



Federal University of Rio de Janeiro

# UFRJ - World Finals

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Contest (1)

```
.bashrc
13 lines
b() {
  g++ -DLOC -O2 -std=c++20 -Wall -W -Wfatal-errors \
    -Wconversion -Wshadow -Wlogical-op \
    -Wfloat-equal -o $1 $1.cpp )
d() {
  b $@ -O0 -g -D_GLIBCXX_DEBUG -fsanitize=address,undefined)
run() {
  $@ && echo start >&2 && time ./ $2)
hash-cpp() {
  sed -n $2' '$3' p' $1 | sed '/^#w/d' | cpp -dD -P \
    -fpreprocessed | tr -d '[:space:]' | md5sum | cut -c-6)
# Other compilation flags:
# -Wformat=2 -Wshift-overflow=2 -Wcast-qual
# -Wcast-align -Wduplicated-cond
# -D_GLIBCXX_DEBUG_PEDANTIC -D_FORTIFY_SOURCE=2
# -fno-sanitize-recover -fstack-protector
# Print optimization info: -fopt-info-all
```

Mathematics (2)

2.1 Recurrences

If  $a_n = c_1a_{n-1} + \dots + c_ka_{n-k}$ , and  $r_1, \dots, r_k$  are distinct roots of  $x^k - c_1x^{k-1} - \dots - c_k$ , there are  $d_1, \dots, d_k$  s.t.

$$a_n = d_1r_1^n + \dots + d_kr_k^n.$$

Non-distinct roots  $r$  become polynomial factors, e.g.  $a_n = (d_1n + d_2)r^n$ .

2.2 Trigonometry

$$\sin(v + w) = \sin v \cos w + \cos v \sin w$$
$$\cos(v + w) = \cos v \cos w - \sin v \sin w$$

$$\tan(v + w) = \frac{\tan v + \tan w}{1 - \tan v \tan w}$$
$$\sin v + \sin w = 2 \sin \frac{v + w}{2} \cos \frac{v - w}{2}$$
$$\cos v + \cos w = 2 \cos \frac{v + w}{2} \cos \frac{v - w}{2}$$

$$(V + W) \tan(v - w)/2 = (V - W) \tan(v + w)/2$$

where  $V, W$  are lengths of sides opposite angles  $v, w$ .

$$a \cos x + b \sin x = r \cos(x - \phi)$$
$$a \sin x + b \cos x = r \sin(x + \phi)$$

where  $r = \sqrt{a^2 + b^2}, \phi = \text{atan2}(b, a)$ .

2.3 Geometry

2.3.1 Triangles

Side lengths:  $a, b, c$

Semiperimeter:  $p = \frac{a + b + c}{2}$

Area:  $A = \sqrt{p(p - a)(p - b)(p - c)}$

.bashrc

Circumradius:  $R = \frac{abc}{4A}$

Inradius:  $r = \frac{A}{p}$

Length of median (divides triangle into two equal-area triangles):  $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

Length of bisector (divides angles in two):  $s_a = \sqrt{bc \left[ 1 - \left( \frac{a}{b + c} \right)^2 \right]}$

Law of sines:  $\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{1}{2R}$

Law of cosines:  $a^2 = b^2 + c^2 - 2bc \cos \alpha$

Law of tangents:  $\frac{a + b}{a - b} = \frac{\tan \frac{\alpha + \beta}{2}}{\tan \frac{\alpha - \beta}{2}}$

Pick's: A polygon on an integer grid strictly containing  $i$  lattice points and having  $b$  lattice points on the boundary has area  $i + \frac{b}{2} - 1$ . (Nothing similar in higher dimensions)

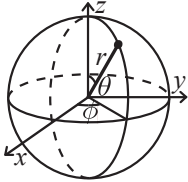
2.3.2 Quadrilaterals

With side lengths  $a, b, c, d$ , diagonals  $e, f$ , diagonals angle  $\theta$ , area  $A$  and magic flux  $F = b^2 + d^2 - a^2 - c^2$ :

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is  $180^\circ$ ,  $ef = ac + bd$ , and  $A = \sqrt{(p - a)(p - b)(p - c)(p - d)}$ .

2.3.3 Spherical coordinates



$$x = r \sin \theta \cos \phi$$
$$y = r \sin \theta \sin \phi$$
$$z = r \cos \theta$$
$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \text{acos}(z/\sqrt{x^2 + y^2 + z^2})$$
$$\phi = \text{atan2}(y, x)$$

2.4 Derivatives/Integrals

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}$$
$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1 - x^2}}$$
$$\frac{d}{dx} \tan x = 1 + \tan^2 x$$
$$\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}$$
$$\int \tan ax = -\frac{\ln |\cos ax|}{a}$$
$$\int x \sin ax = \frac{\sin ax - ax \cos ax}{a^2}$$
$$\int e^{-x^2} = \frac{\sqrt{\pi}}{2} \text{erf}(x)$$
$$\int xe^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1)$$

2.4.1 XOR sum

$$\bigoplus_{x=0}^{n-1} x = \{0, n - 1, 1, n\}[n \bmod 4]$$
$$\bigoplus_{x=l}^{r-1} x = \bigoplus_{a=0}^{r-1} a \oplus \bigoplus_{b=0}^{l-1} b$$

2.5 Sums

$$c^a + c^{a+1} + \dots + c^b = \frac{c^{b+1} - c^a}{c - 1}, c \neq 1$$

$$1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}$$
$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(2n + 1)(n + 1)}{6}$$
$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n + 1)^2}{4}$$
$$1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n + 1)(2n + 1)(3n^2 + 3n - 1)}{30}$$

2.6 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, (-\infty < x < \infty)$$
$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, (-1 < x \leq 1)$$
$$\sqrt{1 + x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \dots, (-1 \leq x \leq 1)$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, (-\infty < x < \infty)$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, (-\infty < x < \infty)$$

2.7 Probability theory

Let  $X$  be a discrete random variable with probability  $p_X(x)$  of assuming the value  $x$ . It will then have an expected value (mean)  $\mu = \mathbb{E}(X) = \sum_x x p_X(x)$  and variance  $\sigma^2 = V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$  where  $\sigma$  is the standard deviation. If  $X$  is instead continuous it will have a probability density function  $f_X(x)$  and the sums above will instead be integrals with  $p_X(x)$  replaced by  $f_X(x)$ .

Expectation is linear:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

For independent  $X$  and  $Y$ ,

$$V(aX + bY) = a^2V(X) + b^2V(Y).$$

2.7.1 Discrete distributions

Binomial distribution

The number of successes in  $n$  independent yes/no experiments, each which yields success with probability  $p$  is  $\text{Bin}(n, p)$ ,  $n = 1, 2, \dots$ ,  $0 \leq p \leq 1$ .

$$p(k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$\mu = np, \sigma^2 = np(1 - p)$$

$\text{Bin}(n, p)$  is approximately  $\text{Po}(np)$  for small  $p$ .

2.8 Markov chains

A *Markov chain* is a discrete random process with the property that the next state depends only on the current state. Let  $X_1, X_2, \dots$  be a sequence of random variables generated by the Markov process. Then there is a transition matrix  $\mathbf{P} = (p_{ij})$ , with  $p_{ij} = \Pr(X_n = i | X_{n-1} = j)$ , and  $\mathbf{p}^{(n)} = \mathbf{P}^n \mathbf{p}^{(0)}$  is the probability distribution for  $X_n$  (i.e.,  $p_i^{(n)} = \Pr(X_n = i)$ ), where  $\mathbf{p}^{(0)}$  is the initial distribution.

$\pi$  is a stationary distribution if  $\pi = \pi \mathbf{P}$ . If the Markov chain is *irreducible* (it is possible to get to any state from any state), then  $\pi_i = \frac{1}{\mathbb{E}(T_i)}$  where  $\mathbb{E}(T_i)$  is the expected time between two visits in state  $i$ .  $\pi_j / \pi_i$  is the expected number of visits in state  $j$  between two visits in state  $i$ .

For a connected, undirected and non-bipartite graph, where the transition probability is uniform among all neighbors,  $\pi_i$  is proportional to node  $i$ 's degree.

A Markov chain is *ergodic* if the asymptotic distribution is independent of the initial distribution. A finite Markov chain is ergodic iff it is irreducible and *aperiodic* (i.e., the gcd of cycle lengths is 1).  $\lim_{k \rightarrow \infty} \mathbf{P}^k = \mathbf{1P}$ .

A Markov chain is an absorbing chain if

- 1. there is at least one absorbing state and
- 2. it is possible to go from any state to at least one absorbing state in a finite number of steps.

A Markov chain is an A-chain if the states can be partitioned into two sets  $\mathbf{A}$  and  $\mathbf{G}$ , such that all states in  $\mathbf{A}$  are absorbing ( $p_{ii} = 1$ ), and all states in  $\mathbf{G}$  leads to an absorbing state in  $\mathbf{A}$ . The probability for absorption in state  $i \in \mathbf{A}$ , when the initial state is  $j$ , is  $a_{ij} = p_{ij} + \sum_{k \in \mathbf{G}} a_{ik} p_{kj}$ . The expected time until absorption, when the initial state is  $i$ , is  $t_i = 1 + \sum_{k \in \mathbf{G}} p_{ki} t_k$ .

Data Structures (3)

pb-ds.h

**Description:** Faster/better hash maps, taken from teapot. And set (not multiset!) with support for finding the n'th element, and finding the index of an element.

**Time:**  $\mathcal{O}(\log N)$

```
<bits/extc++.h>
ef3b3c, 25 lines

const size_t HXOR = std::mt19937_64(time(0))();
template<class T> struct SafeHash {
    size_t operator()(const T& x) const {
        return std::hash<T>()(x ^ T(HXOR));
    }
};
template <typename K, typename V, typename Hash = SafeHash<K>>
using hash_map = __gnu_pbds::gp_hash_table<K, V, Hash>;
template <typename K, typename Hash = SafeHash<K>>
using hash_set = hash_map<K, __gnu_pbds::null_type, Hash>;
template <typename K, typename V, typename Comp = std::less<K>>
using ordered_map = __gnu_pbds::tree<K, V, Comp,
    __gnu_pbds::rb_tree_tag,
    __gnu_pbds::tree_order_statistics_node_update>;
template <typename K, typename Comp = std::less<K>>
using ordered_set = ordered_map<K, __gnu_pbds::null_type, Comp>;
void example() {
    ordered_set<int> t, t2; t.insert(8);
    auto it = t.insert(10).first;
    assert(it == t.lower_bound(9));
    assert(t.order_of_key(10) == 1); // num strictly smaller
    assert(t.order_of_key(11) == 2);
    assert(*t.find_by_order(0) == 8);
    t.join(t2); // assuming T < T2 or T > T2, merge t2 into t
}
```

dsu-rollback.h

**Description:** Disjoint-set data structure with undo.

**Usage:** int t = uf.time(); ...; uf.rollback(t);

**Time:**  $\mathcal{O}(\log(N))$

```
7ddfd1d, 21 lines

struct RollbackUF {
    vector<int> e; vector<pair<int,int>> st;
    RollbackUF(int n) : e(n, -1) {}
    int size(int x) { return -e[find(x)]; }
    int find(int x) { return e[x] < 0 ? x : find(e[x]); }
    int time() { return st.size(); }
    void rollback(int t) { // hash-1
        for (int i = time(); i --> t;)
            e[st[i].first] = st[i].second;
        st.resize(t);
    } // hash-1 = 30bb61
    bool unite(int a, int b) { // hash-2
        a = find(a), b = find(b);
        if (a == b) return false;
        if (e[a] > e[b]) swap(a, b);
        st.push_back({a, e[a]});
        st.push_back({b, e[b]});
        e[a] += e[b]; e[b] = a;
        return true;
    } // hash-2 = a7445e
};
```

monotonic-queue.h

**Description:** Supports pop and push queue-like, and add function adds a constant to all elements currently in the queue.

**Time:**  $\mathcal{O}(1)$

```
0dfbe6, 18 lines

template<typename T, typename Comp> struct monotonic_queue {
    int lo, hi; T S; // hash-1
    deque<pair<T, T>> q;
    monotonic_queue() : lo(0), hi(0), S(0) {}
    void push(T val) {
```

```
while(!q.empty() && Comp()(val, q.back().first + S))
    q.pop_back();
q.emplace_back(val - S, hi++);
}
void pop() {
    if (!q.empty() && q.front().second == lo++) q.pop_front();
} // hash-1 = d8c3c1
void add(T val) { S += val; }
T get_val() const { return q.front().first + S; }
int size() const { return hi-lo; }
};
template<typename T> using min_queue = monotonic_queue<T,
    less_equal<T>>;
template<typename T> using max_queue = monotonic_queue<T,
    greater_equal<T>>;
```

point-context.h

**Description:** Examples of Segment Tree

228ba0, 20 lines

```
struct seg_node {
    int val, int mi, ma;
    seg_node() : mi(INT_MAX), ma(INT_MIN), val(0) {}
    seg_node(int x) : mi(x), ma(x), val(x) {}
    void merge(const seg_node& l, const seg_node& r) {
        val = l.val + r.val;
        mi = min(l.mi, r.mi), ma = max(l.ma, r.ma);
    }
    void update(int x) { mi = ma = val = x; }
    bool acc_min(int& acc, int x) const {
        if (x >= mi) return true;
        if (acc > mi) acc = mi;
        return false;
    }
    bool acc_max(int& acc, int x) const {
        if (x <= ma) return true;
        if (acc < ma) acc = ma;
        return false;
    }
};
```

rec-lazy-segtree.h

**Description:** Segment Tree with Lazy update (half-open interval).

**Time:**  $\mathcal{O}(\lg(N) * Q)$

22037e, 48 lines

```
template<class T> struct segtree_range { // hash-1
    int N; vector<T> ts;
    segtree_range() {}
    segtree_range(int M) : segtree_range(vector<T>(M, T(0))) {}
    template<class Q> segtree_range(const vector<Q>& A) {
        const int N_ = int(A.size());
        N = (1 << __lg(2*N_-1)); ts.resize(2*N);
        for (int i = 0; i < N_; ++i) at(i) = T(A[i]);
        build();
    } // hash-1 = 37f0ae
    T& at(int a) { return ts[a + N]; }
    void build() { for (int a = N; --a; ) merge(a); }
    inline void push(int a) { ts[a].push(ts[2*a], ts[2*a+1]); }
    inline void merge(int a) { ts[a].merge(ts[2*a], ts[2*a+1]); }
    T query(int v, int l, int r, int a, int b) { // hash-2
        if (l >= b || r <= a) return T();
        if (l >= a && r <= b) return ts[v];
        int m = (l + r)/2; push(v); T t;
        t.merge(query(2*v, l, m, a, b), query(2*v+1, m, r, a, b));
        return t;
    } // hash-2 = eb3198
    T query(int a, int b) { return query(1, 0, N, a, b); }
    template<class F, class... Args> void update(int v, int l,
        int r, int a, int b, F f, Args&&... args) { // hash-3
        if (l >= b || r <= a) return;
```

```

    if (l >= a && r <= b && (ts[v].*f)(args...)) return;
    int m = (l + r)/2; push(v);
    update(2*v, l, m, a, b, f, args...);
    update(2*v+1, m, r, a, b, f, args...);
    merge(v);
} // hash-3 = 02f5df
template<class F, class... Args>
void update(int a, int b, F f, Args&&... args) {
    update(l, 0, N, a, b, f, args...);
}
template<class F, class... Args> int find_first(int v, int l,
    int r, int a, int b, F f, Args&&... args) { // hash-4
    if (l >= b || r <= a || !(ts[v].*f)(args...)) return -1;
    if (l + 1 == r) return l;
    int m = (l + r)/2; push(v);
    int cur = find_first(2*v, l, m, a, b, f, args...);
    if (cur == -1)
        cur = find_first(2*v+1, m, r, a, b, f, args...);
    return cur;
} // hash-4 = 68aa30
template<class F, class... Args>
int find_first(int a, int b, F f, Args&&... args) {
    return find_first(l, 0, N, a, b, f, args...);
}
};

```

## lazy-context.h

**Description:** Examples of Segment Tree with Lazy update bd0d51, 172 lines

```

template<typename T = int64_t> struct seg_node { // hash-1
    T val, lz_add, lz_set;
    int sz; bool to_set;
    seg_node(T n = 0) : val(n), lz_add(0), lz_set(0), sz(1),
        to_set(0) {}
    void push(seg_node& l, seg_node& r) {
        if (to_set) {
            l.assign(lz_set), r.assign(lz_set);
            lz_set = 0; to_set = false;
        }
        if (lz_add != 0) {
            l.add(lz_add), r.add(lz_add), lz_add = 0;
        }
    }
    void merge(const seg_node& l, const seg_node& r) {
        sz = l.sz + r.sz; val = l.val + r.val;
    }
    bool add(T v) { // update range a[i] <- a[i] + v
        val += v * sz; lz_add += v; return true;
    }
    bool assign(T v) { // update range a[i] <- v
        val = v * sz; lz_add = 0;
        lz_set = v; to_set = true; return true;
    }
    T get_sum() const { return val; } // sum a[l, r)
}; // hash-1 = 42f915
// update range a[i] <- a[i] + b * (i - s) + c
// assuming b and c are non zero, be careful
// get sum a[l, r)
template<typename T = int64_t> struct seg_node { // hash-2
    T sum, lzB, lzC;
    int sz, idx;
    seg_node(int id = 0, T v = 0, int s = 0, T b = 0, T c = 0) :
        sum(v), lzB(b), lzC(c - s * b), idx(id), sz(1) {}
    void push(seg_node& l, seg_node& r) {
        l.add(lzB, lzC), r.add(lzB, lzC);
        lzB = lzC = 0;
    }
    void merge(const seg_node& l, const seg_node& r) {
        idx = min(l.idx, r.idx), sz = l.sz + r.sz;

```

```

        sum = l.sum + r.sum;
    }
    T sum_idx(T n) const { return n * (n + 1) / 2; }
    bool add(T b, T c) {
        sum += b * (sum_idx(idx + sz) - sum_idx(idx)) + sz * c;
        lzB += b, lzC += c; return true;
    }
    T get_sum() const { return sum; }
}; // hash-2 = ca218b
// update range a[i] <- b * a[i] + c
// get sum a[l, r)
struct seg_node { // hash-3
    int sz; i64 sum, lzB, lzC;
    seg_node() : sz(1), sum(0), lzB(1), lzC(0) {}
    seg_node(i64 v) : sz(1), sum(v), lzB(1), lzC(0) {}
    void push(seg_node& l, seg_node& r) {
        l.add(lzB, lzC), r.add(lzB, lzC);
        lzB = 1, lzC = 0;
    }
    void merge(const seg_node& l, const seg_node& r) {
        sz = l.sz + r.sz, sum = l.sum + r.sum;
    }
    bool add(i64 b, i64 c) {
        sum = (b * sum + c * sz), lzB = (lzB * b);
        lzC = (lzC * b + c); return true;
    }
    i64 get_sum() const { return sum; }
}; // hash-3 = 3fd6ec
// update range a[i] <- min(a[i], b);
// update range a[i] <- max(a[i], b);
// get val a[i]
struct seg_node { // hash-4
    int mn, mx;
    int lz0, lz1;
    seg_node() : mn(INT_MAX), mx(INT_MIN), lz0(INT_MAX), lz1(
        INT_MIN) {}
    void push(seg_node& l, seg_node& r) {
        l.minimize(lz0), l.maximize(lz1);
        r.minimize(lz0), r.maximize(lz1);
        lz0 = INT_MAX, lz1 = INT_MIN;
    }
    void merge(const seg_node& l, const seg_node& r) {
        mn = min(l.mn, r.mn), mx = max(l.mx, r.mx);
    }
    bool minimize(int val) {
        mn = lz0 = min(lz0, val);
        mx = lz1 = min(lz0, lz1); return true;
    }
    bool maximize(int val) {
        mx = lz1 = max(lz1, val);
        mn = lz0 = max(lz0, lz1); return true;
    }
    pair<int, int> get() const { return {mx, mn}; }
}; // hash-4 = a6c9e9

```

```

template<typename T> struct lazy_t { // hash-5
    T a, b, c;
    lazy_t() : a(0), b(-INF), c(+INF) {}
    lazy_t(T a, T b, T c) : a(a), b(b), c(c) {}
    void add(T val) {
        a += val, b += val, c += val;
    }
    void upd_min(T val) {
        if (b > val) b = val;
        if (c > val) c = val;
    }
    void upd_max(T val) {
        if (b < val) b = val;
        if (c < val) c = val;
    }

```

```

    }
}; // hash-5 = 9fb833
template<typename T = int64_t> struct seg_node { // hash-6
    T mi, mi2, ma, ma2, sum;
    lazy_t<T> lz;
    seg_node() : mi(INF), mi2(INF), ma(-INF), ma2(-INF), sum(0),
        cnt_mi(0), cnt_ma(0), sz(0), lz() {}
    seg_node(T n) : mi(n), mi2(INF), ma(n), ma2(-INF), sum(n),
        cnt_mi(1), cnt_ma(1), sz(1), lz() {}
    void push(seg_node& l, seg_node& r) {
        if (!l.can_apply(lz) || !r.can_apply(lz)) return;
        lz = lazy_t<T>();
    }
    bool can_apply(const lazy_t<T>& f) {
        if (!add(f.a) || !upd_max(f.b) || !upd_min(f.c)) return
            false;
        return true;
    } // hash-6 = 7b66a0
    void merge(const seg_node& l, const seg_node& r) { // hash-7
        mi = min(l.mi, r.mi);
        mi2 = min((l.mi == mi) ? l.mi2 : l.mi, (r.mi == mi) ? r.mi2
            : r.mi);
        cnt_mi = ((l.mi == mi) ? l.cnt_mi : 0) + ((r.mi == mi) ? r.
            cnt_mi : 0);
        ma = max(l.ma, r.ma);
        ma2 = max((l.ma == ma) ? l.ma2 : l.ma, (r.ma == ma) ? r.ma2
            : r.ma);
        cnt_ma = ((l.ma == ma) ? l.cnt_ma : 0) + ((r.ma == ma) ? r.
            cnt_ma : 0);
        sum = l.sum + r.sum;
        sz = l.sz + r.sz;
    } // hash-7 = 936019
    bool add(T v) { // a-i = a-i + v // hash-8
        if (v) {
            mi += v;
            if (mi2 < INF) mi2 += v;
            ma += v;
            if (ma2 > -INF) ma2 += v;
            sum += sz * v;
            lz.add(v);
        }
        return true;
    } // hash-8 = 83626c
    bool upd_max(T v) { // a-i = max(a-i, v) // hash-9
        if (v > -INF) {
            if (v >= mi2) return false;
            else if (v > mi) {
                if (ma == mi) ma = v;
                if (ma2 == mi) ma2 = v;
                sum += cnt_mi * (v - mi);
                mi = v;
                lz.upd_max(v);
            }
        }
        return true;
    } // hash-9 = 877a43
    bool upd_min(T v) { // a-i = min(a-i, v) // hash-10
        if (v < INF) {
            if (v <= ma2) return false;
            else if (v < ma) {
                if (ma == mi) mi = v;
                if (mi2 == ma) mi2 = v;
                sum -= cnt_ma * (ma - v);
                ma = v;
                lz.upd_min(v);
            }
        }
        return true;
    }

```

```
    } // hash-10 = 514d75
    T get_sum() const { return sum; } // sum a[l, r)
};
```

sparse-segtree.h

**Description:** Sparse Segment Tree with point update. Doesnt allocate storage for nodes with no data. Use BumpAllocator for better performance!

```
const int SZ = 1<<19;
template<class T> struct node_t {
    T delta = 0; node_t<T>* c[2];
    node_t() { c[0] = c[1] = nullptr; }
    void upd(int pos, T v, int L = 0, int R = SZ-1) { // add v //
        hash-1
        if (L == pos && R == pos) { delta += v; return; }
        int M = (L + R)>>1;
        if (pos <= M) {
            if (!c[0]) c[0] = new node_t();
            c[0]->upd(pos, v, L, M);
        } else {
            if (!c[1]) c[1] = new node_t();
            c[1]->upd(pos, v, M+1, R);
        }
        delta = 0;
        for (int i = 0; i < 2; ++i) if (c[i]) delta += c[i]->delta;
    } // hash-1 = 8c7b01
    T query(int lx, int rx, int L = 0, int R = SZ-1) { // query
        sum of segment // hash-2
        if (rx < L || R < lx) return 0;
        if (lx <= L && R <= rx) return delta;
        int M = (L + R)>>1; T res = 0;
        if (c[0]) res += c[0]->query(lx, rx, L, M);
        if (c[1]) res += c[1]->query(lx, rx, M+1, R);
        return res;
    } // hash-2 = 6aa202
    void upd(int pos, node_t *a, node_t *b, int L = 0, int R = SZ-1) { // hash-3
        if (L != R) {
            int M = (L + R)>>1;
            if (pos <= M) {
                if (!c[0]) c[0] = new node_t();
                c[0]->upd(pos, a ? a->c[0] : nullptr, b ? b->c[0] :
                    nullptr, L, M);
            } else {
                if (!c[1]) c[1] = new node_t();
                c[1]->upd(pos, a ? a->c[1] : nullptr, b ? b->c[1] :
                    nullptr, M+1, R);
            }
        }
        delta = (a ? a->delta : 0)+(b ? b->delta : 0);
    } // hash-3 = 514618
};
```

segtree-2d.h

**Description:** 2D Segment Tree.  
**Time:**  $\mathcal{O}(N \log^2 N)$  of memory,  $\mathcal{O}(\log^2 N)$  per query

```
"sparse-segtree.h" 09098e, 25 lines
template<class T> struct Node {
    node_t<T> seg; Node* c[2];
    Node() { c[0] = c[1] = nullptr; }
    void upd(int x, int y, T v, int L = 0, int R = SZ-1) { //add v
        if (L == x && R == x) { seg.upd(y,v); return; }
        int M = (L+R)>>1;
        if (x <= M) {
            if (!c[0]) c[0] = new Node();
            c[0]->upd(x,y,v,L,M);
        } else {
            if (!c[1]) c[1] = new Node();
            c[1]->upd(x,y,v,M+1,R);
        }
    }
};
```

```
    }
    seg.upd(y,v); // only for addition
    // seg.upd(y,c[0]?&c[0]->seg:nullptr,c[1]?&c[1]->seg:
        nullptr);
    }
    T query(int x1, int x2, int y1, int y2, int L = 0, int R = SZ-1) { // query sum of rectangle
        if (x1 <= L && R <= x2) return seg.query(y1,y2);
        if (x2 < L || R < x1) return 0;
        int M = (L+R)>>1; T res = 0;
        if (c[0]) res += c[0]->query(x1, x2, y1, y2, L, M);
        if (c[1]) res += c[1]->query(x1, x2, y1, y2, M+1, R);
        return res;
    }
};
```

persistent-segtree.h

**Description:** Sparse ( $N$  can be up to  $1e18$ ) persistent segment tree.

```
template<typename T> struct seg_tree {
    T idnt = 0; // identity value
    T f(T l, T r) { return l + r; }
    struct node {
        int l = 0, r = 0;
        T x;
        node(T _x) : x(_x) {}
    };
    int N; vector<node> t;
    seg_tree(int _N) : N(_N) { t.push_back(node(idnt)); }
    int cpy(int v) {
        t.push_back(t[v]);
        return int(t.size()) - 1;
    }
    int upd(int v, int p, T x, int a = 0, int b = -1) { // hash-1
        b = ~b ? b : N - 1;
        int u = cpy(v);
        if (a == b) {
            t[u].x += x;
            return u;
        }
        int m = (a + b) / 2;
        if (p <= m) t[u].l = upd(t[v].l, p, x, a, m);
        else t[u].r = upd(t[v].r, p, x, m+1, b);
        t[u].x = f(t[t[u].l].x, t[t[u].r].x);
        return u;
    } // hash-1 = 9c27d1
    T get(int v, int l, int r, int a = 0, int b = -1) { // hash-2
        b = ~b ? b : N - 1;
        if (!v || l > b || r < a) return idnt;
        if (a >= l && b <= r) return t[v].x;
        int m = (a + b) / 2;
        return f(get(t[v].l, l, r, a, m), get(t[v].r, l, r, m+1, b));
    } // hash-2 = 7d1e05
    T get_kth(int l, int r, int k, int a = 0, int b = -1) { //
        hash-3
        b = ~b ? b : N - 1;
        if (a == b) return a;
        int cnt = t[t[r].l].x - t[t[l].l].x;
        int m = (a + b) / 2;
        if (k <= cnt) return get_kth(t[l].l, t[r].l, k, a, m);
        return get_kth(t[l].r, t[r].r, k-cnt, m+1, b);
    } // hash-3 = 732fd0
};
```

rmq.h

**Description:** Range Minimum/Maximum Queries on an array. Returns  $\min(V[a], V[a + 1], \dots V[b])$  in constant time. Returns a pair that holds the answer, first element is the value and the second is the index.

