Coulomb Branch of $3d$ $\mathcal{N} = 4$ Quiver Gauge Theories

Bachelor of Science Project

Author: Carl Bouchard
Partner: George Smith

Supervisor: Prof. Amihay Hanany
Assessor: Prof. Daniel Waldram

Department of Physics
Imperial College London
2024

Abstract

$3d$ $\mathcal{N} = 4$ quiver gauge theories are of interest due to their compelling properties, such as both branches of the moduli space being hyper-Kähler. In this study, we introduce relevant tools in the analysis of their moduli spaces, including the Hilbert series, the plethystic programme, quiver diagrams, and the monopole formula. Employing these methods, we make explicit calculations for the Coulomb branch of the moduli space for theories such as $U(1)$ with n flavours and the subregular nilpotent orbit of G_2. We investigate properties such as their dimension and their global symmetries.

Declaration

All aspects of this project, including research and calculations, were completed collaboratively with my project partner, with both of us contributing equally to every component. This has also been our first exposure to quiver gauge theories and, more generally, to supersymmetry and Lie algebras.

Acknowledgements

First and foremost, I would like to thank our supervisor, Prof Amihay Hanany, for giving us this incredible opportunity to investigate topics that would, I believe, generally be considered outside the scope of a third-year physicist's repertoire. As such, I am very grateful that he took the time to direct our research and answer our questions.

I would also like to thank my project partner, for without our collaborative efforts I believe neither of us would have managed to push this project as far as we did.

Additionally, I would like to thank Guhesh Kumaran for his continued support and for always being approachable to answer our questions. I would also like to thank Torben Skrzypek for pointing us in the right direction at the beginning of the project, when we faced the question of what the Hilbert series for the Coulomb branch of $U(1)$ with n flavours is, without any prior experiences or additional pointers.

Contents

Chapter 1

Introduction

> ”Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection.”
>
> — Hermann Weyl

Symmetry unveils the most fundamental and intrinsic structures in physics, reaching depths that even Weyl could not have fully envisioned. Shortly after his passing in 1955, the greatest scientific theory of all time - the Standard Model of particle physics - began to take shape[1]. At its core, symmetries give rise to all particles and their interactions. Most prominently, it includes the Poincaré spacetime symmetry, as well as the famous local gauge symmetry

$$G = U(1) \times SU(2) \times SU(3), \tag{1.1}$$

dictating the interactions of gauge bosons (spin-1 fields).

Another fundamental aspect of symmetries is their relation to conservation laws. Or, as Noether's theorem [1] states (somewhat informally): Every continuous symmetry of a system corresponds to a conserved quantity. For example, translational symmetry leads to conservation of momentum and symmetry in time leads to energy conservation.

In the early 1960s, the $SU(3)$ symmetry was shown to accurately describe hadrons of the same spin, leading many physicists to pursue an extension that includes different spins in a $SU(6)$ symmetry. An effort that, although leading to some successes, ultimately failed due to not working in a relativistic setting. Efforts of this kind led to a series of so-called no-go theorems restricting the ways symmetries can combine, giving rise to a theorem by Sidney Coleman and Jeffrey Mandula [2]:

[1] It is important to emphasize that his introduction of the concept of gauge theories laid the very foundation of the standard model. Despite his many significant contributions, he remains underrecognized by the public. (Interestingly, the name Weyl appears, on average, once every four pages.)

Theorem 1 (Coleman-Mandula). *The symmetry group of a theory, under certain assumptions, is a direct product of the Poincaré symmetry and internal symmetries*

$$G = G_{Poincaré} \times G_{Internal}. \tag{1.2}$$

Yet, this theorem has certain limitations. Fundamentally, the Coleman-Mandula theorem is a statement about Lie algebras, but it can be extended to include Lie superalgebras, where instead of having only commutation relations, we can also have anti-commutation relations. This work was done by Haag, Łopuszański and Sohnius [3] and the corresponding theorem now states:

Theorem 2 (Haag-Łopuszański-Sohnius). *The symmetry group of a theory, under certain assumptions, is a direct product of the superPoincaré symmetry and internal symmetries*

$$G = G_{superPoincaré} \times G_{Internal}. \tag{1.3}$$

The superPoincaré algebra allows for additional so-called supercharges. Conceptually, they transform between bosons and fermions

$$Q\,|\text{boson}\rangle = |\text{fermion}\rangle \qquad \text{and} \qquad Q\,|\text{fermion}\rangle = |\text{boson}\rangle\,. \tag{1.4}$$

This is the essence of supersymmetry, every fermion has a bosonic partner and vice versa[2]. For example, the top quarks (fermion) superpartner is the stop squark (boson), and the superpartner of the gluon (boson) is the gluino (fermion).

Considering the stark differences between bosons and fermions, as well as the fact that we have never encountered a superpartner experimentally, one should question why it is even worth thinking about supersymmetry. Most arguments follow the spirit of the quote at the very beginning of this chapter, following a sense of beauty and elegance.

First, we have already seen that supersymmetry bypasses the very restrictive Coleman-Mandula theorem. This alone gives it a sense of importance and intrigue. Another reason to consider it is that it places mathematically interesting restrictions on the dynamics of theories. As such, one can solve certain problems in supersymmetry, which would have been difficult to solve in traditional quantum field theory alone. One example of this is the confinement problem of quarks to hadrons. We never see a quark on its own; they always come in at least pairs, yet we only know this to be true from experiments rather than from the standard model theory. Surprisingly (or perhaps not surprisingly), we can prove confinement analytically in certain supersymmetric gauge theories. Finally, there is more evidence if we look at string theories, which seem to require supersymmetry at some level. Again, this seems to paint a bigger picture. If supersymmetry exists in our universe, then it is broken at our current energy scales and will only emerge once we are able to probe larger energies. Regardless, because of its many intriguing features, it is worthwhile to continue exploring and further our understanding.

[2]Chronologically speaking, supersymmetry started emerging before the Haag-Łopuszański-Sohnius theorem.

In particular, we want to focus on $3d\ \mathcal{N}=4$ quiver gauge theories. These are supersymmetric theories with eight supercharges in $2+1$ dimensions, whose Lagrangian can be expressed as a graph called a quiver. The moduli space of quantum field theories is the space of allowed vacuum expectation values, and in supersymmetry, it usually has a rich structure. This is especially true in $3d\ \mathcal{N}=4$ theories as both branches of the moduli space are hyperKähler manifolds, leading to interesting mathematical consequences (see, for example, Section 4.2.5). The moduli space is also an integral part of studying a theory's quantum dynamics and, hence, of paramount importance. We will put special emphasis on the Coulomb branch of these theories' moduli space and investigate some of its properties.

The report is structured as follows:

- Chapter 2 aims to introduce some tools we frequently use, these include the refined and unrefined Hilbert series, plethystic exponential and logarithm, as well as highest weight generating functions. This chapter is, in a sense, self-contained in that it does not yet make the connection to supersymmetry or any physical theory.

- Chapter 3 introduces supersymmetry in the mix and specifies the realm of theories that we are studying, discussing dimensional reduction from $4d\ \mathcal{N}=2$ theories to $3d\ \mathcal{N}=4$, quiver diagrams and the moduli spaces. In this chapter, we will also explicitly see the connection between our abstractly developed methods and the physical theories.

- Chapter 4 provides a specific look at the Coulomb branch of the moduli space where we go through many relevant examples. It also introduces the monopole formula, which is the main way of studying the Coulomb branch in our studies. We also discuss the perturbative approach, which simplifies the calculation of the monopole formula for complicated quivers.

- Chapter 5 summarises our findings and provides an outlook for future studies.

Additionally, as most of the work requires prior knowledge of, in particular, Lie algebras and supersymmetry, they are (at times extensively) discussed in the appendix, see Appendix A and Appendix B. Appendix A can be viewed as a standalone introduction to Lie algebras for the specific purpose of quiver gauge theories. Further relevant background material, not made direct reference to in the text, can be found in [4, 5].

