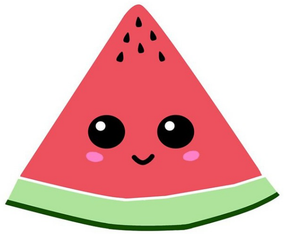
Competições de Algoritmos e Programação UFRJ



Federal University of Rio de Janeiro

UFRJ - World Finals

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2022 and 2023 World Finals - Sharm El-Sheikh

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Contest (1)

Makefile 8 lines

```
CXX = g++
CXXFLAGS = -std=c++17 -O2 -Wall -Wextra -pedantic -Wshadow -
     Wformat=2 -Wfloat-equal -Wconversion -Wlogical-op -Wshift-
     overflow=2 -Wduplicated-cond -Wcast-qual -Wcast-align -Wno
     -unused-result -Wno-sign-conversion
DEBUGFLAGS = -D_GLIBCXX_DEBUG -D_GLIBCXX_DEBUG_PEDANTIC -DLOCAL
      -fsanitize=address -fsanitize=undefined -fno-sanitize-
     recover=all -fstack-protector -D_FORTIFY_SOURCE=2

DEBUG = false
ifeq ($(DEBUG),true)
  CXXFLAGS += $(DEBUGFLAGS)
endif
```

hash.sh 3 lines

```
# Hashes a file, ignoring all whitespace and comments. Use for
# verifying that code was correctly typed.
cpp -dD -P -fpreprocessed | tr -d '[:space:]'| md5sum |cut -c-6
```

hash-cpp.sh 5 lines

```
# Hashes a file, ignoring all whitespace, comments and defines.
     Use for
# verifying that code was correctly typed.
# First do: chmod +x ./hash-cpp.sh
# ./hash-cpp.sh *.cpp start end
sed -n $2','$3' p' $1 | sed '/^#w/d' | cpp -dD -P -
     fpreprocessed | tr -d '[:space:]' | md5sum |cut -c-6
```

Mathematics (2)

2.1 Equations

$$ax^2+bx+c=0 \Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The extremum is given by $x=-b/2a$.

$$\begin{aligned} ax+by=e \\ cx+dy=f \end{aligned} \Rightarrow \begin{aligned} x=\frac{ed-bf}{ad-bc} \\ y=\frac{af-ec}{ad-bc} \end{aligned}$$

In general, given an equation $Ax=b$, the solution to a variable x_i is given by

$$x_i=\frac{\det A_i'}{\det A}$$

where A_i' is A with the i'th column replaced by b.

2.2 Trigonometry

$$\sin(v+w)=\sin v\cos w+\cos v\sin w$$
$$\cos(v+w)=\cos v\cos w-\sin v\sin w$$

$$\tan(v+w)=\frac{\tan v+\tan w}{1-\tan v\tan w}$$
$$\sin v+\sin w=2\sin\frac{v+w}{2}\cos\frac{v-w}{2}$$
$$\cos v+\cos w=2\cos\frac{v+w}{2}\cos\frac{v-w}{2}$$

$$(V+W)\tan(v-w)/2=(V-W)\tan(v+w)/2$$

where V,W are lengths of sides opposite angles v,w.

$$a\cos x+b\sin x=r\cos(x-\phi)$$
$$a\sin x+b\cos x=r\sin(x+\phi)$$

where $r=\sqrt{a^2+b^2}, \phi=\operatorname{atan2}(b,a)$.

2.3 Geometry

2.3.1 Triangles

Side lengths: a,b,c

Semiperimeter: $p=\frac{a+b+c}{2}$

Area: $A=\sqrt{p(p-a)(p-b)(p-c)}$

Circumradius: $R=\frac{abc}{4A}$

Inradius: $r=\frac{A}{p}$

Length of median (divides triangle into two equal-area triangles): $m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

Length of bisector (divides angles in two):

$$s_a=\sqrt{bc\left[1-\left(\frac{a}{b+c}\right)^2\right]}$$

Law of sines: $\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}=\frac{1}{2R}$

Law of cosines: $a^2=b^2+c^2-2bc\cos\alpha$

Law of tangents: $\frac{a+b}{a-b}=\frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}$

Pick's: A polygon on an integer grid strictly containing i lattice points and having b lattice points on the boundary has area $i+\frac{b}{2}-1$. (Nothing similar in higher dimensions)

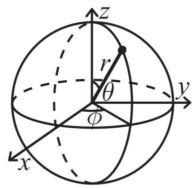
2.3.2 Quadrilaterals

With side lengths a,b,c,d, diagonals e,f, diagonals angle θ, area A and magic flux $F=b^2+d^2-a^2-c^2$:

$$4A=2ef\cdot\sin\theta=F\tan\theta=\sqrt{4e^2f^2-F^2}$$

For cyclic quadrilaterals the sum of opposite angles is $180°$, $ef=ac+bd$, and $A=\sqrt{(p-a)(p-b)(p-c)(p-d)}$.

2.3.3 Spherical coordinates



$$\begin{aligned} x&=r\sin\theta\cos\phi & r&=\sqrt{x^2+y^2+z^2} \\ y&=r\sin\theta\sin\phi & \theta&=\operatorname{acos}(z/\sqrt{x^2+y^2+z^2}) \\ z&=r\cos\theta & \phi&=\operatorname{atan2}(y,x) \end{aligned}$$

2.4 Derivatives/Integrals

$$\frac{d}{dx}\arcsin x=\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x=-\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan x=1+\tan^2 x \qquad \frac{d}{dx}\arctan x=\frac{1}{1+x^2}$$
$$\int\tan ax=-\frac{\ln|\cos ax|}{a} \qquad \int x\sin ax=\frac{\sin ax-ax\cos ax}{a^2}$$
$$\int e^{-x^2}=\frac{\sqrt{\pi}}{2}\operatorname{erf}(x) \qquad \int xe^{ax}dx=\frac{e^{ax}}{a^2}(ax-1)$$

2.4.1 XOR sum

$$\bigoplus_{x=0}^{n-1}x=\{0,n-1,1,n\}[n \bmod 4]$$
$$\bigoplus_{x=l}^{r-1}x=\bigoplus_{a=0}^{r-1}a\oplus\bigoplus_{b=0}^{l-1}b$$

2.5 Series

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots,\ (-\infty<x<\infty)$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots,\ (-1<x\leq 1)$$

$$\sqrt{1+x}=1+\frac{x}{2}-\frac{x^2}{8}+\frac{2x^3}{32}-\frac{5x^4}{128}+\dots,\ (-1\leq x\leq 1)$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots,\ (-\infty<x<\infty)$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots,\ (-\infty<x<\infty)$$

2.6 Probability theory

Let X be a discrete random variable with probability $p_X(x)$ of assuming the value x. It will then have an expected value (mean) $\mu = \mathbb{E}(X) = \sum_x x p_X(x)$ and variance $\sigma^2 = V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$ where σ is the standard deviation. If X is instead continuous it will have a probability density function $f_X(x)$ and the sums above will instead be integrals with $p_X(x)$ replaced by $f_X(x)$.

Expectation is linear:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

For independent X and Y,

$$V(aX + bY) = a^2V(X) + b^2V(Y).$$

2.6.1 Discrete distributions

Binomial distribution

The number of successes in n independent yes/no experiments, each which yields success with probability p is $\text{Bin}(n, p)$, $n = 1, 2, \ldots, 0 \le p \le 1$.

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mu = np, \sigma^2 = np(1-p)$$

$\text{Bin}(n, p)$ is approximately $\text{Po}(np)$ for small p.

2.7 Markov chains

A *Markov chain* is a discrete random process with the property that the next state depends only on the current state. Let $X_1, X_2, \ldots$ be a sequence of random variables generated by the Markov process. Then there is a transition matrix $\mathbf{P} = (p_{ij})$, with $p_{ij} = \Pr(X_n = i | X_{n-1} = j)$, and $\mathbf{p}^{(n)} = \mathbf{P}^n \mathbf{p}^{(0)}$ is the probability distribution for X_n (i.e., $p_i^{(n)} = \Pr(X_n = i)$), where $\mathbf{p}^{(0)}$ is the initial distribution.

π is a stationary distribution if $\pi = \pi\mathbf{P}$. If the Markov chain is *irreducible* (it is possible to get to any state from any state), then $\pi_i = \frac{1}{\mathbb{E}(T_i)}$ where $\mathbb{E}(T_i)$ is the expected time between two visits in state i. π_j/π_i is the expected number of visits in state j between two visits in state i.

For a connected, undirected and non-bipartite graph, where the transition probability is uniform among all neighbors, π_i is proportional to node i's degree.

A Markov chain is *ergodic* if the asymptotic distribution is independent of the initial distribution. A finite Markov chain is ergodic iff it is irreducible and *aperiodic* (i.e., the gcd of cycle lengths is 1). $\lim_{k\to\infty} \mathbf{P}^k = \mathbf{1}\pi$.

A Markov chain is an absorbing chain if
1. there is at least one absorbing state and
2. it is possible to go from any state to at least one absorbing state in a finite number of steps.

A Markov chain is an A-chain if the states can be partitioned into two sets $\mathbf{A}$ and $\mathbf{G}$, such that all states in $\mathbf{A}$ are absorbing ($p_{ii} = 1$), and all states in $\mathbf{G}$ leads to an absorbing state in $\mathbf{A}$. The probability for absorption in state $i \in \mathbf{A}$, when the initial state is j, is $a_{ij} = p_{ij} + \sum_{k\in\mathbf{G}} a_{ik} p_{kj}$. The expected time until absorption, when the initial state is i, is $t_i = 1 + \sum_{k\in\mathbf{G}} p_{ki} t_k$.

Data Structures (3)

order-statistic-tree.h

Description: A set (not multiset!) with support for finding the n'th element, and finding the index of an element.
Time: $\mathcal{O}(\log N)$

<bits/extc++.h> acfa21, 17 lines

```cpp
template <typename K, typename V, typename Comp = std::less<K>>
using ordered_map = __gnu_pbds::tree<
  K, V, Comp,
  __gnu_pbds::rb_tree_tag,
  __gnu_pbds::tree_order_statistics_node_update
>;
template <typename K, typename Comp = std::less<K>>
using ordered_set = ordered_map<K, __gnu_pbds::null_type, Comp
    >;
void example() {
  ordered_set<int> t, t2; t.insert(8);
  auto it = t.insert(10).first;
  assert(it == t.lower_bound(9));
  assert(t.order_of_key(10) == 1); // num strictly smaller
  assert(t.order_of_key(11) == 2);
  assert(*t.find_by_order(0) == 8);
  t.join(t2); // assuming T < T2 or T > T2, merge t2 into t
}
```

dsu-rollback.h

Description: Disjoint-set data structure with undo.
Usage: `int t = uf.time(); ...; uf.rollback(t);`
Time: $\mathcal{O}(\log(N))$

7ddf1d, 21 lines

```cpp
struct RollbackUF {
  vector<int> e; vector<pair<int,int>> st;
  RollbackUF(int n) : e(n, -1) {}
  int size(int x) { return -e[find(x)]; }
  int find(int x) { return e[x] < 0 ? x : find(e[x]); }
  int time() { return st.size(); }
  void rollback(int t) {
    for (int i = time(); i --> t;)
      e[st[i].first] = st[i].second;
    st.resize(t);
  }
  bool unite(int a, int b) {
    a = find(a), b = find(b);
    if (a == b) return false;
    if (e[a] > e[b]) swap(a, b);
    st.push_back({a, e[a]});
    st.push_back({b, e[b]});
    e[a] += e[b]; e[b] = a;
    return true;
  }
```

```cpp
};
```

monotonic-queue.h

Description: Supports pop and push queue-like, and add function adds a constant to all elements currently in the queue.
Time: $\mathcal{O}(1)$

8b6ad8, 18 lines

```cpp
template<typename T, typename Comp> struct monotonic_queue {
  int lo, hi; T S;
  deque<pair<T, T>> q;
  monotonic_queue() : lo(0), hi(0), S(0) {}
  void push(T val) {
    while(!q.empty() && Comp()(val, q.back().first + S))
      q.pop_back();
    q.emplace_back(val - S, hi++);
  }
  void pop() {
    if (!q.empty() && q.front().second == lo++) q.pop_front();
  }
  void add(T val) { S += val; }
  T get_val() const {  return q.front().first + S; }
  int size() const { return hi-lo; }
};
template<typename T> using min_monotonic_queue =
    monotonic_queue<T, std::less_equal<T>>;
template<typename T> using max_monotonic_queue =
    monotonic_queue<T, std::greater_equal<T>>;
```

point-context.h

Description: Examples of Segment Tree

70d417, 32 lines

```cpp
struct seg_node { // bbfc07
  int val, int mi, ma;
  seg_node() : mi(INT_MAX), ma(INT_MIN), val(0) {}
  seg_node(int x) : mi(x), ma(x), val(x) {}
  void merge(const seg_node& l, const seg_node& r) {
    val = l.val + r.val;
    mi = min(l.mi, r.mi), ma = max(l.ma, r.ma);
  }
  void update(int x) {
    mi = ma = val = x;
  }
  bool acc_min(int& acc, int x) const {
    if (x >= mi) return true;
    if (acc > mi) acc = mi;
    return false;
  }
  bool acc_max(int& acc, int x) const {
    if (x <= ma) return true;
    if (acc < ma) acc = ma;
    return false;
  }
};
// 1 + min of [a, N) <= x
auto find_min_right = [&](segtree<seg_node>& sg, int a, int x)
    -> int {
  int acc = INT_MAX;
  return sg.find_first(a, N, &seg_node::acc_min, acc, x);
};
// max of [0, a) >= x
auto find_max_left = [&](segtree<seg_node>& sg, int a, int x)
    -> int {
  int acc = INT_MIN;
  return sg.find_last(0, a, &seg_node::acc_max, acc, x);
};
```

rec-lazy-segtree.h

Description: Segment Tree with Lazy update (half-open interval).
Time: $\mathcal{O}(\lg(N) * Q)$

f00ac4, 63 lines

```cpp
template<class T> struct segtree_range {
  int N; vector<T> ts;
  segtree_range() {}
  segtree_range(int M) : segtree_range(vector<T>(M, T(0))) {}
  template<class Q> segtree_range(const vector<Q>& A) {
    const int N_ = int(A.size());
    N = (1 << __lg(2*N_-1)); ts.resize(2*N);
    for (int i = 0; i < N_; ++i) at(i) = T(A[i]);
    build();
  }
  T& at(int a) { return ts[a + N]; }
  void build() { for (int a = N; --a; ) merge(a); }
  inline void push(int a) { ts[a].push(ts[2*a], ts[2*a+1]); }
  inline void merge(int a) { ts[a].merge(ts[2*a], ts[2*a+1]); }
  template<class Op, class E, class F, class... Args>
  auto query(int v, int l, int r, int a, int b, Op op, E e, F f
      , Args&&... args) {
    if (l >= b || r <= a) return e();
    if (l >= a && r <= b) return (ts[v].*f)(args...);
    int m = (l + r)/2;
    push(v);
    return op(query(2*v, l, m, a, b, op, args...), query(2*v+1,
        m, r, a, b, op, args...));
  }
  template<class Op, class E, class F, class... Args>
  auto query(int a, int b, Op op, E e, F f, Args&&... args) {
    return query(1, 0, N, a, b, op, e, f, args...);
  }
  T query(int v, int l, int r, int a, int b) {
    if (l >= b || r <= a) return T();
    if (l >= a && r <= b) return ts[v];
    int m = (l + r)/2;
    push(v); T t;
    t.merge(query(2*v, l, m, a, b), query(2*v+1, m, r, a, b));
    return t;
  }
  T query(int a, int b) { return query(1, 0, N, a, b); }
  template<class F, class... Args> void update(int v, int l,
      int r, int a, int b, F f, Args&&... args) {
    if (l >= b || r <= a) return;
    if (l >= a && r <= b && (ts[v].*f)(args...)) return;
    int m = (l + r)/2;
    push(v);
    update(2*v, l, m, a, b, f, args...);
    update(2*v+1, m, r, a, b, f, args...);
    merge(v);
  }
  template<class F, class... Args>
  void update(int a, int b, F f, Args&&... args) {
    update(1, 0, N, a, b, f, args...);
  }
  template<class F, class... Args> int find_first(int v, int l,
      int r, int a, int b, F f, Args&&... args) {
    if (l >= b || r <= a || !(ts[v].*f)(args...)) return -1;
    if (l + 1 == r) return l;
    int m = (l + r)/2;
    push(v);
    int cur = find_first(2*v, l, m, a, b, f, args...);
    if (cur == -1)
      cur = find_first(2*v+1, m, r, a, b, f, args...);
    return cur;
  }
  template<class F, class... Args>
  int find_first(int a, int b, F f, Args&&... args) {
    return find_first(1, 0, N, a, b, f, args...);
  }
};
```

lazy-context.h

Description: Examples of Segment Tree with Lazy update bd0d51, 173 lines

```cpp
template<typename T = int64_t> struct seg_node {
  T val, lz_add, lz_set;
  int sz; bool to_set;
  seg_node(T n = 0) : val(n), lz_add(0), lz_set(0), sz(1),
      to_set(0) {}
  void push(seg_node& l, seg_node& r) {
    if (to_set) {
      l.assign(lz_set), r.assign(lz_set);
      lz_set = 0; to_set = false;
    }
    if (lz_add != 0) {
      l.add(lz_add), r.add(lz_add), lz_add = 0;
    }
  }
  void merge(const seg_node& l, const seg_node& r) {
    sz = l.sz + r.sz; val = l.val + r.val;
  }
  bool add(T v) {  // update range a[i] <- a[i] + v
    val += v * sz; lz_add += v; return true;
  }
  bool assign(T v) {   // update range a[i] <- v
    val = v * sz; lz_add = 0;
    lz_set = v; to_set = true; return true;
  }
  T get_sum() const { return val; } // sum a[l, r)
};
// update range a[i] <- a[i] + b * (i - s) + c
// assuming b and c are non zero, be careful
// get sum a[l, r)
template<typename T = int64_t> struct seg_node {
  T sum, lzB, lzC;
  int sz, idx;
  seg_node(int id = 0, T v = 0, int s = 0, T b = 0, T c = 0) :
    sum(v), lzB(b), lzC(c - s * b), idx(id), sz(1) {}
  void push(seg_node& l, seg_node& r) {
    l.add(lzB, lzC), r.add(lzB, lzC);
    lzB = lzC = 0;
  }
  void merge(const seg_node& l, const seg_node& r) {
    idx = min(l.idx, r.idx), sz = l.sz + r.sz;
    sum = l.sum + r.sum;
  }
  T sum_idx(T n) const { return n * (n + 1) / 2; }
  bool add(T b, T c) {
    sum += b * (sum_idx(idx + sz) - sum_idx(idx)) + sz * c;
    lzB += b, lzC += c; return true;
  }
  T get_sum() const { return sum; }
};

// update range a[i] <- b * a[i] + c
// get sum a[l, r)
struct seg_node {
  int sz; i64 sum, lzB, lzC;
  seg_node() : sz(1), sum(0), lzB(1), lzC(0) {}
  seg_node(i64 v) : sz(1), sum(v), lzB(1), lzC(0) {}
  void push(seg_node& l, seg_node& r) {
    l.add(lzB, lzC), r.add(lzB, lzC);
    lzB = 1, lzC = 0;
  }
  void merge(const seg_node& l, const seg_node& r) {
    sz = l.sz + r.sz, sum = l.sum + r.sum;
  }
  bool add(i64 b, i64 c) {
    sum = (b * sum + c * sz), lzB = (lzB * b);
    lzC = (lzC * b + c); return true;
  }
  i64 get_sum() const { return sum; }
};
// update range a[i] <- min(a[i], b);
// update range a[i] <- max(a[i], b);
// get val a[i]
struct seg_node {
  int mn, mx;
  int lz0, lz1;
  seg_node() : mn(INT_MAX), mx(INT_MIN), lz0(INT_MAX), lz1(
      INT_MIN) {}
  void push(seg_node& l, seg_node& r) {
    l.minimize(lz0), l.maximize(lz1);
    r.minimize(lz0), r.maximize(lz1);
    lz0 = INT_MAX, lz1 = INT_MIN;
  }
  void merge(const seg_node& l, const seg_node& r) {
    mn = min(l.mn, r.mn), mx = max(l.mx, r.mx);
  }
  bool minimize(int val) {
    mn = lz0 = min(lz0, val);
    mx = lz1 = min(lz0, lz1); return true;
  }
  bool maximize(int val) {
    mx = lz1 = max(lz1, val);
    mn = lz0 = max(lz0, lz1); return true;
  }
  pair<int, int> get() const { return {mx, mn}; }
};

template<typename T> struct lazy_t {
  T a, b, c;
  lazy_t() : a(0), b(-INF), c(+INF) {}
  lazy_t(T a, T b, T c) : a(a), b(b), c(c) {}
  void add(T val) {
    a += val, b += val, c += val;
  }
  void upd_min(T val) {
    if (b > val) b = val;
    if (c > val) c = val;
  }
  void upd_max(T val) {
    if (b < val) b = val;
    if (c < val) c = val;
  }
};
template<typename T = int64_t> struct seg_node {
  T mi, mi2, ma, ma2, sum;
  T cnt_mi, cnt_ma, sz;
  lazy_t<T> lz;
  seg_node() : mi(INF), mi2(INF), ma(-INF), ma2(-INF), sum(0),
      cnt_mi(0), cnt_ma(0), sz(0), lz() {}
  seg_node(T n) : mi(n), mi2(INF), ma(n), ma2(-INF), sum(n),
      cnt_mi(1), cnt_ma(1), sz(1), lz() {}
  void push(seg_node& l, seg_node& r) {
    if (!l.can_apply(lz) || !r.can_apply(lz)) return;
    lz = lazy_t<T>();
  }
  bool can_apply(const lazy_t<T>& f) {
    if (!add(f.a) || !upd_max(f.b) || !upd_min(f.c)) return
        false;
    return true;
  }
  void merge(const seg_node& l, const seg_node& r) {
    mi = min(l.mi, r.mi);
    mi2 = min((l.mi == mi) ? l.mi2 : l.mi, (r.mi == mi) ? r.mi2
        : r.mi);
    cnt_mi = ((l.mi == mi) ? l.cnt_mi : 0) + ((r.mi == mi) ? r.
        cnt_mi : 0);
```

```
    ma = max(l.ma, r.ma);
    ma2 = max((l.ma == ma) ? l.ma2 : l.ma, (r.ma == ma) ? r.ma2
        : r.ma);
    cnt_ma = ((l.ma == ma) ? l.cnt_ma : 0) + ((r.ma == ma) ? r.
        cnt_ma : 0);
    sum = l.sum + r.sum;
    sz = l.sz + r.sz;
  }
  bool add(T v) { // a_i = a_i + v
    if (v) {
      mi += v;
      if (mi2 < INF) mi2 += v;
      ma += v;
      if (ma2 > -INF) ma2 += v;
      sum += sz * v;
      lz.add(v);
    }
    return true;
  }
  bool upd_max(T v) { // a_i = max(a_i, v)
    if (v > -INF) {
      if (v >= mi2) return false;
      else if (v > mi) {
        if (ma == mi) ma = v;
        if (ma2 == mi) ma2 = v;
        sum += cnt_mi * (v - mi);
        mi = v;
        lz.upd_max(v);
      }
    }
    return true;
  }
  bool upd_min(T v) { // a_i = min(a_i, v)
    if (v < INF) {
      if (v <= ma2) return false;
      else if (v < ma) {
        if (ma == mi) mi = v;
        if (mi2 == ma) mi2 = v;
        sum -= cnt_ma * (ma - v);
        ma = v;
        lz.upd_min(v);
      }
    }
    return true;
  }
  T get_sum() const { return sum; } // sum a[l, r)
};
```

segtree-2d.h

Description: 2D Segment Tree.
Time: $\mathcal{O}\left(N \log^2 N\right)$ of memory, $\mathcal{O}\left(\log^2 N\right)$ per query

"sparse.seg.tree.h" 09098e, 25 lines

```
template<class T> struct Node {
  node_t<T> seg; Node* c[2];
  Node() { c[0] = c[1] = nullptr; }
  void upd(int x, int y, T v, int L = 0, int R = SZ-1) {//add v
    if (L == x && R == x) { seg.upd(y,v); return; }
    int M = (L+R)>>1;
    if (x <= M) {
      if (!c[0]) c[0] = new Node();
      c[0]->upd(x,y,v,L,M);
    } else {
      if (!c[1]) c[1] = new Node();
      c[1]->upd(x,y,v,M+1,R);
    }
    seg.upd(y,v); // only for addition
          // seg.upd(y,c[0]?&c[0]->seg:nullptr,c[1]?&c[1]->
              seg:nullptr);
  }
  T query(int x1, int x2, int y1, int y2, int L = 0, int R = SZ
      -1) { // query sum of rectangle
    if (x1 <= L && R <= x2) return seg.query(y1,y2);
    if (x2 < L || R < x1) return 0;
    int M = (L+R)>>1; T res = 0;
    if (c[0]) res += c[0]->query(x1, x2, y1, y2, L, M);
    if (c[1]) res += c[1]->query(x1, x2, y1, y2, M+1, R);
    return res;
  }
};
```

rmq.h

Description: Range Minimum/Maximum Queries on an array. Returns min(V[a], V[a + 1], ... V[b]) in constant time. Returns a pair that holds the answer, first element is the value and the second is the index.
Usage:
```
rmq_t<pair<int, int>> rmq(values);
// values is a vector of pairs {val(i), index(i)}
rmq.query(inclusive, exclusive);
rmq_t<pair<int, int>, greater<pair<int, int>>> rmq(values)
//max query
```
Time: $\mathcal{O}\left(|V|\log|V| + Q\right)$

8c53c5, 19 lines

```
template<typename T, typename Cmp=less<T>>
struct rmq_t : private Cmp {
  int N = 0;
  vector<vector<T>> table;
  const T& min(const T& a, const T& b) const { return Cmp::
      operator()(a, b) ? a : b; }
  rmq_t() {}
  rmq_t(const vector<T>& values) : N(int(values.size())), table
      (__lg(N) + 1) {
    table[0] = values;
    for (int a = 1; a < int(table.size()); ++a) {
      table[a].resize(N - (1 << a) + 1);
      for (int b = 0; b + (1 << a) <= N; ++b)
        table[a][b] = min(table[a-1][b], table[a-1][b + (1 << (
            a-1))]);
    }
  }
  T query(int a, int b) const {
    int lg = __lg(b - a);
    return min(table[lg][a], table[lg][b - (1 << lg) ]);
  }
};
```

fenwick-tree.h

Description: Computes partial sums a[0] + a[1] + ... + a[pos - 1], and updates single elements a[i], taking the difference between the old and new value.
Time: Both operations are $\mathcal{O}\left(\log N\right)$.