**Usage:** `rmq.t<pair<int, int>> rmq(values);`  
// values is a vector of pairs {val(i), index(i)}  
`rmq.query(inclusive, exclusive);`  
`rmq.t<pair<int, int>, greater<pair<int, int>>> rmq(values)`  
//max query  
**Time:**  $\mathcal{O}(|V| \log |V| + Q)$

```
8c53c5, 19 lines
template<typename T, typename Cmp=less<T>>
struct rmq_t : private Cmp {
    int N = 0;
    vector<vector<T>> table;
    const T& min(const T& a, const T& b) const { return Cmp::
        operator()(a, b) ? a : b; }
    rmq_t() {}
    rmq_t(const vector<T>& values) : N(int(values.size())), table
        (__lg(N) + 1) {
        table[0] = values;
        for (int a = 1; a < int(table.size()); ++a) {
            table[a].resize(N - (1 << a) + 1);
            for (int b = 0; b + (1 << a) <= N; ++b)
                table[a][b] = min(table[a-1][b], table[a-1][b+(1<<(a-1)
                    )]);
        }
    }
    T query(int a, int b) const {
        int lg = __lg(b - a);
        return min(table[lg][a], table[lg][b - (1 << lg) ]);
    }
};
```

fenwick-tree.h

**Description:** Computes partial sums  $a[0] + a[1] + \dots + a[\text{pos} - 1]$ , and updates single elements  $a[i]$ , taking the difference between the old and new value.  
**Time:** Both operations are  $\mathcal{O}(\log N)$ .

```
2ee6d4, 26 lines
template<typename T> struct FT {
    vector<T> s;
    FT(int n) : s(n) {}
    FT(const vector<T>& A) : s(A) {
        const int N = int(s.size());
        for (int a = 0; a < N; ++a)
            if ((a | (a + 1)) < N) s[a | (a + 1)] += s[a];
    }
    void update(int pos, T dif) { // a[pos] += dif
        for (; pos < (int)s.size(); pos |= pos + 1) s[pos] += dif;
    }
    T query(int pos) { // sum of values in [0, pos)
        T res = 0;
        for (; pos > 0; pos &= pos - 1) res += s[pos-1];
        return res;
    }
    // min pos st sum of [0, pos] >= sum. Returns n if no sum
    int lower_bound(T sum) { //is >= sum, or -1 if empty sum is.
        if (sum <= 0) return -1;
        int pos = 0;
        for (int pw = 1 << 25; pw; pw >= 1)
            if (pos + pw <= (int)s.size() && s[pos + pw-1] < sum)
                pos += pw, sum -= s[pos-1];
        return pos;
    }
};
```

fenwick-tree-2d.h

**Description:** Computes sums  $a[i,j]$  for all  $i < I, j < J$ , and increases single elements  $a[i,j]$ . Requires that the elements to be updated are known in advance (call `fakeUpdate()` before `init()`).  
**Time:**  $\mathcal{O}(\log^2 N)$ . (Use persistent segment trees for  $\mathcal{O}(\log N)$ .)

```
"fenwick-tree.h" aebbdc, 25 lines
```



```
template<typename T> struct FT2 {
    vector<vector<int>>> ys; vector<FT<T>> ft;
    FT2(int limx) : ys(limx) {}
    void fakeUpdate(int x, int y) {
        for (; x < (int)ys.size(); x |= x + 1) ys[x].push_back(y);
    }
    void init() {
        for(auto &v : ys){
            sort(v.begin(), v.end());
            v.resize(unique(v.begin(),v.end()) - v.begin());
            ft.emplace_back(v.size());
        }
    }
    int ind(int x, int y) {
        return (int)(lower_bound(ys[x].begin(), ys[x].end(), y) -
            ys[x].begin()); }
    void update(int x, int y, T dif) {
        for (; x < ys.size(); x |= x + 1)
            ft[x].update(ind(x, y), dif);
    }
    T query(int x, int y) {
        T sum = 0;
        for (; x; x &= x - 1) sum += ft[x-1].query(ind(x-1, y));
        return sum;
    }
};
```

mo.h

**Description:** Answer interval or tree path queries by finding an approximate TSP through the queries, and moving from one query to the next by adding/removing points at the ends. If values are on tree edges, change step to add/remove the edge (a,c) and remove the initial add call (but keep in).  
**Time:**  $\mathcal{O}(N\sqrt{Q})$

```
void add(int ind, int end) { ... } // add a[ind] (end = 0 or 1)
void del(int ind, int end) { ... } // remove a[ind]
int calc() { ... } // compute current answer
```

```
vector<int> mo(vector<pair<int, int>> Q) { // hash-1
    int L = 0, R = 0, blk = 350; // ~N/sqrt(Q)
    vector<int> s(int(Q.size())), res = s;
    #define K(x) pair<int, int>(x.first/blk, x.second ^ -(x.first/blk & 1))
    iota(s.begin(), s.end(), 0);
    sort(s.begin(), s.end(), [&](int s, int t){ return K(Q[s]) < K(Q[t]); });
    for (int qi : s) {
        auto q = Q[qi];
        while (L > q.first) add(--L, 0);
        while (R < q.second) add(R++, 1);
        while (L < q.first) del(L++, 0);
        while (R > q.second) del(--R, 1);
        res[qi] = calc();
    }
    return res;
} // hash-1 = d9247c
```

```
vector<int> moTree(vector<array<int, 2>> Q, vector<vector<int>
    >>& ed, int root=0){ // hash-2
    int N = int(ed.size()), pos[2] = {}, blk = 350; // ~N/sqrt(Q)
    vector<int> s(int(Q.size())), res = s, I(N), L(N), R(N), in(N), par(N);
    add(0, 0), in[0] = 1;
    auto dfs = [&](int x, int p, int dep, auto& f) -> void {
        par[x] = p;
        L[x] = N;
        if (dep) I[x] = N++;
        for (int y : ed[x]) if (y != p) f(y, x, !dep, f);
        if (!dep) I[x] = N++;
    };
```

mo line-container matrix range-color implicit-treap

```
        R[x] = N;
    };
    dfs(root, -1, 0, dfs);
    #define K(x) pii(I[x][0]) / blk, I[x][1] ^ -(I[x][0] / blk & 1))
    iota(s.begin(), s.end(), 0);
    sort(s.begin(), s.end(), [&](int s, int t){ return K(Q[s]) < K(Q[t]); });
    for (int qi : s) for (int end = 0; end < 2; ++end) {
        int &a = pos[end], b = Q[qi][end], i = 0;
        #define step(c) { if (in[c]) { del(a, end); in[a] = 0; } \
            else { add(c, end); in[c] = 1; } a = c; }
        while (!(L[b] <= L[a] && R[a] <= R[b]))
            I[i++] = b, b = par[b];
        while (a != b) step(par[a]);
        while (i--) step(I[i]);
        if (end) res[qi] = calc();
    }
    return res;
} // hash-2 = bbf891
```

line-container.h

**Description:** Container where you can add lines of the form  $kx+m$ , and query maximum values at points  $x$ . Useful for dynamic programming (“convex hull trick”).  
**Time:**  $\mathcal{O}(\log N)$

```
struct Line { // hash-1
    mutable ll k, m, p;
    bool operator<(const Line& o) const { return k < o.k; }
    bool operator<(ll x) const { return p < x; }
}; // hash-1 = 7e3ecf
struct LineContainer : multiset<Line, less<>> {
    static const ll inf = LLONG_MAX; //for doubles 1/0
    ll div(ll a, ll b) { //for doubles a/b
        return a / b - ((a ^ b) < 0 && a % b); }
    bool isect(iterator x, iterator y) { // hash-2
        if (y == end()) { x->p = inf; return false; }
        if (x->k == y->k) x->p = x->m > y->m ? inf : -inf;
        else x->p = div(y->m - x->m, x->k - y->k);
        return x->p >= y->p;
    } // hash-2 = ea7809
    void add(ll k, ll m) { // hash-3
        auto z = insert({k, m, 0}); y = z++, x = y;
        while (isect(y, z)) z = erase(z);
        if (x != begin() && isect(--x, y)) isect(x, y = erase(y));
        while ((y = x) != begin() && (--x)->p >= y->p)
            isect(x, erase(y));
    } // hash-3 = 08625f
    ll query(ll x) { // hash-4
        assert(!empty()); auto l = *lower_bound(x);
        return l.k * x + l.m;
    } // hash-4 = d21e2f
};
```

matrix.h

**Description:** Basic operations on square matrices.  
**Usage:** Matrix<int> A(N, vector<int>(N));

```
template <typename T> struct Matrix : vector<vector<T>>> {
    using vector<vector<T>>>::vector;
    using vector<vector<T>>>::size;
    int h() const { return int(size()); }
    int w() const { return int((*this)[0].size()); }
    Matrix operator*(const Matrix& r) const {
        assert(w() == r.h()); Matrix res(h(), vector<T>(r.w()));
        for(int i = 0; i < h(); ++i) for(int j = 0; j < r.w(); ++j)
            for (int k = 0; k < w(); ++k)
                res[i][j] += (*this)[i][k] * r[k][j];
        return res;
    };
```

```
    }
    friend auto operator*(const Matrix<T>& A, const vector<T>& b) {
        int N = int(A.size()), M = int(A[0].size());
        vector<T> y(N);
        for (int i = 0; i < N; ++i)
            for (int j = 0; j < M; ++j) y[i] += A[i][j] * b[j];
        return y;
    }
    Matrix& operator*=(const Matrix& r){return *this= *this * r;}
    Matrix pow(ll n) const {
        assert(h() == w()); assert(n >= 0);
        Matrix x = *this, r(h(), vector<T>(w()));
        for (int i = 0; i < h(); ++i) r[i][i] = T(1);
        while (n) { if (n & 1) r *= x; x *= x; n >>= 1; }
        return r;
    }
};
```

range-color.h

**Description:** RangeColor structure, supports point queries and range updates, if  $C$  isn't `int32_t` change `freq` to `map`  
**Time:**  $\mathcal{O}(\lg(L) * Q)$

```
template<class T, class C> struct RangeColor { // hash-1
    struct Node{
        T lo, hi; C color;
        bool operator<(const Node &n) const { return hi < n.hi; }
    };
    C minInf; set<Node> st; vector<T> freq;
    RangeColor(T first, T last, C maxColor, C iniColor = C(0)) :
        minInf(first - T(1)), freq(maxColor + 1) {
        freq[iniColor] = last - first + T(1);
        st.insert({first, last, iniColor});
    } // hash-1 = b9833d
    C query(T i) { //get color in position i
        return st.upper_bound({T(0), i - T(1), minInf})->color;
    }
    void upd(T a, T b, C x) { //set x in [a, b] // hash-2
        auto p = st.upper_bound({T(0), a - T(1), minInf});
        assert(p != st.end());
        T lo = p->lo, hi = p->hi; C old = p->color;
        freq[old] -= (hi - lo + T(1)); p = st.erase(p);
        if (lo < a)
            freq[old] += (a-lo), st.insert({lo, a-T(1), old});
        if (b < hi)
            freq[old] += (hi-b), st.insert({b+T(1), hi, old});
        while ((p != st.end()) && (p->lo <= b)) {
            lo = p->lo, hi = p->hi; old = p->color;
            freq[old] -= (hi - lo + T(1));
            if (b < hi){
                freq[old] += (hi - b); st.erase(p);
                st.insert({b + T(1), hi, old});
                break;
            } else p = st.erase(p);
        }
        freq[x] += (b - a + T(1)); st.insert({a, b, x});
    } // hash-2 = 7575b8
    T countColor(C x){ return freq[x]; }
};
```

implicit-treap.h

**Description:** A short self-balancing tree. It acts as a sequential container with log-time splits/joins, and is easy to augment with additional data.  
**Time:**  $\mathcal{O}(\log N)$

```
struct node { // hash-1
    int val, p, sz; bool rev;
    array<node*, 2> c{nullptr, nullptr};
    node(int k) : val(k), p(rng()), sz(0), rev(false) {}
};
```

```
~node() {
    delete c[0];
    delete c[1];
}
}; // hash-1 = 0055b5
inline int sz(node *t) { // hash-2
    return (!t ? 0 : t->sz);
} // hash-2 = 99838d
inline void push(node *t) { // hash-3
    if (!t) return;
    if (t->rev) {
        swap(t->c[0], t->c[1]);
        if (t->c[0]) t->c[0]->rev ^= t->rev;
        if (t->c[1]) t->c[1]->rev ^= t->rev;
        t->rev = 0;
    }
} // hash-3 = 910227
inline void pull(node *t) { // hash-4
    if (!t) return;
    push(t); push(t->c[0]); push(t->c[1]);
    t->sz = sz(t->c[0]) + sz(t->c[1]) + 1;
} // hash-4 = 85cde9
inline void split(node *t, node *&a, node *&b, int k) { //k on
    left
    push(t);
    if (!t) a = b = nullptr;
    else if (k <= sz(t->c[0])) {
        split(t->c[0], a, t->c[0], k);
        b = t;
    } else {
        split(t->c[1], t->c[1], b, k-1-sz(t->c[0]));
        a = t;
    }
    pull(t);
}
inline void merge(node *&t, node *a, node *b) { // hash-5
    push(a); push(b);
    if (!a) t = b;
    else if (!b) t = a;
    else if (a->p <= b->p) {
        merge(a->c[1], a->c[1], b);
        t = a;
    } else {
        merge(b->c[0], a, b->c[0]);
        t = b;
    }
    pull(t);
} // hash-5 = 34473a
inline void add(node *&t, node *a, int k) { // hash-6
    push(t);
    if (!t) t = a;
    else if (a->p >= t->p) {
        split(t, a->c[0], a->c[1], k);
        t = a;
    } else if (sz(t->c[0]) >= k) add(t->c[0], a, k);
    else add(t->c[1], a, k-1-sz(t->c[0]));
    pull(t);
} // hash-6 = 58c383
void del(node *&t, int k) { // hash-7
    push(t);
    if (!t) return;
    if (sz(t->c[0]) == k) merge(t, t->c[0], t->c[1]);
    else if (sz(t->c[0]) > k) del(t->c[0], k);
    else del(t->c[1], k);
    pull(t);
} // hash-7 = a097c3
inline void dump_treap(node *t) { // hash-8
    if (!t) return;
    push(t);
```

```
dump_treap(t->c[0]);
cerr << t->val << ' ' ;
dump_treap(t->c[1]);
} // hash-8 = 8675f9

Numerical (4)

poly-interpolate.h
Description: Given n points (x[i], y[i]), computes an n-1-degree polynomial
p that passes through them: p(x) = a[0] * x^0 + ... + a[n - 1] * x^{n-1}. For
numerical precision, pick x[k] = c * cos(k/(n - 1) * pi), k = 0 ... n - 1.
Time: O(n^2)
97a266, 12 lines

typedef vector<double> vd;
vd interpolate(vd x, vd y, int n) {
    vd res(n), temp(n);
    for(int k = 0; k < n-1; ++k) for(int i = k+1; i < n; ++i)
        y[i] = (y[i] - y[k]) / (x[i] - x[k]);
    double last = 0; temp[0] = 1;
    for(int k = 0; k < n; ++k) for(int i = 0; i < n; ++i) {
        res[i] += y[k] * temp[i]; swap(last, temp[i]);
        temp[i] -= last * x[k];
    }
    return res;
}

lagrange.h
Description: Lagrange interpolation over a finite field and some combo stuff
Time: O(N)
".../number-theory/modular-arithmetic.h", ".../number-theory/preparator.h" ad3879, 13
lines

template<typename T> struct interpolator_t {
    vector<T> S;
    interpolator_t(int N): S(N) {}
    T interpolate(const vector<T>& y, T x) { // hash-1
        int N = int(y.size()); int sgn = (N & 1 ? 1 : -1);
        T res = 0, P = 1; S[N - 1] = 1;
        for (int i = N-1; i > 0; --i) S[i-1] = S[i] * (x-i);
        for (int i = 0; i < N; ++i, sgn *= -1, P *= (x - i + 1)) {
            res += y[i] * sgn * P * S[i] * ifact[i] * ifact[N-1-i];
        }
        return res;
    } // hash-1 = 2fb780
};

berlekamp-massey.h
Description: Recovers any n-order linear recurrence relation from the first
2n terms of the recurrence. Useful for guessing linear recurrences after brute-
forcing the first terms. Should work on any field, but numerical stability for
floats is not guaranteed. Output will have size ≤ n.
Usage: BerlekampMassey({0, 1, 1, 3, 5, 11}) // {1, 2}
Time: O(N^2)
".../number-theory/modular-arithmetic.h" 66d78a, 17 lines

template <typename num>
vector<num> BerlekampMassey(const vector<num>& s) {
    int n = int(s.size()), L = 0, m = 0; num b = 1;
    vector<num> C(n), B(n), T; C[0] = B[0] = 1;
    for(int i = 0; i < n; i++) { ++m; // hash-1
        num d = s[i];
        for (int j = 1; j <= L; j++) d += C[j] * s[i - j];
        if (d == 0) continue;
        T = C; num coef = d / b;
        for (int j = m; j < n; j++) C[j] -= coef * B[j - m];
        if (2 * L > i) continue;
        L = i + 1 - L; B = T; b = d; m = 0;
    } // hash-1 = ebf5ec
    C.resize(L + 1); C.erase(C.begin());;
```

```
for (auto& x : C) x = -x;
return C;
}

linear-recurrence.h
Description: Bostan-Mori algorithm. Generates the k'th term of an n-
order linear recurrence S[i] = ∑_j S[i - j - 1]tr[j], given S[0 ... n - 1] and
tr[0 ... n - 1]. Faster than matrix multiplication. Useful together with
Berlekamp-Massey.
Usage: linear_rec({0, 1}, {1, 1}, k) // k'th Fibonacci number
Time: O(n log n log k)
".../number-theory/modular-arithmetic.h" aa7314, 16 lines

template<typename T>
T linear_rec(const vector<T>& S, const vector<T>& tr, ll K) {
    const int N = int(tr.size());
    vector<T> qs(N + 1); qs[0] = 1;
    for (int i = 0; i < N; ++i) qs[i + 1] = -tr[i];
    auto fs = fft.convolve(S, qs); fs.resize(N);
    for (; K; K /= 2) { // hash-1
        auto qneg = qs;
        for (int i = 1; i <= N; i += 2) qneg[i] = -qneg[i];
        fs = fft.convolve(fs, qneg), qs = fft.convolve(qs, qneg);
        for (int i = 0; i < N; ++i)
            fs[i] = fs[2 * i + (K & 1)], qs[i] = qs[2 * i];
        qs[N] = qs[2*N]; fs.resize(N), qs.resize(N+1);
    } // hash-1 = 641c09
    return fs[0];
}

integrate.h
Description: Simple integration of a function over an interval using Simp-
son's rule. The error should be proportional to h^4, although in practice you
will want to verify that the result is stable to desired precision when epsilon
changes.
7bb98e, 7 lines

template<class F>
double quad(double a, double b, F& f, const int n = 1000) {
    double h = (b - a) / 2 / n, v = f(a) + f(b);
    for(int i = 1; i < n*2; ++i)
        v += f(a + i*h) * (i&1 ? 4 : 2);
    return v * h / 3;
}

integrate-adaptive.h
Description: Fast integration using an adaptive Simpson's rule.
Usage: double sphereVolume = quad(-1, 1, [](double x) {
return quad(-1, 1, [&](double y) {
return quad(-1, 1, [&](double z) {
return x*x + y*y + z*z < 1; });});});
cfcd42, 13 lines

typedef double d;
#define S(a,b) (f(a) + 4*f((a+b) / 2) + f(b)) * (b-a) / 6
template <class F>
d rec(F& f, d a, d b, d eps, d S) {
    d c = (a + b) / 2, S1 = S(a, c), S2 = S(c, b), T = S1 + S2;
    if (abs(T - S) <= 15 * eps || b - a < 1e-10)
        return T + (T - S) / 15;
    return rec(f, a, c, eps / 2, S1) + rec(f, c, b, eps / 2, S2);
}
template<class F>
d quad(d a, d b, F f, d eps = 1e-8) {
    return rec(f, a, b, eps, S(a, b));
}

gaussian-elimination.h
Time: O(min(N, M)NM)
".../data-structures/matrix.h" 059077, 61 lines

template<typename T> struct gaussian_elimination {
```

```

int N, M; Matrix<T> A, E;
vector<int> pivot; int rank, nullity, sgn;
gaussian_elimination(const Matrix<T>& A_) : A(A_) { // hash-1
    N = A.size(), M = A[0].size(), E=Matrix<T>(N, vector<T>(N));
    for (int i = 0; i < N; ++i) E[i][i] = 1;
    rank = 0, nullity = M, sgn = 0; pivot.assign(M, -1);
    for (int col = 0, row = 0; col < M && row < N; ++col) {
        int sel = -1;
        for (int i = row; i < N; ++i) if (A[i][col] != 0) {
            sel = i; break;
        }
        if (sel == -1) continue;
        if (sel != row) {
            sgn += 1;
            swap(A[sel], A[row]); swap(E[sel], E[row]);
        }
        for (int i = 0; i < N; ++i) {
            if (i == row) continue;
            T c = A[i][col] / A[row][col];
            for (int j = col; j < M; ++j)
                A[i][j] -= c*A[row][j];
            for (int j = 0; j < N; ++j)
                E[i][j] -= c*E[row][j];
        }
        pivot[col] = row++; ++rank, --nullity;
    }
} // hash-1 = d52725
auto solve(vector<T> b, bool reduced=false) const { // hash-2
    if (reduced == false) b = E * b;
    vector<T> x(M);
    for (int j = 0; j < M; ++j) {
        if (pivot[j] == -1) continue;
        x[j] = b[pivot[j]] / A[pivot[j]][j];
        b[pivot[j]] = 0;
    }
    for (int i = 0; i < N; ++i)
        if (b[i] != 0) return make_pair(false, x);
    return make_pair(true, x);
} // hash-2 = ed96b5
auto kernel_basis() const { // hash-3
    vector<vector<T>> basis; vector<T> e(M);
    for (int j = 0; j < M; ++j) {
        if (pivot[j] != -1) continue;
        e[j] = 1; auto y = solve(A * e, true).second;
        e[j] = 0, y[j] = -1; basis.push_back(y);
    }
    return basis;
} // hash-3 = 126a4f
auto inverse() const { // hash-4
    assert(N == M); assert(rank == N);
    Matrix<T> res(N, vector<T>(N));
    vector<T> e(N);
    for (int i = 0; i < N; ++i) {
        e[i] = 1; auto x = solve(e).second;
        for (int j = 0; j < N; ++j) res[j][i] = x[j];
        e[i] = 0;
    }
    return res;
} // hash-4 = a87fb3
};

```

### linear-solver-z2.h

**Description:** Solves  $Ax = b$  over  $\mathbb{F}_2$ . If there are multiple solutions, one is returned arbitrarily. Returns true, or false if no solutions. Last column of  $a$  is  $b$ .  $c$  is the rank.

**Time:**  $\mathcal{O}(n^2m)$

7a24e1, 24 lines

```

typedef bitset<2010> bs;
bool gauss(vector<bs> a, bs& ans, int n) {

```

```

int m = int(a.size()), c = 0;
bs pos; pos.set();
for (int j = n-1, i; j >= 0; --j) {
    for (i = c; i < m; ++i)
        if (a[i][j]) break;
    if (i == m) continue;
    swap(a[c], a[i]);
    i = c++; pos[j] = 0;
    for (int k = 0; k < m; ++k)
        if (a[k][j] && k != i) a[k] ^= a[i];
    ans = pos;
}
for (int i = 0; i < m; ++i) {
    int ac = 0;
    for (int j = 0; j < n; ++j) {
        if (!a[i][j]) continue;
        if (!pos[j]) pos[j] = 1, ans[j] = ac^a[i][n];
        ac ^= ans[j];
    }
    if (ac != a[i][n]) return false;
}
return true;
}

```

### simplex.h

**Description:** Solves a general linear maximization problem: maximize  $c^T x$  subject to  $Ax \leq b$ ,  $x \geq 0$ .

**Time:**  $\mathcal{O}(NM * \#pivots)$ , where a pivot may be e.g. an edge relaxation.  $\mathcal{O}(2^n)$  in the general case. WARNING- segfaults on empty (size 0) max cx st Ax<=b, x>=0 do 2 phases; 1st check feasibility; 2nd check boundedness and ans

83fec6, 39 lines

```

using dbl = double; using vd = vector<dbl>; // hash-1
vd simplex(vector<vd> A, vd b, vd c) { const dbl E = 1e-9;
    int n = A.size(), m = A[0].size() + 1, r = n, s = m-1;
    auto D = vector<vd>(n+2, vd(m+1));
    vector<int> ix = vector<int>(n + m);
    for (int i = 0; i < n + m; ++i) ix[i] = i;
    for (int i = 0; i < n; ++i) {
        for (int j = 0; j < m-1; ++j) D[i][j] = -A[i][j];
        D[i][m-1] = 1; D[i][m] = b[i];
        if (D[r][m] > D[i][m]) r = i;
    } // hash-1 = 0a7467
    for (int j = 0; j < m-1; ++j) D[n][j] = c[j];
    D[n+1][m-1] = -1; int z = 0;
    for (dbl d;;) { // hash-2
        if (r < n) { swap(ix[s], ix[r + m]);
            D[r][s] = 1.0/D[r][s];
            for (int j = 0; j <= m; ++j) if (j != s)
                D[r][j] *= -D[r][s];
            for (int i = 0; i <= n+1; ++i) if (i != r) {
                for (int j = 0; j <= m; ++j)
                    if (j != s) D[i][j] += D[r][j] * D[i][s];
                D[i][s] *= D[r][s];
            }
        }
        r = -1; s = -1;
        for (int j = 0; j < m; ++j) if (s < 0 || ix[s] > ix[j])
            if (D[n+1][j]>E || D[n+1][j]>-E && D[n][j]>E) s = j;
        if (s < 0) break;
        for (int i = 0; i < n; ++i) if (D[i][s] < -E) {
            if (r<0 || (d = D[r][m]/D[r][s]-D[i][m]/D[i][s]) < -E
                || d < E && ix[r+m] > ix[i+m]) r = i;
        }
        if (r < 0) return vd(); // unbounded
    } // hash-2 = dd3d9c
    if (D[n+1][m] < -E) return vd(); // infeasible
    vd x(m-1);
    for (int i=m; i < n+m; ++i) if (ix[i]<m-1) x[ix[i]]=D[i-m][m];
    dbl result = D[n][m]; return x; // ans: D[n][m]
}

```

```

}

```

### tridiagonal.h

**Description:**  $x = \text{tridiagonal}(d, p, q, b)$  solves the equation system

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} d_0 & p_0 & 0 & 0 & \cdots & 0 \\ q_0 & d_1 & p_1 & 0 & \cdots & 0 \\ 0 & q_1 & d_2 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{n-3} & d_{n-2} & p_{n-2} \\ 0 & 0 & \cdots & q_{n-2} & d_{n-1} & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{pmatrix}.$$

This is useful for solving problems on the type

$$a_i = b_i a_{i-1} + c_i a_{i+1} + d_i, 1 \leq i \leq n,$$

where  $a_0, a_{n+1}, b_i, c_i$  and  $d_i$  are known.  $a$  can then be obtained from

$$\{a_i\} = \text{tridiagonal}(\{1, -1, -1, \dots, -1, 1\}, \{0, c_1, c_2, \dots, c_n\}, \{b_1, b_2, \dots, b_n, 0\}, \{a_0, d_1, d_2, \dots, d_n, a_{n+1}\}).$$

Fails if the solution is not unique.

