return min(table[lg][a], table[lg][b - (1 << lg) ]);
    }
};
```

## fenwick-tree.h
**Description:** Computes partial sums a[0] + a[1] + ... + a[pos - 1], and updates single elements a[i], taking the difference between the old and new value.
**Time:** Both operations are $\mathcal{O}(\log N)$.
<span style="float:right">19f347, 49 lines</span>

```cpp
template<typename T> struct FT { // 8b7639
  vector<T> s;
  FT(int n) : s(n) {}
  FT(const vector<T>& A) : s(A) {
    const int N = int(s.size());
    for (int a = 0; a < N; ++a) {
      if ((a | (a + 1)) < N) s[a | (a + 1)] += s[a];
    }
  }
  void update(int pos, T dif) { // a[pos] += dif
    for (; pos < (int)s.size(); pos |= pos + 1) s[pos] += dif;
  }
  T query(int pos) { // sum of values in [0, pos)
    T res = 0;
    for (; pos > 0; pos &= pos - 1) res += s[pos-1];
    return res;
  }
  int lower_bound(T sum) {// min pos st sum of [0, pos] >= sum
    // Returns n if no sum is >= sum, or -1 if empty sum is.
    if (sum <= 0) return -1;
    int pos = 0;
    for (int pw = 1 << 25; pw; pw >>= 1) {
      if (pos + pw <= (int)s.size() && s[pos + pw-1] < sum)
        pos += pw, sum -= s[pos-1];
    }
    return pos;
  }
};

template<typename T> struct range_layout { // fd83ef
  FT<T> lhs, rhs;
  range_layout(int N = 0) : lhs(N), rhs(N) {}
  range_layout(const vector<T>& A) : lhs(A), rhs(int(A.size()))
      {}
  void update(int pos, T dif) {
    rhs.update(0, dif);
    rhs.update(pos, -dif);
    lhs.update(pos, (pos - 1) * dif);
  }
  void update(int a, int b, T dif) {
    update(a, -dif);
    update(b + 1, dif);
  }
  T query(int pos) {
    return rhs.query(pos + 1) * pos + lhs.query(pos + 1);
  }
  T query(int a, int b) {
    return query(b) - query(a - 1);
  }
};
```

## fenwick-tree-2d.h
**Description:** Computes sums a[i,j] for all i<I, j<J, and increases single elements a[i,j]. Requires that the elements to be updated are known in advance (call fakeUpdate() before init()).
**Time:** $\mathcal{O}(\log^2 N)$. (Use persistent segment trees for $\mathcal{O}(\log N)$.)
<span style="float:right">aebbdc, 25 lines</span>

```cpp
"fenwick-tree.h"
template<typename T> struct FT2 {
  vector<vector<int>> ys; vector<FT<T>> ft;
  FT2(int limx) : ys(limx) {}
  void fakeUpdate(int x, int y) {
    for (; x < (int)ys.size(); x |= x + 1) ys[x].push_back(y);
  }
```

```cpp
  void init() {
    for(auto &v : ys){
      sort(v.begin(), v.end());
      v.resize(unique(v.begin(),v.end()) - v.begin());
      ft.emplace_back(v.size());
    }
  }
  int ind(int x, int y) {
    return (int)(lower_bound(ys[x].begin(), ys[x].end(), y) -
        ys[x].begin()); }
  void update(int x, int y, T dif) {
    for (; x < ys.size(); x |= x + 1)
      ft[x].update(ind(x, y), dif);
  }
  T query(int x, int y) {
    T sum = 0;
    for (; x; x &= x - 1) sum += ft[x-1].query(ind(x-1, y));
    return sum;
  }
};
```

## mo.h
**Description:** Answer interval or tree path queries by finding an approximate TSP through the queries, and moving from one query to the next by adding/removing points at the ends. If values are on tree edges, change step to add/remove the edge $(a, c)$ and remove the initial add call (but keep in).
**Time:** $\mathcal{O}(N\sqrt{Q})$
<span style="float:right">5ef29d, 49 lines</span>

```cpp
void add(int ind, int end) { ... } // add a[ind] (end = 0 or 1)
void del(int ind, int end) { ... } // remove a[ind]
int calc() { ... } // compute current answer

vector<int> mo(vector<pair<int, int>> Q) { // d9247c
  int L = 0, R = 0, blk = 350; // ~N/sqrt(Q)
  vector<int> s(int(Q.size())), res = s;
#define K(x) pair<int, int>(x.first/blk, x.second ^ -(x.first/
    blk & 1))
  iota(s.begin(), s.end(), 0);
  sort(s.begin(), s.end(), [&](int s, int t){ return K(Q[s]) <
      K(Q[t]); });
  for (int qi : s) {
    auto q = Q[qi];
    while (L > q.first) add(--L, 0);
    while (R < q.second) add(R++, 1);
    while (L < q.first) del(L++, 0);
    while (R > q.second) del(--R, 1);
    res[qi] = calc();
  }
  return res;
}

vector<int> moTree(vector<array<int, 2>> Q, vector<vector<int
    >>& ed, int root=0){ // bbf891
  int N = int(ed.size()), pos[2] = {}, blk = 350; // ~N/sqrt(Q)
  vector<int> s(int(Q.size())), res = s, I(N), L(N), R(N), in(N
    ), par(N);
  add(0, 0), in[0] = 1;
  auto dfs = [&](int x, int p, int dep, auto& f) -> void {
    par[x] = p;
    L[x] = N;
    if (dep) I[x] = N++;
    for (int y : ed[x]) if (y != p) f(y, x, !dep, f);
    if (!dep) I[x] = N++;
    R[x] = N;
  };
  dfs(root, -1, 0, dfs);
#define K(x) pii(I[x[0]] / blk, I[x[1]] ^ -(I[x[0]] / blk & 1))
  iota(s.begin(), s.end(), 0);
```

```cpp
  sort(s.begin(), s.end(), [&](int s, int t){ return K(Q[s]) <
      K(Q[t]); });
  for (int qi : s) for (int end = 0; end < 2; ++end) {
    int &a = pos[end], b = Q[qi][end], i = 0;
#define step(c) { if (in[c]) { del(a, end); in[a] = 0; } \
    else { add(c, end); in[c] = 1; } a = c; }
    while (!(L[b] <= L[a] && R[a] <= R[b]))
      I[i++] = b, b = par[b];
    while (a != b) step(par[a]);
    while (i--) step(I[i]);
    if (end) res[qi] = calc();
  }
  return res;
}
```

## line-container.h
**Description:** Container where you can add lines of the form kx+m, and query maximum values at points x. Useful for dynamic programming ("convex hull trick").
**Time:** $\mathcal{O}(\log N)$
<span style="float:right">8b2ace, 29 lines</span>

```cpp
struct Line {
  mutable lint k, m, p;
  bool operator<(const Line& o) const { return k < o.k; }
  bool operator<(lint x) const { return p < x; }
};
struct LineContainer : multiset<Line, less<>> {
  // (for doubles, use inf = 1/.0, div(a,b) = a/b)
  static const lint inf = LLONG_MAX;
  lint div(lint a, lint b) { // floored division
    return a / b - ((a ^ b) < 0 && a % b); }
  bool isect(iterator x, iterator y) {
    if (y == end()) { x->p = inf; return false; }
    if (x->k == y->k) x->p = x->m > y->m ? inf : -inf;
    else x->p = div(y->m - x->m, x->k - y->k);
    return x->p >= y->p;
  }
  void add(lint k, lint m) {
    auto z = insert({k, m, 0}), y = z++, x = y;
    while (isect(y, z)) z = erase(z);
    if (x != begin() && isect(--x, y)) isect(x, y = erase(y));
    while ((y = x) != begin() && (--x)->p >= y->p)
      isect(x, erase(y));
  }
  lint query(lint x) {
    assert(!empty());
    auto l = *lower_bound(x);
    return l.k * x + l.m;
  }
};
```

## matrix.h
**Description:** Basic operations on square matrices.
**Usage:** Matrix<int> A(N, vector<int>(N));
<span style="float:right">a623ec, 40 lines</span>

```cpp
template <typename T> struct Matrix : vector<vector<T>> {
  using vector<vector<T>>::vector;
  using vector<vector<T>>::size;
  int h() const { return int(size()); }
  int w() const { return int((*this)[0].size()); }
  Matrix operator*(const Matrix& r) const {
    assert(w() == r.h());
    Matrix res(h(), vector<T>(r.w()));
    for (int i = 0; i < h(); ++i) {
      for (int j = 0; j < r.w(); ++j) {
        for (int k = 0; k < w(); ++k) {
          res[i][j] += (*this)[i][k] * r[k][j];
        }
      }
    }
  }
```

```cpp
  }
  return res;
}
friend vector<T> operator*(const Matrix<T>& A, const vector<T
    >& b) {
  int N = int(A.size()), M = int(A[0].size());
  vector<T> y(N);
  for (int i = 0; i < N; ++i) {
    for (int j = 0; j < M; ++j) {
      y[i] += A[i][j] * b[j];
    }
  }
  return y;
}
Matrix& operator*=(const Matrix& r) { return *this = *this *
    r; }
Matrix pow(int n) const {
  assert(h() == w());
  Matrix x = *this, r(h(), vector<T>(w()));
  for (int i = 0; i < h(); ++i) r[i][i] = T(1);
  while (n) {
    if (n & 1) r *= x;
    x *= x;
    n >>= 1;
  }
  return r;
}
};
```

## wavelet.h

**Description:** Segment tree on values instead of indices.
**Time:** $\mathcal{O}(\log(n))$

```cpp
struct wavelet_t { // b26328
  struct BitVector { // space: 32N bits
    vector<int> _rank = {0};
    BitVector(vector<char> v = vector<char>()) {
      _rank.reserve(v.size() + 1);
      for (int d : v) _rank.push_back(_rank.back() + d);
    }
    int rank(bool f, int k) { return f ? _rank[k] : (k - _rank[
        k]); }
    int rank(bool f, int l, int r) { return rank(f, r) - rank(f
        , l); }
  };
  /*
    struct BitVector {   // space: 1.5N bits
    vector<ull> v;
    vector<int> _rank;
    BitVector(vector<char> _v = vector<char>()) {
    int n = int(_v.size());
    v = vector<ull>((n + 63) / 64);
    _rank = vector<int>(v.size() + 1);
    for (int i = 0; i < n; i++) {
    if (_v[i]) {
    v[i / 64] |= 1ULL << (i % 64);
    _rank[i / 64 + 1]++;
    }
    }
    for (int i = 0; i < int(v.size()); i++) {
    _rank[i+1] += _rank[i];
    }
    }
    int rank(int k) {
    int a = _rank[k / 64];
    if (k % 64) a += __builtin_popcountll(v[k / 64] << (64 - k
        % 64));
    return a;
    }
```

```cpp
    int rank(bool f, int k) { return f ? rank(k) : k - rank(k)
        ; }
    int rank(bool f, int l, int r) { return rank(f, r) - rank(
        f, l); }
    };
  */
  int n, lg = 1;
  vector<int> mid;
  vector<BitVector> data;
  wavelet_t(vector<int> v = vector<int>()) : n(int(v.size())) {
    int ma = 0;
    for (int x : v) ma = max(ma, x);
    while ((1 << lg) <= ma) lg++;
    mid = vector<int>(lg);
    data = vector<BitVector>(lg);
    for (int lv = lg - 1; lv >= 0; lv--) {
      vector<char> buf;
      vector<vector<int>> nx(2);
      for (int d : v) {
        bool f = (d & (1 << lv)) > 0;
        buf.push_back(f);
        nx[f].push_back(d);
      }
      mid[lv] = int(nx[0].size());
      data[lv] = BitVector(buf);
      v.clear();
      v.insert(v.end(), nx[0].begin(), nx[0].end());
      v.insert(v.end(), nx[1].begin(), nx[1].end());
    }
  }
  pair<int, int> succ(bool f, int a, int b, int lv) {
    int na = data[lv].rank(f, a) + (f ? mid[lv] : 0);
    int nb = data[lv].rank(f, b) + (f ? mid[lv] : 0);
    return {na, nb};
  }
  // count i, s.t. (a <= i < b) && (v[i] < u)
  int rank(int a, int b, int u) {
    if ((1 << lg) <= u) return b - a;
    int ans = 0;
    for (int lv = lg - 1; lv >= 0; lv--) {
      bool f = (u & (1 << lv)) > 0;
      if (f) ans += data[lv].rank(false, a, b);
      tie(a, b) = succ(f, a, b, lv);
    }
    return ans;
  }
  // k-th(0-indexed!) number in v[a..b]
  int select(int a, int b, int k) {
    int u = 0;
    for (int lv = lg - 1; lv >= 0; lv--) {
      int le = data[lv].rank(false, a, b);
      bool f = (le <= k);
      if (f) {
        u += (1 << lv);
        k -= le;
      }
      tie(a, b) = succ(f, a, b, lv);
    }
    return u;
  }
  // k-th(0-indexed!) largest number in v[a..b]
  int large_select(int a, int b, int k) {
    return select(a, b, b - a - k - 1);
  }
  // count i s.t. (a <= i < b) && (x <= v[i] < y)
  int count(int a, int b, int x, int y) {
    return rank(a, b, y) - rank(a, b, x);
  }
  // max v[i] s.t. (a <= i < b) && (v[i] < x)
```

```cpp
  int pre_count(int a, int b, int x) {
    int cnt = rank(a, b, x);
    return cnt == 0 ? -1 : select(a, b, cnt - 1);
  }
  // min v[i] s.t. (a <= i < b) && (x <= v[i])
  int nxt_count(int a, int b, int x) {
    int cnt = rank(a, b, x);
    return cnt == b - a ? -1 : select(a, b, cnt);
  }
};

struct CompressWavelet { // 2447db
  wavelet_t wt;
  vector<int> v, vidx;
  int zip(int x) {
    return int(lower_bound(vidx.begin(), vidx.end(), x) - vidx.
        begin());
  }
  CompressWavelet(vector<int> _v = vector<int>()) : v(_v), vidx
      (v) {
    sort(vidx.begin(), vidx.end());
    vidx.erase(unique(vidx.begin(), vidx.end()), vidx.end());
    for (auto& d : v) d = zip(d);
    wt = Wavelet(v);
  }
  int rank(int a, int b, int u) { return wt.rank(a, b, zip(u));
      }
  int select(int a, int b, int k) { return vidx[wt.select(a, b,
      k)]; }
  int largest(int a, int b, int k) { return wt.large_select(a,
      b, k); }
  int count(int a, int b, int mi, int ma) { return wt.count(a,
      b, mi, ma); }
  int find_max(int a, int b, int x) { return wt.pre_count(a, b,
      x); }
  int find_min(int a, int b, int x) { return wt.nxt_count(a, b,
      x); }
};
```

## range-color.h

**Description:** RangeColor structure, supports point queries and range updates, if $C$ isn't int32_t change $freq$ to map
**Time:** $\mathcal{O}(\lg(L) * Q)$

```cpp
template<class T = int64_t, class C = int32_t> struct
    RangeColor{

  struct Node{
    T left, right;
    C color;
    bool operator < (const Node &n) const{ return right < n.
        right; }
  };

  C minInf;
  set<Node> st;
  vector<T> freq;

  RangeColor(T first, T last, C maxColor, C iniColor = C(0)):
      minInf(first - T(1)), freq(maxColor + 1) {
    freq[iniColor] = last - first + T(1);
    st.insert({first, last, iniColor});
  }
  //get color in position i
  C query(T i){
    auto p = st.upper_bound({T(0), i - T(1), minInf});
    return p->color;
  }
  //set newColor in [a, b]
```

```cpp
  void upd(T a, T b, C newColor){
    auto p = st.upper_bound({T(0), a - T(1), minInf});
    assert(p != st.end());
    T left = p->left, right = p->right;
    C old = p->color;
    freq[old] -= (right - left + T(1));
    p = st.erase(p);
    if (left < a){
      freq[old] += (a - left);
      st.insert({left, a - T(1), old});
    }
    if (b < right){
      freq[old] += (right - b);
      st.insert({b + T(1), right, old});
    }
    while ((p != st.end()) && (p->left <= b)){
      left = p->left, right = p->right;
      old = p->color;
      freq[old] -= (right - left + T(1));
      if (b < right){
        freq[old] += (right - b);
        st.erase(p);
        st.insert({b + T(1), right, old});
        break;
      } else  p = st.erase(p);
    }
    freq[newColor] += (b - a + T(1));
    st.insert({a, b, newColor});
  }
  T countColor(C x){ return freq[x]; }
};
```

## implicit-treap.h

**Description:** A short self-balancing tree. It acts as a sequential container
with log-time splits/joins, and is easy to augment with additional data.
**Time:** $\mathcal{O}(\log N)$

2d0d97, 96 lines

```cpp
mt19937 rng(chrono::steady_clock::now().time_since_epoch().
    count());
struct node {
  int v, p, sz;
  node *l, *r;
  bool rev;
  node(int k) : v(k), p(rng()), l(nullptr), rev(0), r(nullptr),
      sz(0) {}
};
int sz(node *t) {
  if (t == nullptr) return 0;
  return t->sz;
}
void push(node *t) {
  if (t == nullptr) return;
  if (t->rev) {
    swap(t->l, t->r);
    if (t->l != nullptr) t->l->rev ^= t->rev;
    if (t->r != nullptr) t->r->rev ^= t->rev;
    t->rev = 0;
  }
}
void updsz(node *t) {
  if (t == nullptr) return;
  push(t); push(t->l); push(t->r);
  t->sz = sz(t->l) + sz(t->r) + 1;
}
void split(node *t, node *&l, node *&r, int k) { //k on left
  push(t);
  if (t == nullptr) l = r = nullptr;
  else if (k <= sz(t->l)) {
    split(t->l, l, t->l, k);
```

```cpp
    r = t;
  }
  else {
    split(t->r, t->r, r, k-1-sz(t->l));
    l = t;
  }
  updsz(t);
}
void merge(node *&t, node *l, node *r) {
  push(l); push(r);
  if (l == nullptr) t = r;
  else if (r == nullptr) t = l;
  else if (l->p <= r->p) {
    merge(l->r, l->r, r);
    t = l;
  }
  else {
    merge(r->l, l, r->l);
    t = r;
  }
  updsz(t);
}
void add(node *&t, node *c, int k) {
  push(t);
  if (t == nullptr) t = c;
  else if (c->p >= t->p) {
    split(t, c->l, c->r, k);
    t = c;
  }
  else if (sz(t->l) >= k) add(t->l, c, k);
  else add(t->r, c, k-1-sz(t->l));
  updsz(t);
}
void del(node *&t, int k) {
  push(t);
  if (t == nullptr) return;
  if (sz(t->l) == k) merge(t, t->l, t->r);
  else if (sz(t->l) > k) del(t->l, k);
  else del(t->r, k);
  updsz(t);
}
void print(node *t) {
  if (r == nullptr) return;
  print(t->l);
  cout << t->v << ' ';
  print(t->r);
}

int main() {
  node *treap = nullptr;
  while(1) {
    int a;
    cin >> a;
    if (a == 1) {
      int c, d;
      cin >> c >> d;
      node *r = new node(d);
      add(treap, r, c);
    } else if (a == 2) {
      int d;
      cin >> d;
      del(treap, d);
    }
    print(treap);
  }
}
```

## treap.h

**Description:** A short self-balancing tree. It acts as a sequential container
with log-time splits/joins, and is easy to augment with additional data.
**Time:** $\mathcal{O}(\log N)$

cfd4af, 101 lines

```cpp
mt19937 rng(chrono::steady_clock::now().time_since_epoch().
    count());
struct node {
  int k, p, sz;
  node *l, *r;
  node(int k) : k(k), p(rng()), l(nullptr), r(nullptr), sz(0)
      {}
};

int sz(node *t) {
  if (t == nullptr) return 0;
  return t->sz;
}

void updsz(node *t) {
  if (t == nullptr) return;
  t->sz = sz(t->l) + sz(t->r) + 1;
}
void split(node *t, node *&l, node *&r, int k) {
  if (t == nullptr) l = r = nullptr;
  else if (k <= t->k) {
    split(t->l, l, t->l, k);
    r = t;
  }
  else {
    split(t->r, t->r, r, k);
    l = t;
  }
  updsz(t);
}
void merge(node *&t, node *l, node *r) {
  if (l == nullptr) t = r;
  else if (r == nullptr) t = l;
  else if (l->p <= r->p) {
    merge(l->r, l->r, r);
    t = l;
  }
  else {
    merge(r->l, l, r->l);
    t = r;
  }
  updsz(t);
}
void add(node *&t, node *c) {
  if (t == nullptr) t = c;
  else if (c->p >= t->p) {
    split(t, c->l, c->r, c->k);
    t = c;
  }
  else if (c->k <= t->k) add(t->l, c);
  else add(t->r, c);
  updsz(t);
}
void del(int k, node *&t) {
  if (t->k == k) merge(t, t->l, t->r);
  else if (t->k > k) del(k, t->l);
  else del(k, t->r);
  updsz(t);
}
node *find(node *t, int k) {
  if (t == nullptr) return t;
  if (t->k == k) return t;
  if (t->k < k) return find(t->r, k);
  else return find(t->l, k);
```

```cpp
}
int cnt(node *t, int k) { // <= k
  if (t == nullptr) return 0;
  if (t->k <= k) return 1 + sz(t->l) + cnt(t->r, k);
  else cnt(t->l, k);
}
void print(node *r) {
  if (r == nullptr) return;
  cout << r->k << ' ' << r->p << ' ' << r->sz << '\n';
  print(r->l);  print(r->r);
}
node *kth(node *t, int k) {
  if (k == sz(t->l)) return t;
  else if (k < sz(t->l)) return kth(t->l, k);
  else return kth(t->r, k-1-sz(t->l));
}

int main() {
  node *treap = nullptr;
  while(1) {
    int a;
    cin >> a;
    if (a == 1) {
      int c;
      cin >> c;
      node *r = new node(c);
      add(treap, r);
      print(treap);
    } else if (a == 2) {
      int d;
      cin >> d;
      del(d, treap);
      print(treap);
    } else if (a == 3) {
      for (int i = 0; i < sz(treap); ++i)
        cout << kth(treap, i)->k << '\n';
    }
  }
}
```

# Numerical (4)

## polynomial.h

```cpp
struct Poly {
  vector<double> a;
  double operator()(double x) const {
    double val = 0;
    for(int i = a.size(); i--;) (val *= x) += a[i];
    return val;
  }
  void diff() {
    for(int i = 1; i < a.size(); ++i) a[i-1] = i*a[i];
    a.pop_back();
  }
  void divroot(double x0) {
    double b = a.back(), c; a.back() = 0;
    for(int i = a.size()-1; i--;) c = a[i], a[i] = a[i+1]*x0+b,
        b=c;
    a.pop_back();
  }
};
```

## poly-roots.h

**Description:** Finds the real roots to a polynomial.
**Usage:** poly-roots({{2,-3,1}},-1e9,1e9) // solve x^2-3x+2 = 0
**Time:** $\mathcal{O}\left(n^2 \log(1/\epsilon)\right)$

```cpp
vector<double> poly_roots(Poly p, double xmin, double xmax) {
  if ((p.a).size() == 2) { return {-p.a[0]/p.a[1]}; }
  vector<double> ret;
  Poly der = p;
  der.diff();
  auto dr = poly_roots(der, xmin, xmax);
  dr.push_back(xmin-1);
  dr.push_back(xmax+1);
  sort(dr.begin(), dr.end());
  for(int i = 0; i < dr.size()-1; ++i) {
    double l = dr[i], h = dr[i+1];
    bool sign = p(l) > 0;
    if (sign^(p(h) > 0)) {
      for(int it = 0; it < 60; ++it) { // while (h - l > 1e-8)
        double m = (l + h) / 2, f = p(m);
        if ((f <= 0) ^ sign) l = m;
        else h = m;
      }
      ret.push_back((l + h) / 2);
    }
  }
  return ret;
}
```

## poly-interpolate.h

**Description:** Given $n$ points (x[i], y[i]), computes an n-1-degree polynomial $p$ that passes through them: $p(x) = a[0] * x^0 + ... + a[n-1] * x^{n-1}$. For numerical precision, pick $x[k] = c * \cos(k/(n-1) * \pi), k = 0 \ldots n-1$.
**Time:** $\mathcal{O}\left(n^2\right)$

```cpp
typedef vector<double> vd;
vd interpolate(vd x, vd y, int n) {
  vd res(n), temp(n);
  for(int k = 0; k < n-1; ++k) for(int i = k+1; i < n; ++i)
    y[i] = (y[i] - y[k]) / (x[i] - x[k]);
  double last = 0; temp[0] = 1;
  for(int k = 0; k < n; ++k) for(int i = 0; i < n; ++i) {
    res[i] += y[k] * temp[i];
    swap(last, temp[i]);
    temp[i] -= last * x[k];
  }
  return res;
}
```

## lagrange.h

**Description:** Lagrange interpolation over a finite field and some combo stuff
**Time:** $\mathcal{O}(N)$

```cpp
template<typename T> struct Combinatorics {
  vector<T> pref, suff;
  Combinatorics(int N) : pref(N), suff(N) {}
  T interpolate(const vector<T>& y, T x) {
    int n = int(y.size());
    pref[0] = suff[n - 1] = 1;
    for (int i = 0; i + 1 < n; ++i) {
      pref[i + 1] = pref[i] * (x - i);
    }
    for (int i = n - 1; i > 0; --i) {
      suff[i - 1] = suff[i] * (x - i);
    }
    T res = 0;
    for (int i = 0, sgn = (n % 2 ? +1 : -1); i < n; ++i, sgn *=
        -1) {
      res += y[i] * sgn * pref[i] * suff[i] * invFac[i] *
          invFac[n - 1 - i];
    }
    return res;
  }
};
```

```cpp
T C(int n, int k) {
  return k < 0 || n < k ? 0 : fac[n] * invFac[k] * invFac[n -
      k];
}
T S(int n, int k) {
  return k == 0 ? n == 0 : C(n + k - 1, k - 1);
}
};
```

## berlekamp-massey.h

**Description:** Recovers any $n$-order linear recurrence relation from the first $2n$ terms of the recurrence. Useful for guessing linear recurrences after brute-forcing the first terms. Should work on any field, but numerical stability for floats is not guaranteed. Output will have size $\leq n$.
**Usage:** BerlekampMassey({0, 1, 1, 3, 5, 11}) // {1, 2}
**Time:** $\mathcal{O}\left(N^2\right)$

```cpp
template <typename num>
vector<num> BerlekampMassey(const vector<num>& s) {
  int n = int(s.size()), L = 0, m = 0;
  vector<num> C(n), B(n), T;
  C[0] = B[0] = 1;
  num b = 1;
  for(int i = 0; i < n; i++) { ++m;
    num d = s[i];
    for (int j = 1; j <= L; j++) d += C[j] * s[i - j];
    if (d == 0) continue;
    T = C; num coef = d / b;
    for (int j = m; j < n; j++) C[j] -= coef * B[j - m];
    if (2 * L > i) continue;
    L = i + 1 - L; B = T; b = d; m = 0;
  }
  C.resize(L + 1); C.erase(C.begin());
  for (auto& x : C) x = -x;
  return C;
}
```

## linear-recurrence.h

**Description:** Generates the $k$'th term of an $n$-order linear recurrence $S[i] = \sum_j S[i-j-1]tr[j]$, given $S[0 \ldots n-1]$ and $tr[0 \ldots n-1]$. Faster than matrix multiplication. Useful together with Berlekamp–Massey.
**Usage:** linearRec({0, 1}, {1, 1}, k) // k'th Fibonacci number
**Time:** $\mathcal{O}\left(n^2 \log k\right)$

```cpp
template <typename num>
num linearRec(const vector<num>& S, const vector<num>& tr, lint
    k) {
  int n = int(tr.size());
  assert(S.size() >= tr.size());
  auto combine = [&](vector<num> a, vector<num> b) {
    vector<num> res(n * 2 + 1);
    for (int i = 0; i <= n; i++) for (int j = 0; j <= n; j++)
      res[i + j] += a[i] * b[j];
    for (int i = 2 * n; i > n; --i) for (int j = 0; j < n; j++)
      res[i - 1 - j] += res[i] * tr[j];
    res.resize(n + 1);
    return res;
  };
  vector<num> pol(n + 1), e(pol);
  pol[0] = e[1] = 1;
  for (++k; k; k /= 2) {
    if (k % 2) pol = combine(pol, e);
    e = combine(e, e);
  }
  num res = 0;
  for (int i = 0; i < n; i++) res += pol[i + 1] * S[i];
  return res;
}
```

## integrate.h
**Description:** Simple integration of a function over an interval using Simpson's rule. The error should be proportional to $h^4$, although in practice you will want to verify that the result is stable to desired precision when epsilon changes.

7bb98e, 7 lines

```cpp
template<class F>
double quad(double a, double b, F& f, const int n = 1000) {
  double h = (b - a) / 2 / n, v = f(a) + f(b);
  for(int i = 1; i < n*2; ++i)
    v += f(a + i*h) * (i&1 ? 4 : 2);
  return v * h / 3;
}
```

## integrate-adaptive.h
**Description:** Fast integration using an adaptive Simpson's rule.
**Usage:** double sphereVolume = quad(-1, 1, [](double x) {
return quad(-1, 1, [&](double y) {
return quad(-1, 1, [&](double z) {
return x*x + y*y + z*z < 1; });});});

92dd79, 15 lines

```cpp
typedef double d;
#define S(a,b) (f(a) + 4*f((a+b) / 2) + f(b)) * (b-a) / 6

template <class F>
d rec(F& f, d a, d b, d eps, d S) {
  d c = (a + b) / 2;
  d S1 = S(a, c), S2 = S(c, b), T = S1 + S2;
  if (abs(T - S) <= 15 * eps || b - a < 1e-10)
    return T + (T - S) / 15;
  return rec(f, a, c, eps / 2, S1) + rec(f, c, b, eps / 2, S2);
}
template<class F>
d quad(d a, d b, F f, d eps = 1e-8) {
  return rec(f, a, b, eps, S(a, b));
}
```

## gaussian-elimination.h
"../data-structures/matrix.h"                  6f094c, 86 lines