2ee6d4, 26 lines

```
template<typename T> struct FT { // 8b7639
  vector<T> s;
  FT(int n) : s(n) {}
  FT(const vector<T>& A) : s(A) {
    const int N = int(s.size());
    for (int a = 0; a < N; ++a)
      if ((a | (a + 1)) < N) s[a | (a + 1)] += s[a];
  }
  void update(int pos, T dif) { // a[pos] += dif
    for (; pos < (int)s.size(); pos |= pos + 1) s[pos] += dif;
  }
  T query(int pos) { // sum of values in [0, pos)
    T res = 0;
    for (; pos > 0; pos &= pos - 1) res += s[pos-1];
    return res;
  }
  // min pos st sum of [0, pos] >= sum. Returns n if no sum
  int lower_bound(T sum) { //is >= sum, or -1 if empty sum is.
    if (sum <= 0) return -1;
    int pos = 0;
    for (int pw = 1 << 25; pw; pw >>= 1)
      if (pos + pw <= (int)s.size() && s[pos + pw-1] < sum)
        pos += pw, sum -= s[pos-1];
    return pos;
  }
};
```

fenwick-tree-2d.h

Description: Computes sums a[i,j] for all i<I, j<J, and increases single elements a[i,j]. Requires that the elements to be updated are known in advance (call fakeUpdate() before init()).
Time: $\mathcal{O}\left(\log^2 N\right)$. (Use persistent segment trees for $\mathcal{O}\left(\log N\right)$.)

"fenwick-tree.h" aebbdc, 25 lines

```
template<typename T> struct FT2 {
  vector<vector<int>> ys; vector<FT<T>> ft;
  FT2(int limx) : ys(limx) {}
  void fakeUpdate(int x, int y) {
    for (; x < (int)ys.size(); x |= x + 1) ys[x].push_back(y);
  }
  void init() {
    for(auto &v : ys){
      sort(v.begin(), v.end());
      v.resize(unique(v.begin(),v.end()) - v.begin());
      ft.emplace_back(v.size());
    }
  }
  int ind(int x, int y) {
    return (int)(lower_bound(ys[x].begin(), ys[x].end(), y) -
        ys[x].begin()); }
  void update(int x, int y, T dif) {
    for (; x < ys.size(); x |= x + 1)
      ft[x].update(ind(x, y), dif);
  }
  T query(int x, int y) {
    T sum = 0;
    for (; x; x &= x - 1) sum += ft[x-1].query(ind(x-1, y));
    return sum;
  }
};
```

mo.h

Description: Answer interval or tree path queries by finding an approximate TSP through the queries, and moving from one query to the next by adding/removing points at the ends. If values are on tree edges, change `step` to add/remove the edge (a, c) and remove the initial `add` call (but keep `in`).
Time: $\mathcal{O}\left(N\sqrt{Q}\right)$

5ef29d, 49 lines

```
void add(int ind, int end) { ... } // add a[ind] (end = 0 or 1)
void del(int ind, int end) { ... } // remove a[ind]
int calc() { ... } // compute current answer

vector<int> mo(vector<pair<int, int>> Q) { // d9247c
  int L = 0, R = 0, blk = 350; // ~N/sqrt(Q)
  vector<int> s(int(Q.size())), res = s;
#define K(x) pair<int, int>(x.first/blk, x.second ^ -(x.first/
    blk & 1))
  iota(s.begin(), s.end(), 0);
  sort(s.begin(), s.end(), [&](int s, int t){ return K(Q[s]) <
      K(Q[t]); });
  for (int qi : s) {
    auto q = Q[qi];
    while (L > q.first) add(--L, 0);
    while (R < q.second) add(R++, 1);
    while (L < q.first) del(L++, 0);
    while (R > q.second) del(--R, 1);
    res[qi] = calc();
  }
```

```cpp
  return res;
}

vector<int> moTree(vector<array<int, 2>> Q, vector<vector<int
    >>& ed, int root=0){ // bbf891
  int N = int(ed.size()), pos[2] = {}, blk = 350; // ~N/sqrt(Q)
  vector<int> s(int(Q.size())), res = s, I(N), L(N), R(N), in(N
    ), par(N);
  add(0, 0), in[0] = 1;
  auto dfs = [&](int x, int p, int dep, auto& f) -> void {
    par[x] = p;
    L[x] = N;
    if (dep) I[x] = N++;
    for (int y : ed[x]) if (y != p) f(y, x, !dep, f);
    if (!dep) I[x] = N++;
    R[x] = N;
  };
  dfs(root, -1, 0, dfs);
#define K(x) pii(I[x[0]] / blk, I[x[1]] ^ -(I[x[0]] / blk & 1))
  iota(s.begin(), s.end(), 0);
  sort(s.begin(), s.end(), [&](int s, int t){ return K(Q[s]) <
      K(Q[t]); });
  for (int qi : s) for (int end = 0; end < 2; ++end) {
    int &a = pos[end], b = Q[qi][end], i = 0;
#define step(c) { if (in[c]) { del(a, end); in[a] = 0; } \
  else { add(c, end); in[c] = 1; } a = c; }
    while (!(L[b] <= L[a] && R[a] <= R[b]))
      I[i++] = b, b = par[b];
    while (a != b) step(par[a]);
    while (i--) step(I[i]);
    if (end) res[qi] = calc();
  }
  return res;
}
```

line-container.h

Description: Container where you can add lines of the form kx+m, and query maximum values at points x. Useful for dynamic programming ("convex hull trick").
Time: $\mathcal{O}(\log N)$

8b2ace, 27 lines

```cpp
struct Line {
  mutable lint k, m, p;
  bool operator<(const Line& o) const { return k < o.k; }
  bool operator<(lint x) const { return p < x; }
};
struct LineContainer : multiset<Line, less<>> {
  static const lint inf = LLONG_MAX; //for doubles 1/.0
  lint div(lint a, lint b) { //for doubles a/b
    return a / b - ((a ^ b) < 0 && a % b); }
  bool isect(iterator x, iterator y) {
    if (y == end()) { x->p = inf; return false; }
    if (x->k == y->k) x->p = x->m > y->m ? inf : -inf;
    else x->p = div(y->m - x->m, x->k - y->k);
    return x->p >= y->p;
  }
  void add(lint k, lint m) {
    auto z = insert({k, m, 0}), y = z++, x = y;
    while (isect(y, z)) z = erase(z);
    if (x != begin() && isect(--x, y)) isect(x, y = erase(y));
    while ((y = x) != begin() && (--x)->p >= y->p)
      isect(x, erase(y));
  }
  lint query(lint x) {
    assert(!empty()); auto l = *lower_bound(x);
    return l.k * x + l.m;
  }
};
```

matrix.h

Description: Basic operations on square matrices.
Usage: `Matrix<int> A(N, vector<int>(N));`

700522, 28 lines

```cpp
template <typename T> struct Matrix : vector<vector<T>> {
  using vector<vector<T>>::vector;
  using vector<vector<T>>::size;
  int h() const { return int(size()); }
  int w() const { return int((*this)[0].size()); }
  Matrix operator*(const Matrix& r) const {
    assert(w() == r.h()); Matrix res(h(), vector<T>(r.w()));
    for(int i = 0; i < h(); ++i) for(int j = 0; j < r.w(); ++j)
      for (int k = 0; k < w(); ++k)
        res[i][j] += (*this)[i][k] * r[k][j];
    return res;
  }
  friend vector<T> operator*(const Matrix<T>& A, const vector<T
      >& b) {
    int N = int(A.size()), M = int(A[0].size());
    vector<T> y(N);
    for (int i = 0; i < N; ++i)
      for (int j = 0; j < M; ++j) y[i] += A[i][j] * b[j];
    return y;
  }
  Matrix& operator*=(const Matrix& r){return *this= *this * r;}
  Matrix pow(int n) const {
    assert(h() == w()); assert(n >= 0);
    Matrix x = *this, r(h(), vector<T>(w()));
    for (int i = 0; i < h(); ++i) r[i][i] = T(1);
    while (n) { if (n & 1) r *= x; x *= x; n >>= 1; }
    return r;
  }
};
```

range-color.h

Description: RangeColor structure, supports point queries and range updates, if C isn't int32_t change $freq$ to map
Time: $\mathcal{O}(\lg(L) * Q)$

3d860e, 35 lines

```cpp
template<class T, class C> struct RangeColor {
  struct Node{
    T lo, hi; C color;
    bool operator<(const Node &n) const { return hi < n.hi; }
  };
  C minInf; set<Node> st; vector<T> freq;
  RangeColor(T first, T last, C maxColor, C iniColor = C(0)):
      minInf(first - T(1)), freq(maxColor + 1) {
    freq[iniColor] = last - first + T(1);
    st.insert({first, last, iniColor});
  }
  C query(T i) { //get color in position i
    return st.upper_bound({T(0), i - T(1), minInf})->color;
  }
  void upd(T a, T b, C x) { //set x in [a, b]
    auto p = st.upper_bound({T(0), a - T(1), minInf});
    assert(p != st.end());
    T lo = p->lo, hi = p->hi; C old = p->color;
    freq[old] -= (hi - lo + T(1)); p = st.erase(p);
    if (lo < a)
      freq[old] += (a-lo), st.insert({lo, a-T(1), old});
    if (b < hi)
      freq[old] += (hi-b), st.insert({b+T(1), hi, old});
    while ((p != st.end()) && (p->lo <= b)) {
      lo = p->lo, hi = p->hi; old = p->color;
      freq[old] -= (hi - lo + T(1));
      if (b < hi){
        freq[old] += (hi - b); st.erase(p);
        st.insert({b + T(1), hi, old});
        break;
      } else p = st.erase(p);
    }
    freq[x] += (b - a + T(1)); st.insert({a, b, x});
  }
  T countColor(C x){ return freq[x]; }
};
```

implicit-treap.h

Description: A short self-balancing tree. It acts as a sequential container with log-time splits/joins, and is easy to augment with additional data.
Time: $\mathcal{O}(\log N)$

2d0d97, 96 lines

```cpp
mt19937 rng(chrono::steady_clock::now().time_since_epoch().
    count());
struct node {
  int v, p, sz;
  node *l, *r;
  bool rev;
  node(int k) : v(k), p(rng()), l(nullptr), rev(0), r(nullptr),
      sz(0) {}
};
int sz(node *t) {
  if (t == nullptr) return 0;
  return t->sz;
}
void push(node *t) {
  if (t == nullptr) return;
  if (t->rev) {
    swap(t->l, t->r);
    if (t->l != nullptr) t->l->rev ^= t->rev;
    if (t->r != nullptr) t->r->rev ^= t->rev;
    t->rev = 0;
  }
}
void updsz(node *t) {
  if (t == nullptr) return;
  push(t); push(t->l); push(t->r);
  t->sz = sz(t->l) + sz(t->r) + 1;
}
void split(node *t, node *&l, node *&r, int k) { //k on left
  push(t);
  if (t == nullptr) l = r = nullptr;
  else if (k <= sz(t->l)) {
    split(t->l, l, t->l, k);
    r = t;
  }
  else {
    split(t->r, t->r, r, k-1-sz(t->l));
    l = t;
  }
  updsz(t);
}
void merge(node *&t, node *l, node *r) {
  push(l); push(r);
  if (l == nullptr) t = r;
  else if (r == nullptr) t = l;
  else if (l->p <= r->p) {
    merge(l->r, l->r, r);
    t = l;
  }
  else {
    merge(r->l, l, r->l);
    t = r;
  }
  updsz(t);
}
void add(node *&t, node *c, int k) {
  push(t);
  if (t == nullptr) t = c;
  else if (c->p >= t->p) {
```

```
    split(t, c->l, c->r, k);
    t = c;
  }
  else if (sz(t->l) >= k) add(t->l, c, k);
  else add(t->r, c, k-1-sz(t->l));
  updsz(t);
}
void del(node *&t, int k) {
  push(t);
  if (t == nullptr) return;
  if (sz(t->l) == k) merge(t, t->l, t->r);
  else if (sz(t->l) > k) del(t->l, k);
  else del(t->r, k);
  updsz(t);
}
void print(node *t) {
  if (r == nullptr) return;
  print(t->l);
  cout << t->v << ' ';
  print(t->r);
}

int main() {
  node *treap = nullptr;
  while(1) {
    int a;
    cin >> a;
    if (a == 1) {
      int c, d;
      cin >> c >> d;
      node *r = new node(d);
      add(treap, r, c);
    } else if (a == 2) {
      int d;
      cin >> d;
      del(treap, d);
    }
    print(treap);
  }
}
```

Numerical (4)

polynomial.h

84593c, 17 lines

```
struct Poly {
  vector<double> a;
  double operator()(double x) const {
    double val = 0;
    for(int i = a.size(); i--;) (val *= x) += a[i];
    return val;
  }
  void diff() {
    for(int i = 1; i < a.size(); ++i) a[i-1] = i*a[i];
    a.pop_back();
  }
  void divroot(double x0) {
    double b = a.back(), c; a.back() = 0;
    for(int i = a.size()-1; i--;) c = a[i],a[i]=a[i+1]*x0+b, b=
      c;
    a.pop_back();
  }
};
```

poly-roots.h

Description: Finds the real roots to a polynomial.
Usage: `poly_roots({{2,-3,1}},-1e9,1e9) // solve x^2-3x+2 = 0`
Time: $\mathcal{O}\left(n^2 \log(1/\epsilon)\right)$

"Polynomial.h" 49396a, 20 lines

```
vector<double> poly_roots(Poly p, double xmin, double xmax) {
  if ((p.a).size() == 2) { return {-p.a[0]/p.a[1]}; }
  vector<double> ret;
  Poly der = p; der.diff();
  auto dr = poly_roots(der, xmin, xmax);
  dr.push_back(xmin-1); dr.push_back(xmax+1);
  sort(dr.begin(), dr.end());
  for(int i = 0; i < dr.size()-1; ++i) {
    double l = dr[i], h = dr[i+1]; bool sign = p(l) > 0;
    if (sign^(p(h) > 0)) {
      for(int it = 0; it < 60; ++it) { // while (h - l > 1e-8)
        double m = (l + h) / 2, f = p(m);
        if ((f <= 0) ^ sign) l = m;
        else h = m;
      }
      ret.push_back((l + h) / 2);
    }
  }
  return ret;
}
```

poly-interpolate.h

Description: Given n points (x[i], y[i]), computes an n-1-degree polynomial p that passes through them: $p(x) = a[0] * x^0 + \ldots + a[n-1] * x^{n-1}$. For numerical precision, pick $x[k] = c * \cos(k/(n-1) * \pi), k = 0 \ldots n-1$.
Time: $\mathcal{O}\left(n^2\right)$

97a266, 12 lines

```
typedef vector<double> vd;
vd interpolate(vd x, vd y, int n) {
  vd res(n), temp(n);
  for(int k = 0; k < n-1; ++k) for(int i = k+1; i < n; ++i)
    y[i] = (y[i] - y[k]) / (x[i] - x[k]);
  double last = 0; temp[0] = 1;
  for(int k = 0; k < n; ++k) for(int i = 0; i < n; ++i) {
    res[i] += y[k] * temp[i]; swap(last, temp[i]);
    temp[i] -= last * x[k];
  }
  return res;
}
```

lagrange.h

Description: Lagrange interpolation over a finite field and some combo stuff
Time: $\mathcal{O}(N)$

"../number-theory/modular-arithmetic.h", "../number-theory/preparator.h" ad3879, 13 lines

```
template<typename T> struct interpolator_t {
  vector<T> S;
  interpolator_t(int N): S(N) {}
  T interpolate(const vector<T>& y, T x) {
    int N = int(y.size()); int sgn = (N & 1 ? 1 : -1);
    T res = 0, P = 1; S[N - 1] = 1;
    for (int i = N-1; i > 0; --i) S[i-1] = S[i] * (x-i);
    for (int i = 0; i < N; ++i, sgn *= -1, P *= (x - i + 1)) {
      res += y[i] * sgn * P * S[i] * ifact[i] * ifact[N-1-i];
    }
    return res;
  }
};
```

berlekamp-massey.h

Description: Recovers any n-order linear recurrence relation from the first $2n$ terms of the recurrence. Useful for guessing linear recurrences after brute-forcing the first terms. Should work on any field, but numerical stability for floats is not guaranteed. Output will have size $\leq n$.
Usage: `BerlekampMassey({0, 1, 1, 3, 5, 11}) // {1, 2}`
Time: $\mathcal{O}\left(N^2\right)$

"modular-arithmetic.h" 66d78a, 17 lines

```
template <typename num>
vector<num> BerlekampMassey(const vector<num>& s) {
  int n = int(s.size()), L = 0, m = 0; num b = 1;
  vector<num> C(n), B(n), T; C[0] = B[0] = 1;
  for(int i = 0; i < n; i++) { ++m;
    num d = s[i];
    for (int j = 1; j <= L; j++) d += C[j] * s[i - j];
    if (d == 0) continue;
    T = C; num coef = d / b;
    for (int j = m; j < n; j++) C[j] -= coef * B[j - m];
    if (2 * L > i) continue;
    L = i + 1 - L; B = T; b = d; m = 0;
  }
  C.resize(L + 1); C.erase(C.begin());
  for (auto& x : C) x = -x;
  return C;
}
```

linear-recurrence.h

Description: Bostan-Mori algorithm. Generates the k'th term of an n-order linear recurrence $S[i] = \sum_j S[i-j-1]tr[j]$, given $S[0 \ldots n-1]$ and $tr[0 \ldots n-1]$. Faster than matrix multiplication. Useful together with Berlekamp–Massey.
Usage: `linear_rec({0, 1}, {1, 1}, k) // k'th Fibonacci number`
Time: $\mathcal{O}(n \log n \log k)$

"../modular-arithmetic.h" aa7314, 16 lines

```
template<typename T>
T linear_rec(const vector<T>& S, const vector<T>& tr, ll K) {
  const int N = int(tr.size());
  vector<T> qs(N + 1); qs[0] = 1;
  for (int i = 0; i < N; ++i) qs[i + 1] = -tr[i];
  auto fs = fft.convolve(S, qs); fs.resize(N);
  for (; K; K /= 2) {
    auto qneg = qs;
    for (int i = 1; i <= N; i += 2) qneg[i] = -qneg[i];
    fs = fft.convolve(fs, qneg), qs = fft.convolve(qs, qneg);
    for (int i = 0; i < N; ++i)
      fs[i] = fs[2 * i + (K & 1)], qs[i] = qs[2 * i];
    qs[N] = qs[2*N]; fs.resize(N), qs.resize(N+1);
  }
  return fs[0];
}
```

integrate.h

Description: Simple integration of a function over an interval using Simpson's rule. The error should be proportional to h^4, although in practice you will want to verify that the result is stable to desired precision when epsilon changes.

7bb98e, 7 lines

```
template<class F>
double quad(double a, double b, F& f, const int n = 1000) {
  double h = (b - a) / 2 / n, v = f(a) + f(b);
  for(int i = 1; i < n*2; ++i)
    v += f(a + i*h) * (i&1 ? 4 : 2);
  return v * h / 3;
}
```

integrate-adaptive.h

Description: Fast integration using an adaptive Simpson's rule.
Usage:
```
double sphereVolume = quad(-1, 1, [](double x) {
return quad(-1, 1, [&](double y) {
return quad(-1, 1, [&](double z) {
return x*x + y*y + z*z < 1; });});});
```

cfcad2, 13 lines

```
typedef double d;
#define S(a,b) (f(a) + 4*f((a+b) / 2) + f(b)) * (b-a) / 6
```

```
template <class F>
d rec(F& f, d a, d b, d eps, d S) {
  d c = (a + b) / 2, S1 = S(a, c), S2 = S(c, b), T = S1 + S2;
  if (abs(T - S) <= 15 * eps || b - a < 1e-10)
    return T + (T - S) / 15;
  return rec(f, a, c, eps / 2, S1) + rec(f, c, b, eps / 2, S2);
}
template<class F>
d quad(d a, d b, F f, d eps = 1e-8) {
  return rec(f, a, b, eps, S(a, b));
}
```

gaussian-elimination.h

Time: $\mathcal{O}(\min(N, M)NM)$

"../data-structures/matrix.h" a5570d, 61 lines

```
template<typename T> struct gaussian_elimination {
  int N, M; Matrix<T> A, E;
  vector<int> pivot; int rank, nullity, sgn;
  gaussian_elimination(const Matrix<T>& A_) : A(A_) {
    N = A.size(), M = A[0].size(), E=Matrix<T>(N, vector<T>(N))
        ;
    for (int i = 0; i < N; ++i) E[i][i] = 1;
    rank = 0, nullity = M, sgn = 0; pivot.assign(M, -1);
    for (int col = 0, row = 0; col < M && row < N; ++col) {
      int sel = -1;
      for (int i = row; i < N; ++i) if (A[i][col] != 0) {
        sel = i; break;
      }
      if (sel == -1) continue;
      if (sel != row) {
        sgn += 1;
        swap(A[sel], A[row]); swap(E[sel], E[row]);
      }
      for (int i = 0; i < N; ++i) {
        if (i == row) continue;
        T c = A[i][col] / A[row][col];
        for (int j = col; j < M; ++j)
          A[i][j] -= c*A[row][j];
        for (int j = 0; j < N; ++j)
          E[i][j] -= c*E[row][j];
      }
      pivot[col] = row++; ++rank, --nullity;
    }
  }
  pair<bool, vector<T>> solve(vector<T> b, bool reduced = false
      ) const {
    if (reduced == false) b = E * b;
    vector<T> x(M);
    for (int j = 0; j < M; ++j) {
      if (pivot[j] == -1) continue;
      x[j] = b[pivot[j]] / A[pivot[j]][j];
      b[pivot[j]] = 0;
    }
    for (int i = 0; i < N; ++i)
      if (b[i] != 0) return {false, x};
    return {true, x};
  }
  vector<vector<T>> kernel_basis() const {
    vector<vector<T>> basis; vector<T> e(M);
    for (int j = 0; j < M; ++j) {
      if (pivot[j] != -1) continue;
      e[j] = 1; auto y = solve(A * e, true).second;
      e[j] = 0, y[j] = -1; basis.push_back(y);
    }
    return basis;
  }
  Matrix<T> inverse() const {
    assert(N == M); assert(rank == N);
    Matrix<T> res(N, vector<T>(N));
    vector<T> e(N);
    for (int i = 0; i < N; ++i) {
      e[i] = 1; auto x = solve(e).second;
      for (int j = 0; j < N; ++j) res[j][i] = x[j];
      e[i] = 0;
    }
    return res;
  }
};
```

linear-solver-z2.h

Description: Solves $Ax = b$ over $\mathbb{F}_2$. If there are multiple solutions, one is returned arbitrarily. Returns true, or false if no solutions. Last column of a is b. c is the rank.

Time: $\mathcal{O}(n^2 m)$

7a24e1, 24 lines

```
typedef bitset<2010> bs;
bool gauss(vector<bs> a, bs& ans, int n) {
  int m = int(a.size()), c = 0;
  bs pos; pos.set();
  for (int j = n-1, i; j >= 0; --j) {
    for (i = c; i < m; ++i)
      if (a[i][j]) break;
    if (i == m) continue;
    swap(a[c], a[i]);
    i = c++; pos[j] = 0;
    for (int k = 0; k < m; ++k)
      if (a[k][j] && k != i) a[k] ^= a[i];
  } ans = pos;
  for(int i = 0; i < m; ++i) {
    int ac = 0;
    for (int j = 0; j < n ; ++j) {
      if (!a[i][j]) continue;
      if (!pos[j]) pos[j] = 1, ans[j] = ac^a[i][n];
      ac ^= ans[j];
    }
    if (ac != a[i][n]) return false;
  }
  return true;
}
```

simplex.h

Description: Solves a general linear maximization problem: maximize $c^T x$ subject to $Ax \leq b$, $x \geq 0$.