If  $|d_i| > |p_i| + |q_{i-1}|$  for all  $i$ , or  $|d_i| > |p_{i-1}| + |q_i|$ , or the matrix is positive definite, the algorithm is numerically stable and neither tr nor the check for diag[i] == 0 is needed.

**Time:**  $\mathcal{O}(N)$

be9642, 23 lines

```

typedef double T;
vector<T> tridiagonal(vector<T> diag, const vector<T> &super,
    const vector<T> &sub, vector<T> b) {
    int n = b.size(); vector<int> tr(n);
    for (int i = 0; i < n-1; ++i)
        if (abs(diag[i]) < 1e-9 * abs(super[i])) { // diag[i] == 0
            b[i+1] -= b[i] * diag[i+1] / super[i];
            if (i+2 < n) b[i+2] -= b[i] * sub[i+1] / super[i];
            diag[i+1] = sub[i]; tr[++i] = 1;
        } else {
            diag[i+1] -= super[i]*sub[i]/diag[i];
            b[i+1] -= b[i]*sub[i]/diag[i];
        }
    for (int i = n; i--;)
        if (tr[i]) {
            swap(b[i], b[i-1]); diag[i-1] = diag[i];
            b[i] /= super[i-1];
        } else {
            b[i] /= diag[i];
            if (i) b[i-1] -= b[i]*super[i-1];
        }
    return b;
}

```

## 4.1 Fourier transforms

### fast-fourier-transform.h

**Description:** For convolution of complex numbers or more than two vectors: FFT, multiply pointwise, FFT inverse back.

**Time:**  $\mathcal{O}(N \log N)$  with  $N = |A| + |B|$  ( $\sim 1s$  for  $N = 2^{22}$ )

"../number-theory/modular-arithmetic.h"

61aa30, 121 lines

```

template <typename dbl> struct cplx { // hash-1
    dbl x, y; using P = cplx;
    cplx(dbl x_ = 0, dbl y_ = 0) : x(x_), y(y_) {}
    friend P operator+(P a, P b) { return P(a.x+b.x, a.y+b.y); }
    friend P operator-(P a, P b) { return P(a.x-b.x, a.y-b.y); }
    friend P operator*(P a, P b) { return P(a.x*b.x - a.y*b.y, a.x*b.y + a.y*b.x); }
    friend P conj(P a) { return P(a.x, -a.y); }
    friend P inv(P a) { dbl n = (a.x*a.x+a.y*a.y); return P(a.x/n, -a.y/n); }
};
template <typename dbl> struct root_of_unity<cplx<dbl>> {
    static cplx<dbl> f(int k) {

```



```

    static const dbl PI = acos(-1); dbl a = 2*PI/k;
    return cplx<dbl>(cos(a),sin(a));
}
}; // hash-1 = 84c651
inline int nxt_pow2(int s) { return 1 << (s > 1 ? 32 -
    __builtin_clz(s-1) : 0); }
template<typename T> struct root_of_unity {};
// (MOD 3) := (M1:897581057), (M3:985661441), (M5:935329793)
using M0 = modnum<998244353U>; // hash-2
constexpr unsigned primitive_root(unsigned M) {
    if (M == 880803841U) return 26U; // (M2)
    else if (M == 943718401U) return 7U; // (M4)
    else if (M == 918552577U) return 5U; // (M6)
    else return 3U;
}
template<unsigned MOD> struct root_of_unity<modnum<MOD>> {
    static constexpr modnum<MOD> g0 = primitive_root(MOD);
    static modnum<MOD> f(int K) {
        assert((MOD-1)%K == 0); return g0.pow((MOD-1)/K);
    }
}; // hash-2 = de8785
template<typename T> struct FFT {
    vector<T> rt; vector<int> rev;
    FFT() : rt(2, T(1)) {}
    void init(int N) { // hash-3
        N = nxt_pow2(N);
        if (N > int(rt.size())) {
            rev.resize(N); rt.reserve(N);
            for (int a = 0; a < N; ++a)
                rev[a] = (rev[a/2] | ((a&1)*N)) >> 1;
            for (int k = int(rt.size()); k < N; k += 2) {
                rt.resize(2*k);
                T z = root_of_unity<T>::f(2*k);
                for (int a = k/2; a < k; ++a)
                    rt[2*a] = rt[a], rt[2*a+1] = rt[a] * z;
            }
        }
    } // hash-3 = fcd279
    void fft(vector<T>& xs, bool inverse) const { // hash-4
        int N = int(xs.size());
        int s = __builtin_ctz(int(rev.size())/N);
        if (inverse) reverse(xs.begin() + 1, xs.end());
        for (int a = 0; a < N; ++a) {
            if (a < (rev[a] >> s)) swap(xs[a], xs[rev[a] >> s]);
        }
        for (int k = 1; k < N; k += 2) {
            for (int a = 0; a < N; a += 2*k) {
                int u = a, v = u + k;
                for (int b = 0; b < k; ++b, ++u, ++v) {
                    T z = rt[b + k] * xs[v];
                    xs[v] = xs[u] - z, xs[u] = xs[u] + z;
                }
            }
        }
        if (inverse)
            for (int a = 0; a < N; ++a) xs[a] = xs[a] * inv(T(N));
    } // hash-4 = e2ea4a
    vector<T> convolve(vector<T> as, vector<T> bs) { // hash-5
        int N = int(as.size()), M = int(bs.size());
        int K = N + M - 1, S = nxt_pow2(K); init(S);
        if (min(N, M) <= 64) {
            vector<T> res(K);
            for (int u = 0; u < N; ++u) for (int v = 0; v < M; ++v)
                res[u + v] = res[u + v] + as[u] * bs[v];
            return res;
        } else {
            as.resize(S), bs.resize(S);
            fft(as, false); fft(bs, false);
            for (int i = 0; i < S; ++i) as[i] = as[i] * bs[i];

```

```

        fft(as, true); as.resize(K); return as;
    }
    } // hash-5 = e4b903
}; FFT<M0> FFT0;
// T = {unsigned, unsigned long long, modnum<M>}
// Remark: need to satisfy |poly| * mod^2 < \prod_{i} M_i
template<class T, unsigned M0, unsigned M1, unsigned M2,
    unsigned M3, unsigned M4>
T garner(modnum<M0> a0, modnum<M1> a1, modnum<M2> a2, modnum<M3>
    > a3, modnum<M4> a4) { // hash-6
    static const modnum<M1> INV_M0_M1 = modnum<M1>(M0).inv();
    static const modnum<M2> INV_M0M1_M2 = (modnum<M2>(M0) * M1).
        inv();
    // static const modnum<M3> INV_M0M1M2M3 = (modnum<M3>(M0) *
        M1 * M2).inv();
    // static const modnum<M4> INV_M0M1M2M3M4 = (modnum<M4>(M0)
        * M1 * M2 * M3).inv();
    const modnum<M1> b1 = INV_M0_M1 * (a1 - a0.x);
    const modnum<M2> b2 = INV_M0M1_M2 * (a2 - (modnum<M2>(b1.x) *
        M0 + a0.x));
    // const modnum<M3> b3 = INV_M0M1M2M3 * (a3 - ((modnum<M3>(
        b2.x) * M1 + b1.x) * M0 + a0.x));
    // const modnum<M4> b4 = INV_M0M1M2M3M4 * (a4 - (((modnum<M4>
        >(b3.x) * M2 + b2.x) * M1 + b1.x) * M0 + a0.x));
    return (T(b2.x) * M1 + b1.x) * M0 + a0.x;
    // return (((T(b4.x) * M3 + b3.x) * M2 + b2.x) * M1 + b1.x) *
        M0 + a0.x;
} // hash-6 = b93180
// results must be in [-448002610255888384, 448002611254132736]
vector<ll> convolve(const vector<ll>& as, const vector<ll>& bs)
    { // hash-7
    static constexpr unsigned M0 = M0::M, M1 = M1::M;
    static const modnum<M1> INV_M0_M1 = modnum<M1>(M0).inv();
    if (as.empty() || bs.empty()) return {};
    const int len_as = int(as.size()), len_bs = int(bs.size());
    vector<modnum<M0>> as0(len_as), bs0(len_bs);
    for (int i = 0; i < len_as; ++i) as0[i] = as[i];
    for (int i = 0; i < len_bs; ++i) bs0[i] = bs[i];
    const vector<modnum<M0>> cs0 = FFT0.convolve(as0, bs0);
    vector<modnum<M1>> as1(len_as), bs1(len_bs);
    for (int i = 0; i < len_as; ++i) as1[i] = as[i];
    for (int i = 0; i < len_bs; ++i) bs1[i] = bs[i];
    const vector<modnum<M1>> cs1 = FFT1.convolve(as1, bs1);
    vector<ll> cs(len_as + len_bs - 1);
    for (int i = 0; i < len_as + len_bs - 1; ++i) {
        const modnum<M1> d1 = INV_M0_M1 * (cs1[i] - cs0[i].x);
        cs[i] = (d1.x > M1 - d1.x)
            ? (-1ULL - (static_cast<unsigned ll>(M1 - 1U - d1.x) * M0
                + (M0 - 1U - cs0[i].x)))
            : (static_cast<unsigned ll>(d1.x) * M0 + cs0[i].x);
    }
    return cs;
} // hash-7 = a3bae7

```

## fast-subset-transform.h

**Description:** Transform to a basis with fast convolutions of the form  $c[z] = \sum_{z=x \oplus y} a[x] \cdot b[y]$ , where  $\oplus$  is one of AND, OR, XOR. The size of a must be a power of two.

**Time:**  $\mathcal{O}(N \log N)$

```

5b9574, 16 lines
void FST(vector<int> &a, bool inv) {
    for (int n = a.size(), step = 1; step < n; step *= 2) {
        for (int i = 0; i < n; i += 2 * step) for (int j = i; j < i +
            step; ++j) {
            int &u = a[j], &v = a[j + step]; tie(u, v) =
                inv ? pii(v - u, u) : pii(v, u + v); // AND
            inv ? pii(v, u - v) : pii(u + v, u); // OR
            pii(u + v, u - v); // XOR
        }
    }
}

```

```

    }
    if (inv) for (auto &x : a) x /= a.size(); // XOR only
}
vector<int> conv(vector<int> a, vector<int> b) {
    FST(a, 0); FST(b, 0);
    for (int i = 0; i < a.size(); ++i) a[i] *= b[i];
    FST(a, 1); return a;
}

```

## sum-of-powers.h

**Description:** Computes monomials and sum of powers product certain polynomials. Check "General purpose numbers" section for more info. (Mono-

mials)  $pw(x) = x^d$  for a fixed  $d$ .  $\sum_{x=0}^{\infty} r^x f(x)$ . (degree of  $f \leq d$ ).  $\sum_{x=0}^{N-1} r^x f(x)$ .

(degree of  $f \leq d$ ).

"/./number-theory/modular-arithmetic.h", "/lagrange.h" 85dfa0, 33 lines

```

vector<num> get_monomials(int N, long long d) { // hash-1
    vector<int> pfac(N);
    for (int i = 2; i < N; ++i) pfac[i] = i;
    for (int p = 2; p < N; ++p) if (pfac[p] == p)
        for (int m = 2*p; m < N; m += p) if (pfac[m] > p) pfac[m] = p;
    vector<num> pw(N);
    for (int i = 0; i < N; ++i)
        if (i <= 1 || pfac[i] == i) pw[i] = num(i).pow(d);
        else pw[i] = (pw[pfac[i]] * pw[i / pfac[i]]);
    return pw;
} // hash-1 = 966124
num sum_of_power_limit(num r, int d, const vector<num>& fs) {
    // hash-2
    interpolator_t<num> M(d + 2); num s = 1; auto gs = fs;
    for (int x = 0; x <= d; ++x, s *= r) gs[x] *= s;
    num ans = 0, cur_sum = 0; s = 1;
    for (int x = 0; x <= d; ++x, s *= -r) {
        cur_sum += choose(d+1, x) * s; ans += cur_sum * gs[d-x];
    } ans *= (1 - r).pow(-(d + 1));
    return ans;
} // hash-2 = d5b811
num sum_of_power(num r, int d, vector<num>& fs, ll N) { // hash
    -3
    if (r == 0) return (0 < N) ? fs[0] : 0;
    interpolator_t<num> M(d + 2);
    vector<num> gs(d + 2); gs[0] = 0; num s = 1;
    for (int x = 0; x <= d; ++x, s *= r)
        gs[x + 1] = gs[x] + s * fs[x];
    if (r == 1) return M.interpolate(gs, N);
    const num c = sum_of_power_limit(r, d, fs);
    const num r_inv = r.inv(); num w = 1;
    for (int x = 0; x <= d + 1; ++x, w *= r_inv)
        gs[x] = w * (gs[x] - c);
    return c + r.pow(N) * M.interpolate(gs, N);
} // hash-3 = 62a62d

```

## power-series.h

"fast-fourier-transform.h", "/./number-theory/mod-sqrt.h",  
"/./number-theory/preparator.h" 6c18f3, 131 lines

```

using num = modnum<998244353U>; FFT<num> fft_data;
using P = vector<num>;
void shrink(P& P) { while (!P.empty() && !P.back()) P.pop_back(); }
P& operator+=(P& a, const P& b) { // hash-1
    int N = int(a.size()), M = int(b.size());
    a.resize(max(N, M));
    for (int i = 0; i < M; ++i) a[i] += b[i];
    return a;
} // hash-1 = 2bef29
P operator+(const P& a, const P& b) { P r=a; return r+=b; }
P& operator-=(P& a, const P& b) { // hash-2
    int N = int(a.size()), M = int(b.size());

```

```

    a.resize(max(N, M));
    for (int i = 0; i < M; ++i) a[i] -= b[i];
    return a;
} // hash-2 = 6c3395
P operator-(const P& a, const P& b) { P r=a; return r-=b; }
P operator*(const P& a, const P& b) {
    return fft.convolve(a,b); }
P& operator*=(P& a, const P& b){ return a*=b; }
// inverse series mod x^N
P inverse(const P& P, int N) { // hash-3
    if (P.empty()) return {};
    P b({P[0]}), ret = {P[0].inv()};
    for (int len = 1; len < N; len += 2) {
        for (int i = len; i < min(2*len, int(P.size())); ++i)
            b.push_back(P[i]);
        (ret *= P[2] - b * ret).resize(2*len);
    }
    ret.resize(N); return ret;
} // hash-3 = e62bca
P operator/(P a, P b) { // hash-4
    int s = int(a.size()) - int(b.size()) + 1;
    if (s <= 0) return {};
    reverse(a.begin(), a.end()); reverse(b.begin(), b.end());
    a.resize(s); b.resize(s); a *= inverse(std::move(b), s);
    a.resize(s); reverse(a.begin(), a.end());
    return a;
} // hash-4 = 9a630a
P& operator/=(P& a, const P& b){return a = a / b;}
P& operator%=(P& a, const P& b) { // hash-5
    if (a.size() >= b.size()) {
        P c = (a/b) * b; a.resize(b.size()-1);
        for (int i = 0; i < int(a.size()); ++i)
            a[i] = a[i]-c[i];
    }
    shrink(a); return a;
} // hash-5 = 871616
P operator%(const P& a, const P& b) { P r=a; return r%=b; }
P deriv(P P) { // hash-6
    int N = int(P.size());
    if (N == 0) return {};
    P f(N-1);
    for (int x = 1; x < N; ++x) f[x-1] = x * P[x];
    return f;
} // hash-6 = 8c328d
P integ(P P) { // hash-7
    int N = int(P.size());
    if (N == 0) return {};
    P f(N+1); f[1] = 1;
    for (int x = 0; x < N; ++x) f[x+1] = invs[x+1] * P[x];
    return f;
} // hash-7 = bbdd46
// ln(P) mod x^N
P log(const P& P, int N) { // hash-8
    if (P.empty()) return {};
    P f = integ(deriv(P) * inverse(P, N));
    f.resize(N); return f;
} // hash-8 = 76ed43
// exp(P) mod x^N
P exp(P P, int N) { // hash-9
    if (P.empty()) return {1};
    P b({P[0]+1}), ret = {1};
    for (int len = 1; len < N; len += 2) {
        for (int i = len; i < min(2*len, int(P.size())); ++i)
            b.push_back(P[i]);
        (ret *= (b - log(ret, 2*len))).resize(2*len);
    }
    ret.resize(N); return ret;
} // hash-9 = 71a111
P pow(P a, ll e, int N) { // hash-10
```

```

    P b(N);
    if (!e) { b[0] = 1; return b; }
    int p = 0;
    while (e*p < N && a[p].x == 0) ++p;
    if (1LL*e*p >= N) return b;
    num mu = a[p].pow(e), di = a[p].inv();
    int M = e*p; P c(N-M);
    for (int x = 0; x < N-M; ++x) c[x] = a[x+p] * di;
    c = log(c, N-M);
    for (auto& x : c) x = x * e;
    c = exp(c, N-M);
    for (int x = 0; x < int(c.size()); ++x) b[x+M] = c[x]*mu;
    return b;
} // hash-10 = af6c8e
P eval(const P& a, const vector<num>& x) { // hash-11
    int N = int(x.size());
    if (!N) return {};
    vector<P> up(2*N), down(2*N);
    for (int i = 0; i < N; ++i) up[i+N]=P({0-x[i],1});
    for (int i = N-1; i >= 1; --i) up[i]=up[2*i]*up[2*i+1];
    down[1] = a % up[1];
    for (int i = 2; i < 2*N; ++i) down[i]=down[i/2]*up[i];
    vector<num> y(N);
    for (int i = 0; i < N; ++i)
        y[i]= down[i+N].empty() ? 0 : down[i+N][0];
    return y;
} // hash-11 = 28f80c
P interp(const vector<num>&x, const vector<num>&y){ // hash-12
    int N = int(x.size()); assert(N);
    vector<P> up(2*N), down(2*N);
    for (int i = 0; i < N; ++i) up[i+N]=P({0-x[i],1});
    for (int i = N-1; i >= 1; --i) up[i]=up[2*i]*up[2*i+1];
    vector<num> a = eval(deriv(up[1]), x);
    for (int i = 0; i < N; ++i) down[i+N]=P({y[i]*a[i].inv()});
    for (int i = N-1; i >= 1; --i)
        down[i]=down[2*i]*up[2*i+1] + down[2*i+1]*up[2*i];
    return down[1];
} // hash-12 = d81a89
P taylor_shift(P a, int N, int K) { // hash-13
    P P(N), Q = a;
    P[0] = 1;
    for (int i = 1; i < N; ++i) P[i] += P[i-1] * K;
    for (int i = 1; i < N; ++i) P[i] *= ifact[i];
    reverse(P.begin(), P.end());
    for (int i = 1; i < N; ++i) Q[i] *= fact[i];
    P *= Q;
    for (int i = 0; i < N; ++i) P[i] = P[N-1+i] * ifact[i];
    P.resize(N);
    return P;
} // hash-13 = d89033
```

4.1.1 LP formulation

Standard form: maximize  $\mathbf{c}^T \mathbf{x}$  subject to  $\mathbf{Ax} \leq \mathbf{b}$  and  $\mathbf{x} \geq 0$ .  
Dual LP: minimize  $\mathbf{b}^T \mathbf{y}$  subject to  $\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$  and  $\mathbf{y} \geq 0$ .  
 $\bar{\mathbf{x}}$  and  $\bar{\mathbf{y}}$  are optimal if and only if for all  $i \in [1, n]$ , either  $\bar{x}_i = 0$  or  $\sum_{j=1}^m A_{ji} \bar{y}_j = c_i$  holds and for all  $i \in [1, m]$  either  $\bar{y}_i = 0$  or  $\sum_{j=1}^n A_{ij} \bar{x}_j = b_j$  holds.

- 1. In case of minimization, let  $c'_i = -c_i$
- 2.  $\sum_{1 \leq i \leq n} A_{ji} x_i \geq b_j \rightarrow \sum_{1 \leq i \leq n} -A_{ji} x_i \leq -b_j$
- 3.  $\sum_{1 \leq i \leq n} A_{ji} x_i = b_j$ 
  - $\sum_{1 \leq i \leq n} A_{ji} x_i \leq b_j$
  - $\sum_{1 \leq i \leq n} A_{ji} x_i \geq b_j$

4. If  $x_i$  has no lower bound, replace  $x_i$  with  $x_i - x'_i$

4.1.2 Generating functions

A list of generating functions for useful sequences:

$(1, 1, 1, 1, 1, \dots)$	$\frac{1}{1-z}$
$(1, -1, 1, -1, 1, \dots)$	$\frac{1}{1+z}$
$(1, 0, 1, 0, 1, 0, \dots)$	$\frac{1-z^2}{1-z^2}$
$(1, 0, \dots, 0, 1, 0, 1, 0, \dots, 0, 1, 0, \dots)$	$\frac{1}{1-z^2}$
$(1, 2, 3, 4, 5, 6, \dots)$	$\frac{1}{(1-z)^2}$
$(1, \binom{m+1}{m}, \binom{m+2}{m}, \binom{m+3}{m}, \dots)$	$\frac{1}{(1-z)^{m+1}}$
$(1, c, \binom{c+1}{2}, \binom{c+2}{3}, \dots)$	$\frac{1}{(1-z)^c}$
$(1, c, c^2, c^3, \dots)$	$\frac{1}{1-cz}$
$(0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$	$\ln \frac{1}{1-z}$

A neat manipulation trick is:

$$\frac{1}{1-z} G(z) = \sum_n \sum_{k \leq n} g_k z^n$$

Number theory (5)

5.1 Modular arithmetic

modular-arithmetic.h

**Description:** Operators for modular arithmetic.

```

"mod-inv.h" 5a35b8, 27 lines

template<unsigned M_> struct modnum {
    static constexpr unsigned M = M_; using num = modnum;
    using ll = int64_t; using ull = uint64_t; unsigned x;
    num& norm(unsigned a){x = a<M ? a : a-M; return *this;}
    constexpr modnum(ll a = 0) : x(unsigned((a % M)<0 ? a+M : a)) {}
    explicit operator int() const { return x; }
    num& operator+=(const num& a){ return norm(x+a.x); }
    num& operator-=(const num& a){ return norm(x-a.x+M); }
    num& operator*=(const num& a){ return norm(unsigned(ull(x)*a.x % M)); }
    num& operator/=(const num& a){ return (*this *= a.inv()); }
    template<typename T> friend num operator+(T a, const num& b){
        return (num(a) += b); }
    template<typename T> friend num operator-(T a, const num& b){
        return (num(a) -= b); }
    template<typename T> friend num operator*(T a, const num& b){
        return (num(a) *= b); }
    template<typename T> friend num operator/(T a, const num& b){
        return (num(a) /= b); }
    num operator+() const { return *this; }
    num operator-() const { return num() - *this; }
    num pow(ll e) const {
        if (e < 0) { return inv().pow(-e); } num b = x, xe = 1U;
        for (; e >= 1) { if (e & 1) xe *= b; b *= b; }
        return xe;
    }
    num inv() const { return minv(x, M); }
    friend num inv(const num& a) { return a.inv(); }
    explicit operator bool() const { return x; }
    friend bool operator==(const num& a, const num& b){return a.x == b.x;}
    friend bool operator!=(const num& a, const num& b){return a.x != b.x;}
};
```

mod-inv.h  
**Description:** Find  $x$  such that  $ax \equiv 1(\text{mod } m)$ . The inverse only exist if  $a$  and  $m$  are coprimes.

```
template<typename T> T minv(T a, T m) {
    a %= m; assert(a);
    return a == 1 ? 1 : T(m - i64(minv(m, a)) * m / a);
}
```

mod-sum.h  
**Description:** Sums of mod'ed arithmetic progressions.  
 $\text{modsum}(to, c, k, m) = \sum_{i=0}^{to-1} (ki + c) \% m$ . `divsum` is similar but for floored division.  
**Time:**  $\log(m)$ , with a large constant.

```
typedef unsigned long long ull;
ull sumsq(ull to) { return to / 2 * ((to-1) | 1); }
ull divsum(ull to, ull c, ull k, ull m) { // hash-1
    ull res = k / m * sumsq(to) + c / m * to;
    k %= m; c %= m;
    if (k) {
        ull to2 = (to * k + c) / m;
        res += to * to2;
        res -= divsum(to2, m-1 - c, m, k) + to2;
    }
    return res;
} // hash-1 = ae0cdb
lint modsum(ull to, lint c, lint k, lint m) { // hash-2
    c = ((c % m) + m) % m; k = ((k % m) + m) % m;
    return to * c + k * sumsq(to) - m * divsum(to, c, k, m);
} // hash-2 = 29f27f
```

mod-mul.h  
**Description:** Calculate  $a \cdot b \text{ mod } c$  (or  $a^b \text{ mod } c$ ) for  $0 \leq a, b \leq c \leq 7.2 \cdot 10^{18}$ .  
**Time:**  $\mathcal{O}(1)$  for `modmul`,  $\mathcal{O}(\log b)$  for `modpow`

```
typedef unsigned long long ull;
ull modmul(ull a, ull b, ull M) { // hash-1
    lint ret = a * b - M * ull(1.L / M * a * b);
    return ret + M * (ret < 0) - M * (ret >= (lint)M);
} // hash-1 = e1a2a4
ull modpow(ull b, ull e, ull mod) { // hash-2
    ull ans = 1;
    for (; e; b = modmul(b, b, mod), e /= 2)
        if (e & 1) ans = modmul(ans, b, mod);
    return ans;
} // hash-2 = 100b91
```

mod-sqrt.h  
**Description:** Tonelli-Shanks algorithm for modular square roots. Finds  $x$  s.t.  $x^2 = a \text{ (mod } p)$  ( $-x$  gives the other solution).  
**Time:**  $\mathcal{O}(\log^2 p)$  worst case,  $\mathcal{O}(\log p)$  for most  $p$

```
int jacobi(ll a, ll m) { // hash-1
    int s = 1;
    if (a < 0) a = a % m + m;
    for (; m > 1; ) {
        a %= m; if (a == 0) return 0;
        const int r = __builtin_ctzll(a);
        if ((r & 1) && ((m + 2) & 4)) s = -s;
        a >>= r; if (a & m & 2) s = -s;
        swap(a, m);
    } return s;
} // hash-1 = aacc12
vector<ll> mod_sqrt(ll a, ll p) { // hash-2
    if (p == 2) return {a & 1};
    const int j = jacobi(a, p);
    if (j == 0) return {};
    if (j == -1) return {};
    ll b, d;
```

```
while (true) {
    b = xrand() % p; d = (b * b - a) % p;
    if (d < 0) d += p;
    if (jacobi(d, p) == -1) break;
}
ll f0 = b, f1 = 1, g0 = 1, g1 = 0, tmp;
for (ll e = (p + 1) >> 1; e; e >>= 1) {
    if (e & 1) {
        tmp = (g0 * f0 + d * ((g1 * f1) % p)) % p;
        g1 = (g0 * f1 + g1 * f0) % p; g0 = tmp;
    }
    tmp = (f0 * f0 + d * ((f1 * f1) % p)) % p;
    f1 = (2 * f0 * f1) % p; f0 = tmp;
}
return (g0<p-g0) ? vector<ll>{g0,p-g0} : vector<ll>{p-g0,g0};
} // hash-2 = b22da7
```

mod-range.h  
**Description:**  $\min x \geq 0$  s.t.  $l \leq ((ax) \text{ mod } m) \leq r, m > 0, a \geq 0$ .

```
template<typename T> T mod_range(T m, T a, T l, T r) {
    l = max(l, T(0)); r = min(r, m - 1);
    if (l > r) return -1;
    a %= m;
    if (a == 0) return (l > 0) ? -1 : 0;
    const T k = (l + a - 1) / a;
    if (a * k <= r) return k;
    const T y = mod_range(a, m, a * k - r, a * k - 1);
    return (y == -1) ? -1 : ((m * y + r) / a);
}
```