```cpp
template<typename T> struct gaussian_elimination {
  int N, M;
  Matrix<T> A, E;
  vector<int> pivot;
  int rank, nullity, sgn;
  // O(std::min(N, M)NM)
  gaussian_elimination(const Matrix<T>& A_) : A(A_) {
    N = A.size(), M = A[0].size();
    E = Matrix<T>(N, vector<T>(N));
    for (int i = 0; i < N; ++i) {
      E[i][i] = 1;
    }
    rank = 0, nullity = M, sgn = 0;
    pivot.assign(M, -1);
    for (int col = 0, row = 0; col < M && row < N; ++col) {
      int sel = -1;
      for (int i = row; i < N; ++i) {
        if (A[i][col] != 0) {
          sel = i;
          break;
        }
      }
      if (sel == -1) continue;
      if (sel != row) {
        sgn += 1;
        swap(A[sel], A[row]);
        swap(E[sel], E[row]);
      }
      for (int i = 0; i < N; ++i) {
        if (i == row) continue;
```

```cpp
        T c = A[i][col] / A[row][col];
        for (int j = col; j < M; ++j) {
          A[i][j] -= c * A[row][j];
        }
        for (int j = 0; j < N; ++j) {
          E[i][j] -= c * E[row][j];
        }
      }
      pivot[col] = row++;
      ++rank, --nullity;
    }
  }
  // O(N^2 + M)
  pair<bool, vector<T>> solve(vector<T> b, bool reduced = false
    ) const {
    assert(N == b.size());
    if (reduced == false) b = E * b;
    vector<T> x(M);
    for (int j = 0; j < M; ++j) {
      if (pivot[j] == -1) continue;
      x[j] = b[pivot[j]] / A[pivot[j]][j];
      b[pivot[j]] = 0;
    }
    for (int i = 0; i < N; ++i) {
      if (b[i] != 0) return {false, x};
    }
    return {true, x};
  }
  // O(nullity * NM)
  vector<vector<T>> kernel_basis() const {
    vector<vector<T>> basis;
    vector<T> e(M);
    for (int j = 0; j < M; ++j) {
      if (pivot[j] != -1) continue;
      e[j] = 1;
      auto y = solve(A * e, true).second;
      e[j] = 0, y[j] = -1;
      basis.push_back(y);
    }
    return basis;
  }
  // O(N^3)
  Matrix<T> inverse() const {
    assert(N == M); assert(rank == N);
    Matrix<T> res(N, vector<T>(N));
    vector<T> e(N);
    for (int i = 0; i < N; ++i) {
      e[i] = 1;
      auto x = solve(e).second;
      for (int j = 0; j < N; ++j) {
        res[j][i] = x[j];
      }
      e[i] = 0;
    }
    return res;
  }
};
```

## linear-solver-z2.h
**Description:** Solves $Ax = b$ over $\mathbb{F}_2$. If there are multiple solutions, one is returned arbitrarily. Returns true, or false if no solutions. Last column of $a$ is $b$. $c$ is the rank.
**Time:** $\mathcal{O}(n^2 m)$

7a24e1, 26 lines

```cpp
typedef bitset<2010> bs;
bool gauss(vector<bs> a, bs& ans, int n) {
  int m = int(a.size()), c = 0;
  bs pos; pos.set();
  for (int j = n-1, i; j >= 0; --j) {
```

```cpp
    for (i = c; i < m; ++i)
      if (a[i][j]) break;
    if (i == m) continue;
    swap(a[c], a[i]);
    i = c++; pos[j] = 0;
    for (int k = 0; k < m; ++k)
      if (a[k][j] && k != i)
        a[k] ^= a[i];
  }
  ans = pos;
  for(int i = 0; i < m; ++i) {
    int ac = 0;
    for (int j = 0; j < n ; ++j) {
      if (!a[i][j]) continue;
      if (!pos[j]) pos[j] = 1, ans[j] = ac^a[i][n];
      ac ^= ans[j];
    }
    if (ac != a[i][n]) return false;
  }
  return true;
}
```

## simplex.h
**Description:** Solves a general linear maximization problem: maximize $c^T x$ subject to $Ax \leq b$, $x \geq 0$.
**Time:** $\mathcal{O}(NM * \#pivots)$, where a pivot may be e.g. an edge relaxation. $\mathcal{O}(2^n)$ in the general case. WARNING- segfaults on empty (size 0) max cx st Ax<=b, x>=0 do 2 phases; 1st check feasibility; 2nd check boundedness and ans

c3703c, 39 lines

```cpp
vector<double> simplex(vector<vector<double>> A, vector<double>
    b, vector<double> c) {
  int n = A.size(), m = A[0].size() + 1, r = n, s = m-1;
  vector<vector<double>> D = vector<vector<double>>(n+2, vector
      <double>(m+1));
  vector<int> ix = vector<int>(n + m);
  for (int i = 0; i < n + m; ++i) ix[i] = i;
  for (int i = 0; i < n; ++i) {
    for (int j = 0; j < m-1; ++j) D[i][j] = -A[i][j];
    D[i][m - 1] = 1;
    D[i][m] = b[i];
    if (D[r][m] > D[i][m]) r = i;
  }
  for (int j = 0; j < m-1; ++j) D[n][j] = c[j];
  D[n + 1][m - 1] = -1; int z = 0;
  for (double d;;) {
    if (r < n) {
      swap(ix[s], ix[r + m]);
      D[r][s] = 1.0/D[r][s];
      for (int j = 0; j <= m; ++j) if (j != s) D[r][j] *= -D[r
          ][s];
      for (int i = 0; i <= n+1; ++i) if (i != r) {
        for (int j = 0; j <= m; ++j) if (j != s) D[i][j] += D[r
            ][j] * D[i][s];
        D[i][s] *= D[r][s];
      }
    }
    r = -1; s = -1;
    for (int j = 0; j < m; ++j) if (s < 0 || ix[s] > ix[j])
      if (D[n+1][j] > eps || D[n+1][j] > -eps && D[n][j] > eps)
          s = j;
    if (s < 0) break;
    for (int i = 0; i < n; ++i) if (D[i][s] < -eps) {
      if (r < 0 || (d = D[r][m]/D[r][s]-D[i][m]/D[i][s]) < -eps
          || d < eps && ix[r+m] > ix[i+m]) r = i;
    }
    if (r < 0) return vector<double>(); // unbounded
  }
  if (D[n+1][m] < -eps) return vector<double>(); // infeasible
```

```
    vector<double> x(m-1);
    for (int i = m; i < n+m; ++i) if (ix[i] < m-1) x[ix[i]] = D[i
        -m][m];
    double result = D[n][m];
    return x; // ans: D[n][m]
}
```

## tridiagonal.h

**Description:** $x = \mathrm{tridiagonal}(d, p, q, b)$ solves the equation system

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} d_0 & p_0 & 0 & 0 & \cdots & 0 \\ q_0 & d_1 & p_1 & 0 & \cdots & 0 \\ 0 & q_1 & d_2 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{n-3} & d_{n-2} & p_{n-2} \\ 0 & 0 & \cdots & 0 & q_{n-2} & d_{n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{pmatrix}.$$

This is useful for solving problems on the type

$$a_i = b_i a_{i-1} + c_i a_{i+1} + d_i,\ 1 \le i \le n,$$

where $a_0, a_{n+1}, b_i, c_i$ and $d_i$ are known. $a$ can then be obtained from

$$\{a_i\} = \mathrm{tridiagonal}(\{1, -1, -1, ..., -1, 1\}, \{0, c_1, c_2, \ldots, c_n\},$$
$$\{b_1, b_2, \ldots, b_n, 0\}, \{a_0, d_1, d_2, \ldots, d_n, a_{n+1}\}).$$

Fails if the solution is not unique.
If $|d_i| > |p_i| + |q_{i-1}|$ for all $i$, or $|d_i| > |p_{i-1}| + |q_i|$, or the matrix is positive definite, the algorithm is numerically stable and neither `tr` nor the check for `diag[i] == 0` is needed.
**Time:** $\mathcal{O}(N)$

```
typedef double T;
vector<T> tridiagonal(vector<T> diag, const vector<T> &super,
    const vector<T> &sub, vector<T> b) {
  int n = b.size(); vector<int> tr(n);
  for(int i = 0; i < n-1; ++i) {
    if (abs(diag[i]) < 1e-9 * abs(super[i])) { // diag[i] == 0
      b[i+1] -= b[i] * diag[i+1] / super[i];
      if (i+2 < n) b[i+2] -= b[i] * sub[i+1] / super[i];
      diag[i+1] = sub[i]; tr[++i] = 1;
    } else {
      diag[i+1] -= super[i]*sub[i]/diag[i];
      b[i+1] -= b[i]*sub[i]/diag[i];
    }
  }
  for (int i = n; i--;) {
    if (tr[i]) {
      swap(b[i], b[i-1]);
      diag[i-1] = diag[i];
      b[i] /= super[i-1];
    } else {
      b[i] /= diag[i];
      if (i) b[i-1] -= b[i]*super[i-1];
    }
  }
  return b;
}
```

# 4.1 Fourier transforms

## fast-fourier-transform.h

**Description:** fft(a) computes $\hat{f}(k) = \sum_x a[x] \exp(2\pi i \cdot kx/N)$ for all $k$. N must be a power of 2. Useful for convolution: `conv(a, b) = c`, where $c[x] = \sum a[i]b[x-i]$. For convolution of complex numbers or more than two vectors: FFT, multiply pointwise, divide by n, reverse(start+1, end), FFT back. Rounding is safe if $(\sum a_i^2 + \sum b_i^2) \log_2 N < 9 \cdot 10^{14}$ (in practice $10^{16}$; higher for random inputs). Otherwise, use NTT/FFTMod.
**Time:** $\mathcal{O}(N \log N)$ with $N = |A| + |B|$ ($\sim$1s for $N = 2^{22}$)

```
inline int nxt_pow2(int s) { return 1 << (s > 1 ? 32 -
    __builtin_clz(s-1) : 0); }
```

```
template <typename dbl> struct cplx {
  dbl x, y;
  cplx(dbl x_ = 0, dbl y_ = 0) : x(x_), y(y_) { }
  friend cplx operator+(cplx a, cplx b) { return cplx(a.x + b.x
      , a.y + b.y); }
  friend cplx operator-(cplx a, cplx b) { return cplx(a.x - b.x
      , a.y - b.y); }
  friend cplx operator*(cplx a, cplx b) { return cplx(a.x * b.x
      - a.y * b.y, a.x * b.y + a.y * b.x); }
  friend cplx conj(cplx a) { return cplx(a.x, -a.y); }
  friend cplx inv(cplx a) { dbl n = (a.x*a.x+a.y*a.y); return
      cplx(a.x/n,-a.y/n); }
};

template <typename T>
struct root_of_unity {};
template <typename dbl> struct root_of_unity<cplx<dbl>> {
  static cplx<dbl> f(int k) {
    static const dbl PI = acos(-1);
    dbl a = 2*PI/k;
    return cplx<dbl>(cos(a),sin(a));
  }
};

using M0 = modnum<998244353U>;  // g = 3
using M1 = modnum<897581057U>;  // g = 3
using M2 = modnum<880803841U>;  // g = 26
using M3 = modnum<985661441U>;  // g = 3
using M4 = modnum<943718401U>;  // g = 7
using M5 = modnum<935329793U>;  // g = 3
using M6 = modnum<918552577U>;  // g = 5

constexpr unsigned primitive_root(unsigned M) {
  if (M == 880803841U) return 26U;
  else if (M == 943718401U) return 7U;
  else if (M == 918552577U) return 5U;
  else return 3U;
}
template<unsigned MOD> struct root_of_unity<modnum<MOD>> {
  static constexpr modnum<MOD> g0 = primitive_root(MOD);
  static modnum<MOD> f(int K) {
    assert((MOD-1)%K == 0);
    return g0.pow((MOD-1)/K);
  }
};

template<typename T> struct FFT {
  vector<T> rt; vector<int> rev;
  FFT() : rt(2, T(1)) {}
  void init(int N) {
    N = nxt_pow2(N);
    if (N > int(rt.size())) {
      rev.resize(N); rt.reserve(N);
      for (int a = 0; a < N; ++a) {
        rev[a] = (rev[a/2] | ((a&1)*N)) >> 1;
      }
      for (int k = int(rt.size()); k < N; k *= 2) {
        rt.resize(2*k);
        T z = root_of_unity<T>::f(2*k);
        for (int a = k/2; a < k; ++a) {
          rt[2*a] = rt[a];
          rt[2*a+1] = rt[a] * z;
        }
      }
    }
  }
  void fft(vector<T>& xs, bool inverse) const {
    int N = int(xs.size());
    int s = __builtin_ctz(int(rev.size())/N);
```

```
    if (inverse) reverse(xs.begin() + 1, xs.end());
    for (int a = 0; a < N; ++a) {
      if (a < (rev[a] >> s))
        swap(xs[a], xs[rev[a] >> s]);
    }
    for (int k = 1; k < N; k *= 2) {
      for (int a = 0; a < N; a += 2*k) {
        int u = a, v = u + k;
        for (int b = 0; b < k; ++b, ++u, ++v) {
          T z = rt[b + k] * xs[v];
          xs[v] = xs[u] - z;
          xs[u] = xs[u] + z;
        }
      }
    }
    if (inverse) {
      for (int a = 0; a < N; ++a)
        xs[a] = xs[a] * inv(T(N));
    }
  }
  vector<T> convolve(vector<T> as, vector<T> bs) {
    int N = int(as.size()), M = int(bs.size());
    int K = N + M - 1, S = nxt_pow2(K); init(S);
    if (min(N, M) <= 64) {
      vector<T> res(K);
      for (int u = 0; u < N; ++u)
        for (int v = 0; v < M; ++v)
          res[u + v] = res[u + v] + as[u] * bs[v];
      return res;
    } else {
      as.resize(S), bs.resize(S);
      fft(as, false); fft(bs, false);
      for (int i = 0; i < S; ++i) as[i] = as[i] * bs[i];
      fft(as, true); as.resize(K);
      return as;
    }
  }
};

FFT<M0> FFT0; FFT<M1> FFT1;
FFT<M2> FFT2; FFT<M3> FFT3;
FFT<M4> FFT4; FFT<M5> FFT5; FFT<M6> FFT6;

// M0 M1 = 896005221510021121 (> 4.48 * 10^{17}, > 2^58)
// M0 M1 M2 = 789204846662082423367925761 (> 7.892 * 10^26, > 2
    ^89)
// M0 M3 M4 M5 M6 =
    797766583174034668024539679147517452591562753 (> 7.977 *
    10^44, > 2^149)

// T = {unsigned, unsigned long long, modnum<M>}
template<class T, unsigned M0, unsigned M1, unsigned M2>
T garner(modnum<M0> a0, modnum<M1> a1, modnum<M2> a2) {
  static const modnum<M1> INV_M0_M1 = modnum<M1>(M0).inv();
  static const modnum<M2> INV_M0M1_M2 = (modnum<M2>(M0) * M1).
      inv();
  const modnum<M1> b1 = INV_M0_M1 * (a1 - a0.x);
  const modnum<M2> b2 = INV_M0M1_M2 * (a2 - (modnum<M2>(b1.x) *
      M0 + a0.x));
  return (T(b2.x) * M1 + b1.x) * M0 + a0.x;
}
template<class T, unsigned M0, unsigned M1, unsigned M2,
    unsigned M3, unsigned M4>
T garner(modnum<M0> a0, modnum<M1> a1, modnum<M2> a2, modnum<M3
    > a3, modnum<M4> a4) {
  static const modnum<M1> INV_M0_M1 = modnum<M1>(M0).inv();
  static const modnum<M2> INV_M0M1_M2 = (modnum<M2>(M0) * M1).
      inv();
```

```cpp
  static const modnum<M3> INV_M0M1M2_M3 = (modnum<M3>(M0) * M1
      * M2).inv();
  static const modnum<M4> INV_M0M1M2M3_M4 = (modnum<M4>(M0) *
      M1 * M2 * M3).inv();
  const modnum<M1> b1 = INV_M0_M1 * (a1 - a0.x);
  const modnum<M2> b2 = INV_M0M1_M2 * (a2 - (modnum<M2>(b1.x) *
      M0 + a0.x));
  const modnum<M3> b3 = INV_M0M1M2_M3 * (a3 - ((modnum<M3>(b2.x
      ) * M1 + b1.x) * M0 + a0.x));
  const modnum<M4> b4 = INV_M0M1M2M3_M4 * (a4 - (((modnum<M4>(
      b3.x) * M2 + b2.x) * M1 + b1.x) * M0 + a0.x));
  return (((T(b4.x) * M3 + b3.x) * M2 + b2.x) * M1 + b1.x) * M0
      + a0.x;
}

// results must be in [-448002610255888384, 448002611254132736]
vector<long long> convolve(const vector<long long>& as, const
    vector<long long>& bs) {
  static constexpr unsigned M0 = M0::M, M1 = M1::M;
  static const modnum<M1> INV_M0_M1 = modnum<M1>(M0).inv();
  if (as.empty() || bs.empty()) return {};
  const int len_as = int(as.size()), len_bs = int(bs.size());
  vector<modnum<M0>> as0(len_as), bs0(len_bs);
  for (int i = 0; i < len_as; ++i) as0[i] = as[i];
  for (int i = 0; i < len_bs; ++i) bs0[i] = bs[i];
  const vector<modnum<M0>> cs0 = FFT0.convolve(as0, bs0);
  vector<modnum<M1>> as1(len_as), bs1(len_bs);
  for (int i = 0; i < len_as; ++i) as1[i] = as[i];
  for (int i = 0; i < len_bs; ++i) bs1[i] = bs[i];
  const vector<modnum<M1>> cs1 = FFT1.convolve(as1, bs1);
  vector<long long> cs(len_as + len_bs - 1);
  for (int i = 0; i < len_as + len_bs - 1; ++i) {
    const modnum<M1> d1 = INV_M0_M1 * (cs1[i] - cs0[i].x);
    cs[i] = (d1.x > M1 - d1.x)
      ? (-1ULL - (static_cast<unsigned long long>(M1 - 1U - d1.
          x) * M0 + (M0 - 1U - cs0[i].x)))
      : (static_cast<unsigned long long>(d1.x) * M0 + cs0[i].x)
          ;
  }
  return cs;
}
```

### fast-subset-transform.h
**Description:** Transform to a basis with fast convolutions of the form $c[z] = \sum_{z=x\oplus y} a[x] \cdot b[y]$, where $\oplus$ is one of AND, OR, XOR. The size of $a$ must be a power of two.
**Time:** $\mathcal{O}(N \log N)$

```cpp
void FST(vector<int> &a, bool inv) {
  for (int n = a.size(), step = 1; step < n; step *= 2) {
    for (int i = 0; i < n; i += 2 * step) for(int j = i; j < i+
        step; ++j) {
      int &u = a[j], &v = a[j + step]; tie(u, v) =
        inv ? pii(v - u, u) : pii(v, u + v); // AND
        inv ? pii(v, u - v) : pii(u + v, u); // OR
        pii(u + v, u - v);                   // XOR
    }
  }
  if (inv) for(auto &x : a) x /= a.size(); // XOR only
}
vector<int> conv(vector<int> a, vector<int> b) {
  FST(a, 0); FST(b, 0);
  for(int i = 0; i < a.size(); ++i) a[i] *= b[i];
  FST(a, 1); return a;
}
```

### sum-of-powers.h
**Description:** Computes monomials and sum of powers product certain polynomials. Check "General purpose numbers" section for more info. (Monomials) $pw(x) = x^d$ for a fixed $d$. (Sum of power limit) $\sum_{x=0}^{\infty} r^x f(x)$. (degree of $f \le d$). (Sum of powers til n) $\sum_{x=0}^{N} r^x f(x)$. (degree of $f \le d$).

```cpp
vector<num> get_monomials(int N, long long d) {
  vector<int> pfac(N);
  for (int i = 2; i < N; ++i) pfac[i] = i;
  for (int p = 2; p < N; ++p) if (pfac[p] == p) {
    for (int m = 2*p; m < N; m += p) if (pfac[m] > p) pfac[m] =
        p;
  }
  vector<num> pw(N);
  for (int i = 0; i < N; ++i) {
    if (i <= 1 || pfac[i] == i) pw[i] = num(i).pow(d);
    else pw[i] = (pw[pfac[i]] * pw[i / pfac[i]]);
  }
  return pw;
}

num sum_of_power_limit(num r, int d, const vector<num>& fs) {
  Combinatorics<num> M(d + 2);
  vector<num> qs(d + 1); qs[0] = 1;
  for (int x = 1; x <= d; ++x) qs[x] = qs[x - 1] * r;
  num ans = 0, cur_sum = 0;
  for (int x = 0; x <= d; ++x) {
    cur_sum += qs[x] * fs[x];
    ans += cur_sum * invFac[d - x] * invFac[x + 1] * (((d - x)
        & 1) ? -1 : +1) * qs[d - x];
  }
  // ans is equivalent to invFac(d + 1) * dp(d+1), where
  // for all x in [0, d], dp(x + 1) := E(d, d-x) + dp(x) * r,
  //    dp(0) = 0.
  // with E being the eulerian number. Works in O(d^2).
  ans *= (1 - r).pow(-(d + 1)) * fac[d + 1];
  return ans;
}

num sum_of_power(num r, int d, const vector<num>& fs, long long
    N) {
  if (r == 0) return (0 < N) ? fs[0] : 0;
  Combinatorics<num> M(d + 10);
  vector<num> gs(d + 2); gs[0] = 0;
  num rr = 1;
  for (int x = 0; x <= d; ++x) {
    gs[x + 1] = gs[x] + rr * fs[x];
    rr *= r;
  }
  if (r == 1) return M.interpolate(gs, N);
  const num c = sum_of_power_limit(r, d, fs);
  const num r_inv = r.inv();
  num rr_inv = 1;
  for (int x = 0; x <= d + 1; ++x) {
    gs[x] = rr_inv * (gs[x] - c);
    rr_inv *= r_inv;
  }
  return c + r.pow(N) * M.interpolate(gs, N);
}
```

#### 4.1.1 Duality

max $c^T x$ sjt to $Ax \le b$. Dual problem is min $b^T x$ sjt to $A^T x \ge c$. By strong duality, min max value coincides.

#### 4.1.2 Strong duality

Given a linear problem $\Pi_1$: minimize $c^t x$, sjt to $Ax \le b$, $x \ge 0$ we can define the linear problem dual standard $\Pi_2$ like the following: minimize $-b^t y$, sjt to $A^t y \ge c$. If $\Pi_1$ is satisfied then $\Pi_2$ is also satisfied and $c^t x = b^t y$. If $\Pi_1$ is not satisfied and unbounded, then $\Pi_2$ is not satisfied and unbounded. (OBS: Can't be both unbounded!)

#### 4.1.3 Polyominoes

How many free (rotation, reflection), one-sided (rotation) and fixed $n$-ominoes are there?

| n | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| free | 2 | 5 | 12 | 35 | 108 | 369 | 1.285 | 4.655 |
| one-sided | 2 | 7 | 18 | 60 | 196 | 704 | 2.500 | 9.189 |
| fixed | 6 | 19 | 63 | 216 | 760 | 2.725 | 9.910 | 36.446 |

#### 4.1.4 Generating functions

A list of generating functions for useful sequences:

| | |
|---|---|
| $(1, 1, 1, 1, 1, 1, \dots)$ | $\frac{1}{1-z}$ |
| $(1, -1, 1, -1, 1, -1, \dots)$ | $\frac{1}{1+z}$ |
| $(1, 0, 1, 0, 1, 0, \dots)$ | $\frac{1}{1-z^2}$ |
| $(1, 0, \dots, 0, 1, 0, 1, 0, \dots, 0, 1, 0, \dots)$ | $\frac{1}{1-z^2}$ |
| $(1, 2, 3, 4, 5, 6, \dots)$ | $\frac{1}{(1-z)^2}$ |
| $(1, \binom{m+1}{m}, \binom{m+2}{m}, \binom{m+3}{m}, \dots)$ | $\frac{1}{(1-z)^{m+1}}$ |
| $(1, c, \binom{c+1}{2}, \binom{c+2}{3}, \dots)$ | $\frac{1}{(1-z)^c}$ |
| $(1, c, c^2, c^3, \dots)$ | $\frac{1}{1-cz}$ |
| $(0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ | $\ln \frac{1}{1-z}$ |

A neat manipulation trick is:

$$\frac{1}{1-z}G(z) = \sum_n \sum_{k \le n} g_k z^n$$

# Number theory (5)

## 5.1 Modular arithmetic

### modular-arithmetic.h
**Description:** Operators for modular arithmetic.

```cpp
template<unsigned M_> struct modnum {
  static constexpr unsigned M = M_;
  using ll = long long; using ull = unsigned long long;
    unsigned x;

  // ec131c
  constexpr modnum() : x(0U) {}
  constexpr modnum(unsigned a) : x(a % M) {}
  constexpr modnum(ull a) : x(a % M) {}
  constexpr modnum(int a) : x(((a %= static_cast<int>(M)) < 0)
      ? (a + static_cast<int>(M)) : a) {}
  constexpr modnum(ll a) : x(((a %= static_cast<ll>(M)) < 0) ?
      (a + static_cast<ll>(M)) : a) {}
  // da738b
  explicit operator int() const { return x; }
```

```cpp
modnum& operator+=(const modnum& a) { x = ((x += a.x) >= M) ?
    (x - M) : x; return *this; }
modnum& operator-=(const modnum& a) { x = ((x -= a.x) >= M) ?
    (x + M) : x; return *this; }
modnum& operator*=(const modnum& a) { x = unsigned((
    static_cast<ull>(x) * a.x) % M); return *this; }
modnum& operator/=(const modnum& a) { return (*this *= a.inv
    ()); }
// ef603f
modnum operator+(const modnum& a) const { return (modnum(*
    this) += a); }
modnum operator-(const modnum& a) const { return (modnum(*
    this) -= a); }
modnum operator*(const modnum& a) const { return (modnum(*
    this) *= a); }
modnum operator/(const modnum& a) const { return (modnum(*
    this) /= a); }
modnum operator+() const { return *this; }
modnum operator-() const { modnum a; a.x = x ? (M - x) : 0U;
    return a; }
// b8b96c
template<typename T> friend modnum operator+(T a, const
    modnum& b) { return (modnum(a) += b); }
template<typename T> friend modnum operator-(T a, const
    modnum& b) { return (modnum(a) -= b); }
template<typename T> friend modnum operator*(T a, const
    modnum& b) { return (modnum(a) *= b); }
template<typename T> friend modnum operator/(T a, const
    modnum& b) { return (modnum(a) /= b); }

modnum pow(ll e) const {
    if (e < 0) return inv().pow(-e);
    modnum x2 = x, xe = 1U;
    for (; e; e >>= 1) {
        if (e & 1) xe *= x2;
        x2 *= x2;
    }
    return xe;
}
modnum inv() const {
    unsigned a = x, b = M; int y = 1, z = 0;
    while (a) {
        const unsigned q = (b/a), c = (b - q*a);
        b = a, a = c; const int w = z - static_cast<int>(q) * y;
        z = y, y = w;
    } assert(b == 1U); return modnum(z);
}
friend modnum inv(const modnum& a) { return a.inv(); }

explicit operator bool() const { return x; }
friend bool operator==(const modnum& a, const modnum& b) {
    return a.x == b.x; }
friend bool operator!=(const modnum& a, const modnum& b) {
    return a.x != b.x; }
friend ostream &operator<<(ostream& os, const modnum& a) {
    return os << a.x; }
friend istream &operator>>(istream& in, modnum& n) { ull v_;
    in >> v_; n = modnum(v_); return in; }
};
```

## mod-inv.h
**Description:** Find $x$ such that $ax \equiv 1 \pmod m$. The inverse only exist if $a$ and $m$ are coprimes.
<div align="right">811ad8, 20 lines</div>

```cpp
template<typename T>
T modinv(T a) { // a^-1 (mod 2^64)
    assert(a & 1);
    T b = ((a << 1) + a) * ((a << 1) + a);
    b *= 2 - a * b;
```

```cpp
    b *= 2 - a * b;
    b *= 2 - a * b;
    b *= 2 - a * b;
    return b;
}
template<typename T>
T modinv(T a, T m) { // a^-1 (mod m)
    assert(m > 0);
    if (m == 1) return 0;
    a %= m;
    if (a < 0) a += m;
    assert(a != 0);
    if (a == 1) return 1;
    return m - modinv(m, a) * m/a;
}
```

## mod-sum.h
**Description:** Sums of mod'ed arithmetic progressions.
$\mathrm{modsum}(to, c, k, m) = \sum_{i=0}^{to-1}(ki + c)\%m$. divsum is similar but for floored division.
**Time:** $\log(m)$, with a large constant.
<div align="right">decfb8, 17 lines</div>

```cpp
typedef unsigned long long ull;
ull sumsq(ull to) { return to / 2 * ((to-1) | 1); }
ull divsum(ull to, ull c, ull k, ull m) {
    ull res = k / m * sumsq(to) + c / m * to;
    k %= m; c %= m;
    if (k) {
        ull to2 = (to * k + c) / m;
        res += to * to2;
        res -= divsum(to2, m-1 - c, m, k) + to2;
    }
    return res;
}
lint modsum(ull to, lint c, lint k, lint m) {
    c = ((c % m) + m) % m;
    k = ((k % m) + m) % m;
    return to * c + k * sumsq(to) - m * divsum(to, c, k, m);
}
```

## mod-mul.h
**Description:** Calculate $a \cdot b \mod c$ (or $a^b \mod c$) for $0 \le a, b \le c \le 7.2 \cdot 10^{18}$.
**Time:** $\mathcal{O}(1)$ for modmul, $\mathcal{O}(\log b)$ for modpow
<div align="right">59afa8, 11 lines</div>

```cpp
typedef unsigned long long ull;
ull modmul(ull a, ull b, ull M) {
    lint ret = a * b - M * ull(1.L / M * a * b);
    return ret + M * (ret < 0) - M * (ret >= (lint)M);
}
ull modpow(ull b, ull e, ull mod) {
    ull ans = 1;
    for (; e; b = modmul(b, b, mod), e /= 2)
        if (e & 1) ans = modmul(ans, b, mod);
    return ans;
}
```

## mod-sqrt.h
**Description:** Tonelli-Shanks algorithm for modular square roots. Finds $x$ s.t. $x^2 = a \pmod p$ ($-x$ gives the other solution).
**Time:** $\mathcal{O}(\log^2 p)$ worst case, $\mathcal{O}(\log p)$ for most $p$
<div align="right">829f86, 50 lines</div>

