MSA 8200 Predictive Analytics
Week2: Autoregressive Model

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Spring 2020

ARMA (Autoregressive Moving Average)

- AR (Autoregressive)
- MA (Moving Average)
- ARMA (Autoregressive Moving Average)

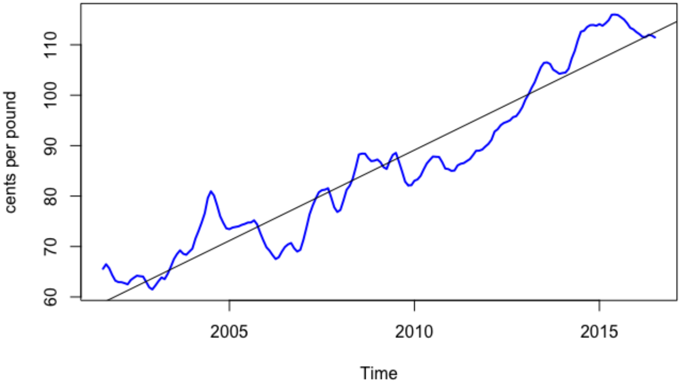
Goal:

- ACF and PACF plots for order determination.
- Parameter Estimation
- Forcasting

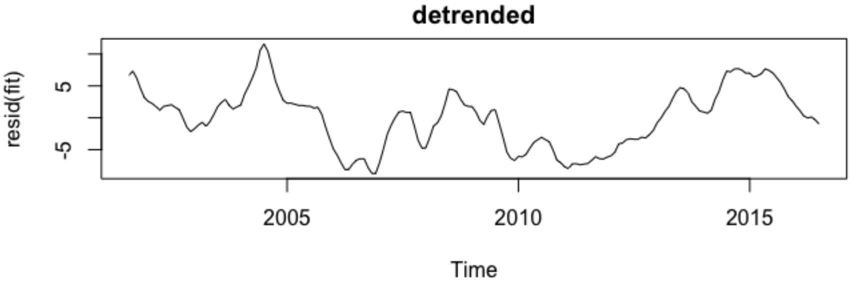
AR model: chicken price

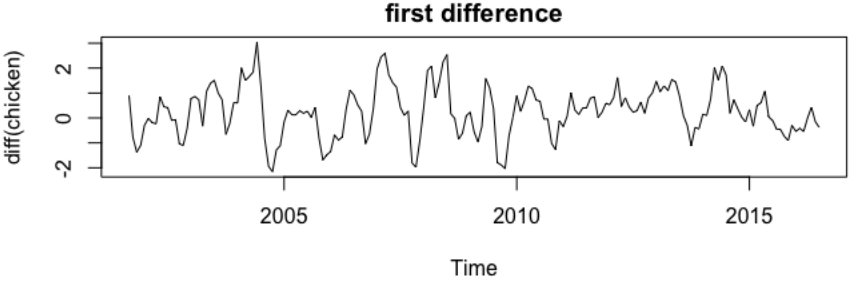
Example: Chicken price:

- monthly price (per pound) of a chicken
- mid-2001 to mide-2016 (180 months)