## 5.2 Primality

sieve.h  
**Description:** Prime sieve for generating all primes up to a certain limit.  $\text{pfac}[i]$  is the lowest prime factor of  $i$ . Also useful if you need to compute any multiplicative function.  
**Time:**  $\mathcal{O}(N)$

```
vector<int> run_sieve(int N) {
    vector<int> pfac(N+1), primes, mu(N+1,-1), phi(N+1);
    primes.reserve(N+1); mu[1] = phi[1] = 1;
    for (int i = 2; i <= N; ++i) {
        if (!pfac[i])
            pfac[i] = i, phi[i] = i-1, primes.push_back(i);
        for (int p : primes) {
            if (p > N/i) break;
            pfac[p * i] = p; mu[p * i] *= mu[i];
            phi[p * i] = phi[i] * phi[p];
            if (i % p == 0) {
                mu[p * i] = 0; phi[p * i] = phi[i] * p;
                break;
            }
        }
    } return primes;
}
```

miller-rabin.h  
**Description:** Deterministic Miller-Rabin primality test. Guaranteed to work for numbers up to  $2^{64}$ ; for larger numbers, extend A randomly.  
**Time:** 7 times the complexity of  $a^b \text{ mod } c$ .

```
"mod-mul.h"
bool isPrime(ull n) { // hash-1
    if (n < 2 || n % 6 % 4 != 1) return (n | 1) == 3;
    vector<ull> A = {2, 325, 9375, 28178, 450775, 9780504,
        1795265022};
    ull s = __builtin_ctzll(n-1), d = n >> s; // hash-1 = 8cef21
    for(ull a : A) { // ^ count trailing zeroes
        ull p = modpow(a % n, d, n), i = s;
```

```
while (p != 1 && p != n - 1 && a % n && i--)
    p = modmul(p, p, n);
    if (p != n-1 && i != s) return 0;
}
return 1;
}
```

pollard-rho.h  
**Description:** Pollard-rho randomized factorization algorithm. Returns prime factors of a number, in arbitrary order (e.g. 2299 -> {11, 19, 11}).  
**Time:**  $\mathcal{O}\left(n^{1/4}\right)$ , less for numbers with small factors.

```
"mod-mul.h", "extended-euclid.h", "miller-rabin.h"
ull pollard(ull n) { // hash-1
    auto f = [n](ull x, ull k) { return modmul(x, x, n) + k; };
    ull x = 0, y = 0, t = 30, prd = 2, i = 1, q;
    while (t++ % 40 || gcd(prd, n) == 1) {
        if (x == y) x = ++i, y = f(x, i);
        if ((q = modmul(prd, max(x,y) - min(x,y), n))) prd = q;
        x = f(x, i), y = f(f(y, i), i);
    }
    return gcd(prd, n);
} // hash-1 = 8875ba
vector<ull> factor(ull n) { // hash-2
    if (n == 1) return {};
    if (isPrime(n)) return {n};
    ull x = pollard(n); auto l = factor(x), r = factor(n/x);
    l.insert(l.end(), r.begin(), r.end());
    return l;
} // hash-2 = 04f1a4
```

## 5.3 Divisibility

extended-euclid.h  
**Description:** Finds two integers  $x$  and  $y$ , such that  $ax + by = \text{gcd}(a, b)$ . If  $a$  and  $b$  are coprime, then  $x$  is the inverse of  $a \text{ (mod } b)$ .

```
template<typename T>
T egcd(T a, T b, T &x, T &y) {
    if (!a) { x = 0, y = 1; return b; }
    T g = egcd(b % a, a, y, x);
    x -= y * (b/a); return g;
}
```

division-lemma.h  
**Description:** This lemma let us exploit the fact tha the sequence (harmonic on integer division) has at most  $2\sqrt{N}$  distinct elements, so we can iterate through every possible value of  $\lfloor \frac{N}{i} \rfloor$ , using the fact that the greatest integer  $j$  satisfying  $\lfloor \frac{N}{i} \rfloor = \lfloor \frac{N}{j} \rfloor$  is  $\lfloor \frac{N}{\lfloor \frac{N}{i} \rfloor} \rfloor$ . This one computes the  $\sum_{i=1}^N \lfloor \frac{N}{i} \rfloor$ .

```
Time:  $\mathcal{O}\left(\sqrt{N}\right)$ 
int res = 0;
for (int a = 1, b; a <= N; a = b + 1) { // floor
    b = N / (N / a);
    // quotient (N/a) and there are (b - a + 1) elements
    int l = b - a + 1, r = a + b; // l * r / 2 = sum(i, j)
    if (l & 1) r /= 2;
    else l /= 2;
    res += l * r * (N / a);
}
// [1, N), need to deal with case where a = N separately
for (int a = 1, b; a < N; a = b + 1) { // ceil
    const int k = (N - 1) / a + 1; // quotient k
    b = (N - 1) / (k - 1);
    int cnt = b - a + 1; // occur cnt times on interval [a, b)
}
```

divisors.h

**Description:** Generate all factors of  $n$  given it's prime factorization.

Time:  $\mathcal{O}\left(\frac{\sqrt{N}}{\log N}\right)$

"prime-factors.h"	5de75c, 14 lines
-------------------	------------------

```
template<typename T> vector<T> get_divisors(T N) {
    auto factors = prime_factorize(N);
    vector<T> ans; ans.reserve(int(sqrtl(N) + 1));
    auto dfs = [&](auto&& self, auto& ans, T val, int d)->void {
        auto& [P, E] = factors[d];
        if (d == int(factors.size())) ans.push_back(val);
        else {
            T X = 1;
            for (int pw = 0; pw <= E; ++pw, X *= P)
                self(self, ans, val * X, d + 1);
        }
    }; dfs(dfs, ans, 1, 0);
    return ans;
}
```

phi-function.h

da7671, 6 lines
-----------------

```
const int n = int(1e5)*5; vector<int> phi(n);
void calculatePhi() {
    for(int i = 0; i < n; ++i) phi[i] = i&1 ? i : i/2;
    for(int i = 3; i < n; i += 2) if (phi[i] == i)
        for(int j = i; j < n; j += i) phi[j] -= phi[j]/i;
}
```

discrete-log.h

**Description:** Returns the smallest  $x \geq 0$  s.t.  $a^x = b \pmod m$ , or  $-1$  if no such  $x$  exists. `modLog(a,l,m)` can be used calculate the order of  $a$ . Assumes that  $0^0 = 1$ .

Time:  $\mathcal{O}(\sqrt{m})$

"extended-euclid.h"	62fc5e, 15 lines
---------------------	------------------

```
template<typename T> T modLog(T a, T b, T m) {
    T k = 1, it = 0, g; // hash-1
    while ((g = gcd(a, m)) != 1) {
        if (b == k) return it;
        if (b % g) return -1;
        b /= g; m /= g; ++it; k = k * a / g % m;
    } // hash-1 = 416572
    T n = sqrtl(m) + 1, f = 1, j = 1; // hash-2
    unordered_map<T, T> A;
    while (j <= n)
        f = f * a % m, A[f * b % m] = j++;
    for(int i = 1; i <= n; ++i) if (A.count(k = k * f % m))
        return n * i - A[k] + it;
    return -1; // hash-2 = 25c7bf
}
```

prime-counting.h

**Description:** Count the number of primes up to  $N$ . Also useful for sum of primes.

Time:  $\mathcal{O}(N^{3/4}/\log N)$

513314, 21 lines
------------------

```
struct primes_t {
    vector<ll> dp, w;
    ll pi(ll N) { // hash-1
        dp.clear(), w.clear();
        const int sqrtN = int(sqrt(N));
        for (ll a = 1, b; a <= N; a = b+1)
            b = N / (N / a), w.push_back(N/a);
        auto get = [&](ll x) {
            if (x <= sqrtN) return int(x-1);
            return int(w.size() - N/x);
        }; // hash-1 = a785c9
        reverse(w.begin(), w.end()); dp.reserve(w.size());
        for (auto& x : w) dp.push_back(x-1); // hash-2
    };
};
```

--	--

```
for (ll i = 2; i*i <= N; ++i) {
    if (dp[i-1] == dp[i-2]) continue;
    for (int j = int(w.size()-1; w[j] >= i*i; --j)
        dp[j] -= dp[get(w[j]/i)] - dp[i-2];
    }
    return dp.back(); // hash-2 = c6b848
};
```

5.4 Chinese remainder theorem

chinese-remainder.h

**Description:** Chinese Remainder Theorem. `crt(a, m, b, n)` computes  $x$  such that  $x \equiv a \pmod m$ ,  $x \equiv b \pmod n$ . If  $|a| < m$  and  $|b| < n$ ,  $x$  will obey  $0 \leq x < \text{lcm}(m,n)$ . Assumes  $mn < 2^{62}$ .

Time:  $\mathcal{O}(\log(\text{LCM}(m)))$

"extended-euclid.h"	ece59a, 7 lines
---------------------	-----------------

```
pair<ll, ll> crt(ll a, ll m, ll b, ll n) {
    if (n > m) swap(a, b), swap(m, n);
    ll x, y, g = egcd(m, n, x, y);
    if ((a - b) % g != 0) return {0, -1};
    x = (b - a) % n * x % n / g * m + a;
    return {x + (x < 0 ? m*n/g : 0), m*n/g};
}
```

5.5 Fractions

fractions.h

**Description:** Template that helps deal with frtions.

596163, 28 lines

--	--

```
template<typename num> struct fraction_t {
    num p, q; using fr = fraction_t; // hash-1
    fraction_t() : p(0), q(1) {}
    fraction_t(num _n, num _d = 1): p(_n), q(_d){
        num g = gcd(p, q); p /= g, q /= g;
        if (q < 0) p *= -1, q *= -1; assert(q != 0);
    } // hash-1 = 2306e9
    friend bool operator<(const fr& l, const fr& r){ // hash-2
        return l.p*r.q < r.p*l.q;}
    friend bool operator==(const fr& l, const fr& r){return l.p
        == r.p && l.q == r.q;}
    friend bool operator!=(const fr& l, const fr& r){return !(l
        == r);}
    friend fr operator+(const fr& l, const fr& r){
        num g = gcd(l.q, r.q);
        return fr(r.q / g * l.p + l.q / g * r.p, l.q / g * r.q);
    } // hash-2 = 604bf5
    friend fr operator-(const fr& l, const fr& r) { // hash-3
        num g = gcd(l.q, r.q);
        return fr( r.q / g * l.p - l.q / g * r.p, l.q / g * r.q);
    }
    friend fr operator*(const fr& l, const fr& r){
        return fr(l.p*r.p, l.q*r.q);}
    friend fr operator/(const fr& l, const fr& r){
        return l*fr(r.q,r.p);} // hash-3 = 9d4fb0
    friend fr& operator+=(fr& l, const fr& r){return l=l+r;}
    friend fr& operator-=(fr& l, const fr& r){return l=l-r;}
    template<class T> friend fr& operator*=(fr& l, const T& r){
        return l=l*r;}
    template<class T> friend fr& operator/=(fr& l, const T& r){
        return l=l/r;}
};
```

continued-fractions.h

**Description:** Given  $N$  and a real number  $x \geq 0$ , finds the closest rational approximation  $p/q$  with  $p, q \leq N$ . It will obey  $|p/q - x| \leq 1/qN$ .

For consecutive convergents,  $p_{k+1}q_k - q_{k+1}p_k = (-1)^k$ .  $(p_k/q_k)$  alternates between  $> x$  and  $< x$ . If  $x$  is rational,  $y$  eventually becomes  $\infty$ ; if  $x$  is the root of a degree 2 polynomial the  $a$ 's eventually become cyclic.

Time:  $\mathcal{O}(\log N)$

3dfe17, 18 lines
------------------

```
typedef double dbl; // for N ~ 1e7; long double for N ~ 1e9
pair<ll, ll> approximate(dbl x, ll N) {
    ll LP = 0, LQ = 1, P = 1, Q = 0, inf = ll(1e18); dbl y = x;
    for (;;) {
        ll lim = min(P ? (N-LP) / P : inf, Q ? (N-LQ) / Q : inf),
            a = (ll)floor(y), b = min(a, lim),
            NP = b*P + LP, NQ = b*Q + LQ;
        if (a > b) { // hash-1
            // If b > a/2, we have a semi-convergent that gives us a
            // better approximation; if b = a/2, we *may* have one.
            // Return {P, Q} here for a more canonical approximation.
            return (abs(x - (dbl)NP / (dbl)NQ) < abs(x - (dbl)P / (
                dbl)Q)) ?
                make_pair(NP, NQ) : make_pair(P, Q);
        }
        if (abs(y = 1/(y - (dbl)a)) > 3*N) return {NP, NQ};
        LP = P; P = NP; LQ = Q; Q = NQ;
    } // hash-1 = bfee6a
}
```

frac-binary-search.h

**Description:** Given  $f$  and  $N$ , finds the smallest fraction  $p/q \in [0, 1]$  such that  $f(p/q)$  is true, and  $p, q \leq N$ . You may want to throw an exception from  $f$  if it finds an exact solution, in which case  $N$  can be removed.

**Usage:** `fracBS([f](Frac f) { return f.p>=3*f.q; }, 10);` // {1,3}  
**Time:**  $\mathcal{O}(\log(N))$

74ffd8, 20 lines

--	--

```
struct Frac { ll p, q; };
template<class F> Frac fracBS(F f, ll N) {
    bool dir = 1, A = 1, B = 1;
    Frac left{0, 1}, right{1, 1}; //right{1,0} to search (0,N]
    assert(!f(left)); assert(f(right));
    while (A || B) { // hash-1
        ll adv = 0, step = 1; // move right if dir, else left
        for (int si = 0; step; (step *= 2) >= si) {
            adv += step;
            Frac mid{left.p * adv + right.p, left.q * adv + right.q};
            if (abs(mid.p) > N || mid.q > N || dir == !f(mid)) {
                adv -= step; si = 2;
            }
        }
        right.p += left.p * adv; right.q += left.q * adv;
        dir = !dir; swap(left, right);
        A = B; B = !adv;
    } // hash-1 = dedbc8
    return dir ? right : left;
}
```

5.5.1 Pythagorean Triples

The Pythagorean triples are uniquely generated by

$$a = k \cdot (m^2 - n^2), \quad b = k \cdot (2mn), \quad c = k \cdot (m^2 + n^2),$$

with  $m > n > 0$ ,  $k > 0$ ,  $m \perp n$ , and either  $m$  or  $n$  even.

5.6 Primes

$p = 962592769$  is such that  $2^{21} \mid p - 1$ , which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than 1 000 000.

5.6.1 Prime counting function ( $\pi(x)$ )

The prime counting function is asymptotic to  $\frac{x}{\log x}$ , by the prime number theorem.



x	10	10 <sup>2</sup>	10 <sup>3</sup>	10 <sup>4</sup>	10 <sup>5</sup>	10 <sup>6</sup>	10 <sup>7</sup>	10 <sup>8</sup>
π(x)	4	25	168	1.229	9.592	78.498	664.579	5.761.455

5.6.2 Sum of primes

For any multiplicative  $f$ :

$$S(n,p) = S(n,p-1) - f(p) \cdot (S(n/p,p-1) - S(p-1,p-1))$$

5.6.3 Moebius Function

$$\mu(n) = \begin{cases} 0 & n \text{ is not square free} \\ 1 & n \text{ has even number of prime factors} \\ -1 & n \text{ has odd number of prime factors} \end{cases}$$

Moebius Inversion:

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d)g(n/d)$$

Other useful formulas/forms:

$$\sum_{d|n} \phi(d) = n$$

$$\sum_{\substack{i < n \\ \gcd(i,n)=1}} i = n \frac{\phi(n)}{2}$$

$$\sum_{a=1}^n \sum_{b=1}^n [\gcd(a,b) = 1] = \sum_{d=1}^n \mu(d) \lfloor \frac{n}{d} \rfloor^2$$

$$\sum_{a=1}^n \sum_{b=1}^n \gcd(a,b) = \sum_{d=1}^n d \sum_{d|x} \lfloor \frac{n}{x} \rfloor^2 \mu(\frac{x}{d})$$

$$\sum_{a=1}^n \sum_{b=a}^n \gcd(a,b) = \sum_{d=1}^n \sum_{d|x} \phi(\frac{x}{d}) d$$

$$\sum_{a=1}^n \sum_{b=1}^n \text{lcm}(a,b) = \sum_{d=1}^n \mu(d) d \sum_{d|x} x \binom{\lfloor \frac{n}{x} \rfloor + 1}{2}$$

$$\sum_{a=1}^n \sum_{b=a+1}^n \text{lcm}(a,b) = \sum_{d=1}^n \sum_{d|x \wedge x > d} \phi(\frac{x}{d}) \frac{x^2}{2d}$$

$$\sum_{a \in S} \sum_{b \in S} \gcd(a,b) = \sum_{d=1}^n (\sum_{x|d} \frac{d}{x} \mu(x)) (\sum_{d|v} \text{freq}[v])^2$$

$$\sum_{a \in S} \sum_{b \in S} \text{lcm}(a,b) = \sum_{d=1}^n \frac{1}{d} (\sum_{x|d} x \mu(x)) (\sum_{d|v} v \cdot \text{freq}[v])^2$$

$$\sum_{d|n} \mu(d) = [n = 1] \text{ (very useful)}$$

$$g(n) = \sum_{n|d} f(d) \Leftrightarrow f(n) = \sum_{n|d} \mu(d/n)g(d)$$

$$g(n) = \sum_{1 \leq m \leq n} f(\lfloor \frac{n}{m} \rfloor) \Leftrightarrow f(n) = \sum_{1 \leq m \leq n} \mu(m)g(\lfloor \frac{n}{m} \rfloor)$$

5.6.4 Dirichlet Convolution

Given a function  $f(x)$ , let

$$(f * g)(x) = \sum_{d|x} g(d)f(x/d)$$

If the partial sums  $s_{f * g}(n), s_g(n)$  can be computed in  $O(1)$  and  $s_f(1 \dots n^{2/3})$  can be computed in  $O(n^{2/3})$  then all  $s_f(\frac{n}{d})$  can as well. Use

$$s_{f * g}(n) = \sum_{d=1}^n g(d)s_f(n/d).$$

$$\implies s_f(n) = \frac{s_{f * g}(n) - \sum_{d=2}^n g(d)s_f(n/d)}{g(1)}$$

- 1. If  $f(x) = \mu(x)$  then  $g(x) = 1, (f * g)(x) = (x == 1)$ , and  $s_f(n) = 1 - \sum_{i=2}^n s_f(n/i)$
- 2. If  $f(x) = \phi(x)$  then  $g(x) = 1, (f * g)(x) = x$ , and  $s_f(n) = \frac{n(n+1)}{2} - \sum_{i=2}^n s_f(n/i)$

dirichlet-convolution.h

**Description:** Dirichlet convolution. Change  $f, g$ s and  $fgs$  accordingly. This example calculates  $\phi(N)$ .

**Time:**  $\mathcal{O}(N^{\frac{2}{3}})$  eac754, 21 lines

```
template<typename T, typename V> struct mertens {
    V N; T inv; // ~ N^{2/3}
    vector<V> fs; vector<T> psum;
    unordered_map<V, T> mapa;
    V f(V x) { return fs[x]; } // hash-1
    T gs(V x) { return x; }
    T fgs(V x) { return T(x) * (x + 1) / 2; }
    mertens(V M, const vector<V>& F) : N(M+1), fs(F), psum(M+1) {
        inv = gs(1);
        for (V a = 0; a + 1 < N; ++a)
            psum[a + 1] = f(a + 1) + psum[a];
    } // hash-1 = ceee32
    T query(V x) { // hash-2
        if (x < N) return psum[x];
        if (mapa.find(x) != mapa.end()) return mapa[x];
        T ans = fgs(x);
        for (V a = 2, b; a <= x; a = b + 1)
            b = x / (x/a), ans -= (gs(b)-gs(a-1)) * query(x/a);
        return mapa[x] = (ans / inv);
    } // hash-2 = 43d0b6
};
```

Combinatorial (6)

6.1 Permutations

6.1.1 Factorial

$n$	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24	120	720	5040	40320	362880	3628800
$\frac{n}{n!}$	1	11	12	13	14	15	16	17		
$\frac{n!}{n}$	4.0e7	4.8e8	6.2e9	8.7e10	1.3e12	2.1e13	3.6e14			
$\frac{n}{n!}$	20	25	30	40	50	100	150	171		
$n!$	2e18	2e25	3e32	8e47	3e64	9e157	6e262	>DBL_MAX		

int-perm.h

**Description:** Permutation -> integer conversion. (Not order preserving.)

**Time:**  $\mathcal{O}(n)$  06f786, 6 lines

```
int permToInt(vector<int>& v) {
    int use = 0, i = 0, r = 0;
    for (auto &x : v) r=r* ++i + __builtin_popcount(use&-(1<=x)),
        use |= 1 <= x; // (note: minus, not ~!)
    return r;
}
```

6.1.2 Binomials

- Sum of every element in the  $n$ -th row of pascal triangle is  $2^n$ .
- The product of the elements in each row is  $\frac{(n+1)^n}{n!}$
- $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$
- In a row  $p$  where  $p$  is a prime number, all the terms in that row except the 1s are multiples of  $p$
- To count odd terms in row  $n$ , convert  $n$  to binary. Let  $x$  be the number of 1s in the binary representation. Then the number of odd terms will be  $2^x$
- Every entry in row  $2^n - 1$  is odd

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$$
$$\sum_{k=0}^n \binom{n}{k} k = n 2^{n-1}$$
$$\sum_{m=0}^n \binom{m}{j} \binom{n-m}{k-j} = \binom{n+1}{k+1}$$
$$\binom{n}{k} \binom{n-k}{v} = \binom{n}{v} \binom{n-v}{k}$$
$$\sum_{j=0}^m \binom{m}{j}^2 = \binom{2m}{m}$$

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$
$$\sum_{j=0}^k \binom{m}{j} \binom{n-m}{k-j} = \binom{n}{k}$$
$$\sum_{r=0}^k m \binom{n+r}{r} = \binom{n+m+1}{k+1}$$
$$\sum_{m=k}^n \binom{m}{k} = \binom{n+1}{k+1}$$
$$2 \sum_{i=L}^R \binom{n}{i} - \binom{n}{L} - \binom{n}{R} = \sum_{i=L+1}^R \binom{n+1}{i}$$

lucas.h

**Description:** Lucas' thm: Let  $n, m$  be non-negative integers and  $p$  a prime. Write  $n = n_k p^k + \dots + n_1 p + n_0$  and  $m = m_k p^k + \dots + m_1 p + m_0$ . Then  $\binom{n}{m} \equiv \prod_{i=0}^k \binom{n_i}{m_i} \pmod{p}$ . fact and ifact must hold pre-computed factorials / inverse factorials, e.g. from ModInv.h.

**Time:**  $\mathcal{O}(\log_p m)$

rolling-binomial.h

**Description:**  $\binom{n}{k} \pmod{m}$  in time proportional to the difference between  $(n, k)$  and the previous  $(n, k)$ .

**Time:**  $\mathcal{O}(n)$  de359e, 12 lines

```
struct Bin {
    int N = 0, K = 0; ll r = 1;
    void m(int a, int b) { r = r * a % mod * invs[b] % mod; }
    ll choose(int n, int k) {
        if (k > n || k < 0) return 0;
        while(N < n) ++N, m(N, N - K);
        while(K < k) ++K, m(N - K + 1, K);
        while(K > k) m(K, N - K + 1), --K;
        while(N > n) m(N - K, N), --N;
        return r;
    }
};
```

multinomial.h

**Description:** Computes  $\binom{k_1 + \dots + k_n}{k_1, k_2, \dots, k_n} = \frac{(\sum k_i)!}{k_1! k_2! \dots k_n!}$ . 864cdb, 7 lines

```
lint multinomial(vector<int>& v) {
    lint c = 1, m = v.empty() ? 1 : v[0];
    for (int i = 1 < v.size(); ++i)
        for (int j = 0; j < v[i]; ++j)
            c = c * ++m / (j+1);
    return c;
}
```



6.1.3 The twelvefold way (from Stanley)

How many functions  $f: N \rightarrow X$  are there?

$N$	$X$	Any $f$	Injective	Surjective
dist.	dist.	$x^n$	$\frac{x!}{(x-n)!}$	$x!\{x\}^n$
indist.	dist.	$\binom{x+n-1}{n}$	$\binom{x}{n}$	$\binom{n-1}{n-x}$
dist.	indist.	$\{1\}^n + \dots + \{x\}^n$	$[n \leq x]$	$\{x\}^n$
indist.	indist.	$p_1(n) + \dots + p_x(n)$	$[n \leq x]$	$p_x(n)$

Where  $\binom{a}{b} = \frac{1}{b!}(a)_b$ ,  $p_x(n)$  is the number of ways to partition the integer  $n$  using  $x$  summand and  $\{x\}^n$  is the number of ways to partition a set of  $n$  elements into  $x$  subsets (aka Stirling number of the second kind).

6.1.4 Burnside

Given a group  $G$  of symmetries and a set  $X$ , the number of elements of  $X$  up to symmetry equals

$$\frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where  $X^g$  are the elements fixed by  $g$  ( $g.x = x$ ).

If  $f(n)$  counts “configurations” (of some sort) of length  $n$ , we can ignore rotational symmetry using  $G = \mathbb{Z}_n$  to get

$$g(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\gcd(n, k)) = \frac{1}{n} \sum_{k|n} f(k) \phi(n/k)$$

6.1.5 Cycles

Let the number of  $n$ -permutations whose cycle lengths all belong to the set  $S$  be denoted by  $g_S(n)$

$$\sum_{n=0}^\infty g_S(n) \frac{x^n}{n!} = \exp \left( \sum_{n \in S} \frac{x^n}{n} \right)$$

6.1.6 Involutions

An involution is a permutation with maximum cycle length 2, and it is its own inverse.

$$a(n) = a(n-1) + (n-1)a(n-2), a(0) = a(1) = 1.$$

1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, 35696, 140152

6.1.7 Derangements

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1) + D(n-2)) = nD(n-1) + (-1)^n = \left\lfloor \frac{n!}{e} \right\rfloor$$

6.2 Partitions and subsets

6.2.1 Partition function

Number of ways of writing  $n$  as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1, \quad p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} p(n - k(3k-1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

$\frac{n}{p(n)}$	0	1	2	3	4	5	6	7	8	9	20	50	100
	1	1	2	3	5	7	11	15	22	30	627	$\sim 2e5$	$\sim 2e8$

partitions.h

3af1e7, 14 lines

```
vector<int64_t> prep(int N) {
    vector<int64_t> dp(N); dp[0] = 1;
    for (int n = 1; n < N; ++n) {
        int64_t sum = 0;
        for (int k = 0, l = 1, m = n - 1; ; ) {
            sum += dp[m]; if ((m -= (k += 1)) < 0) break;
            sum += dp[m]; if ((m -= (l += 2)) < 0) break;
            sum -= dp[m]; if ((m -= (k += 1)) < 0) break;
            sum -= dp[m]; if ((m -= (l += 2)) < 0) break;
        }
        if ((sum % M) < 0) sum += M;
        dp[n] = sum;
    } return dp;
}
```

6.3 General purpose numbers

6.3.1 Bernoulli numbers

EGF of Bernoulli numbers is  $B(t) = \frac{t}{e^t-1}$  (FFT-able).

$$B[0, \dots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots]$$

Sums of powers:

$$\sum_{i=1}^n n^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k (n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\begin{aligned} \sum_{i=m}^\infty f(i) &= \int_m^\infty f(x)dx - \sum_{k=1}^\infty \frac{B_k}{k!} f^{(k-1)}(m) \\ &\approx \int_m^\infty f(x)dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m)) \end{aligned}$$

6.3.2 Stirling numbers of the first kind

Number of permutations on  $n$  items with  $k$  cycles.

$$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k), c(0, 0) = 1$$

$$\sum_{k=0}^n c(n, k) x^k = x(x+1) \dots (x+n-1)$$

$$c(8, k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$$

$$c(n, 2) = 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \dots$$

6.3.3 Eulerian numbers

Number of permutations  $\pi \in S_n$  in which exactly  $k$  elements are greater than the previous element.  $k$  j:s s.t.  $\pi(j) > \pi(j+1)$ ,  $k+1$  j:s s.t.  $\pi(j) \geq j$ ,  $k$  j:s s.t.  $\pi(j) > j$ .

$$E(n, k) = (n-k)E(n-1, k-1) + (k+1)E(n-1, k)$$

$$E(n, 0) = E(n, n-1) = 1$$

$$E(n, k) = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n$$

6.3.4 Stirling numbers of the second kind

Partitions of  $n$  distinct elements into exactly  $k$  groups.