```cpp
uint xrand() {
    static uint x = 314159265, y = 358979323, z = 846264338, w =
        327950288;
    uint t = x ^ x << 11; x = y; y = z; z = w; return w = w ^ w
        >> 19 ^ t ^ t >> 8;
}

// Jacobi symbol (a/m)
```

```cpp
//   m > 0, m: odd
int jacobi(int64_t a, int64_t m) {
    int s = 1;
    if (a < 0) a = a % m + m;
    for (; m > 1; ) {
        a %= m;
        if (a == 0) return 0;
        const int r = __builtin_ctzll(a);
        if ((r & 1) && ((m + 2) & 4)) s = -s;
        a >>= r;
        if (a & m & 2) s = -s;
        swap(a, m);
    }
    return s;
}

// sqrt(a) (mod p)
//   p: prime, p < 2^31, -p^2 <= a <= P^2
//   (b + sqrt(b^2 - a))^((p+1)/2) in F_p(sqrt(b^2 - a))
vector<int64_t> mod_sqrt(int64_t a, int64_t p) {
    if (p == 2) return {a & 1};
    const int j = jacobi(a, p);
    if (j == 0) return {0};
    if (j == -1) return {};
    int64_t b, d;
    for (; ; ) {
        b = xrand() % p;
        d = (b * b - a) % p;
        if (d < 0) d += p;
        if (jacobi(d, p) == -1) break;
    }
    int64_t f0 = b, f1 = 1, g0 = 1, g1 = 0, tmp;
    for (int64_t e = (p + 1) >> 1; e; e >>= 1) {
        if (e & 1) {
            tmp = (g0 * f0 + d * ((g1 * f1) % p)) % p;
            g1 = (g0 * f1 + g1 * f0) % p;
            g0 = tmp;
        }
        tmp = (f0 * f0 + d * ((f1 * f1) % p)) % p;
        f1 = (2 * f0 * f1) % p;
        f0 = tmp;
    }
    return (g0 < p - g0) ? vector<int64_t>{g0, p - g0} : vector<
        int64_t>{p - g0, g0};
}
```

## mod-range.h
**Description:** min $x \ge 0$ s.t. $l \le ((ax) \mod m) \le r$, $m > 0$, $a \ge 0$.
<div align="right">eb665e, 11 lines</div>

```cpp
template<typename T> T mod_range(T m, T a, T l, T r) {
    l = max(l, T(0));
    r = min(r, m - 1);
    if (l > r) return -1;
    a %= m;
    if (a == 0) return (l > 0) ? -1 : 0;
    const T k = (l + a - 1) / a;
    if (a * k <= r) return k;
    const T y = mod_range(a, m, a * k - r, a * k - l);
    return (y == -1) ? -1 : ((m * y + r) / a);
}
```

## 5.2  Primality

### sieve.h
**Description:** Prime sieve for generating all primes up to a certain limit. $pfac[i]$ is the lowest prime factor of $i$. Also useful if you need to compute any multiplicative function.
**Time:** $\mathcal{O}(N)$
<div align="right">a76cb9, 24 lines</div>

```cpp
vector<int> run_sieve(int N) {
```

```
    vector<int> pfac(N + 1);
    vector<int> primes; primes.reserve(N+1);
    vector<int> mu(N + 1, -1); mu[1] = 1;
    vector<int> phi(N + 1); phi[1] = 1;
    for (int i = 2; i <= N; ++i) {
        if (!pfac[i]) {
            pfac[i] = i; primes.push_back(i);
            phi[i] = i - 1;
        }
        for (int p : primes) {
            if (p > N/i) break;
            pfac[p * i] = p;
            mu[p * i] *= mu[i];
            phi[p * i] = phi[i] * phi[p];
            if (i % p == 0) {
                mu[p * i] = 0;
                phi[p * i] = phi[i] * p;
                break;
            }
        }
    }
    return primes;
}
```

### miller-rabin.h

**Description:** Deterministic Miller-Rabin primality test. Guaranteed to work for numbers up to $2^{64}$; for larger numbers, extend A randomly.
**Time:** 7 times the complexity of $a^b \mod c$.

`"mod-mul.h"`                      bbee97, 12 lines

```
bool isPrime(ull n) {
    if (n < 2 || n % 6 % 4 != 1) return (n | 1) == 3;
    vector<ull> A = {2, 325, 9375, 28178, 450775, 9780504,
        1795265022};
    ull s = __builtin_ctzll(n-1), d = n >> s;
    for(ull a : A) {   // ^ count trailing zeroes
        ull p = modpow(a % n, d, n), i = s;
        while (p != 1 && p != n - 1 && a % n && i--)
            p = modmul(p, p, n);
        if (p != n-1 && i != s) return 0;
    }
    return 1;
}
```

### pollard-rho.h

**Description:** Pollard-rho randomized factorization algorithm. Returns prime factors of a number, in arbitrary order (e.g. 2299 -> {11, 19, 11}).
**Time:** $\mathcal{O}\left(n^{1/4}\right)$, less for numbers with small factors.

`"mod-mul.h"`, `"extended-euclid.h"`, `"miller-rabin.h"`     6bf31f, 18 lines

```
ull pollard(ull n) {
    auto f = [n](ull x, ull k) { return modmul(x, x, n) + k; };
    ull x = 0, y = 0, t = 30, prd = 2, i = 1, q;
    while (t++ % 40 || gcd(prd, n) == 1) {
        if (x == y) x = ++i, y = f(x, i);
        if ((q = modmul(prd, max(x,y) - min(x,y), n))) prd = q;
        x = f(x, i), y = f(f(y, i), i);
    }
    return gcd(prd, n);
}
vector<ull> factor(ull n) {
    if (n == 1) return {};
    if (isPrime(n)) return {n};
    ull x = pollard(n);
    auto l = factor(x), r = factor(n / x);
    l.insert(l.end(), r.begin(), r.end());
    return l;
}
```

## 5.3 Divisibility

### extended-euclid.h

**Description:** Finds two integers $x$ and $y$, such that $ax + by = \gcd(a, b)$. If you just need gcd, use the built in __gcd instead. If $a$ and $b$ are coprime, then $x$ is the inverse of $a \pmod{b}$.
                             8b62c4, 6 lines

```
template<typename T>
T egcd(T a, T b, T &x, T &y) {
    if (!a) { x = 0, y = 1; return b; }
    T g = egcd(b % a, a, y, x);
    x -= y * (b/a); return g;
}
```

### division-lemma.h

**Description:** This lemma let us exploit the fact tha the sequence (harmonic on integer division) has at most $2\sqrt{N}$ distinct elements, so we can iterate through every possible value of $\lfloor\frac{N}{i}\rfloor$, using the fact that the greatest integer $j$ satisfying $\lfloor\frac{N}{i}\rfloor = \lfloor\frac{N}{j}\rfloor$ is $\lfloor\frac{N}{\lfloor\frac{N}{i}\rfloor}\rfloor$. This one computes the $\sum_{i=1}^{N}\lfloor\frac{N}{i}\rfloor i$.

**Time:** $\mathcal{O}\left(\sqrt{N}\right)$
                                 b2c1ab, 27 lines

```
// floor(N/a) = K
// <=> K <= N/a < K + 1
// <=> K/(K+1) < a <= N/K
// <=> floor(N/(K+1)) < a <= floor(N/K)
int res = 0;
for (int a = 1, b; a <= N; a = b + 1) {
    b = N / (N / a);
    // for all i in [a, b] since they all have the same quotient
    //     (N / a)
    // and there are (b - a + 1) elements in this interval
    int l = b - a + 1, r = a + b; // l * r / 2 = sum(i, j)
    if (l & 1) r /= 2;
    else l /= 2;
    res += l * r * (N / a);
}

// ceil(N/a) = K
// <=> K-1 < N/a <= K
// <=> N/K <= a < N/(K-1)
// <=> ceil(N/K) <= a < ceil(N/(K-1))
// ceil(N/a) = floor((N-1)/a) + 1

// [1, N), need to deal with case where a = N separately
for (int a = 1, b; a < N; a = b + 1) {
    const int k = (N - 1) / a + 1; // quotient k
    b = (N - 1) / (k - 1);
    int cnt = b - a + 1; // occur cnt times on interval [a, b]
}
```

### divisors.h

**Description:** Generate all factors of $n$ given it's prime factorization.
**Time:** $\mathcal{O}\left(\frac{\sqrt{N}}{\log N}\right)$

`"prime-factors.h"`                          0526d1, 15 lines

```
template<typename T> vector<T> get_divisors(T N) {
    auto factors = prime_factorize(N);
    vector<T> ans; ans.reserve(int(sqrtl(N) + 1));
    auto dfs = [&](auto&& self, auto& ans, T val, int depth) ->
        void {
        auto& [P, E] = factors[depth];
        if (depth == int(factors.size())) ans.push_back(val);
        else {
            T X = 1;
            for (int pw = 0; pw <= E; ++pw, X *= P) {
                self(self, ans, val * X, depth + 1);
            }
        }
    }
```

```
    }; dfs(dfs, ans, 1, 0);
    return ans;
}
```

### phi-function.h

**Description:** *Euler's totient* or *Euler's phi* function is defined as $\phi(n) := \#$ of positive integers $\leq n$ that are coprime with $n$. The *cototient* is $n - \phi(n)$. $\phi(1) = 1$, $p$ prime $\Rightarrow \phi(p^k) = (p-1)p^{k-1}$, $m, n$ coprime $\Rightarrow \phi(mn) = \phi(m)\phi(n)$. If $n = p_1^{k_1}p_2^{k_2}...p_r^{k_r}$ then $\phi(n) = (p_1-1)p_1^{k_1-1}...(p_r-1)p_r^{k_r-1}$. $\phi(n) = n \cdot \prod_{p|n}(1 - 1/p)$. $\sum_{d|n}\phi(d) = n$, $\sum_{1\leq k\leq n, \gcd(k,n)=1} k = n\phi(n)/2, n > 1$
**Euler's thm:** $a, n$ coprime $\Rightarrow a^{\phi(n)} \equiv 1 \pmod{n}$.
**Fermat's little thm:** $p$ prime $\Rightarrow a^{p-1} \equiv 1 \pmod{p}$ $\forall a$.
                        da7671, 7 lines

```
const int n = int(1e5)*5;
vector<int> phi(n);
void calculatePhi() {
    for(int i = 0; i < n; ++i) phi[i] = i&1 ? i : i/2;
    for(int i = 3; i < n; i += 2) if (phi[i] == i)
        for(int j = i; j < n; j += i) phi[j] -= phi[j]/i;
}
```

### discrete-log.h

**Description:** Returns the smallest $x >= 0$ s.t. $a^x = b \pmod{m}$, or $-1$ if no such $x$ exists. modLog(a,1,m) can be used calculate the order of $a$. Assumes that $0^0 = 1$.
**Time:** $\mathcal{O}\left(\sqrt{m}\right)$

`"extended-euclid.h"`                      6c6eb0, 18 lines

```
template<typename T> T modLog(T a, T b, T m) {
    T k = 1, it = 0, g;
    while ((g = gcd(a, m)) != 1) {
        if (b == k) return it;
        if (b % g) return -1;
        b /= g; m /= g; ++it;
        k = k * a / g % m;
    }
    T n = sqrtl(m) + 1, f = 1, j = 1;
    unordered_map<T, T> A;
    while (j <= n) {
        f = f * a % m;
        A[f * b % m] = j++;
    }
    for(int i = 1; i <= n; ++i) if (A.count(k = k * f % m))
        return n * i - A[k] + it;
    return -1;
}
```

### prime-counting.h

**Description:** Count the number of primes up to $x$. Also useful for sum of primes.
**Time:** $\mathcal{O}\left(n^{3/4}/\log n\right)$
                                6fa7c7, 54 lines

```
using ll = int64_t;
int isqrt(ll n) {
    return sqrtl(n);
}

ll count_primes(const ll N) {
    if (N <= 1) return 0;
    if (N == 2) return 1;
    const int v = isqrt(N);
    int s = (v + 1) / 2;
    vector<int> smalls(s);
    for (int i = 1; i < s; i++) smalls[i] = i;
    vector<int> roughs(s);
    for (int i = 0; i < s; i++) roughs[i] = 2 * i + 1;
    vector<ll> larges(s);
```

```cpp
  for (int i = 0; i < s; i++) larges[i] = (N / (2 * i + 1) - 1)
       / 2;
  vector<bool> skip(v + 1);
  const auto divide = [](ll n, ll d) -> int { return (double)n
       / d;};
  const auto half = [](int n) -> int { return (n - 1) >> 1;};
  int pc = 0;
  for (int p = 3; p <= v; p += 2) if (!skip[p]) {
    int q = p * p;
    if ((ll)q * q > N) break;
    skip[p] = true;
    for (int i = q; i <= v; i += 2 * p) skip[i] = true;
    int ns = 0;
    for (int k = 0; k < s; k++) {
      int i = roughs[k];
      if (skip[i]) continue;
      ll d = (ll)i * p;
      larges[ns] = larges[k] - (d <= v ? larges[smalls[d >> 1]
          - pc) : smalls[half(divide(N, d))]) + pc;
      roughs[ns++] = i;
    }
    s = ns;
    for (int i = half(v), j = ((v / p) - 1) | 1; j >= p; j -=
        2) {
      int c = smalls[j >> 1] - pc;
      for (int e = (j * p) >> 1; i >= e; i--) smalls[i] -= c;
    }
    pc++;
  }
  larges[0] += (ll)(s + 2 * (pc - 1)) * (s - 1) / 2;
  for (int k = 1; k < s; k++) larges[0] -= larges[k];
  for (int l = 1; l < s; l++) {
    ll q = roughs[l];
    ll M = N / q;
    int e = smalls[half(M / q)] - pc;
    if (e < l + 1) break;
    ll t = 0;
    for (int k = l + 1; k <= e; k++)
      t += smalls[half(divide(M, roughs[k]))];
    larges[0] += t - (ll)(e - l) * (pc + l - 1);
  }
  return larges[0] + 1;
}
```

## 5.4   Chinese remainder theorem

### chinese-remainder.h
**Description:** Chinese Remainder Theorem. `crt(a, m, b, n)` computes $x$ such that $x \equiv a \pmod{m}$, $x \equiv b \pmod{n}$. If $|a| < m$ and $|b| < n$, $x$ will obey $0 \le x < \text{lcm}(m, n)$. Assumes $mn < 2^{62}$.
**Time:** $\mathcal{O}(n \log(LCM(m)))$

<div style="text-align: right">"extended-euclid.h"    ecbf25, 14 lines</div>

```cpp
template<typename T>
pair<T, T> crt(const vector<T>& a, const vector<T>& m) {
  int N = int(a.size());
  T r = 0, md = 1, x, y;
  for (int i = 0; i < N; ++i) {
    T g = egcd(md, m[i], x = 0, y = 0);
    T im = x;
    if ((a[i] - r) % g) return {0, -1};
    T tmp = (a[i] - r) / g * im % (m[i] / g);
    r += md * tmp;
    md *= m[i] / g;
  }
  return {(r % md + md) % md, md};
}
```

## 5.5   Fractions

### fractions.h
**Description:** Template that helps deal with fractions.

<div style="text-align: right">df1f1d, 31 lines</div>

```cpp
template<typename num = long long>
struct frac {
  num n, d;
  frac() : n(0), d(1) { }
  frac(num _n, num _d = 1): n(_n), d(_d){
    num g = gcd(n, d); n /= g, d /= g;
    if (d < 0) n *= -1, d *= -1;
    assert(d != 0);
  }
  friend bool operator<(const frac& l, const frac& r) { return
      l.n * r.d < r.n * l.d; }
  friend bool operator==(const frac& l, const frac& r) { return
      l.n == r.n && l.d == r.d; }
  friend bool operator!=(const frac& l, const frac& r) { return
      !(l == r); }
  friend frac operator+(const frac& l, const frac& r) {
    num g = gcd(l.d, r.d);
    return frac( r.d / g * l.n + l.d / g * r.n, l.d / g * r.d); }
  friend frac operator-(const frac& l, const frac& r) {
    num g = gcd(l.d, r.d);
    return frac( r.d / g * l.n - l.d / g * r.n, l.d / g * r.d); }
  friend frac operator*(const frac& l, const frac& r) { return
      frac(l.n * r.n, l.d * r.d); }
  friend frac operator/(const frac& l, const frac& r) { return
      l * frac(r.d, r.n); }
  friend frac& operator+=(frac& l, const frac& r) { return l =
      l+r; }
  friend frac& operator-=(frac& l, const frac& r) { return l =
      l-r; }
  template<class T> friend frac& operator*=(frac& l, const T& r
      ) { return l = l*r; }
  template<class T> friend frac& operator/=(frac& l, const T& r
      ) { return l = l/r; }
  friend ostream& operator<<(ostream& strm, const frac& a) {
    strm << a.n << "/" << a.d;
    return strm;
  }
};
```

### continued-fractions.h
**Description:** Given $N$ and a real number $x \ge 0$, finds the closest rational approximation $p/q$ with $p, q \le N$. It will obey $|p/q - x| \le 1/qN$.
For consecutive convergents, $p_{k+1}q_k - q_{k+1}p_k = (-1)^k$. ($p_k/q_k$ alternates between $> x$ and $< x$.) If $x$ is rational, $y$ eventually becomes $\infty$; if $x$ is the root of a degree 2 polynomial the $a$'s eventually become cyclic.
**Time:** $\mathcal{O}(\log N)$

<div style="text-align: right">61608b, 21 lines</div>

```cpp
typedef double dbl; // for N ~ 1e7; long double for N ~ 1e9
pair<lint, lint> approximate(dbl x, lint N) {
  lint LP = 0, LQ = 1, P = 1, Q = 0, inf = LLONG_MAX; dbl y = x
      ;
  for (;;) {
    lint lim = min(P ? (N-LP) / P : inf, Q ? (N-LQ) / Q : inf),
        a = (lint)floor(y), b = min(a, lim),
        NP = b*P + LP, NQ = b*Q + LQ;
    if (a > b) {
      // If b > a/2, we have a semi-convergent that gives us a
      // better approximation; if b = a/2, we *may* have one.
      // Return {P, Q} here for a more canonical approximation.
      return (abs(x - (dbl)NP / (dbl)NQ) < abs(x - (dbl)P / (
          dbl)Q)) ?
        make_pair(NP, NQ) : make_pair(P, Q);
    }
```

```cpp
    if (abs(y = 1/(y - (dbl)a)) > 3*N) {
      return {NP, NQ};
    }
    LP = P; P = NP;
    LQ = Q; Q = NQ;
  }
}
```

### frac-binary-search.h
**Description:** Given $f$ and $N$, finds the smallest fraction $p/q \in [0, 1]$ such that $f(p/q)$ is true, and $p, q \le N$. You may want to throw an exception from $f$ if it finds an exact solution, in which case $N$ can be removed.
**Usage:** `fracBS([](Frac f) { return f.p>=3*f.q; }, 10); // {1,3}`
**Time:** $\mathcal{O}(\log(N))$

<div style="text-align: right">f83d46, 23 lines</div>

```cpp
struct Frac { lint p, q; };
template<class F>
Frac fracBS(F f, lint N) {
  bool dir = 1, A = 1, B = 1;
  Frac left{0, 1}, right{1, 1}; // Set right to 1/0 to search
      (0, N]
  assert(!f(left)); assert(f(right));
  while (A || B) {
    lint adv = 0, step = 1; // move right if dir, else left
    for (int si = 0; step; (step *= 2) >>= si) {
      adv += step;
      Frac mid{left.p * adv + right.p, left.q * adv + right.q};
      if (abs(mid.p) > N || mid.q > N || dir == !f(mid)) {
        adv -= step; si = 2;
      }
    }
    right.p += left.p * adv;
    right.q += left.q * adv;
    dir = !dir;
    swap(left, right);
    A = B; B = !!adv;
  }
  return dir ? right : left;
}
```

#### 5.5.1   Bézout's identity
For $a \ne$, $b \ne 0$, then $d = gcd(a, b)$ is the smallest positive integer for which there are integer solutions to

$$ax + by = d$$

If $(x, y)$ is one solution, then all solutions are given by

$$\left( x + \frac{kb}{\gcd(a,b)}, y - \frac{ka}{\gcd(a,b)} \right), \quad k \in \mathbb{Z}$$

#### 5.5.2   Pythagorean Triples
The Pythagorean triples are uniquely generated by

$$a = k \cdot (m^2 - n^2), \; b = k \cdot (2mn), \; c = k \cdot (m^2 + n^2),$$

with $m > n > 0$, $k > 0$, $m \perp n$, and either $m$ or $n$ even.

#### 5.5.3   Primitive Roots
It only exists when $n$ is $2, 4, p^k, 2p^k$, where $p$ odd prime. If $g$ is a primitive root, all primitive roots are of the form $g^k$ where $k, \phi(p)$ are coprime (hence there are $\phi(\phi(p))$ primitive roots).

#### 5.5.4 Chicken McNugget theorem

Let $x$ and $y$ be two coprime integers, the greater integer that can't be written in the form of $ax + by$ is $\frac{(x-1)(y-1)}{2}$.

### 5.6 Primes

$p = 962592769$ is such that $2^{21} \mid p - 1$, which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than $1\,000\,000$.

Primitive roots exist modulo any prime power $p^a$, except for $p = 2, a > 2$, and there are $\phi(\phi(p^a))$ many. For $p = 2, a > 2$, the group $\mathbb{Z}_{2^a}^{\times}$ is instead isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$.

#### 5.6.1 Wilson's theorem

Let $n > 1$. Then $n | (n-1)! + 1$ iff $n$ is prime.

#### 5.6.2 Wolstenholme's theorem

Let $p > 3$ be a prime number. Then its numerator $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$ is divisible by $p^2$.

#### 5.6.3 Prime counting function ($\pi(x)$)

The prime counting function is asymptotic to $\frac{x}{\log x}$, by the prime number theorem.

| x | 10 | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ | $10^7$ | $10^8$ |
|---|---|---|---|---|---|---|---|---|
| $\pi(x)$ | 4 | 25 | 168 | 1.229 | 9.592 | 78.498 | 664.579 | 5.761.455 |

#### 5.6.4 Sum of primes

For any multiplicative $f$:

$$S(n, p) = S(n, p-1) - f(p) \cdot (S(n/p, p-1) - S(p-1, p-1))$$

#### 5.6.5 Moebius Function

$$\mu(n) = \begin{cases} 0 & n \text{ is not square free} \\ 1 & n \text{ has even number of prime factors} \\ -1 & n \text{ has odd number of prime factors} \end{cases}$$

Moebius Inversion:

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d) g(n/d)$$

Other useful formulas/forms:

$\sum_{d|n} \phi(d) = n$

$\sum_{\substack{i<n \\ \gcd(i,n)=1}} i = n \frac{\phi(n)}{2}$

$\sum_{a=1}^{n} \sum_{b=1}^{n} [\gcd(a,b) = 1] = \sum_{d=1}^{n} \mu(d) \lfloor \frac{n}{d} \rfloor^2$

$\sum_{a=1}^{n} \sum_{b=1}^{n} \gcd(a,b) = \sum_{d=1}^{n} d \sum_{d|x} \lfloor \frac{n}{x} \rfloor^2 \mu(\frac{x}{d})$

$\sum_{a=1}^{n} \sum_{b=a}^{n} \gcd(a,b) = \sum_{d=1}^{n} \sum_{d|x} \phi(\frac{x}{d}) d$

$\sum_{a=1}^{n} \sum_{b=1}^{n} \text{lcm}(a,b) = \sum_{d=1}^{n} \mu(d) d \sum_{d|x} x \left( \frac{\lfloor \frac{n}{x} \rfloor + 1}{2} \right)^2$

$\sum_{a=1}^{n} \sum_{b=a+1}^{n} \text{lcm}(a,b) = \sum_{d=1}^{n} \sum_{d|x} \phi(\frac{x}{d}) \frac{x^2}{2d}$

$\sum_{a \in S} \sum_{b \in S} \gcd(a,b) = \sum_{d=1}^{n} (\sum_{x|d} \frac{d}{x} \mu(x)) (\sum_{d|v} \text{freq}[v])^2$

$\sum_{a \in S} \sum_{b \in S} \text{lcm}(a,b) = \sum_{d=1}^{n} (\sum_{x|d} \frac{x}{d} \mu(x)) (\sum_{v \in S, d|v} v)^2$

$\sum_{d|n} \mu(d) = [n = 1]$ (very useful)

$g(n) = \sum_{n|d} f(d) \Leftrightarrow f(n) = \sum_{n|d} \mu(d/n) g(d)$

$g(n) = \sum_{1 \le m \le n} f(\lfloor \frac{n}{m} \rfloor) \Leftrightarrow f(n) = \sum_{1 \le m \le n} \mu(m) g(\lfloor \frac{n}{m} \rfloor)$

#### 5.6.6 Dirichlet Convolution

Given a function $f(x)$, let

$$(f * g)(x) = \sum_{d|x} g(d) f(x/d)$$

If the partial sums $s_{f*g}(n), s_g(n)$ can be computed in $O(1)$ and $s_f(1 \ldots n^{2/3})$ can be computed in $O\left(n^{2/3}\right)$ then all $s_f\left(\frac{n}{d}\right)$ can as well. Use

$$s_{f*g}(n) = \sum_{d=1}^{n} g(d) s_f(n/d).$$

$$\implies s_f(n) = \frac{s_{f*g}(n) - \sum_{d=2}^{n} g(d) s_f(n/d)}{g(1)}$$

1. If $f(x) = \mu(x)$ then $g(x) = 1, (f * g)(x) = (x == 1)$, and $s_f(n) = 1 - \sum_{i=2}^{n} s_f(n/i)$

2. If $f(x) = \phi(x)$ then $g(x) = 1, (f * g)(x) = x$, and $s_f(n) = \frac{n(n+1)}{2} - \sum_{i=2}^{n} s_f(n/i)$

**dirichlet-convolution.h**
**Description:** Dirichlet convolution. Change $f$, $gs$ and $fgs$ accordingly. This example calculates $\phi(N)$.
**Time:** $\mathcal{O}\left(N^{\frac{2}{3}}\right)$

```cpp
template<typename T, typename V> struct dirichlet_convolution {
  V N; // ~ N^{2/3}
  T inv;
  vector<V> fs; // can be any multiplicative function
  vector<T> psum;
  unordered_map<V, T> mapa;
  V f(V x) { return fs[x]; }
  T gs(V x) { return x; }
  T fgs(V x) { return T(x) * (x + 1) / 2; }
  dirichlet_convolution(V _N, const vector<V>& F) : N(_N + 1),
      fs(F), psum(_N + 1) {
    inv = gs(1);
    for (V a = 0; a + 1 < N; ++a) {
      psum[a + 1] = f(a + 1) + psum[a];
    }
  }
  T query(V x) {
    if (x < N) return psum[x];
    if (mapa.find(x) != mapa.end()) return mapa[x];
    T ans = fgs(x);
    for (V a = 2, b; a <= x; a = b + 1) {
      b = x / (x / a);
      ans -= (gs(b) - gs(a - 1)) * query(x / a);
    }
    return mapa[x] = (ans / inv);
  }
};
```

#### 5.6.7 Estimates

$$\sum_{d|n} d = O(n \log \log n)$$

The number of divisors of $n$ is at most around 100 for $n < 5e4$, 500 for $n < 1e7$, 2000 for $n < 1e10$, 200\,000 for $n < 1e19$.

# Combinatorial (6)

### 6.1 Permutations

#### 6.1.1 Factorial

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $n!$ | 1 | 2 | 6 | 24 | 120 | 720 | 5040 | 40320 | 362880 | 3628800 |

| $n$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|
| $n!$ | 4.0e7 | 4.8e8 | 6.2e9 | 8.7e10 | 1.3e12 | 2.1e13 | 3.6e14 |

| $n$ | 20 | 25 | 30 | 40 | 50 | 100 | 150 | 171 |
|---|---|---|---|---|---|---|---|---|
| $n!$ | 2e18 | 2e25 | 3e32 | 8e47 | 3e64 | 9e157 | 6e262 | >DBL_MAX |

**int-perm.h**
**Description:** Permutation -> integer conversion. (Not order preserving.)
**Time:** $\mathcal{O}(n)$

```cpp
int permToInt(vector<int>& v) {
  int use = 0, i = 0, r = 0;
  for (auto &x : v) r=r * ++i + __builtin_popcount(use & -(1 <<
      x)),
    use |= 1 << x;              // (note: minus, not ~!)
  return r;
}
```

#### 6.1.2 Binomials

- Sum of every element in the $n$-th row of pascal triangle is $2^n$.

- The product of the elements in each row is $\frac{(n+1)^n}{n!}$

- $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

- In a row $p$ where $p$ is a prime number, all the terms in that row except the 1s are multiples of $p$

- To count odd terms in row $n$, convert $n$ to binary. Let $x$ be the number of 1s in the binary representation. Then the number of odd terms will be $2^x$

- Every entry in row $2^n - 1$ is odd

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

$$\sum_{k=0}^{n}\binom{n}{k}k = n2^{n-1}$$

$$\sum_{m=0}^{n}\binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1}$$

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

$$\sum_{j=0}^{m}\binom{m}{j}^2 = \binom{2m}{m}$$

$$2\sum_{i=L}^{R}\binom{n}{i} - \binom{n}{L} - \binom{n}{R} = \sum_{i=L+1}^{R}\binom{n+1}{i}$$

$$\binom{n}{k}\binom{n-k}{v} = \binom{n}{v}\binom{n-v}{k}$$

$$\sum_{j=0}^{k}\binom{m}{j}\binom{n-m}{k-j} = \binom{n}{k}$$

$$\sum_{r=0}^{n}m\binom{n+r}{r} = \binom{n+m+1}{m}$$

$$\sum_{m=k}^{n}\binom{m}{k} = \binom{n+1}{k+1}$$

## rolling-binomial.h

**Description:** $\binom{n}{k}$ (mod $m$) in time proportional to the difference between $(n,k)$ and the previous $(n,k)$.

`"../number-theory/preparator.h"`      d087bf, 14 lines

