MSA 8200 Predictive Analytics
Week5: SARIMA and Case Study

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Outline

- Integrated models for nonstationary data (ARIMA).
- Cointegration.
- Multiplicative seasonal ARIMA models.
- Walmart sales prediction example.
- Regression with correlated errors.

Integrated models

Random walk:

$$x_t = x_{t-1} + w_t$$

- $\nabla x_t = w_t$ is stationary.
- First order difference.

Integrated models

A nonstationary trend + a zero mean stationary component:

$$x_t = \mu_t + y_t$$

1. $\mu_t = \beta_0 + \beta_1 t$

$$\nabla x_t = x_t - x_{t-1} = \beta_1 + y_t - y_{t-1} = \beta_1 + \nabla y_t$$

2. $\mu_t = \mu_{t-1} + \gamma_t$, γ_t stationary.

$$\nabla x_t = \gamma_t + \nabla y_t$$

3. $\mu_t = \sum_{j=0}^{k} \beta_j t^j \Rightarrow \nabla^k x_t$ stationary.

Integrated models

Example:

- $x_t = \mu_t + y_t$

$$\mu_t = \mu_{t-1} + \gamma_t \text{ and } \gamma_t = \gamma_{t-1} + e_t$$

- e_t stationary.

- $\nabla x_t = \gamma_t + \nabla y_t$: not stationary.
- $\nabla^2 x_t = e_t + \nabla^2 y_t$: stationary.

Such process is called integrated ARMA model or ARIMA model.

Integrated models

Definition 3.11 A process x_t is said to be ARIMA(p, d, q) if

- $\nabla^d x_t = (1 - B)^d x_t$ is ARMA(p, q).
- In general, it can be written as $\phi(B)(1 - B)^d x_t = \theta(B) w_t$.

Integrated models – Unit Root test

Unit Root test.

Integrated models – Cointegration

What is cointegration?

How to use it in practice?

- Trading strategy. https://quant.stackexchange.com/questions/3270/what-are-the-applications-of-cointegration

Multiplicative seasonal ARIMA models

- Dependence occures most strongly at multiples of seasonal lag s.
- For example, monthly economic data, s=12.
- Quarterly data, s = 4.

Pure seasonal autoregressive moving average model: ARMA$(P, Q)_s$

$$\Phi_P(B^s)x_t = \Theta_Q(B^s)w_t$$

- Seasonal AR operator: $\Phi_P(B^s) = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \ldots - \Phi_P B^{Ps}$.
- Seasonal MA operator: $\Theta_Q(B^s) = 1 + \Theta_1 B^s + \Theta_2 B^{2s} + \ldots + \Theta_Q B^{Qs}$.

Multiplicative seasonal ARIMA models

Example 3.46 Seasonal AR(1) series

$$(1 - \Phi B^{12})x_t = w_t$$

- $x_t = \Phi x_{t-12} + w_t$
- Estimation and forcasting is similar to AR(1) model.
- $|\Phi| < 1$

Simulated 3 years of data with $\Phi = 0.9$:

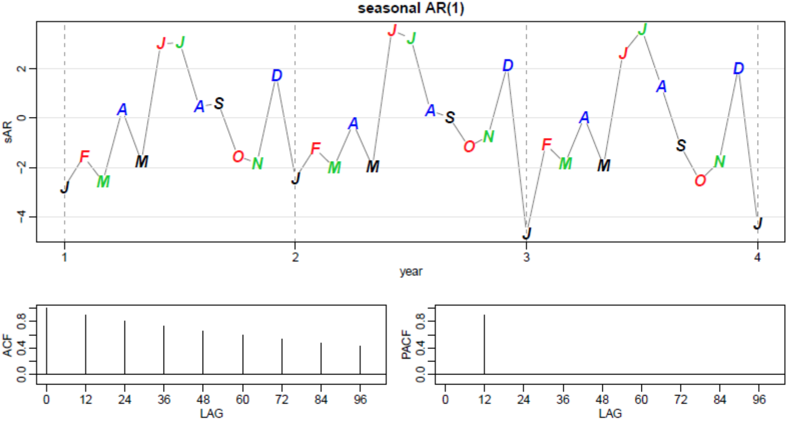

Fig. 3.20. *Data generated from a seasonal ($s = 12$) AR(1), and the true ACF and PACF of the model* $x_t = .9x_{t-12} + w_t$.

Multiplicative seasonal ARIMA models

First order seasonal ($d = 12$) MA model

$$x_t = w_t + \Theta w_{t-12}$$

- $\gamma(0) = (1 + \Theta^2)\sigma^2$
- $\gamma(\pm 12) = \Theta\sigma^2$
- $\gamma(h) = 0$ otherwise.
- The only non-zero correlation is $\rho(\pm 12) = \Theta/(1 + \Theta^2)$.