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

$$S(n, 1) = S(n, n) = 1$$

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n$$

6.3.5 Bell numbers

Total number of partitions of  $n$  distinct elements.  $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \dots$

$$\mathcal{B}_{n+1} = \sum_{k=0}^n \binom{n}{k} \mathcal{B}_k$$

Also possible to calculate using Stirling numbers of the second kind,

$$B_n = \sum_{k=0}^n S(n, k)$$

If  $p$  is prime:

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod{p}$$

6.3.6 Labeled unrooted trees

# on  $n$  vertices:  $n^{n-2}$   
# on  $k$  existing trees of size  $n_i$ :  $n_1 n_2 \dots n_k n^{k-2}$   
# with degrees  $d_i$ :  $(n-2)! / ((d_1-1)! \dots (d_n-1)!)$  # forests with exactly  $k$  rooted trees:

$$\binom{n}{k} k \cdot n^{n-k-1}$$

6.3.7 Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1, \quad C_{n+1} = \frac{2(2n+1)}{n+2} C_n, \quad C_{n+1} = \sum C_i C_{n-i}$$

$$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$$

- sub-diagonal monotone paths in a  $n \times n$  grid.
- strings with  $n$  pairs of parenthesis, correctly nested.
- binary trees with  $n+1$  leaves (0 or 2 children) or  $2n+1$  elements.
- ordered trees with  $n+1$  vertices.
- # ways a convex polygon with  $n+2$  sides can be cut into triangles by connecting vertices with straight lines.
- permutations of  $[n]$  with no 3-term increasing subsequence.

6.3.8 Super Catalan numbers

The number of monotonic lattice paths of a  $nxn$  grid that do not touch the diagonal.

$$S(n) = \frac{3(2n-3)S(n-1) - (n-3)S(n-2)}{n}$$
$$S(1) = S(2) = 1$$

1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859

6.3.9 Motzkin numbers

Number of ways of drawing any number of nonintersecting chords among  $n$  points on a circle. Number of lattice paths from  $(0, 0)$  to  $(n, 0)$  never going below the  $x$ -axis, using only steps NE, E, SE.

$$M(n) = \frac{3(n-1)M(n-2) + (2n+1)M(n-1)}{n+2}$$

$$M(0) = M(1) = 1$$

1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634

6.3.10 Narayana numbers

Number of lattice paths from  $(0,0)$  to  $(2n,0)$  never going below the  $x$ -axis, using only steps NE and SE, and with  $k$  peaks.

$$N(n,k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$$

$$N(n,1) = N(n,n) = 1$$

$$\sum_{k=1}^n N(n,k) = C_n$$

1, 1, 1, 1, 3, 1, 1, 6, 6, 1, 1, 10, 20, 10, 1, 1, 15, 50

6.3.11 Schroder numbers

Number of lattice paths from  $(0, 0)$  to  $(n, n)$  using only steps N,NE,E, never going above the diagonal. Number of lattice paths from  $(0, 0)$  to  $(2n, 0)$  using only steps NE, SE and double east EE, never going below the  $x$ -axis. Twice the Super Catalan number, except for the first term.

1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098

6.3.12 Triangles

Given rods of length 1, ...,  $n$ ,

$$T(n) = \frac{1}{24} \left\{ \begin{array}{ll} n(n-2)(2n-5) & n \text{ even} \\ (n-1)(n-3)(2n-1) & n \text{ odd} \end{array} \right\}$$

is the number of distinct triangles (positive are) that can be constructed, i.e., the # of 3-subsets of  $[n]$  s.t.  $x \leq y \leq z$  and  $z \neq x+y$ .

6.4 Fibonacci

$$Fib(x+y) = Fib(x+1)Fib(y) + Fib(x)Fib(y-1)$$

$$Fib(n+1)Fib(n-1) - Fib(n)^2 = (-1)^n$$

$$Fib(2n-1) = Fib(n)^2 - Fib(n-1)^2$$

$$\sum_{i=0}^n Fib(i) = Fib(n+2) - 1$$

$$\sum_{i=0}^n Fib(i)^2 = Fib(n)Fib(n+1)$$

$$\sum_{i=0}^n Fib(i)^3 = \frac{Fib(n)Fib(n+1)^2 - (-1)^n Fib(n-1) + 1}{2}$$

6.5 Linear Recurrences

$$F_i = \sum_{j=1}^K C_j F_{i-j} + D$$
$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ C_K & C_{K-1} & C_{K-2} & C_{K-3} & \dots & C_1 & 1 \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_{K-1} \\ F_D \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_K \\ F_D \end{bmatrix}$$

6.6 Game Theory

A game can be reduced to Nim if it is a finite impartial game. Nim and its variants include:

6.6.1 Nim

Let  $X = \bigoplus_{i=1}^n x_i$ , then  $(x_i)_{i=1}^n$  is a winning position iff  $X \neq 0$ . Find a move by picking  $k$  such that  $x_k > x_k \oplus X$ .

6.6.2 Misère Nim

Regular Nim, except that the last player to move *loses*. Play regular Nim until there is only one pile of size larger than 1, reduce it to 0 or 1 such that there is an odd number of piles. The second player wins  $(a_1, \dots, a_n)$  if 1) there is a pile  $a_i > 1$  and  $\bigoplus_{i=1}^n a_i = 0$  or 2) all  $a_i \leq 1$  and  $\bigoplus_{i=1}^n a_i = 1$ .

6.6.3 Staircase Nim

Stones are moved down a staircase and only removed from the last pile.  $(x_i)_{i=1}^n$  is an  $L$ -position if  $(x_{2i-1})_{i=1}^{n/2}$  is (i.e. only look at odd-numbered piles).

nim-product.h

**Description:** Product of nimbers is associative, commutative, and distributive over addition (xor). Forms finite field of size  $2^{2^k}$ . Application: Given 1D coin turning games  $G_1, G_2$   $G_1 \times G_2$  is the 2D coin turning game defined as follows. If turning coins at  $x_1, x_2, \dots, x_m$  is legal in  $G_1$  and  $y_1, y_2, \dots, y_n$  is legal in  $G_2$ , then turning coins at all positions  $(x_i, y_j)$  is legal assuming that the coin at  $(x_m, y_n)$  goes from heads to tails. Then the Grundy function  $g(x, y)$  of  $G_1 \times G_2$  is  $g_1(x) \times g_2(y)$ .  
**Time:**  $64^2$  xors per multiplication, memorize to speed up.

```
ull nim_prod[64][64];
ull nim_prod2(int i, int j) { // hash-1
    if (nim_prod[i][j]) return nim_prod[i][j];
    if ((i & j) == 0) return nim_prod[i][j] = 1ull << (i|j);
    int a = (i&j) & ~(i&j);
    return nim_prod[i][j] = nim_prod2(i ^ a, j) ^ nim_prod2((i ^
        a) | (a-1), (j ^ a) | (i & (a-1)));
} // hash-1 = 65b711
void all_nim_prod() { // hash-2
    for (int i = 0; i < 64; i++)
        for (int j = 0; j < 64; j++)
            if ((i & j) == 0) nim_prod[i][j] = 1ull << (i|j);
            else {
```

```
int a = (i&j) & ~(i&j);
nim_prod[i][j] = nim_prod[i ^ a][j] ^ nim_prod[(i ^ a)
    | (a-1)][(j ^ a) | (i & (a-1))];
}
} // hash-2 = f46d0f
ull get_nim_prod(ull x, ull y) { // hash-3
    ull res = 0;
    for (int i = 0; i < 64 && (x >> i); ++i)
        if ((x >> i) & 1)
            for (int j = 0; j < 64 && (y >> j); ++j)
                if ((y >> j) & 1) res ^= nim_prod2(i, j);
    return res;
} // hash-3 = c5b053
```

Graph (7)

7.1 Fundamentals

euler-walk.h

**Description:** Eulerian undirected/directed path/cycle algorithm. Input should be a vector of (dest, global edge index), where for undirected graphs, forward/backward edges have the same index. Returns a list of nodes in the Eulerian path/cycle with src at both start and end, or empty list if no cycle/path exists. To get edge indices back, add .second to s and ret.  
**Time:**  $\mathcal{O}(V + E)$

```
vector<int> eulerWalk(vector<vector<pii>>& gr, int nedges, int
    src=0) {
    int n = gr.size();
    vector<int> D(n), its(n), eu(nedges), ret, s = {src};
    D[src]++; // to allow Euler paths, not just cycles
    while (!s.empty()) { // hash-1
        int x = s.back(), y, e, &it = its[x], end = int(gr[x].size
            ());
        if (it == end){ ret.push_back(x); s.pop_back(); continue; }
        tie(y, e) = gr[x][it++];
        if (!eu[e])
            D[x]--, D[y]++, eu[e] = 1, s.push_back(y);
    } // hash-1 = f50118
    for(auto &x : D) if (x < 0 || int(ret.size()) != nedges+1)
        return {};
    return {ret.rbegin(), ret.rend()};
}
```

7.2 DFS algorithms

tarjan.h

**Description:** Finds all strongly connected components in a directed graph.  
**Usage:** scc.t s(g); s.solve([&](const vector<int>& cc) {...}); visits all components in reverse topological order.  
**Time:**  $\mathcal{O}(E + V)$

```
struct scc_t {
    int n, t, scc_num;
    vector<vector<int>>> adj;
    vector<int> low, id, stk, in_stk, cc_id;
    scc_t(const auto& g) : n(int(g.size())), t(0), scc_num(0),
        adj(g), low(n,-1), id(n,-1), in_stk(n, false), cc_id(n) {}
    template<class F> void dfs(int cur, F f) { // hash-1
        id[cur] = low[cur] = t++;
        stk.push_back(cur); in_stk[cur] = true;
        for (int nxt : adj[cur])
            if (id[nxt] == -1)
                dfs(nxt, f), low[cur] = min(low[cur], low[nxt]);
            else if (in_stk[nxt])
                low[cur] = min(low[cur], id[nxt]);
        if (low[cur] == id[cur]) {
            vector<int> cc; cc.reserve(stk.size());
```

```

    while (true) {
        int v = stk.back(); stk.pop_back();
        in_stk[v] = false;
        cc.push_back(v); cc_id[v] = scc_num;
        if (v == cur) break;
    } f(cc); scc_num++;
}
} // hash-1 = b8aa46
template<class F> void solve(F f) {
    stk.reserve(n);
    for (int r = 0; r < n; ++r) if (id[r] == -1) dfs(r, f);
}
};

```

## bcc.h

**Description:** Finds all biconnected components in an undirected graph. In a biconnected component there are at least two distinct paths between any two nodes or the component is a bridge. Note that a node can be in several components. *blockcut* constructs the block cut tree of given graph. The first nodes represents the blocks, the others represents the articulation points.

**Usage:** int e\_id = 0; vector<pair<int, int>> g(N);  
for (auto [a,b] : edges) {  
g[a].emplace\_back(b, e\_id);  
g[b].emplace\_back(a, e\_id++); }  
bcc\_t b(g); b.solve([&](const vector<int>& edges\_id) {...});  
**Time:**  $\mathcal{O}(E + V)$

```

struct bcc_t {
    int n, t;
    vector<vector<pii>> adj;
    vector<int> low, id, stk, is_art;
    bcc_t(const vector<vector<pii>> &g) : n(int(g.size())),
    t(0), adj(g), low(n,-1), id(n,-1), is_art(n) {}
    template<class F> void dfs(int cur, int e_par, F f){
        id[cur] = low[cur] = t++;
        stk.push_back(e_par); int c = 0;
        for (auto [nxt, e_id] : adj[cur]) {
            if (id[nxt] == -1) { // hash-1
                dfs(nxt, e_id, f);
                low[cur] = min(low[cur], low[nxt]); c++;
                if (low[nxt] < id[cur]) continue;
                is_art[cur] = true;
                auto top = find(stk.rbegin(), stk.rend(), e_id);
                vector<int> cc(stk.rbegin(), next(top));
                f(cc); stk.resize(stk.size()-cc.size());
            }
            else if (e_id != e_par) {
                low[cur] = min(low[cur], id[nxt]);
                if (id[nxt] < id[cur]) stk.push_back(e_id);
            } // hash-1 = ea7101
        } if(e_par == -1) is_art[cur] = (c > 1) ? true : false;
    }
    template<class F> void solve(F f) {
        stk.reserve(n);
        for(int r = 0; r < n; ++r) if(id[r] == -1) dfs(r,-1,f);
    }
    auto blockcut(const vector<pii> &edges){
        vector<vector<int>> cc; vector<int> cc_id(n,-1);
        solve( [&](const vector<int> &c) { // hash-2
            set<int> vc;
            for(int e : c){
                auto [a, b] = edges[e];
                cc_id[a] = cc_id[b] = int(cc.size());
                vc.insert(a); vc.insert(b);
            } cc.emplace_back(vc.begin(), vc.end());
        } ); // hash-2 = 4e2783
        for(int a = 0; a < n; a++) if(is_art[a])
            cc_id[a] = int(cc.size()), cc.push_back({a});
        int bcc_num = int(cc.size()); // hash-3
    }
};

```

```

vector<vector<int>> tree(bcc_num);
for(int c = 0; c < bcc_num && 1<int(cc[c].size()); ++c)
    for(int a : cc[c]) if(is_art[a]) {
        tree[c].push_back(cc_id[a]);
        tree[cc_id[a]].push_back(c);
    } return make_tuple(cc_id, cc, tree);
} // hash-3 = 4b5ad9
};

```

## 2sat.h

**Description:** Calculates a valid assignment to boolean variables a, b, c,... to a 2-SAT problem, so that an expression of the type  $(a \vee b) \wedge (\neg a \vee c) \wedge (d \vee \neg b) \wedge \dots$  becomes true, or reports that it is unsatisfiable. Negated variables are represented by bit-inversions ( $\sim x$ ).

**Usage:** TwoSat ts(number of boolean variables);  
ts.either(0, ~3); // Var 0 is true or var 3 is false  
ts.set.value(2); // Var 2 is true  
ts.at.most.one({0, ~1, 2}); //  $\leq 1$  of vars 0, ~1 and 2 are true  
ts.solve(); // Returns true iff it is solvable  
ts.values[0..N-1] holds the assigned values to the vars  
**Time:**  $\mathcal{O}(N + E)$ , where N is the number of boolean variables, and E is the number of clauses.

```

"tarjan.h" 74e46b, 36 lines
struct TwoSat { // hash-1
    int N;
    vector<vector<int>> gr;
    vector<int> values; // 0 = false, 1 = true
    TwoSat(int n = 0) : N(n), gr(2*n) {}
    int add_var() { // (optional)
        gr.emplace_back(); gr.emplace_back();
        return N++;
    } // hash-1 = cc8a86
    void either(int f, int j) { // hash-2
        f = max(2*f, -1-2*f); j = max(2*j, -1-2*j);
        gr[f].push_back(j^1); gr[j].push_back(f^1);
    } // hash-2 = 516db0
    void implies(int f, int j) { either(~f, j); }
    void set_value(int x) { either(x, x); }
    void at_most_one(const vector<int>& li) { // (optional)
        if (int(li.size()) <= 1) return;
        int cur = ~li[0];
        for (int i = 2; i < int(li.size()); ++i) {
            int next = add_var();
            either(cur, ~li[i]); either(cur, next);
            either(~li[i], next); cur = ~next;
        } either(cur, ~li[1]);
    }
    bool solve() { // hash-3
        scc_t s(gr);
        s.solve([](const vector<int> &v){ return; } );
        values.assign(N, -1);
        for (int i = 0; i < N; ++i)
            if (s.cc_id[2*i] == s.cc_id[2*i+1]) return 0;
        for (int i = 0; i < N; ++i)
            if (s.cc_id[2*i] < s.cc_id[2*i+1]) values[i] = false;
            else values[i] = true;
        return 1;
    } // hash-3 = 91930f
};

```

## 7.3 Trees

### centroid-decomposition.h

```

struct centroid_tree { // hash-1
    int N;
    vector<pair<int, int>> max_sub;
    vector<vector<int>> dist, adj;
};

```

```

vector<int> depth, par;
vector<bool> deleted;
centroid_tree(const auto& G) : N(G.size()), adj(G), depth(N),
    par(N,-1), deleted(N), dist(__lg(N)+1, vector(N,0)),
    max_sub(N) {
    rec(0, -1, 0);
} // hash-1 = 05eea5
int dfs(int cur, int prv) { // hash-2
    int sz = 1;
    max_sub[cur] = {0, -1};
    for (int nxt : adj[cur]) {
        if (nxt == prv || deleted[nxt]) continue;
        int cur_sz = dfs(nxt, cur);
        sz += cur_sz;
        max_sub[cur] = max(max_sub[cur], {cur_sz, nxt});
    }
    return sz;
} // hash-2 = 57a49c
void layer(int cur, int prv, int lvl) { // hash-3
    for (int nxt : adj[cur]) {
        if (nxt == prv || deleted[nxt]) continue;
        dist[lvl][nxt] = dist[lvl][cur] + 1;
        layer(nxt, cur, lvl);
    }
} // hash-3 = 6e1d27
void rec(int cur, int prv, int d) { // hash-4
    int sz = dfs(cur, prv);
    while(max_sub[cur].first > sz/2) cur = max_sub[cur].second;
    depth[cur] = d;
    par[cur] = prv;
    layer(cur, prv, d);
    deleted[cur] = true;
    for (int nxt : adj[cur]) {
        if (deleted[nxt]) continue;
        rec(nxt, cur, d + 1);
    }
} // hash-4 = 24f90d
};

```

## heavylight.h

**Description:** Compress Tree: Given a subset S of nodes, computes the compress tree and returns a list of edges representing a tree rooted at 0.

**Time:**  $\mathcal{O}((\log N)^2)$

```

d3dcdc, 77 lines
template<bool use_edges> struct hld_t {
    int N, T{};
    vector<vector<int>> adj;
    vector<int> sz, depth, chain, par, in, out, preorder;
    hld_t() {}
    hld_t(const vector<vector<int>>& G, int r = 0) : N(int(G.size())
    ),
    adj(G), sz(N), depth(N), chain(N), par(N), in(N), out(N),
    preorder(N) { dfs_sz(r); chain[r] = r; dfs_hld(r); }
    void dfs_sz(int cur) { // hash-1
        sz[cur] = 1;
        for (auto& nxt : adj[cur]) {
            par[nxt] = cur; depth[nxt] = 1 + depth[cur];
            adj[nxt].erase(find(adj[nxt].begin(), adj[nxt].end(), cur
            ));
            dfs_sz(nxt); sz[cur] += sz[nxt];
            if (sz[nxt] > sz[adj[cur][0]]) swap(nxt, adj[cur][0]);
        }
    } // hash-1 = 87f84a
    void dfs_hld(int cur) { // hash-2
        in[cur] = T++; preorder[in[cur]] = cur;
        for (auto& nxt : adj[cur]) {
            chain[nxt] = (nxt == adj[cur][0] ? chain[cur] : nxt);
            dfs_hld(nxt);
        } out[cur] = T;
    }
};

```

```

} // hash-2 = b2413a
int lca(int a, int b) { // hash-3
    while (chain[a] != chain[b]) {
        if (in[a] < in[b]) swap(a, b);
        a = par[chain[a]];
    } return (in[a] < in[b] ? a : b);
} // hash-3 = 672231
bool is_ancestor(int a, int b) { return in[a] <= in[b] && in[
    b] < out[a]; }
int climb(int a, int k) { // hash-4
    if (depth[a] < k) return -1;
    int d = depth[a] - k;
    while (depth[chain[a]] > d) a = par[chain[a]];
    return preorder[in[a] - depth[a] + d];
} // hash-4 = 268fc4
int kth_on_path(int a, int b, int K) { // hash-5
    int m = lca(a, b);
    int x = depth[a] - depth[m], y = depth[b] - depth[m];
    if (K > x + y) return -1;
    return (x > K ? climb(a, K) : climb(b, x + y - K));
} // hash-5 = 34f729
// bool is true if path should be reversed (only for
// noncommutative operations)
const auto& get_path(int a, int b) const { // hash-6
    static vector<tuple<bool, int, int>> L, R;
    L.clear(); R.clear();
    while (chain[a] != chain[b]) {
        if (depth[chain[a]] > depth[chain[b]]) {
            L.push_back({true, in[chain[a]], in[a] + 1});
            a = par[chain[a]];
        } else {
            R.push_back({false, in[chain[b]], in[b] + 1});
            b = par[chain[b]];
        }
    }
    if (depth[a] > depth[b])
        L.push_back({true, in[b] + use_edges, in[a] + 1});
    else R.push_back({false, in[a] + use_edges, in[b] + 1});
    L.insert(L.end(), R.rbegin(), R.rend());
    return L;
} // hash-6 = 5b653a
auto get_subtree(int a) const {
    return make_pair(in[a] + use_edges, in[a] + sz[a]);
}
auto compress(vector<int> s) { // hash-7
    auto cmp = [&](int a, int b){ return in[a] < in[b]; };
    sort(s.begin(), s.end(), cmp); int m = int(s.size())-1;
    for (int i = 0; i < m; ++i)
        s.push_back(lca(s[i], s[i+1]));
    sort(s.begin(), s.end(), cmp);
    s.erase(unique(s.begin(), s.end(), cmp), s.end());
    vector<pii> ret = { {0, s[0]} };
    for (int i = 0; i + 1 < int(s.size()); ++i)
        ret.emplace_back(lca(s[i], s[i+1]), s[i+1]);
    return ret;
} // hash-7 = a035c4
};

```

## tree-isomorphism.h

**Time:**  $\mathcal{O}(N \log(N))$

a4fc1, 38 lines

```

struct tree_t { // hash-1
    vector<int> cen, sz;
    vector<vector<int>> adj;
    tree_t(vector<vector<int>>& g):cen(2), sz(g.size()), adj(g){}
    int dfs_sz(int v, int p) {
        sz[v] = 1;
        for (int u : adj[v]) if (u != p)
            sz[v] += dfs_sz(u, v);
        return sz[v];
    }
};

```

```

} // hash-1 = f1f64f
int dfs(int tsz, int v, int p) { // hash-2
    for (int u : adj[v]) if (u != p) {
        if (2*sz[u] <= tsz) continue;
        return dfs(tsz, u, v);
    } return cen[0] = v;
} // hash-2 = d35be1
void find_cenroid(int v) { // hash-3
    int tsz = dfs_sz(v, -1);
    cen[1] = dfs(tsz, v, -1);
    for (int u : adj[cen[0]]) if (2*sz[u] == tsz)
        cen[1] = u;
} // hash-3 = 6d03ed
int hash_it(int v, int p = -1) { // hash-4
    static map<vector<int>, int> val;
    vector<int> offset;
    for (int u : adj[v]) if (u != p)
        offset.push_back(hash_it(u, v));
    sort(offset.begin(), offset.end());
    if (!val.count(offset)) val[offset] = int(val.size());
    return val[offset];
} // hash-4 = f8d4d1
ll get_hash(int v = 0) { // hash-5
    find_cenroid(v);
    ll x = hash_it(cen[0]), y = hash_it(cen[1]);
    if (x > y) swap(x, y);
    return (x << 30) + y;
} // hash-5 = 387158
};

```

## 7.4 Network flow

### dinitz.h

**Description:** Flow algorithm with complexity  $\mathcal{O}(VE \log U)$  where  $U = \max|cap|$ .  $\mathcal{O}(\min(E^{1/2}, V^{2/3})E)$  if  $U = 1$ ;  $\mathcal{O}(\sqrt{VE})$  for bipartite matching. To obtain each partition  $A$  and  $B$  of the cut look at  $lvl$ , for  $v \in A$ ,  $lvl[v] > 0$ , for  $u \in B$ ,  $lvl[u] = 0$ .

de8d6b, 44 lines

```

template<typename T = int> struct Dinitz { // hash-1
    struct edge_t { int to, rev; T c, f; };
    vector<vector<edge_t>> adj;
    vector<int> lvl, ptr, q;
    Dinitz(int n) : lvl(n), ptr(n), q(n), adj(n) {}
    inline void addEdge(int a, int b, T c, T rcap = 0) {
        adj[a].push_back({b, (int)adj[b].size(), c, 0});
        adj[b].push_back({a, (int)adj[a].size() - 1, rcap, 0});
    } // hash-1 = 5a0d77
    T dfs(int v, int t, T f) { // hash-2
        if (v == t || !f) return f;
        for (int &i = ptr[v]; i < int(adj[v].size()); ++i) {
            edge_t &e = adj[v][i];
            if (lvl[e.to] == lvl[v] + 1)
                if (T p = dfs(e.to, t, min(f, e.c - e.f)) {
                    e.f += p, adj[e.to][e.rev].f -= p;
                    return p;
                }
        } return 0;
    } // hash-2 = 8ffe6b
    T maxflow(int s, int t) { // hash-3
        T flow = 0; q[0] = s;
        for (int L = 0; L < 31; ++L) do { // consider L = 30
            lvl = vector<int>(q.size());
            int qi = 0, qe = lvl[s] = 1;
            while (qi < qe && !lvl[t]) {
                int v = q[qi++];
                for (edge_t &e : adj[v])
                    if (!lvl[e.to] && (e.c - e.f) >> (30 - L))
                        q[qi++] = e.to, lvl[e.to] = lvl[v] + 1;
            }
            while (T p = dfs(s, t, numeric_limits<T>::max()/4)) flow += p;
        }
    }
};

```

```

    } while (lvl[t]);
    return flow;
} // hash-3 = db2141
bool leftOfMinCut(int v) { return bool(lvl[v] != 0); }
auto minCut(int s, int t) { // hash-4
    T cost = maxflow(s, t);
    vector<edge_t> cut;
    for (int i = 0; i < int(adj.size()); ++i) for (edge_t &e :
        adj[i])
        if (lvl[i] && !lvl[e.to]) cut.push_back(e);
    return make_pair(cost, cut);
} // hash-4 = 1843d5
};