```cpp
using i64 = int64_t;
const int mod = int(1e9) + 7;
struct Bin {
    int N = 0,  K = 0; i64 r = 1;
    void m(int a, int b) { r = r * a % mod * invs[b] % mod; }
    i64 choose(int n, int k) {
        if (k > n || k < 0) return 0;
        while(N < n) ++N, m(N, N - K);
        while(K < k) ++K, m(N - K + 1, K);
        while(K > k) m(K, N - K + 1), --K;
        while(N > n) m(N - K, N), --N;
        return r;
    }
};
```

## lucas.h

**Description:** Lucas' thm: Let $n, m$ be non-negative integers and $p$ a prime. Write $n = n_k p^k + \ldots + n_1 p + n_0$ and $m = m_k p^k + \ldots + m_1 p + m_0$. Then $\binom{n}{m} \equiv \prod_{i=0}^{k}\binom{n_i}{m_i}$ (mod $p$). fact and ifact must hold pre-computed factorials / inverse factorials, e.g. from ModInv.h.
**Time:** $\mathcal{O}\left(\log_p m\right)$

`"../number-theory/preparator.h"`      c55480, 10 lines

```cpp
ll chooseModP(ll n, ll m, int p) {
    assert(m < 0 || m > n);
    ll c = 1;
    for (; m > 0; n /= p, m /= p) {
        lint n0 = n % p, m0 = m % p;
        if (n0 < m0) return 0;
        c = c * ((((fac[n0] * invFac[m0]) % p) * invFac[n0 - m0]) %
            p) % p;
    }
    return c;
}
```

## multinomial.h

**Description:** Computes $\binom{k_1 + \cdots + k_n}{k_1, k_2, \ldots, k_n} = \frac{(\sum k_i)!}{k_1! k_2! \ldots k_n!}$.
     864cdb, 7 lines

```cpp
lint multinomial(vector<int>& v) {
    lint c = 1, m = v.empty() ? 1 : v[0];
    for (int i = 1; i < v.size(); ++i)
        for (int j = 0; j < v[i]; ++j)
            c = c * ++m / (j+1);
    return c;
}
```

### 6.1.3   Involutions

An involution is a permutation with maximum cycle length 2, and it is its own inverse.

$$a(n) = a(n-1) + (n-1)a(n-2)$$

$$a(0) = a(1) = 1$$

---

1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, 35696, 140152

### 6.1.4   Cycles

Let the number of $n$-permutations whose cycle lengths all belong to the set $S$ be denoted by $g_S(n)$

$$\sum_{n=0}^{\infty}g_S(n)\frac{x^n}{n!} = \exp\left(\sum_{n\in S}\frac{x^n}{n}\right)$$

### 6.1.5   Inclusion-Exclusion Principle

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then $A_1 \cup A_2 \cup \ldots \cup A_n$ is

$$\left|\bigcup_{i=1}^{n}A_i\right| = \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ I \neq \emptyset}}(-1)^{|I|+1}\left|\bigcap_{i\in I}A_i\right|$$

### 6.1.6   The twelvefold way (from Stanley)

How many functions $f\colon N \to X$ are there?

| $N$ | $X$ | Any $f$ | Injective | Surjective |
|---|---|---|---|---|
| dist. | dist. | $x^n$ | $\frac{x!}{(x-n)!}$ | $x!\{{n \atop x}\}$ |
| indist. | dist. | $\binom{x+n-1}{n}$ | $\binom{x}{n}$ | $\binom{n-1}{n-x}$ |
| dist. | indist. | $\{{n \atop 1}\} + \ldots + \{{n \atop x}\}$ | $[n \le x]$ | $\{{n \atop k}\}$ |
| indist. | indist. | $p_1(n) + \ldots p_x(n)$ | $[n \le x]$ | $p_x(n)$ |

Where $\binom{a}{b} = \frac{1}{b!}(a)_b$, $p_x(n)$ is the number of ways to partition the integer $n$ using $x$ summand and $\{{n \atop x}\}$ is the number of ways to partition a set of $n$ elements into $x$ subsets (aka Stirling number of the second kind).

### 6.1.7   Burnside

Given a group $G$ of symmetries and a set $X$, the number of elements of $X$ *up to symmetry* equals

$$\frac{1}{|G|}\sum_{g\in G}|X^g|,$$

where $X^g$ are the elements fixed by $g$ ($g.x = x$).

If $f(n)$ counts "configurations" (of some sort) of length $n$, we can ignore rotational symmetry using $G = \mathbb{Z}_n$ to get

$$g(n) = \frac{1}{n}\sum_{k=0}^{n-1}f(\gcd(n,k)) = \frac{1}{n}\sum_{k|n}f(k)\phi(n/k).$$

### 6.1.8   Derangements

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1)+D(n-2)) = nD(n-1) + (-1)^n = \left\lfloor\frac{n!}{e}\right\rfloor$$

---

## 6.2   Partitions and subsets

### 6.2.1   Partition function

Number of ways of writing $n$ as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1, \; p(n) = \sum_{k\in\mathbb{Z}\setminus\{0\}}(-1)^{k+1}p(n - k(3k-1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 20 | 50 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p(n)$ | 1 | 1 | 2 | 3 | 5 | 7 | 11 | 15 | 22 | 30 | 627 | ~2e5 | ~2e8 |

## 6.3   General purpose numbers

### 6.3.1   Bernoulli numbers

EGF of Bernoulli numbers is $B(t) = \frac{t}{e^t-1}$ (FFT-able).
$B[0, \ldots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \ldots]$

Sums of powers:

$$\sum_{i=1}^{n}n^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k(n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\sum_{i=m}^{\infty}f(i) = \int_{m}^{\infty}f(x)dx - \sum_{k=1}^{\infty}\frac{B_k}{k!}f^{(k-1)}(m)$$

$$\approx \int_{m}^{\infty}f(x)dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m))$$

### 6.3.2   Stirling numbers of the first kind

Number of permutations on $n$ items with $k$ cycles.

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k), c(0,0) = 1$$

$$\sum_{k=0}^{n}c(n,k)x^k = x(x+1)\ldots(x+n-1)$$

$c(8,k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$
$c(n,2) = 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \ldots$

### 6.3.3   Eulerian numbers

Number of permutations $\pi \in S_n$ in which exactly $k$ elements are greater than the previous element. $k$ $j$:s s.t. $\pi(j) > \pi(j+1)$, $k+1$ $j$:s s.t. $\pi(j) \ge j$, $k$ $j$:s s.t. $\pi(j) > j$.

$$E(n,k) = (n-k)E(n-1,k-1) + (k+1)E(n-1,k)$$

$$E(n,0) = E(n,n-1) = 1$$

$$E(n,k) = \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k+1-j)^n$$

### 6.3.4   Stirling numbers of the second kind

Partitions of $n$ distinct elements into exactly $k$ groups.

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

$$S(n,1) = S(n,n) = 1$$

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

### 6.3.5 Bell numbers

Total number of partitions of $n$ distinct elements. $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \ldots$.

$$\mathcal{B}_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_k$$

Also possible to calculate using Stirling numbers of the second kind,

$$B_n = \sum_{k=0}^{n} S(n, k)$$

If $p$ is prime:

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod{p}$$

### 6.3.6 Labeled unrooted trees

\# on $n$ vertices: $n^{n-2}$
\# on $k$ existing trees of size $n_i$: $n_1 n_2 \cdots n_k n^{k-2}$
\# with degrees $d_i$: $(n-2)!/((d_1 - 1)! \cdots (d_n - 1)!)$ \# forests with exactly $k$ rooted trees:

$$\binom{n}{k} k \cdot n^{n-k-1}$$

.

### 6.3.7 Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)! n!}$$

$$C_0 = 1, \ C_{n+1} = \frac{2(2n+1)}{n+2} C_n, \ C_{n+1} = \sum C_i C_{n-i}$$

$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$

- sub-diagonal monotone paths in a $n \times n$ grid.
- strings with $n$ pairs of parenthesis, correctly nested.
- binary trees with with $n+1$ leaves (0 or 2 children) or $2n + 1$ elements.
- ordered trees with $n+1$ vertices.
- \# ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.
- permutations of $[n]$ with no 3-term increasing subsequence.

### 6.3.8 Super Catalan numbers

The number of monotonic lattice paths of a $n$x$n$ grid that do not touch the diagonal.

$$S(n) = \frac{3(2n-3)S(n-1) - (n-3)S(n-2)}{n}$$

$$S(1) = S(2) = 1$$

$1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859$

### 6.3.9 Motzkin numbers

Number of ways of drawing any number of nonintersecting chords among $n$ points on a circle. Number of lattice paths from $(0, 0)$ to $(n, 0)$ never going below the $x$-axis, using only steps NE, E, SE.

$$M(n) = \frac{3(n-1)M(n-2) + (2n+1)M(n-1)}{n+2}$$

$$M(0) = M(1) = 1$$

$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634$

### 6.3.10 Narayana numbers

Number of lattice paths from $(0,0)$ to $(2n,0)$ never going below the $x$-axis, using only steps NE and SE, and with $k$ peaks.

$$N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$$

$$N(n, 1) = N(n, n) = 1$$

$$\sum_{k=1}^{n} N(n, k) = C_n$$

$1, 1, 1, 1, 3, 1, 1, 6, 6, 1, 1, 10, 20, 10, 1, 1, 15, 50$

### 6.3.11 Schroder numbers

Number of lattice paths from $(0, 0)$ to $(n, n)$ using only steps N,NE,E, never going above the diagonal. Number of lattice paths from $(0, 0)$ to $(2n, 0)$ using only steps NE, SE and double east EE, never going below the x-axis. Twice the Super Catalan number, except for the first term.
$1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098$

### 6.3.12 Triangles

Given rods of length 1, ..., $n$,

$$T(n) = \frac{1}{24} \left\{ \begin{array}{ll} n(n-2)(2n-5) & n \text{ even} \\ (n-1)(n-3)(2n-1) & n \text{ odd} \end{array} \right\}$$

is the number of distinct triangles (positive are) that can be constructed, i.e., the \# of 3-subsets of $[n]$ s.t. $x \le y \le z$ and $z \ne x + y$.

## 6.4 Fibonacci

$Fib(x + y) = Fib(x + 1)Fib(y) + Fib(x)Fib(y - 1)$

$Fib(n + 1)Fib(n - 1) - Fib(n)^2 = (-1)^n$

$Fib(2n - 1) = Fib(n)^2 - Fib(n - 1)^2$

$\sum_{i=0}^{n} Fib(i) = Fib(n + 2) - 1$

$\sum_{i=0}^{n} Fib(i)^2 = Fib(n)Fib(n + 1)$

$\sum_{i=0}^{n} Fib(i)^3 = \frac{Fib(n)Fib(n+1)^2 - (-1)^n Fib(n-1) + 1}{2}$

## 6.5 Linear Recurrences

(i) $F_n = F_{n-1} + F_{n-2}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

(ii) $F_i = \sum_{j=1}^{K} C_j F_{i-j}$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & . & 0 \\ 0 & 0 & 1 & 0 & . & 0 \\ 0 & 0 & 0 & 1 & . & 0 \\ . & . & . & . & & . \\ C_K & C_{K-1} & C_{K-2} & C_{K-3} & . & C_1 \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ . \\ F_{K-1} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ . \\ F_K \end{bmatrix}$$

(iii) $F_i = \sum_{j=1}^{K} C_j F_{i-j} + D$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 1 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 1 & . & 0 & 0 \\ . & . & . & . & & . & . \\ C_K & C_{K-1} & C_{K-2} & C_{K-3} & . & C_1 & 1 \\ 0 & 0 & 0 & 0 & . & 0 & 1 \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ . \\ F_{K-1} \\ D \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ . \\ F_K \\ D \end{bmatrix}$$

## 6.6 Game Theory

A game can be reduced to Nim if it is a finite impartial game. Nim and its variants include:

### 6.6.1 Nim

Let $X = \bigoplus_{i=1}^{n} x_i$, then $(x_i)_{i=1}^{n}$ is a winning position iff $X \ne 0$. Find a move by picking $k$ such that $x_k > x_k \oplus X$.

### 6.6.2 Misère Nim

Regular Nim, except that the last player to move *loses*. Play regular Nim until there is only one pile of size larger than 1, reduce it to 0 or 1 such that there is an odd number of piles. The second player wins $(a_1, \ldots, a_n)$ if 1) there is a pile $a_i > 1$ and $\oplus_{i=1}^{n} a_i = 0$ or 2) all $a_i \le 1$ and $\oplus_{i=1}^{n} a_i = 1$.

### 6.6.3 Staircase Nim

Stones are moved down a staircase and only removed from the last pile. $(x_i)_{i=1}^{n}$ is an $L$-position if $(x_{2i-1})_{i=1}^{n/2}$ is (i.e. only look at odd-numbered piles).

### 6.6.4 Moore's Nim$_k$

The player may remove from at most $k$ piles (Nim = Nim$_1$). Expand the piles in base 2, do a carry-less addition in base $k + 1$ (i.e. the number of ones in each column should be divisible by $k + 1$).

### 6.6.5 Dim$^+$

The number of removed stones must be a divisor of the pile size. The Sprague-Grundy function is $k + 1$ where $2^k$ is the largest power of 2 dividing the pile size.

### 6.6.6 Aliquot Game

Same as above, except the divisor should be proper (hence 1 is also a terminal state, but watch out for size 0 piles). Now the Sprague-Grundy function is just $k$.

### 6.6.7   Nim (at most half)

Write $n + 1 = 2^m y$ with $m$ maximal, then the Sprague-Grundy function of $n$ is $(y - 1)/2$.

### 6.6.8   Lasker's Nim

Players may alternatively split a pile into two new non-empty piles. $g(4k + 1) = 4k + 1$, $g(4k + 2) = 4k + 2$, $g(4k + 3) = 4k + 4$, $g(4k + 4) = 4k + 3$ $(k \geq 0)$.

### 6.6.9   Hackenbush on Trees

A tree with stalks $(x_i)_{i=1}^n$ may be replaced with a single stalk with length $\bigoplus_{i=1}^n x_i$.

**nim-product.cpp**
**Description:** Product of nimbers is associative, commutative, and distributive over addition (xor). Forms finite field of size $2^{2^k}$. Application: Given 1D coin turning games $G_1, G_2$ $G_1 \times G_2$ is the 2D coin turning game defined as follows. If turning coins at $x_1, x_2, \ldots, x_m$ is legal in $G_1$ and $y_1, y_2, \ldots, y_n$ is legal in $G_2$, then turning coins at all positions $(x_i, y_j)$ is legal assuming that the coin at $(x_m, y_n)$ goes from heads to tails. Then the grundy function $g(x, y)$ of $G_1 \times G_2$ is $g_1(x) \times g_2(y)$.
**Time:** $64^2$ xors per multiplication, memorize to speed up.    38fb87, 28 lines

```cpp
using ull = uint64_t;
ull nim_prod[64][64];
ull nim_prod2(int i, int j) {
  if (nim_prod[i][j]) return nim_prod[i][j];
  if ((i & j) == 0) return nim_prod[i][j] = 1ull << (i|j);
  int a = (i&j) & -(i&j);
  return nim_prod[i][j] = nim_prod2(i ^ a, j) ^ nim_prod2((i ^
      a) | (a-1), (j ^ a) | (i & (a-1)));
}

void all_nim_prod() {
  for (int i = 0; i < 64; i++) {
    for (int j = 0; j < 64; j++) {
      if ((i & j) == 0) nim_prod[i][j] = 1ull << (i|j);
      else {
        int a = (i&j) & -(i&j);
        nim_prod[i][j] = nim_prod[i ^ a][j] ^ nim_prod[(i ^ a)
            | (a-1)][(j ^ a) | (i & (a-1))];
      }
    }
  }
}
ull get_nim_prod(ull x, ull y) {
  ull res = 0;
  for (int i = 0; i < 64 && (x >> i); ++i)
    if ((x >> i) & 1)
      for (int j = 0; j < 64 && (y >> j); ++j)
        if ((y >> j) & 1) res ^= nim_prod2(i, j);
  return res;
}
```

**partitions.h**
   378a72, 16 lines

```cpp
const int M = 998244353;
vector<int64_t> prep(int N) {
  vector<int64_t> dp(N); dp[0] = 1;
  for (int n = 1; n < N; ++n) {
    int64_t sum = 0;
    for (int k = 0, l = 1, m = n - 1; ;) {
      sum += dp[m]; if ((m -= (k += 1)) < 0) break;
      sum += dp[m]; if ((m -= (l += 2)) < 0) break;
      sum -= dp[m]; if ((m -= (k += 1)) < 0) break;
      sum -= dp[m]; if ((m -= (l += 2)) < 0) break;
    }
    if ((sum %= M) < 0) sum += M;
    dp[n] = sum;
  }
  return dp;
}
```

# Graph (7)

## 7.1   Fundamentals

**euler-walk.h**
**Description:** Eulerian undirected/directed path/cycle algorithm. Input should be a vector of (dest, global edge index), where for undirected graphs, forward/backward edges have the same index. Returns a list of nodes in the Eulerian path/cycle with src at both start and end, or empty list if no cycle/path exists. To get edge indices back, add .second to s and ret.
**Time:** $\mathcal{O}(V + E)$    c1cf41, 16 lines

```cpp
using pii = pair<int,int>;
vector<int> eulerWalk(vector<vector<pii>>& gr, int nedges, int
    src=0) {
  int n = gr.size();
  vector<int> D(n), its(n), eu(nedges), ret, s = {src};
  D[src]++; // to allow Euler paths, not just cycles
  while (!s.empty()) {
    int x = s.back(), y, e, &it = its[x], end = int(gr[x].size
        ());
    if (it == end){ ret.push_back(x); s.pop_back(); continue; }
    tie(y, e) = gr[x][it++];
    if (!eu[e]) {
      D[x]--, D[y]++;
      eu[e] = 1; s.push_back(y);
    }}
  for(auto &x : D) if (x < 0 || int(ret.size()) != nedges+1)
      return {};
  return {ret.rbegin(), ret.rend()};
}
```

## 7.2   Network flow

**push-relabel.h**
**Description:** Push-relabel using the highest label selection rule and the gap heuristic. Quite fast in practice. To obtain the actual flow, look at positive values only. id can be used to restore each edge and its amount of flow used.
**Time:** $\mathcal{O}\left(V^2\sqrt{E}\right)$ Better for dense graphs - Slower than Dinic (in practice)    387a0c, 42 lines

```cpp
template<typename flow_t = int> struct PushRelabel {
  struct edge_t { int dest, back; flow_t f, c; };
  vector<vector<edge_t>> g;
  vector<flow_t> ec;
  vector<edge_t*> cur;
  vector<vector<int>> hs; vector<int> H;
  PushRelabel(int n) : g(n), ec(n), cur(n), hs(2*n), H(n) {}
  void addEdge(int s, int t, flow_t cap, flow_t rcap = 0) { //
      d58501
    if (s == t) return;
    g[s].push_back({t, (int)g[t].size(), 0, cap});
    g[t].push_back({s, (int)g[s].size()-1, 0, rcap});
  }
  void addFlow(edge_t& e, flow_t f) { // 2f7969
    edge_t &back = g[e.dest][e.back];
    if (!ec[e.dest] && f) hs[H[e.dest]].push_back(e.dest);
    e.f += f; e.c -= f; ec[e.dest] += f;
    back.f -= f; back.c += f; ec[back.dest] -= f;
  }
  flow_t maxflow(int s, int t) { // 21100c
    int v = int(g.size()); H[s] = v; ec[t] = 1;
    vector<int> co(2*v); co[0] = v-1;
    for(int i = 0; i < v; ++i) cur[i] = g[i].data();
    for(auto& e : g[s]) addFlow(e, e.c);
    for (int hi = 0;;) {
      while (hs[hi].empty()) if (!hi--) return -ec[s];
      int u = hs[hi].back(); hs[hi].pop_back();
      while (ec[u] > 0)  // discharge u
        if (cur[u] == g[u].data() + g[u].size()) {
          H[u] = 1e9;
          for(auto &e : g[u]) if (e.c && H[u] > H[e.dest]+1)
            H[u] = H[e.dest]+1, cur[u] = &e;
          if (++co[H[u]], !--co[hi] && hi < v)
            for(int i = 0; i < v; ++i) if (hi < H[i] && H[i] <
                v)
              --co[H[i]], H[i] = v + 1;
          hi = H[u];
        } else if (cur[u]->c && H[u] == H[cur[u]->dest]+1)
          addFlow(*cur[u], min(ec[u], cur[u]->c));
        else ++cur[u];
    }
  }
  bool leftOfMinCut(int a) { return H[a] >= int(g.size()); }
};
```

**dinitz.h**
**Description:** Flow algorithm with complexity $O(VE \log U)$ where $U = \max|cap|$. $O(\min(E^{1/2}, V^{2/3})E)$ if $U = 1$; $O(\sqrt{V}E)$ for bipartite matching. To obtain each partition $A$ and $B$ of the cut look at $lvl$, for $v \subset A$, $lvl[v] > 0$, for $u \subset B$, $lvl[u] = 0$.    7c5dcd, 75 lines

```cpp
template<typename T = int> struct Dinitz {
  struct edge_t { int to, rev; T c, f; };
  vector<vector<edge_t>> adj;
  vector<int> lvl, ptr, q;
  Dinitz(int n) : lvl(n), ptr(n), q(n), adj(n) {}
  inline void addEdge(int a, int b, T c, T rcap = 0) { // 694
      aae
    adj[a].push_back({b, (int)adj[b].size(), c, 0});
    adj[b].push_back({a, (int)adj[a].size() - 1, rcap, 0});
  }
  T dfs(int v, int t, T f) { // 8ffe6b
    if (v == t || !f) return f;
    for (int &i = ptr[v]; i < int(adj[v].size()); ++i) {
      edge_t &e = adj[v][i];
      if (lvl[e.to] == lvl[v] + 1)
        if (T p = dfs(e.to, t, min(f, e.c - e.f))) {
          e.f += p, adj[e.to][e.rev].f -= p;
          return p;
        }
    }
    return 0;
  }
  T maxflow(int s, int t) { // db2141
    T flow = 0; q[0] = s;
    for (int L = 0; L < 31; ++L) do { // 'int L=30' maybe
        faster for random data
      lvl = ptr = vector<int>(q.size());
      int qi = 0, qe = lvl[s] = 1;
      while (qi < qe && !lvl[t]) {
        int v = q[qi++];
        for (edge_t &e : adj[v])
          if (!lvl[e.to] && (e.c - e.f) >> (30 - L))
            q[qe++] = e.to, lvl[e.to] = lvl[v] + 1;
      }
      while (T p = dfs(s, t, numeric_limits<T>::max()/4)) flow
          += p;
    } while (lvl[t]);
    return flow;
  }
```

```cpp
  }
  bool leftOfMinCut(int v) { return bool(lvl[v] != 0); }
  pair<T, vector<pair<int,int>>> minCut(int s, int t) { // 727
      b22
    T cost = maxflow(s,t);
    vector<pair<int,int>> cut;
    for (int i = 0; i < int(adj.size()); i++) for(edge_t &e :
        adj[i])
      if (lvl[i] && !lvl[e.to]) cut.push_back({i, e.to});
    return {cost, cut};
  }
};

struct flow_demand_t {
  int src, sink;
  vector<int> d;
  Dinitz<int> flower;
  flow_demand_t(int N) : src(N + 1), sink(N + 2), d(N + 3),
      flower(N + 3) {}
  void add_edge(int a, int b, int demand, int cap) {
    d[a] -= demand;
    d[b] += demand;
    flower.addEdge(a, b, cap - demand);
  }
  int get_flow() {
    const int INF = std::numeric_limits<int>::max():
    int x = 0, y = 0;
    flower.add_edge(N, N-1, INF);
    for (int i = 0; i <= N; ++i) {
      if (d[i] < 0) {
        flower.add_edge(i, sink, -d[i]);
        x += -d[i];
      }
      if (d[i] > 0) {
        flower.add_edge(src, i, d[i]);
        y += d[i];
      }
    }
    bool has_circulation = (flower.maxflow(src, sink) == x && x
        == y);
    if (!has_circulation) return -1;
    return flower.maxflow(N-1, N);
  }
};
```

## min-cost-max-flow.h

**Description:** Min-cost max-flow.
**Time:** $\mathcal{O}\left(F(V + E)logV\right)$, being F the amount of flow.
<span style="float:right">62f2a8, 101 lines</span>

```cpp
// Minimum cost flow by successive shortest paths.
// Assumes that there exists no negative-cost cycle.
// TODO: Check the range of intermediate values.
template<class flow_t, class cost_t> struct min_cost {
  // Watch out when using types other than int and long long.
  static constexpr flow_t FLOW_EPS = 1e-10L;
  static constexpr flow_t FLOW_INF = std::numeric_limits<flow_t
      >::max();
  static constexpr cost_t COST_EPS = 1e-10L;
  static constexpr cost_t COST_INF = std::numeric_limits<cost_t
      >::max();

  int n, m;
  vector<int> ptr, nxt, zu;
  vector<flow_t> capa;
  vector<cost_t> cost;

  explicit min_cost(int n_) : n(n_), m(0), ptr(n_, -1) {}
  void add_edge(int u, int v, flow_t w, cost_t c) { // d482f5
    assert(0 <= u); assert(u < n);
```

```cpp
    assert(0 <= v); assert(v < n);
    assert(0 <= w);
    nxt.push_back(ptr[u]); zu.push_back(v); capa.push_back(w);
        cost.push_back( c); ptr[u] = m++;
    nxt.push_back(ptr[v]); zu.push_back(u); capa.push_back(0);
        cost.push_back(-c); ptr[v] = m++;
  }
};

vector<cost_t> pot, dist;
vector<bool> vis;
vector<int> pari;

// cost slopes[j] per flow when flows[j] <= flow <= flows[j +
//   1]
vector<flow_t> flows;
vector<cost_t> slopes;

// Finds a shortest path from s to t in the residual graph.
// O((n + m) log m) time.
//    Assumes that the members above are set.
//    The distance to a vertex might not be determined if it
//    is >= dist[t].
//    You can pass t = -1 to find a shortest path to each
//    vertex.
void shortest(int s, int t) { // e9bb0d
  using Entry = pair<cost_t, int>;
  priority_queue<Entry, vector<Entry>, std::greater<Entry>>
      que;
  for (int u = 0; u < n; ++u) { dist[u] = COST_INF; vis[u] =
      false; }
  for (que.emplace(dist[s] = 0, s); !que.empty(); ) {
    const cost_t c = que.top().first;
    const int u = que.top().second;
    que.pop();
    if (vis[u]) continue;
    vis[u] = true;
    if (u == t) return;
    for (int i = ptr[u]; ~i; i = nxt[i]) if (capa[i] >
        FLOW_EPS) {
      const int v = zu[i];
      const cost_t cc = c + cost[i] + pot[u] - pot[v];
      if (dist[v] > cc) { que.emplace(dist[v] = cc, v); pari[
          v] = i; }
    }
  }
}

// Finds a minimum cost flow from s to t of amount min{(max
//   flow), limFlow}.
//    Bellman-Ford takes O(n m) time, or O(m) time if there is
//    no negative-cost
//    edge, or cannot stop if there exists a negative-cost
//    cycle.
//    min{(max flow), limFlow} shortest paths if Flow is an
//    integral type.
// d9868f
pair<flow_t, cost_t> run(int s, int t, flow_t limFlow =
    FLOW_INF) {
  assert(0 <= s); assert(s < n);
  assert(0 <= t); assert(t < n);
  assert(s != t);
  assert(0 <= limFlow);
  pot.assign(n, 0);
  for (; ; ) {
    bool upd = false;
    for (int i = 0; i < m; ++i) if (capa[i] > FLOW_EPS) {
      const int u = zu[i ^ 1], v = zu[i];
      const cost_t cc = pot[u] + cost[i];
```

```cpp
      if (pot[v] > cc + COST_EPS) { pot[v] = cc; upd = true;
          }
    }
    if (!upd) break;
  }
  dist.resize(n);
  vis.resize(n);
  pari.resize(n);
  flows.clear(); flows.push_back(0);
  slopes.clear();
  flow_t flow = 0;
  cost_t cost = 0;
  for (; flow < limFlow; ) {
    shortest(s, t);
    if (!vis[t]) break;
    for (int u = 0; u < n; ++u) pot[u] += min(dist[u], dist[t
        ]);
    flow_t f = limFlow - flow;
    for (int v = t; v != s; ) {
      const int i = pari[v]; if (f > capa[i]) { f = capa[i];
          } v = zu[i ^ 1];
    }
    for (int v = t; v != s; ) {
      const int i = pari[v]; capa[i] -= f; capa[i ^ 1] += f;
          v = zu[i ^ 1];
    }
    flow += f;
    cost += f * (pot[t] - pot[s]);
    flows.push_back(flow); slopes.push_back(pot[t] - pot[s]);
  }
  return make_pair(flow, cost);
}
};
```

## 7.3 Matching

### hopcroft-karp.h

**Description:** Fast bipartite matching algorithm. Graph $g$ should be a list of neighbors of the left partition, and $btoa$ should be a vector full of -1's of the same size as the right partition. Returns the size of the matching. $btoa[i]$ will be the match for vertex $i$ on the right side, or $-1$ if it's not matched.
**Usage:** `vector<int> btoa(m, -1); hopcroftKarp(g, btoa);`
**Time:** $\mathcal{O}\left(\sqrt{V}E\right)$
<span style="float:right">d9a55d, 35 lines</span>

```cpp
using vi = vector<int>;
bool dfs(int a, int L, const vector<vi> &g, vi &btoa, vi &A, vi
    &B) {
  if (A[a] != L) return 0;
  A[a] = -1;
  for(auto &b : g[a]) if (B[b] == L + 1) {
    B[b] = 0;
    if (btoa[b] == -1 || dfs(btoa[b], L+1, g, btoa, A, B))
      return btoa[b] = a, 1;
  }
  return 0;
}

int hopcroftKarp(const vector<vi> &g, vi &btoa) {
  int res = 0;
  vector<int> A(g.size()), B(int(btoa.size())), cur, next;
  for (;;) {
    fill(A.begin(), A.end(), 0), fill(B.begin(), B.end(), 0);
    cur.clear();
    for(auto &a : btoa) if (a != -1) A[a] = -1;
    for (int a = 0; a < g.size(); ++a) if (A[a] == 0) cur.
        push_back(a);
    for (int lay = 1;; ++lay) {
      bool islast = 0; next.clear();
      for(auto &a : cur) for(auto &b : g[a]) {
        if (btoa[b] == -1) B[b] = lay, islast = 1;
```