Time: $\mathcal{O}(NM * \#pivots)$, where a pivot may be e.g. an edge relaxation. $\mathcal{O}(2^n)$ in the general case. WARNING- segfaults on empty (size 0) max cx st Ax<=b, x>=0 do 2 phases; 1st check feasibility; 2nd check boundedness and ans

83fec6, 39 lines

```
using dbl = double; using vd = vector<dbl>;
vd simplex(vector<vd> A, vd b, vd c) { const dbl E = 1e-9;
  int n = A.size(), m = A[0].size() + 1, r = n, s = m-1;
  auto D = vector<vd>(n+2, vd(m+1));
  vector<int> ix = vector<int>(n + m);
  for (int i = 0; i < n + m; ++i) ix[i] = i;
  for (int i = 0; i < n; ++i) {
    for (int j = 0; j < m-1; ++j) D[i][j] = -A[i][j];
    D[i][m - 1] = 1; D[i][m] = b[i];
    if (D[r][m] > D[i][m]) r = i;
  }
  for (int j = 0; j < m-1; ++j) D[n][j] = c[j];
  D[n + 1][m - 1] = -1; int z = 0;
  for (dbl d;;) {
    if (r < n) { swap(ix[s], ix[r + m]);
      D[r][s] = 1.0/D[r][s];
      for (int j = 0; j <= m; ++j) if (j != s)
        D[r][j] *= -D[r][s];
      for (int i = 0; i <= n+1; ++i) if (i != r) {
        for (int j = 0; j <= m; ++j)
          if (j != s) D[i][j] += D[r][j] * D[i][s];
        D[i][s] *= D[r][s];
      }
    }
    r = -1; s = -1;
    for (int j = 0; j < m; ++j) if (s < 0 || ix[s] > ix[j])
      if (D[n+1][j]>E || D[n+1][j]>-E && D[n][j]>E) s = j;
    if (s < 0) break;
    for (int i = 0; i < n; ++i) if (D[i][s] < -E) {
      if (r<0 || (d = D[r][m]/D[r][s]-D[i][m]/D[i][s]) < -E
        || d < E && ix[r+m] > ix[i+m]) r = i;
    }
    if (r < 0) return vd(); // unbounded
  }
  if (D[n+1][m] < -E) return vd(); // infeasible
  vd x(m-1);
  for(int i=m; i < n+m; ++i) if(ix[i]<m-1) x[ix[i]]=D[i-m][m];
  dbl result = D[n][m]; return x; // ans: D[n][m]
}
```

tridiagonal.h

Description: $x = \text{tridiagonal}(d, p, q, b)$ solves the equation system

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} d_0 & p_0 & 0 & 0 & \cdots & 0 \\ q_0 & d_1 & p_1 & 0 & \cdots & 0 \\ 0 & q_1 & d_2 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{n-3} & d_{n-2} & p_{n-2} \\ 0 & 0 & \cdots & 0 & q_{n-2} & d_{n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{pmatrix}.$$

This is useful for solving problems on the type

$$a_i = b_i a_{i-1} + c_i a_{i+1} + d_i,\ 1 \leq i \leq n,$$

where a_0, a_{n+1}, b_i, c_i and d_i are known. a can then be obtained from

$$\begin{aligned}\{a_i\} = \text{tridiagonal}(&\{1, -1, -1, ..., -1, 1\}, \{0, c_1, c_2, \dots, c_n\}, \\ &\{b_1, b_2, \dots, b_n, 0\}, \{a_0, d_1, d_2, \dots, d_n, a_{n+1}\}).\end{aligned}$$

Fails if the solution is not unique.

If $|d_i| > |p_i| + |q_{i-1}|$ for all i, or $|d_i| > |p_{i-1}| + |q_i|$, or the matrix is positive definite, the algorithm is numerically stable and neither `tr` nor the check for `diag[i] == 0` is needed.

Time: $\mathcal{O}(N)$

be9642, 23 lines

```
typedef double T;
vector<T> tridiagonal(vector<T> diag, const vector<T> &super,
    const vector<T> &sub, vector<T> b) {
  int n = b.size(); vector<int> tr(n);
  for(int i = 0; i < n-1; ++i)
    if (abs(diag[i]) < 1e-9 * abs(super[i])) { // diag[i] == 0
      b[i+1] -= b[i] * diag[i+1] / super[i];
      if (i+2 < n) b[i+2] -= b[i] * sub[i+1] / super[i];
      diag[i+1] = sub[i]; tr[++i] = 1;
    } else {
      diag[i+1] -= super[i]*sub[i]/diag[i];
      b[i+1] -= b[i]*sub[i]/diag[i];
    }
  for (int i = n; i--;)
    if (tr[i]) {
      swap(b[i], b[i-1]); diag[i-1] = diag[i];
      b[i] /= super[i-1];
    } else {
      b[i] /= diag[i];
      if (i) b[i-1] -= b[i]*super[i-1];
    }
  return b;
}
```

4.1 Fourier transforms

fast-fourier-transform.h

Description: For convolution of complex numbers or more than two vectors: FFT, multiply pointwise, FFT inverse back.
Time: $\mathcal{O}(N \log N)$ with $N = |A| + |B|$ ($\sim 1s$ for $N = 2^{22}$)

366399, 123 lines

```
inline int nxt_pow2(int s) { return 1 << (s > 1 ? 32 - __builtin_clz(s-1) : 0); }
template <typename T> struct root_of_unity {};

template <typename dbl> struct cplx {
  dbl x, y; using P = cplx;
  cplx(dbl x_ = 0, dbl y_ = 0) : x(x_), y(y_) { }
  friend P operator+(P a, P b) { return P(a.x+b.x, a.y+b.y); }
  friend P operator-(P a, P b) { return P(a.x-b.x, a.y-b.y); }
  friend P operator*(P a, P b) { return P(a.x*b.x - a.y*b.y, a.x*b.y + a.y*b.x); }
  friend P conj(P a) { return P(a.x, -a.y); }
  friend P inv(P a) { dbl n = (a.x*a.x+a.y*a.y); return P(a.x/n,-a.y/n); }
};
template <typename dbl> struct root_of_unity<cplx<dbl>> {
  static cplx<dbl> f(int k) {
    static const dbl PI = acos(-1); dbl a = 2*PI/k;
    return cplx<dbl>(cos(a),sin(a));
  }
};

//(MOD,3) := (M1:897581057),(M3:985661441),(M5:935329793)
using M0 = modnum<998244353U>;
constexpr unsigned primitive_root(unsigned M) {
  if (M == 880803841U) return 26U; // (M2)
  else if (M == 943718401U) return 7U; // (M4)
  else if (M == 918552577U) return 5U; // (M6)
  else return 3U;
}
template<unsigned MOD> struct root_of_unity<modnum<MOD>> {
  static constexpr modnum<MOD> g0 = primitive_root(MOD);
  static modnum<MOD> f(int K) {
    assert((MOD-1)%K == 0); return g0.pow((MOD-1)/K);
  }
};
template<typename T> struct FFT {
  vector<T> rt; vector<int> rev;
  FFT() : rt(2, T(1)) {}
  void init(int N) {
    N = nxt_pow2(N);
    if (N > int(rt.size())) {
      rev.resize(N); rt.reserve(N);
      for (int a = 0; a < N; ++a)
        rev[a] = (rev[a/2] | ((a&1)*N)) >> 1;
      for (int k = int(rt.size()); k < N; k *= 2) {
        rt.resize(2*k);
        T z = root_of_unity<T>::f(2*k);
        for (int a = k/2; a < k; ++a)
          rt[2*a] = rt[a], rt[2*a+1] = rt[a] * z;
      }
    }
  }
  void fft(vector<T>& xs, bool inverse) const {
    int N = int(xs.size());
    int s = __builtin_ctz(int(rev.size())/N);
    if (inverse) reverse(xs.begin() + 1, xs.end());
    for (int a = 0; a < N; ++a) {
      if (a < (rev[a] >> s)) swap(xs[a], xs[rev[a] >> s]);
    }
    for (int k = 1; k < N; k *= 2) {
      for (int a = 0; a < N; a += 2*k) {
        int u = a, v = u + k;
        for (int b = 0; b < k; ++b, ++u, ++v) {
          T z = rt[b + k] * xs[v];
          xs[v] = xs[u] - z, xs[u] = xs[u] + z;
        }
      }
    }
    if (inverse)
      for (int a = 0; a < N; ++a) xs[a] = xs[a] * inv(T(N));
  }
  vector<T> convolve(vector<T> as, vector<T> bs) {
    int N = int(as.size()), M = int(bs.size());
    int K = N + M - 1, S = nxt_pow2(K); init(S);
    if (min(N, M) <= 64) {
      vector<T> res(K);
      for (int u = 0; u < N; ++u) for (int v = 0; v < M; ++v)
        res[u + v] = res[u + v] + as[u] * bs[v];
      return res;
    } else {
      as.resize(S), bs.resize(S);
      fft(as, false); fft(bs, false);
      for (int i = 0; i < S; ++i) as[i] = as[i] * bs[i];
      fft(as, true); as.resize(K); return as;
    }
  }
}; FFT<M0> FFT0;

// T = {unsigned, unsigned long long, modnum<M>}
template<class T, unsigned M0, unsigned M1, unsigned M2,
    unsigned M3, unsigned M4>
T garner(modnum<M0> a0, modnum<M1> a1, modnum<M2> a2, modnum<M3> a3, modnum<M4> a4) {
  static const modnum<M1> INV_M0_M1 = modnum<M1>(M0).inv();
  static const modnum<M2> INV_M0M1_M2 = (modnum<M2>(M0) * M1).inv();
  // static const modnum<M3> INV_M0M1M2_M3 = (modnum<M3>(M0) * M1 * M2).inv();
  // static const modnum<M4> INV_M0M1M2M3_M4 = (modnum<M4>(M0) * M1 * M2 * M3).inv();
  const modnum<M1> b1 = INV_M0_M1 * (a1 - a0.x);
  const modnum<M2> b2 = INV_M0M1_M2 * (a2 - (modnum<M2>(b1.x) * M0 + a0.x));
  // const modnum<M3> b3 = INV_M0M1M2_M3 * (a3 - ((modnum<M3>(b2.x) * M1 + b1.x) * M0 + a0.x));
  // const modnum<M4> b4 = INV_M0M1M2M3_M4 * (a4 - (((modnum<M4>(b3.x) * M2 + b2.x) * M1 + b1.x) * M0 + a0.x));
  return (T(b2.x) * M1 + b1.x) * M0 + a0.x;
  // return (((T(b4.x) * M3 + b3.x) * M2 + b2.x) * M1 + b1.x) * M0 + a0.x;
}
// results must be in [-448002610255888384, 448002611254132736]
vector<long long> convolve(const vector<long long>& as, const
    vector<long long>& bs) {
  static constexpr unsigned M0 = M0::M, M1 = M1::M;
  static const modnum<M1> INV_M0_M1 = modnum<M1>(M0).inv();
  if (as.empty() || bs.empty()) return {};
  const int len_as = int(as.size()), len_bs = int(bs.size());
  vector<modnum<M0>> as0(len_as), bs0(len_bs);
  for (int i = 0; i < len_as; ++i) as0[i] = as[i];
  for (int i = 0; i < len_bs; ++i) bs0[i] = bs[i];
  const vector<modnum<M0>> cs0 = FFT0.convolve(as0, bs0);
  vector<modnum<M1>> as1(len_as), bs1(len_bs);
  for (int i = 0; i < len_as; ++i) as1[i] = as[i];
  for (int i = 0; i < len_bs; ++i) bs1[i] = bs[i];
  const vector<modnum<M1>> cs1 = FFT1.convolve(as1, bs1);
  vector<long long> cs(len_as + len_bs - 1);
  for (int i = 0; i < len_as + len_bs - 1; ++i) {
    const modnum<M1> d1 = INV_M0_M1 * (cs1[i] - cs0[i].x);
    cs[i] = (d1.x > M1 - d1.x)
      ? (-1ULL - (static_cast<unsigned long long>(M1 - 1U - d1.x) * M0 + (M0 - 1U - cs0[i].x)))
      : (static_cast<unsigned long long>(d1.x) * M0 + cs0[i].x);
  }
  return cs;
}
```

fast-subset-transform.h

Description: Transform to a basis with fast convolutions of the form $c[z] = \sum_{z=x\oplus y} a[x] \cdot b[y]$, where $\oplus$ is one of AND, OR, XOR. The size of a must be a power of two.
Time: $\mathcal{O}(N \log N)$

5b9574, 16 lines

```
void FST(vector<int> &a, bool inv) {
  for (int n = a.size(), step = 1; step < n; step *= 2) {
    for (int i = 0; i < n; i += 2 * step) for(int j = i; j < i+step; ++j) {
      int &u = a[j], &v = a[j + step]; tie(u, v) =
        inv ? pii(v - u, u) : pii(v, u + v); // AND
        inv ? pii(v, u - v) : pii(u + v, u); // OR
        pii(u + v, u - v);                   // XOR
    }
  }
  if (inv) for(auto &x : a) x /= a.size(); // XOR only
}
vector<int> conv(vector<int> a, vector<int> b) {
  FST(a, 0); FST(b, 0);
  for(int i = 0; i < a.size(); ++i) a[i] *= b[i];
  FST(a, 1); return a;
}
```

sum-of-powers.h

Description: Computes monomials and sum of powers product certain polynomials. Check "General purpose numbers" section for more info. (Monomials) $pw(x) = x^d$ for a fixed d. $\sum_{x=0}^{\infty} r^x f(x)$. (degree of $f \le d$). $\sum_{x=0}^{N-1} r^x f(x)$. (degree of $f \le d$).

"../number-theory/modular-arithmetic.h", "/lagrange.h" 85dfa0, 33 lines

```
vector<num> get_monomials(int N, long long d) {
  vector<int> pfac(N);
  for (int i = 2; i < N; ++i) pfac[i] = i;
  for (int p = 2; p < N; ++p) if (pfac[p] == p)
    for(int m = 2*p; m < N; m += p) if(pfac[m] > p) pfac[m]=p;
  vector<num> pw(N);
  for (int i = 0; i < N; ++i)
    if (i <= 1 || pfac[i] == i) pw[i] = num(i).pow(d);
    else pw[i] = (pw[pfac[i]] * pw[i / pfac[i]]);
  return pw;
}
num sum_of_power_limit(num r, int d, const vector<num>& fs) {
  interpolator_t<num> M(d + 2); num s = 1; auto gs = fs;
  for (int x = 0; x <= d; ++x, s *= r) gs[x] *= s;
  num ans = 0, cur_sum = 0; s = 1;
  for (int x = 0; x <= d; ++x, s *= -r) {
    cur_sum += choose(d+1, x) * s; ans += cur_sum * gs[d-x];
  } ans *= (1 - r).pow(-(d + 1));
  return ans;
}
num sum_of_power(num r, int d, vector<num>& fs, ll N) {
  if (r == 0) return (0 < N) ? fs[0] : 0;
  interpolator_t<num> M(d + 2);
  vector<num> gs(d + 2); gs[0] = 0; num s = 1;
  for (int x = 0; x <= d; ++x, s *= r)
    gs[x + 1] = gs[x] + s * fs[x];
  if (r == 1) return M.interpolate(gs, N);
  const num c = sum_of_power_limit(r, d, fs);
```

```
  const num r_inv = r.inv(); num w = 1;
  for (int x = 0; x <= d + 1; ++x, w *= r_inv)
    gs[x] = w * (gs[x] - c);
  return c + r.pow(N) * M.interpolate(gs, N);
}
```

4.1.1 Duality

max $c^T x$ sjt to $Ax \leq b$. Dual problem is min $b^T x$ sjt to $A^T x \geq c$. By strong duality, min max value coincides.

4.1.2 Generating functions

A list of generating functions for useful sequences:

$(1, 1, 1, 1, 1, 1, \dots)$	$\frac{1}{1-z}$
$(1, -1, 1, -1, 1, -1, \dots)$	$\frac{1}{1+z}$
$(1, 0, 1, 0, 1, 0, \dots)$	$\frac{1}{1-z^2}$
$(1, 0, \dots, 0, 1, 0, 1, 0, \dots, 0, 1, 0, \dots)$	$\frac{1}{1-z^2}$
$(1, 2, 3, 4, 5, 6, \dots)$	$\frac{1}{(1-z)^2}$
$(1, \binom{m+1}{m}, \binom{m+2}{m}, \binom{m+3}{m}, \dots)$	$\frac{1}{(1-z)^{m+1}}$
$(1, c, \binom{c+1}{2}, \binom{c+2}{3}, \dots)$	$\frac{1}{(1-z)^c}$
$(1, c, c^2, c^3, \dots)$	$\frac{1}{1-cz}$
$(0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$	$\ln \frac{1}{1-z}$

A neat manipulation trick is:

$$\frac{1}{1-z} G(z) = \sum_n \sum_{k \leq n} g_k z^n$$

Number theory (5)

5.1 Modular arithmetic

modular-arithmetic.h

Description: Operators for modular arithmetic.

"mod-inv.h" 3c7e89, 31 lines

```
template<unsigned M_> struct modnum {
  static constexpr unsigned M = M_; using num = modnum;
  using ll = int64_t; using ull = uint64_t; unsigned x;
  num& norm(unsigned a){x = a<M ? a : a-M;return *this;}
  constexpr modnum(ll a = 0U) : x(unsigned((a %= ll(M)) < 0 ? a
      + ll(M) : a)) {}
  explicit operator int() const { return x; }
  num& operator+=(const num& a){ return norm(x+a.x); }
  num& operator-=(const num& a){ return norm(x-a.x+M); }
  num& operator*=(const num& a){ x = unsigned(ull(x)*a.x%M);
      return *this; }
  num& operator/=(const num& a){ return (*this *= a.inv());}
  num operator+(const num& a) const {return (num(*this) += a);}
  num operator-(const num& a) const {return (num(*this) -= a);}
  num operator*(const num& a) const {return (num(*this) *= a);}
  num operator/(const num& a) const {return (num(*this) /= a);}
  template<typename T> friend num operator+(T a, const num& b){
      return (num(a) += b); }
  template<typename T> friend num operator-(T a, const num& b){
      return (num(a) -= b); }
  template<typename T> friend num operator*(T a, const num& b){
      return (num(a) *= b); }
  template<typename T> friend num operator/(T a, const num& b){
      return (num(a) /= b); }
  num operator+() const { return *this; }
  num operator-() const { return num() - *this; }
  num pow(ll e) const {
    if (e < 0) { return inv().pow(-e); } num b = x, xe = 1U;
    for (; e; e >>= 1) { if (e & 1) xe *= b; b *= b; }
    return xe;
  }
  num inv() const { return minv(x, M); }
  friend num inv(const num& a) { return a.inv(); }
  explicit operator bool() const { return x; }
  friend bool operator==(const num& a, const num& b){return a.x
      == b.x;}
  friend bool operator!=(const num& a, const num& b){return a.x
      != b.x;}
};
```

mod-inv.h

Description: Find x such that $ax \equiv 1 (\text{mod } m)$. The inverse only exist if a and m are coprimes.

48d5fb, 4 lines

```
int minv(int a, int m) {
  a %= m; assert(a);
  return a == 1 ? 1 : int(m - int64_t(minv(m, a)) * m / a);
}
```

mod-sum.h

Description: Sums of mod'ed arithmetic progressions.
modsum(to, c, k, m) $= \sum_{i=0}^{to-1} (ki + c) \% m$. divsum is similar but for floored division.
Time: $\log(m)$, with a large constant.

decfb8, 16 lines

```
typedef unsigned long long ull;
ull sumsq(ull to) { return to / 2 * ((to-1) | 1); }
ull divsum(ull to, ull c, ull k, ull m) {
  ull res = k / m * sumsq(to) + c / m * to;
  k %= m; c %= m;
  if (k) {
    ull to2 = (to * k + c) / m;
    res += to * to2;
    res -= divsum(to2, m-1 - c, m, k) + to2;
  }
  return res;
}
lint modsum(ull to, lint c, lint k, lint m) {
  c = ((c % m) + m) % m; k = ((k % m) + m) % m;
  return to * c + k * sumsq(to) - m * divsum(to, c, k, m);
}
```

mod-mul.h

Description: Calculate $a \cdot b \bmod c$ (or $a^b \bmod c$) for $0 \leq a, b \leq c \leq 7.2 \cdot 10^{18}$.
Time: $\mathcal{O}(1)$ for modmul, $\mathcal{O}(\log b)$ for modpow

59afa8, 11 lines

```
typedef unsigned long long ull;
ull modmul(ull a, ull b, ull M) {
  lint ret = a * b - M * ull(1.L / M * a * b);
  return ret + M * (ret < 0) - M * (ret >= (lint)M);
}
ull modpow(ull b, ull e, ull mod) {
  ull ans = 1;
  for (; e; b = modmul(b, b, mod), e /= 2)
    if (e & 1) ans = modmul(ans, b, mod);
  return ans;
}
```

mod-sqrt.h

Description: Tonelli-Shanks algorithm for modular square roots. Finds x s.t. $x^2 = a \pmod p$ ($-x$ gives the other solution).
Time: $\mathcal{O}(\log^2 p)$ worst case, $\mathcal{O}(\log p)$ for most p

abffbe, 33 lines

```
int jacobi(ll a, ll m) { // Jacobi symbol (a/m)
  int s = 1;
  if (a < 0) a = a % m + m;
  for (; m > 1; ) {
    a %= m; if (a == 0) return 0;
    const int r = __builtin_ctzll(a);
    if ((r & 1) && ((m + 2) & 4)) s = -s;
    a >>= r; if (a & m & 2) s = -s;
    swap(a, m);
  } return s;
}
vector<ll> mod_sqrt(ll a, ll p) {
  if (p == 2) return {a & 1};
  const int j = jacobi(a, p);
  if (j == 0) return {0};
  if (j == -1) return {};
  ll b, d;
  while (true) {
    b = xrand() % p; d = (b * b - a) % p;
    if (d < 0) d += p;
    if (jacobi(d, p) == -1) break;
  }
  ll f0 = b, f1 = 1, g0 = 1, g1 = 0, tmp;
  for (ll e = (p + 1) >> 1; e; e >>= 1) {
    if (e & 1) {
      tmp = (g0 * f0 + d * ((g1 * f1) % p)) % p;
      g1 = (g0 * f1 + g1 * f0) % p; g0 = tmp;
    }
    tmp = (f0 * f0 + d * ((f1 * f1) % p)) % p;
    f1 = (2 * f0 * f1) % p; f0 = tmp;
  }
  return (g0<p-g0) ? vector<ll>{g0,p-g0} : vector<ll>{p-g0,g0};
}
```

mod-range.h

Description: min $x \geq 0$ s.t. $l \leq ((ax) \mod m) \leq r$, $m > 0$, $a \geq 0$.

eb665e, 10 lines

```
template<typename T> T mod_range(T m, T a, T l, T r) {
  l = max(l, T(0)); r = min(r, m - 1);
  if (l > r) return -1;
  a %= m;
  if (a == 0) return (l > 0) ? -1 : 0;
  const T k = (l + a - 1) / a;
  if (a * k <= r) return k;
  const T y = mod_range(a, m, a * k - r, a * k - l);
  return (y == -1) ? -1 : ((m * y + r) / a);
}
```

5.2 Primality

sieve.h

Description: Prime sieve for generating all primes up to a certain limit. $pfac[i]$ is the lowest prime factor of i. Also useful if you need to compute any multiplicative function.
Time: $\mathcal{O}(N)$

a0b9c3, 17 lines

```
vector<int> run_sieve(int N) {
  vector<int> pfac(N+1), primes, mu(N+1,-1), phi(N+1);
  primes.reserve(N+1); mu[1] = phi[1] = 1;
  for (int i = 2; i <= N; ++i) {
    if (!pfac[i])
      pfac[i] = i, phi[i] = i-1, primes.push_back(i);
    for (int p : primes) {
      if (p > N/i) break;
      pfac[p * i] = p; mu[p * i] *= mu[i];
      phi[p * i] = phi[i] * phi[p];
      if (i % p == 0) {
        mu[p * i] = 0; phi[p * i] = phi[i] * p;
        break;
      }
    }
  } return primes;
```

```
}
```

miller-rabin.h

Description: Deterministic Miller-Rabin primality test. Guaranteed to work for numbers up to 2^{64}; for larger numbers, extend A randomly.
Time: 7 times the complexity of $a^b \mod c$.

"mod-mul.h" bbee97, 12 lines

```
bool isPrime(ull n) {
  if (n < 2 || n % 6 % 4 != 1) return (n | 1) == 3;
  vector<ull> A = {2, 325, 9375, 28178, 450775, 9780504,
       1795265022};
  ull s = __builtin_ctzll(n-1), d = n >> s;
  for(ull a : A) {   // ^ count trailing zeroes
    ull p = modpow(a % n, d, n), i = s;
    while (p != 1 && p != n - 1 && a % n && i--)
      p = modmul(p, p, n);
    if (p != n-1 && i != s) return 0;
  }
  return 1;
}
```

pollard-rho.h

Description: Pollard-rho randomized factorization algorithm. Returns prime factors of a number, in arbitrary order (e.g. 2299 -> {11, 19, 11}).
Time: $\mathcal{O}\left(n^{1/4}\right)$, less for numbers with small factors.