```

### min-cost-max-flow.h

**Description:** Min-cost max-flow. Assumes there is no negative cycle.

**Time:**  $\mathcal{O}(F(V + E) \log V)$ , being  $F$  the amount of flow.

36ddeb, 57 lines

```

template<class flow_t, class cost_t> struct min_cost {
    static constexpr flow_t FLOW_EPS = flow_t(1e-10);
    static constexpr flow_t FLOW_INF = numeric_limits<flow_t>::
        max();
    static constexpr cost_t COST_EPS = cost_t(1e-10);
    static constexpr cost_t COST_INF = numeric_limits<cost_t>::
        max();
    int n, m{}; vector<int> ptr, nxt, zu;
    vector<flow_t> capa; vector<cost_t> cost;
    min_cost(int N) : n(N), ptr(n, -1), dist(n), vis(n), pari(n) {}
    void add_edge(int u, int v, flow_t w, cost_t c) {
        nxt.push_back(ptr[u]); zu.push_back(v); capa.push_back(w);
        cost.push_back(c); ptr[u] = m++;
        nxt.push_back(ptr[v]); zu.push_back(u); capa.push_back(0);
        cost.push_back(-c); ptr[v] = m++;
    }
    vector<cost_t> pot, dist; vector<bool> vis; vector<int> pari;
    vector<flow_t> flows; vector<cost_t> slopes;
    // You can pass t = -1 to find a shortest
    void shortest(int s, int t) { // path to each vertex. // hash-1
        using E = pair<cost_t, int>;
        priority_queue<E, vector<E>, greater<E>> que;
        for (int u = 0; u < n; ++u) { dist[u] = COST_INF; vis[u] = false; }
        for (que.emplace(dist[s] = 0, s); !que.empty(); ) {
            const cost_t c = que.top().first;
            const int u = que.top().second; que.pop();
            if (vis[u]) continue;
            vis[u] = true; if (u == t) return;
            for (int i = ptr[u]; ~i; i = nxt[i]) if (capa[i] >
                FLOW_EPS) {
                const int v = zu[i];
                const cost_t cc = c + cost[i] + pot[u] - pot[v];
                if (dist[v] > cc) { que.emplace(dist[v] = cc, v); pari[v] = i; }
            }
        }
    } // hash-1 = 89f16a
    auto run(int s, int t, flow_t limFlow = FLOW_INF) { // hash-2
        pot.assign(n, 0); flows = {0}; slopes.clear();
        while (true) {
            bool upd = false;
            for (int i = 0; i < m; ++i) if (capa[i] > FLOW_EPS) {
                const int u = zu[i], v = zu[i];
                const cost_t cc = pot[u] + cost[i];
                if (pot[v] > cc + COST_EPS) { pot[v] = cc; upd = true; }
            } if (!upd) break;
        }
        flow_t flow = 0; cost_t tot_cost = 0;
        while (flow < limFlow) {
            shortest(s, t); flow_t f = limFlow - flow;
            if (!vis[t]) break;
            for (int u = 0; u < n; ++u) pot[u] += min(dist[u], dist[t]);
        }
    }
};

```



```
for (int v = t; v != s; ) { const int i = pari[v];
    if (f > capa[i]) { f = capa[i]; } v = zu[i^1];
}
for (int v = t; v != s; ) { const int i = pari[v];
    capa[i] -= f; capa[i^1] += f; v = zu[i^1];
}
flow += f; tot_cost += f * (pot[t] - pot[s]);
flows.push_back(flow); slopes.push_back(pot[t] - pot[s]);
} return make_pair(flow, tot_cost);
} // hash-2 = 285527
};
```

## 7.5 Matching

### hopcroft-karp.h

**Description:** Fast bipartite matching algorithm. Graph  $g$  should be a list of neighbors of the left partition, and  $btoa$  should be a vector full of -1's of the same size as the right partition. Returns the size of the matching.  $btoa[i]$  will be the match for vertex  $i$  on the right side, or -1 if it's not matched.  
**Usage:** vector<int> btoa(m, -1); hopcroftKarp(g, btoa);

**Time:**  $\mathcal{O}(\sqrt{VE})$

d9a55d, 35 lines

```
using vi = vector<int>;
bool dfs(int a, int L, const vector<vi> &g, vi &btoa, vi &A, vi
    &B) { // hash-1
    if (A[a] != L) return 0;
    A[a] = -1;
    for(auto &b : g[a]) if (B[b] == L + 1) {
        B[b] = 0;
        if (btoa[b] == -1 || dfs(btoa[b], L+1, g, btoa, A, B))
            return btoa[b] = a, 1;
    }
    return 0;
} // hash-1 = 2beb8d
int hopcroftKarp(const vector<vi> &g, vi &btoa) { // hash-2
    int res = 0;
    vector<int> A(g.size()), B(int(btoa.size()), cur, next;
    for (;;) {
        fill(A.begin(), A.end(), 0), fill(B.begin(), B.end(), 0);
        cur.clear();
        for(auto &a : btoa) if (a != -1) A[a] = -1;
        for (int a = 0; a < g.size(); ++a) if (A[a] == 0) cur.
            push_back(a);
        for (int lay = 1;; ++lay) {
            bool islast = 0; next.clear();
            for(auto &a : cur) for(auto &b : g[a]) {
                if (btoa[b] == -1) B[b] = lay, islast = 1;
                else if (btoa[b] != a && !B[b])
                    B[b] = lay, next.push_back(btoa[b]);
            }
            if (islast) break;
            if (next.empty()) return res;
            for(auto &a : next) A[a] = lay;
            cur.swap(next);
        }
        for(int a = 0; a < int(g.size()); ++a)
            res += dfs(a, 0, g, btoa, A, B);
    }
} // hash-2 = ea37fa
```

### bipartite-matching.h

**Description:** Fast Kuhn! Simple maximum cardinality bipartite matching algorithm. Better than hopcroftKarp in practice. Worst case is  $\mathcal{O}(VE)$  on an hairy tree. Shuffling the edges and vertices ordering should break some worst-case inputs.

**Time:**  $\Omega(VE)$

1b4d72, 31 lines

```
struct bm_t {
    int N, M, T;
```

```
vector<vector<int>> adj;
vector<int> match, seen;
bm_t(int a, int b) : N(a), M(a+b), T(0), adj(M),
    match(M, -1), seen(M, -1) {}
void add_edge(int a, int b) { adj[a].push_back(b + N); }
bool dfs(int cur) { // hash-1
    if (seen[cur] == T) return false;
    seen[cur] = T;
    for (int nxt : adj[cur]) if (match[nxt] == -1) {
        match[nxt] = cur, match[cur] = nxt;
        return true;
    }
    for (int nxt : adj[cur]) if (dfs(match[nxt])) {
        match[nxt] = cur, match[cur] = nxt;
        return true;
    }
    return false;
} // hash-1 = 85af97
int solve() { // hash-2
    int res = 0;
    for (int cur = 1; cur; ) {
        cur = 0; ++T;
        for (int i = 0; i < N; ++i) if (match[i] == -1)
            cur += dfs(i);
        res += cur;
    }
    return res;
} // hash-2 = ff3448
};
```

### weighted-matching.h

**Description:** Given a weighted bipartite graph, matches every node on the left with a node on the right such that no nodes are in two matchings and the sum of the edge weights is minimal. Takes cost[N][M], where cost[i][j] = cost for L[i] to be matched with R[j] and returns (min cost, match), where L[i] is matched with R[match[i]]. Negate costs for max cost.

**Time:**  $\mathcal{O}(N^2M)$

f0ea90, 28 lines

```
pair<int, vector<int>> hungarian(const vector<vector<int>> &a){
    if (a.empty()) return {0, {}};
    int n = a.size() + 1, m = a[0].size() + 1;
    vector<int> u(n), v(m), p(m), ans(n - 1);
    for(int i = 1; i < n; ++i) { // hash-1
        p[0] = i; int j0 = 0; // add "dummy" worker 0
        vector<int> dist(m, INT_MAX), pre(m, -1);
        vector<bool> done(m + 1);
        do {
            done[j0] = true;
            int i0 = p[j0], j1, delta = INT_MAX;
            for(int j = 1; j < m; ++j) if (!done[j]) {
                auto cur = a[i0-1][j-1] - u[i0] - v[j];
                if (cur < dist[j]) dist[j] = cur, pre[j] = j0;
                if (dist[j] < delta) delta = dist[j], j1 = j;
            }
            for(int j = 0; j < m; ++j)
                if (done[j]) u[p[j]] += delta, v[j] -= delta;
                else dist[j] -= delta;
            j0 = j1;
        } while (p[j0]);
        while (j0) { // update alternating path
            int j1 = pre[j0]; p[j0] = p[j1], j0 = j1;
        }
    }
    // hash-1 = 6dbf35
    for(int j = 1; j < m; ++j) if (p[j]) ans[p[j]-1] = j-1;
    return {-v[0], ans}; // min cost
}
```

### general-matching.h

**Description:** Maximum Matching for general graphs (undirected and non bipartite) using Edmond's Blossom Algorithm.

**Time:**  $\mathcal{O}(EV^2)$

e5db8e, 47 lines

```
struct blossom_t { // hash-1
    int N, M; vector<vector<int>> adj;
    vector<int> match, ts, ps; vector<array<int, 2>> fs;
    blossom_t(auto& G) : N(int(G.size())), M(0), adj(G), match(N
        ,-1), ts(N,-1), ps(N,-1), fs(N,{-1,-1}) {}
    int root(int a) {
        return (ts[a] != M || !~ps[a]) ? a : (ps[a] = root(ps[a]));
    } // hash-1 = abfc73
    void rematch(int a, int b) { // hash-2
        const int w = match[a]; match[a] = b; auto [x, y] = fs[a];
        if (~w && match[w] == a) {
            if (~y) rematch(x, y), rematch(y, x);
            else match[w] = x, rematch(x, w);
        }
    } // hash-2 = b389e7
    bool augment(int src) { // hash-3
        vector<int> bfs = {src}; bfs.reserve(N);
        ts[src] = M; ps[src] = -1; fs[src] = {-1,-1};
        for (int z = 0; z < int(bfs.size()); ++z) {
            int cur = bfs[z];
            for (int nxt : adj[cur]) if (nxt != src) {
                if (match[nxt] == -1) {
                    match[nxt] = cur; rematch(cur, nxt); return true;
                }
                if (ts[nxt] == M) {
                    int a = root(cur), b = root(nxt), m = src;
                    if (a == b) continue;
                    while (a != src || b != src) {
                        if (b != src) swap(a, b);
                        if (fs[a][0]==cur&&fs[a][1]==nxt) { m = a; break; }
                        fs[a] = {cur, nxt}; a = root(fs[match[a]][0]);
                    }
                    for (const int r : {root(cur), root(nxt)})
                        for (int v = r; v != m; v = root(fs[match[v]][0]))
                            ts[v] = M, ps[v] = m, bfs.push_back(v);
                }
            }
            else if (ts[match[nxt]] != M) {
                fs[nxt] = {-1, -1}; ts[match[nxt]] = M;
                ps[match[nxt]] = nxt; fs[match[nxt]] = {cur, -1};
                bfs.push_back(match[nxt]);
            }
        }
    } // hash-3 = e2482c
    int run() {
        for(int v = 0; v < N; ++v) if(!~match[v]) M += augment(v);
        return M;
    }
};
```

### max-independent-set.h

**Description:** To obtain a maximum independent set of a graph, find a max clique of the complement. If the graph is bipartite, see MinimumVertex-Cover.

## 7.6 Heuristics

### maximal-cliques.h

**Description:** Runs a callback for all maximal cliques in a graph (given as a symmetric bitset matrix; self-edges not allowed). Possible optimization: on the top-most recursion level, ignore 'cands', and go through nodes in order of increasing degree, where degrees go down as nodes are removed.

**Time:**  $\mathcal{O}(3^{n/3})$ , much faster for sparse graphs

57e107, 11 lines



```
typedef bitset<128> B;
template<class F>
void cliques(vector<B> &eds, F f, B P = ~B(), B X={}, B R={}) {
    if (!P.any()) { if (!X.any()) f(R); return; }
    auto q = (P | X)._Find_first();
    auto cands = P & ~eds[q];
    for(int i = 0; i < eds.size(); ++i) if (cands[i]) {
        R[i] = 1; cliques(eds, f, P & eds[i], X & eds[i], R);
        R[i] = P[i] = 0; X[i] = 1;
    }
}
```

### chromatic-number.h

**Description:** Compute the chromatic number of a graph. Minimum number of colors needed to paint the graph in a way s.t. if two vertices share an edge, they must have distinct colors.

**Time:**  $\mathcal{O}\left(N2^N\right)$

dd49e4, 26 lines

```
template<class T> int min_colors(int N, const T& gr) {
    vector<int> adj(N);
    for (int a = 0; a < N; ++a) // hash-1
        for (int b = a + 1; b < N; ++b) {
            if (!gr[a][b]) continue;
            adj[a] |= (1 << b); adj[b] |= (1 << a);
        } // hash-1 = 1612da
    static vector<unsigned> dp(1 << N), buf(1 << N), w(1 << N);
    for (int mask = 0; mask < (1 << N); ++mask) { // hash-2
        bool ok = true;
        for (int i = 0; i < N; ++i) if (mask & 1 << i)
            if (adj[i] & mask) ok = false;
        if (ok) dp[mask]++;
        buf[mask] = 1;
        w[mask] = __builtin_popcount(mask) % 2 == N % 2 ? 1 : -1;
    } // hash-2 = 4efbee
    for (int i = 0; i < N; ++i) // hash-3
        for (int mask = 0; mask < (1 << N); ++mask)
            if (!(mask & 1 << i)) dp[mask^(1 << i)] += dp[mask];
    for (int colors = 1; colors <= N; ++colors) {
        unsigned S = 0;
        for (int mask = 0; mask < (1 << N); ++mask)
            S += (buf[mask] * dp[mask]) * w[mask];
        if (S) return colors;
    } assert(false); // hash-3 = 1e2316
}
```

### cycle-counting.h

**Description:** Counts 3 and 4 cycles

**Time:**  $\mathcal{O}\left(E\sqrt{E}\right)$

1cf947, 31 lines

```
using vi = vector<int>;
int count_cycles(const vector<vi>& adj, const vi& deg) {
    const int N = int(adj.size());
    vi idx(N), loc(N); iota(idx.begin(), idx.end(), 0);
    sort(idx.begin(), idx.end(), [&](const int& a, const int& b)
        { return deg[a] < deg[b]; });
    for (int i = 0; i < N; ++i) loc[idx[i]] = i;
    vector<vi> gr(N);
    for (int a = 0; a < N; ++a) for (int b : adj[a])
        if (loc[a] < loc[b]) gr[a].push_back(b);
    int cycle3 = 0, cycle4 = 0; // hash-1
    {
        vector<bool> seen(N, false);
        for (int a = 0; a < N; ++a) {
            for (int b : gr[a]) seen[b] = true;
            for (int b : gr[a]) for (int c : gr[b])
                if (seen[c]) cycle3 += 1;
            for (int b : gr[a]) seen[b] = false;
        }
    }
```

```
    }
    {
        vi cnt(N);
        for (int a = 0; a < N; ++a) {
            for (int b : adj[a]) for (int c : gr[b])
                if (loc[a] < loc[c]) {
                    cycle4 += cnt[c];
                    cnt[c]++;
                }
            for (int b : adj[a]) for (int c : gr[b]) cnt[c] = 0;
        }
    }
    return cycle3; // hash-1 = 586f3b
}
```

### edge-coloring.h

**Description:** Given a simple, undirected graph with max degree  $D$ , computes a  $(D + 1)$ -coloring of the edges such that no neighboring edges share a color. ( $D$ -coloring is NP-hard, but can be done for bipartite graphs by repeated matchings of max-degree nodes.)

**Time:**  $\mathcal{O}(NM)$

13a6e5, 27 lines

```
vector<int> misra_gries(int N, vector<pair<int, int>> eds) {
    const int M = int(eds.size());
    vector<int> cc(N + 1), ret(M), fan(N), free(N), loc;
    for (auto e : eds) ++cc[e.first], ++cc[e.second];
    int u, v, ncols = *max_element(cc.begin(), cc.end()) + 1;
    vector<vector<int>> adj(N, vi(ncols, -1)); // hash-1
    for (auto e : eds) {
        tie(u, v) = e; fan[0] = v; loc.assign(ncols, 0);
        int at = u, end = u, d, c = free[u], ind = 0, i = 0;
        while (d = free[v], !loc[d] && (v = adj[u][d]) != -1)
            loc[d] = ++ind, cc[ind] = d, fan[ind] = v;
        cc[loc[d]] = c;
        for (int cd = d; at != -1; cd ^= c ^ d, at = adj[at][cd])
            swap(adj[at][cd], adj[end = at][cd ^ c ^ d]);
        while (adj[fan[i]][d] != -1) {
            int left = fan[i], right = fan[++i], e = cc[i];
            adj[u][e] = left; adj[left][e] = u;
            adj[right][e] = -1; free[right] = e;
        } // hash-1 = ed5ec6
        adj[u][d] = fan[i]; adj[fan[i]][d] = u;
        for (int y : {fan[0], u, end})
            for (int& z = free[y] = 0; adj[y][z] != -1; z++);
    }
    for (int i = 0; i < M; ++i)
        for (tie(u, v) = eds[i]; adj[u][ret[i]] != v;) ++ret[i];
    return ret;
}
```

## 7.7 Other

### manhattan-mst.h

**Description:** Given  $N$  points, returns up to  $4*N$  edges, which are guaranteed to contain a minimum spanning tree for the graph with edge weights  $w(p,q) = |p.x - q.x| + |p.y - q.y|$ . Edges are in the form (distance, src, dst). Use a standard MST algorithm on the result to find the final MST.

**Time:**  $\mathcal{O}(N\log N)$

<dsu.h> de8170, 24 lines

```
typedef Point<int> P;
pair<vector<array<int, 3>>, int> manhattanMST(vector<P> ps) {
    vector<int> id(ps.size()); iota(id.begin(), id.end(), 0);
    vector<array<int, 3>> edges; // hash-1
    for(int k = 0; k < 4; ++k) {
        sort(id.begin(), id.end(), [&](int i, int j) {
            return (ps[i]-ps[j]).x < (ps[j]-ps[i]).y;});
        map<int, int> sweep;
        for(auto& i : id) {
            for (auto it = sweep.lower_bound(-ps[i].y);
                it != sweep.end(); sweep.erase(it++)) {
```

```
                int j = it->second; P d = ps[i] - ps[j];
                if (d.y > d.x) break;
                edges.push_back({d.y + d.x, i, j});
            } sweep[-ps[i].y] = i;
        }
        if (k & 1) for(auto& p : ps) p.x = -p.x;
        else for(auto& p : ps) swap(p.x, p.y);
    } // hash-1 = b02b12
    sort(edges.begin(), edges.end());
    UF uf(ps.size()); int cost = 0;
    for (auto e: edges) if (uf.unite(e[1], e[2])) cost += e[0];
    return {edges, cost};
}
```

### directed-mst.h

**Description:** Edmonds' algorithm for finding the weight of the minimum spanning tree/arborescence of a directed graph, given a root node. If no MST exists, returns -1.

**Time:**  $\mathcal{O}(E\log V)$

../data-structures/dsu-rollback.h" b2f135, 56 lines

```
struct Edge { int a, b; ll w; };
struct Node {
    Edge key; // hash-1
    Node *l, *r; ll delta;
    void prop() {
        key.w += delta;
        if (l) l->delta += delta;
        if (r) r->delta += delta;
        delta = 0;
    }
    Edge top() { prop(); return key; }
};
Node *merge(Node *a, Node *b) {
    if (!a || !b) return a ?: b;
    a->prop(), b->prop();
    if (a->key.w > b->key.w) swap(a, b);
    swap(a->l, (a->r = merge(b, a->r)));
    return a;
} // hash-1 = 245213
void pop(Node*& a) { a->prop(); a = merge(a->l, a->r); }
auto dmst(int n, int r, vector<Edge>& g) { // hash-2
    RollbackUF uf(n);
    vector<Node*> heap(n);
    for (Edge e : g) heap[e.b] = merge(heap[e.b], new Node(e));
    ll res = 0;
    vector<int> seen(n, -1), path(n), par(n);
    seen[r] = r;
    vector<Edge> Q(n), in(n, {-1,-1}), comp;
    deque<tuple<int, int, vector<Edge>>> cycs;
    for(int s = 0; s < n; ++s) {
        int u = s, qi = 0, w;
        while (seen[u] < 0) {
            if (!heap[u]) return make_pair(-1L, vector<int>{});
            Edge e = heap[u]->top();
            heap[u]->delta -= e.w, pop(heap[u]);
            Q[qi] = e, path[qi++] = u, seen[u] = s;
            res += e.w, u = uf.find(e.a);
            if (seen[u] == s) {
                Node* cyc = 0;
                int end = qi, time = uf.time();
                do cyc = merge(cyc, heap[w = path[--qi]]);
                while (uf.unite(u, w));
                u = uf.find(u), heap[u] = cyc, seen[u] = -1;
                cycs.push_front({u, time, {Q[qi], &Q[end]}});
            }
        }
        for(int i = 0; i < qi; ++i) in[uf.find(Q[i].b)] = Q[i];
    }
    for (auto& [u,t,comp] : cycs) { // restore sol (optional)
```

```

uf.rollback(t); Edge inEdge = in[u];
for(auto& e : comp) in[uf.find(e.b)] = e;
in[uf.find(inEdge.b)] = inEdge;
}
for(int i = 0; i < n; ++i) par[i] = in[i].a;
return make_pair(res, par);
} // hash-2 = 222439

```

## 7.8 Theorems

### 7.8.1 Euler's theorem

Let  $V$ ,  $A$  and  $F$  be the number of vertices, edges and faces of connected planar graph,  $V - A + F = 2$ .

### 7.8.2 Konig's theorem

In any bipartite graph the following holds: Maximum matching and minimum vertex cover have the same cardinality. Minimum edge cover, minimum path cover and maximum independent set have the same cardinality  $N - |M|$ .

### 7.8.3 Construct minimum vertex cover from maximum matching on bipartite graph

1. Redirect every edge:  $y \rightarrow x$  if  $(x, y) \in M$ ,  $x \rightarrow y$  otherwise.
2. DFS from unmatched vertices in  $X$ .
3.  $x \in X$  is chosen iff  $x$  is unvisited.
4.  $y \in Y$  is chosen iff  $y$  is visited.

### 7.8.4 Minimum weight edge cover

1. For each  $v \in V$  create a copy  $v'$ , and connect  $u' \rightarrow v'$  with weight  $w(u, v)$ .
2. Connect  $v \rightarrow v'$  with weight  $2\mu(v)$ , where  $\mu(v)$  is the cost of the cheapest edge incident to  $v$ .
3. Find the minimum weight perfect matching on  $G'$ .

### 7.8.5 Maximum/Minimum flow with lower bound / Circulation problem

1. Construct super source  $S$  and sink  $T$ .
2. For each edge  $(x, y, l, u)$ , connect  $x \rightarrow y$  with capacity  $u - l$ .
3. For each vertex  $v$ , denote by  $in(v)$  the difference between the sum of incoming lower bounds and the sum of outgoing lower bounds.
4. If  $in(v) > 0$ , connect  $S \rightarrow v$  with capacity  $in(v)$ , otherwise, connect  $v \rightarrow T$  with capacity  $-in(v)$ .

- To maximize, connect  $t \rightarrow s$  with capacity  $\infty$  (skip this in circulation problem), and let  $f$  be the maximum flow from  $S$  to  $T$ . If  $f \neq \sum_{v \in V, in(v) > 0} in(v)$ , there's no solution. Otherwise, the maximum flow from  $s$  to  $t$  is the answer.
  - To minimize, let  $f$  be the maximum flow from  $S$  to  $T$ . Connect  $t \rightarrow s$  with capacity  $\infty$  and let the flow from  $S$  to  $T$  be  $f'$ . If  $f + f' \neq \sum_{v \in V, in(v) > 0} in(v)$ , there's no solution. Otherwise,  $f'$  is the answer.
5. The solution of each edge  $e$  is  $l_e + f_e$ , where  $f_e$  corresponds to the flow of edge  $e$  on the graph.

### 7.8.6 Minimum cost cyclic flow

1. Construct super source  $S$  and sink  $T$

2. For each edge  $(x, y, c)$ , connect  $x \rightarrow y$  with  $(cost, cap) = (c, 1)$  if  $c > 0$ , otherwise connect  $y \rightarrow x$  with  $(cost, cap) = (-c, 1)$
3. For each edge with  $c < 0$ , sum these cost as  $K$ , then increase  $d(y)$  by 1, decrease  $d(x)$  by 1
4. For each vertex  $v$  with  $d(v) > 0$ , connect  $S \rightarrow v$  with  $(cost, cap) = (0, d(v))$
5. For each vertex  $v$  with  $d(v) < 0$ , connect  $v \rightarrow T$  with  $(cost, cap) = (0, -d(v))$
6. Flow from  $S$  to  $T$ , the answer is the cost of the flow  $C + K$

### 7.8.7 Maximum density induced subgraph

1. Binary search on answer, suppose we're checking answer  $T$
2. Construct a max flow model, let  $K$  be the sum of all weights
3. Connect source  $s \rightarrow v$ ,  $v \in G$  with capacity  $K$
4. For each edge  $(u, v, w)$  in  $G$ , connect  $u \rightarrow v$  and  $v \rightarrow u$  with capacity  $w$
5. For  $v \in G$ , connect it with sink  $v \rightarrow t$  with capacity  $K + 2T - (\sum_{e \in E(v)} w(e)) - 2w(v)$
6.  $T$  is a valid answer if the maximum flow  $f < K|V|$

### 7.8.8 Project selection problem

1. If  $p_v > 0$ , create edge  $(s, v)$  with capacity  $p_v$ ; otherwise, create edge  $(v, t)$  with capacity  $-p_v$ .
2. Create edge  $(u, v)$  with capacity  $w$  with  $w$  being the cost of choosing  $u$  without choosing  $v$ .
3. The mincut is equivalent to the maximum profit of a subset of projects.

### 7.8.9 0/1 quadratic programming

$$\sum_x c_x x + \sum_y c_y \bar{y} + \sum_{xy} c_{xy} x \bar{y} + \sum_{xyx'y'} c_{xyx'y'} (x \bar{y} + x' \bar{y}')$$

can be minimized by the mincut of the following graph:

1. Create edge  $(x, t)$  with capacity  $c_x$  and create edge  $(s, y)$  with capacity  $c_y$ .
2. Create edge  $(x, y)$  with capacity  $c_{xy}$ .
3. Create edge  $(x, y')$  and edge  $(x', y')$  with capacity  $c_{xyx'y'}$ .

### 7.8.10 Eulerian Cycles

The number of Eulerian cycles in a *directed* graph  $G$  is:  $t_w(G) \prod_{v \in G} (\deg v - 1)!$ , where  $t_w(G)$  is the number of arborescences ("directed spanning" tree) rooted at  $w$  (Check Number of Spanning Trees)

### 7.8.11 Number of Spanning Trees

Define Laplacian Matrix as  $L = D - A$ ,  $D$  being a Diagonal Matrix with  $D_{i,i} = \text{indeg}(i)$  and  $A$  an Adjacency Matrix. Create an  $N \times N$  Laplacian matrix  $\text{mat}$ , and for each edge  $a \rightarrow b \in G$ , do  $\text{mat}[a][b]--$ ,  $\text{mat}[b][b]++$  (and  $\text{mat}[b][a]--$ ,  $\text{mat}[a][a]++$  if  $G$  is undirected). Remove the  $i$ th row and column and take the determinant; this yields the number of directed spanning trees rooted at  $i$  (if  $G$  is undirected, remove any row/column).

### 7.8.12 Tutte Matrix

- A graph has a perfect matching iff the *Tutte* matrix has a non-zero determinant.
- The rank of the *Tutte* matrix is equal to twice the size of the maximum matching. The maximum cost matching can be found by polynomial interpolation.