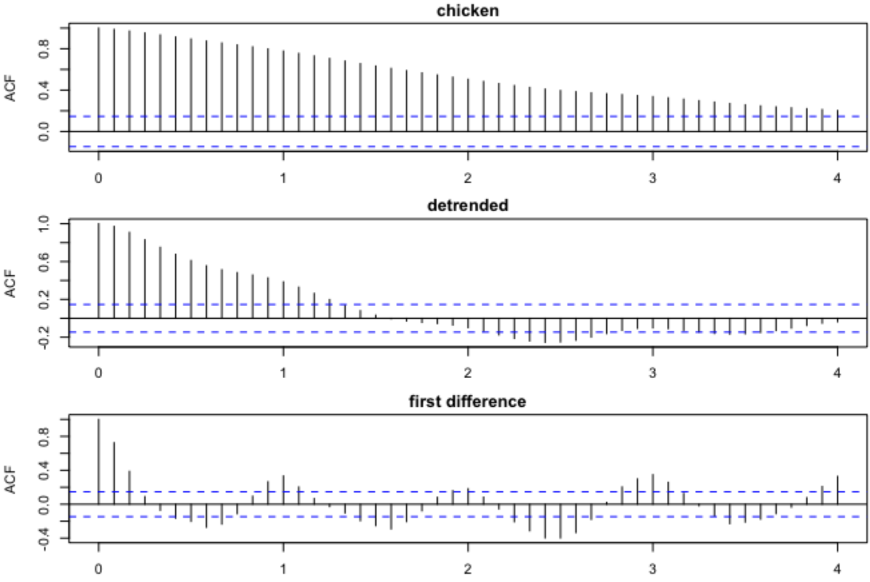


AR model: chicken price

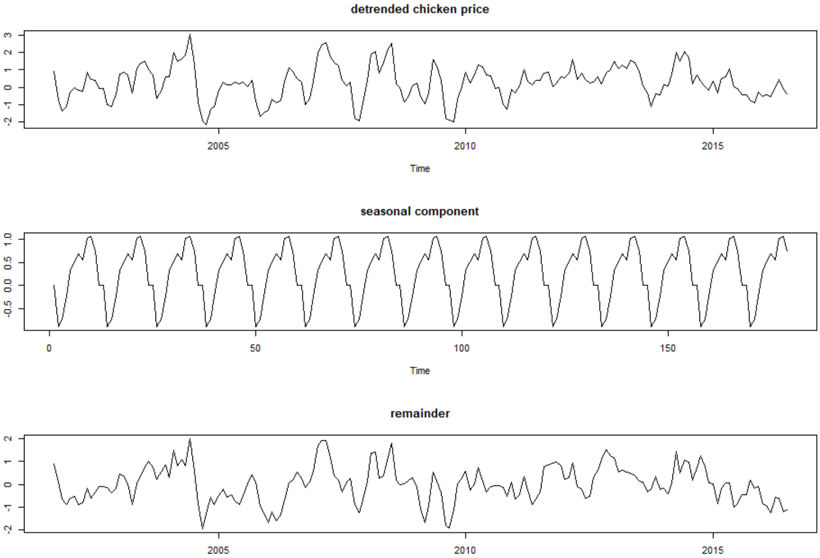




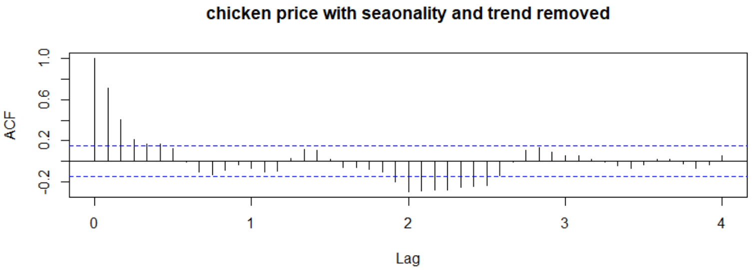
AR model: chicken price

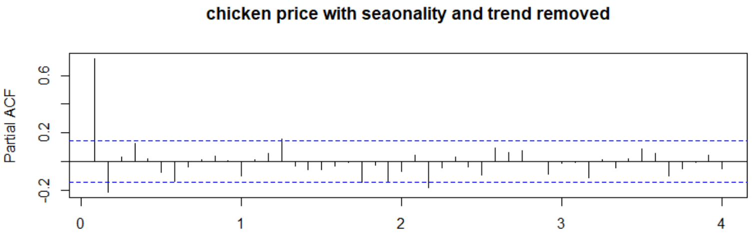


AR model: chicken price



AR model: chicken price





AR model: chicken price

what regression model to fit?

AR model

- Model
  - what will an AR time series look like?
  - what will the acf/pacf look like?
  - any constraints?
- Estimation
  - Method of Moments: Yule-Walker equations.
  - Maximum likelihood estimator.

AR model

Autoregressive model:

- Explain the current value x_t using the past values $x_{t-1}, \ldots$.

AR model

Example:

- A girl moving randomly on a giant chess board
- Next position of the girl is only dependent on the last position

Source: https://www.analyticsvidhya.com/blog/2015/12/complete-tutorial-time-series-modeling/

AR model

Example:

- You are in another room, not able to see the girl.
- You want to predit her position. How accurate will you be?