"mod-mul.h", "extended-euclid.h", "miller-rabin.h" 6bf31f, 17 lines

```
ull pollard(ull n) {
  auto f = [n](ull x, ull k) { return modmul(x, x, n) + k; };
  ull x = 0, y = 0, t = 30, prd = 2, i = 1, q;
  while (t++ % 40 || gcd(prd, n) == 1) {
    if (x == y) x = ++i, y = f(x, i);
    if ((q = modmul(prd, max(x,y) - min(x,y), n))) prd = q;
    x = f(x, i), y = f(f(y, i), i);
  }
  return gcd(prd, n);
}
vector<ull> factor(ull n) {
  if (n == 1) return {};
  if (isPrime(n)) return {n};
  ull x = pollard(n); auto l = factor(x), r = factor(n/x);
  l.insert(l.end(), r.begin(), r.end());
  return l;
}
```

5.3 Divisibility

extended-euclid.h

Description: Finds two integers x and y, such that $ax + by = \gcd(a, b)$. If a and b are coprime, then x is the inverse of $a \pmod b$.

8b62c4, 6 lines

```
template<typename T>
T egcd(T a, T b, T &x, T &y) {
  if (!a) { x = 0, y = 1; return b; }
  T g = egcd(b % a, a, y, x);
  x -= y * (b/a); return g;
}
```

division-lemma.h

Description: This lemma let us exploit the fact tha the sequence (harmonic on integer division) has at most $2\sqrt{N}$ distinct elements, so we can iterate through every possible value of $\lfloor \frac{N}{i} \rfloor$, using the fact that the greatest integer j satisfying $\lfloor \frac{N}{i} \rfloor = \lfloor \frac{N}{j} \rfloor$ is $\lfloor \frac{N}{\lfloor \frac{N}{i} \rfloor} \rfloor$. This one computes the $\sum_{i=1}^{N} \lfloor \frac{N}{i} \rfloor i$.
Time: $\mathcal{O}\left(\sqrt{N}\right)$

b2c1ab, 15 lines

```
int res = 0;
for (int a = 1, b; a <= N; a = b + 1) { // floor
  b = N / (N / a);
  // quotient (N/a) and there are (b - a + 1) elements
  int l = b - a + 1, r = a + b; // l * r / 2 = sum(i, j)
  if (l & 1) r /= 2;
  else l /= 2;
  res += l * r * (N / a);
}
// [1, N), need to deal with case where a = N separately
for (int a = 1, b; a < N; a = b + 1) { // ceil
  const int k = (N - 1) / a + 1; // quotient k
  b = (N - 1) / (k - 1);
  int cnt = b - a + 1; // occur cnt times on interval [a, b]
}
```

divisors.h

Description: Generate all factors of n given it's prime factorization.
Time: $\mathcal{O}\left(\frac{\sqrt{N}}{\log N}\right)$

"prime-factors.h" 5de75c, 14 lines

```
template<typename T> vector<T> get_divisors(T N) {
  auto factors = prime_factorize(N);
  vector<T> ans; ans.reserve(int(sqrtl(N) + 1));
  auto dfs = [&](auto&& self, auto& ans, T val, int d) -> void{
    auto& [P, E] = factors[d];
    if (d == int(factors.size())) ans.push_back(val);
    else {
      T X = 1;
      for (int pw = 0; pw <= E; ++pw, X *= P)
        self(self, ans, val * X, d + 1);
    }
  }; dfs(dfs, ans, 1, 0);
  return ans;
}
```

phi-function.h

da7671, 6 lines

```
const int n = int(1e5)*5; vector<int> phi(n);
void calculatePhi() {
  for(int i = 0; i < n; ++i) phi[i] = i&1 ? i : i/2;
  for(int i = 3; i < n; i += 2) if (phi[i] == i)
    for(int j = i; j < n; j += i) phi[j] -= phi[j]/i;
}
```

discrete-log.h

Description: Returns the smallest $x >= 0$ s.t. $a^x = b \pmod m$, or -1 if no such x exists. modLog(a,1,m) can be used calculate the order of a. Assumes that $0^0 = 1$.
Time: $\mathcal{O}\left(\sqrt{m}\right)$

"extended-euclid.h" 62fc5e, 15 lines

```
template<typename T> T modLog(T a, T b, T m) {
  T k = 1, it = 0, g;
  while ((g = gcd(a, m)) != 1) {
    if (b == k) return it;
    if (b % g) return -1;
    b /= g; m /= g; ++it; k = k * a / g % m;
  }
  T n = sqrtl(m) + 1, f = 1, j = 1;
  unordered_map<T, T> A;
  while (j <= n)
    f = f * a % m, A[f * b % m] = j++;
  for(int i = 1; i <= n; ++i) if (A.count(k = k * f % m))
    return n * i - A[k] + it;
  return -1;
}
```

prime-counting.h

Description: Count the number of primes up to N. Also useful for sum of primes.
Time: $\mathcal{O}(N^{3/4}/\log N)$

c26239, 20 lines

```
struct primes_t {
  vector<ll> dp, w;
  ll pi(ll N) {
    const int sqrtN = int(sqrt(N));
    for (ll a = 1, b; a <= N; a = b+1)
      b = N / (N / a), w.push_back(N/a);
    auto get = [&](ll x) {
      if (x <= sqrtN) return int(x-1);
      return int(w.size() - N/x);
    };
    reverse(w.begin(), w.end()); dp.reserve(w.size());
    for (auto& x : w) dp.push_back(x-1);
    for (ll i = 2; i*i <= N; ++i) {
      if (dp[i-1] == dp[i-2]) continue;
      for (int j = int(w.size())-1; w[j] >= i*i; --j)
        dp[j] -= dp[get(w[j]/i)] - dp[i-2];
    }
    return dp.back();
  }
};
```

5.4 Chinese remainder theorem

chinese-remainder.h

Description: Chinese Remainder Theorem. crt(a, m, b, n) computes x such that $x \equiv a \pmod m$, $x \equiv b \pmod n$. If $|a| < m$ and $|b| < n$, x will obey $0 \le x < \text{lcm}(m, n)$. Assumes $mn < 2^{62}$.
Time: $\mathcal{O}\left(\log(LCM(m))\right)$

"extended-euclid.h" ece59a, 7 lines

```
pair<ll, ll> crt(ll a, ll m, ll b, ll n) {
  if (n > m) swap(a, b), swap(m, n);
  ll x, y, g = egcd(m, n, x, y);
  if ((a - b) % g != 0) return {0, -1};
  x = (b - a) % n * x % n / g * m + a;
  return {x + (x < 0 ? m*n/g : 0), m*n/g};
}
```

5.5 Fractions

fractions.h

Description: Template that helps deal with frtions.

596163, 28 lines

```
template<typename num> struct fraction_t {
  num p, q; using fr = fraction_t;
  fraction_t() : p(0), q(1) { }
  fraction_t(num _n, num _d = 1): p(_n), q(_d){
    num g = gcd(p, q); p /= g, q /= g;
    if (q < 0) p *= -1, q *= -1; assert(q != 0);
  }
  friend bool operator<(const fr& l, const fr& r){
    return l.p*r.q < r.p*l.q;}
  friend bool operator==(const fr& l, const fr& r){return l.p
       == r.p && l.q == r.q;}
  friend bool operator!=(const fr& l, const fr& r){return !(l
       == r);}
  friend fr operator+(const fr& l, const fr& r){
    num g = gcd(l.q, r.q);
    return fr(r.q / g * l.p + l.q / g * r.p, l.q / g * r.q);
  }
  friend fr operator-(const fr& l, const fr& r) {
    num g = gcd(l.q, r.q);
    return fr( r.q / g * l.p - l.q / g * r.p, l.q / g * r.q);
  }
  friend fr operator*(const fr& l, const fr& r){
    return fr(l.p*r.p, l.q*r.q);}
  friend fr operator/(const fr& l, const fr& r){
    return l*fr(r.q,r.p);}
  friend fr& operator+=(fr& l, const fr& r){return l=l+r;}
  friend fr& operator-=(fr& l, const fr& r){return l=l-r;}
```

```
	template<class T> friend fr& operator*=(fr& l, const T& r){
		return l=l*r;}
	template<class T> friend fr& operator/=(fr& l, const T& r){
		return l=l/r;}
};
```

continued-fractions.h

Description: Given N and a real number $x \geq 0$, finds the closest rational approximation p/q with $p, q \leq N$. It will obey $|p/q - x| \leq 1/qN$.
For consecutive convergents, $p_{k+1}q_k - q_{k+1}p_k = (-1)^k$. ($p_k/q_k$ alternates between $> x$ and $< x$.) If x is rational, y eventually becomes ∞; if x is the root of a degree 2 polynomial the a's eventually become cyclic.
Time: $\mathcal{O}(\log N)$

3dfe17, 18 lines

```
typedef double dbl; // for N ~ 1e7; long double for N ~ 1e9
pair<ll, ll> approximate(dbl x, ll N) {
  ll LP = 0, LQ = 1, P = 1, Q = 0, inf = ll(1e18); dbl y = x;
  for (;;) {
    ll lim = min(P ? (N-LP) / P : inf, Q ? (N-LQ) / Q : inf),
       a = (ll)floor(y), b = min(a, lim),
       NP = b*P + LP, NQ = b*Q + LQ;
    if (a > b) {
      // If b > a/2, we have a semi-convergent that gives us a
      // better approximation; if b = a/2, we *may* have one.
      // Return {P, Q} here for a more canonical approximation.
      return (abs(x - (dbl)NP / (dbl)NQ) < abs(x - (dbl)P / (
           dbl)Q)) ?
        make_pair(NP, NQ) : make_pair(P, Q);
    }
    if (abs(y = 1/(y - (dbl)a)) > 3*N) return {NP, NQ};
    LP = P; P = NP; LQ = Q; Q = NQ;
  }
}
```

frac-binary-search.h

Description: Given f and N, finds the smallest fraction $p/q \in [0, 1]$ such that $f(p/q)$ is true, and $p, q \leq N$. You may want to throw an exception from f if it finds an exact solution, in which case N can be removed.
Usage: `fracBS([](Frac f) { return f.p>=3*f.q; }, 10); // {1,3}`
Time: $\mathcal{O}(\log(N))$

74ffd8, 20 lines

```
struct Frac { ll p, q; };
template<class F> Frac fracBS(F f, ll N) {
  bool dir = 1, A = 1, B = 1;
  Frac left{0, 1}, right{1, 1};//right{1,0} to search (0,N]
  assert(!f(left)); assert(f(right));
  while (A || B) {
    ll adv = 0, step = 1; // move right if dir, else left
    for (int si = 0; step; (step *= 2) >>= si) {
      adv += step;
      Frac mid{left.p * adv + right.p, left.q * adv + right.q};
      if (abs(mid.p) > N || mid.q > N || dir == !f(mid)) {
        adv -= step; si = 2;
      }
    }
    right.p += left.p * adv; right.q += left.q * adv;
    dir = !dir; swap(left, right);
    A = B; B = !!adv;
  }
  return dir ? right : left;
}
```

5.5.1 Bézout's identity

For $a \neq, b \neq 0$, then $d = gcd(a, b)$ is the smallest positive integer for which there are integer solutions to

$$ax + by = d$$

If (x, y) is one solution, then all solutions are given by

$$\left(x + \frac{kb}{\gcd(a,b)}, y - \frac{ka}{\gcd(a,b)}\right), \quad k \in \mathbb{Z}$$

5.5.2 Pythagorean Triples

The Pythagorean triples are uniquely generated by

$$a = k \cdot (m^2 - n^2), \;\; b = k \cdot (2mn), \;\; c = k \cdot (m^2 + n^2),$$

with $m > n > 0$, $k > 0$, $m \perp n$, and either m or n even.

5.6 Primes

$p = 962592769$ is such that $2^{21} \mid p - 1$, which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than 1 000 000.

Primitive roots exist modulo any prime power p^a, except for $p = 2, a > 2$, and there are $\phi(\phi(p^a))$ many. For $p = 2, a > 2$, the group $\mathbb{Z}_{2^a}^\times$ is instead isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$.

5.6.1 Prime counting function ($\pi(x)$)

The prime counting function is asymptotic to $\frac{x}{\log x}$, by the prime number theorem.

x	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\pi(x)$	4	25	168	1.229	9.592	78.498	664.579	5.761.455

5.6.2 Sum of primes

For any multiplicative f:

$$S(n, p) = S(n, p-1) - f(p) \cdot (S(n/p, p-1) - S(p-1, p-1))$$

5.6.3 Moebius Function

$$\mu(n) = \begin{cases} 0 & n \text{ is not square free} \\ 1 & n \text{ has even number of prime factors} \\ -1 & n \text{ has odd number of prime factors} \end{cases}$$

Moebius Inversion:

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d) g(n/d)$$

Other useful formulas/forms:

$\sum_{d|n} \phi(d) = n$

$\sum_{\substack{i<n \\ \gcd(i,n)=1}} i = n\frac{\phi(n)}{2}$

$\sum_{a=1}^n \sum_{b=1}^n [\gcd(a,b) = 1] = \sum_{d=1}^n \mu(d) \lfloor \frac{n}{d} \rfloor^2$

$\sum_{a=1}^n \sum_{b=1}^n \gcd(a,b) = \sum_{d=1}^n d \sum_{d|x}^n \lfloor \frac{n}{x} \rfloor^2 \mu(\frac{x}{d})$

$\sum_{a=1}^n \sum_{b=a}^n \gcd(a,b) = \sum_{d=1}^n \sum_{d|x}^n \phi(\frac{x}{d}) d$

$\sum_{a=1}^n \sum_{b=1}^n \text{lcm}(a,b) = \sum_{d=1}^n \mu(d) d \sum_{d|x}^n x \left(\frac{\lfloor \frac{n}{x} \rfloor + 1}{2}\right)^2$

$\sum_{a=1}^n \sum_{b=a+1}^n \text{lcm}(a,b) = \sum_{d=1}^n \sum_{d|x}^n \phi(\frac{x}{d}) \frac{x^2}{2d}$

$\sum_{a \in S} \sum_{b \in S} \gcd(a,b) = \sum_{d=1}^n (\sum_{x|d} \frac{d}{x} \mu(x)) (\sum_{d|v} \text{freq[v]})^2$

$\sum_{a \in S} \sum_{b \in S} \text{lcm}(a,b) = \sum_{d=1}^n (\sum_{x|d} \frac{x}{d} \mu(x)) (\sum_{v \in S, d|v} v)^2$

$\sum_{d|n} \mu(d) = [n = 1]$ (very useful)

$g(n) = \sum_{n|d} f(d) \Leftrightarrow f(n) = \sum_{n|d} \mu(d/n) g(d)$

$g(n) = \sum_{1 \leq m \leq n} f(\lfloor \frac{n}{m} \rfloor) \Leftrightarrow f(n) = \sum_{1 \leq m \leq n} \mu(m) g(\lfloor \frac{n}{m} \rfloor)$

5.6.4 Dirichlet Convolution

Given a function $f(x)$, let

$$(f * g)(x) = \sum_{d|x} g(d) f(x/d)$$

If the partial sums $s_{f*g}(n), s_g(n)$ can be computed in $O(1)$ and $s_f(1 \ldots n^{2/3})$ can be computed in $O\left(n^{2/3}\right)$ then all $s_f\left(\frac{n}{d}\right)$ can as well. Use

$$s_{f*g}(n) = \sum_{d=1}^n g(d) s_f(n/d).$$

$$\implies s_f(n) = \frac{s_{f*g}(n) - \sum_{d=2}^n g(d) s_f(n/d)}{g(1)}$$

1. If $f(x) = \mu(x)$ then $g(x) = 1, (f * g)(x) = (x == 1)$, and $s_f(n) = 1 - \sum_{i=2}^n s_f(n/i)$
2. If $f(x) = \phi(x)$ then $g(x) = 1, (f * g)(x) = x$, and $s_f(n) = \frac{n(n+1)}{2} - \sum_{i=2}^n s_f(n/i)$

dirichlet-convolution.h

Description: Dirichlet convolution. Change f, gs and fgs accordingly. This example calculates $\phi(N)$.
Time: $\mathcal{O}\left(N^{\frac{2}{3}}\right)$

eac754, 21 lines

```
template<typename T, typename V> struct mertens {
  V N; T inv; // ~ N^{2/3}
  vector<V> fs; vector<T> psum;
  unordered_map<V, T> mapa;
  V f(V x) { return fs[x]; }
  T gs(V x) { return x; }
  T fgs(V x) { return T(x) * (x + 1) / 2; }
  mertens(V M, const vector<V>& F) : N(M+1), fs(F), psum(M+1){
    inv = gs(1);
    for (V a = 0; a + 1 < N; ++a)
      psum[a + 1] = f(a + 1) + psum[a];
  }
  T query(V x) {
    if (x < N) return psum[x];
    if (mapa.find(x) != mapa.end()) return mapa[x];
    T ans = fgs(x);
    for (V a = 2, b; a <= x; a = b + 1)
      b = x/(x/a), ans -= (gs(b)-gs(a-1)) * query(x/a);
```

```
    return mapa[x] = (ans / inv);
  }
};
```

Combinatorial (6)

6.1 Permutations

6.1.1 Factorial

n	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24	120	720	5040	40320	362880	3628800

n	11	12	13	14	15	16	17
$n!$	4.0e7	4.8e8	6.2e9	8.7e10	1.3e12	2.1e13	3.6e14

n	20	25	30	40	50	100	150	171
$n!$	2e18	2e25	3e32	8e47	3e64	9e157	6e262	>DBL_MAX

int-perm.h

Description: Permutation -> integer conversion. (Not order preserving.)
Time: $\mathcal{O}(n)$

06f786, 6 lines

```
int permToInt(vector<int>& v) {
  int use = 0, i = 0, r = 0;
  for (auto &x : v) r=r* ++i +__builtin_popcount(use&-(1<<x)),
    use |= 1 << x;                 // (note: minus, not ~!)
  return r;
}
```

6.1.2 Binomials

- Sum of every element in the n-th row of pascal triangle is 2^n.
- The product of the elements in each row is $\frac{(n+1)^n}{n!}$
- $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$
- In a row p where p is a prime number, all the terms in that row except the 1s are multiples of p
- To count odd terms in row n, convert n to binary. Let x be the number of 1s in the binary representation. Then the number of odd terms will be 2^x
- Every entry in row $2^n - 1$ is odd

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \qquad (x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

$$\sum_{k=0}^{n}\binom{n}{k}k = n2^{n-1} \qquad \sum_{j=0}^{k}\binom{m}{j}\binom{n-m}{k-j} = \binom{n}{k}$$

$$\sum_{m=0}^{n}\binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1} \qquad \sum_{r=0} m\binom{n+r}{r} = \binom{n+m+1}{m}$$

$$\binom{n}{k}\binom{n-k}{v} = \binom{n}{v}\binom{n-v}{k} \qquad \sum_{m=k}^{n}\binom{m}{k} = \binom{n+1}{k+1}$$

$$\sum_{j=0}^{m}\binom{m}{j}^2 = \binom{2m}{m} \qquad 2\sum_{i=L}^{R}\binom{n}{i} - \binom{n}{L} - \binom{n}{R} = \sum_{i=L+1}^{R}\binom{n+1}{i}$$

lucas.h

Description: Lucas' thm: Let n, m be non-negative integers and p a prime. Write $n = n_k p^k + ... + n_1 p + n_0$ and $m = m_k p^k + ... + m_1 p + m_0$. Then $\binom{n}{m} \equiv \prod_{i=0}^{k}\binom{n_i}{m_i} \pmod{p}$. fact and ifact must hold pre-computed factorials / inverse factorials, e.g. from ModInv.h.
Time: $\mathcal{O}(\log_p m)$

rolling-binomial.h

Description: $\binom{n}{k} \pmod{m}$ in time proportional to the difference between (n, k) and the previous (n, k).

"../number-theory/preparator.h" de359e, 12 lines

```
struct Bin {
  int N = 0,  K = 0; ll r = 1;
  void m(int a, int b) { r = r * a % mod * invs[b] % mod; }
  ll choose(int n, int k) {
    if (k > n || k < 0) return 0;
    while(N < n) ++N, m(N, N - K);
    while(K < k) ++K, m(N - K + 1, K);
    while(K > k) m(K, N - K + 1), --K;
    while(N > n) m(N - K, N), --N;
    return r;
  }
};
```

multinomial.h

Description: Computes $\binom{k_1 + \cdots + k_n}{k_1, k_2, \ldots, k_n} = \frac{(\sum k_i)!}{k_1! k_2! \ldots k_n!}$.

864cdb, 7 lines

```
lint multinomial(vector<int>& v) {
  lint c = 1, m = v.empty() ? 1 : v[0];
  for (int i = 1 < v.size(); ++i)
    for (int j = 0; j < v[i]; ++j)
      c = c * ++m / (j+1);
  return c;
}
```

6.1.3 The twelvefold way (from Stanley)

How many functions $f\colon N \to X$ are there?

N	X	Any f	Injective	Surjective
dist.	dist.	x^n	$\frac{x!}{(x-n)!}$	$x!\left\{ n \atop x \right\}$
indist.	dist.	$\binom{x+n-1}{n}$	$\binom{x}{n}$	$\binom{n-1}{n-x}$
dist.	indist.	$\left\{ n \atop 1 \right\} + \ldots + \left\{ n \atop x \right\}$	$[n \le x]$	$\left\{ n \atop k \right\}$
indist.	indist.	$p_1(n) + \ldots p_x(n)$	$[n \le x]$	$p_x(n)$

Where $\binom{a}{b} = \frac{1}{b!}(a)_b$, $p_x(n)$ is the number of ways to partition the integer n using x summand and $\left\{ n \atop x \right\}$ is the number of ways to partition a set of n elements into x subsets (aka Stirling number of the second kind).

6.1.4 Burnside

Given a group G of symmetries and a set X, the number of elements of X *up to symmetry* equals

$$\frac{1}{|G|}\sum_{g \in G}|X^g|,$$

where X^g are the elements fixed by g $(g.x = x)$.

If $f(n)$ counts "configurations" (of some sort) of length n, we can ignore rotational symmetry using $G = \mathbb{Z}_n$ to get

$$g(n) = \frac{1}{n}\sum_{k=0}^{n-1} f(\gcd(n,k)) = \frac{1}{n}\sum_{k|n} f(k)\phi(n/k)$$

6.1.5 Cycles

Let the number of n-permutations whose cycle lenghts all belong to the set S be denoted by $g_S(n)$

$$\sum_{n=0}^{\infty} g_S(n)\frac{x^n}{n!} = \exp\left(\sum_{n \in S}\frac{x^n}{n}\right)$$

6.1.6 Involutions

An involution is a permutation with maximum cycle length 2, and it is its own inverse.

$a(n) = a(n-1) + (n-1)a(n-2), a(0) = a(1) = 1.$

$1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, 35696, 140152$

6.1.7 Derangements

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1) + D(n-2)) = nD(n-1) + (-1)^n = \left\lfloor \frac{n!}{e} \right\rceil$$

6.2 Partitions and subsets

6.2.1 Partition function

Number of ways of writing n as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1,\ p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} p(n - k(3k-1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

n	0	1	2	3	4	5	6	7	8	9	20	50	100
$p(n)$	1	1	2	3	5	7	11	15	22	30	627	~2e5	~2e8

partitions.h

3af1e7, 14 lines

```
vector<int64_t> prep(int N) {
  vector<int64_t> dp(N); dp[0] = 1;
  for (int n = 1; n < N; ++n) {
    int64_t sum = 0;
    for (int k = 0, l = 1, m = n - 1; ;) {
      sum += dp[m]; if ((m -= (k += 1)) < 0) break;
      sum += dp[m]; if ((m -= (l += 2)) < 0) break;
      sum -= dp[m]; if ((m -= (k += 1)) < 0) break;
      sum -= dp[m]; if ((m -= (l += 2)) < 0) break;
    }
    if ((sum %= M) < 0) sum += M;
    dp[n] = sum;
  } return dp;
}
```

6.3 General purpose numbers

6.3.1 Bernoulli numbers

EGF of Bernoulli numbers is $B(t) = \frac{t}{e^t - 1}$ (FFT-able). $B[0, \ldots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \ldots]$

Sums of powers:

$$\sum_{i=1}^{n} n^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k(n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\sum_{i=m}^{\infty} f(i) = \int_m^{\infty} f(x)dx - \sum_{k=1}^{\infty}\frac{B_k}{k!}f^{(k-1)}(m)$$

$$\approx \int_m^{\infty} f(x)dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m))$$

6.3.2 Stirling numbers of the first kind

Number of permutations on n items with k cycles.

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k), c(0,0) = 1$$
$$\textstyle\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\ldots(x+n-1)$$

$c(8,k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$
$c(n,2) = 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \ldots$

6.3.3 Eulerian numbers

Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j:s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j:s s.t. $\pi(j) \geq j$, k j:s s.t. $\pi(j) > j$.

$$E(n,k) = (n-k)E(n-1,k-1) + (k+1)E(n-1,k)$$
$$E(n,0) = E(n,n-1) = 1$$
$$E(n,k) = \sum_{j=0}^{k}(-1)^j \binom{n+1}{j}(k+1-j)^n$$

6.3.4 Stirling numbers of the second kind

Partitions of n distinct elements into exactly k groups.

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$
$$S(n,1) = S(n,n) = 1$$
$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

6.3.5 Bell numbers

Total number of partitions of n distinct elements. $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \ldots$.

$$\mathcal{B}_{n+1} = \sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k$$

Also possible to calculate using Stirling numbers of the second kind,

$$B_n = \sum_{k=0}^{n} S(n,k)$$

If p is prime:

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod{p}$$

6.3.6 Labeled unrooted trees

\# on n vertices: n^{n-2}
\# on k existing trees of size n_i: $n_1 n_2 \cdots n_k n^{k-2}$
\# with degrees d_i: $(n-2)!/((d_1-1)!\cdots(d_n-1)!)$ # forests with exactly k rooted trees:

$$\binom{n}{k}k \cdot n^{n-k-1}$$

.