## Geometry (8)

### 8.1 Geometry

all-geometry.h

**Description:** Geometry 2D Library

d3692c, 606 lines

```

template<class T> int sgn(T x) { return (x > 0) - (x < 0); }
template<class T> // hash-1
struct Point {
    typedef Point P;
    T x, y;
    explicit Point(T x=0, T y=0) : x(x), y(y) {}
    bool operator<(P p) const { return tie(x,y) < tie(p.x,p.y); }
    bool operator==(P p) const { return tie(x,y)==tie(p.x,p.y); }
    P operator+(P p) const { return P(x+p.x, y+p.y); }
    P operator-(P p) const { return P(x-p.x, y-p.y); }
    P operator*(T d) const { return P(x*d, y*d); }
    P operator/(T d) const { return P(x/d, y/d); }
    T dot(P p) const { return x*p.x + y*p.y; }
    T cross(P p) const { return x*p.y - y*p.x; }
    T cross(P a, P b) const { return (a-*this).cross(b-*this); }
    T dist2() const { return x*x + y*y; }
    double dist() const { return sqrt((double)dist2()); }
    // angle to x-axis in interval [-pi, pi]
    double angle() const { return atan2(y, x); }
    P unit() const { return *this/dist(); } // makes dist()=1
    P perp() const { return P(-y, x); } // rotates +90 degrees
    P normal() const { return perp().unit(); }
    // returns point rotated 'a' radians ccw around the origin
    P rotate(double a) const {
        return P(x*cos(a)-y*sin(a), x*sin(a)+y*cos(a)); }
};
using P = Point<double>; // hash-1 = 96024b
// signed distance between point p and line (a, b)
template<class P> // hash-2
double lineDist(const P& a, const P& b, const P& p) {
    return (double)(b-a).cross(p-a)/(b-a).dist();
} // hash-2 = f6bf6b
// shortest distance between point p and line segment (s, e)
double segDist(P& s, P& e, P& p) { // hash-3
    if (s==e) return (p-s).dist();
    auto d = (e-s).dist2(), t = min(d, max(.0, (p-s).dot(e-s)));
    return ((p-s)*d-(e-s)*t).dist()/d;
} // hash-3 = ae751a
// intersection between two segments: Returns either the
// intersection point or the line segment (s,e)
// if there is infinitely many. Empty vector if no intersection
template<class P> // hash-4
vector<P> segInter(P a, P b, P c, P d) {
    auto oa = c.cross(d, a), ob = c.cross(d, b),
    oc = a.cross(b, c), od = a.cross(b, d);
    // Checks if intersection is single non-endpoint point.
    if (sgn(oa) * sgn(ob) < 0 && sgn(oc) * sgn(od) < 0)
        return {(a * ob - b * oa) / (ob - oa)};
    set<P> s;
    if (onSegment(c, d, a)) s.insert(a);

```

```

    if (onSegment(c, d, b)) s.insert(b);
    if (onSegment(a, b, c)) s.insert(c);
    if (onSegment(a, b, d)) s.insert(d);
    return {s.begin(), s.end()};
} // hash-4 = f6be16
// Same as above but returns only true or false
template<class P> // hash-5
bool segmentIntersectionQ(P s1, P e1, P s2, P e2) {
    if (e1 == s1) {
        if (e2 == s2) return e1 == e2;
        swap(s1, s2); swap(e1, e2);
    }
    P v1 = e1 - s1, v2 = e2 - s2, d = s2 - s1;
    auto a = v1.cross(v2), a1 = d.cross(v1), a2 = d.cross(v2);
    if (a == 0) { // parallel
        auto b1 = s1.dot(v1), c1 = e1.dot(v1),
            b2 = s2.dot(v1), c2 = e2.dot(v1);
        return !a1 && max(b1, min(b2, c2)) <= min(c1, max(b2, c2));
    }
    if (a < 0) { a = -a; a1 = -a1; a2 = -a2; }
    return (0 <= a1 && a1 <= a && 0 <= a2 && a2 <= a);
} // hash-5 = 1ff4ba
// {1, P} if there is a intersection, {-1, (0,0)} if
// there is infinitely, {0, (0,0)} otherwise.
template<class P> // hash-6
pair<int, P> lineInter(P s1, P e1, P s2, P e2) {
    auto d = (e1 - s1).cross(e2 - s2);
    if (d == 0) // if parallel
        return {-(s1.cross(e1, s2) == 0), P(0, 0)};
    auto p = s2.cross(e1, e2), q = s2.cross(e2, s1);
    return {1, (s1 * p + e1 * q) / d};
} // hash-6 = a01f81
// Projects point P onto line ab. Refl=true to get reflection
// of point P across line ab instead.
template<class P> // hash-7
P lineProj(P a, P b, P p, bool refl=false) {
    P v = b - a;
    return p - v.perp() * (1 + refl) * v.cross(p - a) / v.dist2();
} // hash-7 = b5562d
// 1/0/-1 <=> left/on line/right
template<class P> // hash-8
int sideOf(P s, P e, P p) { return sgn(s.cross(e, p)); }
template<class P>
int sideOf(const P& s, const P& e, const P& p, double eps) {
    auto a = (e - s).cross(p - s);
    double l = (e - s).dist() * eps;
    return (a > l) - (a < -l);
} // hash-8 = 3af81c
// True iff P lies on the line segment (s, e)
template<class P> bool onSegment(P s, P e, P p) { // hash-9
    return p.cross(s, e) == 0 && (s - p).dot(e - p) <= 0;
} // hash-9 = c597e8
// Linear transformation which takes line p0p1 to q0q1
P linearTransformation(const P& p0, const P& p1,
    const P& q0, const P& q1, const P& r) { // hash-10
    P dp = p1 - p0, dq = q1 - q0, num(dp.cross(dq), dp.dot(dq));
    return q0 + P((r - p0).cross(num), (r - p0).dot(num)) / dp.dist2();
} // hash-10 = 45ea01
// Useful utilities for dealing with angles of rays from origin

template<class P> // hash-11
bool sameDir(P s, P t) {
    return s.cross(t) == 0 && s.dot(t) > 0;
}

template<class P> // checks 180 <= s..t < 360?
bool isReflex(P s, P t) {
    auto c = s.cross(t);
    return c ? (c < 0) : (s.dot(t) < 0);
}

```

```

// operator < (s,t) for angles in [base, base+2pi)
template<class P>
bool angleCmp(P base, P s, P t) {
    int r = isReflex(base, s) - isReflex(base, t);
    return r ? (r < 0) : (0 < s.cross(t));
}

// is x in [s,t] taken ccw? 1/0/-1 for in/border/out
template<class P>
int angleBetween(P s, P t, P x) {
    if (sameDir(x, s) || sameDir(x, t)) return 0;
    return angleCmp(s, x, t) ? 1 : -1; } // hash-11 = 6edd25
// hash-12
int half(P p) { return p.y != 0 ? sgn(p.y) : -sgn(p.x); }
bool angle_cmp(P a, P b) { int A = half(a), B = half(b);
    return A == B ? a.cross(b) > 0 : A < B; } // hash-12 = 9a6894
// out is the pair of points at which two circles intersect
// hash-13
bool circleInter(P a, P b, double r1, double r2, pair<P, P>* out) {
    if (a == b) { assert(r1 != r2); return false; }
    P vec = b - a;
    double d2 = vec.dist2(), sum = r1 + r2, dif = r1 - r2,
        p = (d2 + r1*r1 - r2*r2) / (d2*2), h2 = r1*r1 - p*p*d2;
    if (sum*sum < d2 || dif*dif > d2) return false;
    P mid = a + vec*p, per = vec.perp() * sqrt(fmax(0, h2) / d2);
    *out = {mid + per, mid - per};
    return true;
} // hash-13 = c64785
// Finds the external tangents of two circles, or internal if
// r2 is negated, .first and .second gives the tangency point
// at circle 1 and 2 respec.
template<class P> // hash-14
vector<pair<P, P>> tangents(P c1, double r1, P c2, double r2) {
    P d = c2 - c1;
    double dr = r1 - r2, d2 = d.dist2(), h2 = d2 - dr * dr;
    if (d2 == 0 || h2 < 0) return {};
    vector<pair<P, P>> out;
    for (double sign : {-1, 1}) {
        P v = (d * dr + d.perp() * sqrt(h2) * sign) / d2;
        out.push_back({c1 + v * r1, c2 + v * r2});
    }
    if (h2 == 0) out.pop_back();
    return out;
} // hash-14 = b0153d
// radius of the circle going through points A, B and C
double ccRadius(const P& A, const P& B, const P& C) { // hash-15
    return (B - A).dist() * (C - B).dist() * (A - C).dist() /
        abs((B - A).cross(C - A)) / 2;
}

// Center of the circle above
P ccCenter(const P& A, const P& B, const P& C) {
    P b = C - A, c = B - A;
    return A + (b*c.dist2() - c*b.dist2()).perp() / b.cross(c) / 2;
} // hash-15 = c98102
// min enclosing circle ~ O(N)
pair<P, double> mec(vector<P> ps) { // hash-16
    shuffle(ps.begin(), ps.end(), mt19937(time(0)));
    P o = ps[0];
    double r = 0, EPS = 1 + 1e-8;
    for (int i = 0; i < ps.size(); ++i)
        if ((o - ps[i]).dist() > r * EPS) {
            o = ps[i], r = 0;
            for (int j = 0; j < i; ++j) if ((o - ps[j]).dist() > r * EPS) {
                o = (ps[i] + ps[j]) / 2;
                r = (o - ps[i]).dist();
                for (int k = 0; k < j; ++k)
                    if ((o - ps[k]).dist() > r * EPS) {
                        o = ccCenter(ps[i], ps[j], ps[k]);
                        r = (o - ps[i]).dist();
                    }
            }
        }
}

```

```

    }
}

return {o, r};
} // hash-16 = 8ab87f
// Intersection between circle and line ab, returns 0/1/2
// intersections
template<class P> // hash-17
vector<P> circleLine(P c, double r, P a, P b) {
    double h2 = r*r - a.cross(b, c)*a.cross(b, c)/(b-a).dist2();
    if (h2 < 0) return {};
    P p = lineProj(a, b, c), h = (b-a).unit() * sqrt(h2);
    if (h2 == 0) return {p};
    return {p - h, p + h};
} // hash-17 = debf86
template<class P> // intersection area of two circles // hash-18
double circleCircleArea(P c, double cr, P d, double dr) {
    if (cr < dr) swap(c, d), swap(cr, dr);
    auto A = [&](double r, double h) {
        return r*r*acos(h/r) - h*sqrt(r*r-h*h);
    };
    auto l = (c - d).dist(), a = (l*l + cr*cr - dr*dr)/(2*l);
    if (l - cr - dr >= 0) return 0; // far away
    if (l - cr + dr <= 0) return M_PI*dr*dr;
    if (l - cr >= 0) return A(cr, a) + A(dr, l-a);
    else return A(cr, a) + M_PI*dr*dr - A(dr, a-l);
} // hash-18 = 8bf2b6
// area of intersection between a circle and a ccw polygon
#define arg(p, q) atan2(p.cross(q), p.dot(q)) // (conc or conv)
double circlePoly(P c, double r, vector<P> ps) { // hash-19
    auto tri = [&](P p, P q) {
        auto r2 = r * r / 2;
        P d = q - p;
        auto a = d.dot(p)/d.dist2(), b = (p.dist2() - r*r)/d.dist2();
        auto det = a * a - b;
        if (det <= 0) return arg(p, q) * r2;
        auto s = max(0., -a*sqrt(det)), t = min(1., -a*sqrt(det));
        if (t < 0 || 1 <= s) return arg(p, q) * r2;
        P u = p + d * s, v = p + d * t;
        return arg(p, u) * r2 + u.cross(v)/2 + arg(v, q) * r2;
    };
    auto sum = 0.0;
    for (int i = 0; i < ps.size(); ++i)
        sum += tri(ps[i] - c, ps[(i + 1) % ps.size()] - c);
    return sum;
} // hash-19 = c6ad22
// True if P lies within the polygon. If strict=true returns
// false for points on the boundary.
template<class P> // hash-20
bool inPolygon(vector<P> &p, P a, bool strict = true) {
    int cnt = 0, n = p.size();
    for (int i = 0; i < n; ++i) {
        P q = p[(i + 1) % n];
        if (onSegment(p[i], q, a)) return !strict; // change to
            // -1 if u need to detect points in the boundary
            // or: if (segDist(p[i], q, a) <= eps) return !strict;
        cnt ^= ((a.y < p[i].y) - (a.y < q.y)) * a.cross(p[i], q) > 0;
    }
    return cnt;
} // hash-20 = f9442d
template<class T> // hash-21
T polygonArea(vector<Point<T>> &v) {
    T a = v.back().cross(v[0]);
    for (int i = 0; i < v.size() - 1; ++i)
        a += v[i].cross(v[i+1]);
    return abs(a) / 2.0;
} // hash-21 = 3bcaa4
Point<T> polygonCentroid(vector<Point<T>> &v) { // hash-22
    Point<T> cent(0, 0); T area = 0;
    for (int i = 0; i < v.size(); ++i) {

```

```

    int j = (i+1) % (v.size()); T a = cross(v[i], v[j]);
    cent += a * (v[i] + v[j]);
    area += a;
}
return cent/area/(T)3;
} // not tested!!! // hash-22 = 218bc9
// Center of mass of a polygon
P polygonCenter(const vector<P>& v) { // hash-23
    P res(0, 0); double A = 0;
    for (int i = 0, j = v.size() - 1; i < v.size(); j = ++i) {
        res = res + (v[i] + v[j]) * v[j].cross(v[i]);
        A += v[j].cross(v[i]);
    }
    return res / A / 3;
} // hash-23 = 26a00f
// Returns vertices of the polygon to the left of the cut (s,e)
vector<P> polygonCut(const vector<P>& poly, P s, P e){
    vector<P> res; // hash-24
    for(int i = 0; i < poly.size(); ++i) {
        P cur = poly[i], prev = i ? poly[i-1] : poly.back();
        bool side = s.cross(e, cur) < 0;
        if (side != (s.cross(e, prev) < 0))
            res.push_back(lineInter(s, e, cur, prev).second);
        if (side) res.push_back(cur);
    }
    return res;
} // hash-24 = 120d5b
// no collinear points allowed
vector<P> convexHull(vector<P> pts) { // hash-25
    if (pts.size() <= 1) return pts;
    sort(pts.begin(), pts.end());
    vector<P> h(pts.size()+1);
    int s = 0, t = 0;
    for (int it = 2; it--; s = --t, reverse(pts.begin(), pts.end()))
        for (P p : pts) {
            while (t >= s + 2 && h[t-2].cross(h[t-1], p) <= 0) t--;
            h[t++] = p;
        }
    return {h.begin(), h.begin() + t - (t == 2 && h[0] == h[1])};
} // hash-25 = 3612d7
// Two points with the max distance on convex hull (ccw, no
array<P, 2> hullDiameter(vector<P> S) { // duplicate/collinear
    int n = S.size(), j = n < 2 ? 0 : 1; // hash-26
    pair<int, array<P, 2>> res({0, {S[0], S[0]}});
    for(int i = 0; i < j; ++i)
        for (; j = (j + 1) % n) {
            res = max(res, {(S[i] - S[j]).dist2(), {S[i], S[j]}});
            if ((S[(j + 1) % n] - S[j]).cross(S[i + 1] - S[i]) >= 0)
                break;
        }
    return res.second;
} // hash-26 = a17dbf
// Checks whether P lies inside a convex hull.
bool inHull(const vector<P> &l, P p, bool strict = true) {
    int a = 1, b = l.size() - 1, r = !strict; // hash-27
    if (l.size() < 3) return r && onSegment(l[0], l.back(), p);
    if (sideOf(l[0], l[a], l[b]) > 0) swap(a, b);
    if (sideOf(l[0], l[a], p) >= r || sideOf(l[0], l[b], p) <= -r)
        return false;
    while (abs(a - b) > 1) {
        int c = (a + b) / 2;
        (sideOf(l[0], l[c], p) > 0 ? b : a) = c;
    }
    return sgn(l[a].cross(l[b], p)) < r;
} // hash-27 = d6ec66
// O(N^3M) // hash-28
vector<P> minkowski_sum(vector<P> A, vector<P> B) {
    if (int(A.size()) > int(B.size())) swap(A, B);

```

```

    if (A.empty()) return {};
    if (int(A.size()) == 1) {
        for (auto& b : B) b = b + A.front();
        return B;
    }
    rotate(A.begin(), min_element(A.begin(), A.end()), A.end());
    rotate(B.begin(), min_element(B.begin(), B.end()), B.end());
    A.push_back(A[0]); A.push_back(A[1]);
    B.push_back(B[0]); B.push_back(B[1]);
    const int N = int(A.size()), M = int(B.size());
    vector<P> ans; ans.reserve(N+M);
    for (int i = 0, j = 0; i+2 < N || j+2 < M; i++) {
        ans.push_back(A[i] + B[j]);
        auto sgn = (A[i+1] - A[i]).cross(B[j+1] - B[j]);
        i += (sgn >= 0); j += (sgn <= 0);
    }
    return ans;
} // hash-28 = 71a25a
// Intersection between a line and a convex polygon (given ccw)
typedef array<P, 2> Line; // hash-29
#define cmp(i, j) sgn(dir.perp().cross(poly[(i)%n]-poly[(j)%n]))
#define extr(i) cmp(i + 1, i) >= 0 && cmp(i, i - 1 + n) < 0
int extrVertex(vector<P>& poly, P dir) {
    int n = poly.size(), left = 0, right = n;
    if (extr(0)) return 0;
    while (left + 1 < right) {
        int m = (left + right) / 2;
        if (extr(m)) return m;
        int ls = cmp(left + 1, left), ms = cmp(m + 1, m);
        (ls < ms || (ls == ms && ls == cmp(left, m)) ? right : left)
            = m;
    }
    return left;
} // hash-29 = d1d6b5
#define cmpL(i) sgn(line[0].cross(poly[i], line[1])) // hash-30
array<int, 2> lineHull(Line line, vector<P>& poly) {
    int endA = extrVertex(poly, (line[0] - line[1]).perp());
    int endB = extrVertex(poly, (line[1] - line[0]).perp());
    if (cmpL(endA) < 0 || cmpL(endB) > 0)
        return {-1, -1};
    array<int, 2> res;
    for(int i = 0; i < 2; ++i) {
        int left = endB, right = endA, n = poly.size();
        while ((left + 1) % n != right) {
            int m = ((left + right + (left < right ? 0 : n)) / 2) % n;
            (cmpL(m) == cmpL(endB) ? left : right) = m;
        }
        res[i] = (left + !cmpL(right)) % n;
        swap(endA, endB);
    }
    if (res[0] == res[1]) return {res[0], -1};
    if (!cmpL(res[0]) && !cmpL(res[1]))
        switch ((res[0] - res[1] + sz(poly) + 1) % poly.size()) {
            case 0: return {res[0], res[0]};
            case 2: return {res[1], res[1]};
        }
    return res;
} // hash-30 = 324dbc
// Halfplane intersection area
#define eps 1e-8
using P = Point<double>;
struct line_t { // hash-31
    P P1, P2;
    // Right hand side of the ray P1 -> P2
    explicit line_t(P a = P(), P b = P()) : P1(a), P2(b) {};
    P intoP(line_t y) {
        pair<int, P> r = lineInter(P1, P2, y.P1, y.P2);

```

```

        assert(r.first == 1);
        return r.second;
    }
    P dir() { return P2 - P1; }
    bool contains(P x) {
        return (P2 - P1).cross(x - P1) < eps;
    }
    bool out(P x) { return !contains(x); }
}; // hash-31 = f8b605
bool mycmp(P a, P b) { // hash-32
    // return atan2(a.y, a.x) < atan2(b.y, b.x);
    if (a.x * b.x < 0) return a.x < 0;
    if (abs(a.x) < eps) {
        if (abs(b.x) < eps) return a.y > 0 && b.y < 0;
        if (b.x < 0) return a.y > 0;
        if (b.x > 0) return true;
    }
    if (abs(b.x) < eps) {
        if (a.x < 0) return b.y < 0;
        if (a.x > 0) return false;
    }
    return a.cross(b) > 0;
} // hash-32 = 71c36e
bool cmp(line_t a, line_t b){return mycmp(a.dir(), b.dir());}
double intersection_area(vector<line_t> b) { // hash-33
    sort(b.begin(), b.end(), cmp);
    int n = int(b.size());
    int q = 1, h = 0, i;
    vector<line_t> c(b.size() + 10);
    for (i = 0; i < n; i++) {
        while (q < h && b[i].out(c[h].intoP(c[h - 1]))) h--;
        while (q < h && b[i].out(c[q].intoP(c[q + 1]))) q++;
        c[++h] = b[i];
        if (q < h && abs(c[h].dir().cross(c[h - 1].dir())) < eps) {
            if (c[h].dir().dot(c[h - 1].dir()) > 0) {
                h--;
                if (b[i].out(c[h].P1)) c[h] = b[i];
            } else {
                // The area is either 0 or infinite. If you have a
                // bounding box, then the area is definitely 0.
                return 0;
            }
        }
    }
    while (q < h-1 && c[q].out(c[h].intoP(c[h - 1]))) h--;
    while (q < h-1 && c[h].out(c[q].intoP(c[q + 1]))) q++;
    // Intersection is empty. This is sometimes different from
    // the case when the intersection area is 0.
    if (h - q <= 1) return 0;
    c[h + 1] = c[q];
    vector<P> s;
    for (i = q; i <= h; i++) s.push_back(c[i].intoP(c[i+1]));
    s.push_back(s[0]);
    double ans = 0; int m = int(s.size());
    for (i = 0; i+1 < m; i++) ans += s[i].cross(s[i+1]);
    return ans/2;
} // hash-33 = e9dda6
// Closes pair of points. O(N log N)
pair<P, P> closest(vector<P> v) { // hash-34
    assert(v.size() > 1);
    set<P> S;
    sort(v.begin(), v.end(), [](P a, P b) { return a.y < b.y; });
    pair<int64_t, pair<P, P>> ret{LLONG_MAX, {P(), P()}};
    int j = 0;
    for (P &p : v) {
        P d(1 + (int64_t)sqrt(ret.first), 0);
        while (v[j].y <= p.y - d.x) S.erase(v[j++]);
        auto lo = S.lower_bound(p - d), hi = S.upper_bound(p + d);
        for (; lo != hi; ++lo)

```



```

        ret = min(ret, {(x*lo - p).dist2(), {x*lo, p}});
        S.insert(p);
    }
    return ret.second;
} // hash-34 = 32b14f
// Rectangle union area
struct seg_node{ // hash-35
    int val, cnt, lz;
    seg_node(int n = INF, int c = 0): val(n), cnt(c), lz(0) {}
    void push(seg_node& l, seg_node& r){
        if(lz){
            l.add(lz); r.add(lz); lz = 0;
        }
    }
    void merge(const seg_node& l, const seg_node& r){
        if(l.val < r.val) val = l.val, cnt = l.cnt;
        else if(l.val > r.val) val = r.val, cnt = r.cnt;
        else val = l.val, cnt = l.cnt + r.cnt;
    }
    void add(int n){
        val += n; lz += n;
    }
    int get_sum(){ return (val ? 0 : cnt); }
};
// x1 y1 x2 y2
lint solve(const vector<array<int, 4>>&v){
    vector<int>ys;
    for(auto& [a, b, c, d] : v){
        ys.push_back(b); ys.push_back(d);
    }
    sort(ys.begin(), ys.end());
    ys.erase(unique(ys.begin(), ys.end(), ys.end()));
    vector<array<int, 4>>e;
    for(auto [a, b, c, d] : v){
        b = int(lower_bound(ys.begin(), ys.end(), b) - ys.begin());
        d = int(lower_bound(ys.begin(), ys.end(), d) - ys.begin());
        e.push_back({a, b, d, 1}); e.push_back({c, b, d, -1});
    }
    sort(e.begin(), e.end()); int m = (int)ys.size();
    segtree_range<seg_node>seg(m-1);
    for(int i=0; i<m-1; i++) seg.at(i) = seg_node(0, ys[i+1] - ys[i]);
    seg.build();
    int last = INT_MIN, total = ys[m-1] - ys[0]; lint ans = 0;
    for(auto [x, y1, y2, c] : e){
        ans += (lint)(total - seg.query(0, m-1).get_sum()) * (x - last);
        last = x; seg.update(y1, y2, &seg_node::add, c);
    }
    return ans;
} // hash-35 = 276218
// Delaunay Triangulation. O(N^2). Each circumcircle contains
// none of the input points. If any three points are colinear
// or any four are on the same circle, behavior is undefined.
template<class P, class F> // hash-36
void delaunay(vector<P>& ps, F trifu) {
    if (ps.size() == 3){ int d=(ps[0].cross(ps[1], ps[2]) < 0);
        trifu(0,1+d,2-d); }
    vector<P3> p3;
    for(auto &p : ps) p3.emplace_back(p.x, p.y, p.dist2());
    if (ps.size()>3) for(auto &t: hull3d(p3)) if((p3[t.b]-p3[t.a]).
        cross(p3[t.c]-p3[t.a]).dot(P3(0,0,1)) < 0)
        trifu(t.a, t.c, t.b);
} // hash-36 = f6175a
// GEOMETRY 3D
template<class T> struct Point3D { // hash-37
    typedef Point3D P;
    typedef const P& R;

```

```

    T x, y, z;
    explicit Point3D(T x=0, T y=0, T z=0) : x(x), y(y), z(z) {}
    bool operator<(R p) const {
        return tie(x, y, z) < tie(p.x, p.y, p.z); }
    bool operator==(R p) const {
        return tie(x, y, z) == tie(p.x, p.y, p.z); }
    P operator+(R p) const { return P(x+p.x, y+p.y, z+p.z); }
    P operator-(R p) const { return P(x-p.x, y-p.y, z-p.z); }
    P operator*(T d) const { return P(x*d, y*d, z*d); }
    P operator/(T d) const { return P(x/d, y/d, z/d); }
    T dot(R p) const { return x*p.x + y*p.y + z*p.z; }
    P cross(R p) const {
        return P(y*p.z - z*p.y, z*p.x - x*p.z, x*p.y - y*p.x);
    } // hash-37 = f914db
    T dist2() const { return x*x + y*y + z*z; } // hash-38
    double dist() const { return sqrt((double)dist2()); }
    //Azimuthal angle (longitude) to x-axis in interval [-pi, pi]
    double phi() const { return atan2(y, x); }
    //Zenith angle (latitude) to the z-axis in interval [0, pi]
    double theta() const { return atan2(sqrt(x*x+y*y), z); }
    P unit() const { return *this/(T)dist(); } //makes dist()==1
    //returns unit vector normal to *this and p
    P normal(P p) const { return cross(p).unit(); }
    //returns point rotated 'angle' radians ccw around axis
    P rotate(double angle, P axis) const {
        double s = sin(angle), c = cos(angle); P u = axis.unit();
        return u*dot(u)*(1-c) + (*this)*c - cross(u)*s;
    }
}; // hash-38 = c9d029
// polyhedron formula. Faces should point outwards.
template<class V, class L>
double signed_poly_volume(const V &p, const L &trilist) {
    double v = 0;
    for(auto &i : trilist) v += p[i.a].cross(p[i.b]).dot(p[i.c]);
    return v / 6;
}
// ConvexHull3D
typedef Point3D<double> P3;
struct PR { // hash-39
    void ins(int x) { (a == -1 ? a : b) = x; }
    void rem(int x) { (a == x ? a : b) = -1; }
    int cnt() { return (a != -1) + (b != -1); }
    int a, b;
}; // hash-39 = cf7c9e
struct F { P3 q; int a, b, c; };
vector<F> hull3d(const vector<P3>& A) { // hash-40
    assert(A.size() >= 4);
    vector<vector<PR>> E(A.size(), vector<PR>(A.size(), {-1, -1}));
};
#define E(x,y) E[f.x][f.y]
vector<F> FS;
auto mf = [&](int i, int j, int k, int l) {
    P3 q = (A[j] - A[i]).cross((A[k] - A[i]));
    if (q.dot(A[l]) > q.dot(A[i]))
        q = q * -1;
    F f{q, i, j, k};
    E(a,b).ins(k); E(a,c).ins(j); E(b,c).ins(i);
    FS.push_back(f);
};
for(int i=0; i<4; i++) for(int j=i+1; j<4; j++) for(k=j+1; k<4; k++)
    mf(i, j, k, 6 - i - j - k); // hash-40 = 80e001
for(int i=4; i<A.size(); ++i) { // hash-41
    for(int j=0; j<FS.size(); ++j) {
        F f = FS[j];
        if(f.q.dot(A[i]) > f.q.dot(A[f.a])) {
            E(a,b).rem(f.c);
            E(a,c).rem(f.b);
            E(b,c).rem(f.a);
            swap(FS[j--], FS.back());

```

```

        FS.pop_back();
    }
}
int nw = FS.size();
for(int j=0; j<nw; j++) {
    F f = FS[j];
#define C(a, b, c) if (E(a,b).cnt() != 2) mf(f.a, f.b, i, f.c);
    C(a, b, c); C(a, c, b); C(b, c, a);
}
}
for(auto &it: FS) if ((A[it.b] - A[it.a]).cross(
    A[it.c] - A[it.a]).dot(it.q) <= 0) swap(it.c, it.b);
return FS;
}; // hash-41 = 52653c

```

## Strings (9)

### kmp.h

**Description:** failure[x] computes the length of the longest prefix of s that ends at x, other than s[0...x] itself (abacaba -> -1,0,0,1,0,1,2,3). Can be used to find all occurrences of a pattern in a text.