Chapter 2

Methods in Quiver Gauge Theories

In this chapter, we introduce the foundational tools and methods we will frequently rely on in our subsequent studies of quiver gauge theories. Initially, we examine these concepts in an isolated environment without any direct physical applications. In the next chapter, Chapter 3, we will make the connection between the mathematical framework and the physical theories explicit by showing how the space of permissible vacuum expectation values in a supersymmetric theory can be viewed as an algebraic variety. Part of the material here is compiled from [6, 7, 8, 9, 10] supplemented with additional explanations and examples.

2.1 Hilbert Series

The objects that we are interested in studying are affine algebraic varieties, which we will study using their coordinate rings. We begin with defining the objects of interest.

Definition 1. An **affine algebraic set** is the set of solutions in an algebraically closed field k of a system of polynomial equations with coefficients in k. So, if $f_1, ..., f_m$ are polynomials with coefficients in k, they define an affine algebraic set

$$\mathcal{V}(f_1, ..., f_m) = \{(a_1, ..., a_n) \in k^n \mid f_1(a_1, ..., a_n) = ... = f_m(a_1, ..., a_n) = 0\} . \quad (2.1)$$

Definition 2. An **affine variety** is an affine algebraic set that is not the union of two proper affine algebraic subsets, i.e. it is irreducible.

To study these objects, it will be useful to look at functions defined over them. We define the coordinate ring:

Definition 3. Given an affine variety $\mathcal{V} \subseteq k^n$, the coordinate ring $k[\mathcal{V}]$ is defined to be the set of polynomials

$$f|_{\mathcal{V}} : \mathcal{V} \to k, \quad (2.2)$$

where $f \in k[x_1, x_2, ..., x_n]$. We say that $k[\mathcal{V}]$ is the ring of polynomials on $\mathcal{V}$.

The ring of polynomials can be graded by degree, and we can decompose it into rings of different polynomial degrees.

Definition 4. A **graded polynomial ring** $\mathcal{R}$ is defined as a ring that can be written as a direct sum decomposition

$$\mathcal{R} = \bigoplus_i \mathcal{R}_i, \tag{2.3}$$

where $\mathcal{R}_i$ consist of polynomials of degree i.

Example 1. Consider the coordinate ring $\mathbb{C}[x]$ of complex polynomials, and we may grade them by degree:

Degree i	Monomials in $\mathcal{R}_i$
0	1
1	x
2	x^2
...	...

We list the monomials as they are the generators of the graded piece, and in the next section, we will be interested in counting them.

Example 2. A more complicated example is the affine variety $\mathcal{V} = \mathbb{C}^2/\mathbb{Z}_2$. For the coordinate ring, we now have two complex variables x_1 and x_2, but we also have the group $\mathbb{Z}_2$ acting on the polynomial functions. The action of this group is taking the positive elements to their negative counterparts and vice versa:

$$(x_1, x_2) \to (-x_1, -x_2). \tag{2.4}$$

We are hence only interested in invariant monomials, so if we grade by the coordinate ring $\mathbb{C}^2[x_1, x_2]/\mathbb{Z}_2$, we get:

Degree i	Invariant Monomials in $\mathcal{R}_i$
0	1
1	-
2	x_1^2, x_1x_2, x_2^2
...	...

Something worth mentioning at this point is that we have three generators,

$$X = x_1^2, \quad Y = x_2^2, \quad Z = x_1x_2, \tag{2.5}$$

which can generate the whole ring and are related by

$$XY = Z^2. \tag{2.6}$$

Alternatively, one could have generated the complete ring from just those two pieces of information.

2.1.1 Unrefined Hilbert Series

We can now develop some machinery to investigate the coordinate rings. The first tool allows us to count the generators of each graded piece, we define:

Definition 5. The **Hilbert series** of a graded ring $\mathcal{R}$ is defined as

$$\mathrm{HS}(t) = \sum_{i=0}^{\infty} \dim_{\mathbb{C}}(\mathcal{R}_i) t^i, \tag{2.7}$$

where $\mathcal{R}_i$ is the graded piece of $\mathcal{R}$, and t is called the fugacity, which acts as a dummy counting variable.

In addition to the form in Eq. (2.7), we can also write this as

$$\mathrm{HS}(t) = \frac{Q(t)}{(1-t^\alpha)^d}, \tag{2.8}$$

where $d = \dim_{\mathbb{C}}(\mathcal{V})$ and $Q(t)$ is some polynomial which satisfies $Q(1) \neq 0$, i.e. the order of the pole at $t = 1$ gives us the dimension of the space that we are investigating. This form is very instructive and will be used frequently throughout.

Example 3. Expanding on Example 1, we can count the different monomials at each degree for the coordinate ring $\mathbb{C}[x]$. Since the monomials are the generators of the graded piece, the number of monomials is the dimension of the graded piece.

Degree i	Monomials in $\mathcal{R}_i$	$\dim_{\mathbb{C}}(\mathcal{R}_i)$
0	1	1
1	x	1
2	x^2	1
...	...	...
i	x^i	1

So, the Hilbert series is simply

$$\begin{aligned}\mathrm{HS}(t) &= 1 + t + t^2 + ... \\ &= \sum_{i=0}^{\infty} t^i.\end{aligned} \tag{2.9}$$

Or in quotient form

$$\mathrm{HS}(t) = \frac{1}{1-t}. \tag{2.10}$$

In this form we also immediately see that the dimension of our space is 1.

Example 4. Extending this to multiple dimensions is straightforward. We consider the coordinate ring $\mathbb{C}^2[x_1, x_2]$. Again, enumerating possible monomials at each degree and counting them, we get:

Degree i	Monomials in $\mathcal{R}_i$	$\dim_{\mathbb{C}}(\mathcal{R}_i)$
0	1	1
1	x_1, x_2	2
2	x_1^2, x_1x_2, x_2^2	3
...	...	...
i	x_1^i, ..., x_2^i	$i+1$

We have the following Hilbert series

$$\mathrm{HS}(t) = 1 + 2t + 3t^2 + ... = \frac{1}{(1-t)^2}, \tag{2.11}$$

and therefore $\dim_{\mathbb{C}}(\mathbb{C}^2) = 2$, as expected.

If $\mathcal{V}$ is a complete intersection, we also have the following form of the Hilbert series

$$\mathrm{HS}(t) = \frac{\prod_j (1-t^{b_j})^{r_j}}{\prod_i (1-t^{a_i})^{g_i}}, \tag{2.12}$$

where g_i is the number of generators and a_i is their degree, r_j is the number of relations between generators and b_j is their degree. If the relations have themselves relations, then $\mathcal{V}$ cannot be a complete intersection, and we will not have this form available to us.

Example 5. Consider the affine variety $\mathbb{C}^2/\mathbb{Z}_2$ from Example 2, where we already identified the generators. Now, we only need to list their symmetric products at each degree and count them.

Degree i	Monomials in $\mathcal{R}_i$	$\dim_{\mathbb{C}}(\mathcal{R}_i)$
0	1	1
1	$-$	0
2	x_1^2, x_1x_2, x_2^2	3
3	$-$	0
4	x_1^4, $x_1^3x_2$, $x_1^2x_2^2$, $x_1x_2^3$, x_2^4	5
...	...	...
$2i$	x_1^{2i}, ..., x_2^{2i}	$2i+1$
$2i+1$	$-$	0

Then, it can be shown using simple algebra that

$$\mathrm{HS}(t) = \sum_{i=0}^{\infty} (2i+1)t^{2i} = \frac{1+t^2}{(1-t^2)^2}. \tag{2.13}$$

Alternatively, to retrieve the dimension of the space, we can rewrite this as,

$$\mathrm{HS}(t) = \frac{1}{(1+t)(1-t)^2}. \tag{2.14}$$

Thus, we have that $\dim(\mathbb{C}^2/\mathbb{Z}_2) = 2$. And lastly, using the form for a complete intersection, we have

$$\mathrm{HS}(t) = \frac{1-t^4}{(1-t^2)^3}. \tag{2.15}$$

The last equation verifies what we already found in Example 2; we have 3 generators at degree 2 and 1 relation at degree 4.