Source: https://www.analyticsvidhya.com/blog/2015/12/complete-tutorial-time-series-modeling/

AR model – a special case: random walk

A special case of AR model: (random walk)

$$x_t = x_{t-1} + w_t$$

- Is it stationary?

$$x_t = x_0 + w_1 + \ldots + w_t$$

- $E(x_t) = E(x_0) + E(w_1) + \ldots + E(w_t) = E(x_0)$
- $Var(x_t) = Var(x_0) + Var(w_1) + \ldots + Var(w_t) = t \times \sigma_w^2$

AR model – Introduced coefficient

Let's spice up things a bit:

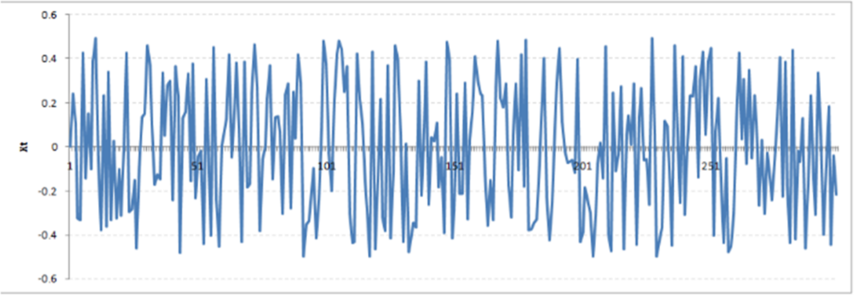
$$x_t = \phi x_{t-1} + w_t$$

- What does it look like?
- Is it stationary?

AR model – Introduced coefficient

Set $\phi = 0$ in an AR(1) model:

$$x_t = w_t$$

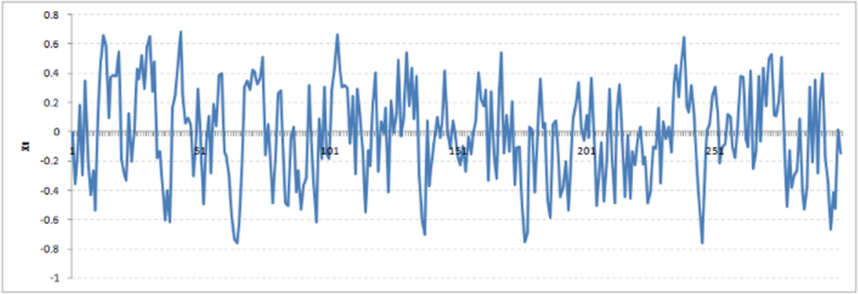


Source: https://www.analyticsvidhya.com/blog/2015/12/complete-tutorial-time-series-modeling/

AR model – Introduced coefficient

Set $\phi = 0.5$ in an AR(1) model:

$$x_t = 0.5x_{t-1} + w_t$$

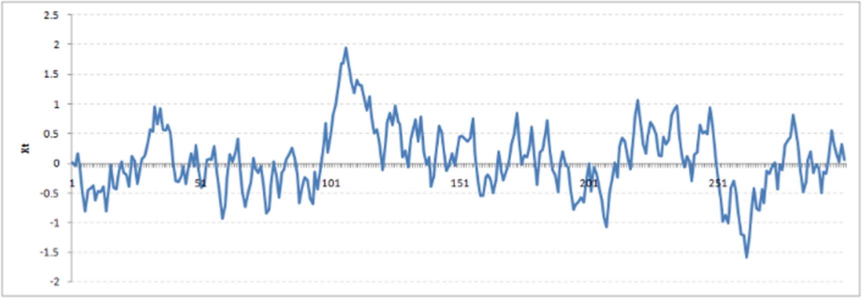


Source: https://www.analyticsvidhya.com/blog/2015/12/complete-tutorial-time-series-modeling/

AR model – Introduced coefficient

Set $\phi = 0.9$ in an AR(1) model:

$$x_t = 0.9x_{t-1} + w_t$$

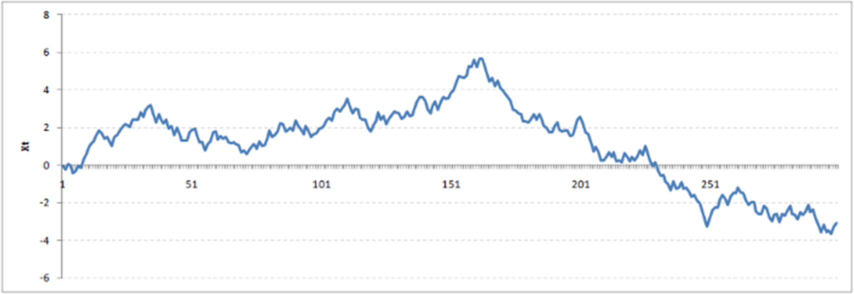


Source: https://www.analyticsvidhya.com/blog/2015/12/complete-tutorial-time-series-modeling/