6.3.7 Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$
$$C_0 = 1,\ C_{n+1} = \frac{2(2n+1)}{n+2}C_n,\ C_{n+1} = \sum C_i C_{n-i}$$

$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$

- sub-diagonal monotone paths in a $n \times n$ grid.
- strings with n pairs of parenthesis, correctly nested.
- binary trees with with $n+1$ leaves (0 or 2 children) or $2n+1$ elements.
- ordered trees with $n+1$ vertices.
- # ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.
- permutations of $[n]$ with no 3-term increasing subsequence.

6.3.8 Super Catalan numbers

The number of monotonic lattice paths of a nxn grid that do not touch the diagonal.

$$S(n) = \frac{3(2n-3)S(n-1) - (n-3)S(n-2)}{n}$$
$$S(1) = S(2) = 1$$

1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859

6.3.9 Motzkin numbers

Number of ways of drawing any number of nonintersecting chords among n points on a circle. Number of lattice paths from (0, 0) to (n, 0) never going below the x-axis, using only steps NE, E, SE.

$$M(n) = \frac{3(n-1)M(n-2) + (2n+1)M(n-1)}{n+2}$$
$$M(0) = M(1) = 1$$

1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634

6.3.10 Narayana numbers

Number of lattice paths from (0,0) to ($2n$,0) never going below the x-axis, using only steps NE and SE, and with k peaks.

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$
$$N(n,1) = N(n,n) = 1$$
$$\sum_{k=1}^{n} N(n,k) = C_n$$

1, 1, 1, 1, 3, 1, 1, 6, 6, 1, 1, 10, 20, 10, 1, 1, 15, 50

6.3.11 Schroder numbers

Number of lattice paths from (0, 0) to (n, n) using only steps N,NE,E, never going above the diagonal. Number of lattice paths from (0, 0) to ($2n$, 0) using only steps NE, SE and double east EE, never going below the x-axis. Twice the Super Catalan number, except for the first term.
1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098

6.3.12 Triangles

Given rods of length 1, ..., n,

$$T(n) = \frac{1}{24}\left\{\begin{array}{ll} n(n-2)(2n-5) & n \text{ even} \\ (n-1)(n-3)(2n-1) & n \text{ odd} \end{array}\right\}$$

is the number of distinct triangles (positive are) that can be constructed, i.e., the # of 3-subsets of $[n]$ s.t. $x \leq y \leq z$ and $z \neq x + y$.

6.4 Fibonacci

$Fib(x+y) = Fib(x+1)Fib(y) + Fib(x)Fib(y-1)$

$Fib(n+1)Fib(n-1) - Fib(n)^2 = (-1)^n$

$Fib(2n-1) = Fib(n)^2 - Fib(n-1)^2$

$\sum_{i=0}^{n} Fib(i) = Fib(n+2) - 1$

$\sum_{i=0}^{n} Fib(i)^2 = Fib(n)Fib(n+1)$

$\sum_{i=0}^{n} Fib(i)^3 = \frac{Fib(n)Fib(n+1)^2 - (-1)^n Fib(n-1) + 1}{2}$

6.5 Linear Recurrences

$F_i = \sum_{j=1}^{K} C_j F_{i-j} + D$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & : & 0 & 0 \\ 0 & 0 & 1 & 0 & : & 0 & 0 \\ 0 & 0 & 0 & 1 & : & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & : & \cdot & \cdot \\ C_K & C_{K-1} & C_{K-2} & C_{K-3} & : & C_1 & 1 \\ 0 & 0 & 0 & 0 & . & 0 & 1 \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ \cdot \\ F_{K-1} \\ D \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ \cdot \\ F_K \\ D \end{bmatrix}$$

6.6 Game Theory

A game can be reduced to Nim if it is a finite impartial game. Nim and its variants include:

6.6.1 Nim

Let $X = \bigoplus_{i=1}^{n} x_i$, then $(x_i)_{i=1}^{n}$ is a winning position iff $X \neq 0$. Find a move by picking k such that $x_k > x_k \oplus X$.

6.6.2 Misere Nim

Regular Nim, except that the last player to move *loses*. Play regular Nim until there is only one pile of size larger than 1, reduce it to 0 or 1 such that there is an odd number of piles. The second player wins $(a_1, \ldots, a_n)$ if 1) there is a pile $a_i > 1$ and $\oplus_{i=1}^{n} a_i = 0$ or 2) all $a_i \leq 1$ and $\oplus_{i=1}^{n} a_i = 1$.

6.6.3 Staircase Nim

Stones are moved down a staircase and only removed from the last pile. $(x_i)_{i=1}^n$ is an L-position if $(x_{2i-1})_{i=1}^{n/2}$ is (i.e. only look at odd-numbered piles).

nim-product.cpp

Description: Product of nimbers is associative, commutative, and distributive over addition (xor). Forms finite field of size 2^{2^k}. Application: Given 1D coin turning games G_1, G_2 $G_1 \times G_2$ is the 2D coin turning game defined as follows. If turning coins at $x_1, x_2, \ldots, x_m$ is legal in G_1 and $y_1, y_2, \ldots, y_n$ is legal in G_2, then turning coins at all positions (x_i, y_j) is legal assuming that the coin at (x_m, y_n) goes from heads to tails. Then the grundy function $g(x, y)$ of $G_1 \times G_2$ is $g_1(x) \times g_2(y)$.

Time: 64^2 xors per multiplication, memorize to speed up. f55947, 24 lines

```cpp
ull nim_prod[64][64];
ull nim_prod2(int i, int j) {
  if (nim_prod[i][j]) return nim_prod[i][j];
  if ((i & j) == 0) return nim_prod[i][j] = 1ull << (i|j);
  int a = (i&j) & -(i&j);
  return nim_prod[i][j] = nim_prod2(i ^ a, j) ^ nim_prod2((i ^
      a) | (a-1), (j ^ a) | (i & (a-1)));
}
void all_nim_prod() {
  for (int i = 0; i < 64; i++)
    for (int j = 0; j < 64; j++)
      if ((i & j) == 0) nim_prod[i][j] = 1ull << (i|j);
      else {
        int a = (i&j) & -(i&j);
        nim_prod[i][j] = nim_prod[i ^ a][j] ^ nim_prod[(i ^ a)
            | (a-1)][(j ^ a) | (i & (a-1))];
      }
}
ull get_nim_prod(ull x, ull y) {
  ull res = 0;
  for (int i = 0; i < 64 && (x >> i); ++i)
    if ((x >> i) & 1)
      for (int j = 0; j < 64 && (y >> j); ++j)
        if ((y >> j) & 1) res ^= nim_prod2(i, j);
  return res;
}
```

Graph (7)

7.1 Fundamentals

euler-walk.h

Description: Eulerian undirected/directed path/cycle algorithm. Input should be a vector of (dest, global edge index), where for undirected graphs, forward/backward edges have the same index. Returns a list of nodes in the Eulerian path/cycle with src at both start and end, or empty list if no cycle/path exists. To get edge indices back, add .second to s and ret.

Time: $\mathcal{O}(V + E)$ 643df6, 14 lines

```cpp
vector<int> eulerWalk(vector<vector<pii>>& gr, int nedges, int
    src=0) {
  int n = gr.size();
  vector<int> D(n), its(n), eu(nedges), ret, s = {src};
  D[src]++; // to allow Euler paths, not just cycles
  while (!s.empty()) {
    int x = s.back(), y, e, &it = its[x], end = int(gr[x].size
        ());
    if (it == end){ ret.push_back(x); s.pop_back(); continue; }
    tie(y, e) = gr[x][it++];
    if (!eu[e])
      D[x]--, D[y]++, eu[e] = 1, s.push_back(y);
  }
  for(auto &x : D) if (x < 0 || int(ret.size()) != nedges+1)
      return {};
  return {ret.rbegin(), ret.rend()};
}
```

7.2 Network flow

dinitz.h

Description: Flow algorithm with complexity $O(VE\log U)$ where $U = \max|\text{cap}|$. $O(\min(E^{1/2}, V^{2/3})E)$ if $U = 1$; $O(\sqrt{V}E)$ for bipartite matching. To obtain each partition A and B of the cut look at lvl, for $v \subset A$, $lvl[v] > 0$, for $u \subset B$, $lvl[u] = 0$. 80673b, 66 lines

```cpp
template<typename T = int> struct Dinitz {
  struct edge_t { int to, rev; T c, f; };
  vector<vector<edge_t>> adj;
  vector<int> lvl, ptr, q;
  Dinitz(int n) : lvl(n), ptr(n), q(n), adj(n) {}
  inline void addEdge(int a, int b, T c, T rcap = 0) { // 694
      aae
    adj[a].push_back({b, (int)adj[b].size(), c, 0});
    adj[b].push_back({a, (int)adj[a].size() - 1, rcap, 0});
  }
  T dfs(int v, int t, T f) { // 8ffe6b
    if (v == t || !f) return f;
    for (int &i = ptr[v]; i < int(adj[v].size()); ++i) {
      edge_t &e = adj[v][i];
      if (lvl[e.to] == lvl[v] + 1)
        if (T p = dfs(e.to, t, min(f, e.c - e.f))) {
          e.f += p, adj[e.to][e.rev].f -= p;
          return p;
        }
    } return 0;
  }
  T maxflow(int s, int t) { // db2141
    T flow = 0; q[0] = s;
    for (int L = 0; L < 31; ++L) do { // 'int L=30' maybe
        faster for random data
      lvl = ptr = vector<int>(q.size());
      int qi = 0, qe = lvl[s] = 1;
      while (qi < qe && !lvl[t]) {
        int v = q[qi++];
        for (edge_t &e : adj[v])
          if (!lvl[e.to] && (e.c - e.f) >> (30 - L))
            q[qe++] = e.to, lvl[e.to] = lvl[v] + 1;
      }
      while (T p = dfs(s, t, numeric_limits<T>::max()/4)) flow
          += p;
    } while (lvl[t]);
    return flow;
  }
  bool leftOfMinCut(int v) { return bool(lvl[v] != 0); }
  pair<T, vector<pair<int,int>>> minCut(int s, int t) { // 727
      b22
    T cost = maxflow(s,t);
    vector<pair<int,int>> cut;
    for (int i = 0; i < int(adj.size()); i++) for(edge_t &e :
        adj[i])
      if (lvl[i] && !lvl[e.to]) cut.push_back({i, e.to});
    return {cost, cut};
  }
};
struct flow_demand_t {
  int src, sink;
  vector<int> d; Dinitz<int> flower;
  flow_demand_t(int N) : src(N + 1), sink(N + 2), d(N + 3),
      flower(N + 3) {}
  void add_edge(int a, int b, int demand, int cap) {
    d[a] -= demand; d[b] += demand;
    flower.addEdge(a, b, cap - demand);
  }
  int get_flow() {
    int x = 0, y = 0;
    flower.add_edge(N, N-1, numeric_limits<int>::max());
    for (int i = 0; i <= N; ++i) {
      if (d[i] < 0)
        flower.add_edge(i, sink, -d[i]), x += -d[i];
      if (d[i] > 0)
        flower.add_edge(src, i, d[i]), y += d[i];
    }
    bool has_circulation=(flower.maxflow(src,sink)==x && x==y);
    if (!has_circulation) return -1;
    return flower.maxflow(N-1, N);
  }
};
```

min-cost-max-flow.h

Description: Min-cost max-flow. Assumes there is no negative cycle.

Time: $\mathcal{O}(F(V + E)logV)$, being F the amount of flow. 3b2abb, 58 lines

```cpp
template<class flow_t, class cost_t> struct min_cost {
  static constexpr flow_t FLOW_EPS = 1e-10L;
  static constexpr flow_t FLOW_INF = numeric_limits<flow_t>::
      max();
  static constexpr cost_t COST_EPS = 1e-10L;
  static constexpr cost_t COST_INF = numeric_limits<cost_t>::
      max();
  int n, m{}; vector<int> ptr, nxt, zu;
  vector<flow_t> capa; vector<cost_t> cost;
  min_cost(int N) : n(N),ptr(n,-1),dist(n),vis(n),pari(n) {}
  void add_edge(int u, int v, flow_t w, cost_t c) { // d482f5
    nxt.push_back(ptr[u]); zu.push_back(v); capa.push_back(w);
        cost.push_back(c); ptr[u] = m++;
    nxt.push_back(ptr[v]); zu.push_back(u); capa.push_back(0);
        cost.push_back(-c); ptr[v] = m++;
  }
  vector<cost_t> pot, dist; vector<bool> vis; vector<int> pari;
  vector<flow_t> flows; vector<cost_t> slopes;
  // You can pass t = -1 to find a shortest
  void shortest(int s, int t) { // path to each vertex.
    using E = pair<cost_t, int>;
    priority_queue<E, vector<E>, greater<E>> que;
    for(int u = 0; u < n; ++u){dist[u]=COST_INF; vis[u]=false;}
    for (que.emplace(dist[s] = 0, s); !que.empty(); ) {
      const cost_t c = que.top().first;
      const int u = que.top().second; que.pop();
      if (vis[u]) continue;
      vis[u] = true; if (u == t) return;
      for (int i = ptr[u]; ~i; i = nxt[i]) if (capa[i] >
          FLOW_EPS) {
        const int v = zu[i];
        const cost_t cc = c + cost[i] + pot[u] - pot[v];
        if(dist[v] > cc){que.emplace(dist[v]=cc,v);pari[v]=i;}
      }
    }
  }
  pair<flow_t, cost_t> run(int s, int t, flow_t limFlow =
      FLOW_INF) {
    pot.assign(n,0); flows = {0}; slopes.clear();
    while (true) {
      bool upd = false;
      for (int i = 0; i < m; ++i) if (capa[i] > FLOW_EPS) {
        const int u = zu[i ^ 1], v = zu[i];
        const cost_t cc = pot[u] + cost[i];
        if(pot[v] > cc + COST_EPS) { pot[v] = cc; upd = true; }
      } if (!upd) break;
    }
    flow_t flow = 0; cost_t cost = 0;
    while (flow < limFlow) {
```

```
      shortest(s, t);
      if (!vis[t]) break;
      for(int u = 0; u < n; ++u)pot[u] += min(dist[u],dist[t]);
      flow_t f = limFlow - flow;
      for (int v = t; v != s; ) { const int i = pari[v];
        if (f > capa[i]) { f = capa[i]; } v = zu[i^1];
      }
      for (int v = t; v != s; ) { const int i = pari[v];
        capa[i] -= f; capa[i^1] += f; v = zu[i^1];
      }
      flow += f; cost += f * (pot[t] - pot[s]);
      flows.push_back(flow); slopes.push_back(pot[t] - pot[s]);
    } return {flow, cost};
  }
};
```

7.3 Matching

hopcroft-karp.h

Description: Fast bipartite matching algorithm. Graph g should be a list of neighbors of the left partition, and *btoa* should be a vector full of -1's of the same size as the right partition. Returns the size of the matching. $btoa[i]$ will be the match for vertex i on the right side, or -1 if it's not matched.
Usage: `vector<int> btoa(m, -1); hopcroftKarp(g, btoa);`
Time: $\mathcal{O}\left(\sqrt{V}E\right)$ d9a55d, 35 lines

```
using vi = vector<int>;
bool dfs(int a, int L, const vector<vi> &g, vi &btoa, vi &A, vi
      &B) {
  if (A[a] != L) return 0;
  A[a] = -1;
  for(auto &b : g[a]) if (B[b] == L + 1) {
    B[b] = 0;
    if (btoa[b] == -1 || dfs(btoa[b], L+1, g, btoa, A, B))
      return btoa[b] = a, 1;
  }
  return 0;
}
int hopcroftKarp(const vector<vi> &g, vi &btoa) {
  int res = 0;
  vector<int> A(g.size()), B(int(btoa.size())), cur, next;
  for (;;) {
    fill(A.begin(), A.end(), 0), fill(B.begin(), B.end(), 0);
    cur.clear();
    for(auto &a : btoa) if (a != -1) A[a] = -1;
    for (int a = 0; a < g.size(); ++a) if (A[a] == 0) cur.
         push_back(a);
    for (int lay = 1;; ++lay) {
      bool islast = 0; next.clear();
      for(auto &a : cur) for(auto &b : g[a]) {
        if (btoa[b] == -1) B[b] = lay, islast = 1;
        else if (btoa[b] != a && !B[b])
          B[b] = lay, next.push_back(btoa[b]);
      }
      if (islast) break;
      if (next.empty()) return res;
      for(auto &a : next) A[a] = lay;
      cur.swap(next);
    }
    for(int a = 0; a < int(g.size()); ++a)
      res += dfs(a, 0, g, btoa, A, B);
  }
}
```

bipartite-matching.h

Description: Fast Kuhn! Simple maximum cardinality bipartite matching algorithm. Better than hopcroftKarp in practice. Worst case is $O(VE)$ on an hairy tree. Shuffling the edges and vertices ordering should break some worst-case inputs.
Time: $\Omega(VE)$ 1b4d72, 31 lines

```
struct bm_t {
  int N, M, T;
  vector<vector<int>> adj;
  vector<int> match, seen;
  bm_t(int a, int b) : N(a), M(a+b), T(0), adj(M),
  match(M, -1), seen(M, -1) {}
  void add_edge(int a, int b) { adj[a].push_back(b + N); }
  bool dfs(int cur) {
    if (seen[cur] == T) return false;
    seen[cur] = T;
    for (int nxt : adj[cur]) if (match[nxt] == -1) {
      match[nxt] = cur, match[cur] = nxt;
      return true;
    }
    for (int nxt : adj[cur]) if (dfs(match[nxt])) {
      match[nxt] = cur, match[cur] = nxt;
      return true;
    }
    return false;
  }
  int solve() {
    int res = 0;
    for (int cur = 1; cur; ) {
      cur = 0; ++T;
      for (int i = 0; i < N; ++i) if (match[i] == -1)
        cur += dfs(i);
      res += cur;
    }
    return res;
  }
};
```

weighted-matching.h

Description: Given a weighted bipartite graph, matches every node on the left with a node on the right such that no nodes are in two matchings and the sum of the edge weights is minimal. Takes cost[N][M], where cost[i][j] = cost for L[i] to be matched with R[j] and returns (min cost, match), where L[i] is matched with R[match[i]]. Negate costs for max cost.
Time: $\mathcal{O}\left(N^2M\right)$ f0ea90, 28 lines

```
pair<int, vector<int>> hungarian(const vector<vector<int>> &a){
  if (a.empty()) return {0, {}};
  int n = a.size() + 1, m = a[0].size() + 1;
  vector<int> u(n), v(m), p(m), ans(n - 1);
  for(int i = 1; i < n; ++i) {
    p[0] = i; int j0 = 0; // add "dummy" worker 0
    vector<int> dist(m, INT_MAX), pre(m, -1);
    vector<bool> done(m + 1);
    do {
      done[j0] = true;
      int i0 = p[j0], j1, delta = INT_MAX;
      for(int j = 1; j < m; ++j) if (!done[j]) {
        auto cur = a[i0-1][j-1] - u[i0] - v[j];
        if (cur < dist[j]) dist[j] = cur, pre[j] = j0;
        if (dist[j] < delta) delta = dist[j], j1 = j;
      }
      for(int j = 0; j < m; ++j)
        if (done[j]) u[p[j]] += delta, v[j] -= delta;
        else dist[j] -= delta;
      j0 = j1;
    } while (p[j0]);
    while (j0) { // update alternating path
      int j1 = pre[j0]; p[j0] = p[j1], j0 = j1;
    }
  }
  for(int j = 1; j < m; ++j) if (p[j]) ans[p[j]-1] = j-1;
  return {-v[0], ans}; // min cost
}
```

general-matching.h

Description: Maximum Matching for general graphs (undirected and non bipartite) using Edmond's Blossom Algorithm.
Time: $\mathcal{O}\left(EV^2\right)$ e5db8e, 47 lines

```
struct blossom_t {
  int N, M; vector<vector<int>> adj;
  vector<int> match, ts, ps; vector<array<int, 2>> fs;
  blossom_t(auto& G) : N(int(G.size())), M(0), adj(G), match(N
       ,-1), ts(N,-1), ps(N,-1), fs(N,{-1,-1}) {}
  int root(int a) {
    return (ts[a] != M || !~ps[a]) ? a : (ps[a] = root(ps[a]));
  }
  void rematch(int a, int b) {
    const int w = match[a]; match[a] = b; auto [x, y] = fs[a];
    if (~w && match[w] == a) {
      if (~y) rematch(x, y), rematch(y, x);
      else match[w] = x, rematch(x, w);
    }
  }
  bool augment(int src) {
    vector<int> bfs = {src}; bfs.reserve(N);
    ts[src] = M; ps[src] = -1; fs[src] = {-1,-1};
    for (int z = 0; z < int(bfs.size()); ++z) {
      int cur = bfs[z];
      for (int nxt : adj[cur]) if (nxt != src) {
        if (match[nxt] == -1) {
          match[nxt] = cur; rematch(cur, nxt); return true;
        }
        if (ts[nxt] == M) {
          int a = root(cur), b = root(nxt), m = src;
          if (a == b) continue;
          while (a != src || b != src) {
            if (b != src) swap(a, b);
            if (fs[a][0]==cur&&fs[a][1]==nxt) { m = a; break; }
            fs[a] = {cur, nxt}; a = root(fs[match[a]][0]);
          }
          for (const int r : {root(cur), root(nxt)})
            for (int v = r; v != m; v = root(fs[match[v]][0]))
              ts[v] = M, ps[v] = m, bfs.push_back(v);
        } else if (ts[match[nxt]] != M) {
          fs[nxt] = {-1, -1}; ts[match[nxt]] = M;
          ps[match[nxt]] = nxt; fs[match[nxt]] = {cur, -1};
          bfs.push_back(match[nxt]);
        }
      }
    } return false;
  }
  int run() {
    for(int v = 0; v < N; ++v) if(!~match[v]) M += augment(v);
    return M;
  }
};
```

max-independent-set.h

Description: To obtain a maximum independent set of a graph, find a max clique of the complement. If the graph is bipartite, see MinimumVertex-Cover.