2.1.2 Refined Hilbert Series

We can refine a Hilbert series to include additional information about the global symmetry of the affine variety.

Definition 6. The **refined Hilbert series** is defined as

$$\mathrm{HS}_{ref}(t) = \sum_{i=0}^{\infty} \chi([n_1, ..., n_r]_G) t^i, \tag{2.16}$$

where $\chi([n_1, ..., n_r]_G)$ is the character of the representation with the highest weight Dynkin label $[n_1, ..., n_r]_G$ of a group G (see Appendix A for more on Dynkin labels).

This can then also be used to retrieve the unrefined Hilbert series by

$$\mathrm{HS}_{unref}(t) = \sum_{i=0}^{\infty} \dim\left(\chi([n_1, ..., n_r]_G)\right) t^i. \tag{2.17}$$

And since this notation is getting quite cumbersome, we usually drop the χ for brevity and write

$$\chi([n_1, ..., n_r]_G) = [n_1, ..., n_r], \tag{2.18}$$

where we occasionally also drop the group label if it is obvious from the context. Also, depending on the context, we may explicitly label a Hilbert series as refined and unrefined or just leave away the subscript altogether. An interesting feature of the refined Hilbert series is that the t^2 term gives us the character of the adjoint representation of the global symmetry, whereas the other terms might consist of multiple different characters of the global symmetry.

Example 6. We again consider the space $\mathbb{C}^2/\mathbb{Z}_2$ as we did in Example 5, but now we want to refine the Hilbert series. All the monomials in the ring were of the form $x_1^a x_2^b$ where $a + b$ is even. So we can sum over some fugacities t_1 and t_2 in the following way:

$$\begin{aligned} \mathrm{HS}(t_1, t_2) &= \sum_{\substack{a,b \\ a+b \text{ even}}} t_1^a t_2^b. \\ &= \frac{1 + t_1 t_2}{(1 - t_1^2)(1 - t_2^2)}, \end{aligned} \tag{2.19}$$

where we have split the sum into even and odd parts to arrive at the last line. If we now apply the mapping $t_1 \to tx$ and $t_2 \to t/x$, we get

$$\begin{aligned} \mathrm{HS}(t) &= \frac{1 - t^4}{(1 - t^2 x^2)(1 - t^2)(1 - t^2/x^2)} \\ &= 1 + (x^2 + 1 + x^{-2})t^2 + (x^4 + x^2 + 1 + x^{-2} + x^{-4})t^4 + ... \\ &= 1 + [2]_{SU(2)} t^2 + [4]_{SU(2)} t^4 + ... \end{aligned} \tag{2.20}$$

In the second to last line, we have recognised the characters of $SU(2)$ representations (see Appendix A.7). Lastly, we may write this as

$$\mathrm{HS}(t) = \sum_{i=0}^{\infty} [2i]_{SU(2)} t^{2i}, \tag{2.21}$$

which we will occasionally refer to as the **character expansion**. If we were to now set $x = 1$ in the above, we would retrieve the unrefined Hilbert series as found in Example 5.

2.2 Plethystics

We now develop tools that allow for further analysis of the algebraic varieties, particularly for generating Hilbert series and extracting information from them. Following [11, 12], we have:

Definition 7. The **plethystic exponential** of a polynomial $f(t)$ is defined as

$$\mathrm{PE}[f(t)] \equiv \exp\left(\sum_{k=1}^{\infty} \frac{f(t^k) - f(0)}{k}\right), \tag{2.22}$$

and for a multivariate function $f(t_1, ..., t_n)$ this generalises straightforwardly

$$\mathrm{PE}[f(t_1, ..., t_n)] \equiv \exp\left(\sum_{k=1}^{\infty} \frac{f(t_1^k, ..., t_n^k) - f(0, ..., 0)}{k}\right). \tag{2.23}$$

A very useful result is that

$$\mathrm{PE}[f(t) + g(t)] = \mathrm{PE}[f(t)]\,\mathrm{PE}[g(t)]. \tag{2.24}$$

Usually, we will not need to use the definition of the plethystic exponential in our calculations, but rather we will use the following result: If $f(t)$ can be written as a power series of the form $f(t) = \sum_{n=0}^{k} a_n t^n$, where k is some upper limit which could be ∞, then the plethystic exponential can be written as

$$\mathrm{PE}[f(t)] = \mathrm{PE}\left[\sum_{n=0}^{k} a_n t^n\right] = \prod_{n=0}^{k} \frac{1}{(1-t^n)^{a_n}}. \tag{2.25}$$

The plethystic exponential is especially relevant in the calculation of the Hilbert series of the Higgs branch as it computes symmetric products. But we will not focus on this any further here.

Definition 8. The **plethystic logarithm** is given by

$$\mathrm{PL}[f(t)] = \mathrm{PE}^{-1}[f(t)] = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(f(t^n)), \tag{2.26}$$

where $\mu(n)$ is the Möbius function defined as

$$\mu(n) = \begin{cases} 0, & \text{if } n \text{ has repeated prime factors} \\ 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n \text{ is a product of } k \text{ distinct primes} \end{cases} . \tag{2.27}$$

A useful property of the Möbius function is that

$$\sum_{d|n} \mu(d) = \delta_{1,n}. \tag{2.28}$$

And similarly to Eq. (2.24), we have the following for the plethystic logarithm:

$$\mathrm{PL}[f(t)g(t)] = \mathrm{PL}[f(t)] + \mathrm{PL}[g(t)]. \tag{2.29}$$

Since the plethystic logarithm is the inverse function of the plethystic exponential, we can invert Eq. (2.25) to get

$$\mathrm{PL}\left[\prod_{n=0}^{k} \frac{1}{(1-t^n)^{a_n}}\right] = \sum_{n=0}^{k} a_n t^n. \tag{2.30}$$

By recalling the form of the Hilbert series of a complete intersection, see Eq. (2.12), we can readily extract all the information about the generators and relations by applying the plethystic logarithm to it:

$$\begin{aligned} \mathrm{PL}[\mathrm{HS}(t)] &= \mathrm{PL}\left[\frac{\prod_j (1-t^{b_j})^{r_j}}{\prod_i (1-t^{a_i})^{g_i}}\right] \\ &= \sum_i g_i t^{a_i} - \sum_j r_j t^{b_j}. \end{aligned} \tag{2.31}$$

If the plethystic logarithm has an infinite amount of terms, the Hilbert series can not be a complete intersection, and there exist so-called syzygies, which are relations between relations of the generators.

Example 7. Building on Example 5, we have seen that the coordinate ring $\mathbb{C}^2[x_1, x_2]/\mathbb{Z}_2$ has 3 generators of degree 2 and 1 relation at degree 4. We already alluded to this being enough information to compute the Hilbert series, and we now make this explicit. We recall that the generators are

$$X = x_1^2, \quad Y = x_2^2, \quad Z = x_1 x_2, \tag{2.32}$$

and the relation is

$$XY = Z^2. \tag{2.33}$$

Then, using the plethystic exponential, the unrefined Hilbert series is

$$\mathrm{HS}(t) = \mathrm{PE}[3t^2 - t^4] = \frac{1-t^4}{(1-t^2)^3}. \tag{2.34}$$

Example 8. Alternatively, if we already have the Hilbert series and merely want to investigate it in more detail, we can apply the plethystic logarithm. In Example 6, we calculated the refined Hilbert series in its fractional form

$$\mathrm{HS}(t,x) = \frac{1-t^4}{(1-t^2x^2)(1-t^2)(1-t^2/x^2)}. \tag{2.35}$$

Hence, the plethystic logarithm of this is

$$\begin{aligned}\mathrm{PL}[\mathrm{HS}(t,x)] &= x^2t^2 + t^2 + x^{-2}t^2 - t^4 \\ &= [2]_{SU(2)}t^2 - [0]_{SU(2)}t^4.\end{aligned} \tag{2.36}$$

This tells us that the $\dim([2]_{SU(2)}) = 3$ generators transform under the adjoint representation of $SU(2)$ and the $\dim([0]_{SU(2)}) = 1$ relation transforms in the trivial representation of $SU(2)$.