AR model – Introduced coefficient

Set $\phi = 1$ in an AR(1) model:

$$x_t = x_{t-1} + w_t$$



Source: https://www.analyticsvidhya.com/blog/2015/12/complete-tutorial-time-series-modeling/

AR model

An autoregressive model of order p, denoted as $AR(p)$:

$$x_t = \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + w_t$$

- $w_t \overset{iid}{\sim} Wn(o, \sigma_w^2)$
- $\phi_1, \ldots, \phi_p$ constants and $\phi_p \neq 0$.
- For simplicity, assume $E(x_t) = 0$

AR model – AR(p)

An autoregressive model of order p, denoted as AR(p):

$$x_t = \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + w_t$$

Move all the items related with x to the left hand side:

$$\begin{aligned} & x_t - \phi_1 x_{t-1} - \ldots - \phi_p x_{t-p} = w_t \\ \Rightarrow & (1 - \phi_1 B - \ldots - \phi_p B^p) x_t = w_t \end{aligned}$$

Define $\Phi(B) \triangleq 1 - \phi_1 B - \ldots - \phi_p B^p$, the above can be written as

$$\Phi(B) x_t = w_t.$$

AR model

- Model
  - what will an AR time series look like?
  - what will the acf/pacf look like?
  - any constraints?
- Estimation
  - Method of Moments: Yule-Walker equations.
  - Maximum likelihood estimator.

AR model – AR(1)

Let's start from the first order model AR(1).

$$
\begin{aligned}
x_t &= \phi x_{t-1} + w_t \\
&= \phi(\phi x_{t-2} + w_{t-1}) + w_t = \phi^2 x_{t-2} + \phi w_{t-1} + w_t \\
&\vdots \\
&= \phi^k x_{t-k} + \phi^{k-1} w_{t-k+1} + \phi^{k-2} w_{t-k+2} + \ldots + \phi w_{t-1} + w_t \\
&= \phi^k x_{t-k} + \sum_{j=0}^{k-1} \phi^j w_{t-j}
\end{aligned}
$$

- If $|\phi| < 1$, let $k \to \infty$, we have $\phi^k \to 0$.
- So, $x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}$.

ARMA model – AR(1)

$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}$ is called the stationary solution of the AR(1) model.

- $E(x_t) = \sum_{j=0}^{\infty} \phi^j E(w_{t-j}) = 0$.
- $\gamma(0) = Var(x_t) = \sigma_w^2/(1-\phi^2)$.
- $\gamma(h) = Cov(x_{t+h}, x_t) = \sigma_w^2 \phi^h/(1-\phi^2)$.
- $\rho(h) = \gamma(h)/\gamma(0) = \phi^h$ – decay with h.

Note: sum of a geometric series for $\phi^2 < 1$:

- $1 + \phi^2 + \phi^4 + \ldots = (1-\phi^2)^{-1}$
- $1 + \phi^2 + \phi^4 + \ldots + \phi^{2(k-1)} = (1-\phi^{2k})/(1-\phi^2)$

AR model – AR

Explosive AR model

- AR(1) $x_t = \phi x_{t-1} + w_t$
- $|\phi| < 1 \Rightarrow$ stationary.
- $\phi = 1 \Rightarrow$ random walk. $\Rightarrow$ not stationary!
- How about $|\phi| > 1$?
- The same procedure in page 19 will not work because $|\phi|^k \to \infty$ for large n.
- Also, $\sum_{j=1}^{k-1} \phi^j w_{t-j}$ will not converge as $k \to \infty$.

AR model – AR

AR(1):

$$x_t = \phi x_{t-1} + w_t$$

- $|\phi| < 1$: stationary and causal:

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}.$$

- $\phi = 1$: random walk:

$$x_t = w_t + w_{t-1} + \ldots + w_1 + \ldots.$$

- $|\phi| > 1$: stationary but future dependent (explosive):

$$x_t = -\sum_{j=1}^{\infty} \phi^{-j} w_{t+j}.$$

Test for Unit Root:

$$x_t = \phi x_{t-1} + w_t$$

We want to test for whether: $\phi = 1$, so rewrite the AR(1) model:

$$\Delta(x_t) = x_t - x_{t-1} = (\phi - 1)x_{t-1} + w_t$$

And test for whether $(\phi - 1) = 0$.