7.4 DFS algorithms

centroid-decomposition.h

Description: Divide and Conquer on Trees. dd21a1, 65 lines

```
template<typename T> struct centroid_t {
  int N;
  vector<vector<int>> adj;
  vector<vector<int>> dist; // dist to all ancestors
  vector<bool> blocked; // processed centroid
```

```
  vector<int> sz, depth, parent; // centroid parent
  centroid_t(int _n) : N(_n), adj(_n), dist(32 - __builtin_clz(
      _n), vector<int>(_n)),
  blocked(_n), sz(_n), depth(_n), parent(_n) {}
  void add_edge(int a, int b) {
    adj[a].push_back(b); adj[b].push_back(a);
  }
  void dfs_sz(int cur, int prv) {
    sz[cur] = 1;
    for (int nxt : adj[cur]) {
      if (nxt == prv || blocked[nxt]) continue;
      dfs_sz(nxt, cur); sz[cur] += sz[nxt];
    }
  }
  int find(int cur, int prv, int tsz) {
    for (int nxt : adj[cur])
      if (!blocked[nxt] && nxt != prv && 2*sz[nxt] > tsz)
        return find(nxt, cur, tsz);
    return cur;
  }
  void dfs_dist(int cur, int prv, int layer, int d) {
    dist[layer][cur] = d;
    for (int nxt : adj[cur]) {
      if (blocked[nxt] || nxt == prv) continue;
      dfs_dist(nxt, cur, layer, d + 1);
    }
  }
  void get_path(int cur, int prv, int d,vector<int>& cur_path){
    cur_path.push_back(d);
    for (int nxt : adj[cur]) {
      if (nxt == prv || blocked[nxt]) continue;
      get_path(nxt, cur, d + 1, cur_path);
    }
  }
  // solve for each subtree (cnt := # of paths of length K
  // that goes through vertex cur)
  T solve_subtree(int cur, int prv, int K) {
    vector<T> dp(sz[prv] + 1); dp[0] = 1;
    T cnt = 0;
    for (int nxt : adj[cur]) {
      if (blocked[nxt]) continue;
      vector<int> path; get_path(nxt, cur, 1, path);
      for (int d : path) {
        if (d > K || K - d > sz[prv]) continue;
        cnt += dp[K - d];
      }
      for (int d : path) dp[d] += 1;
    } return cnt;
  }
  T decompose(int cur, int K, int layer=0, int prv_root = -1){
    dfs_sz(cur, -1);
    int root = find(cur, cur, sz[cur]);
    blocked[root] = true; depth[root] = layer;
    parent[root] = prv_root; dfs_dist(root, root, layer, 0);
    T res = solve_subtree(root, cur, K);
    for (int nxt : adj[root]) {
      if (blocked[nxt]) continue;
      res += decompose(nxt, K, layer + 1, root);
    } return res;
  }
};
```

tarjan.h

Description: Finds all strongly connected components in a directed graph.
Usage: `scc_t s(g); s.solve([&](const vector<int>& cc) {...});`
`visits all components in reverse topological order.`
Time: $\mathcal{O}(E+V)$

50f8c4, 29 lines

```
struct scc_t {
    int n, t, scc_num;
    vector<vector<int>> adj;
    vector<int> low, id, stk, in_stk, cc_id;
    scc_t(const vector<vector<int>>& g) : n(int(g.size())), t
        (0), scc_num(0),
    adj(g), low(n,-1), id(n,-1), in_stk(n, false), cc_id(n) {}
    template<class F> void dfs(int cur, F f) {
        id[cur] = low[cur] = t++;
        stk.push_back(cur); in_stk[cur] = true;
        for (int nxt : adj[cur])
            if (id[nxt] == -1)
                dfs(nxt, f), low[cur] = min(low[cur], low[nxt])
                    ;
            else if (in_stk[nxt])
                low[cur] = min(low[cur], id[nxt]);
        if (low[cur] == id[cur]) {
            vector<int> cc; cc.reserve(stk.size());
            while (true) {
                int v = stk.back(); stk.pop_back();
                in_stk[v] = false;
                cc.push_back(v); cc_id[v] = scc_num;
                if (v == cur) break;
            } f(cc); scc_num++;
        }
    }
    template<class F> void solve(F f) {
        stk.reserve(n);
        for (int r = 0; r < n; ++r) if (id[r] == -1) dfs(r, f);
    }
};
```

bcc.h

Description: Finds all biconnected components in an undirected graph. In a biconnected component there are at least two distinct paths between any two nodes or the component is a bridge. Note that a node can be in several components. *blockcut* constructs the block cut tree of given graph. The first nodes represents the blocks, the others represents the articulation points.
Usage:
```
int e_id = 0; vector<pair<int, int>> g(N);
for (auto [a,b] : edges) {
g[a].emplace_back(b, e_id);
g[b].emplace_back(a, e_id++); }
bcc_t b(g); b.solve([&](const vector<int>& edges_id) {...});
```
Time: $\mathcal{O}(E+V)$

2f0210, 50 lines

```
struct bcc_t{
    int n, t;
    vector<vector<pii>> adj;
    vector<int> low, id, stk, is_art;
    bcc_t(const vector<vector<pii>> &g) : n(int(g.size())),
    t(0), adj(g), low(n,-1), id(n,-1), is_art(n) {}
    template<class F> void dfs(int cur, int e_par, F f){
        id[cur] = low[cur] = t++;
        stk.push_back(e_par); int c = 0;
        for (auto [nxt, e_id] : adj[cur]) {
            if (id[nxt] == -1) {
                dfs(nxt, e_id, f);
                low[cur] = min(low[cur], low[nxt]); c++;
                if (low[nxt] < id[cur]) continue;
                is_art[cur] = true;
                auto top =find(stk.rbegin(), stk.rend(), e_id);
                vector<int> cc(stk.rbegin(), next(top));
                f(cc); stk.resize(stk.size()-cc.size());
            }
            else if (e_id != e_par) {
                low[cur] = min(low[cur], id[nxt]);
                if (id[nxt] < id[cur]) stk.push_back(e_id);
            }
        } if(e_par == -1) is_art[cur] = (c > 1) ? true : false;
    }
    template<class F> void solve(F f) {
        stk.reserve(n);
        for(int r = 0; r < n; ++r) if(id[r] == -1) dfs(r,-1,f);
    }
    auto blockcut(const vector<pii> &edges){
        vector<vector<int>> cc; vector<int> cc_id(n);
        solve( [&](const vector<int> &c) {
            set<int> vc;
            for(int e : c){
                auto [a, b] = edges[e];
                cc_id[a] = cc_id[b] = int(cc.size());
                vc.insert(a); vc.insert(b);
            } cc.emplace_back(vc.begin(), vc.end());
        } );
        for(int a = 0; a < n; a++) if(is_art[a])
            cc_id[a] = int(cc.size()), cc.push_back({a});
        int bcc_num = int(cc.size());
        vector<vector<int>> tree(bcc_num);
        for(int c = 0; c < bcc_num && 1<int(cc[c].size()); ++c)
            for(int a : cc[c]) if(is_art[a]) {
                tree[c].push_back(cc_id[a]);
                tree[cc_id[a]].push_back(c);
            } return make_tuple(cc_id, cc, tree);
    }
};
```

2sat.h

Description: Calculates a valid assignment to boolean variables a, b, c,... to a 2-SAT problem, so that an expression of the type (a||b)&&(!a||c)&&(d||!b)&&... becomes true, or reports that it is unsatisfiable. Negated variables are represented by bit-inversions (~x).
Usage:
```
TwoSat ts(number of boolean variables);
ts.either(0, ~3); // Var 0 is true or var 3 is false
ts.set_value(2); // Var 2 is true
ts.at_most_one({0,~1,2}); // <= 1 of vars 0, ~1 and 2 are true
ts.solve(); // Returns true iff it is solvable
ts.values[0..N-1] holds the assigned values to the vars
```
Time: $\mathcal{O}(N+E)$, where N is the number of boolean variables, and E is the number of clauses.

"tarjan.h" 74e46b, 36 lines

```
struct TwoSat {
    int N;
    vector<vector<int>> gr;
    vector<int> values; // 0 = false, 1 = true
    TwoSat(int n = 0) : N(n), gr(2*n) {}
    int add_var() { // (optional)
        gr.emplace_back(); gr.emplace_back();
        return N++;
    }
    void either(int f, int j) {
        f = max(2*f, -1-2*f); j = max(2*j, -1-2*j);
        gr[f].push_back(j^1); gr[j].push_back(f^1);
    }
    void implies(int f, int j) { either(~f, j); }
    void set_value(int x) { either(x, x); }
    void at_most_one(const vector<int>& li) { // (optional)
        if (int(li.size()) <= 1) return;
        int cur = ~li[0];
        for (int i = 2; i < int(li.size()); ++i) {
            int next = add_var();
            either(cur, ~li[i]); either(cur, next);
            either(~li[i], next); cur = ~next;
        } either(cur, ~li[1]);
    }
    bool solve() {
        scc_t s(gr);
        s.solve([](const vector<int> &v){ return; } );
        values.assign(N, -1);
        for (int i = 0; i < N; ++i)
```

```
        if (s.cc_id[2*i] == s.cc_id[2*i+1]) return 0;
      for (int i = 0; i < N; ++i)
        if (s.cc_id[2*i] < s.cc_id[2*i+1]) values[i] =
            false;
        else values[i] = true;
      return 1;
    }
};
```

7.5 Heuristics

maximal-cliques.h

Description: Runs a callback for all maximal cliques in a graph (given as a symmetric bitset matrix; self-edges not allowed). Possible optimization: on the top-most recursion level, ignore 'cands', and go through nodes in order of increasing degree, where degrees go down as nodes are removed.

Time: $\mathcal{O}\left(3^{n/3}\right)$, much faster for sparse graphs

57e107, 11 lines

```
typedef bitset<128> B;
template<class F>
void cliques(vector<B> &eds, F f, B P = ~B(), B X={}, B R={}) {
  if (!P.any()) { if (!X.any()) f(R); return; }
  auto q = (P | X)._Find_first();
  auto cands = P & ~eds[q];
  for(int i = 0; i < eds.size(); ++i) if (cands[i]) {
    R[i] = 1; cliques(eds, f, P & eds[i], X & eds[i], R);
    R[i] = P[i] = 0; X[i] = 1;
  }
}
```

chromatic-number.h

Description: Compute the chromatic number of a graph. Minimum number of colors needed to paint the graph in a way s.t. if two vertices share an edge, they must have distinct colors.

Time: $\mathcal{O}\left(N2^N\right)$

dd49e4, 26 lines

```
template<class T> int min_colors(int N, const T& gr) {
  vector<int> adj(N);
  for (int a = 0; a < N; ++a)
    for (int b = a + 1; b < N; ++b) {
      if (!gr[a][b]) continue;
      adj[a] |= (1 << b); adj[b] |= (1 << a);
    }
  static vector<unsigned> dp(1 << N), buf(1 << N), w(1 << N);
  for (int mask = 0; mask < (1 << N); ++mask) {
    bool ok = true;
    for (int i = 0; i < N; ++i) if (mask & 1 << i)
      if (adj[i] & mask) ok = false;
    if (ok) dp[mask]++;
    buf[mask] = 1;
    w[mask] = __builtin_popcount(mask) % 2 == N % 2 ? 1 : -1;
  }
  for (int i = 0; i < N; ++i)
    for (int mask = 0; mask < (1 << N); ++mask)
      if (!(mask & 1 << i)) dp[mask^(1 << i)] += dp[mask];
  for (int colors = 1; colors <= N; ++colors) {
    unsigned S = 0;
    for (int mask = 0; mask < (1 << N); ++mask)
      S += (buf[mask] *= dp[mask]) * w[mask];
    if (S) return colors;
  } assert(false);
}
```

cycle-counting.cpp

Description: Counts 3 and 4 cycles

Time: $\mathcal{O}\left(E\sqrt{E}\right)$

1cf947, 31 lines

```
using vi = vector<int>;
int count_cycles(const vector<vi>& adj, const vi& deg) {
  const int N = int(adj.size());
  vi idx(N), loc(N); iota(idx.begin(), idx.end(), 0);
  sort(idx.begin(), idx.end(), [&](const int& a, const int& b)
      { return deg[a] < deg[b]; });
  for (int i = 0; i < N; ++i) loc[idx[i]] = i;
  vector<vi> gr(N);
  for (int a = 0; a < N; ++a) for (int b : adj[a])
    if (loc[a] < loc[b]) gr[a].push_back(b);
  int cycle3 = 0, cycle4 = 0;
  {
    vector<bool> seen(N, false);
    for (int a = 0; a < N; ++a) {
      for (int b : gr[a]) seen[b] = true;
      for (int b : gr[a]) for (int c : gr[b])
        if (seen[c]) cycle3 += 1;
      for (int b : gr[a]) seen[b] = false;
    }
  }
  {
    vi cnt(N);
    for (int a = 0; a < N; ++a) {
      for (int b : adj[a]) for (int c : gr[b])
        if (loc[a] < loc[c]) {
          cycle4 += cnt[c];
          cnt[c]++;
        }
      for (int b : adj[a]) for (int c : gr[b]) cnt[c] = 0;
    }
  } return cycle3;
}
```

edge-coloring.h

Description: Given a simple, undirected graph with max degree D, computes a $(D+1)$-coloring of the edges such that no neighboring edges share a color. (D-coloring is NP-hard, but can be done for bipartite graphs by repeated matchings of max-degree nodes.)

Time: $\mathcal{O}(NM)$

13a6e5, 27 lines

```
vector<int> misra_gries(int N, vector<pair<int, int>> eds) {
  const int M = int(eds.size());
  vector<int> cc(N + 1), ret(M), fan(N), free(N), loc;
  for (auto e : eds) ++cc[e.first], ++cc[e.second];
  int u, v, ncols = *max_element(cc.begin(), cc.end()) + 1;
  vector<vector<int>> adj(N, vi(ncols, -1));
  for (auto e : eds) {
    tie(u, v) = e; fan[0] = v; loc.assign(ncols, 0);
    int at = u, end = u, d, c = free[u], ind = 0, i = 0;
    while (d = free[v], !loc[d] && (v = adj[u][d]) != -1)
      loc[d] = ++ind, cc[ind] = d, fan[ind] = v;
    cc[loc[d]] = c;
    for (int cd = d; at != -1; cd ^= c ^ d, at = adj[at][cd])
      swap(adj[at][cd], adj[end = at][cd ^ c ^ d]);
    while (adj[fan[i]][d] != -1) {
      int left = fan[i], right = fan[++i], e = cc[i];
      adj[u][e] = left; adj[left][e] = u;
      adj[right][e] = -1; free[right] = e;
    }
    adj[u][d] = fan[i]; adj[fan[i]][d] = u;
    for (int y : {fan[0], u, end})
      for (int& z = free[y] = 0; adj[y][z] != -1; z++);
  }
  for (int i = 0; i < M; ++i)
    for (tie(u, v) = eds[i]; adj[u][ret[i]] != v;) ++ret[i];
  return ret;
}
```

7.6 Trees

heavylight.h

Time: $\mathcal{O}\left((\log N)^2\right)$

"../data-structures/lazy-segtree.h" 934c23, 62 lines

```
template<bool use_edges> struct hld_t {
  int N, T{};
  vector<vector<int>> adj;
  vector<int> sz, depth, chain, par, in, out, preorder;
  hld_t() {}
  hld_t(const vector<vector<int>>& G, int r = 0) : N(int(G.size
      ())),
  adj(G), sz(N), depth(N), chain(N), par(N), in(N), out(N),
  preorder(N) { dfs_sz(r); chain[r] = r; dfs_hld(r); }
  void dfs_sz(int cur) {
    sz[cur] = 1;
    for (auto& nxt : adj[cur]) {
      par[nxt] = cur; depth[nxt] = 1 + depth[cur];
      adj[nxt].erase(find(adj[nxt].begin(), adj[nxt].end(), cur
          ));
      dfs_sz(nxt); sz[cur] += sz[nxt];
      if (sz[nxt] > sz[adj[cur][0]]) swap(nxt, adj[cur][0]);
    }
  }
  void dfs_hld(int cur) {
    in[cur] = T++; preorder[in[cur]] = cur;
    for (auto& nxt : adj[cur]) {
      chain[nxt] = (nxt == adj[cur][0] ? chain[cur] : nxt);
      dfs_hld(nxt);
    } out[cur] = T;
  }
  int lca(int a, int b) {
    while (chain[a] != chain[b]) {
      if (in[a] < in[b]) swap(a, b);
      a = par[chain[a]];
    } return (in[a] < in[b] ? a : b);
  }
  bool is_ancestor(int a, int b) { return in[a] <= in[b] && in[
      b] < out[a]; }
  int climb(int a, int k) {
    if (depth[a] < k) return -1;
    int d = depth[a] - k;
    while (depth[chain[a]] > d) a = par[chain[a]];
    return preorder[in[a] - depth[a] + d];
  }
  int kth_on_path(int a, int b, int K) {
    int m = lca(a, b);
    int x = depth[a] - depth[m], y = depth[b] - depth[m];
    if (K > x + y) return -1;
    return (x > K ? climb(a, K) : climb(b, x + y - K));
  }
  // bool is true if path should be reversed (only for
      noncommutative operations)
  const vector<tuple<bool, int, int>>& get_path(int a, int b)
      const {
    static vector<tuple<bool, int, int>> L, R;
    L.clear(); R.clear();
    while (chain[a] != chain[b])
      if (depth[chain[a]] > depth[chain[b]]) {
        L.push_back({true, in[chain[a]], in[a] + 1});
        a = par[chain[a]];
      } else {
        R.push_back({false, in[chain[b]], in[b] + 1});
        b = par[chain[b]];
      }
    if (depth[a] > depth[b])
      L.push_back({true, in[b] + use_edges, in[a] + 1});
    else R.push_back({false, in[a] + use_edges, in[b] + 1});
    L.insert(L.end(), R.rbegin(), R.rend());
```

```
    return L;
  }
};
```

compress-tree.h

Description: Given a rooted tree and a subset S of nodes, compute the minimal subtree that contains all the nodes by adding all (at most $|S|-1$) pairwise LCA's and compressing edges. Returns a list of (par, orig_index) representing a tree rooted at 0. The root points to itself.
Time: $\mathcal{O}(|S|\log|S|)$

"LCA.h" ae0a91, 19 lines

```
vector<pair<int,int>> compressTree(lca_t &lca, const vector<int
    >& subset) {
  static vector<int> rev; rev.resize(lca.time.size());
  vector<int> li = subset, &T = lca.time;
  auto cmp = [&](int a, int b) { return T[a] < T[b]; };
  sort(li.begin(), li.end(), cmp);
  int m = li.size()-1;
  for (int i = 0; i < m; ++i) {
    int a = li[i], b = li[i+1];
    li.push_back(lca.lca(a, b));
  }
  sort(li.begin(), li.end(), cmp);
  li.erase(unique(li.begin(), li.end()), li.end());
  for (int i = 0; i < int(li.size()); ++i) rev[li[i]] = i;
  vector<pair<int,int>> ret = {{0, li[0]}};
  for (int i = 0; i < li.size()-1; ++i) {
    int a = li[i], b = li[i+1];
    ret.emplace_back(rev[lca.lca(a, b)], b);
  } return ret;
}
```

tree-isomorphism.h

Time: $\mathcal{O}(N\log(N))$

a4f6c1, 38 lines

```
struct tree_t {
  vector<int> cen, sz;
  vector<vector<int>> adj;
  tree_t(vector<vector<int>>& g):cen(2), sz(g.size()), adj(g){}
  int dfs_sz(int v, int p) {
    sz[v] = 1;
    for (int u : adj[v]) if (u != p)
      sz[v] += dfs_sz(u, v);
    return sz[v];
  }
  int dfs(int tsz, int v, int p) {
    for (int u : adj[v]) if (u != p) {
      if (2*sz[u] <= tsz) continue;
      return dfs(tsz, u, v);
    } return cen[0] = v;
  }
  void find_cenroid(int v) {
    int tsz = dfs_sz(v, -1);
    cen[1] = dfs(tsz, v, -1);
    for (int u : adj[cen[0]]) if (2*sz[u] == tsz)
      cen[1] = u;
  }
  int hash_it(int v, int p = -1) {
      static map<vector<int>, int> val;
    vector<int> offset;
    for (int u : adj[v]) if (u != p)
      offset.push_back(hash_it(u, v));
    sort(offset.begin(), offset.end());
    if (!val.count(offset))  val[offset] = int(val.size());
    return val[offset];
  }
  ll get_hash(int v = 0) {
    find_cenroid(v);
    ll x = hash_it(cen[0]), y=hash_it(cen[1]);
    if (x > y) swap(x, y);
    return (x << 30) + y;
  }
};
```

7.6.1 Sqrt Decomposition

HLD generally suffices. If not, here are some common strategies:

- Rebuild the tree after every $\sqrt{N}$ queries.
- Consider vertices with $>$ or $<$ $\sqrt{N}$ degree separately.
- For subtree updates, note that there are $O(\sqrt{N})$ distinct sizes among child subtrees of any vertex.

Block Tree: Use a DFS to split edges into contiguous groups of size $\sqrt{N}$ to $2\sqrt{N}$.

7.7 Other

manhattan-mst.h

Description: Given N points, returns up to 4*N edges, which are guaranteed to contain a minimum spanning tree for the graph with edge weights $w(p,q) = |p.x - q.x| + |p.y - q.y|$. Edges are in the form (distance, src, dst). Use a standard MST algorithm on the result to find the final MST.
Time: $\mathcal{O}(NlogN)$

<dsu.h> de8170, 24 lines

```
typedef Point<int> P;
pair<vector<array<int, 3>>, int> manhattanMST(vector<P> ps) {
  vector<int> id(ps.size()); iota(id.begin(), id.end(), 0);
  vector<array<int, 3>> edges;
  for(int k = 0; k < 4; ++k) {
    sort(id.begin(), id.end(), [&](int i, int j) {
      return (ps[i]-ps[j]).x < (ps[j]-ps[i]).y;});
    map<int, int> sweep;
    for(auto& i : id) {
      for (auto it = sweep.lower_bound(-ps[i].y);
        it != sweep.end(); sweep.erase(it++)) {
        int j = it->second; P d = ps[i] - ps[j];
        if (d.y > d.x) break;
        edges.push_back({d.y + d.x, i, j});
      } sweep[-ps[i].y] = i;
    }
    if (k & 1) for(auto& p : ps) p.x = -p.x;
    else for(auto& p : ps) swap(p.x, p.y);
  }
  sort(edges.begin(), edges.end());
  UF uf(ps.size()); int cost = 0;
  for (auto e: edges) if (uf.unite(e[1], e[2])) cost += e[0];
  return {edges, cost};
}
```

directed-mst.h

Description: Edmonds' algorithm for finding the weight of the minimum spanning tree/arborescence of a directed graph, given a root node. If no MST exists, returns -1.
Time: $\mathcal{O}(E\log V)$

"../data-structures/dsu-rollback.h" dedbb2, 56 lines

```
struct Edge { int a, b; ll w; };
struct Node {
  Edge key;
  Node *l, *r; ll delta;
  void prop() {
    key.w += delta;
    if (l) l->delta += delta;
    if (r) r->delta += delta;
    delta = 0;
  }
  Edge top() { prop(); return key; }
};
Node *merge(Node *a, Node *b) {
  if (!a || !b) return a ?: b;
  a->prop(), b->prop();
  if (a->key.w > b->key.w) swap(a, b);
  swap(a->l, (a->r = merge(b, a->r)));
  return a;
}
void pop(Node*& a) { a->prop(); a = merge(a->l, a->r); }
pair<ll, vector<int>> dmst(int n, int r, vector<Edge>& g) {
  RollbackUF uf(n);
  vector<Node*> heap(n);
  for (Edge e : g) heap[e.b] = merge(heap[e.b], new Node{e});
  ll res = 0;
  vector<int> seen(n, -1), path(n), par(n);
  seen[r] = r;
  vector<Edge> Q(n), in(n, {-1,-1}), comp;
  deque<tuple<int, int, vector<Edge>>> cycs;
  for(int s = 0; s < n; ++s) {
    int u = s, qi = 0, w;
    while (seen[u] < 0) {
      if (!heap[u]) return {-1,{}};
      Edge e = heap[u]->top();
      heap[u]->delta -= e.w, pop(heap[u]);
      Q[qi] = e, path[qi++] = u, seen[u] = s;
      res += e.w, u = uf.find(e.a);
      if (seen[u] == s) {
        Node* cyc = 0;
        int end = qi, time = uf.time();
        do cyc = merge(cyc, heap[w = path[--qi]]);
        while (uf.unite(u, w));
        u = uf.find(u), heap[u] = cyc, seen[u] = -1;
        cycs.push_front({u, time, {&Q[qi], &Q[end]}});
      }
    }
    for(int i = 0; i < qi; ++i) in[uf.find(Q[i].b)] = Q[i];
  }
  for (auto& [u,t,comp] : cycs) { // restore sol (optional)
    uf.rollback(t); Edge inEdge = in[u];
    for (auto& e : comp) in[uf.find(e.b)] = e;
    in[uf.find(inEdge.b)] = inEdge;
  }
  for(int i = 0; i < n; ++i) par[i] = in[i].a;
  return {res, par};
}
```

7.8 Theorems

7.8.1 Euler's theorem

Let V, A and F be the number of vertices, edges and faces of connected planar graph, $V - A + F = 2$

7.8.2 Menger's theorem

- Vertices: A graph is k-connected iff all pairwise vertices are connected to at least k internally disjoint paths.
- Edges: A graph is called k-edge-connected if the removal of at least k edges of the graph keeps it connected. A graph is k-edge-connected iff for all pairwise vertices u and v, exist k paths which link u to v without sharing an edge.

7.8.3 Matching

In any bipartite graph the following holds: Maximum matching and minimum vertex cover have the same cardinality. Minimum edge cover and minimum path cover have the same cardinality as the complement of the maximum matching.

7.8.4 Eulerian Cycles

The number of Eulerian cycles in a *directed* graph G is: $t_w(G)\prod_{v\in G}(\deg v-1)!$, where $t_w(G)$ is the number of arborescences ("directed spanning" tree) rooted at w (Check Number of Spanning Trees)

7.8.5 Dilworth's theorem

For any partially ordered set, the sizes of the max antichain and of the min chain decomposition are equal. Equivalent to Konig's theorem on the bipartite graph (U,V,E) where $U=V=S$ and (u,v) is an edge when $u<v$. Those vertices outside the min vertex cover in both U and V form a max antichain

Maximum Weight Closure

Given a vertex-weighted directed graph G. Turn the graph into a flow network, adding weight ∞ to each edge. Add vertices S,T. For each vertex v of weight w, add edge (S,v,w) if $w\geq 0$, or edge $(v,T,-w)$ if $w<0$. Sum of positive weights minus minimum $S-T$ cut is the answer. Vertices reachable from S are in the closure. The maximum-weight closure is the same as the complement of the minimum-weight closure on the graph with edges reversed.

7.8.6 Maximum Weighted Independent Set in a Bipartite Graph

This is the same as the minimum weighted vertex cover. Solve this by constructing a flow network with edges $(S,u,w(u))$ for $u\in L$, $(v,T,w(v))$ for $v\in R$ and (u,v,∞) for $(u,v)\in E$. The minimum S,T-cut is the answer. Vertices adjacent to a cut edge are in the vertex cover.

7.8.7 Number of Spanning Trees

Define Laplacian Matrix as $L=D-A$, D being a Diagonal Matrix with $D_{i,i}=deg(i)$ and A an Adjacency Matrix. Create an $N\times N$ Laplacian matrix `mat`, and for each edge $a\to b\in G$, do `mat[a][b]--, mat[b][b]++` (and `mat[b][a]--, mat[a][a]++` if G is undirected). Remove the ith row and column and take the determinant; this yields the number of directed spanning trees rooted at i (if G is undirected, remove any row/column).

7.8.8 Tutte Matrix

- A graph has a perfect matching iff the *Tutte* matrix has a non-zero determinant.
- The rank of the *Tutte* matrix is equal to twice the size of the maximum matching. The maximum cost matching can be found by polynomial interpolation.