2.3 Highest Weight Generating Function

The highest weight generating function (HWG) [13] is a neat way to encapsulate all the information in a refined Hilbert series in a more compact form. We do the following mapping

$$[n_1, ..., n_r]_G \leftrightarrow \prod_{i=1}^{r} \mu_i^{n_i} = \mu_1^{n_1}...\mu_r^{n_r}. \tag{2.37}$$

So, for example, the adjoint representation of G_2 becomes $[0,1] \leftrightarrow \mu_1^0\mu_2^1 = \mu_2$. To calculate a complete HWG, we often compute the first few terms of the refined Hilbert series to establish the global symmetry as well as any character patterns and then make an educated guess for the HWG. We can verify the guess by checking the dimensions of the associated characters with the unrefined Hilbert series, which is usually significantly easier to calculate.

Example 9. We can again consider the affine variety $\mathcal{V} = \mathbb{C}^2/\mathbb{Z}_2$, where we already established the refined Hilbert series to be

$$\mathrm{HS}(t,x) = \sum_{n=0}^{\infty}[2n]_{SU(2)}t^{2n}. \tag{2.38}$$

Now, we can construct the highest weight generating function by mapping $[2n] \leftrightarrow \mu^{2n}$ such that

$$\begin{aligned}\mathrm{HWG}(t,\mu) &= \sum_{n=0}^{\infty}\mu^{2n}t^{2n} \\ &= \frac{1}{1-\mu^2t^2}.\end{aligned} \tag{2.39}$$

One important thing to note is that computing the plethystic logarithm gets rid of the information about the relation

$$\mathrm{PL}[\mathrm{HWG}(t,\mu)] = \mu^2t^2, \tag{2.40}$$

nevertheless, the HWG turns out to be very useful.

Example 10. As a final and complete example, we consider the affine variety $\mathcal{V} = \mathbb{C}^3/\mathbb{Z}_3$. And similarly to what we had in Example 2, the action of $\mathbb{Z}_3$ on $\mathbb{C}^3$ is

$$x_i \leftrightarrow \alpha_n x_i, \tag{2.41}$$

where $\alpha_n^3 = 1$, i.e. it is a 3rd root of unity. And the condition on a monomial $x_1^a x_2^b x_3^c$ becomes

$$a + b + c = 0 \text{ mod } 3. \tag{2.42}$$

Again, we can construct monomials at each degree,

Degree i	Monomials in $\mathcal{R}_i$	$\dim_{\mathbb{C}}(\mathcal{R}_i)$
0	1	1
1	–	0
2	–	0
3	$x_i^3\,(3), x_i^2 x_j\,(2\times 3), x_1x_2x_3$	10
4	–	0
5	–	0
6	$x_i^6\,(3)$, $x_i^5 x_j\,(2\times 3)$, $x_i^4 x_j^2\,(2\times 3)$, $x_i^4 x_j x_k\,(3)$, $x_i^3 x_j^3\,(3)$, $x_i^3 x_j^2 x_k\,(2\times 3)$, $x_1^2 x_2^2 x_3^2$	28
...	...	...

We see that the first few terms of the unrefined Hilbert series are

$$\mathrm{HS}(t) = 1 + 10t^3 + 28t^6 + \dots . \tag{2.43}$$

The factor of 10 in front of the t^2 term also already gives us a hint at what the global symmetry might be. We will continue by putting the monomial in a sum to compute the refined Hilbert series

$$\mathrm{HS}(t_1,t_2,t_3) = \sum_{\substack{a,b,c=0 \\ a+b+c=0 \text{ mod } 3}}^{\infty} t_1^a t_2^b t_3^c. \tag{2.44}$$

By summing around the origin and making the substitutions

$$t_1 = t x_1, \quad t_2 = t x_2/x_1, \quad t/x_2, \tag{2.45}$$

we arrive at the following refined Hilbert series

$$\begin{aligned}\mathrm{HS}(t,x_1,x_2) = {} & 1 \\ & + \left(x_1^3 + \frac{x_1^2}{x_2} + x_1x_2 + \frac{x_1}{x_2^2} + 1 + \frac{1}{x_2^3} + \frac{x_2^2}{x_1} + \frac{1}{x_1x_2} + \frac{x_2}{x_1^2} + \frac{x_2^3}{x_1^3}\right) t^2 \\ & + \mathcal{O}(t^6).\end{aligned} \tag{2.46}$$

The t^2 term can be recognised as the character of the adjoint representation of $SU(3)$ with highest weight $[3,0]$. By checking the dimensions of different representations of $SU(3)$ and comparing them to the unrefined Hilbert series, we can see that the Hilbert series has to be

$$\mathrm{HS}(t,x_1,x_2) = \sum_{n=0}^{\infty} [3n,0]_{SU(3)} t^{3n}. \tag{2.47}$$

Using the Weyl dimension formula for $\mathfrak{su}(3)$, see Appendix A,

$$\dim(V_{[m_1,m_2]}) = \frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2), \tag{2.48}$$

with $m_1 = 3n$ and $m_2 = 0$, we can recompute the unrefined Hilbert series as

$$\begin{aligned}
\mathrm{HS}(t) &= \sum_{n=0}^{\infty} \dim([3n,0]_{SU(3)})t^{3n} \\
&= \sum_{n=0}^{\infty} \frac{(3n+1)(3n+2)}{2} t^{3n} \\
&= \frac{1+7t^3+t^6}{(1-t^3)^3}.
\end{aligned} \tag{2.49}$$

And the plethystic logarithm is

$$\mathrm{PL}[\mathrm{HS}(t)] = 10t^3 - 27t^6 + \mathcal{O}(t^9), \tag{2.50}$$

hence, there are 10 generators at degree 3 (essentially what we found when we constructed the monomials), and they satisfy 27 relations of degree 6.

Lastly, we calculate the highest weight generating function to be

$$\mathrm{HWG}(t,\mu_1,\mu_2) = \sum_{n=0}^{\infty} \mu_1^{3n} t^{3n} = \frac{1}{1-\mu_1^3 t^3}. \tag{2.51}$$

Chapter 3

$3d$ $\mathcal{N} = 4$ Quiver Gauge Theories

As already seen in Chapter 1, there are many intriguing reasons to study supersymmetric theories, in particular $3d$ $\mathcal{N} = 4$ theories. Here, we will establish how exactly these theories can be constructed by dimensional reduction from the more familiar $4d$ $\mathcal{N} = 2$ theories. We also make the connection between the moduli space of a supersymmetric theory and algebraic varieties so that we may apply the methods developed in the previous chapter to our physical theories. In Section 3.1, we summarise the branching rule, roughly following [6]. The discussion of quiver diagrams is partly based on [6, 14, 15, 16]. And lastly, the section on the moduli spaces is inspired by [17, 18, 16].