- This is called a Dickey-Fuller test

Reference:
http://www.ams.sunysb.edu/∼zhu/ams586/UnitRoot_ADF.pdf

AR model – AR(p)

Example: Find the stationary solution of AR(1) model: $x_t = 0.9x_{t-1} + w_t$.

$$
\begin{aligned}
x_t &= 0.9x_{t-1} + w_t \\
&= 0.9(0.9x_{t-2} + w_{t-1}) + w_t = 0.9^2 x_{t-2} + 0.9w_{t-1} + w_t \\
&\vdots \\
&= 0.9^k x_{t-k} + \sum_{j=0}^{k-1} 0.9^j w_{t-j} \\
&= \sum_{j=0}^{\infty} 0.9^j w_{t-j}
\end{aligned}
$$

ARMA model – AR

The techniques of iterating back works well for AR(1) but not for AR(p), $p > 1$.

- A more general technique:

$$\begin{aligned}\Phi(B)x_t &= w_t \\ \Rightarrow x_t &= \sum_{j=0}^{\infty} \psi_j w_{t-j} = \Psi(B)w_t\end{aligned}$$

- But how can we get $\Psi(B)$ from $\Phi(B)$?
- $\Phi(B)\Psi(B)w_t = w_t \Rightarrow \Phi(B)\Psi(B) = 1$.
- That is, $(1 - \phi B)(1 + \psi_1 B + \psi_2 B^2 + \ldots + \psi_j B^j + \ldots) = 1$.

ARMA model – AR

$$
\begin{aligned}
(1-\phi B)(1+\psi_1 B+\psi_2 B^2+\ldots++\psi_j B^j+\ldots) &= 1 \\
\Rightarrow 1+(\psi_1-\phi)B+(\psi_2-\psi_1\phi)B^2+\ldots+(\psi_j-\psi_{j-1}\phi)B^j+\ldots &= 1 \\
\Rightarrow \psi_j = \psi_{j-1}\phi &
\end{aligned}
$$

Since $\phi_0 = 1$, we have

- $\psi_j = \phi^j$.

ARMA model – AR

Another way to think about it:

$$
\begin{aligned}
\Phi(B)x_t &= w_t \\
\Rightarrow \Phi^{-1}(B)\Phi(B)x_t &= \Phi^{-1}(B)w_t \\
x_t &= \Phi^{-1}(B)w_t
\end{aligned}
$$

- $\Phi(B) = 1 - \phi B$.
- So, $\Phi^{-1}(B) = 1 + \phi B + \phi^2 B^2 + \ldots + \phi^j B^j + \ldots$.
- Note, we always have $1 + a + a^2 + a^3 + \ldots = (1-a)^{-1}$ for $|a| < 1$.

ARMA model – Causal AR

An AR(1) model $(1 - \phi B)x_t = w_t$ is causal if

- $|\phi| < 1$;
- or in other words if $\phi(z) = 1 - \phi z = 0$ has root $|z| > 1$.

An AR(2) model $(1 - \phi_1 B - \phi_2 B^2)x_t = w_t$ is causal if

- the two roots of $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 = 0$ satisfy $|z_1| > 1$ and $|z_2| > 1$.

ARMA model – Causal AR

Check whether the following AR(2) model is causal

$$x_t = 1.5x_{t-1} - 0.75x_{t-2} + w_t$$

- $\phi(z) = 1 - 1.5z + 0.75z^2 = 0$
- $z_1 = 1 + i/\sqrt{3}$ and $z_2 = 1 - i/\sqrt{3}$

Estimation - Method of moment

$$x_t = \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + w_t$$

- Multiply both side by x_{t-h} and take expectation.
- Method of moments for AR: Yule-Walker equations.

$$\begin{cases} \gamma(h) = \phi_1\gamma(h-1) + \ldots + \phi_p\gamma(h-p), h = 1, \ldots, p. \\ \sigma_w^2 = \gamma(0) - \phi_1\gamma(1) - \ldots - \phi_p\gamma(p) \end{cases}$$

- It relates AR model parameters to the autocovariance.