Geometry (8)

8.1 Geometry

all-geometry.h

Description: Geometry 2D Library

5355d9, 543 lines

```
template <class T> int sgn(T x) { return (x > 0) - (x < 0); }
template<class T>
struct Point {
  typedef Point P;
  T x, y;
  explicit Point(T x=0, T y=0) : x(x), y(y) {}
  bool operator<(P p) const { return tie(x,y) < tie(p.x,p.y); }
  bool operator==(P p) const { return tie(x,y)==tie(p.x,p.y); }
  P operator+(P p) const { return P(x+p.x, y+p.y); }
  P operator-(P p) const { return P(x-p.x, y-p.y); }
  P operator*(T d) const { return P(x*d, y*d); }
  P operator/(T d) const { return P(x/d, y/d); }
  T dot(P p) const { return x*p.x + y*p.y; }
  T cross(P p) const { return x*p.y - y*p.x; }
  T cross(P a, P b) const { return (a-*this).cross(b-*this); }
  T dist2() const { return x*x + y*y; }
  double dist() const { return sqrt((double)dist2()); }
  // angle to x-axis in interval [-pi, pi]
  double angle() const { return atan2(y, x); }
  P unit() const { return *this/dist(); } // makes dist()=1
  P perp() const { return P(-y, x); } // rotates +90 degrees
  P normal() const { return perp().unit(); }
  // returns point rotated 'a' radians ccw around the origin
  P rotate(double a) const {
    return P(x*cos(a)-y*sin(a),x*sin(a)+y*cos(a)); }
};

using P = Point<double>;

// signed distance between point p and line (a, b)
template<class P>
double lineDist(const P& a, const P& b, const P& p) {
  return (double)(b-a).cross(p-a)/(b-a).dist();
}
// shortest distance between point p and line segment (s, e)
double segDist(P& s, P& e, P& p) {
  if (s==e) return (p-s).dist();
  auto d = (e-s).dist2(), t = min(d,max(.0,(p-s).dot(e-s)));
  return ((p-s)*d-(e-s)*t).dist()/d;
}
// intersection between two segments: Returns either the
// intersection point or the line segment (s,e)
// if there is infinitely many. Empty vector if no intersection
  .
template<class P> vector<P> segInter(P a, P b, P c, P d) {
  auto oa = c.cross(d, a), ob = c.cross(d, b),
     oc = a.cross(b, c), od = a.cross(b, d);
  // Checks if intersection is single non-endpoint point.
  if (sgn(oa) * sgn(ob) < 0 && sgn(oc) * sgn(od) < 0)
    return {(a * ob - b * oa) / (ob - oa)};
  set<P> s;
  if (onSegment(c, d, a)) s.insert(a);
  if (onSegment(c, d, b)) s.insert(b);
  if (onSegment(a, b, c)) s.insert(c);
  if (onSegment(a, b, d)) s.insert(d);
  return {s.begin(), s.end()};
}
// Same as above but returns only true or false
template<class P>
bool segmentIntersectionQ(P s1, P e1, P s2, P e2) {
  if (e1 == s1) {
    if (e2 == s2) return e1 == e2;
    swap(s1,s2); swap(e1,e2);
  }
  P v1 = e1-s1, v2 = e2-s2, d = s2-s1;
  auto a = v1.cross(v2), a1 = d.cross(v1), a2 = d.cross(v2);
  if (a == 0) { // parallel
    auto b1 = s1.dot(v1), c1 = e1.dot(v1),
       b2 = s2.dot(v1), c2 = e2.dot(v1);
    return !a1 && max(b1,min(b2,c2)) <= min(c1,max(b2,c2));
  }
  if (a < 0) { a = -a; a1 = -a1; a2 = -a2; }
  return (0 <= a1 && a1 <= a && 0 <= a2 && a2 <= a);
}
// {1, P} if there is a intersection, {-1, (0,0)} if
// there is infinitely, {0,(0,0)} otherwise.
template<class P>
pair<int, P> lineInter(P s1, P e1, P s2, P e2) {
  auto d = (e1 - s1).cross(e2 - s2);
  if (d == 0) // if parallel
    return {-(s1.cross(e1, s2) == 0), P(0, 0)};
  auto p = s2.cross(e1, e2), q = s2.cross(e2, s1);
  return {1, (s1 * p + e1 * q) / d};
}
// Projects point P onto line ab. Refl=true to get reflection
// of point P across line ab instead.
template<class P>
P lineProj(P a, P b, P p, bool refl=false) {
  P v = b - a;
  return p - v.perp()*(1+refl)*v.cross(p-a)/v.dist2();
}
// 1/0/-1 <=> left/on line/right
template<class P>
int sideOf(P s, P e, P p) { return sgn(s.cross(e, p)); }
template<class P>
int sideOf(const P& s, const P& e, const P& p, double eps) {
  auto a = (e-s).cross(p-s);
  double l = (e-s).dist()*eps;
  return (a > l) - (a < -l);
}
// True iff P lies on the line segment (s, e)
template<class P> bool onSegment(P s, P e, P p) {
  return p.cross(s, e) == 0 && (s - p).dot(e - p) <= 0;
}
// Linear transformation which takes line p0p1 to q0q1
P linearTransformation(const P& p0, const P& p1,
  const P& q0, const P& q1, const P& r) {
  P dp = p1-p0, dq = q1-q0, num(dp.cross(dq), dp.dot(dq));
  return q0 + P((r-p0).cross(num), (r-p0).dot(num))/dp.dist2();
}
// Useful utilities for dealing with angles of rays from origin
  .
template <class P>
bool sameDir(P s, P t) {
  return s.cross(t) == 0 && s.dot(t) > 0;
}
template <class P> // checks 180 <= s..t < 360?
bool isReflex(P s, P t) {
  auto c = s.cross(t);
  return c ? (c < 0) : (s.dot(t) < 0);
}
// operator < (s,t) for angles in [base,base+2pi)
template <class P>
bool angleCmp(P base, P s, P t) {
  int r = isReflex(base, s) - isReflex(base, t);
  return r ? (r < 0) : (0 < s.cross(t));
```

```cpp
}
// is x in [s,t] taken ccw? 1/0/-1 for in/border/out
template <class P>
int angleBetween(P s, P t, P x) {
  if (sameDir(x, s) || sameDir(x, t)) return 0;
  return angleCmp(s, x, t) ? 1 : -1;
}
int half(P p) { return p.y != 0 ? sgn(p.y) : -sgn(p.x); }
bool angle_cmp(P a, P b) { int A = half(a), B = half(b);
  return A == B ? a.cross(b) > 0 : A < B; }
// out is the pair of points at which two circles intersect
bool circleInter(P a,P b,double r1,double r2,pair<P, P>* out) {
  if (a == b) { assert(r1 != r2); return false; }
  P vec = b - a;
  double d2 = vec.dist2(), sum = r1+r2, dif = r1-r2,
       p = (d2 + r1*r1 - r2*r2)/(d2*2), h2 = r1*r1 - p*p*d2;
  if (sum*sum < d2 || dif*dif > d2) return false;
  P mid = a + vec*p, per = vec.perp() * sqrt(fmax(0, h2) / d2);
  *out = {mid + per, mid - per};
  return true;
}
// Finds the external tangents of two circles, or internal if
// r2 is negated, .first and .second gives the tangency point
// at circle 1 and 2 respec.
template<class P>
vector<pair<P, P>> tangents(P c1, double r1, P c2, double r2) {
  P d = c2 - c1;
  double dr = r1 - r2, d2 = d.dist2(), h2 = d2 - dr * dr;
  if (d2 == 0 || h2 < 0)  return {};
  vector<pair<P, P>> out;
  for (double sign : {-1, 1}) {
    P v = (d * dr + d.perp() * sqrt(h2) * sign) / d2;
    out.push_back({c1 + v * r1, c2 + v * r2});
  }
  if (h2 == 0) out.pop_back();
  return out;
}
// radius of the circle going through points A, B and C
double ccRadius(const P& A, const P& B, const P& C) {
  return (B-A).dist()*(C-B).dist()*(A-C).dist()/
    abs((B-A).cross(C-A))/2;
}
// Center of the circle above
P ccCenter(const P& A, const P& B, const P& C) {
  P b = C-A, c = B-A;
  return A + (b*c.dist2()-c*b.dist2()).perp()/b.cross(c)/2;
}
// min enclosing circle
pair<P, double> mec(vector<P> ps) { // ~O(N)
  shuffle(ps.begin(),ps.end(), mt19937(time(0)));
  P o = ps[0];
  double r = 0, EPS = 1 + 1e-8;
  for(int i = 0; i < ps.size(); ++i)
    if ((o - ps[i]).dist() > r * EPS) {
      o = ps[i], r = 0;
      for(int j = 0; j < i; ++j) if ((o-ps[j]).dist()>r*EPS){
        o = (ps[i] + ps[j]) / 2;
        r = (o - ps[i]).dist();
        for(int k = 0; k < j; ++k)
          if ((o - ps[k]).dist() > r * EPS) {
            o = ccCenter(ps[i], ps[j], ps[k]);
            r = (o - ps[i]).dist();
          }
      }
    }
  return {o, r};
}
// Intersection between circle and line ab, returns 0/1/2
    intersections
template<class P>
vector<P> circleLine(P c, double r, P a, P b) {
  double h2 = r*r - a.cross(b,c)*a.cross(b,c)/(b-a).dist2();
  if (h2 < 0) return {};
  P p = lineProj(a, b, c), h = (b-a).unit() * sqrt(h2);
  if (h2 == 0) return {p};
  return {p - h, p + h};
}
template<class P> // intersection area of two circles
double circleCircleArea(P c, double cr, P d, double dr) {
  if (cr < dr) swap(c, d), swap(cr, dr);
  auto A = [&](double r, double h) {
    return r*r*acos(h/r)-h*sqrt(r*r-h*h);
  };
  auto l = (c - d).dist(), a = (l*l + cr*cr - dr*dr)/(2*l);
  if (l - cr - dr >= 0) return 0; // far away
  if (l - cr + dr <= 0) return M_PI*dr*dr;
  if (l - cr >= 0) return A(cr, a) + A(dr, l-a);
  else return A(cr, a) + M_PI*dr*dr - A(dr, a-l);
}
// area of intersection between a circle and a ccw polygon
#define arg(p, q) atan2(p.cross(q), p.dot(q)) // (conc or conv)
double circlePoly(P c, double r, vector<P> ps) {
  auto tri = [&](P p, P q) {
    auto r2 = r * r / 2;
    P d = q - p;
    auto a = d.dot(p)/d.dist2(), b = (p.dist2()-r*r)/d.dist2();
    auto det = a * a - b;
    if (det <= 0) return arg(p, q) * r2;
    auto s = max(0., -a-sqrt(det)), t = min(1., -a+sqrt(det));
    if (t < 0 || 1 <= s) return arg(p, q) * r2;
    P u = p + d * s, v = p + d * t;
    return arg(p,u) * r2 + u.cross(v)/2 + arg(v,q) * r2;
  };
  auto sum = 0.0;
  for (int i = 0; i < ps.size(); ++i)
    sum += tri(ps[i] - c, ps[(i + 1) % ps.size()] - c);
  return sum;
}
// True if P lies within the polygon. If strict=true returns
// false for points on the boundary.
template<class P>
bool inPolygon(vector<P> &p, P a, bool strict = true) {
  int cnt = 0, n = p.size();
  for(int i = 0; i < n; ++i) {
    P q = p[(i + 1) % n];
    if (onSegment(p[i], q, a)) return !strict; // change to
          // -1 if u need to detect points in the boundary
    //or: if (segDist(p[i], q, a) <= eps) return !strict;
    cnt ^= ((a.y<p[i].y) - (a.y<q.y)) * a.cross(p[i], q) > 0;
  }
  return cnt;
}
template<class T>
T polygonArea(vector<Point<T>> &v) {
  T a = v.back().cross(v[0]);
  for(int i = 0; i < v.size()-1; ++i)
    a += v[i].cross(v[i+1]);
  return abs(a)/2.0;
}
Point<T> polygonCentroid(vector<Point<T>> &v) { // not tested
  Point<T> cent(0,0); T area = 0;
  for(int i = 0; i < v.size(); ++i) {
    int j = (i+1) % (v.size()); T a = cross(v[i], v[j]);
    cent += a * (v[i] + v[j]);
    area += a;
  }
  return cent/area/(T)3;
}
// Center of mass of a polygon
P polygonCenter(const vector<P>& v) {
  P res(0, 0); double A = 0;
  for (int i = 0, j = v.size() - 1; i < v.size(); j = ++i) {
    res = res + (v[i] + v[j]) * v[j].cross(v[i]);
    A += v[j].cross(v[i]);
  }
  return res / A / 3;
}
// Returns vertices of the polygon to the left of the cut (s,e)
vector<P> polygonCut(const vector<P>& poly, P s, P e) {
  vector<P> res;
  for(int i = 0; i < poly.size(); ++i) {
    P cur = poly[i], prev = i ? poly[i-1] : poly.back();
    bool side = s.cross(e, cur) < 0;
    if (side != (s.cross(e, prev) < 0))
      res.push_back(lineInter(s, e, cur, prev).second);
    if (side) res.push_back(cur);
  }
  return res;
}
// no collinear points allowed
vector<P> convexHull(vector<P> pts) {
  if (pts.size() <= 1) return pts;
  sort(pts.begin(), pts.end());
  vector<P> h(pts.size()+1);
  int s = 0, t = 0;
  for (int it = 2; it--; s = --t, reverse(pts.begin(), pts.end
      ()))
    for (P p : pts) {
      while (t >= s + 2 && h[t-2].cross(h[t-1], p) <= 0) t--;
      h[t++] = p;
    }
  return {h.begin(), h.begin() + t - (t == 2 && h[0] == h[1])};
}
// Two points with the max distance on convex hull (ccw, no
array<P, 2> hullDiameter(vector<P> S) { // duplicate/collinear)
  int n = S.size(), j = n < 2 ? 0 : 1;
  pair<lint, array<P, 2>> res({0, {S[0], S[0]}});
  for(int i = 0; i < j; ++i)
    for (;; j = (j + 1) % n) {
      res = max(res, {(S[i] - S[j]).dist2(), {S[i], S[j]}});
      if ((S[(j + 1) % n] - S[j]).cross(S[i + 1] - S[i]) >= 0)
        break;
    }
  return res.second;
}
// Checks whether P lies inside a convex hull.
bool inHull(const vector<P> &l, P p, bool strict = true) {
  int a = 1, b = l.size() - 1, r = !strict;
  if (l.size() < 3) return r && onSegment(l[0], l.back(), p);
  if (sideOf(l[0], l[a], l[b]) > 0) swap(a, b);
  if (sideOf(l[0], l[a], p) >= r || sideOf(l[0], l[b], p)<= -r)
    return false;
  while (abs(a - b) > 1) {
    int c = (a + b) / 2;
    (sideOf(l[0], l[c], p) > 0 ? b : a) = c;
  }
  return sgn(l[a].cross(l[b], p)) < r;
}
// O(N+M)
vector<P> minkowski_sum(vector<P> A, vector<P> B) {
  if (int(A.size()) > int(B.size())) swap(A, B);
  if (A.empty()) return {};
  if (int(A.size()) == 1) {
    for (auto& b : B) b = b + A.front();
    return B;
  }
  rotate(A.begin(), min_element(A.begin(), A.end()), A.end());
```

```
  rotate(B.begin(), min_element(B.begin(), B.end()), B.end());
  A.push_back(A[0]); A.push_back(A[1]);
  B.push_back(B[0]); B.push_back(B[1]);
  const int N = int(A.size()), M = int(B.size());
  vector<P> ans; ans.reserve(N+M);
  for (int i = 0, j = 0; i+2 < N || j+2 < M;) {
    ans.push_back(A[i] + B[j]);
    auto sgn = (A[i+1] - A[i]).cross(B[j+1] - B[j]);
    i += (sgn >= 0); j += (sgn <= 0);
  }
  return ans;
}
// Union area of polygons, must be given in ccw order
double rat(P a, P b) { return sgn(b.x) ? a.x/b.x : a.y/b.y; }
double polyUnion(vector<vector<P>>& poly) { // ~O(N^2)
  double ret = 0;
  for(int i = 0; i < poly.size(); ++i)
    for(int v = 0; v < poly[i].size(); ++v) {
      P A = poly[i][v], B = poly[i][(v + 1) % poly[i].size()];
      vector<pair<double, int>> segs = {{0, 0}, {1, 0}};
      for(int j = 0; j < poly.size(); ++j) if (i != j) {
        for(int u = 0; u < poly[j]; ++u) {
          P C = poly[j][u], D = poly[j][(u + 1) % poly[j].size
            ()];
          int sc = sideOf(A, B, C), sd = sideOf(A, B, D);
          if (sc != sd) {
            double sa = C.cross(D, A), sb = C.cross(D, B);
            if (min(sc, sd) < 0)
              segs.emplace_back(sa / (sa - sb), sgn(sc - sd));
          } else if (!sc && !sd && j<i && sgn((B-A).dot(D-C))
            >0){
            segs.emplace_back(rat(C - A, B - A), 1);
            segs.emplace_back(rat(D - A, B - A), -1);
          }
        }
      }
    }
    sort(segs.begin(), segs.end());
    for(auto& s : segs) s.first = min(max(s.first, 0.0), 1.0);
    double sum = 0;
    int cnt = segs[0].second;
    for(int j = 1; j < segs.size(); ++j) {
      if (!cnt) sum += segs[j].first - segs[j - 1].first;
      cnt += segs[j].second;
    }
    ret += A.cross(B) * sum;
  }
  return ret / 2;
}
// Intersection between a line and a convex polygon (given ccw)
  .
typedef array<P, 2> Line;
#define cmp(i,j) sgn(dir.perp().cross(poly[(i)%n]-poly[(j)%n]))
#define extr(i) cmp(i + 1, i) >= 0 && cmp(i, i - 1 + n) < 0
int extrVertex(vector<P>& poly, P dir) {
  int n = poly.size(), left = 0, right = n;
  if (extr(0)) return 0;
  while (left + 1 < right) {
    int m = (left + right) / 2;
    if (extr(m)) return m;
    int ls = cmp(left + 1, left), ms = cmp(m + 1, m);
    (ls < ms || (ls == ms && ls == cmp(left, m)) ? right : left
      ) = m;
  }
  return left;
}

#define cmpL(i) sgn(line[0].cross(poly[i], line[1]))
array<int, 2> lineHull(Line line, vector<P>& poly) {
  int endA = extrVertex(poly, (line[0] - line[1]).perp());
  int endB = extrVertex(poly, (line[1] - line[0]).perp());
  if (cmpL(endA) < 0 || cmpL(endB) > 0)
    return {-1, -1};
  array<int, 2> res;
  for(int i = 0; i < 2; ++i) {
    int left = endB, right = endA, n = poly.size();
    while ((left + 1) % n != right) {
      int m = ((left + right + (left < right ? 0 : n)) / 2) % n
        ;
      (cmpL(m) == cmpL(endB) ? left : right) = m;
    }
    res[i] = (left + !cmpL(right)) % n;
    swap(endA, endB);
  }
  if (res[0] == res[1]) return {res[0], -1};
  if (!cmpL(res[0]) && !cmpL(res[1]))
    switch ((res[0] - res[1] + sz(poly) + 1) % poly.size()) {
      case 0: return {res[0], res[0]};
      case 2: return {res[1], res[1]};
    }
  return res;
}
// Halfplace intersection area
#define eps 1e-8
typedef Point<double> P;
struct Line {
  P P1, P2;
  // Right hand side of the ray P1 -> P2
  explicit Line(P a = P(), P b = P()) : P1(a), P2(b) {};
  P intpo(Line y) {
    pair<int, P> r = lineInter(P1, P2, y.P1, y.P2);
    assert(r.first == 1);
    return r.second;
  }
  P dir() { return P2 - P1; }
  bool contains(P x) {
    return (P2 - P1).cross(x - P1) < eps;
  }
  bool out(P x) { return !contains(x); }
};
template<class T>
bool mycmp(Point<T> a, Point<T> b) {
  // return atan2(a.y, a.x) < atan2(b.y, b.x);
  if (a.x * b.x < 0) return a.x < 0;
  if (abs(a.x) < eps) {
    if (abs(b.x) < eps) return a.y > 0 && b.y < 0;
    if (b.x < 0) return a.y > 0;
    if (b.x > 0) return true;
  }
  if (abs(b.x) < eps) {
    if (a.x < 0) return b.y < 0;
    if (a.x > 0) return false;
  }
  return a.cross(b) > 0;
}
bool cmp(Line a, Line b) { return mycmp(a.dir(), b.dir()); }
double Intersection_Area(vector <Line> b) {
  sort(b.begin(), b.end(), cmp);
  int n = b.size();
  int q = 1, h = 0, i;
  vector<Line> c(b.size() + 10);
  for (i = 0; i < n; i++) {
    while (q < h && b[i].out(c[h].intpo(c[h - 1]))) h--;
    while (q < h && b[i].out(c[q].intpo(c[q + 1]))) q++;
    c[++h] = b[i];
    if (q < h && abs(c[h].dir().cross(c[h - 1].dir())) < eps) {
      if (c[h].dir().dot(c[h - 1].dir()) > 0) {
        h--;
        if (b[i].out(c[h].P1))  c[h] = b[i];
      }else {
        // The area is either 0 or infinite.
        // If you have a bounding box,
        return 0; // then the area is definitely 0.
      }
    }
  }
  while (q < h-1 && c[q].out(c[h].intpo(c[h - 1]))) h--;
  while (q < h-1 && c[h].out(c[q].intpo(c[q + 1]))) q++;
  // Intersection is empty. This is sometimes different from
  // the case when the intersection area is 0.
  if (h - q <= 1) return 0;
  c[h + 1] = c[q];
  vector<P> s;
  for (i = q; i <= h; i++) s.push_back(c[i].intpo(c[i + 1]));
  s.push_back(s[0]);
  double ans = 0;
  for (i = 0; i < (int) s.size()-1; i++) ans += s[i].cross(s[i
    + 1]);
  return ans / 2;
}
// Closes pair of points. O(N logN)
pair<P, P> closest(vector<P> v) {
  assert(v.size() > 1);
  set<P> S;
  sort(v.begin(), v.end(), [](P a, P b) { return a.y < b.y; });
  pair<int64_t, pair<P, P>> ret{LLONG_MAX, {P(), P()}};
  int j = 0;
  for(P &p : v) {
    P d{1 + (int64_t)sqrt(ret.first), 0};
    while (v[j].y <= p.y - d.x) S.erase(v[j++]);
    auto lo = S.lower_bound(p - d), hi = S.upper_bound(p + d);
    for (; lo != hi; ++lo)
      ret = min(ret, {(*lo - p).dist2(), {*lo, p}});
    S.insert(p);
  }
  return ret.second;
}
// Rectangle union area
struct seg_node{
  int val, cnt, lz;
  seg_node(int n = INF, int c = 0): val(n), cnt(c), lz(0) {}
  void push(seg_node& l, seg_node& r){
    if(lz){
      l.add(lz); r.add(lz); lz = 0;
    }
  }
  void merge(const seg_node& l, const seg_node& r){
    if(l.val < r.val) val = l.val, cnt = l.cnt;
    else if(l.val > r.val) val = r.val, cnt = r.cnt;
    else val = l.val, cnt = l.cnt + r.cnt;
  }
  void add(int n){
    val += n; lz += n;
  }
  int get_sum(){ return (val ? 0 : cnt); }
};
// x1 y1 x2 y2
lint solve(const vector<array<int, 4>>&v){
  vector<int>ys;
  for(auto& [a, b, c, d] : v){
    ys.push_back(b); ys.push_back(d);
  }
  sort(ys.begin(), ys.end());
  ys.erase(unique(ys.begin(), ys.end()), ys.end());
  vector<array<int, 4>>e;
  for(auto [a, b, c, d] : v){
    b = int(lower_bound(ys.begin(), ys.end(), b) - ys.begin());
    d = int(lower_bound(ys.begin(), ys.end(), d) - ys.begin());
```

```
    e.push_back({a, b, d, 1}); e.push_back({c, b, d, -1});
  }
  sort(e.begin(), e.end()); int m = (int)ys.size();
  segtree_range<seg_node>seg(m-1);
  for(int i=0;i<m-1;i++) seg.at(i) = seg_node(0, ys[i+1] - ys[i
      ]);
  seg.build();
  int last = INT_MIN, total = ys[m-1] - ys[0]; lint ans = 0;
  for(auto [x, y1, y2, c] : e){
    ans += (lint)(total - seg.query(0, m-1).get_sum()) * (x -
        last);
    last = x; seg.update(y1, y2, &seg_node::add, c);
  }
  return ans;
}
```

Strings (9)

kmp.h

Description: failure[x] computes the length of the longest prefix of s that ends at x, other than s[0...x] itself (abacaba -> -1,0,0,1,0,1,2,3). Can be used to find all occurrences of a pattern in a text.
Time: $\mathcal{O}(n)$

4eb73b, 9 lines

```
vector<int> prefix_function(const string& S) {
  vector<int> fail = {-1}; fail.reserve(S.size());
  for (int i = 0; i < int(S.size()); ++i) {
    int j = fail.back();
    while (j != -1 && S[i] != S[j]) j = fail[j];
    fail.push_back(j+1);
  }
  return fail;
}
```

duval.h

Description: A string is called simple (or a Lyndon word), if it is strictly smaller than any of its own nontrivial suffixes.
Time: $\mathcal{O}(N)$