```cpp
      else if (btoa[b] != a && !B[b])
        B[b] = lay, next.push_back(btoa[b]);
      }
      if (islast) break;
      if (next.empty()) return res;
      for(auto &a : next) A[a] = lay;
      cur.swap(next);
    }
    for(int a = 0; a < int(g.size()); ++a)
      res += dfs(a, 0, g, btoa, A, B);
  }
}
```

## bipartite-matching.h

**Description:** Fast Kuhn! Simple maximum cardinality bipartite matching algorithm. Better than hopcroftKarp in practice. Worst case is $O(VE)$ on an hairy tree. Shuffling the edges and vertices ordering should break some worst-case inputs.

**Time:** $\Omega(VE)$

19101e, 42 lines

```cpp
struct bipartite_matching {
  int N, M, T;
  vector<vector<int>> adj;
  vector<int> match, seen;
  bipartite_matching(int a, int b) : N(a), M(a + b), adj(M),
  match(M, -1), seen(M, -1), T(0) {}
  void add_edge(int a, int b) {
    assert(0 <= a && a < N && b + N < M && N <= b + N);
    adj[a].push_back(b + N);
  }
  void shuffle_edges() { // useful to break some hairy tests
    mt19937 rng(chrono::steady_clock::now().time_since_epoch().
         count());
    for (auto& cur : adj)
      shuffle(cur.begin(), cur.end(), rng);
  }
  bool dfs(int cur) {
    if (seen[cur] == T) return false;
    seen[cur] = T;
    for (int nxt : adj[cur])
      if (match[nxt] == -1) {
        match[nxt] = cur, match[cur] = nxt;
        return true;
      }
    for (int nxt : adj[cur])
      if (dfs(match[nxt])) {
        match[nxt] = cur, match[cur] = nxt;
        return true;
      }
    return false;
  }
  int solve() {
    int res = 0;
    while (true) {
      int cur = 0; ++T;
      for (int i = 0; i < N; ++i)
        if (match[i] == -1) cur += dfs(i);
      if (cur == 0) break;
      else res += cur;
    }
    return res;
  }
};
```

## weighted-matching.h

**Description:** Given a weighted bipartite graph, matches every node on the left with a node on the right such that no nodes are in two matchings and the sum of the edge weights is minimal. Takes cost[N][M], where cost[i][j] = cost for L[i] to be matched with R[j] and returns (min cost, match), where L[i] is matched with R[match[i]]. Negate costs for max cost.

**Time:** $\mathcal{O}\left(N^2 M\right)$

7a2392, 31 lines

```cpp
pair<int, vector<int>> hungarian(const vector<vector<int>> &a)
    {
  if (a.empty()) return {0, {}};
  int n = a.size() + 1, m = a[0].size() + 1;
  vector<int> u(n), v(m), p(m), ans(n - 1);
  for(int i = 1; i < n; ++i) {
    p[0] = i;
    int j0 = 0; // add "dummy" worker 0
    vector<int> dist(m, INT_MAX), pre(m, -1);
    vector<bool> done(m + 1);
    do {
      done[j0] = true;
      int i0 = p[j0], j1, delta = INT_MAX;
      for(int j = 1; j < m; ++j) if (!done[j]) {
        auto cur = a[i0-1][j-1] - u[i0] - v[j];
        if (cur < dist[j]) dist[j] = cur, pre[j] = j0;
        if (dist[j] < delta) delta = dist[j], j1 = j;
      }
      for(int j = 0; j < m; ++j) {
        if (done[j]) u[p[j]] += delta, v[j] -= delta;
        else dist[j] -= delta;
      }
      j0 = j1;
    } while (p[j0]);
    while (j0) { // update alternating path
      int j1 = pre[j0];
      p[j0] = p[j1], j0 = j1;
    }
  }
  for(int j = 1; j < m; ++j) if (p[j]) ans[p[j]-1] = j-1;
  return {-v[0], ans}; // min cost
}
```

## general-matching.h

**Description:** Maximum Matching for general graphs (undirected and non bipartite) using Edmond's Blossom Algorithm.

**Time:** $\mathcal{O}\left(EV^2\right)$

0b82ee, 68 lines

```cpp
struct blossom_t {
  int t, n; // 1-based indexing!!
  vector<vector<int>> edges;
  vector<int> seen, parent, og, match, aux, Q;
  blossom_t(int _n) : n(_n), edges(n+1), seen(n+1),
  parent(n+1), og(n+1), match(n+1), aux(n+10), t(0) {}
  void addEdge(int u, int v) {
    edges[u].push_back(v);
    edges[v].push_back(u);
  }
  void augment(int u, int v) {
    int pv = v, nv; // flip states of edges on u-v path
    do {
      pv = parent[v]; nv = match[pv];
      match[v] = pv; match[pv] = v;
      v = nv;
    } while(u != pv);
  }
  int lca(int v, int w) { // find LCA in O(dist)
    ++t;
    while (1) {
      if (v) {
        if (aux[v] == t) return v; aux[v] = t;
        v = og[parent[match[v]]];
      }
```

```cpp
      swap(v, w);
    }
  }
  void blossom(int v, int w, int a) {
    while (og[v] != a) {
      parent[v] = w; w = match[v]; // go other way around cycle
      if(seen[w] == 1) Q.push_back(w), seen[w] = 0;
      og[v] = og[w] = a;       // merge into supernode
      v = parent[w];
    }
  }
  bool bfs(int u) {
    for (int i = 1; i <= n; ++i) seen[i] = -1, og[i] = i;
    Q = vector<int>(); Q.push_back(u); seen[u] = 0;
    for(int i = 0; i < Q.size(); ++i) {
      int v = Q[i];
      for(auto &x : edges[v]) {
        if (seen[x] == -1) {
          parent[x] = v; seen[x] = 1;
          if (!match[x]) return augment(u, x), true;
          Q.push_back(match[x]); seen[match[x]] = 0;
        } else if (seen[x] == 0 && og[v] != og[x]) {
          int a = lca(og[v], og[x]);
          blossom(x, v, a); blossom(v, x, a);
        }
      }
    }
    return false;
  }
  int solve() {
    int ans = 0; // find random matching (not necessary,
    vector<int> V(n-1); iota(V.begin(), V.end(), 1);// constant
           improvement)
    shuffle(V.begin(), V.end(), mt19937(0x94949));
    for(auto &x : V) if(!match[x])
      for(auto &y : edges[x]) if (!match[y]) {
        match[x] = y, match[y] = x;
        ++ans; break;
      }
    for (int i = 1; i <= n; ++i)
      if (!match[i] && bfs(i)) ++ans;
    return ans;
  }
};
```

## max-independent-set.h

**Description:** To obtain a maximum independent set of a graph, find a max clique of the complement. If the graph is bipartite, see MinimumVertex-Cover.

## 7.4 DFS algorithms

### centroid-decomposition.h

**Description:** Divide and Conquer on Trees.

dd21a1, 75 lines

```cpp
template<typename T> struct centroid_t {
  int N;
  vector<vector<int>> adj;
  vector<vector<int>> dist; // dist to all ancestors
  vector<bool> blocked; // processed centroid
  vector<int> sz, depth, parent; // centroid parent
  centroid_t(int _n) : N(_n), adj(_n), dist(32 - __builtin_clz(
     _n), vector<int>(_n)),
  blocked(_n), sz(_n), depth(_n), parent(_n) {}
  void add_edge(int a, int b) {
    adj[a].push_back(b);
    adj[b].push_back(a);
  }
  void dfs_sz(int cur, int prv) {
```

```cpp
    sz[cur] = 1;
    for (int nxt : adj[cur]) {
        if (nxt == prv || blocked[nxt]) continue;
        dfs_sz(nxt, cur);
        sz[cur] += sz[nxt];
    }
}
int find(int cur, int prv, int tsz) {
    for (int nxt : adj[cur])
        if (!blocked[nxt] && nxt != prv && 2*sz[nxt] > tsz)
            return find(nxt, cur, tsz);
    return cur;
}
void dfs_dist(int cur, int prv, int layer, int d) {
    dist[layer][cur] = d;
    for (int nxt : adj[cur]) {
        if (blocked[nxt] || nxt == prv) continue;
        dfs_dist(nxt, cur, layer, d + 1);
    }
}
void get_path(int cur, int prv, int d, vector<int>& cur_path)
    {
    cur_path.push_back(d);
    for (int nxt : adj[cur]) {
        if (nxt == prv || blocked[nxt]) continue;
        get_path(nxt, cur, d + 1, cur_path);
    }
}
// solve for each subtree (cnt := # of paths of length K
// that goes through vertex cur)
T solve_subtree(int cur, int prv, int K) {
    vector<T> dp(sz[prv] + 1); dp[0] = 1;
    T cnt = 0;
    for (int nxt : adj[cur]) {
        if (blocked[nxt]) continue;
        vector<int> path;
        get_path(nxt, cur, 1, path);
        for (int d : path) {
            if (d > K || K - d > sz[prv]) continue;
            cnt += dp[K - d];
        }
        for (int d : path) dp[d] += 1;
    }
    return cnt;
}
T decompose(int cur, int K, int layer = 0, int prv_root = -1)
    {
    dfs_sz(cur, -1);
    int root = find(cur, cur, sz[cur]);
    blocked[root] = true;
    depth[root] = layer;
    parent[root] = prv_root;
    dfs_dist(root, root, layer, 0);

    T res = solve_subtree(root, cur, K);

    for (int nxt : adj[root]) {
        if (blocked[nxt]) continue;
        res += decompose(nxt, K, layer + 1, root);
    }
    return res;
}
};
```

### tarjan.h

**Description:** Finds strongly connected components in a directed graph. If vertices $u, v$ belong to the same component, we can reach $u$ from $v$ and vice versa.

**Usage:** scc(graph, [&](vi& v) { ... }) visits all components in reverse topological order. comp[i] holds the component index of a node (a component only has edges to components with lower index). ncomps will contain the number of components.
**Time:** $\mathcal{O}(E + V)$                                    cd5271, 38 lines

```cpp
using G = vector<vector<int>>;
vector<int> val, comp, z, cont;
int Time, ncomps;
template<class G, class F> int dfs(int j, G& g, F& f) {
    int low = val[j] = ++Time, x; z.push_back(j);
    for (auto e : g[j]) if (comp[e] < 0)
        low = min(low, val[e] ?: dfs(e,g,f));
    if (low == val[j]) {
        do {
            x = z.back(); z.pop_back();
            comp[x] = ncomps;
            cont.push_back(x);
        } while (x != j);
        f(cont); cont.clear();
        ncomps++;
    }
    return val[j] = low;
}
template<class G, class F> void scc(G& g, F f) {
    int n = int(g.size());
    val.assign(n, 0); comp.assign(n, -1);
    Time = ncomps = 0;
    for(int i = 0; i < n; ++i) if (comp[i] < 0) dfs(i, g, f);
}
pair<G, G> make_scc_dag(G &g){
    G vertOfComp;
    scc(g, [&](const vector<int> &vert){
        vertOfComp.push_back(vert);
    } );
    G dag(ncomps);
    for(int u=0; u < int(g.size()); u++)
        for(int v:g[u])
            if(comp[u] != comp[v])
                dag[ comp[u] ].push_back(comp[v]);
    for(int u=0; u<ncomps; u++)
        dag[u].resize( distance( dag[u].begin(), unique(dag[u].
            begin(), dag[u].end()) ) );
    return { dag, vertOfComp };
}
```

### bcc.h

**Description:** Finds all biconnected components in an undirected graph, and runs a callback for the edges in each. In a biconnected component there are at least two distinct paths between any two nodes. Note that a node can be in several components. An edge which is not in a component is a bridge, i.e., not part of any cycle. $make_bcc_tree$ constructs the block cut tree of given graph. The first $comps.size()$ nodes represents the blocks, the others represents the cut vertices.

**Usage:** int eid = 0; ed.resize(N);
for each edge (a,b) {
ed[a].emplace_back(b, eid);
ed[b].emplace_back(a, eid++); }
bicomps([&](const vi& edgelist) {...});
**Time:** $\mathcal{O}(E + V)$                                    74bf8b, 78 lines

```cpp
vector<int> num, st, stk;
vector<vector<int>> two_edge_cc; // two-edge-connected
                                    components
vector<vector<pii>> ed;
int Time;
template<class F> int dfs(int at, int par, F& f) { // ba3883
    int me = num[at] = ++Time, e, y, top = me;
    stk.push_back(at);
    for(auto &pa : ed[at]) if (pa.second != par) {
```

```cpp
        tie(y, e) = pa;
        if (num[y]) {
            top = min(top, num[y]);
            if (num[y] < me) st.push_back(e);
        } else {
            int si = int(st.size());
            int up = dfs(y, e, f);
            top = min(top, up);
            if (up == me) {
                st.push_back(e);
                f(vector<int>(st.begin() + si, st.end()));
                st.resize(si);
            }
            else if (up < me) st.push_back(e);
            else { f({e}); /* e is a bridge */ }
        }
    }
    if (top >= num[at]) {
        vector<int> cur_two_edge_cc;
        while (stk.back() != at) {
            cur_two_edge_cc.push_back(stk.back());
            stk.pop_back();
        }
        cur_two_edge_cc.push_back(stk.back());
        stk.pop_back();
        two_edge_cc.push_back(cur_two_edge_cc);
    }
    return top;
}

template<class F> void bicomps(F f) { // c44d89
    Time = 0;
    st.resize(0);
    num.assign(ed.size(), 0);
    for(int i = 0; i < int(ed.size()); ++i)
        if (!num[i]) dfs(i, -1, f);
}

using vvi = vector<vector<int>>;
tuple<vvi, vvi, vector<int>> make_bcc_tree(const vector<pii> &
    edges){ // c6742c
    int nart = 0, ncomp = 0, n = int(ed.size());
    vector<int> inv(n);
    vvi comps;
    bicomps([&](const vector<int> &eid){
        ncomp++;
        set<int> cur;
        for(int e: eid){
            cur.insert(edges[e].first);
            cur.insert(edges[e].second);
        }
        comps.push_back(vector<int>(cur.begin(), cur.end()));
        for(int v: cur)
            inv[v]++;
    } );
    vector<int> art;
    for(int u = 0; u < n; u++)
        if(inv[u] > 1){
            inv[u] = nart++;
            art.push_back(u);
        } else inv[u] = -1;
    vvi tree(ncomp + nart);
    for(int c = 0; c < ncomp; c++)
        for(int u: comps[c])
            if(inv[u] != -1){
                tree[ c ].push_back( ncomp + inv[u] );
                tree[ ncomp + inv[u] ].push_back( c );
            }
```

```cpp
      return {tree, comps, art};
}
```

## 2sat.h
**Description:** Calculates a valid assignment to boolean variables a, b, c,... to a 2-SAT problem, so that an expression of the type $(a\|\|b)\&\&(!a\|\|c)\&\&(d\|\|!b)\&\&...$ becomes true, or reports that it is unsatisfiable. Negated variables are represented by bit-inversions ($\sim$x).
**Usage:** TwoSat ts(number of boolean variables);
ts.either(0, ~3); // Var 0 is true or var 3 is false
ts.set_value(2); // Var 2 is true
ts.at_most_one({0,~1,2}); // <= 1 of vars 0, ~1 and 2 are true
ts.solve(); // Returns true iff it is solvable
ts.values[0..N-1] holds the assigned values to the vars
**Time:** $\mathcal{O}(N + E)$, where N is the number of boolean variables, and E is the number of clauses.

"tarjan.h"        4552aa, 59 lines
```cpp
struct TwoSat {
  int N;
  vector<vector<int>> gr;
  vector<int> values; // 0 = false, 1 = true
  TwoSat(int n = 0) : N(n), gr(2*n) {}
  int add_var() { // (optional)
    gr.emplace_back();
    gr.emplace_back();
    return N++;
  }
  void either(int f, int j) {
    f = max(2*f, -1-2*f);
    j = max(2*j, -1-2*j);
    gr[f].push_back(j^1);
    gr[j].push_back(f^1);
  }
  void implies(int f, int j) { either(~f, j); }
  void set_value(int x) { either(x, x); }
  void at_most_one(const vector<int>& li) { // (optional)
    if (int(li.size()) <= 1) return;
    int cur = ~li[0];
    for (int i = 2; i < int(li.size()); ++i) {
      int next = add_var();
      either(cur, ~li[i]);
      either(cur, next);
      either(~li[i], next);
      cur = ~next;
    }
    either(cur, ~li[1]);
  }
  bool solve() {
    scc(gr, [](const vector<int> &v){ return; } );
    values.assign(N, -1);
    for (int i = 0; i < N; ++i) if (comp[2*i] == comp[2*i+1])
        return 0;
    for (int i = 0; i < N; ++i){
      if (comp[2*i] < comp[2*i+1]) values[i] = false;
      else values[i] = true;
    }
    // to minimize (to maximize change < to >) number of
    //    variables true (need graph to be symmetric if a => b
    //    then ~a => ~b)
    /*
       vector<pair<int, int>>cnt(2*N);
       for (int i = 0; i < N; ++i){
       if (comp[2*i] < comp[2*i+1]) cnt[comp[2*i]].st++;
       else cnt[comp[2*i+1]].nd++;
       }
       for (int i = 0; i < N; ++i){
       if (comp[2*i] < comp[2*i+1])
       if( cnt[comp[2*i]].st < cnt[comp[2*i]].nd ) values[i] =
           true; //change here
       else values[i] = false;
       }
       else{
       if( cnt[comp[2*i+1]].st < cnt[comp[2*i+1]].nd ) values[i
          ] = false; //change here
       else values[i] = true;
       }
       }
     */
    return 1;
  }
};
```

## 7.5 Heuristics

### maximal-cliques.h
**Description:** Runs a callback for all maximal cliques in a graph (given as a symmetric bitset matrix; self-edges not allowed). Possible optimization: on the top-most recursion level, ignore 'cands', and go through nodes in order of increasing degree, where degrees go down as nodes are removed.
**Time:** $\mathcal{O}(3^{n/3})$, much faster for sparse graphs

     57e107, 12 lines
```cpp
typedef bitset<128> B;
template<class F>
void cliques(vector<B> &eds, F f, B P = ~B(), B X={}, B R={}) {
  if (!P.any()) { if (!X.any()) f(R); return; }
  auto q = (P | X)._Find_first();
  auto cands = P & ~eds[q];
  for(int i = 0; i < eds.size(); ++i) if (cands[i]) {
    R[i] = 1;
    cliques(eds, f, P & eds[i], X & eds[i], R);
    R[i] = P[i] = 0; X[i] = 1;
  }
}
```

### chromatic-number.h
**Description:** Compute the chromatic number of a graph. Minimum number of colors needed to paint the graph in a way s.t. if two vertices share an edge, they must have distinct colors.
**Time:** $\mathcal{O}(N2^N)$

     ea44b7, 33 lines
```cpp
template<class T> int min_colors(int N, const T& gr) {
  vector<int> adj(N);
  for (int a = 0; a < N; ++a) {
    for (int b = a + 1; b < N; ++b) {
      if (!gr[a][b]) continue;
      adj[a] |= (1 << b);
      adj[b] |= (1 << a);
    }
  }
  static vector<unsigned> dp(1 << N), buf(1 << N), w(1 << N);
  for (int mask = 0; mask < (1 << N); ++mask) {
    bool ok = true;
    for (int i = 0; i < N; ++i) if (mask & 1 << i) {
      if (adj[i] & mask) ok = false;
    }
    if (ok) dp[mask]++;
    buf[mask] = 1;
    w[mask] = __builtin_popcount(mask) % 2 == N % 2 ? 1 : -1;
  }
  for (int i = 0; i < N; ++i) {
    for (int mask = 0; mask < (1 << N); ++mask) if (!(mask & 1
        << i)) {
      dp[mask^(1 << i)] += dp[mask];
    }
  }
  for (int colors = 1; colors <= N; ++colors) {
    unsigned S = 0;
    for (int mask = 0; mask < (1 << N); ++mask) {
      S += (buf[mask] *= dp[mask]) * w[mask];
    }
    if (S) return colors;
  }
  assert(false);
}
```

### cycle-counting.cpp
**Description:** Counts 3 and 4 cycles
**Time:** $\mathcal{O}(E\sqrt{E})$

     77cb6f, 47 lines
```cpp
int count_cycles(const vector<vector<int>>& adj, const vector<
    int>& deg) {
  const int N = int(adj.size());
  vector<int> idx(N), loc(N);
  iota(idx.begin(), idx.end(), 0);
  sort(idx.begin(), idx.end(), [&](const int& a, const int& b)
      { return deg[a] < deg[b]; });
  for (int i = 0; i < N; ++i) loc[idx[i]] = i;

  vector<vector<int>> gr(N);
  for (int a = 0; a < N; ++a) {
    for (int b : adj[a]) {
      if (loc[a] < loc[b]) gr[a].push_back(b);
    }
  }

  int cycle3 = 0;
  {
    vector<bool> seen(N, false);
    for (int a = 0; a < N; ++a) {
      for (int b : gr[a]) seen[b] = true;
      for (int b : gr[a]) {
        for (int c : gr[b]) {
          if (seen[c]) {
            cycle3 += 1;
          }
        }
      }
      for (int b : gr[a]) seen[b] = false;
    }
  }
  int cycle4 = 0;
  {
    vector<int> cnt(N);
    for (int a = 0; a < N; ++a) {
      for (int b : adj[a]) {
        for (int c : gr[b]) {
          if (loc[a] < loc[c]) {
            cycle4 += cnt[c];
            cnt[c]++;
          }
        }
      }
      for (int b : adj[a]) for (int c : gr[b]) cnt[c] = 0;
    }
  }

  return cycle3;
}
```

### edge-coloring.h
**Description:** Given a simple, undirected graph with max degree $D$, computes a $(D+1)$-coloring of the edges such that no neighboring edges share a color. ($D$-coloring is NP-hard, but can be done for bipartite graphs by repeated matchings of max-degree nodes.)
**Time:** $\mathcal{O}(NM)$

     13a6e5, 32 lines
```cpp
vector<int> misra_gries(int N, vector<pair<int, int>> eds) {
```

```cpp
  const int M = int(eds.size());
  vector<int> cc(N + 1), ret(M), fan(N), free(N), loc;
  for (auto e : eds) ++cc[e.first], ++cc[e.second];
  int u, v, ncols = *max_element(cc.begin(), cc.end()) + 1;
  vector<vector<int>> adj(N, vi(ncols, -1));
  for (auto e : eds) {
    tie(u, v) = e;
    fan[0] = v;
    loc.assign(ncols, 0);
    int at = u, end = u, d, c = free[u], ind = 0, i = 0;
    while (d = free[v], !loc[d] && (v = adj[u][d]) != -1)
      loc[d] = ++ind, cc[ind] = d, fan[ind] = v;
    cc[loc[d]] = c;
    for (int cd = d; at != -1; cd ^= c ^ d, at = adj[at][cd])
      swap(adj[at][cd], adj[end = at][cd ^ c ^ d]);
    while (adj[fan[i]][d] != -1) {
      int left = fan[i], right = fan[++i], e = cc[i];
      adj[u][e] = left;
      adj[left][e] = u;
      adj[right][e] = -1;
      free[right] = e;
    }
    adj[u][d] = fan[i];
    adj[fan[i]][d] = u;
    for (int y : {fan[0], u, end})
      for (int& z = free[y] = 0; adj[y][z] != -1; z++);
  }
  for (int i = 0; i < M; ++i)
    for (tie(u, v) = eds[i]; adj[u][ret[i]] != v;) ++ret[i];
  return ret;
}
```

## 7.6  Trees

### lca.h
**Description:** Data structure for computing lowest common ancestors and build Euler Tour in a tree. Edges should be an adjacency list of the tree, either directed or undirected.
**Time:** $\mathcal{O}(N \log N + Q + Q \log)$

<div align="right">010d1c, 21 lines</div>

```cpp
struct lca_t {
  int T = 0;
  vector<int> time, path, walk, depth;
  rmq_t<int> rmq;
  lca_t(vector<vector<int>> &edges) : time(int(edges.size())),
  depth(time), rmq((dfs(edges,0,-1), walk)) {}
  void dfs(vector<vector<int>> &edges, int v, int p) {
    time[v] = T++;
    for(int u : edges[v]) {
      if (u == p) continue;
      depth[u] = depth[v] + 1;
      path.push_back(v), walk.push_back(time[v]);
      dfs(edges, u, v);
    }
  }
  int lca(int a, int b) {
    if (a == b) return a;
    tie(a, b) = minmax(time[a], time[b]);
    return path[rmq.query(a, b)];
  }
};
```

### compress-tree.h
**Description:** Given a rooted tree and a subset S of nodes, compute the minimal subtree that contains all the nodes by adding all (at most $|S|-1$) pairwise LCA's and compressing edges. Returns a list of (par, orig_index) representing a tree rooted at 0. The root points to itself.
**Time:** $\mathcal{O}(|S| \log |S|)$

<div align="right">"LCA.h"       ae0a91, 20 lines</div>

```cpp
vector<pair<int,int>> compressTree(lca_t &lca, const vector<int
    >& subset) {
  static vector<int> rev; rev.resize(lca.time.size());
  vector<int> li = subset, &T = lca.time;
  auto cmp = [&](int a, int b) { return T[a] < T[b]; };
  sort(li.begin(), li.end(), cmp);
  int m = li.size()-1;
  for (int i = 0; i < m; ++i) {
    int a = li[i], b = li[i+1];
    li.push_back(lca.lca(a, b));
  }
  sort(li.begin(), li.end(), cmp);
  li.erase(unique(li.begin(), li.end()), li.end());
  for (int i = 0; i < int(li.size()); ++i) rev[li[i]] = i;
  vector<pair<int,int>> ret = {{0, li[0]}};
  for (int i = 0; i < li.size()-1; ++i) {
    int a = li[i], b = li[i+1];
    ret.emplace_back(rev[lca.lca(a, b)], b);
  }
  return ret;
}
```

### heavylight.h
**Description:** Decomposes a tree into vertex disjoint heavy paths and light edges such that the path from any leaf to the root contains at most log(n) light edges. Code supports commutative segtree modifications/queries on paths, edges and subtrees. Takes as input the full adjacency list with pairs of (vertex, value). USE_EDGES being true means that values are stored in the edges and are initialized with the adjacency list, otherwise values are stored in the nodes and are initialized to the T defaults value.
**Time:** $\mathcal{O}\left((\log N)^2\right)$

<div align="right">"../data-structures/lazy-segtree.h"       053b35, 70 lines</div>

```cpp
template<bool use_edges> struct HLD_t {
  int N, T{};
  vector<vector<int>> adj;
  vector<int> sz, depth, chain, par, in, out, preorder;
  HLD_t() {}
  HLD_t(const vector<vector<int>>& G, int r = 0) : N(int(G.size
      ())), adj(G),
  sz(N), depth(N), chain(N), par(N, -1), in(N), out(N),
      preorder(N) {
    dfs_sz(r); chain[r] = r; dfs_hld(r); }
  void dfs_sz(int cur) {
    if (~par[cur]) {
      adj[cur].erase(find(adj[cur].begin(), adj[cur].end(), par
          [cur]));
    }
    sz[cur] = 1;
    for (auto& nxt : adj[cur]) {
      par[nxt] = cur; depth[nxt] = 1 + depth[cur];
      dfs_sz(nxt); sz[cur] += sz[nxt];
      if (sz[nxt] > sz[adj[cur][0]]) swap(nxt, adj[cur][0]);
    }
  }
  void dfs_hld(int cur) {
    in[cur] = T++;
    preorder[in[cur]] = cur;
    for (auto& nxt : adj[cur]) {
      chain[nxt] = (nxt == adj[cur][0] ? chain[cur] : nxt);
      dfs_hld(nxt);
    }
    out[cur] = T;
  }
  int lca(int a, int b) {
    while (chain[a] != chain[b]) {
      if (in[a] < in[b]) swap(a, b);
      a = par[chain[a]];
    }
    return (in[a] < in[b] ? a : b);
```

```cpp
  }
  bool is_ancestor(int a, int b) { return in[a] <= in[b] && in[
      b] < out[a]; }
  int climb(int a, int k) {
    if (depth[a] < k) return -1;
    int d = depth[a] - k;
    while (depth[chain[a]] > d) a = par[chain[a]];
    return preorder[in[a] - depth[a] + d];
  }
  int kth_on_path(int a, int b, int K) {
    int m = lca(a, b);
    int x = depth[a] - depth[m], y = depth[b] - depth[m];
    if (K > x + y) return -1;
    return (x > K ? climb(a, K) : climb(b, x + y - K));
  }
  // bool is true if path should be reversed (only for
      noncommutative operations)
  const vector<tuple<bool, int, int>>& get_path(int a, int b)
      const {
    static vector<tuple<bool, int, int>> L, R;
    L.clear(); R.clear();
    while (chain[a] != chain[b]) {
      if (depth[chain[a]] > depth[chain[b]]) {
        L.push_back({true, in[chain[a]], in[a] + 1});
        a = par[chain[a]];
      } else {
        R.push_back({false, in[chain[b]], in[b] + 1});
        b = par[chain[b]];
      }
    }
    if (depth[a] > depth[b]) {
      L.push_back({true, in[b] + use_edges, in[a] + 1});
    } else {
      R.push_back({false, in[a] + use_edges, in[b] + 1});
    }
    L.insert(L.end(), R.rbegin(), R.rend());
    return L;
  }
};
```

### tree-isomorphism.h
**Time:** $\mathcal{O}(N \log(N))$

<div align="right">92e59f, 51 lines</div>

```cpp
map<vector<int>, int> delta;
struct tree_t {
  int n;
  pair<int,int> centroid;
  vector<vector<int>> edges;
  vector<int> sz;
  tree_t(vector<vector<int>>& graph) :
    edges(graph), sz(edges.size()) {}
  int dfs_sz(int v, int p) {
    sz[v] = 1;
    for (int u : edges[v]) {
      if (u == p) continue;
      sz[v] += dfs_sz(u, v);
    }
    return sz[v];
  }
  int dfs(int tsz, int v, int p) {
    for (int u : edges[v]) {
      if (u == p) continue;
      if (2*sz[u] <= tsz) continue;
      return dfs(tsz, u, v);
    }
    return centroid.first = v;
  }
  pair<int,int> find_centroid(int v) {
    int tsz = dfs_sz(v, -1);
```