Yule-Walker equations.

$$\begin{cases} \gamma(h) = \phi_1\gamma(h-1) + \ldots + \phi_p\gamma(h-p), h = 1, \ldots, p. \\ \sigma_w^2 = \gamma(0) - \phi_1\gamma(1) - \ldots - \phi_p\gamma(p) \end{cases}$$

- Matrix notation: $\Gamma_p \underline{\phi} = \underline{\gamma}_p$, $\sigma_w^2 = \gamma(0) - \underline{\phi}' \underline{\gamma}_p$.
- Yule-Walker estimates:

$$\hat{\underline{\phi}} = \hat{\Gamma}_p^{-1}\hat{\gamma}_p = \hat{R}_p^{-1}\underline{\rho}_p, \ \hat{\sigma}_w^2 = \hat{\underline{\gamma}}(0)[1 - \underline{\rho}_p' \hat{R}_p^{-1} \underline{\rho}_p]$$

$$\sqrt{n}(\hat{\underline{\phi}} - \underline{\phi}) \xrightarrow{d} \mathcal{N}(0, \sigma_w^2 \Gamma_p^{-1})$$

Estimation - Maximum likelihood

$$x_t = \phi(x_{t-1}) + w_t,\ |\phi| < 1,\ w_t \overset{iid}{\sim} \mathcal{N}(0, \sigma_w^2)$$

- $L(\phi, \sigma_w^2) = f(x_1, x_2, \dots, x_n|, \phi, \sigma_w^2) = f(x_1)f(x_2|x_1)\dots f(x_n|x_{n-1})$
- $f(x_t|x_{t-1}) = f_w[(x_t) - \phi(x_{t-1})];$
- $x_1 = \sum_{j=0}^{\infty} \phi^j w_{1-j} \Rightarrow x_1 \sim \mathcal{N}(0, \sigma_w^2/(1-\phi^2))$

$$L(\mu, \phi, \sigma_w^2) = f(x_1)f_w[(x_2) - \phi(x_1)]\dots f_w[(x_n) - \phi(x_{n-1})]$$

- The maximum likelihood estimator is obtained by maximizing the above likelihood function.

Estimation - Method of moment: Example

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t$$

- $\hat{\gamma}(0) = 8.903$, $\hat{\rho}(1) = 0.849$ and $\hat{\rho}(2) = 0.519$.
- Yule-Walker equations:

$$\begin{cases} \gamma(1) = \phi_1\gamma(0) + \phi_2\gamma(-1) \\ \gamma(2) = \phi_1\gamma(1) + \phi_2\gamma(0) \\ \sigma_w^2 = \gamma(0) - \phi_1\gamma(1) - \phi_2\gamma(2) \end{cases}$$

- Yule-Walker estimates:

$$\hat{\phi} = \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix} = \begin{bmatrix} 1 & .849 \\ .849 & 1 \end{bmatrix}^{-1} \begin{pmatrix} .849 \\ .519 \end{pmatrix} = \begin{pmatrix} 1.463 \\ -.723 \end{pmatrix}$$

$$\hat{\sigma}_w^2 = 8.903 \left[1 - (.849, .519) \begin{pmatrix} 1.463 \\ -.723 \end{pmatrix} \right] = 1.187$$

Examples: Look at AR model

$$x_t = \phi x_{t-1} + w_t$$

- $\phi = 0.9$.
- $\phi = -0.9$.
- $\phi = 1$.