19ec0f, 24 lines

```
template<typename T>
pair<int, vector<string>> duval(int n, const T &s) {
  // s += s  // if you need to know the min cyclic string
  vector<string> factors;
  int i = 0, ans = 0;
  while (i < n) { // until n/2 to find min cyclic string
    ans = i; int j = i + 1, k = i;
    while (j < n + n && !(s[j % n] < s[k % n])) {
      if (s[k % n] < s[j % n]) k = i;
      else k++;
      j++;
    }
    while (i <= k) {
      factors.push_back(s.substr(i, j-k));
      i += j - k;
    }
  }
  return {ans, factors};
  // returns 0-indexed position of the least cyclic shift
  // min cyclic string will be s.substr(ans, n/2)
}
template<typename T>pair<int,vector<string>> duval(const T &s){
  return duval((int)s.size(), s);
}
```

z-algorithm.h

Description: z[x] computes the length of the longest common prefix of s[i:] and s, except z[0] = 0. (abacaba -> 0010301)
Time: $\mathcal{O}(n)$

7c8c64, 13 lines

```
vector<int> Z(const string& S) {
  vector<int> z(S.size()); int l = -1, r = -1;
  for(int i = 1; i < int(S.size()); ++i) {
    z[i] = i >= r ? 0 : min(r - i, z[i - l]);
    while (i + z[i] < int(S.size()) && S[i + z[i]] == S[z[i]])
      z[i]++;
    if (i + z[i] > r) l = i, r = i + z[i];
  } return z;
}
vector<int> get_prefix(string a, string b) {
  string str = a + '@' + b; vector<int> k = z(str);
  return vector<int>(k.begin() + int(a.size())+1, k.end());
}
```

manacher.h

Description: For each position in a string, computes p[0][i] = half length of longest even palindrome around pos i, p[1][i] = longest odd (half rounded down).
Time: $\mathcal{O}(N)$

87e1f0, 12 lines

```
array<vector<int>, 2> manacher(const string &s) {
  int n = s.size();
  array<vector<int>, 2> p = {vector<int>(n+1), vector<int>(n)};
  for(int z = 0; z < 2; ++z) for(int i=0,l=0,r=0; i < n; i++) {
    int t = r-i+!z;
    if (i<r) p[z][i] = min(t, p[z][l+t]);
    int L = i-p[z][i], R = i+p[z][i]-!z;
    while (L>=1 && R+1<n && s[L-1] == s[R+1])
      p[z][i]++, L--, R++;
    if (R > r) l = L, r = R;
  } return p;
}
```

min-rotation.h

Description: Finds the lexicographically smallest rotation of a string.
Usage: `rotate(v.begin(), v.begin()+min_rotation(v), v.end());`
Time: $\mathcal{O}(N)$

2a08fd, 7 lines

```
int min_rotation(string s) {
  int a=0, N=s.size(); s += s;
  for(int b = 0; b < N; ++b) for(int i =0; i < N; ++i) {
    if (a+i == b || s[a+i] < s[b+i]) {b += max(0, i-1); break;}
    if (s[a+i] > s[b+i]) { a = b; break; }
  } return a;
}
```

aho-corasick.h

61f5e3, 36 lines

```
const int sigma = 26;
array<int, sigma> init;
for (int i = 0; i < sigma; i++) init[i] = -1;
vector<array<int, sigma>> trie(1, init);
vector<int> out(1, -1), parent(n, -1), ids(n);
for (int i = 0; i < n; i++) {
  int cur = 0;
  for (char ch : s[i]) {
    int c = ch - 'a';
    if (trie[cur][c] == -1) {
      trie[cur][c] = (int)trie.size();
      trie.push_back(init); out.push_back(-1);
    }
    cur = trie[cur][c];
  }
  if (out[cur] == -1) out[cur] = i;
  ids[i] = out[cur];
}
vector<int> bfs,f(trie.size()); bfs.reserve(trie.size());
for (int c = 0; c < sigma; c++)
  if (trie[0][c] == -1) trie[0][c] = 0;
  else bfs.push_back(trie[0][c]);
for (int z = 0; z < (int)bfs.size() ; z++) {
  int cur = bfs[z];
  for (int c = 0; c < sigma; c++) {
    if (trie[cur][c] == -1)
      trie[cur][c] = trie[f[cur]][c];
    else {
      int nxt = trie[cur][c];
      int fail = trie[f[cur]][c];
      if (out[nxt] == -1) out[nxt] = out[fail];
      else parent[out[nxt]] = out[fail];
      f[nxt] = fail; bfs.push_back(nxt);
    }
  }
}
```

suffix-array.h

Description: Builds suffix array for a string, first element is the size of the string. The `lcp` function calculates longest common prefixes for neighbouring strings in suffix array. The returned vector is of size $n + 1$.
Time: $\mathcal{O}(N \log N)$ where N is the length of the string for creation of the SA. $\mathcal{O}(N)$ for longest common prefixes.

<../data-structures/rmq.h>, <../various/random-numbers.h> d31829, 66 lines

```
struct suffix_array_t {
  int N, H; vector<int> sa, invsa, lcp;
  rmq_t<pair<int, int>> rmq;
  bool cmp(int a, int b) { return invsa[a+H] < invsa[b+H]; }
  void ternary_sort(int a, int b) {
    if (a == b) return;
    int md = sa[a+rng() % (b-a)], lo = a, hi = b;
    for (int i = a; i < b; ++i) if (cmp(sa[i], md))
      swap(sa[i], sa[lo++]);
    for (int i = b-1; i >= lo; --i) if (cmp(md, sa[i]))
      swap(sa[i], sa[--hi]);
    ternary_sort(a, lo);
    for (int i = lo; i < hi; ++i) invsa[sa[i]] = hi-1;
    if (hi-lo == 1) sa[lo] = -1;
    ternary_sort(hi, b);
  }
  suffix_array_t() {}
  template<typename I>
  suffix_array_t(I begin, I end): N(int(end-begin)+1), sa(N) {
    vector<int> v(begin, end); v.push_back(INT_MIN);
    invsa = v; iota(sa.begin(), sa.end(), 0);
    H = 0; ternary_sort(0, N);
    for(H = 1; H <= N; H *= 2) for(int j=0, i=j; i!=N; i=j)
        if (sa[i] < 0) {
          while (j < N && sa[j] < 0) j += -sa[j];
          sa[i] = -(j - i);
        } else {j = invsa[sa[i]] + 1; ternary_sort(i, j);}
    for (int i = 0; i < N; ++i) sa[invsa[i]] = i;
    lcp.resize(N-1); int K = 0;
    for (int i = 0; i < N-1; ++i) {
      if(invsa[i] > 0) while(v[i+K] == v[sa[invsa[i]-1]+K])++K;
      lcp[invsa[i]-1] = K; K = max(K - 1, 0);
    }
    vector<pair<int, int>> lcp_index(N-1);
    for (int i = 0; i < N-1; ++i) lcp_index[i] = {lcp[i], 1+i};
    rmq = rmq_t<pair<int, int>>(std::move(lcp_index));
  }
  auto rmq_query(int a, int b) const {return rmq.query(a,b);}
  auto get_split(int a, int b) const {return rmq.query(a,b-1);}
  int get_lcp(int a, int b) const {
    if (a == b) return N - a;
    a = invsa[a], b = invsa[b];
    if (a > b) swap(a, b);
    return rmq_query(a, b).first;
  }
```

```
};
vector<vector<int>> ch(2*N+1); int V = 0;
vector<array<int, 2>> sa_range(2*N+1);
vector<int> leaves(N+1), par(2*N+1), depth(2*N+1);
auto dfs = [&](auto&& self, int lo, int hi, int prv)-> void{
  int cur = V++; par[cur] = prv;
  if (prv != -1) ch[prv].push_back(cur);
  sa_range[cur] = {lo, hi};
  if (hi - lo == 1) {
    leaves[us.sa[lo]] = cur;
    depth[cur] = N-us.sa[lo] + 1;
  } else {
    int d = us.get_split(lo, hi).first;
    depth[cur] = d; int mi = lo;
    while (hi - mi >= 2) {
      auto [nd, nmi] = us.get_split(mi, hi);
      if (nd != d) break;
      self(self, mi, nmi, cur); mi = nmi;
    } self(self, mi, hi, cur);
  }
}; dfs(dfs, 0, N+1, -1);
```

suffix-automaton.h

Description: Suffix automaton

defb60, 33 lines

```
template<int offset = 'a'> struct array_state {
  array<int, 26> as;
  array_state() { fill(begin(as), end(as), ~0); }
  int& operator[](char c) { return as[c - offset]; }
  int count(char c) { return (~as[c - offset] ? 1 : 0); }
};
template<typename C, typename state = map<C, int>> struct
    suffix_automaton {
  struct node_t {
    int len, link; int64_t cnt; state next;
  };
  int N, cur; vector<node_t> nodes;
  suffix_automaton() : N(1), cur(0), nodes{node_t{0, -1, 0,
      {}}} {}
  node_t& operator[](int v) { return nodes[v]; };
  void append(C c) {
    int v = cur; cur = N++;
    nodes.push_back(node_t{nodes[v].len + 1, 0, 1, {}});
    for (; ~v && !nodes[v].next.count(c); v = nodes[v].link)
      nodes[v].next[c] = cur;
    if (~v) {
      const int u = nodes[v].next[c];
      if (nodes[v].len + 1 == nodes[u].len) {
        nodes[cur].link = u;
      } else {
        const int clone = N++;
        nodes.push_back(nodes[u]);
        nodes[clone].len = nodes[v].len + 1;
        nodes[u].link = nodes[cur].link = clone;
        for (; ~v && nodes[v].next[c] == u; v = nodes[v].link)
          nodes[v].next[c] = clone;
      }
    }
  }
};
```

9.1 Suffix Automaton

9.1.1 Number of different substrings

Is the number of paths in the automaton starting at the root. $d(v) = 1 + \sum_{v \to w} d(w)$

9.1.2 Total length of different substrings

Is the sum of children answers and paths starting at each children. $ans(v) = \sum_{v \to w} d(w) + ans(w)$

9.1.3 Lexicographically K-th substring

Is the K-th lexicographically path, so you can search using the number of paths from each state

9.1.4 Smallest cyclic shift

Construct for string $S + S$. Greedily search the minimal character.

9.1.5 Number of occurrences

For each state not created by cloning, initialize $cnt(v) = 1$. Then, just do a dfs to calculate $cnt(v)$, with $cnt(link(v)) += cnt(v)$

9.1.6 First occurence position

When we create a new state cur do $first(pos) = len(cur) - 1$. When we clone q as $clone$ do $first(clone) = first(q)$. Answer is $first(v) - size(P) + 1$, where v is the state of string P

9.1.7 All occurence positions

From $first(v)$ do a dfs using suffix link, from $link(u)$ go to u.

Various (10)

10.1 Intervals

interval-container.h

Description: Add and remove intervals from a set of disjoint intervals. Will merge the added interval with any overlapping intervals in the set when adding. Intervals are [inclusive, exclusive).

Time: $\mathcal{O}(\log N)$

edce47, 20 lines

```
set<pii>::iterator addInterval(set<pii> &is, int L, int R) {
  if (L == R) return is.end();
  auto it = is.lower_bound({L, R}), before = it;
  while (it != is.end() && it->first <= R) {
    R = max(R, it->second);
    before = it = is.erase(it);
  }
  if (it != is.begin() && (--it)->second >= L) {
    L = min(L, it->first); R = max(R, it->second);
    is.erase(it);
  } return is.insert(before, {L,R});
}
void removeInterval(set<pii> &is, int L, int R) {
  if (L == R) return;
  auto it = addInterval(is, L, R);
  auto r2 = it->second;
  if (it->first == L) is.erase(it);
  else (int&)it->second = L;
  if (R != r2) is.emplace(R, r2);
}
```

interval-cover.h

Description: Compute indices of smallest set of intervals covering another interval. Intervals should be [inclusive, exclusive). To support [inclusive, inclusive], change (A) to add `|| R.empty()`. Returns empty set on failure (or if G is empty).

Time: $\mathcal{O}(N \log N)$

133eb4, 17 lines

```
template<class T>
vector<int> cover(pair<T, T> G, vector<pair<T, T>> I) {
  vector<int> S(I.size()), R;
  iota(S.begin(), S.end(), 0);
  sort(S.begin(), S.end(), [&](int a, int b) {
    return I[a] < I[b]; });
  T cur = G.first; int at = 0;
  while (cur < G.second) { // (A)
    pair<T, int> mx = {cur, -1};
    while (at < I.size() && I[S[at]].first <= cur) {
      mx = max(mx, {I[S[at]].second, S[at]});
      at++;
    }
    if (mx.second == -1) return {};
    cur = mx.first; R.push_back(mx.second);
  } return R;
}
```

constant-intervals.h

Description: Split a monotone function on [from, to) into a minimal set of half-open intervals on which it has the same value. Runs a callback g for each such interval.

Usage: `constantIntervals(0, sz(v), [&](int x){return v[x];}, [&](int lo, int hi, T val){...});`

Time: $\mathcal{O}\left(k \log \frac{n}{k}\right)$

753a4c, 17 lines

```
template<class F, class G, class T>
void rec(int from, int to, F& f, G& g, int& i, T& p, T q) {
  if (p == q) return;
  if (from == to) {
    g(i, to, p); i = to; p = q;
  } else {
    int mid = (from + to) >> 1;
    rec(from, mid, f, g, i, p, f(mid));
    rec(mid+1, to, f, g, i, p, q);
  }
}
template<class F, class G>
void constantIntervals(int from, int to, F f, G g) {
  if (to <= from) return;
  int i = from; auto p = f(i), q = f(to-1);
  rec(from, to-1, f, g, i, p, q); g(i, to, q);
}
```

10.2 Misc. algorithms

ternary-search.h

Description: Find the smallest i in $[a, b]$ that maximizes $f(i)$, assuming that $f(a) < \dots < f(i) \geq \dots \geq f(b)$. To reverse which of the sides allows non-strict inequalities, change the $<$ marked with (A) to $<=$, and reverse the loop at (B). To minimize f, change it to $>$, also at (B). If you are dealing with real numbers, you'll need to pick $m_1 = (2a + b)/3.0$ and $m_2 = (a + 2b)/3.0$. Consider setting a constant number of iterations for the search, usually [200, 300] iterations are sufficient for problems with error limit as 10^{-6}.

Usage: `int ind = ternSearch(0,n-1,[&](int i){return a[i];});`

Time: $\mathcal{O}(\log(b - a))$

35ef73, 11 lines

```
template<class F> int ternSearch(int a, int b, F f) {
  assert(a <= b);
  while (b - a >= 5) {
    int mid = (a + b) / 2;
    if (f(mid) < f(mid+1)) a = mid; // (A)
    else b = mid+1;
  }
  for(int i = a+1; i <= b; ++i)
    if (f(a) < f(i)) a = i; // (B)
  return a;
}
```

count-triangles.h

Description: Counts x, y >= 0 such that Ax + By <= C.

fbb812, 6 lines

```
ll count_triangle(ll A, ll B, ll C) {
  if (C < 0) return 0;
  if (A > B) swap(A, B);
  ll p = C / B, k = B / A, d = (C - p * B) / A;
  return count_triangle(B - k * A, A, C - A * (k * p + d + 1))
      + (p + 1) * (d + 1) + k * p * (p + 1) / 2;
}
```

floyd-cycle.h

Description: Detect loop in a list. Consider using mod template to avoid overflow.

Time: $\mathcal{O}(n)$

b456ab, 9 lines

```
template<class F> pair<int,int> find(int x0, F f) {
  int t = f(x0), h = f(t), mu = 0, lam = 1;
  while (t != h) t = f(t), h = f(f(h));
  h = x0;
  while (t != h) t = f(t), h = f(h), ++mu;
  h = f(t);
  while (t != h) h = f(h), ++lam;
  return {mu, lam};
}
```

10.3 Dynamic programming

divide-and-conquer-dp.h

Description: Given $a[i] = \min_{lo(i) \le k < hi(i)}(f(i,k))$ where the (minimal) optimal k increases with i, computes $a[i]$ for $i = L..R-1$.

Time: $\mathcal{O}((N + (hi - lo)) \log N)$

c7f2d1, 29 lines

```
struct DP { // Modify at will:
  vector<int>a, freq;
  vector<ll>old, cur;
  ll cnt; int lcur, rcur;
  DP(const vector<int>&_a, int n): a(_a), freq(n), old(n+1,
      linf), cur(n+1, linf), cnt(0), lcur(0), rcur(0){}
  int lo(int ind) { return 0; }
  int hi(int ind) { return ind; }
  void add(int k, int c){ cnt += freq[a[k]]++; }
  void del(int k, int c){ cnt -= --freq[a[k]]; }
  ll C(int l, int r){
    while(lcur > l) add(--lcur, 0);
    while(rcur < r) add(rcur++, 1);
    while(lcur < l) del(lcur++, 0);
    while(rcur > r) del(--rcur, 1);
    return cnt;
  }
  ll f(int ind, int k) { return old[k] + C(k, ind); }
  void store(int ind, int k, ll v) { cur[ind] = v; }
  void rec(int L, int R, int LO, int HI) {
    if (L >= R) return;
    int mid = (L + R) >> 1;
    pair<ll, int> best(LLONG_MAX, LO);
    for(int k = max(LO,lo(mid)); k <= min(HI,hi(mid)); ++k)
      best = min(best, make_pair(f(mid, k), k));
    store(mid, best.second, best.first);
    rec(L, mid, LO, best.second);
    rec(mid+1, R, best.second, HI);
  }
};
```

knuth-dp.h

Description: When doing DP on intervals: $a[i][j] = \min_{i<k<j}(a[i][k] + a[k][j]) + f(i,j)$, where the (minimal) optimal k increases with both i and j, one can solve intervals in increasing order of length, and search $k = p[i][j]$ for $a[i][j]$ only between $p[i][j-1]$ and $p[i+1][j]$. This is known as Knuth DP. Sufficient criteria for this are if $f(b,c) \le f(a,d)$ and $f(a,c) + f(b,d) \le f(a,d) + f(b,c)$ for all $a \le b \le c \le d$. Consider also: LineContainer (ch. Data structures), monotone queues, ternary search.

Time: $\mathcal{O}(N^2)$

digit-dp.h

Description: Compute how many # between 1 and N have K distinct digits in the base L without leading zeros;

Usage: auto hex_to_dec = [&](char c) -> int {
return ('A' <= c && c <= 'F' ? (10 + c - 'A') : (c - '0'));
};
digit_dp<modnum<int(1e9) + 7>, hex_to_dec>(N, K);

Time: $\mathcal{O}(NK)$

8138af, 26 lines

```
template<typename T, class F> T digit_dp(const string& S, int K
    , F& L) {
  const int base = 16, len = int(S.size());
  vector<bool> w(base);
  vector<vector<T>> dp(len + 1, vector<T>(base + 2));
  int cnt = 0;
  for (int d = 0; d < len; ++d) {
    // adding new digit to numbers with prefix < s
    for (int x = 0; x <= base; ++x) {
      dp[d + 1][x] += dp[d][x] * x;
      dp[d + 1][x + 1] += dp[d][x] * (base - x);
    }
    // adding strings whith prefix only 0's and last digit != 0
    if (d) dp[d + 1][1] += (base - 1);
    // adding prefix equal to s and last digit < s, first digit
        cannot be 0
    for (int x = 0; x < L(S[d]); ++x) {
      if (d == 0 && x == 0) continue;
      if (w[x]) dp[d + 1][cnt] += 1;
      else dp[d + 1][cnt + 1] += 1;
    }
    // marking if the last digit appears in the prefix of s
    if (w[L(S[d])] == false)
      w[L(S[d])] = true, cnt++;
  }
  // adding string k
  dp[len][cnt] += 1; return dp[len][K];
}
```

knapsack-bounded.h

Description: You are given n types of items, each items has a weight and a quantity. Is possible to fill a knapsack with capacity X using any subset of items?

Time: $\mathcal{O}(W \cdot N)$

b24799, 11 lines

```
auto solve(vi weight, vi cnt, int X) {
  vector<int> dp(X+1, 0);
  for (int i = 0; i < N; ++i)
    for (int j = X-weight[i]; j >= 0; --j) {
      if (!dp[j]) continue;
      int k = cnt[i], s = j + weight[i];
      while (k > 0 && s <= X && !dp[s])
        dp[s] = 1, --k, s += weight[i];
    }
  return dp[X];
}
```

knapsack-bounded-costs.h

Description: You are given N types of items, you have $cnt[i]$ items of i-th type, and each item of i-th type $weight[i]$ and $cost[i]$. What is the maximal cost you can get by picking some items weighing exactly X in total?

Time: $\mathcal{O}(N \cdot W)$

"../data-structures/monotonic-queue.h" 03d171, 14 lines

```
auto solve(vi weight, vi cost, vi cnt, int X) {
  vector<int> dp(X+1, 0); int N = int(weight.size());
  vector<max_monotonic_queue<int>> M(X+1);
  for (int i = 0; i < N; ++i) {
    for (int j = 0; j < min(X+1, weight[i]); ++j) M[j] =
        max_monotonic_queue<int>();
    for (int j = 0; j <= X; ++j) {
      auto& que = M[j % weight[i]];
      if (que.size() > cnt[i]) que.pop();
      que.add(cost[i]);
      que.push(dp[j]);
      dp[j] = que.get_val();
    }
  } return dp[X];
}
```

two-max-equal-sum.h

Description: Two maximum equal sum disjoint subsets, s[i] = 0 if v[i] wasn't selected, s[i] = 1 if v[i] is in the first subset and s[i] = 2 if v[i] is in the second subset

Time: $\mathcal{O}(n * S)$

d66110, 15 lines

```
pair<int, vector<int>> twoMaxEqualSumDS(vector<int> &v){
  const int n = int(v.size());
  const int sum = accumulate(v.begin(), v.end(), 0);
  vector<int> dp(2*sum + 1, INT_MIN/2), newdp(2*sum + 1), s(n);
  vector<vector<int>> rec(n, vector<int>(2*sum + 1));
  int i; dp[sum] = 0;
  for(i = 0; i < n; i++, swap(dp, newdp))
    for(int a, b, d = v[i]; d <= 2*sum - v[i]; d++){
      newdp[d] = max({dp[d], a = dp[d - v[i]] + v[i], b = dp[d
          + v[i]]});
      rec[i][d] = newdp[d] == a ? 1 : newdp[d] == b ? 2 : 0;
    }
  for(int j = i-1, d = sum; j >= 0 ; j--)
    d += (s[j] = rec[j][d]) ? s[j] == 2 ? v[j] : -v[j] : 0;
  return {dp[sum], s};
}
```

10.4 Optimization tricks

10.4.1 Bit hacks

- `c = x&-x, r = x+c; (((r^x) >> 2)/c) | r` is the next number after `x` with the same number of bits set.
- `rep(b,0,K) rep(i,0,(1 << K)) if (i & 1 << b) D[i] += D[i^(1 << b)];` computes all sums of subsets.

hashmap.h

Description: Faster/better hash maps, taken from CF

09a72f, 17 lines

```
#include<bits/extc++.h>
struct splitmix64_hash {
  static uint64_t splitmix64(uint64_t x) {
    x += 0x9e3779b97f4a7c15;
    x = (x^(x >> 30)) * 0xbf58476d1ce4e5b9;
    x = (x^(x >> 27)) * 0x94d049bb133111eb;
    return x^(x >> 31);
  }
  size_t operator()(uint64_t x) const {
```

```
    static const uint64_t FIXED_RANDOM = std::chrono::
        steady_clock::now().time_since_epoch().count();
    return splitmix64(x + FIXED_RANDOM);
  }
};
template <typename K, typename V, typename Hash =
    splitmix64_hash>
using hash_map = __gnu_pbds::gp_hash_table<K, V, Hash>;
template <typename K, typename Hash = splitmix64_hash>
using hash_set = hash_map<K, __gnu_pbds::null_type, Hash>;
```

10.5 Bit Twiddling Hack

hacks.h

829b7d, 21 lines

```
// Iterate on non-empty submasks of a bitmask.
for (int s = m; s > 0; s = (m & (s - 1)))
// Iterate on non-zero bits of a bitset B.
for (int j = B._Find_next(0); j < MAXV; j = B._Find_next(j))
ll next_perm(ll v) { // compute next perm i.e.
  ll t = v | (v-1);  // 00111,01011,01101,10011 ...
  return (t + 1) | (((~t & -~t) - 1)>>(__builtin_ctz(v) + 1));
}
template<typename F>  // All subsets of size k of {0..N-1}
void iterate_k_subset(ll N, ll k, F f){
  ll mask = (1ll << k) - 1;
  while (!(mask & 1ll<<N)) { f(mask);
    ll t = mask | (mask-1);
    mask = (t+1)|(((~t & -~t)-1)>>(__builtin_ctzll(mask)+1));
  }
}
template<typename F> // All subsets of set
void iterate_mask_subset(ll set, F f){ ll mask = set;
  do  f(mask), mask = (mask-1) & set;
  while (mask != set);
}
```

10.6 Random Numbers

random-numbers.h

Description: An example on the usage of generator and distribution. Use shuffle instead of random shuffle.

b96f2b, 8 lines

```
mt19937 rng(random_device{}());
mt19937_64 rng(chrono::steady_clock::now().time_since_epoch().
    count());
shuffle(permutation.begin(), permutation.end(), rng);
uniform_int_distribution<int> uid(1, 100); //[1, 100]
unsigned xrand() {
  static unsigned x = 314159265, y = 358979323, z = 846264338,
      w = 327950288;
  unsigned t = x ^ x << 11; x = y; y = z; z = w; return w = w ^
       w >> 19 ^ t ^ t >> 8;
}
```