First order seasonal ($d = 12$) AR model

$$x_t = \Phi x_{t-12} + w_t$$

- $\gamma(0) = \sigma^2/(1 - \Phi^2)$
- $\gamma(\pm 12k) = \sigma^2 \Phi^k/(1 - \Phi^2)$
- $\gamma(h) = 0$ otherwise.
- The only non-zero correlations are $\rho(\pm 12k) = \Theta^k,\ k = 0, 1, 2, \ldots$.

Multiplicative seasonal ARIMA models

Table 3.3. Behavior of the ACF and PACF for Pure SARMA Models

	$\text{AR}(P)_s$	$\text{MA}(Q)_s$	$\text{ARMA}(P, Q)_s$
ACF*	Tails off at lags ks, $k = 1, 2, \ldots,$	Cuts off after lag Qs	Tails off at lags ks
PACF*	Cuts off after lag Ps	Tails off at lags ks $k = 1, 2, \ldots,$	Tails off at lags ks

*The values at nonseasonal lags $h \neq ks$, for $k = 1, 2, \ldots$, are zero.

Multiplicative seasonal ARIMA models

General multiplicative seasonal ARMA, ARMA$(p, q) \times (P, Q)_s$

$$\Phi_P(B^s)\phi(B)x_t = \Theta_Q(B^s)\theta(B)w_t$$

Example 3.47: A mixed seasonal model

ARMA$(0, 1) \times (1, 0)_{12}$ model:

$$x_t = \Phi x_{t-12} + w_t + \theta w_{t-1}$$

Multiplicative seasonal ARIMA models

ARMA$(0,1) \times (1,0)_{12}$ model:

$$x_t = \Phi x_{t-12} + w_t + \theta w_{t-1}$$

- Take the variance of both side: $\gamma(0) = \Phi^2\gamma(0) + \sigma_w^2 + \theta^2\sigma_w^2$

$$\gamma(0) = \frac{1+\theta^2}{1-\Phi^2}\sigma_w^2$$

- Multiply both side by x_{t-1} and take exp: $\gamma(1) = \Phi\gamma(11) + \theta\sigma_w^2$.
- Multiply both side by x_{t-h} $(h \geq 2)$ and take exp: $\gamma(h) = \Phi\gamma(h-12)$.

- $\rho(12h) = \Phi^h$.
- $\rho(12h-1) = \rho(12h+1) = \frac{\theta}{1+\theta^2}\Phi^h$.
- $\rho(h) = 0$ otherwise.

Multiplicative seasonal ARIMA models

ARMA$(0,1) \times (1,0)_{12}$ model:

$$x_t = \Phi x_{t-12} + w_t + \theta w_{t-1}$$

- Setting: $\Phi = 0.8$ and $\theta = -0.5$.

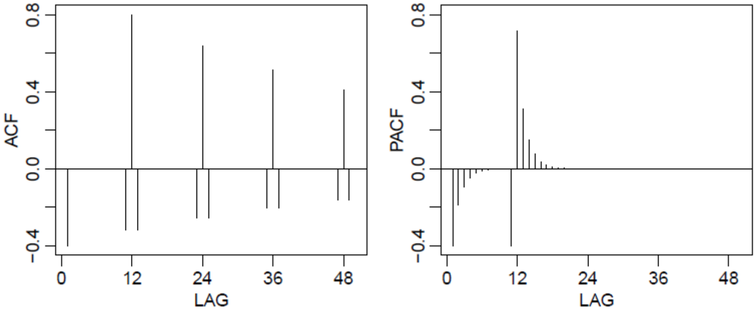

Fig. 3.21. *ACF and PACF of the mixed seasonal ARMA model* $x_t = .8x_{t-12} + w_t - .5w_{t-1}$.

Multiplicative seasonal ARIMA models

Seasonal persistence: when the process is nearly periodic.

- For example, the average monthly temperature over the years.
- Each January would be approximately the same. Each Febuary ...
- Model average monthly temperature (x_t):

$$x_t = S_t + w_t, w_t \sim \text{ white noise.}$$

- S_t the seasonal component, varies a litttle:

$$S_t = S_{t-12} + \nu_t, \nu_t \sim \text{ white noise.}$$

- ACF decays very slowly at lags $h = 12k$.

Multiplicative seasonal ARIMA models

Seasonal persistence: when the process is nearly periodic.

- S_t the seasonal component, varies a litttle:

$$S_t = S_{t-12} + \nu_t, \nu_t \sim \text{ white noise.}$$

- Subtract the effect of successive years from each other (a seasonal difference of order 1):

$$(1 - B^{12})x_t = x_t - x_{t-12} = \nu + w_t - w_{t-12}$$

- $MA(1)_{12}$, ACF has a peak only at lag 12.