```
    centroid.second = dfs(tsz, v, -1);
    for (int u : edges[centroid.first]) {
      if (2*sz[u] == tsz)
        centroid.second = u;
    }
    return centroid;
  }
  int hash_it(int v, int p) {
    vector<int> offset;
    for (int u : edges[v]) {
      if (u == p) continue;
      offset.push_back(hash_it(u, v));
    }
    sort(offset.begin(), offset.end());
    if (!delta.count(offset))
      delta[offset] = int(delta.size());
    return delta[offset];
  }
  lint get_hash(int v = 0) {
    pair<int,int> cent = find_centroid(v);
    lint x = hash_it(cent.first, -1), y = hash_it(cent.second,
        -1);
    if (x > y) swap(x, y);
    return (x << 30) + y;
  }
};
```

### 7.6.1   Sqrt Decomposition

HLD generally suffices. If not, here are some common
strategies:

- Rebuild the tree after every $\sqrt{N}$ queries.

- Consider vertices with $>$ or $< \sqrt{N}$ degree separately.

- For subtree updates, note that there are $O(\sqrt{N})$ distinct
  sizes among child subtrees of any vertex.

**Block Tree:** Use a DFS to split edges into contiguous groups of
size $\sqrt{N}$ to $2\sqrt{N}$.

**Mo's Algorithm for Tree Paths:** Maintain an array of
vertices where each one appears twice, once when a DFS enters
the vertex (st) and one when the DFS exists (en). For a tree
path $u \leftrightarrow v$ such that st[u]<st[v],

- If $u$ is an ancestor of $v$, query [st[u],st[v]].

- Otherwise, query [en[u],st[v]] and consider $lca(u,v)$
  separately.

## 7.7   Other

### manhattan-mst.h
**Description:** Given N points, returns up to 4*N edges, which are guaran-
teed to contain a minimum spanning tree for the graph with edge weights
$w(p,q) = |p.x - q.x| + |p.y - q.y|$. Edges are in the form (distance, src, dst).
Use a standard MST algorithm on the result to find the final MST.
**Time:** $\mathcal{O}(NlogN)$

<dsu.h>                                              de8170, 28 lines
```
typedef Point<int> P;
pair<vector<array<int, 3>>, int> manhattanMST(vector<P> ps) {
  vector<int> id(ps.size());
  iota(id.begin(), id.end(), 0);
```

```
  vector<array<int, 3>> edges;
  for(int k = 0; k < 4; ++k) {
    sort(id.begin(), id.end(), [&](int i, int j) {
      return (ps[i]-ps[j]).x < (ps[j]-ps[i]).y;});
    map<int, int> sweep;
    for(auto& i : id) {
      for (auto it = sweep.lower_bound(-ps[i].y);
        it != sweep.end(); sweep.erase(it++)) {
        int j = it->second;
        P d = ps[i] - ps[j];
        if (d.y > d.x) break;
        edges.push_back({d.y + d.x, i, j});
      }
      sweep[-ps[i].y] = i;
    }
    if (k & 1) for(auto& p : ps) p.x = -p.x;
    else for(auto& p : ps) swap(p.x, p.y);
  }
  sort(edges.begin(), edges.end());
  UF uf(ps.size());
  int cost = 0;
  for (auto e: edges) if (uf.unite(e[1], e[2])) cost += e[0];
  return {edges, cost};
}
```

### directed-mst.h
**Description:** Edmonds' algorithm for finding the weight of the minimum
spanning tree/arborescence of a directed graph, given a root node. If no
MST exists, returns -1.
**Time:** $\mathcal{O}(E \log V)$

"../data-structures/dsu-rollback.h"                  dedbb2, 59 lines
```
struct Edge { int a, b; ll w; };
struct Node {
  Edge key;
  Node *l, *r;
  ll delta;
  void prop() {
    key.w += delta;
    if (l) l->delta += delta;
    if (r) r->delta += delta;
    delta = 0;
  }
  Edge top() { prop(); return key; }
};
Node *merge(Node *a, Node *b) {
  if (!a || !b) return a ?: b;
  a->prop(), b->prop();
  if (a->key.w > b->key.w) swap(a, b);
  swap(a->l, (a->r = merge(b, a->r)));
  return a;
}
void pop(Node*& a) { a->prop(); a = merge(a->l, a->r); }

pair<ll, vector<int>> dmst(int n, int r, vector<Edge>& g) {
  RollbackUF uf(n);
  vector<Node*> heap(n);
  for (Edge e : g) heap[e.b] = merge(heap[e.b], new Node{e});
  ll res = 0;
  vector<int> seen(n, -1), path(n), par(n);
  seen[r] = r;
  vector<Edge> Q(n), in(n, {-1,-1}), comp;
  deque<tuple<int, int, vector<Edge>>> cycs;
  for(int s = 0; s < n; ++s) {
    int u = s, qi = 0, w;
    while (seen[u] < 0) {
      if (!heap[u]) return {-1,{}};
      Edge e = heap[u]->top();
      heap[u]->delta -= e.w, pop(heap[u]);
      Q[qi] = e, path[qi++] = u, seen[u] = s;
```

```
      res += e.w, u = uf.find(e.a);
      if (seen[u] == s) {
        Node* cyc = 0;
        int end = qi, time = uf.time();
        do cyc = merge(cyc, heap[w = path[--qi]]);
        while (uf.unite(u, w));
        u = uf.find(u), heap[u] = cyc, seen[u] = -1;
        cycs.push_front({u, time, {&Q[qi], &Q[end]}});
      }
    }
    for(int i = 0; i < qi; ++i) in[uf.find(Q[i].b)] = Q[i];
  }
  for (auto& [u,t,comp] : cycs) { // restore sol (optional)
    uf.rollback(t);
    Edge inEdge = in[u];
    for (auto& e : comp) in[uf.find(e.b)] = e;
    in[uf.find(inEdge.b)] = inEdge;
  }
  for(int i = 0; i < n; ++i) par[i] = in[i].a;
  return {res, par};
}
```

## 7.8   Theorems

### 7.8.1   Landau
There are a tournament with outdegree $d_1 \leq d_2 \leq \ldots \leq d_n$
iff:

- $d_1 + d_2 + \ldots + d_n = \binom{n}{2}$

- $d_1 + d_2 + \ldots + d_k \geq \binom{k}{2} \quad \forall 1 \leq k \leq n.$

In order to build, lets make 1 point to $2, 3, \ldots, d_1 + 1$ and we
follow recursively

### 7.8.2   Euler's theorem
Let $V$, $A$ and $F$ be the number of vertices, edges and faces of
connected planar graph, $V - A + F = 2$

### 7.8.3   Eulerian Cycles
The number of Eulerian cycles in a *directed* graph $G$ is:

$$t_w(G) \prod_{v \in G} (\deg v - 1)!,$$

where $t_w(G)$ is the number of arborescences ("directed spanning"
tree) rooted at $w$ (Check Number of Spanning Trees)

### 7.8.4   Dilworth's theorem
For any partially ordered set, the sizes of the max antichain and
of the min chain decomposition are equal. Equivalent to Konig's
theorem on the bipartite graph $(U, V, E)$ where $U = V = S$ and
$(u, v)$ is an edge when $u < v$. Those vertices outside the min
vertex cover in both $U$ and $V$ form a max antichain

### 7.8.5 König-Egervary theorem

For Bipartite Graphs, the number of edges in the maximum matching is greater than or equal the number of vertices in the minimum cover

### Maximum Weight Closure

Given a vertex-weighted directed graph $G$. Turn the graph into a flow network, adding weight $\infty$ to each edge. Add vertices $S, T$. For each vertex $v$ of weight $w$, add edge $(S, v, w)$ if $w \geq 0$, or edge $(v, T, -w)$ if $w < 0$. Sum of positive weights minus minimum $S - T$ cut is the answer. Vertices reachable from $S$ are in the closure. The maximum-weight closure is the same as the complement of the minimum-weight closure on the graph with edges reversed.

### 7.8.6 Maximum Weighted Independent Set in a Bipartite Graph

This is the same as the minimum weighted vertex cover. Solve this by constructing a flow network with edges $(S, u, w(u))$ for $u \in L$, $(v, T, w(v))$ for $v \in R$ and $(u, v, \infty)$ for $(u, v) \in E$. The minimum $S, T$-cut is the answer. Vertices adjacent to a cut edge are in the vertex cover.Num

### 7.8.7 Tutte-Berge formula

The theorem states that the size of a maximum matching of a graph $G = (V, E)$ equals $\frac{1}{2} \min_{U \subseteq V} (|U| - \text{odd}(G - U) + |V|)$, where $\text{odd}(H)$ counts how many of the connected components of the graph $H$ have an odd number of vertices.

### 7.8.8 Tutte's theorem

A graph $G = (V, A)$ has a perfect matching iff for all subset $U$ of $V$, the induced subgraph by $V \setminus U$ has at most $|U|$ connected components with odd number of vertices.

### 7.8.9 Number of Spanning Trees

Define Laplacian Matrix as $L = D - A$, $D$ being a Diagonal Matrix with $D_{i,i} = deg(i)$ and $A$ an Adjacency Matrix. Create an $N \times N$ Laplacian matrix `mat`, and for each edge $a \to b \in G$, do `mat[a][b]--`, `mat[b][b]++` (and `mat[b][a]--`, `mat[a][a]++` if $G$ is undirected). Remove the $i$th row and column and take the determinant; this yields the number of directed spanning trees rooted at $i$ (if $G$ is undirected, remove any row/column).

### 7.8.10 Tutte Matrix

- A graph has a perfect matching iff the *Tutte* matrix has a non-zero determinant.
- The rank of the *Tutte* matrix is equal to twice the size of the maximum matching. The maximum cost matching can be found by polynomial interpolation.

### 7.8.11 Menger's theorem

- Vertices: A graph is k-connected iff all pairwise vertices are connected to at least $k$ internally disjoint paths.

- Edges: A graph is called $k$-edge-connected if the removal of at least $k$ edges of the graph keeps it connected. A graph is $k$-edge-connected iff for all pairwise vertices $u$ and $v$, exist $k$ paths which link $u$ to $v$ without sharing an edge.

# Geometry (8)

## 8.1 Geometric primitives

### Point.h
**Description:** Class to handle points in the plane. T can be e.g. double or long long. (Avoid int.)

<span style="float:right">f90ade, 27 lines</span>

```cpp
template <class T> int sgn(T x) { return (x > 0) - (x < 0); }
template<class T>
struct Point {
    typedef Point P;
    T x, y;
    explicit Point(T x=0, T y=0) : x(x), y(y) {}
    bool operator<(P p) const { return tie(x,y) < tie(p.x,p.y); }
    bool operator==(P p) const { return tie(x,y)==tie(p.x,p.y); }
    P operator+(P p) const { return P(x+p.x, y+p.y); }
    P operator-(P p) const { return P(x-p.x, y-p.y); }
    P operator*(T d) const { return P(x*d, y*d); }
    P operator/(T d) const { return P(x/d, y/d); }
    T dot(P p) const { return x*p.x + y*p.y; }
    T cross(P p) const { return x*p.y - y*p.x; }
    T cross(P a, P b) const { return (a-*this).cross(b-*this); }
    T dist2() const { return x*x + y*y; }
    double dist() const { return sqrt((double)dist2()); }
    // angle to x-axis in interval [-pi, pi]
    double angle() const { return atan2(y, x); }
    P unit() const { return *this/dist(); } // makes dist()=1
    P perp() const { return P(-y, x); } // rotates +90 degrees
    P normal() const { return perp().unit(); }
    // returns point rotated 'a' radians ccw around the origin
    P rotate(double a) const {
        return P(x*cos(a)-y*sin(a),x*sin(a)+y*cos(a)); }
};
using P = Point<double>;
```

### LineDistance.h
**Description:**
Returns the signed distance between point p and the line containing points a and b. Positive value on left side and negative on right as seen from a towards b. a==b gives nan. P is supposed to be Point<T> or Point3D<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long. Using Point3D will always give a non-negative distance.

<span style="float:right">f6bf6b, 4 lines</span>

```cpp
template<class P>
double lineDist(const P& a, const P& b, const P& p) {
    return (double)(b-a).cross(p-a)/(b-a).dist();
}
```

### SegmentDistance.h
**Description:**
Returns the shortest distance between point p and the line segment from point s to e.

**Usage:** `Point<double> a, b(2,2), p(1,1);`
`bool onSegment = segDist(a,b,p) < 1e-10;`

<span style="float:right">"Point.h"    ae751a, 5 lines</span>

```cpp
double segDist(P& s, P& e, P& p) {
    if (s==e) return (p-s).dist();
    auto d = (e-s).dist2(), t = min(d,max(.0,(p-s).dot(e-s)));
    return ((p-s)*d-(e-s)*t).dist()/d;
}
```

### SegmentIntersection.h
**Description:**
If a unique intersection point between the line segments going from s1 to e1 and from s2 to e2 exists then it is returned. If no intersection point exists an empty vector is returned. If infinitely many exist a vector with 2 elements is returned, containing the endpoints of the common line segment. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or long long.
**Usage:** `vector<P> inter = segInter(s1,e1,s2,e2);`
`if (sz(inter)==1)`
`cout << "segments intersect at " << inter[0] << endl;`

<span style="float:right">"Point.h", "OnSegment.h"    f6be16, 13 lines</span>

```cpp
template<class P> vector<P> segInter(P a, P b, P c, P d) {
    auto oa = c.cross(d, a), ob = c.cross(d, b),
         oc = a.cross(b, c), od = a.cross(b, d);
    // Checks if intersection is single non-endpoint point.
    if (sgn(oa) * sgn(ob) < 0 && sgn(oc) * sgn(od) < 0)
        return {(a * ob - b * oa) / (ob - oa)};
    set<P> s;
    if (onSegment(c, d, a)) s.insert(a);
    if (onSegment(c, d, b)) s.insert(b);
    if (onSegment(a, b, c)) s.insert(c);
    if (onSegment(a, b, d)) s.insert(d);
    return {s.begin(), s.end()};
}
```
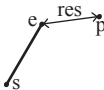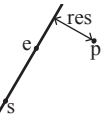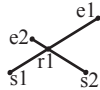
### SegmentIntersectionQ.h
**Description:** Like segmentIntersection, but only returns true/false. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or long long.

<span style="float:right">"Point.h"    1ff4ba, 16 lines</span>

```cpp
template<class P>
bool segmentIntersectionQ(P s1, P e1, P s2, P e2) {
    if (e1 == s1) {
        if (e2 == s2) return e1 == e2;
        swap(s1,s2); swap(e1,e2);
    }
    P v1 = e1-s1, v2 = e2-s2, d = s2-s1;
    auto a = v1.cross(v2), a1 = d.cross(v1), a2 = d.cross(v2);
    if (a == 0) { // parallel
        auto b1 = s1.dot(v1), c1 = e1.dot(v1),
             b2 = s2.dot(v1), c2 = e2.dot(v1);
        return !a1 && max(b1,min(b2,c2)) <= min(c1,max(b2,c2));
    }
    if (a < 0) { a = -a; a1 = -a1; a2 = -a2; }
    return (0 <= a1 && a1 <= a && 0 <= a2 && a2 <= a);
}
```

### LineIntersection.h

**Description:**
If a unique intersection point of the lines going through s1,e1 and s2,e2 exists {1, point} is returned. If no intersection point exists {0, (0,0)} is returned and if infinitely many exists {-1, (0,0)} is returned. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or ll.
**Usage:** `auto res = lineInter(s1,e1,s2,e2);`
`if (res.first == 1)`
`cout << "intersection point at " << res.second << endl;`

<span style="float:right">a01f81, 8 lines</span>
"Point.h"

```cpp
template<class P>
pair<int, P> lineInter(P s1, P e1, P s2, P e2) {
    auto d = (e1 - s1).cross(e2 - s2);
    if (d == 0) // if parallel
        return {-(s1.cross(e1, s2) == 0), P(0, 0)};
    auto p = s2.cross(e1, e2), q = s2.cross(e2, s1);
    return {1, (s1 * p + e1 * q) / d};
}
```

## LineProjectionReflection.h
**Description:** Projects point p onto line ab. Set refl=true to get reflection of point p across line ab insted. The wrong point will be returned if P is an integer point and the desired point doesn't have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow.

<span style="float:right">b5562d, 5 lines</span>
"Point.h"

```cpp
template<class P>
P lineProj(P a, P b, P p, bool refl=false) {
    P v = b - a;
    return p - v.perp()*(1+refl)*v.cross(p-a)/v.dist2();
}
```

## SideOf.h
**Description:** Returns where *p* is as seen from *s* towards *e*. 1/0/-1 ⇔ left/on line/right. If the optional argument *eps* is given 0 is returned if *p* is within distance *eps* from the line. P is supposed to be Point<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long.
**Usage:** `bool left = sideOf(p1,p2,q)==1;`

<span style="float:right">3af81c, 9 lines</span>
"Point.h"

```cpp
template<class P>
int sideOf(P s, P e, P p) { return sgn(s.cross(e, p)); }

template<class P>
int sideOf(const P& s, const P& e, const P& p, double eps) {
    auto a = (e-s).cross(p-s);
    double l = (e-s).dist()*eps;
    return (a > l) - (a < -l);
}
```

## OnSegment.h
**Description:** Returns true iff p lies on the line segment from s to e. Use `(segDist(s,e,p)<=epsilon)` instead when using Point<double>.
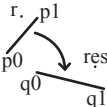
<span style="float:right">c597e8, 3 lines</span>
"Point.h"

```cpp
template<class P> bool onSegment(P s, P e, P p) {
    return p.cross(s, e) == 0 && (s - p).dot(e - p) <= 0;
}
```

## LinearTransformation.h
**Description:**
Apply the linear transformation (translation, rotation and scaling) which takes line p0-p1 to line q0-q1 to point r.

<span style="float:right">03a306, 6 lines</span>
"Point.h"

```cpp
typedef Point<double> P;
P linearTransformation(const P& p0, const P& p1,
    const P& q0, const P& q1, const P& r) {
    P dp = p1-p0, dq = q1-q0, num(dp.cross(dq), dp.dot(dq));
    return q0 + P((r-p0).cross(num), (r-p0).dot(num))/dp.dist2();
}
```

## Angle.h
**Description:** A class for ordering angles (as represented by int points and a number of rotations around the origin). Useful for rotational sweeping. Sometimes also represents points or vectors.
**Usage:** `vector<Angle> v = {w[0], w[0].t360() ...};` // sorted
`int j = 0; rep(i,0,n) { while (v[j] < v[i].t180()) ++j; }`
`// sweeps j such that (j-i) represents the number of positively`
`// oriented triangles with vertices at 0 and i`

<span style="float:right">0f0602, 34 lines</span>

```cpp
struct Angle {
    int x, y;
    int t;
    Angle(int x, int y, int t=0) : x(x), y(y), t(t) {}
    Angle operator-(Angle b) const { return {x-b.x, y-b.y, t}; }
    int half() const {
        assert(x || y);
        return y < 0 || (y == 0 && x < 0);
    }
    Angle t90() const { return {-y, x, t + (half() && x >= 0)}; }
    Angle t180() const { return {-x, -y, t + half()}; }
    Angle t360() const { return {x, y, t + 1}; }
};
bool operator<(Angle a, Angle b) {
    // add a.dist2() and b.dist2() to also compare distances
    return make_tuple(a.t, a.half(), a.y * (ll)b.x) <
           make_tuple(b.t, b.half(), a.x * (ll)b.y);
}
// Given two points, this calculates the smallest angle between
// them, i.e., the angle that covers the defined line segment.
pair<Angle, Angle> segmentAngles(Angle a, Angle b) {
    if (b < a) swap(a, b);
    return (b < a.t180() ?
        make_pair(a, b) : make_pair(b, a.t360()));
}
Angle operator+(Angle a, Angle b) { // point a + vector b
    Angle r(a.x + b.x, a.y + b.y, a.t);
    if (a.t180() < r) r.t--;
    return r.t180() < a ? r.t360() : r;
}
Angle angleDiff(Angle a, Angle b) { // angle b - angle a
    int tu = b.t - a.t; a.t = b.t;
    return {a.x*b.x + a.y*b.y, a.x*b.y - a.y*b.x, tu - (b < a)};
}
```

## AngleCmp.h
**Description:** Useful utilities for dealing with angles of rays from origin. OK for integers, only uses cross product. Doesn't support (0,0).

<span style="float:right">6edd25, 22 lines</span>

```cpp
template <class P>
bool sameDir(P s, P t) {
    return s.cross(t) == 0 && s.dot(t) > 0;
}
// checks 180 <= s..t < 360?
template <class P>
bool isReflex(P s, P t) {
    auto c = s.cross(t);
    return c ? (c < 0) : (s.dot(t) < 0);
}
// operator < (s,t) for angles in [base,base+2pi)
template <class P>
bool angleCmp(P base, P s, P t) {
    int r = isReflex(base, s) - isReflex(base, t);
    return r ? (r < 0) : (0 < s.cross(t));
}
```

```cpp
    const P& q0, const P& q1, const P& r) {
    P dp = p1-p0, dq = q1-q0, num(dp.cross(dq), dp.dot(dq));
    return q0 + P((r-p0).cross(num), (r-p0).dot(num))/dp.dist2();
}
```

```cpp
// is x in [s,t] taken ccw? 1/0/-1 for in/border/out
template <class P>
int angleBetween(P s, P t, P x) {
    if (sameDir(x, s) || sameDir(x, t)) return 0;
    return angleCmp(s, x, t) ? 1 : -1;
}
```

# 8.2 Circles

## CircleIntersection.h
**Description:** Computes the pair of points at which two circles intersect. Returns false in case of no intersection.

<span style="float:right">c64785, 10 lines</span>
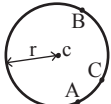"Point.h"

```cpp
bool circleInter(P a,P b,double r1,double r2,pair<P, P>* out) {
    if (a == b) { assert(r1 != r2); return false; }
    P vec = b - a;
    double d2 = vec.dist2(), sum = r1+r2, dif = r1-r2,
        p = (d2 + r1*r1 - r2*r2)/(d2*2), h2 = r1*r1 - p*p*d2;
    if (sum*sum < d2 || dif*dif > d2) return false;
    P mid = a + vec*p, per = vec.perp() * sqrt(fmax(0, h2) / d2);
    *out = {mid + per, mid - per};
    return true;
}
```

## CircleTangents.h
**Description:** Finds the external tangents of two circles, or internal if r2 is negated. Can return 0, 1, or 2 tangents – 0 if one circle contains the other (or overlaps it, in the internal case, or if the circles are the same); 1 if the circles are tangent to each other (in which case .first = .second and the tangent line is perpendicular to the line between the centers). .first and .second give the tangency points at circle 1 and 2 respectively. To find the tangents of a circle with a point set r2 to 0.

<span style="float:right">b0153d, 13 lines</span>
"Point.h"

```cpp
template<class P>
vector<pair<P, P>> tangents(P c1, double r1, P c2, double r2) {
    P d = c2 - c1;
    double dr = r1 - r2, d2 = d.dist2(), h2 = d2 - dr * dr;
    if (d2 == 0 || h2 < 0) return {};
    vector<pair<P, P>> out;
    for (double sign : {-1, 1}) {
        P v = (d * dr + d.perp() * sqrt(h2) * sign) / d2;
        out.push_back({c1 + v * r1, c2 + v * r2});
    }
    if (h2 == 0) out.pop_back();
    return out;
}
```

## Circumcircle.h
**Description:**

The circumcircle of a triangle is the circle intersecting all three vertices. ccRadius returns the radius of the circle going through points A, B and C and ccCenter returns the center of the same circle.

<span style="float:right">c98102, 8 lines</span>
"Point.h"

```cpp
double ccRadius(const P& A, const P& B, const P& C) {
    return (B-A).dist()*(C-B).dist()*(A-C).dist()/
        abs((B-A).cross(C-A))/2;
}
P ccCenter(const P& A, const P& B, const P& C) {
    P b = C-A, c = B-A;
    return A + (b*c.dist2()-c*b.dist2()).perp()/b.cross(c)/2;
}
```

## MinimumEnclosingCircle.h
**Description:** Computes the minimum circle that encloses a set of points.
**Time:** expected $\mathcal{O}(n)$

<span style="float:right">8ab87f, 19 lines</span>
"circumcircle.h"

```cpp
pair<P, double> mec(vector<P> ps) {
  shuffle(ps.begin(),ps.end(), mt19937(time(0)));
  P o = ps[0];
  double r = 0, EPS = 1 + 1e-8;
  for(int i = 0; i < ps.size(); ++i)
    if ((o - ps[i]).dist() > r * EPS) {
      o = ps[i], r = 0;
      for(int j = 0; j < i; ++j) if ((o - ps[j]).dist() > r *
          EPS) {
        o = (ps[i] + ps[j]) / 2;
        r = (o - ps[i]).dist();
        for(int k = 0; k < j; ++k)
          if ((o - ps[k]).dist() > r * EPS) {
            o = ccCenter(ps[i], ps[j], ps[k]);
            r = (o - ps[i]).dist();
          }
      }
    }
  return {o, r};
}
```

## CircleUnion.h
**Description:** Computes the circles union total area

<span style="float:right">fd65da, 86 lines</span>

```cpp
struct CircleUnion {
    static const int maxn = 1e5 + 5;
    const double PI = acos((double)-1.0);
    int n;
    double x[maxn], y[maxn], r[maxn];
    int covered[maxn];
    vector<pair<double, double>> seg, cover;
    double arc, pol;
    inline int sign(double x) {return x < -EPS ? -1 : x > EPS;}
    inline int sign(double x, double y) {return sign(x - y);}
    inline double sqr(const double x) {return x * x;}
    inline double dist(double x1, double y1, double x2, double
        y2) {return sqrt(sqr(x1 - x2) + sqr(y1 - y2));}
    inline double angle(double A, double B, double C) {
        double val = (sqr(A) + sqr(B) - sqr(C)) / (2 * A * B);
        if (val < -1) val = -1;
        if (val > +1) val = +1;
        return acos(val);
    }
    CircleUnion() {
        n = 0;
        seg.clear(), cover.clear();
        arc = pol = 0;
    }
    void init() {
        n = 0;
        seg.clear(), cover.clear();
        arc = pol = 0;
    }
    void add(double xx, double yy, double rr) {
        x[n] = xx, y[n] = yy, r[n] = rr, covered[n] = 0, n++;
    }
    void getarea(int i, double lef, double rig) {
        arc += 0.5 * r[i] * r[i] * (rig - lef - sin(rig - lef))
            ;
        double x1 = x[i] + r[i] * cos(lef), y1 = y[i] + r[i] *
            sin(lef);
        double x2 = x[i] + r[i] * cos(rig), y2 = y[i] + r[i] *
            sin(rig);
        pol += x1 * y2 - x2 * y1;
    }
    double calc() {
        for (int i = 0; i < n; i++)
            for (int j = 0; j < i; j++)
```

```cpp
            if (!sign(x[i] - x[j]) && !sign(y[i] - y[j]) &&
                !sign(r[i] - r[j])) {
              r[i] = 0.0;
              break;
            }
    for (int i = 0; i < n; i++)
        for (int j = 0; j < n; j++)
            if (i != j && sign(r[j] - r[i]) >= 0 && sign(
                dist(x[i], y[i], x[j], y[j]) - (r[j] - r[i
                ])) <= 0) {
                covered[i] = 1;
                break;
            }
    for (int i = 0; i < n; i++) {
        if (sign(r[i]) && !covered[i]) {
            seg.clear();
            for (int j = 0; j < n; j++)
                if (i != j) {
                    double d = dist(x[i], y[i], x[j], y[j])
                        ;
                    if (sign(d - (r[j] + r[i])) >= 0 ||
                        sign(d - abs(r[j] - r[i])) <= 0)
                        continue;
                    double alpha = atan2(y[j] - y[i], x[j]
                        - x[i]);
                    double beta = angle(r[i], d, r[j]);
                    pair<double, double> tmp(alpha - beta,
                        alpha + beta);
                    if (sign(tmp.first) <= 0 && sign(tmp.
                        second) <= 0)
                        seg.push_back(pair<double, double
                            >(2 * PI + tmp.first, 2 * PI +
                            tmp.second));
                    else if (sign(tmp.first) < 0) {
                        seg.push_back(pair<double, double
                            >(2 * PI + tmp.first, 2 * PI))
                            ;
                        seg.push_back(pair<double, double
                            >(0, tmp.second));
                    }
                    else seg.push_back(tmp);
                }
            sort(seg.begin(), seg.end());
            double rig = 0;
            for (vector<pair<double, double> >::iterator
                iter = seg.begin(); iter != seg.end();
                iter++) {
                if (sign(rig - iter->first) >= 0)
                    rig = max(rig, iter->second);
                else {
                    getarea(i, rig, iter->first);
                    rig = iter->second;
                }
            }
            if (!sign(rig)) arc += r[i] * r[i] * PI;
            else getarea(i, rig, 2 * PI);
        }
    }
    return pol / 2.0 + arc;
    }
} ccu;
```

## CircleLine.h
**Description:** Finds the intersection between a circle and a line. Returns a vector of either 0, 1, or 2 intersection points. P is intended to be Point<double>

<span style="float:right">"Point.h", "lineDistance.h", "LineProjectionReflection.h"    debf86, 8 lines</span>

```cpp
template<class P>
vector<P> circleLine(P c, double r, P a, P b) {
```

```cpp
    double h2 = r*r - a.cross(b,c)*a.cross(b,c)/(b-a).dist2();
    if (h2 < 0) return {};
    P p = lineProj(a, b, c), h = (b-a).unit() * sqrt(h2);
    if (h2 == 0) return {p};
    return {p - h, p + h};
}
```

## CircleCircleArea.h
**Description:** Calculates the area of the intersection of 2 circles

<span style="float:right">8bf2b6, 12 lines</span>

```cpp
template<class P>
double circleCircleArea(P c, double cr, P d, double dr) {
    if (cr < dr) swap(c, d), swap(cr, dr);
    auto A = [&](double r, double h) {
        return r*r*acos(h/r)-h*sqrt(r*r-h*h);
    };
    auto l = (c - d).dist(), a = (l*l + cr*cr - dr*dr)/(2*l);
    if (l - cr - dr >= 0) return 0; // far away
    if (l - cr + dr <= 0) return M_PI*dr*dr;
    if (l - cr >= 0) return A(cr, a) + A(dr, l-a);
    else return A(cr, a) + M_PI*dr*dr - A(dr, a-l);
}
```

## CirclePolygonIntersection.h
**Description:** Returns the area of the intersection of a circle with a ccw polygon.
**Time:** $\mathcal{O}(n)$

<span style="float:right">"Point.h"    cf9deb, 18 lines</span>