3.1 From $4d$ $\mathcal{N} = 2$ to $3d$ $\mathcal{N} = 4$

In Appendix B, we discuss supermultiplets and field contents of $4d$ $\mathcal{N} = 1$ and $4d$ $\mathcal{N} = 2$ theories, the latter of which can be used to construct $3d$ $\mathcal{N} = 4$ theories. This is because both $4d$ $\mathcal{N} = 2$ and $3d$ $\mathcal{N} = 4$ theories have 8 supercharges. The $4d$ $\mathcal{N} = 2$ representations can be decomposed to give a $3d$ $\mathcal{N} = 4$ representations, by the following rules:

$$\begin{aligned} [1,1]_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)} &\rightarrow [2]_{\mathfrak{su}(2)} + [0]_{\mathfrak{su}(2)} \\ [1,0]_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)} &\rightarrow [1]_{\mathfrak{su}(2)} \\ [0,1]_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)} &\rightarrow [1]_{\mathfrak{su}(2)} \\ [0,0]_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)} &\rightarrow [0]_{\mathfrak{su}(2)} \end{aligned} \tag{3.1}$$

In $4d$ $\mathcal{N} = 2$, we had:

Hypermultiplet	Vector Multiplet
Complex scalars: φ_1, φ_2	Complex scalar: ϕ
Spinors: ψ_1, ψ_2	Spinors: χ_1, χ_2
	Gauge boson: A_μ $(\mu = 0, 1, 2, 3)$

which now, in $3d$ $\mathcal{N} = 4$, becomes:

Hypermultiplet	Vector Multiplet
Complex scalars: $\varphi_1 + i\varphi_2, \varphi_3 + i\varphi_4$	Complex scalar: $\phi_1 + i\phi_2$
Spinors: ψ_1, ψ_2	Spinors: χ_1, χ_2
	Gauge boson: A_μ $(\mu = 0, 1, 2,)$
	Real scalar: ϕ_3

In particular, we are interested in the real scalar degrees of freedom, which are:

Hypermultiplet	Vector Multiplet
Real scalars: $\varphi_1, \varphi_2, \varphi_3, \varphi_4$	Real scalars: ϕ_1, ϕ_2, ϕ_3
	Dual photon: γ

Where the dual photon γ comes

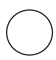
The vector A_μ defines a two-form field strength $F = dA$, which is Hodge-dual to the one-form $\star F$. The one-form can be interpreted as the exterior derivative of a scalar field γ, which we call the dual photon [14, 6]. Effectively, this gives us another scalar degree of freedom[1].

3.2 Quiver Diagrams

We now have finally come to explain the word "quiver". A quiver gauge theory is a supersymmetric theory that can be described using a so-called **quiver diagram**, or **quiver** for short. Under supersymmetry, Lagrangians are heavily restricted, and as such, we can translate them into a graph to which one can apply graph theoretic operations.

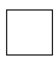
In $3d$ $\mathcal{N} = 4$ we have the following:

- A **round node** representing a gauge group:

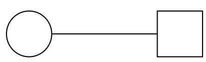
  

  Vector multiplets transform under the adjoint representation of the gauge group.

- A **square node** representing the flavour group:

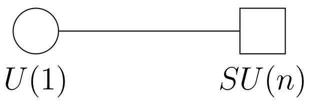
- A **line** connecting two nodes:

[1] This is also the first hint at $3d$ mirror symmetry, where one can interchange vector and hypermultiplets to get another consistent theory. For more, see [19, 20].

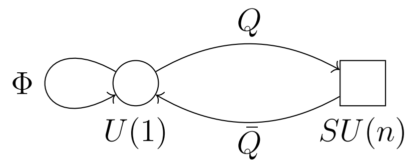
Lines represent the hypermultiplets, which transform under the bifundamental representation of the neighbouring nodes.

For example, consider the following quiver:



Its gauge group is $U(1)$, and its flavour group is $SU(n)$. Also, there are n hypermultiplets transforming in the bifundamental representation of $U(1) \times SU(n)$.

Additionally, we want to write our $3d$ $\mathcal{N} = 4$ quivers in the $3d$ $\mathcal{N} = 2$ formalism as this will give us an easy tool to extract the superpotential. This mapping is always possible. We have that the hypermultiplet decomposes into a chiral and an anti-chiral multiplet, Q and $\tilde{Q}$ respectively, and the vector multiplet decomposes into a vector multiplet and a chiral multiplet, Φ. Applying this to the quiver above leads to the following:



We can easily read off the superpotential by looking at the largest loop:

$$W = \mathrm{Tr}(Q \cdot \Phi \cdot \tilde{Q}). \tag{3.2}$$

3.3 Moduli Spaces

If we have a number of chiral superfields Φ^a and set the Kähler potential to $K(\Phi^a, \bar{\Phi}^a) = \sum_a \bar{\Phi}^a \Phi^a$, then one would find that the scalar potential is given by

$$V(\varphi^a, \bar{\varphi}^a) = \sum_a \left| \frac{\partial W}{\partial \varphi^a} \right|^2 . \tag{3.3}$$

In a supersymmetric ground state $|\Omega\rangle$, the energy must vanish $H|\Omega\rangle = 0$ and hence the scalar potential must vanish too. This implies that

$$\frac{\partial W}{\partial \varphi^a} = 0 \tag{3.4}$$

for all scalar fields φ^a. The holomorphic function $W(\varphi^a)$ is typically a polynomial over the complex numbers $\mathbb{C}$. This means that Eq. (3.4) forms a system of polynomials, and the space of solutions is referred to as the **moduli space** $\mathcal{M}$.

We can consider an example of this. Suppose we have three fields X, Y, Z with the superpotential given by $W(X, Y, Z) = XYZ$, then we have the following

$$\frac{\partial W}{\partial X} = YZ = 0, \qquad \frac{\partial W}{\partial Y} = XZ = 0, \qquad \frac{\partial W}{\partial Z} = XY = 0. \tag{3.5}$$

This system has 3 sets of solutions, which define our moduli space

$$\mathcal{M} = \{X = Y = 0\} \cup \{Y = Z = 0\} \cup \{X = Z = 0\}. \tag{3.6}$$

The structure of our moduli space is three complex planes meeting at the origin, see Figure 3.1. They are each parameterised by the expectation value of the respective non-vanishing field.

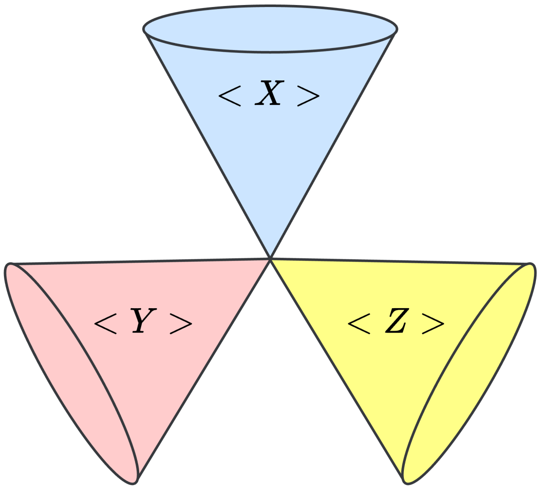


Figure 3.1: Moduli space of superpotential $W(X, Y, Z) = XYZ$. Solutions have the form of three complex planes meeting at the origin.

In general, a moduli space can be viewed as an algebraic variety of the form

$$\mathcal{M} = \frac{\{\text{gauge invariant and holomorphic monomials}\}}{\{\text{algebraic relations}\}}. \tag{3.7}$$

This makes the circle (or ring) complete in making the connection to our earlier discussions in Chapter 2. We can also use all the tools developed in the study of moduli spaces of supersymmetric theories.

For $3d$ $\mathcal{N} = 4$ theories, the moduli space can split into 3 different branches:

- **Higgs Branch** $\mathcal{M}_{\mathcal{H}}$: All scalar **vacuum expectation values**, or VEVs, in the vector multiplet vanish, and the VEVs in the hypermultiplet parameterise the Higgs branch.