Multiplicative seasonal ARIMA models

Definition 3.12 Multiplicative seasonal autoregressive integrated moving average (SARIMA)

$$\Phi_P(B^s)\phi(B)\nabla_s^D\nabla^d x_t = \delta + \Theta_Q(B^s)\theta(B)w_t$$

- ARIMA$(p, d, q) \times (P, D, Q)_s$.

Example 3.49 Air passengers.

- Monthly totals of international airline passengers, 1949 to 1960.

Case Study: Walmart Sales Prediction

Walmart recruiting competition:

- https://www.kaggle.com/c/walmart-recruiting-store-sales-forecasting

https://rpubs.com/spillai/walmart_store_sales_forecast

Regression with autocorrelated errors

$$y_t = \sum_{j=1}^{r} \beta_j z_{tj} + x_t$$

- If x_t is white Gaussian noise such that

$$\gamma_x(s,t) = 0 \text{ for } s \neq t \text{ and } \gamma_x(t,t) = \sigma^2$$

  OLS will provide the efficient estimators.
- Otherwise, weighted least squares.

Regression with autocorrelated errors

$$\underline{y} = Z\underline{\beta} + \underline{x}$$

- $\underline{x} \sim \mathcal{N}(0, \Gamma)$, $\Gamma = \{\gamma_x(s, t)\}$.

$$\Gamma^{-1/2}\underline{y} = \Gamma^{-1/2}Z\underline{\beta} + \Gamma^{-1/2}\underline{x}$$
$$\underline{y}^* = Z^*\underline{\beta} + \underline{\delta}$$

- $\underline{\delta} = \Gamma^{-1/2}\underline{x} \sim \mathcal{N}(0, I)$.
- The weighted least square estimator is

$$\underline{\hat{\beta}}_w = ((Z^*)'Z^*)^{-1}(Z^*)'\underline{y}* = (Z'\Gamma^{-1}Z)^{-1}(Z'\Gamma^{-1}\underline{y})$$

Regression with autocorrelated errors

In the time series setting,

$$y_t = \sum_{j=1}^{r} \beta_j z_{tj} + x_t$$

- If we know the model for x_t:

$$\phi(B)x_t = w_t, \ (\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p)$$

$$\Rightarrow \phi(B)y_t = \sum_{j=1}^{r} \beta_j \phi(B) z_{tj} + \phi(B)x_t$$

$$y_t^* = \sum_{j=1}^{r} \beta_j z_{tj}^* + w_t$$

- Least square estimator minimizes:

$$S(\phi, B) = \sum_{t=1}^{n} w_t^2 = \sum_{t=1}^{n} [\phi(B)y_t - \sum_{j=1}^{r} \beta_j \phi(B) z_{tj}]^2$$

Regression with autocorrelated errors

The general process:

1. First run an ordinary regression (as if the errors are uncorrelated.) Obtain the residuals, $\hat{x}_t = y_t - \sum_{j=1}^{r} \hat{\beta}_j z_{tj}$.
2. Identify the ARMA model for the residuals $\hat{x}_t$.
3. Run weighted least squares (or MLE) on the regression model with autocorrelated error structure.
4. Model diagnostics.

Example 3.45 Regression with two time series.

$$R_t = \beta_0 + \beta_1 S_{t-6} + \beta_2 D_{t-6} + \beta_3 D_{t-6} S_{t-6} + w_t$$

- R_t recruitment series.
- S_t the SOI index series.
- D_t adummy variable.

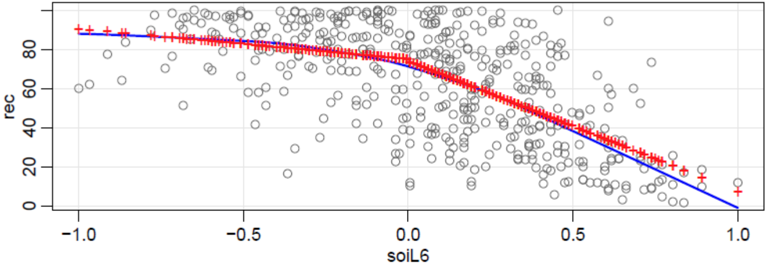

Fig. 2.10. Display for Example 2.9: Plot of Recruitment (R_t) vs SOI lagged 6 months (S_{t-6}) with the fitted values of the regression as points (+) and a lowess fit (—).

Regression with autocorrelated errors

Example Stock Prediction using dividend. https://www.econometrics-with-r.org/14-9-can-you-beat-the-market-part-ii.html

Reading Materials

- Ch 3.6 (Integrated model)
- Ch 3.9 (SARIMA)
- Ch 3.8 p.142-145 (Regression with Autocorrelated Errors)
- https://medium.com/@arneeshaima/walmart-sales-data-analysis-sales-prediction-using-multiple-linear-regression-in-r-programming-adb14afd56fb