```cpp
#define arg(p, q) atan2(p.cross(q), p.dot(q))
double circlePoly(P c, double r, vector<P> ps) {
    auto tri = [&](P p, P q) {
        auto r2 = r * r / 2;
        P d = q - p;
        auto a = d.dot(p)/d.dist2(), b = (p.dist2()-r*r)/d.dist2();
        auto det = a * a - b;
        if (det <= 0) return arg(p, q) * r2;
        auto s = max(0., -a-sqrt(det)), t = min(1., -a+sqrt(det));
        if (t < 0 || 1 <= s) return arg(p, q) * r2;
        P u = p + d * s, v = p + d * t;
        return arg(p,u) * r2 + u.cross(v)/2 + arg(v,q) * r2;
    };
    auto sum = 0.0;
    for (int i = 0; i < ps.size(); ++i)
        sum += tri(ps[i] - c, ps[(i + 1) % ps.size()] - c);
    return sum;
}
```

# 8.3 Polygons

## InsidePolygon.h
**Description:** Returns true if p lies within the polygon. If strict is true, it returns false for points on the boundary. The algorithm uses products in intermediate steps so watch out for overflow.
**Usage:** vector<P> v = {P{4,4}, P{1,2}, P{2,1}};
bool in = inPolygon(v, P{3, 3}, false);
**Time:** $\mathcal{O}(n)$

<span style="float:right">"Point.h", "OnSegment.h", "SegmentDistance.h"    f9442d, 12 lines</span>

```cpp
template<class P>
bool inPolygon(vector<P> &p, P a, bool strict = true) {
    int cnt = 0, n = p.size();
    for(int i = 0; i < n; ++i) {
        P q = p[(i + 1) % n];
        if (onSegment(p[i], q, a)) return !strict; // change to
            // -1 if u need to detect points in the boundary
        //or: if (segDist(p[i], q, a) <= eps) return !strict;
        cnt ^= ((a.y<p[i].y) - (a.y<q.y)) * a.cross(p[i], q) > 0;
    }
    return cnt;
}
```

## PolygonArea.h

**Description:** Returns the area of a polygon. Clockwise enumeration gives negative area. Watch out for overflow if using int as T!

"Point.h"     3794ee, 17 lines

```cpp
template<class T>
T polygonArea(vector<Point<T>> &v) {
  T a = v.back().cross(v[0]);
  for(int i = 0; i < v.size()-1; ++i)
    a += v[i].cross(v[i+1]);
  return abs(a)/2.0;
}

Point<T> polygonCentroid(vector<Point<T>> &v) { // not tested
  Point<T> cent(0,0); T area = 0;
  for(int i = 0; i < v.size(); ++i) {
    int j = (i+1) % (v.size()); T a = cross(v[i], v[j]);
    cent += a * (v[i] + v[j]);
    area += a;
  }
  return cent/area/(T)3;
}
```

## PolygonCenter.h

**Description:** Returns the center of mass for a polygon.

**Time:** $\mathcal{O}(n)$
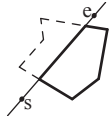
"Point.h"     26a00f, 8 lines

```cpp
P polygonCenter(const vector<P>& v) {
  P res(0, 0); double A = 0;
  for (int i = 0, j = v.size() - 1; i < v.size(); j = ++i) {
    res = res + (v[i] + v[j]) * v[j].cross(v[i]);
    A += v[j].cross(v[i]);
  }
  return res / A / 3;
}
```

## PolygonCut.h

**Description:**
Returns a vector with the vertices of a polygon with everything to the left of the line going from s to e cut away.

**Usage:** `vector<P> p = ...;`
`p = polygonCut(p, P(0,0), P(1,0));`

"Point.h", "lineIntersection.h"     7df36f, 11 lines

```cpp
vector<P> polygonCut(const vector<P>& poly, P s, P e) {
  vector<P> res;
  for(int i = 0; i < poly.size(); ++i) {
    P cur = poly[i], prev = i ? poly[i-1] : poly.back();
    bool side = s.cross(e, cur) < 0;
    if (side != (s.cross(e, prev) < 0))
      res.push_back(lineInter(s, e, cur, prev).second);
    if (side) res.push_back(cur);
  }
  return res;
}
```

## ConvexHull.h

**Description:**
Returns a vector of indices of the convex hull in counterclockwise order. Points on the edge of the hull between two other points are not considered part of the hull.

**Time:** $\mathcal{O}(n \log n)$

"Point.h"     3612d7, 12 lines

```cpp
vector<P> convexHull(vector<P> pts) {
  if (pts.size() <= 1) return pts;
  sort(pts.begin(), pts.end());
  vector<P> h(pts.size()+1);
  int s = 0, t = 0;
  for (int it = 2; it--; s = --t, reverse(pts.begin(), pts.end()))
    for (P p : pts) {
      while (t >= s + 2 && h[t-2].cross(h[t-1], p) <= 0) t--;
      h[t++] = p;
    }
  return {h.begin(), h.begin() + t - (t == 2 && h[0] == h[1])};
}
```

## HullDiameter.h

**Description:** Returns the two points with max distance on a convex hull (ccw, no duplicate/colinear points).

0e0c1f, 11 lines

```cpp
array<P, 2> hullDiameter(vector<P> S) {
  int n = S.size(), j = n < 2 ? 0 : 1;
  pair<lint, array<P, 2>> res({0, {S[0], S[0]}});
  for(int i = 0; i < j; ++i)
    for (;; j = (j + 1) % n) {
      res = max(res, {(S[i] - S[j]).dist2(), {S[i], S[j]}});
      if ((S[(j + 1) % n] - S[j]).cross(S[i + 1] - S[i]) >= 0)
        break;
    }
  return res.second;
}
```

## PointInsideHull.h

**Description:** Determine whether a point t lies inside a convex hull (CCW order, with no colinear points). Returns true if point lies within the hull. If strict is true, points on the boundary aren't included.

**Time:** $\mathcal{O}(\log N)$

"Point.h", "sideOf.h", "OnSegment.h"     7b8514, 12 lines

```cpp
bool inHull(const vector<P> &l, P p, bool strict = true) {
  int a = 1, b = l.size() - 1, r = !strict;
  if (l.size() < 3) return r && onSegment(l[0], l.back(), p);
  if (sideOf(l[0], l[a], l[b]) > 0) swap(a, b);
  if (sideOf(l[0], l[a], p) >= r || sideOf(l[0], l[b], p)<= -r)
    return false;
  while (abs(a - b) > 1) {
    int c = (a + b) / 2;
    (sideOf(l[0], l[c], p) > 0 ? b : a) = c;
  }
  return sgn(l[a].cross(l[b], p)) < r;
}
```

## PolyUnion.h

**Description:** Calculates the area of the union of $n$ polygons (not necessarily convex). The points within each polygon must be given in CCW order. Guaranteed to be precise for integer coordinates up to 3e7. If epsilons are needed, add them in sideOf as well as the definition of rat.

**Time:** $\mathcal{O}(N^2)$, where $N$ is the total number of points

"Point.h", "sideOf.h"     a45bd4, 33 lines

```cpp
double rat(P a, P b) { return sgn(b.x) ? a.x/b.x : a.y/b.y; }
double polyUnion(vector<vector<P>>& poly) {
  double ret = 0;
  for(int i = 0; i < poly.size(); ++i)
    for(int v = 0; v < poly[i].size(); ++v) {
      P A = poly[i][v], B = poly[i][(v + 1) % poly[i].size()];
      vector<pair<double, int>> segs = {{0, 0}, {1, 0}};
      for(int j = 0; j < poly.size(); ++j) if (i != j) {
        for(int u = 0; u < poly[j]; ++u) {
          P C = poly[j][u], D = poly[j][(u + 1) % poly[j].size
            ()];
          int sc = sideOf(A, B, C), sd = sideOf(A, B, D);
          if (sc != sd) {
            double sa = C.cross(D, A), sb = C.cross(D, B);
            if (min(sc, sd) < 0)
              segs.emplace_back(sa / (sa - sb), sgn(sc - sd));
          } else if (!sc && !sd && j<i && sgn((B-A).dot(D-C))
            >0){
            segs.emplace_back(rat(C - A, B - A), 1);
            segs.emplace_back(rat(D - A, B - A), -1);
          }
        }
      }
      sort(segs.begin(), segs.end());
      for(auto& s : segs) s.first = min(max(s.first, 0.0), 1.0);
      double sum = 0;
      int cnt = segs[0].second;
      for(int j = 1; j < segs.size(); ++j) {
        if (!cnt) sum += segs[j].first - segs[j - 1].first;
        cnt += segs[j].second;
      }
      ret += A.cross(B) * sum;
    }
  return ret / 2;
}
```

## LineHullIntersection.h

**Description:** Line-convex polygon intersection. The polygon must be ccw and have no colinear points. lineHull(line, poly) returns a pair describing the intersection of a line with the polygon: ● $(-1, -1)$ if no collision, ● $(i, -1)$ if touching the corner $i$, ● $(i, i)$ if along side $(i, i + 1)$, ● $(i, j)$ if crossing sides $(i, i+1)$ and $(j, j+1)$. In the last case, if a corner $i$ is crossed, this is treated as happening on side $(i, i+1)$. The points are returned in the same order as the line hits the polygon. extrVertex returns the point of a hull with the max projection onto a line.

**Time:** $\mathcal{O}(N + Q \log n)$

"Point.h"     65ebb6, 39 lines

```cpp
typedef array<P, 2> Line;
#define cmp(i,j) sgn(dir.perp().cross(poly[(i)%n]-poly[(j)%n]))
#define extr(i) cmp(i + 1, i) >= 0 && cmp(i, i - 1 + n) < 0
int extrVertex(vector<P>& poly, P dir) {
  int n = poly.size(), left = 0, right = n;
  if (extr(0)) return 0;
  while (left + 1 < right) {
    int m = (left + right) / 2;
    if (extr(m)) return m;
    int ls = cmp(left + 1, left), ms = cmp(m + 1, m);
    (ls < ms || (ls == ms && ls == cmp(left, m)) ? right : left
      ) = m;
  }
  return left;
}

#define cmpL(i) sgn(line[0].cross(poly[i], line[1]))
array<int, 2> lineHull(Line line, vector<P>& poly) {
  int endA = extrVertex(poly, (line[0] - line[1]).perp());
  int endB = extrVertex(poly, (line[1] - line[0]).perp());
  if (cmpL(endA) < 0 || cmpL(endB) > 0)
    return {-1, -1};
  array<int, 2> res;
  for(int i = 0; i < 2; ++i) {
    int left = endB, right = endA, n = poly.size();
    while ((left + 1) % n != right) {
      int m = ((left + right + (left < right ? 0 : n)) / 2) % n
        ;
      (cmpL(m) == cmpL(endB) ? left : right) = m;
    }
    res[i] = (left + !cmpL(right)) % n;
    swap(endA, endB);
  }
  if (res[0] == res[1]) return {res[0], -1};
  if (!cmpL(res[0]) && !cmpL(res[1]))
    switch ((res[0] - res[1] + sz(poly) + 1) % poly.size()) {
      case 0: return {res[0], res[0]};
      case 2: return {res[1], res[1]};
    }
```

```
        }
    return res;
}
```

## HalfPlane.h
**Description:** Halfplane intersection area
`"Point.h"`, `"lineIntersection.h"`           c0a94b, 70 lines

```cpp
#define eps 1e-8
typedef Point<double> P;

struct Line {
  P P1, P2;
  // Right hand side of the ray P1 -> P2
  explicit Line(P a = P(), P b = P()) : P1(a), P2(b) {};
  P intpo(Line y) {
    pair<int, P> r = lineInter(P1, P2, y.P1, y.P2);
    assert(r.first == 1);
    return r.second;
  }
  P dir() { return P2 - P1; }
  bool contains(P x) {
    return (P2 - P1).cross(x - P1) < eps;
  }
  bool out(P x) { return !contains(x); }
};

template<class T>
bool mycmp(Point<T> a, Point<T> b) {
  // return atan2(a.y, a.x) < atan2(b.y, b.x);
  if (a.x * b.x < 0) return a.x < 0;
  if (abs(a.x) < eps) {
    if (abs(b.x) < eps) return a.y > 0 && b.y < 0;
    if (b.x < 0) return a.y > 0;
    if (b.x > 0) return true;
  }
  if (abs(b.x) < eps) {
    if (a.x < 0) return b.y < 0;
    if (a.x > 0) return false;
  }
  return a.cross(b) > 0;
}

bool cmp(Line a, Line b) { return mycmp(a.dir(), b.dir()); }

double Intersection_Area(vector <Line> b) {
  sort(b.begin(), b.end(), cmp);
  int n = b.size();
  int q = 1, h = 0, i;
  vector<Line> c(b.size() + 10);
  for (i = 0; i < n; i++) {
    while (q < h && b[i].out(c[h].intpo(c[h - 1]))) h--;
    while (q < h && b[i].out(c[q].intpo(c[q + 1]))) q++;
    c[++h] = b[i];
    if (q < h && abs(c[h].dir().cross(c[h - 1].dir())) < eps) {
      if (c[h].dir().dot(c[h - 1].dir()) > 0) {
        h--;
        if (b[i].out(c[h].P1))  c[h] = b[i];
      }else {
        // The area is either 0 or infinite.
        // If you have a bounding box, then the area is
        //     definitely 0.
        return 0;
      }
    }
  }
  while (q < h-1 && c[q].out(c[h].intpo(c[h - 1]))) h--;
  while (q < h-1 && c[h].out(c[q].intpo(c[q + 1]))) q++;
  // Intersection is empty. This is sometimes different from
  //   the case when
```

```cpp
  // the intersection area is 0.
  if (h - q <= 1) return 0;
  c[h + 1] = c[q];
  vector<P> s;
  for (i = q; i <= h; i++) s.push_back(c[i].intpo(c[i + 1]));
  s.push_back(s[0]);
  double ans = 0;
  for (i = 0; i < (int) s.size()-1; i++) ans += s[i].cross(s[i
      + 1]);
  return ans / 2;
}
```

## 8.4 Misc. Point Set Problems

### ClosestPair.h
**Description:** Finds the closest pair of points.
**Time:** $\mathcal{O}\left(n \log n\right)$
`"Point.h"`         32b14f, 16 lines

```cpp
pair<P, P> closest(vector<P> v) {
  assert(v.size() > 1);
  set<P> S;
  sort(v.begin(), v.end(), [](P a, P b) { return a.y < b.y; });
  pair<int64_t, pair<P, P>> ret{LLONG_MAX, {P(), P()}};
  int j = 0;
  for(P &p : v) {
    P d{1 + (int64_t)sqrt(ret.first), 0};
    while (v[j].y <= p.y - d.x) S.erase(v[j++]);
    auto lo = S.lower_bound(p - d), hi = S.upper_bound(p + d);
    for (; lo != hi; ++lo)
      ret = min(ret, {(*lo - p).dist2(), {*lo, p}});
    S.insert(p);
  }
  return ret.second;
}
```

### KdTree.h
**Description:** KD-tree (2d, can be extended to 3d)
`"Point.h"`         915562, 63 lines

```cpp
typedef long long T;
typedef Point<T> P;
const T INF = numeric_limits<T>::max();

bool on_x(const P& a, const P& b) { return a.x < b.x; }
bool on_y(const P& a, const P& b) { return a.y < b.y; }

struct Node {
  P pt; // if this is a leaf, the single point in it
  T x0 = INF, x1 = -INF, y0 = INF, y1 = -INF; // bounds
  Node *first = 0, *second = 0;

  T distance(const P& p) { // min squared distance to a point
    T x = (p.x < x0 ? x0 : p.x > x1 ? x1 : p.x);
    T y = (p.y < y0 ? y0 : p.y > y1 ? y1 : p.y);
    return (P(x,y) - p).dist2();
  }

  Node(vector<P>&& vp) : pt(vp[0]) {
    for (P p : vp) {
      x0 = min(x0, p.x); x1 = max(x1, p.x);
      y0 = min(y0, p.y); y1 = max(y1, p.y);
    }
    if (vp.size() > 1) {
      // split on x if the box is wider than high (not best
      //     heuristic...)
      sort(vp.begin(),vp.end(), x1 - x0 >= y1 - y0 ? on_x :
          on_y);
      // divide by taking half the array for each child (not
      // best performance with many duplicates in the middle)
      int half = vp.size()/2;
```

```cpp
      first = new Node({vp.begin(), vp.begin() + half});
      second = new Node({vp.begin() + half, vp.end()});
    }
  }
};

struct KDTree {
  Node* root;
  KDTree(const vector<P>& vp) : root(new Node({vp.begin(), vp.
      end()})) {}

  pair<T, P> search(Node *node, const P& p) {
    if (!node->first) {
      // uncomment if we should not find the point itself:
      // if (p == node->pt) return {INF, P()};
      return make_pair((p - node->pt).dist2(), node->pt);
    }

    Node *f = node->first, *s = node->second;
    T bfirst = f->distance(p), bsec = s->distance(p);
    if (bfirst > bsec) swap(bsec, bfirst), swap(f, s);

    // search closest side first, other side if needed
    auto best = search(f, p);
    if (bsec < best.first)
      best = min(best, search(s, p));
    return best;
  }

  // find nearest point to a point, and its squared distance
  // (requires an arbitrary operator< for Point)
  pair<T, P> nearest(const P& p) {
    return search(root, p);
  }
};
```

### DelaunayTriangulation.h
**Description:** Computes the Delaunay triangulation of a set of points. Each circumcircle contains none of the input points. If any three points are colinear or any four are on the same circle, behavior is undefined.
**Time:** $\mathcal{O}\left(n^2\right)$
`"Point.h"`, `"3dHull.h"`         f6175a, 10 lines

```cpp
template<class P, class F>
void delaunay(vector<P>& ps, F trifun) {
  if (ps.size() == 3) { int d = (ps[0].cross(ps[1], ps[2]) < 0)
      ;
    trifun(0,1+d,2-d); }
  vector<P3> p3;
  for(auto &p : ps) p3.emplace_back(p.x, p.y, p.dist2());
  if (ps.size() > 3) for(auto &t: hull3d(p3)) if ((p3[t.b]-p3[t
      .a]).
    cross(p3[t.c]-p3[t.a]).dot(P3(0,0,1)) < 0)
    trifun(t.a, t.c, t.b);
}
```

### FastDelaunay.h
**Description:** Fast Delaunay triangulation. There must be no duplicate points. If all points are on a line, no triangles will be returned. Should work for doubles as well, though there may be precision issues in 'circ'. Returns triangles in order {t[0][0], t[0][1], t[0][2], t[1][0], ... }, all counter-clockwise.
**Time:** $\mathcal{O}\left(n \log n\right)$
`"Point.h"`         a1f392, 89 lines

```cpp
typedef Point<ll> P;
typedef struct Quad* Q;
typedef __int128_t lll; // (can be ll if coords are < 2e4)
P arb(LLONG_MAX,LLONG_MAX); // not equal to any other point

struct Quad {
```

```
bool mark; Q o, rot; P p;
P F() { return r()->p; }
Q r() { return rot->o; }
Q prev() { return rot->o->rot; }
Q next() { return rot->r()->o->rot; }
};
bool circ(P p, P a, P b, P c) { // is p in the circumcircle?
  lll p2 = p.dist2(), A = a.dist2()-p2,
    B = b.dist2()-p2, C = c.dist2()-p2;
  return p.cross(a,b)*C + p.cross(b,c)*A + p.cross(c,a)*B > 0;
}
Q makeEdge(P orig, P dest) {
  Q q0 = new Quad{0,0,0,orig}, q1 = new Quad{0,0,0,arb},
    q2 = new Quad{0,0,0,dest}, q3 = new Quad{0,0,0,arb};
  q0->o = q0; q2->o = q2; // 0-0, 2-2
  q1->o = q3; q3->o = q1; // 1-3, 3-1
  q0->rot = q1; q1->rot = q2;
  q2->rot = q3; q3->rot = q0;
  return q0;
}
void splice(Q a, Q b) {
  swap(a->o->rot->o, b->o->rot->o); swap(a->o, b->o);
}
Q connect(Q a, Q b) {
  Q q = makeEdge(a->F(), b->p);
  splice(q, a->next());
  splice(q->r(), b);
  return q;
}

pair<Q,Q> rec(const vector<P>& s) {
  if (sz(s) <= 3) {
    Q a = makeEdge(s[0], s[1]), b = makeEdge(s[1], s.back());
    if (sz(s) == 2) return { a, a->r() };
    splice(a->r(), b);
    auto side = s[0].cross(s[1], s[2]);
    Q c = side ? connect(b, a) : 0;
    return {side < 0 ? c->r() : a, side < 0 ? c : b->r() };
  }

#define H(e) e->F(), e->p
#define valid(e) (e->F().cross(H(base)) > 0)
  Q A, B, ra, rb;
  int half = (sz(s) + 1) / 2;
  tie(ra, A) = rec({s.begin(), s.begin() + half});
  tie(B, rb) = rec({s.begin() + half, s.end()});
  while ((B->p.cross(H(A)) < 0 && (A = A->next())) ||
    (A->p.cross(H(B)) > 0 && (B = B->r()->o)));
  Q base = connect(B->r(), A);
  if (A->p == ra->p) ra = base->r();
  if (B->p == rb->p) rb = base;

#define DEL(e, init, dir) Q e = init->dir; if (valid(e)) \
  while (circ(e->dir->F(), H(base), e->F())) { \
    Q t = e->dir; \
    splice(e, e->prev()); \
    splice(e->r(), e->r()->prev()); \
    e = t; \
  }
  for (;;) {
    DEL(LC, base->r(), o); DEL(RC, base, prev());
    if (!valid(LC) && !valid(RC)) break;
    if (!valid(LC) || (valid(RC) && circ(H(RC), H(LC))))
      base = connect(RC, base->r());
    else
      base = connect(base->r(), LC->r());
  }
  return { ra, rb };
}
```

```
vector<P> triangulate(vector<P> pts) {
  sort(pts.begin(), pts.end());  assert(unique(pts.begin(), pts
    .end()) == pts.end());
  if (pts.size() < 2) return {};
  Q e = rec(pts).first;
  vector<Q> q = {e};
  int qi = 0;
  while (e->o->F().cross(e->F(), e->p) < 0) e = e->o;
#define ADD { Q c = e; do { c->mark = 1; pts.push_back(c->p); \
  q.push_back(c->r()); c = c->next(); } while (c != e); }
  ADD; pts.clear();
  while (qi < sz(q)) if (!(e = q[qi++])->mark) ADD;
  return pts;
}
```

## RectangleUnionArea.h

**Description:** Sweep line algorithm that calculates area of union of rectangles in the form $[x1, x2] \times [y1, y2]$

**Usage:** vector<pair<int,int>, pair<int,int>> rectangles;
rectangles.push_back({{x1, x2}, {y1, y2}});
lint result = area(rectangles);                    529ff1, 51 lines

```
struct seg_node{
  int val, cnt, lz;
  seg_node(int n = INT_MAX, int c = 0): val(n), cnt(c), lz(0)
    {}
  void push(seg_node& l, seg_node& r){
    if(lz){
      l.add(lz);
      r.add(lz);
      lz = 0;
    }
  }
  void merge(const seg_node& l, const seg_node& r){
    if(l.val < r.val) val = l.val, cnt = l.cnt;
    else if(l.val > r.val) val = r.val, cnt = r.cnt;
    else val = l.val, cnt = l.cnt + r.cnt;
  }
  void add(int n){
    val += n;
    lz += n;
  }
  int get_sum(){ return (val ? 0 : cnt); }
};
// x1 y1 x2 y2
lint solve(const vector<array<int, 4>>&v){
  vector<int>ys;
  for(auto& [a, b, c, d] : v){
    ys.push_back(b);
    ys.push_back(d);
  }
  sort(ys.begin(), ys.end());
  ys.erase(unique(ys.begin(), ys.end()), ys.end());
  vector<array<int, 4>>e;
  for(auto [a, b, c, d] : v){
    b = int(lower_bound(ys.begin(), ys.end(), b) - ys.begin());
    d = int(lower_bound(ys.begin(), ys.end(), d) - ys.begin());
    e.push_back({a, b, d, 1});
    e.push_back({c, b, d, -1});
  }
  sort(e.begin(), e.end());
  int m = (int)ys.size();
  segtree_range<seg_node>seg(m-1);
  for(int i=0;i<m-1;i++) seg.at(i) = seg_node(0, ys[i+1] - ys[i
    ]);
  seg.build();
  int last = INT_MIN, total = ys[m-1] - ys[0];
  lint ans = 0;
```

```
  for(auto [x, y1, y2, c] : e){
    ans += (lint)(total - seg.query(0, m-1).get_sum()) * (x -
      last);
    last = x;
    seg.update(y1, y2, &seg_node::add, c);
  }
  return ans;
}
```

## 8.5   3D

### PolyhedronVolume.h

**Description:** Magic formula for the volume of a polyhedron. Faces should point outwards.                    832599, 6 lines

```
template<class V, class L>
double signed_poly_volume(const V &p, const L &trilist) {
  double v = 0;
  for(auto &i : trilist) v += p[i.a].cross(p[i.b]).dot(p[i.c]);
  return v / 6;
}
```

### Point3D.h

**Description:** Class to handle points in 3D space. T can be e.g. double or long long.                    8058ae, 32 lines

```
template<class T> struct Point3D {
  typedef Point3D P;
  typedef const P& R;
  T x, y, z;
  explicit Point3D(T x=0, T y=0, T z=0) : x(x), y(y), z(z) {}
  bool operator<(R p) const {
    return tie(x, y, z) < tie(p.x, p.y, p.z); }
  bool operator==(R p) const {
    return tie(x, y, z) == tie(p.x, p.y, p.z); }
  P operator+(R p) const { return P(x+p.x, y+p.y, z+p.z); }
  P operator-(R p) const { return P(x-p.x, y-p.y, z-p.z); }
  P operator*(T d) const { return P(x*d, y*d, z*d); }
  P operator/(T d) const { return P(x/d, y/d, z/d); }
  T dot(R p) const { return x*p.x + y*p.y + z*p.z; }
  P cross(R p) const {
    return P(y*p.z - z*p.y, z*p.x - x*p.z, x*p.y - y*p.x); }
  T dist2() const { return x*x + y*y + z*z; }
  double dist() const { return sqrt((double)dist2()); }
  //Azimuthal angle (longitude) to x-axis in interval [-pi, pi]
  double phi() const { return atan2(y, x); }
  //Zenith angle (latitude) to the z-axis in interval [0, pi]
  double theta() const { return atan2(sqrt(x*x+y*y),z); }
  P unit() const { return *this/(T)dist(); } //makes dist()=1
  //returns unit vector normal to *this and p
  P normal(P p) const { return cross(p).unit(); }
  //returns point rotated 'angle' radians ccw around axis
  P rotate(double angle, P axis) const {
    double s = sin(angle), c = cos(angle); P u = axis.unit();
    return u*dot(u)*(1-c) + (*this)*c - cross(u)*s;
  }
};
```

### 3dHull.h

**Description:** Computes all faces of the 3-dimension hull of a point set. *No four points must be coplanar*, or else random results will be returned. All faces will point outwards.

**Time:** $\mathcal{O}(n^2)$
"Point3D.h"                    3ed613, 48 lines

```
typedef Point3D<double> P3;

struct PR {
  void ins(int x) { (a == -1 ? a : b) = x; }
```

```cpp
    void rem(int x) { (a == x ? a : b) = -1; }
    int cnt() { return (a != -1) + (b != -1); }
    int a, b;
};

struct F { P3 q; int a, b, c; };

vector<F> hull3d(const vector<P3>& A) {
    assert(A.size() >= 4);
    vector<vector<PR>> E(A.size(), vector<PR>(A.size(), {-1, -1})
        );
#define E(x,y) E[f.x][f.y]
    vector<F> FS;
    auto mf = [&](int i, int j, int k, int l) {
        P3 q = (A[j] - A[i]).cross((A[k] - A[i]));
        if (q.dot(A[l]) > q.dot(A[i]))
            q = q * -1;
        F f{q, i, j, k};
        E(a,b).ins(k); E(a,c).ins(j); E(b,c).ins(i);
        FS.push_back(f);
    };
    for(int i=0;i<4;i++) for(int j=i+1;j<4;j++) for(k=j+1;k<4;k
        ++)
        mf(i, j, k, 6 - i - j - k);
    for(int i=4; i<A.size();++i) {
        for(int j=0;j<FS.size();++j) {
            F f = FS[j];
            if(f.q.dot(A[i]) > f.q.dot(A[f.a])) {
                E(a,b).rem(f.c);
                E(a,c).rem(f.b);
                E(b,c).rem(f.a);
                swap(FS[j--], FS.back());
                FS.pop_back();
            }
        }
        int nw = FS.size();
        for(int j=0;j<nw;j++) {
            F f = FS[j];
#define C(a, b, c) if (E(a,b).cnt() != 2) mf(f.a, f.b, i, f.c);
            C(a, b, c); C(a, c, b); C(b, c, a);
        }
    }
    for(auto &it: FS) if ((A[it.b] - A[it.a]).cross(
        A[it.c] - A[it.a]).dot(it.q) <= 0) swap(it.c, it.b);
    return FS;
};
```

## SphericalDistance.h
**Description:** Returns the shortest distance on the sphere with radius radius between the points with azimuthal angles (longitude) f1 ($\phi_1$) and f2 ($\phi_2$) from x axis and zenith angles (latitude) t1 ($\theta_1$) and t2 ($\theta_2$) from z axis. All angles measured in radians. The algorithm starts by converting the spherical coordinates to cartesian coordinates so if that is what you have you can use only the two last rows. dx*radius is then the difference between the two points in the x direction and d*radius is the total distance between the points.
<span style="float:right">611f07, 8 lines</span>

```cpp
double sphericalDistance(double f1, double t1,
    double f2, double t2, double radius) {
    double dx = sin(t2)*cos(f2) - sin(t1)*cos(f1);
    double dy = sin(t2)*sin(f2) - sin(t1)*sin(f1);
    double dz = cos(t2) - cos(t1);
    double d = sqrt(dx*dx + dy*dy + dz*dz);
    return radius*2*asin(d/2);
}
```

# Strings (9)

## kmp.h
**Description:** failure[x] computes the length of the longest prefix of s that ends at x, other than s[0...x] itself (abacaba -> 0010123). Can be used to find all occurrences of a pattern in a text.
**Time:** $\mathcal{O}(n)$
<span style="float:right">ebacdf, 40 lines</span>