- **Coulomb Branch** $\mathcal{M}_{\mathcal{C}}$: Here, the opposite happens, and the VEVs in the vector multiplet parameterise the Coulomb branch. This branch also receives quantum corrections, whereas the Higgs branch does not.

- **Mixed Branch**: There can be mixed branches where some VEVs in the vector multiplet and some VEVs in the hypermultiplet are non-zero. This case can usually be omitted as tools developed for the Higgs or Coulomb branch can be applied.

Both branches are **hyper-Kähler** manifolds, which gives them nice properties that we will later make use of. For more on hyper-Kähler manifolds, see [21].

Chapter 4

Coulomb Branch

4.1 Monopole Formula

The Coulomb branch receives quantum corrections and has historically been difficult to compute. Fortunately, we can now use the so-called **monopole formula** developed in [22]. The monopole formula counts all gauge invariant monopole operators in the chiral ring of the Coulomb branch graded by their charges under the global symmetry of the theory. The monopole operators are charged under the R-symmetry of the theory, and the **R-charge** is given by

$$\Delta(m) = \Delta_V(m) + \Delta_H(m) = -\sum_{\alpha \in \Phi_+} |\alpha(m)| + \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i \in \mathcal{R}_i} |\rho_i(m)|, \tag{4.1}$$

where Φ_+ is the set of positive roots of the gauge group of the theory and $\mathcal{R}_i$ are the weights of the hypermultiplet representations for n hypermultiplets. The first term here is due to the vector multiple, and the second is due to the hypermultiplet. In theories termed "good" and "ugly", as defined in [23], the R-charge coincides with the **conformal dimension**. There are also "bad" theories, but we will not consider them further.

The complete monopole formula is given by

$$\mathrm{HS}_G(t, z) = \sum_{\vec{m} \in \Gamma^*_{\hat{G}}/\mathcal{W}_{\hat{G}}} P_G(t, \vec{m})\, t^{2\Delta(\vec{m})}, \tag{4.2}$$

where we are summing over the magnetic charges $\vec{m}$ in the Weyl chamber of the Langland (GNO) dual group $\hat{G}$ of the gauge group G. Additionally, the factor $P_G(t, m)$ is called the **classical dressing factor** and ensures that we are counting dressed monopole operators rather than bare monopole operators. The classical factor is given by

$$P_G(t, m) = \prod_{i=1}^{r} \frac{1}{1 - t^{2d_i(m)}}, \tag{4.3}$$

where r is the rank of the gauge group and $d_i(m)$ is the degree of the Casimir invariants.

There exists another symmetry, also called **hidden** or **topological symmetry**, which cannot be seen by the fields in the Lagrangian (they are not charged under the symmetry), but monopole operators can carry charge under this symmetry. To keep track of each of the operator's charges, we refine the monopole formula to

$$\mathrm{HS}_G(t,z) = \sum_{\vec{m}\in\Gamma^*_{\hat{G}}/\mathcal{W}_{\hat{G}}} z^{J(\vec{m})}\, P_G(t,\vec{m})\, t^{2\Delta(\vec{m})}, \tag{4.4}$$

where

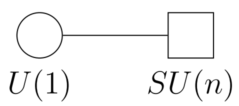
$$J(m) = \sum_{i=1}^{\mathrm{rank}(G)} m_i \tag{4.5}$$

is the topological charge, and z is the fugacity, which keeps track of the charges.

4.2 Examples

4.2.1 $U(1)$ with n flavours

We begin by considering a simple quiver:



Here, we have a $U(1)$ gauge symmetry and a $SU(n)$ flavour symmetry; this theory is also known as SQED.

The magnetic charge of the $U(1)$ gauge symmetry is labelled by a single integer $m \in \mathbb{Z}$. We also note that $U(1)$ does not have a root system since it is abelian. Therefore, the conformal dimension is

$$\begin{aligned}\Delta(m) &= -\sum_{\alpha\in\Phi_+} |\alpha(m)| + \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i\in\mathcal{R}} |\rho_i(m)| \\ &= \frac{1}{2}\sum_{i=1}^{n} |1\cdot m| \\ &= \frac{n}{2}|m|,\end{aligned} \tag{4.6}$$

since the weight of the fundamental representation of $U(1)$ is 1. Lastly, the classical factor is given by

$$P_{U(1)}(t) = \frac{1}{1-t^2}. \tag{4.7}$$

We can now combine the above to calculate the unrefined Hilbert series by using Eq. (4.2),

$$\begin{aligned}\mathrm{HS}_{\mathrm{unref}}(t) &= \frac{1}{1-t^2}\sum_{m\in\mathbb{Z}} t^{n|m|} \\ &= \frac{1-t^{2n}}{(1-t^n)^2(1-t^2)},\end{aligned} \tag{4.8}$$

where, to get to the last line, we have split the sum into positive and negative values of m. Alternatively, one could have computed the refined Hilbert series using Eq. (4.4),

$$\begin{aligned} \mathrm{HS}_{\mathrm{ref}}(t,z) &= \frac{1}{1-t^2}\sum_{m\in\mathbb{Z}} z^m t^{n|m|} \\ &= \frac{1-t^{2n}}{(1-t^n z)(1-t^2)(1-t^n/z)}. \end{aligned} \tag{4.9}$$

If we want to retrieve the unrefined Hilber series, we just need to substitute $z=1$.

From the above, we can get the dimension of the Coulomb branch by rewriting the unrefined Hilbert series in a form where we can immediately see the order of the singularity at $t=1$:

$$\mathrm{HS}_{\mathrm{unref}}(t) = \frac{1+t^n}{(1-t^2)^2(1+t^2+\ldots+t^{n-2})} \tag{4.10}$$

Therefore, we have that $\dim(\mathcal{M}_\mathcal{C}) = 2$. At this point, we may also compute the plethystic logarithm, although it becomes more interesting shortly when we fix n. The plethystic logarithm of the refined Hilbert series is

$$\mathrm{PL}[\mathrm{HS}_{\mathrm{ref}}(t,x)] = t^2 + (z+\frac{1}{z})\, t^n - t^{2n}. \tag{4.11}$$

To balance the quiver (in the sense of [24]), we choose $n=2$ such that the refined Hilbert series becomes

$$\begin{aligned} \mathrm{HS}_{\mathrm{ref}}(t,x) &= \frac{1-t^4}{(1-t^2 z)(1-t^2)(1-t^2/z)} \\ &= 1 + (z+1+\frac{1}{z})\, t^2 + (z^2+z+1+\frac{1}{z}+\frac{1}{z^2})\, t^4 + \mathcal{O}(t^6) \end{aligned} \tag{4.12}$$

where the coefficient of the t^2 term, after the substitution $z=1$, gives us the dimension of the adjoint representation of the global symmetry. After the fugacity mapping $z \to x^2$, we have

$$\begin{aligned} \mathrm{HS}_{\mathrm{ref}}(t,x) &= 1 + (x^2+1+\frac{1}{x^2})\, t^2 + (x^4+x^2+1+\frac{1}{x^2}+\frac{1}{x^4})\, t^4 + \mathcal{O}(t^6) \\ &= 1 + [2]_{SU(2)}\, t^2 + [4]_{SU(2)}\, t^4 + \mathcal{O}(t^6) \\ &= \sum_{n=0}^{\infty} [2n]\, t^{2n}, \end{aligned} \tag{4.13}$$

where we have recognised the characters of $SU(2)$ irreps and labelled them by their highest weight Dynkin label[1]. Applying the same fugacity mapping to the plethystic logarithm, now with $n=2$,

$$\mathrm{PL}[\mathrm{HS}_{\mathrm{ref}}(t,x)] = (x^2+1+\frac{1}{x^2})\, t^2 - t^4 = [2]\, t^2 - [0]\, t^4, \tag{4.14}$$