**Time:**  $\mathcal{O}(n)$

4eb73b, 9 lines

```

vector<int> prefix_function(const string& S) {
    vector<int> fail = {-1}; fail.reserve(S.size());
    for (int i = 0; i < int(S.size()); ++i) {
        int j = fail.back();
        while (j != -1 && S[i] != S[j]) j = fail[j];
        fail.push_back(j+1);
    }
    return fail;
}

```

### z-algorithm.h

**Description:** z[x] computes the length of the longest common prefix of s[i:] and s, except z[0] = 0. (abacaba -> 0010301)

**Time:**  $\mathcal{O}(n)$

7c8c64, 13 lines

```

vector<int> Z(const string& S) { // hash-1
    vector<int> z(S.size()); int l = -1, r = -1;
    for(int i = 1; i < int(S.size()); ++i) {
        z[i] = i >= r ? 0 : min(r - i, z[i - l]);
        while (i + z[i] < int(S.size()) && S[i + z[i]] == S[z[i]])
            z[i]++;
        if (i + z[i] > r) l = i, r = i + z[i];
    } return z;
} // hash-1 = 049391
vector<int> get_prefix(string a, string b) {
    string str = a + '@' + b; vector<int> k = z(str);
    return vector<int>(k.begin() + int(a.size())+1, k.end());
}

```

### manacher.h

**Description:** For each position in a string, computes  $p[0][i]$  = half length of longest even palindrome around pos i,  $p[1][i]$  = longest odd (half rounded down).

**Time:**  $\mathcal{O}(N)$

87e1f0, 12 lines

```

array<vector<int>, 2> manacher(const string &s) {
    int n = s.size();
    array<vector<int>, 2> p = {vector<int>(n+1), vector<int>(n)};
    for(int z = 0; z < 2; ++z) for(int i=0, l=0, r=0; i < n; i++) {
        int t = r-i+!z;
        if (i<r) p[z][i] = min(t, p[z][l+t]);
        int L = i-p[z][i], R = i+p[z][i]-!z;
        while (L>=1 && R+1<n && s[L-1] == s[R+1])
            p[z][i]++, L--, R++;
        if (R > r) l = L, r = R;
    }
}

```



```
    } return p;
}
```

### min-rotation.h

**Description:** Finds the lexicographically smallest rotation of a string.

**Usage:** rotate(v.begin(), v.begin()+min.rotation(v), v.end());

**Time:**  $\mathcal{O}(N)$

2a08fd, 7 lines

```
int min_rotation(string s) {
    int a=0, N=s.size(); s += s;
    for(int b = 0; b < N; ++b) for(int i = 0; i < N; ++i) {
        if (a+i == b || s[a+i] < s[b+i]) {b += max(0, i-1); break;}
        if (s[a+i] > s[b+i]) {a = b; break;}
    } return a;
}
```

### rolling-hash.h

**Description:** Self-explanatory methods for string hashing.

db642a, 36 lines

```
// Arithmetic mod  $2^{64}-1$ . 2x slower than mod  $2^{64}$ 
typedef uint64_t ull;
struct H { // hash-1
    ull x; H(ull x=0) : x(x) {}
    H operator+(H o) { return x + o.x + (x + o.x < x); }
    H operator-(H o) { return *this + ~o.x; }
    H operator*(H o) { auto m = (__uint128_t)x * o.x;
        return H((ull)m) + (ull)(m >> 64); }
    ull get() const { return x + !~x; }
    bool operator==(H o) const { return get() == o.get(); }
    bool operator<(H o) const { return get() < o.get(); }
}; // hash-1 = bff6be7
static const H C = (1ll)1e11+3; // (order ~ 3e9; random also ok)
struct HashInterval { // hash-2
    vector<H> ha, pw;
    HashInterval(string& str) : ha(int(str.size()+1), pw(ha) {
        pw[0] = 1;
        for(int i = 0; i < int(str.size()); ++i)
            ha[i+1] = ha[i] * C + str[i],
            pw[i+1] = pw[i] * C;
    }
    H operator()(int a, int b) { // hash [a, b)
        return ha[b] - ha[a] * pw[b - a];
    }
}; // hash-2 = 2ddba2
vector<H> get_hashes(string& str, int length) { // hash-3
    if (int(str.size()) < length) return {};
    H h = 0, pw = 1;
    for(int i = 0; i < length; ++i)
        h = h * C + str[i], pw = pw * C;
    vector<H> ret = {h};
    for(int i = length; i < int(str.size()); ++i)
        ret.push_back(h = h * C + str[i] - pw * str[i-length]);
    return ret;
} // hash-3 = d21d2a
H hash_all(string& s){H h{}; for(char c:s) h=h*C+c;return h;}
```

### aho-corasick.h

61f5e3, 36 lines

```
const int sigma = 26; // hash-1
array<int, sigma> init;
for (int i = 0; i < sigma; i++) init[i] = -1;
vector<array<int, sigma>> trie(1, init);
vector<int> out(1, -1), parent(n, -1), ids(n);
for (int i = 0; i < n; i++) {
    int cur = 0;
    for (char ch : s[i]) {
        int c = ch - 'a';
        if (trie[cur][c] == -1) {
            trie[cur][c] = (int)trie.size();
```

```
            trie.push_back(init); out.push_back(-1);
        }
        cur = trie[cur][c];
    }
    if (out[cur] == -1) out[cur] = i;
    ids[i] = out[cur];
} // hash-1 = c8c436
vector<int> bfs, f(trie.size()); bfs.reserve(trie.size());
for (int c = 0; c < sigma; c++) // hash-2
    if (trie[0][c] == -1) trie[0][c] = 0;
    else bfs.push_back(trie[0][c]);
for (int z = 0; z < (int)bfs.size(); z++) {
    int cur = bfs[z];
    for (int c = 0; c < sigma; c++) {
        if (trie[cur][c] == -1)
            trie[cur][c] = trie[f[cur]][c];
        else {
            int nxt = trie[cur][c];
            int fail = trie[f[cur]][c];
            if (out[nxt] == -1) out[nxt] = out[fail];
            else parent[out[nxt]] = out[fail];
            f[nxt] = fail; bfs.push_back(nxt);
        }
    }
} // hash-2 = 2188a7
```

### suffix-array.h

**Description:** Builds suffix array for a string, first element is the size of the string. The lcp function calculates longest common prefixes for neighbour-strings in suffix array. The returned vector is of size  $n + 1$ .

**Time:**  $\mathcal{O}(N \log N)$  where  $N$  is the length of the string for creation of the SA.  $\mathcal{O}(N)$  for longest common prefixes.

&lt;../data-structures/rmq.h&gt;, &lt;../various/random-numbers.h&gt; d31829, 66 lines

```
struct suffix_array_t { // hash-1
    int N, H; vector<int> sa, invsa, lcp;
    rmq_t<pair<int, int>> rmq;
    bool cmp(int a, int b) { return invsa[a+H] < invsa[b+H]; }
    void ternary_sort(int a, int b) {
        if (a == b) return;
        int md = sa[a+rng() % (b-a)], lo = a, hi = b;
        for (int i = a; i < b; ++i) if (cmp(sa[i], md))
            swap(sa[i], sa[lo++]);
        for (int i = b-1; i >= lo; --i) if (cmp(md, sa[i]))
            swap(sa[i], sa[--hi]);
        ternary_sort(a, lo);
        for (int i = lo; i < hi; ++i) invsa[sa[i]] = hi-1;
        if (hi-lo == 1) sa[lo] = -1;
        ternary_sort(hi, b);
    }
    suffix_array_t() {} // hash-1 = 310a12
    template<typename I> // hash-2
    suffix_array_t(I begin, I end) : N(int(end-begin)+1), sa(N) {
        vector<int> v(begin, end); v.push_back(INT_MIN);
        invsa = v; iota(sa.begin(), sa.end(), 0);
        H = 0; ternary_sort(0, N);
        for(H = 1; H <= N; H *= 2) for(int j=0, i=j; i!=N; i=j)
            if (sa[i] < 0) {
                while (j < N && sa[j] < 0) j += -sa[j];
                sa[i] = -(j - i);
            } else {j = invsa[sa[i]] + 1; ternary_sort(i, j);}
        for (int i = 0; i < N; ++i) sa[invsa[i]] = i;
        lcp.resize(N-1); int K = 0;
        for (int i = 0; i < N-1; ++i) {
            if (invsa[i] > 0) while (v[i+K] == v[sa[invsa[i]-1]+K]) ++K;
            lcp[invsa[i]-1] = K; K = max(K - 1, 0);
        }
        vector<pair<int, int>> lcp_index(N-1);
        for (int i = 0; i < N-1; ++i) lcp_index[i] = {lcp[i], 1+i};
        rmq = rmq_t<pair<int, int>>(std::move(lcp_index));
```

```
    } // hash-2 = ef483d
    auto rmq_query(int a, int b) const {return rmq.query(a,b);}
    auto get_split(int a, int b) const {return rmq.query(a,b-1);}
    int get_lcp(int a, int b) const { // hash-3
        if (a == b) return N - a;
        a = invsa[a], b = invsa[b];
        if (a > b) swap(a, b);
        return rmq_query(a, b).first;
    } // hash-3 = 6877f0
};
vector<vector<int>>> ch(2*N+1); int V = 0;
vector<array<int, 2>> sa_range(2*N+1);
vector<int> leaves(N+1), par(2*N+1), depth(2*N+1); // hash-4
auto dfs = [&](auto&& self, int lo, int hi, int prv)-> void{
    int cur = V++; par[cur] = prv;
    if (prv != -1) ch[prv].push_back(cur);
    sa_range[cur] = {lo, hi};
    if (hi - lo == 1) {
        leaves[us.sa[lo]] = cur;
        depth[cur] = N-us.sa[lo] + 1;
    } else {
        int d = us.get_split(lo, hi).first;
        depth[cur] = d; int mi = lo;
        while (hi - mi >= 2) {
            auto [nd, nmi] = us.get_split(mi, hi);
            if (nd != d) break;
            self(self, mi, nmi, cur); mi = nmi;
        } self(self, mi, hi, cur);
    }
}; dfs(dfs, 0, N+1, -1); // hash-4 = a4f48a
```

### suffix-automaton.h

**Description:** Suffix automaton

defb60, 33 lines

```
template<int offset = 'a'> struct array_state { // hash-1
    array<int, 26> as;
    array_state() { fill(begin(as), end(as), ~0); }
    int& operator[](char c) { return as[c - offset]; }
    int count(char c) { return (~as[c - offset] ? 1 : 0); }
}; // hash-1 = e60fd5
template<typename C, typename state = map<C, int>>> struct
    suffix_automaton { // hash-2
    struct node_t {
        int len, link; int64_t cnt; state next;
    };
    int N, cur; vector<node_t> nodes;
    suffix_automaton() : N(1), cur(0), nodes{node_t{0, -1, 0, {}}}{
        {}
    }
    node_t& operator[](int v) { return nodes[v]; };
    void append(C c) {
        int v = cur; cur = N++;
        nodes.push_back(node_t{nodes[v].len + 1, 0, 1, {}});
        for (; ~v && !nodes[v].next.count(c); v = nodes[v].link)
            nodes[v].next[c] = cur;
        if (~v) {
            const int u = nodes[v].next[c];
            if (nodes[v].len + 1 == nodes[u].len) {
                nodes[cur].link = u;
            } else {
                const int clone = N++;
                nodes.push_back(nodes[u]);
                nodes[clone].len = nodes[v].len + 1;
                nodes[u].link = nodes[cur].link = clone;
                for (; ~v && nodes[v].next[c] == u; v = nodes[v].link)
                    nodes[v].next[c] = clone;
            }
        }
    }
}; // hash-2 = f4569b
```

## 9.1 Suffix Automaton

### 9.1.1 Number of different substrings

Is the number of paths in the automaton starting at the root.

$$d(v) = 1 + \sum_{v \rightarrow w} d(w)$$

### 9.1.2 Total length of different substrings

Is the sum of children answers and paths starting at each

children.  $ans(v) = \sum_{v \rightarrow w} d(w) + ans(w)$

### 9.1.3 Lexicographically $K$ -th substring

Is the  $K$ -th lexicographically path, so you can search using the

number of paths from each state

### 9.1.4 Smallest cyclic shift

Construct for string  $S + S$ . Greedily search the minimal

character.

### 9.1.5 Number of occurrences

For each state not created by cloning, initialize  $cnt(v) = 1$ . Then,

just do a dfs to calculate  $cnt(v)$ , with

$cnt(link(v)) + = cnt(v)$

### 9.1.6 First occurence position

When we create a new state  $cur$  do  $first(pos) = len(cur) - 1$ .

When we clone  $q$  as  $clone$  do  $first(clone) = first(q)$ . Answer is

$first(v) - size(P) + 1$ , where  $v$  is the state of string  $P$

### 9.1.7 All occurence positions

From  $first(v)$  do a dfs using suffix link, from  $link(u)$  go to  $u$ .

## Various (10)

## 10.1 Intervals

interval-container.h

**Description:** Add and remove intervals from a set of disjoint intervals. Will merge the added interval with any overlapping intervals in the set when adding. Intervals are [inclusive, exclusive).

**Time:**  $\mathcal{O}(\log N)$

```
set<pii>::iterator addInterval(set<pii> &is, int L, int R) {
    if (L == R) return is.end(); // hash-1
    auto it = is.lower_bound({L, R}), before = it;
    while (it != is.end() && it->first <= R) {
        R = max(R, it->second);
        before = it = is.erase(it);
    }
    if (it != is.begin() && (--it)->second >= L) {
        L = min(L, it->first); R = max(R, it->second);
        is.erase(it);
    } return is.insert(before, {L,R});
} // hash-1 = 52519d
void removeInterval(set<pii> &is, int L, int R) { // hash-2
    if (L == R) return;
    auto it = addInterval(is, L, R);
    auto r2 = it->second;
    if (it->first == L) is.erase(it);
    else (int&)it->second = L;
    if (R != r2) is.emplace(R, r2);
} // hash-2 = 0594c1
```

interval-cover.h

**Description:** Compute indices of smallest set of intervals covering another interval. Intervals should be [inclusive, exclusive). To support [inclusive, inclusive], change (A) to add || R.empty(). Returns empty set on failure (or if G is empty).

**Time:**  $\mathcal{O}(N \log N)$

```
template<class T>
vector<int> cover(pair<T, T> G, vector<pair<T, T>> I) {
    vector<int> S(I.size()), R;
    iota(S.begin(), S.end(), 0);
    sort(S.begin(), S.end(), [&](int a, int b) {
        return I[a] < I[b]; });
    T cur = G.first; int at = 0;
    while (cur < G.second) { // (A)
        pair<T, int> mx = {cur, -1};
        while (at < I.size() && I[S[at]].first <= cur) {
            mx = max(mx, {I[S[at]].second, S[at]});
            at++;
        }
        if (mx.second == -1) return {};
        cur = mx.first; R.push_back(mx.second);
    } return R;
}
```

constant-intervals.h

**Description:** Split a monotone function on [from, to) into a minimal set of half-open intervals on which it has the same value. Runs a callback g for each such interval.

**Usage:** constantIntervals(0, sz(v), [&](int x){ return v[x];}, [&](int lo, int hi, T val){...});

**Time:**  $\mathcal{O}(k \log \frac{n}{k})$

```
template<class F, class G, class T> // hash-1
void rec(int from, int to, F& f, G& g, int& i, T& p, T q) {
    if (p == q) return;
    if (from == to) {
        g(i, to, p); i = to; p = q;
    } else {
        int mid = (from + to) >> 1;
        rec(from, mid, f, g, i, p, f(mid));
        rec(mid+1, to, f, g, i, p, q);
    }
} // hash-1 = 69b73b
template<class F, class G> // hash-2
void constantIntervals(int from, int to, F f, G g) {
    if (to <= from) return;
    int i = from; auto p = f(i), q = f(to-1);
    rec(from, to-1, f, g, i, p, q); g(i, to, q);
} // hash-2 = 856088
```

## 10.2 Misc. algorithms

ternary-search.h

**Description:** Find the smallest  $i$  in  $[a, b]$  that maximizes  $f(i)$ , assuming that  $f(a) < \dots < f(i) \geq \dots \geq f(b)$ . To reverse which of the sides allows non-strict inequalities, change the  $<$  marked with (A) to  $<=$ , and reverse the loop at (B). To minimize  $f$ , change it to  $>$ , also at (B). If you are dealing with real numbers, you'll need to pick  $m_1 = (2a+b)/3.0$  and  $m_2 = (a+2b)/3.0$ . Consider setting a constant number of iterations for the search, usually [200, 300] iterations are sufficient for problems with error limit as  $10^{-6}$ .

**Usage:** int ind = ternSearch(0,n-1,[&](int i){return a[i];});

**Time:**  $\mathcal{O}(\log(b-a))$

```
template<class F> int ternSearch(int a, int b, F f) {
    assert(a <= b);
    while (b - a >= 5) {
        int mid = (a + b) / 2;
        if (f(mid) < f(mid+1)) a = mid; // (A)
        else b = mid+1;
    }
```

```
}
for(int i = a+1; i <= b; ++i)
    if (f(a) < f(i)) a = i; // (B)
return a;
}
```

count-triangles.h

**Description:** Counts  $x, y \geq 0$  such that  $Ax + By \leq C$ .

```
ll count_triangle(ll A, ll B, ll C) {
    if (C < 0) return 0;
    if (A > B) swap(A, B);
    ll p = C / B, k = B / A, d = (C - p * B) / A;
    return count_triangle(B - k * A, A, C - A * (k * p + d + 1))
        + (p + 1) * (d + 1) + k * p * (p + 1) / 2;
}
```

floyd-cycle.h

**Description:** Detect loop in a list. Consider using mod template to avoid overflow.

**Time:**  $\mathcal{O}(n)$

```
template<class F> pair<int,int> find(int x0, F f) {
    int t = f(x0), h = f(t), mu = 0, lam = 1;
    while (t != h) t = f(t), h = f(f(h));
    h = x0;
    while (t != h) t = f(t), h = f(h), ++mu;
    h = f(t);
    while (t != h) h = f(h), ++lam;
    return {mu, lam};
}
```

## 10.3 Dynamic programming

divide-and-conquer-dp.h

**Description:** Given  $a[i] = \min_{lo(i) \leq k < hi(i)} (f(i, k))$  where the (minimal) optimal  $k$  increases with  $i$ , computes  $a[i]$  for  $i = L..R - 1$ .

**Time:**  $\mathcal{O}((N + (hi - lo)) \log N)$

```
struct DP { // hash-1
    vector<int>a, freq;
    vector<ll>old, cur;
    ll cnt; int lcur, rcur;
    DP(const vector<int>&a, int n): a(_a), freq(n), old(n+1,
        linf), cur(n+1, linf), cnt(0), lcur(0), rcur(0){}
    int lo(int ind) { return 0; }
    int hi(int ind) { return ind; }
    void add(int k, int c){ cnt += freq[a[k]]++; }
    void del(int k, int c){ cnt -= --freq[a[k]]; }
    ll C(int l, int r){
        while(lcur > l) add(--lcur, 0);
        while(rcur < r) add(rcur++, 1);
        while(lcur < l) del(lcur++, 0);
        while(rcur > r) del(--rcur, 1);
        return cnt;
    } // hash-1 = 7cabd1
    ll f(int ind, int k) { return old[k] + C(k, ind); } // hash-2
    void store(int ind, int k, ll v) { cur[ind] = v; }
    void rec(int L, int R, int LO, int HI) {
        if (L >= R) return;
        int mid = (L + R) >> 1;
        pair<ll, int> best(LLONG_MAX, LO);
        for(int k = max(LO, lo(mid)); k <= min(HI, hi(mid)); ++k)
            best = min(best, make_pair(f(mid, k), k));
        store(mid, best.second, best.first);
        rec(L, mid, LO, best.second);
        rec(mid+1, R, best.second, HI);
    }
}; // hash-2 = 52b5d3
```

knuth-dp.h

**Description:** When doing DP on intervals:  $a[i][j] = \min_{i < k < j} (a[i][k] + a[k][j]) + f(i, j)$ , where the (minimal) optimal  $k$  increases with both  $i$  and  $j$ , one can solve intervals in increasing order of length, and search  $k = p[i][j]$  for  $a[i][j]$  only between  $p[i][j - 1]$  and  $p[i + 1][j]$ . This is known as Knuth DP. Sufficient criteria for this are if  $f(b, c) \leq f(a, d)$  and  $f(a, c) + f(b, d) \leq f(a, d) + f(b, c)$  for all  $a \leq b \leq c \leq d$ . Consider also: LineContainer (ch. Data structures), monotone queues, ternary search.

**Time:**  $\mathcal{O}(N^2)$

digit-dp.h

**Description:** Compute how many # between 1 and  $N$  have  $K$  distinct digits in the base  $L$  without leading zeros;

**Usage:** auto hex.to\_dec = [&](char c) -> int { return ('A' <= c && c <= 'F' ? (10 + c - 'A') : (c - '0'))};

digit.dp<modnum<int(1e9) + 7>, hex.to\_dec>(N, K);

**Time:**  $\mathcal{O}(NK)$

```
template<typename T, class F> T digit_dp(const string& S, int K, F& L) {
    const int base = 16, len = int(S.size());
    vector<bool> w(base);
    vector<vector<T>> dp(len + 1, vector<T>(base + 2));
    int cnt = 0;
    for (int d = 0; d < len; ++d) {
        // adding new digit to numbers with prefix < s
        for (int x = 0; x <= base; ++x) {
            dp[d + 1][x] += dp[d][x] * x;
            dp[d + 1][x + 1] += dp[d][x] * (base - x);
        }
        // adding strings whith prefix only 0's and last digit != 0
        if (d) dp[d + 1][1] += (base - 1);
        // adding prefix equal to s and last digit < s, first digit cannot be 0
        for (int x = 0; x < L(S[d]); ++x) {
            if (d == 0 && x == 0) continue;
            if (w[x]) dp[d + 1][cnt] += 1;
            else dp[d + 1][cnt + 1] += 1;
        }
        // marking if the last digit appears in the prefix of s
        if (w[L(S[d])]) == false)
            w[L(S[d])] = true, cnt++;
    }
    // adding string k
    dp[len][cnt] += 1; return dp[len][K];
}
```

knapsack-bounded.h

**Description:** You are given  $n$  types of items, each items has a weight and a quantity. Is possible to fill a knapsack with capacity  $X$  using any subset of items?

**Time:**  $\mathcal{O}(W \cdot N)$

```
auto solve(vi weight, vi cnt, int X) {
    vector<int> dp(X+1, 0);
    for (int i = 0; i < N; ++i)
        for (int j = X-weight[i]; j >= 0; --j) {
            if (!dp[j]) continue;
            int k = cnt[i], s = j + weight[i];
            while (k > 0 && s <= X && !dp[s])
                dp[s] = 1, --k, s += weight[i];
        }
    return dp[X];
}
```

knapsack-bounded-costs.h

**Description:** You are given  $N$  types of items, you have  $cnt[i]$  items of  $i$ -th type, and each item of  $i$ -th type  $weight[i]$  and  $cost[i]$ . What is the maximal cost you can get by picking some items weighing exactly  $X$  in total?

**Time:**  $\mathcal{O}(N \cdot W)$

```
"/data-structures/monotonic-queue.h"03d171, 15 lines
```

```
auto solve(vi weight, vi cost, vi cnt, int X) {
    vector<int> dp(X+1, 0); int N = int(weight.size());
    vector<max_monotonic_queue<int>> M(X+1);
    for (int i = 0; i < N; ++i) {
        for (int j = 0; j < min(X+1, weight[i]); ++j)
            M[j] = max_monotonic_queue<int>();
        for (int j = 0; j <= X; ++j) {
            auto& que = M[j % weight[i]];
            if (que.size() > cnt[i]) que.pop();
            que.add(cost[i]);
            que.push(dp[j]);
            dp[j] = que.get_val();
        }
    } return dp[X];
}
```

two-max-equal-sum.h

**Description:** Two maximum equal sum disjoint subsets,  $s[i] = 0$  if  $v[i]$  wasn't selected,  $s[i] = 1$  if  $v[i]$  is in the first subset and  $s[i] = 2$  if  $v[i]$  is in the second subset

**Time:**  $\mathcal{O}(n * S)$

```
auto twoMaxEqualSumDS(const vector<int> &v){
    int sum=accumulate(v.begin(), v.end(), 0), n=int(v.size());
    vector<int> old(2*sum + 1, INT_MIN/2), dp(2*sum + 1), s(n);
    vector<vector<int>> rec(n, vector<int>(2*sum + 1));
    int i; old[sum] = 0;
    for(i = 0; i < n; ++i, swap(old, dp)){
        for(int a, b, d = v[i]; d <= 2*sum - v[i]; d++){
            dp[d] = max(old[d], a = old[d - v[i]] + v[i]);
            dp[d] = max(dp[d], b = old[d + v[i]]);
            rec[i][d] = dp[d] == a ? 1 : dp[d] == b ? 2 : 0;
        }
        for(int j = i-1, d = sum; j >= 0; --j)
            d+=(s[j] = rec[j][d]) ? s[j] == 2 ? v[j] : - v[j] : 0;
        return make_pair(old[sum], s);
    }
}
```

## 10.4 Optimization tricks

### 10.4.1 Bit hacks

- $c = x \& -x$ ,  $r = x + c$ ;  $((r \wedge x) \gg 2) / c$  |  $r$  is the next number after  $x$  with the same number of bits set.
- $\text{rep}(b, 0, K) \text{ rep}(i, 0, (1 \ll K))$  if  $(i \& 1 \ll b) \text{ D}[i] += \text{D}[i \wedge (1 \ll b)]$ ; computes all sums of subsets.

## 10.5 Bit Twiddling Hack

hacks.h

```
// Iterate on non-empty submasks of a bitmask.
for (int s = m; s > 0; s = (m & (s - 1)))
// Iterate on non-zero bits of a bitset B. // hash-1
for (int j = B._Find_next(0); j < MAXV; j = B._Find_next(j))
11 next_perm(11 v) { // compute next perm i.e.
    11 t = v | (v-1); // 00111,01011,01101,10011 ...
    return (t + 1) | (((~t & ~t) - 1)>>(__builtin_ctz(v) + 1));
} // hash-1 = 2392b3
template<typename F> // All subsets of size k of {0..N-1}
void iterate_k_subset(11 N, 11 k, F f){ // hash-2
```

```
11 mask = (111 << k) - 1;
while (!(mask & 111<<N)) { f(mask);
    11 t = mask | (mask-1);
    mask = (t+1)|(((~t & ~t)-1)>>(__builtin_ctz11(mask)+1));
}
} // hash-2 = 87802c
template<typename F> // All subsets of set // hash-3
void iterate_mask_subset(11 set, F f){ 11 mask = set;
    do f(mask), mask = (mask-1) & set;
    while (mask != set);
} // hash-3 = 3b9428
```

## 10.6 Random Numbers

random-numbers.h

**Description:** An example on the usage of generator and distribution. Use shuffle instead of random shuffle.

```
a36db2, 7 lines
mt19937_64 rng(chrono::steady_clock::now().time_since_epoch()).
    count();
shuffle(permutation.begin(), permutation.end(), rng);
uniform_int_distribution<int> uid(1, 100); //[1, 100]
unsigned xrand() {
    static unsigned x = 314159265, y = 358979323, z = 846264338,
        w = 327950288;
    unsigned t = x ^ x << 11; x = y; y = z; z = w; return w = w ^
        w >> 19 ^ t ^ t >> 8;
}
```