```cpp
template<typename T> struct kmp_t {
    vector<T> word; vector<int> failure;
    template<typename I> kmp_t(I begin, I end) {
        for (I iter = begin; iter != end; ++iter) word.push_back(*
            iter);
        int n = int(word.size()); failure.resize(n+1, 0);
        for (int s = 2; s <= n; ++s) {
            failure[s] = failure[s-1];
            while (failure[s] > 0 && word[failure[s]] != word[s-1])
                failure[s] = failure[failure[s]];
            if (word[failure[s]] == word[s-1]) failure[s] += 1;
        }
    }
    vector<int> matches_in(const vector<T> &text) {
        vector<int> result; int s = 0;
        for (int i = 0; i < int(text.size()); ++i) {
            while (s > 0 && word[s] != text[i]) s = failure[s];
            if (word[s] == text[i]) s += 1;
            if (s == int(word.size())) {
                result.push_back(i-int(word.size())+1);
                s = failure[s];
            }
        }
        return result;
    }
    template<int K = 26, char offset = 'a'>
    auto build_automaton() {
        word.push_back(offset + K);
        vector<array<int, K>> table(word.size());
        for (int a = 0; a < int(word.size()); ++a) {
            for (int b = 0; b < K; ++b) {
                if (a > 0 && offset + b != word[a])
                    table[a][b] = table[failure[a]][b];
                else
                    table[a][b] = a + (offset + b == word[a]);
            }
        }
        return table;
    }
};
```

## duval.h
**Description:** A string is called simple (or a Lyndon word), if it is strictly smaller than any of its own nontrivial suffixes.
**Time:** $\mathcal{O}(N)$
<span style="float:right">d9b2cb, 27 lines</span>

```cpp
template <typename T>
pair<int, vector<string>> duval(int n, const T &s) {
    assert(n >= 1);
    // s += s  //uncomment if you need to know the min cyclic
        string
    vector<string> factors; // strings here are simple and in non
        —inc order
    int i = 0, ans = 0;
    while (i < n) { // until n/2 to find min cyclic string
        ans = i;
        int j = i + 1, k = i;
        while (j < n + n && !(s[j % n] < s[k % n])) {
            if (s[k % n] < s[j % n]) k = i;
            else k++;
            j++;
        }
        while (i <= k) {
```

```cpp
            factors.push_back(s.substr(i, j-k));
            i += j - k;
        }
    }
    return {ans, factors};
    // returns 0—indexed position of the least cyclic shift
    // min cyclic string will be s.substr(ans, n/2)
}
template <typename T>
pair<int, vector<string>> duval(const T &s) {
    return duval((int) s.size(), s);
}
```

## z-algorithm.h
**Description:** z[x] computes the length of the longest common prefix of s[i:] and s, except z[0] = 0. (abacaba -> 0010301)
**Time:** $\mathcal{O}(n)$
<span style="float:right">7c8c64, 16 lines</span>

```cpp
vector<int> Z(const string& S) {
    vector<int> z(S.size());
    int l = -1, r = -1;
    for(int i = 1; i < int(S.size()); ++i) {
        z[i] = i >= r ? 0 : min(r - i, z[i - l]);
        while (i + z[i] < int(S.size()) && S[i + z[i]] == S[z[i]])
            z[i]++;
        if (i + z[i] > r) l = i, r = i + z[i];
    }
    return z;
}
vector<int> get_prefix(string a, string b) {
    string str = a + '@' + b;
    vector<int> k = z(str);
    return vector<int>(k.begin() + int(a.size())+1, k.end());
}
```

## manacher.h
**Description:** For each position in a string, computes p[0][i] = half length of longest even palindrome around pos i, p[1][i] = longest odd (half rounded down).
**Time:** $\mathcal{O}(N)$
<span style="float:right">87e1f0, 13 lines</span>

```cpp
array<vector<int>, 2> manacher(const string &s) {
    int n = s.size();
    array<vector<int>, 2> p = {vector<int>(n+1), vector<int>(n)};
    for(int z = 0; z < 2; ++z) for (int i=0,l=0,r=0; i < n; i++)
        {
        int t = r-i+!z;
        if (i<r) p[z][i] = min(t, p[z][l+t]);
        int L = i-p[z][i], R = i+p[z][i]-!z;
        while (L>=1 && R+1<n && s[L-1] == s[R+1])
            p[z][i]++, L--, R++;
        if (R > r) l = L, r = R;
    }
    return p;
}
```

## min-rotation.h
**Description:** Finds the lexicographically smallest rotation of a string.
**Usage:** rotate(v.begin(), v.begin()+min_rotation(v), v.end());
**Time:** $\mathcal{O}(N)$
<span style="float:right">2a08fd, 8 lines</span>

```cpp
int min_rotation(string s) {
    int a=0, N=s.size(); s += s;
    for(int b = 0; b < N; ++b) for(int i =0; i < N; ++i) {
        if (a+i == b || s[a+i] < s[b+i]) {b += max(0, i-1); break;}
        if (s[a+i] > s[b+i]) { a = b; break; }
    }
    return a;
}
```

## aho-corasick.h

<div style="text-align:right">5159f5, 41 lines</div>

```cpp
const int sigma = 26;
array<int, sigma> init;
for (int i = 0; i < sigma; i++) init[i] = -1;
vector<array<int, sigma>> trie(1, init);
vector<int> out(1, -1), parent(n, -1), ids(n);
for (int i = 0; i < n; i++) {
  int cur = 0;
  for (char ch : s[i]) {
    int c = ch - 'a';
    if (trie[cur][c] == -1) {
      trie[cur][c] = (int)trie.size();
      trie.push_back(init);
      out.push_back(-1);
    }
    cur = trie[cur][c];
  }
  if (out[cur] == -1) out[cur] = i;
  ids[i] = out[cur];
}
vector<int> bfs; bfs.reserve(trie.size());
vector<int> f(trie.size());
for (int c = 0; c < sigma; c++) {
  if (trie[0][c] == -1) trie[0][c] = 0;
  else bfs.push_back(trie[0][c]);
}
for (int z = 0; z < (int)bfs.size() ; z++) {
  int cur = bfs[z];
  for (int c = 0; c < sigma; c++) {
    if (trie[cur][c] == -1) {
      trie[cur][c] = trie[f[cur]][c];
    }
    else {
      int nxt = trie[cur][c];
      int fail = trie[f[cur]][c];
      if (out[nxt] == -1) out[nxt] = out[fail];
      else parent[out[nxt]] = out[fail];
      f[nxt] = fail;
      bfs.push_back(nxt);
    }
  }
}
```

## suffix-array.h

**Description:** Builds suffix array for a string, first element is the size of the string. The `lcp` function calculates longest common prefixes for neighbouring strings in suffix array. The returned vector is of size $n + 1$.

**Time:** $\mathcal{O}(N \log N)$ where $N$ is the length of the string for creation of the SA. $\mathcal{O}(N)$ for longest common prefixes.

<../data-structures/rmq.h>

<div style="text-align:right">5393da, 47 lines</div>

```cpp
mt19937 rng(chrono::steady_clock::now().time_since_epoch().
    count());
struct suffix_array_t {
  int N, H; vector<int> sa, invsa, lcp;
  rmq_t<pair<int, int>> rmq;
  bool cmp(int a, int b) { return invsa[a + H] < invsa[b + H];
    }
  void ternary_sort(int a, int b) {
    if (a == b) return;
    int md = sa[a+rng() % (b-a)], lo = a, hi = b;
    for (int i = a; i < b; ++i) if (cmp(sa[i], md)) swap(sa[i],
        sa[lo++]);
    for (int i = b-1; i >= lo; --i) if (cmp(md, sa[i])) swap(sa
        [i], sa[--hi]);
    ternary_sort(a, lo);
    for (int i = lo; i < hi; ++i) invsa[sa[i]] = hi-1;
    if (hi-lo == 1) sa[lo] = -1;
    ternary_sort(hi, b);
```

```cpp
  }
  suffix_array_t() {}
  template<typename I>
  suffix_array_t(I begin, I end): N(int(end - begin)+1), sa(N)
      {
    vector<int> v(begin, end); v.push_back(INT_MIN);
    invsa = v; iota(sa.begin(), sa.end(), 0);
    H = 0; ternary_sort(0, N);
    for (H = 1; H <= N; H *= 2)
      for (int j = 0, i = j; i != N; i = j)
        if (sa[i] < 0) {
          while (j < N && sa[j] < 0) j += -sa[j];
          sa[i] = -(j - i);
        }
        else { j = invsa[sa[i]] + 1; ternary_sort(i, j); }
    for (int i = 0; i < N; ++i) sa[invsa[i]] = i;
    lcp.resize(N-1); int K = 0;
    for (int i = 0; i < N-1; ++i) {
      if (invsa[i] > 0) while (v[i + K] == v[sa[invsa[i] - 1] +
          K]) ++K;
      lcp[invsa[i]-1] = K; K = max(K - 1, 0);
    }
    vector<pair<int, int>> lcp_index(N-1);
    for (int i = 0; i < N-1; ++i) lcp_index[i] = {lcp[i], 1 + i
        };
    rmq = rmq_t<pair<int, int>>(std::move(lcp_index));
  }
  pair<int, int> rmq_query(int a, int b) const { return rmq.
      query(a, b); }
  pair<int, int> get_split(int a, int b) const { return rmq.
      query(a, b-1); }
  int get_lcp(int a, int b) const {
    if (a == b) return N - a;
    a = invsa[a], b = invsa[b];
    if (a > b) swap(a, b);
    return rmq_query(a + 1, b + 1).first;
  }
};
```

## suffix-automaton.h

**Description:** Suffix automaton

<div style="text-align:right">c4406e, 38 lines</div>

```cpp
template<int offset = 'a'> struct array_state {
  array<int, 26> as;
  array_state() { fill(begin(as), end(as), ~0); }
  int& operator[](char c) { return as[c - offset]; }
  int count(char c) { return (~as[c - offset] ? 1 : 0); }
};

template<typename Char, typename state = map<Char, int>> struct
    suffix_automaton {
  struct node_t {
    int len, link; int64_t cnt;
    state next;
  };
  int N, cur;
  vector<node_t> nodes;
  suffix_automaton() : N(1), cur(0), nodes{node_t{0, -1, 0,
      {}}} {}
  node_t& operator[](int v) { return nodes[v]; };
  void append(Char c) {
    int v = cur; cur = N++;
    nodes.push_back(node_t{nodes[v].len + 1, 0, 1, {}});
    for (; ~v && !nodes[v].next.count(c); v = nodes[v].link) {
      nodes[v].next[c] = cur;
    }
    if (~v) {
      const int u = nodes[v].next[c];
      if (nodes[v].len + 1 == nodes[u].len) {
```

```cpp
        nodes[cur].link = u;
      } else {
        const int clone = N++;
        nodes.push_back(nodes[u]);
        nodes[clone].len = nodes[v].len + 1;
        nodes[u].link = nodes[cur].link = clone;
        for (; ~v && nodes[v].next[c] == u; v = nodes[v].link)
          {
          nodes[v].next[c] = clone;
        }
      }
    }
  }
};
```

## 9.1 Suffix Automaton

### 9.1.1 Number of different substrings

Is the number of paths in the automaton starting at the root.

$$d(v) = 1 + \sum_{v \to w} d(w)$$

### 9.1.2 Total lenght of different substrings

Is the sum of children answers and paths starting at each children.

$$ans(v) = \sum_{v \to w} d(w) + ans(w)$$

### 9.1.3 Lexicographically $K$-th substring

Is the $K$-th lexicographically path, so you can search using the number of paths from each state

### 9.1.4 Smallest cyclic shift

Construct for string $S + S$. Greedily search the minimal character.

### 9.1.5 Number of occurrences

For each state not created by cloning, initialize $cnt(v) = 1$. Then, just do a dfs to calculate $cnt(v)$

$$cnt(link(v)) + = cnt(v)$$

### 9.1.6 First occurence position

When we create a new state $cur$ do $first(pos) = len(cur) - 1$. When we clone $q$ as $clone$ do $first(clone) = first(q)$. Answer is $first(v) - size(P) + 1$, where $v$ is the state of string $P$

### 9.1.7 All occurence positions

From $first(v)$ do a dfs using suffix link, from $link(u)$ go to $u$.

# <u>Various</u> (10)

## 10.1    Intervals

### interval-container.h
**Description:**  Add and remove intervals from a set of disjoint intervals. Will merge the added interval with any overlapping intervals in the set when adding. Intervals are [inclusive, exclusive).
**Time:** $\mathcal{O}(\log N)$                                    f47dfb, 22 lines

```cpp
set<pair<int,int>>::iterator addInterval(set<pair<int,int>> &is
    , int L, int R) {
  if (L == R) return is.end();
  auto it = is.lower_bound({L, R}), before = it;
  while (it != is.end() && it->first <= R) {
    R = max(R, it->second);
    before = it = is.erase(it);
  }
  if (it != is.begin() && (--it)->second >= L) {
    L = min(L, it->first);
    R = max(R, it->second);
    is.erase(it);
  }
  return is.insert(before, {L,R});
}
void removeInterval(set<pair<int,int>> &is, int L, int R) {
  if (L == R) return;
  auto it = addInterval(is, L, R);
  auto r2 = it->second;
  if (it->first == L) is.erase(it);
  else (int&)it->second = L;
  if (R != r2) is.emplace(R, r2);
}
```

### interval-cover.h
**Description:** Compute indices of smallest set of intervals covering another interval. Intervals should be [inclusive, exclusive). To support [inclusive, inclusive], change (A) to add || R.empty(). Returns empty set on failure (or if G is empty).
**Time:** $\mathcal{O}(N \log N)$                                    133eb4, 19 lines

```cpp
template<class T>
vector<int> cover(pair<T, T> G, vector<pair<T, T>> I) {
  vector<int> S(I.size()), R;
  iota(S.begin(), S.end(), 0);
  sort(S.begin(), S.end(), [&](int a, int b) { return I[a] < I[
      b]; });
  T cur = G.first;
  int at = 0;
  while (cur < G.second) { // (A)
    pair<T, int> mx = {cur, -1};
    while (at < I.size() && I[S[at]].first <= cur) {
      mx = max(mx, {I[S[at]].second, S[at]});
      at++;
    }
    if (mx.second == -1) return {};
    cur = mx.first;
    R.push_back(mx.second);
  }
  return R;
}
```

### constant-intervals.h
**Description:** Split a monotone function on [from, to) into a minimal set of half-open intervals on which it has the same value. Runs a callback g for each such interval.
**Usage:** constantIntervals(0, sz(v), [&](int x){
return v[x];}, [&](int lo, int hi, T val){...});
**Time:** $\mathcal{O}\left(k \log \frac{n}{k}\right)$                                    753a4c, 19 lines

```cpp
template<class F, class G, class T>
void rec(int from, int to, F& f, G& g, int& i, T& p, T q) {
  if (p == q) return;
  if (from == to) {
    g(i, to, p);
    i = to; p = q;
  } else {
    int mid = (from + to) >> 1;
    rec(from, mid, f, g, i, p, f(mid));
    rec(mid+1, to, f, g, i, p, q);
  }
}
template<class F, class G>
void constantIntervals(int from, int to, F f, G g) {
  if (to <= from) return;
  int i = from; auto p = f(i), q = f(to-1);
  rec(from, to-1, f, g, i, p, q);
  g(i, to, q);
}
```

## 10.2    Misc. algorithms

### basis-manager.h
**Description:**  A list of basis values sorted in decreasing order, where each value has a unique highest bit.
                                    d5bcd3, 41 lines

```cpp
const int BITS = 60;

template<typename T> struct xor_basis {
  int N = 0;
  array<T, BITS> basis;

  T min_value(T start) const {
    if (N == BITS) return 0;
    for (int i = 0; i < N; ++i)
      start = min(start, start ^ basis[i]);
    return start;
  }
  T max_value(T start = 0) const {
    if (N == BITS) return ((T) 1 << BITS) - 1;
    for (int i = 0; i < N; ++i)
      start = max(start, start ^ basis[i]);
    return start;
  }
  bool add(T x) {
    x = min_value(x);
    if (x == 0) return false;
    basis[N++] = x;
    // Insertion sort.
    for (int k = N - 1; k > 0 && basis[k] > basis[k - 1]; k--)
      swap(basis[k], basis[k - 1]);
    return true;
  }
  void merge(const xor_basis<T>& other) {
    for (int i = 0; i < other.n && N < BITS; i++)
      add(other.basis[i]);
  }
  void merge(const xor_basis<T>& a, const xor_basis<T>& b) {
    if (a.N > b.N) {
      *this = a;
      merge(b);
    } else {
      *this = b;
      merge(a);
    }
  }
};
```

### ternary-search.h
**Description:** Find the smallest i in $[a, b]$ that maximizes $f(i)$, assuming that $f(a) < \ldots < f(i) \geq \cdots \geq f(b)$. To reverse which of the sides allows non-strict inequalities, change the $<$ marked with (A) to $<=$, and reverse the loop at (B). To minimize $f$, change it to $>$, also at (B). If you are dealing with real numbers, you'll need to pick $m_1 = (2a+b)/3.0$ and $m_2 = (a+2b)/3.0$. Consider setting a constant number of iterations for the search, usually $[200, 300]$ iterations are sufficient for problems with error limit as $10^{-6}$.
**Usage:** int ind = ternSearch(0,n-1,[&](int i){return a[i];});
**Time:** $\mathcal{O}(\log(b - a))$                                    35ef73, 12 lines

```cpp
template<class F>
int ternSearch(int a, int b, F f) {
  assert(a <= b);
  while (b - a >= 5) {
    int mid = (a + b) / 2;
    if (f(mid) < f(mid+1)) a = mid; // (A)
    else b = mid+1;
  }
  for(int i = a+1; i <= b; ++i)
    if (f(a) < f(i)) a = i; // (B)
  return a;
}
```

### count-triangles.h
**Description:** Counts x, y >= 0 such that Ax + By <= C.                                    8d67b3, 8 lines

```cpp
lint count_triangle(lint A, lint B, lint C) {
  if (C < 0) return 0;
  if (A > B) swap(A, B);
  lint p = C / B;
  lint k = B / A;
  lint d = (C - p * B) / A;
  return count_triangle(B - k * A, A, C - A * (k * p + d + 1))
      + (p + 1) * (d + 1) + k * p * (p + 1) / 2;
}
```

### floyd-cycle.h
**Description:** Detect loop in a list. Consider using mod template to avoid overflow.
**Time:** $\mathcal{O}(n)$                                    b456ab, 10 lines

```cpp
template<class F>
pair<int,int> find(int x0, F f) {
  int t = f(x0), h = f(t), mu = 0, lam = 1;
  while (t != h) t = f(t), h = f(f(h));
  h = x0;
  while (t != h) t = f(t), h = f(h), ++mu;
  h = f(t);
  while (t != h) h = f(h), ++lam;
  return {mu, lam};
}
```

## 10.3    Dynamic programming

### divide-and-conquer-dp.h
**Description:** Given $a[i] = \min_{lo(i) \leq k < hi(i)}(f(i, k))$ where the (minimal) optimal $k$ increases with $i$, computes $a[i]$ for $i = L..R - 1$.
**Time:** $\mathcal{O}((N + (hi - lo)) \log N)$                                    2cef33, 50 lines

```cpp
struct DP { // Modify at will:
  int lo(int ind) { return 0; }
  int hi(int ind) { return ind; }
  lint f(int ind, int k) { return dp[ind][k]; }
  void store(int ind, int k, lint v) { res[ind] = {k, v}; }
  void rec(int L, int R, int LO, int HI) {
    if (L >= R) return;
    int mid = (L + R) >> 1;
    pair<lint, int> best(LLONG_MAX, LO);
    for(int k = max(LO,lo(mid)); k <= min(HI,hi(mid)); ++k)
```

```cpp
    best = min(best, make_pair(f(mid, k), k));
    store(mid, best.second, best.first);
    rec(L, mid, LO, best.second);
    rec(mid+1, R, best.second, HI);
  }
  void solve(int L, int R) { rec(L, R, INT_MIN, INT_MAX); }
};
```

```cpp
struct DP { // Modify at will:
  vector<int>a, freq;
  vector<lint>old, cur;
  lint cnt;
  int lcur, rcur;
  DP(const vector<int>&_a, int n): a(_a), freq(n), old(n+1,
      linf), cur(n+1, linf), cnt(0), lcur(0), rcur(0){}
  int lo(int ind) { return 0; }
  int hi(int ind) { return ind; }
  void add(int k, int c){ cnt += freq[a[k]]++; }
  void del(int k, int c){ cnt -= --freq[a[k]]; }
  lint C(int l, int r){
    while(lcur > l) add(--lcur, 0);
    while(rcur < r) add(rcur++, 1);
    while(lcur < l) del(lcur++, 0);
    while(rcur > r) del(--rcur, 1);
    return cnt;
  }
  lint f(int ind, int k) { return old[k] + C(k, ind); }

  void store(int ind, int k, lint v) { cur[ind] = v; }
  void rec(int L, int R, int LO, int HI) {
    if (L >= R) return;
    int mid = (L + R) >> 1;
    pair<lint, int> best(LLONG_MAX, LO);
    for(int k = max(LO,lo(mid)); k <= min(HI,hi(mid)); ++k)
      best = min(best, make_pair(f(mid, k), k));

    store(mid, best.second, best.first);
    rec(L, mid, LO, best.second);
    rec(mid+1, R, best.second, HI);
  }
};
```

## knuth-dp.h

**Description:** When doing DP on intervals: $a[i][j] = \min_{i<k<j}(a[i][k] + a[k][j]) + f(i, j)$, where the (minimal) optimal $k$ increases with both $i$ and $j$, one can solve intervals in increasing order of length, and search $k = p[i][j]$ for $a[i][j]$ only between $p[i][j - 1]$ and $p[i + 1][j]$. This is known as Knuth DP. Sufficient criteria for this are if $f(b, c) \leq f(a, d)$ and $f(a, c) + f(b, d) \leq f(a, d) + f(b, c)$ for all $a \leq b \leq c \leq d$. Consider also: LineContainer (ch. Data structures), monotone queues, ternary search.
**Time:** $\mathcal{O}(N^2)$

## digit-dp.h

**Description:** Compute how many # between 1 and $N$ have $K$ distinct digits in the base $L$ without leading zeros;
**Usage:** auto hex_to_dec = [&](char c) -> int {
return ('A' <= c && c <= 'F' ? (10 + c - 'A') : (c - '0'));
};
digit_dp<modnum<int(1e9) + 7>, hex_to_dec>(N, K);
**Time:** $\mathcal{O}(NK)$

8eca47, 32 lines
```cpp
template<typename T, class F> T digit_dp(const string& S, int K
    , F& L) {
  const int base = 16;
  const int len = int(S.size());

  vector<bool> w(base);
  vector<vector<T>> dp(len + 1, vector<T>(base + 2));
```

```cpp
  int cnt = 0;
  for (int d = 0; d < len; ++d) {
    // adding new digit to numbers wiht prefix < s
    for (int x = 0; x <= base; ++x) {
      dp[d + 1][x] += dp[d][x] * x;
      dp[d + 1][x + 1] += dp[d][x] * (base - x);
    }
    // adding strings whith prefix only 0's and last digit != 0
    if (d) dp[d + 1][1] += (base - 1);
    // adding prefix equal to s and last digit < s, first digit
    //    cannot be 0
    for (int x = 0; x < L(S[d]); ++x) {
      if (d == 0 && x == 0) continue;
      if (w[x]) dp[d + 1][cnt] += 1;
      else dp[d + 1][cnt + 1] += 1;
    }
    // marking if the last digit appears in the prefix of s
    if (w[L(S[d])] == false) {
      w[L(S[d])] = true;
      cnt++;
    }
  }
  // adding string k
  dp[len][cnt] += 1;
  return dp[len][K];
}
```

## knapsack-bounded.h

**Description:** You are given $n$ types of items, each items has a weight and a quantity. Is possible to fill a knapsack with capacity k using any subset of items?
**Time:** $\mathcal{O}(Wn)$

9bddad, 15 lines
```cpp
vector<int> how_many(n+1), dp(k+1);
dp[0] = 1;
for (int i = 1; i <= n; ++i) cin >> how_many[i];
for (int i = 1; i <= n; ++i) {
  for (int j = k-items[i]; j >= 0; --j) {
    if (dp[j]) {
      int x = 1;
      while (x <= how_many[i] &&
        j + x*items[i] <= k && !dp[j + x*items[i]]) {
        dp[j + x*items[i]] = 1;
        ++x;
      }
    }
  }
}
```

## knapsack-bounded-costs.h

**Description:** You are given $n$ types of items, you have $e[i]$ items of $i$-th type, and each item of $i$-th type weight $w[i]$ and cost $c[i]$. What is the minimal cost you can get by picking some items weighing at most $W$ in total?
**Time:** $\mathcal{O}(Wn)$

3ade3c, 28 lines
```cpp
<MinQueue.h>
const int maxn = 1000;
const int maxm = 100000;
const int inf = 0x3f3f3f3f;

minQueue<int> q[maxm];

array<int, maxm> dp; // the minimum cost dp[i] I need to pay in
    //   order to fill the knapsack with total weight i
int w[maxn], e[maxn], c[maxn]; // weight, number, cost

int main() {
  int n, m;
  cin >> n >> m;
```

```cpp
  for (int i = 1; i <= n; i++) cin >> w[i] >> c[i] >> e[i];
  for (int i = 1; i <= m; i++) dp[i] = inf;
  for (int i = 1; i <= n; i++) {
    for (int j = 0; j < w[i]; j++) q[j].clear();
    for (int j = 0; j <= m; j++) {
      minQueue<int> &mq = q[j % w[i]];
      if (mq.size() > e[i]) mq.pop();
      mq.add(c[i]);
      mq.push(dp[j]);
      dp[j] = mq.getMin();
    }
  }
  cout << "Minimum value i can pay putting a total weight " <<
      m << " is " << dp[m] << '\n';
  for (int i = 0; i <= m; i++) cout << dp[i] << " " << i << '\n
      ';
  cout << "\n";
}
```

## two-max-equal-sum.h

**Description:** Two maximum equal sum disjoint subsets, s[i] = 0 if v[i] wasn't selected, s[i] = 1 if v[i] is in the first subset and s[i] = 2 if v[i] is in the second subset
**Time:** $\mathcal{O}(n * S)$

d66110, 15 lines
```cpp
pair<int, vector<int>> twoMaxEqualSumDS(vector<int> &v){
  const int n = int(v.size());
  const int sum = accumulate(v.begin(), v.end(), 0);
  vector<int> dp(2*sum + 1, INT_MIN/2), newdp(2*sum + 1), s(n);
  vector<vector<int>> rec(n, vector<int>(2*sum + 1));
  int i; dp[sum] = 0;
  for(i = 0; i < n; i++, swap(dp, newdp))
    for(int a, b, d = v[i]; d <= 2*sum - v[i]; d++){
      newdp[d] = max({dp[d], a = dp[d - v[i]] + v[i], b = dp[d
          + v[i]]});
      rec[i][d] = newdp[d] == a ? 1 : newdp[d] == b ? 2 : 0;
    }
  for(int j = i-1, d = sum; j >= 0 ; j--)
    d += (s[j] = rec[j][d]) ? s[j] == 2 ? v[j] : -v[j] : 0;
  return {dp[sum], s};
}
```

## 10.4   Optimization tricks
### 10.4.1   Bit hacks

- c = x&-x, r = x+c; (((r^x) >> 2)/c) | r is the next number after x with the same number of bits set.

- rep(b,0,K) rep(i,0,(1 << K)) if (i & 1 << b) D[i] += D[i^(1 << b)]; computes all sums of subsets.

## hashmap.h

**Description:** Faster/better hash maps, taken from CF

09a72f, 19 lines
```cpp
#include<bits/extc++.h>
struct splitmix64_hash {
  static uint64_t splitmix64(uint64_t x) {
    x += 0x9e3779b97f4a7c15;
    x = (x^(x >> 30)) * 0xbf58476d1ce4e5b9;
    x = (x^(x >> 27)) * 0x94d049bb133111eb;
    return x^(x >> 31);
  }
  size_t operator()(uint64_t x) const {
    static const uint64_t FIXED_RANDOM = std::chrono::
        steady_clock::now().time_since_epoch().count();
    return splitmix64(x + FIXED_RANDOM);
```

```
  }
};

template <typename K, typename V, typename Hash =
    splitmix64_hash>
using hash_map = __gnu_pbds::gp_hash_table<K, V, Hash>;

template <typename K, typename Hash = splitmix64_hash>
using hash_set = hash_map<K, __gnu_pbds::null_type, Hash>;
```

## 10.5   Bit Twiddling Hack

hacks.h
829b7d, 22 lines

```
// Iterate on non-empty submasks of a bitmask.
for (int s = m; s > 0; s = (m & (s - 1)))
// Iterate on non-zero bits of a bitset B.
for (int j = B._Find_next(0); j < MAXV; j = B._Find_next(j))

ll next_perm(ll v) { // compute next perm i.e.
  ll t = v | (v-1);  // 00111,01011,01101,10011 ...
  return (t + 1) | (((~t & -~t) - 1) >> (__builtin_ctz(v) + 1))
      ;
}
template<typename F>  // All subsets of size k of {0..N-1}
void iterate_k_subset(ll N, ll k, F f){
  ll mask = (1ll << k) - 1;
  while (!(mask & 1ll<<N)) { f(mask);
    ll t = mask | (mask-1);
    mask = (t+1) | (((~t & -~t) - 1) >> (__builtin_ctzll(mask)
        +1));
  }
}
template<typename F> // All subsets of set
void iterate_mask_subset(ll set, F f){ ll mask = set;
  do  f(mask), mask = (mask-1) & set;
  while (mask != set);
}
```

## 10.6   Random Numbers

random-numbers.h
**Description:** An example on the usage of generator and distribution. Use shuffle instead of random shuffle.
b28ebe, 9 lines

```
mt19937 rng(random_device{}());
mt19937_64 rng(chrono::steady_clock::now().time_since_epoch().
    count());
shuffle(permutation.begin(), permutation.end(), rng);
uniform_int_distribution<int> uid(1, 100); //  [1, 100]
    inclusive!
uniform_real_distribution<double> urd(1, 100);
unsigned xrand() {
  static unsigned x = 314159265, y = 358979323, z = 846264338,
      w = 327950288;
  unsigned t = x ^ x << 11; x = y; y = z; z = w; return w = w ^
      w >> 19 ^ t ^ t >> 8;
}
```