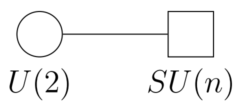
[1] We may note that this is the same Hilbert series as for $\mathbb{C}^2/\mathbb{Z}_2$.

where we again have $SU(2)$ characters. This also tells us, by substituting $x = 1$, that there are 3 generators at degree 2 and 1 relation at degree 4, as well as that the generators transform in the adjoint representation of $SU(2)$ and the relation transforms in the trivial representation of $SU(2)$. Finally, we can also compute the highest weight generating function

$$\mathrm{HS}(t,x) = \sum_{n=0}^{\infty} [2n]\, t^{2n} \quad \rightarrow \quad \mathrm{HWG}(t,\mu) = \sum_{n=0}^{\infty} \mu^{2n} t^{2n} = \frac{1}{1-\mu^2 t^2}. \tag{4.15}$$

4.2.2 $U(2)$ with n flavours

We can extend our previous discussion by considering the gauge group of $U(2)$:



Since $\mathrm{rank}(U(2)) = 2$, we have two magnetic charges, and we can write them as $\vec{m} = (m_1, m_2)$. The group $U(2)$ can be decomposed into $U(1) \times SU(2)$, and by combining their root systems we have that

$$\Phi_{U(2)} = \Phi_{U(1)} \cup \Phi_{SU(2)} = \Phi_{SU(2)} = \{\alpha, -\alpha\}. \tag{4.16}$$

By keeping in mind that the length of the roots is $\sqrt{2}$, we can choose a basis such that $\alpha = (1, -1)$. Then, the vector contribution to the conformal dimension becomes

$$\begin{aligned}
\Delta_{\mathrm{V}}(\vec{m}) &= -\sum_{\alpha \in \Phi_+} |\alpha(\vec{m})| \\
&= -\left| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \right| \\
&= -|m_1 - m_2|.
\end{aligned} \tag{4.17}$$

The weights of the fundamental representation of $U(2)$ are $(1,0)$ and $(0,1)$, and therefore the matter contribution is

$$\begin{aligned}
\Delta_{\mathrm{H}}(\vec{m}) &= \frac{1}{2} \sum_{i=1}^{n} \sum_{\rho_i \in \mathcal{R}} |\rho_i(\vec{m})| \\
&= \frac{n}{2} \left(\left| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \right| + \left| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \right| \right) \\
&= \frac{n}{2} (|m_1| + |m_2|).
\end{aligned} \tag{4.18}$$

The last missing piece is the classical factor, which is given by

$$P_{U(2)}(\vec{m}, t) = \begin{cases} \frac{1}{(1-t^2)(1-t^4)} & \text{if } m_1 = m_2 \\ \frac{1}{(1-t^2)^2} & \text{if } m_1 \neq m_2 \end{cases}. \tag{4.19}$$

We also note that we only need to sum over $m_1 \geq m_2 \in \mathbb{Z}$ since the Weyl group of the GNO of $U(2)$ is the permutation group S_2. Hence,

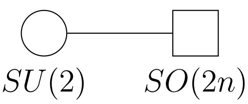
$$\begin{aligned}
\mathrm{HS}(t,z) &= \sum_{m_1 \geq m_2 \in \mathbb{Z}} z^{m_1+m_2} P_{U(2)} t^{2\Delta(\vec{m})} \\
&= \frac{(1-t^{2n})(1-t^{2n-2})}{(1-t^2)(1-t^n z)(1-t^n/z)(1-t^4)(1-t^{n-2}z)(1-t^{n-2}/z)},
\end{aligned} \tag{4.20}$$

and also

$$\mathrm{PL}[\mathrm{HS}(t,z)] = t^2 + (z+\frac{1}{z})t^n + t^4 + (z+\frac{1}{z})t^{n-2} - t^{2n} - t^{2n-2}. \tag{4.21}$$

4.2.3 $SU(2)$ with n flavours

Another simple example is $SU(2)$ with n flavours, with quiver diagram given by:



The rank of $SU(2)$ is 1, and therefore, we have only one magnetic charge m. The positive root of $SU(2)$ is $\alpha = 2$, and therefore, the weight of the fundamental weight is $\rho = 1$. The conformal dimension is then readily computed to be

$$\begin{aligned}
\Delta(m) &= -\sum_{\alpha \in \Phi} |\alpha(m)| + \frac{1}{2}\sum_{i=1}^{n} \sum_{\rho_i \in \mathcal{R}} |\rho_i(m)| \\
&= -|2 \cdot m| + \frac{n}{2}(|1 \cdot m| + |1 \cdot m|) \\
&= -2|m| + n|m|.
\end{aligned} \tag{4.22}$$

The group $SU(2)$ can be broken into $U(1) \times U(1)$ in the inside of the Weyl chamber or will remain $SU(2)$ at the border. Since the Weyl group of $SU(2)$ is $\mathbb{Z}_2$, the symmetry is broken for $m > 0$ and unbroken for $m = 0$, leading to a classical factor of

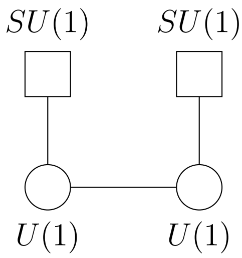
$$P_{SU(2)}(t,m) = \begin{cases} \frac{1}{(1-t^4)} & \text{if } m = 0 \\ \frac{1}{(1-t^2)} & \text{if } m > 0 \end{cases} . \tag{4.23}$$

A simple calculation shows that the refined Hilbert series is

$$\begin{aligned}
\mathrm{HS}(t,z) &= \sum_{m=0}^{\infty} z^m P_{SU(2)} t^{2\Delta(m)} \\
&= \frac{1 - z^2 t^{2n-2}}{(1-t^4)(1-zt^{2n-4})(1-zt^{2n-2})}.
\end{aligned} \tag{4.24}$$

4.2.4 Minimal nilpotent orbit of A_2

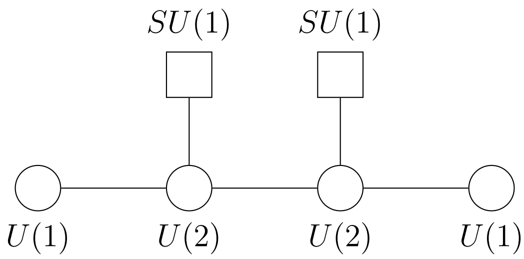
We will now consider a more advanced example. The quiver of the minimal nilpotent orbit of A_2 is given by:



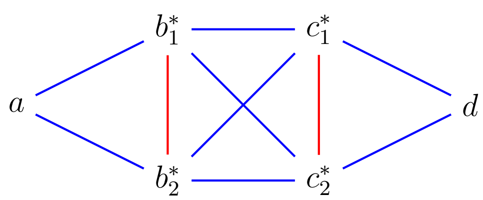
We will now employ a trick to compute the conformal dimension instead of going through the representation theory every time we can generalise our results from earlier. We will turn to an even more complex quiver to illustrate this, but we will not do any further computations of its properties.

Aside: Simplifying the calculation of the conformal dimension

Consider the quiver of the supra minimal nilpotent orbit of A_4:



We know from before that each $U(1)$ is associated with a single magnetic charge, and each $U(2)$ with 2 magnetic charges. We can draw a new diagram that symbolises this:



where we have cross-connected the magnetic charges in blue if a line exists connecting them in the quiver diagram. We also connect charges if they belong to the same gauge node, in which case we use a red line. Additionally, if the gauge node connects to a flavour node, we have marked the corresponding charges with an asterisk. We have the following prescription for constructing the conformal dimension:

- For a blue line connecting two charges a and b, we add $\frac{1}{2}|a-b|$.
- For a charge a^*, i.e. a charge connected to a flavour node $SU(n)$, we add $\frac{n}{2}|a|$.
- For a red line connecting two charges a_1 and a_2, we add $-|a_1-a_2|$.