Examples – Plots of AR(1) models

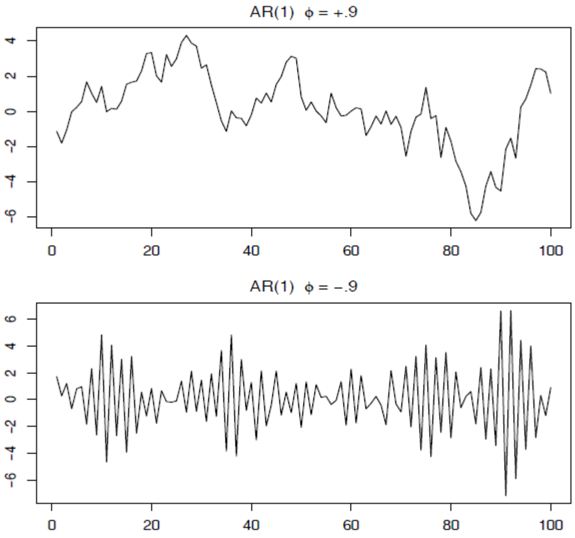
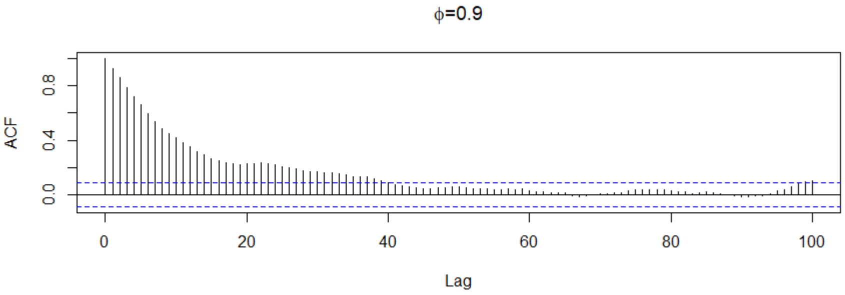
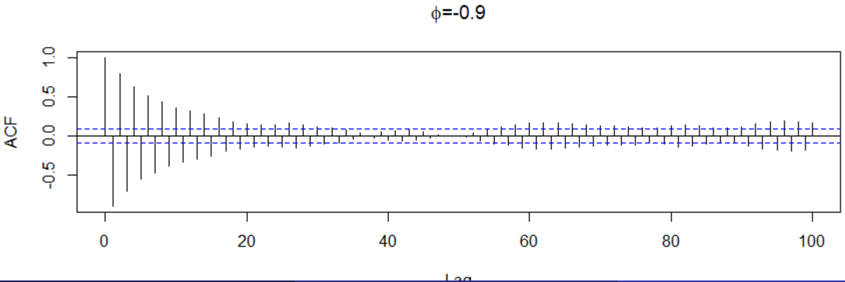


Fig. 3.1. Simulated AR(1) models: $\phi = .9$ (top); $\phi = -.9$ (bottom).

AR model

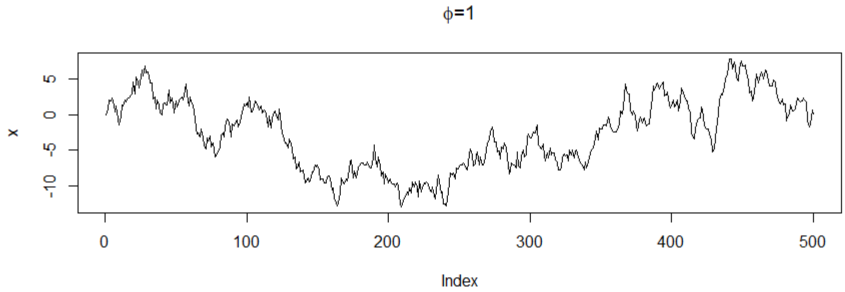


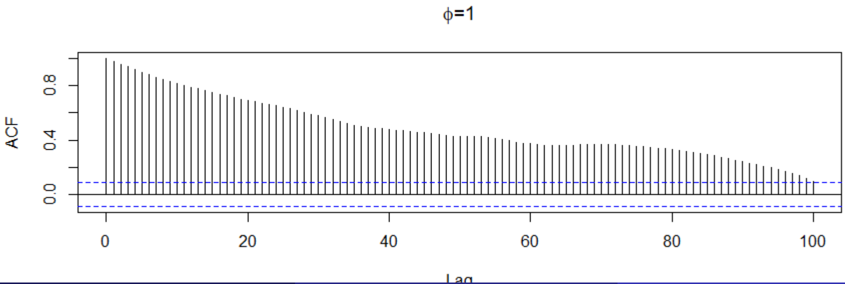
ϕ=0.9
ACF
0.0
0.4
0.8
0
20
40
60
80
100
Lag



ϕ=-0.9
ACF
-0.5
0.0
0.5
1.0
0
20
40
60
80
100

Examples – Plot of random walk





Examples

Example 3.27: Simulation, Estimation and Causality

$$x_t = 1.5x_{t-1} - .75x_{t-2} + w_t \tag{1}$$

To do:

- Generate 144 samples from the model.
- Check causality of the model.
- Model estimation and inference.

Examples

Example: Chicken price data

- Different types of estimation.
- Model diagnosis.

Reading material

https://www.analyticsvidhya.com/blog/2015/12/complete-tutorial-time-series-modeling/

- Ch 3.1 p.75-79
- Ch 3.5 p.113-114