For the above, we have

$$\Delta(\vec{m}) = \frac{1}{2}(|a-b_1| + |a-b_2| + |b_1| + |b_2| + |b_1-c_1| + |b_1-c_2| + |b_2-c_1| \\ + |b_2-c_2| + |c_1| + |c_2||c_1-d| + |c_2-d|) - |b_1-b_2| - |c_1-c_2|. \quad (4.25)$$

Returning to the quiver at hand, we use the strategy developed above to find that

$$\Delta(\vec{m}) = \frac{1}{2}(|a| + |a-b| + |b|), \tag{4.26}$$

where we have adopted labelling analogous to the example above. For the dressing factor, we have

$$P_{U^2(1)} = \frac{1}{(1-t^2)^2}. \tag{4.27}$$

Calculating the refined Hilbert series now becomes increasingly challenging, and we often fall back on other strategies as opposed to direct calculation. To demonstrate this, we begin by computing the unrefined Hilbert series

$$\begin{aligned} \mathrm{HS}(t) &= \frac{1}{(1-t^2)^2} \sum_{a,b\in\mathbb{Z}} t^{|a|+|a-b|+|b|} \\ &= \frac{1+4t^2+t^4}{(1-t^2)^4} \\ &= 1 + 8t^2 + 27t^4 + \mathcal{O}(t^6). \end{aligned} \tag{4.28}$$

And, as we have seen before, the t^2 term gives the dimension of the adjoint representation of the global symmetry. Hence, 8 is already suggesting $SU(3)$.

If we also compute the first few terms of the refined Hilbert series

$$\begin{aligned} \mathrm{HS}(t, z_1, z_2) &= \frac{1}{(1-t^2)^2} \sum_{a,b\in\mathbb{Z}} z_1^a\, z_2^b\, t^{|a|+|a-b|+|b|} \\ &= 1 + \left(2 + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2 + \frac{1}{z_1 z_2}\right) t^2 + \mathcal{O}(t^4). \end{aligned} \tag{4.29}$$

We can now confirm our suspicions of the global symmetry. We define a Cartan fugacity mapping by

$$z_i \to x_1^{C_{i1}} ... x_n^{C_{in}}, \tag{4.30}$$

where n is the number of z fugacities. By recalling the Cartan matrix for $SU(3)$,

$$C_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{4.31}$$

we have that

$$z_1 \to \frac{x_1^2}{x_2} \quad \text{and} \quad z_2 \to \frac{x_2^2}{x_1}. \tag{4.32}$$

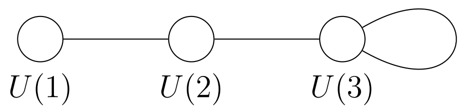
And substituting this back into the refined Hilbert series,

$$\begin{aligned}\mathrm{HS}(t,x_1,x_2) &= 1+\left(2+\frac{x_1^2}{x_2}+\frac{x_2}{x_1^2}+\frac{x_2^2}{x_1}+\frac{x_1}{x_2^2}+x_1x_2+\frac{1}{x_1x_2}\right)t^2+\mathcal{O}(t^4).\\ &= 1+[1,1]\,t^2+[2,2]\,t^4+[3,3]\,t^6+\mathcal{O}(t^8)\\ &= \sum_{n=0}^{\infty}[n,n]\,t^{2n},\end{aligned} \tag{4.33}$$

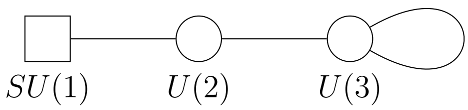
where to get to the second line, we have recognised the characters of $SU(3)$, confirming our initial guess.

4.2.5 Subregular nilpotent orbit of G_2

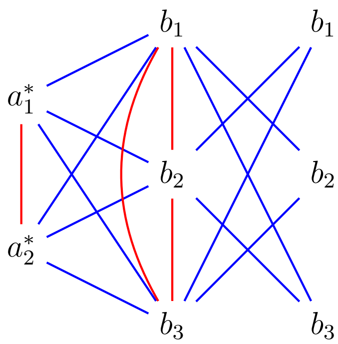
Consider the quiver for the subregular nilpotent orbit of G_2:



This quiver has two new features: it does not have a flavour node, and the $U(3)$ gauge node has an additional loop connecting to itself. The additional loop simply stands for a hypermultiplet transforming in the adjoint representation of $U(3)$. If we were to proceed as before with this new piece of information, the monopole formula would diverge. This is because there is an unaccounted-for $U(1)$ symmetry acting on the Coulomb branch due to the missing flavour node. Luckily, we can easily account for it by ungauging one of the gauge nodes, i.e. turn a $U(k)$ gauge node into a $SU(k)$ flavour node. The new quiver we get from this is:



Using the new quiver diagram, we can draw the magnetic charge diagram:



where $a_{1,2}$ correspond to the $U(2)$ gauge node and $b_{1,2,3}$ to the $U(3)$ node. We also have an additional set of $b_{1,2,3}$, which do not come from another gauge node but simply stand for the self-connection (loop) of the $U(3)$ gauge node. Now we can write down the conformal dimension,

$$\begin{aligned}\Delta(\vec{m}) = \frac{1}{2}(&|a_1 - b_1| + |a_1 - b_2| + |a_1 - b_3| + |a_2 - b_1| + |a_2 - b_2| + |a_2 - b_3| \\ &+ |a_1| + |a_2| + 2|b_1 - b_2| + 2|b_2 - b_3| + 2|b_1 - b_3|) - |a_1 - a_2| \\ &\qquad - |b_1 - b_2| - |b_2 - b_3| - |b_1 - b_3| \\ = \frac{1}{2}(&|a_1 - b_1| + |a_1 - b_2| + |a_1 - b_3| + |a_2 - b_1| + |a_2 - b_2| + |a_2 - b_3| \\ &+ |a_1| + |a_2|) - |a_1 - a_2|.\end{aligned} \tag{4.34}$$

The dressing factors for $U(2)$ is

$$P_{U(2)} = \begin{cases} \frac{1}{(1-t^2)(1-t^4)} & \text{if } a_1 = a_2 \\ \frac{1}{(1-t^2)^2} & \text{if } a_1 < a_2 \end{cases} \tag{4.35}$$

and for $U(3)$,

$$P_{U(3)} = \begin{cases} \frac{1}{(1-t^2)(1-t^4)(1-t^6)} & \text{if } b_1 = b_2 = b_3 \\ \frac{1}{(1-t^2)^2(1-t^4)} & \text{if } b_1 < b_2 = b_3 \text{ or } b_1 = b_2 < b_3 \ . \\ \frac{1}{(1-t^2)^3} & \text{if } b_1 < b_2 < b_3 \end{cases} \tag{4.36}$$

This means that we have 6 different cases that we each need to treat differently, and our complete Hilbert series would be of the form

$$\mathrm{HS}(t) = \sum_{\substack{a_1 \le a_2 \\ \in \mathbb{Z}}} \sum_{\substack{b_1 \le b_2 \le b_3 \\ \in \mathbb{Z}}} P_{U(2)} P_{U(3)} t^{2\Delta(a_1,\ldots,b_3)}. \tag{4.37}$$

One approach would be to find all 6 cases for a general $P_i = P_{U(2)}P_{U(3)}$; in each of the different cases, one can then simplify the conformal dimension. Therefore, one could split the complete Hilbert series into 6 cases, each a little simpler than the initial[2].

Alternatively, we can use the so-called **perturbative approach**, which allows us to compute the exact solution by only summing around the origin instead of over all integers. This also enables us to use Python, or any other programming language for that matter, to compute the sum in a for-loop with if-statements that take care of all the different cases.

We need 2 more pieces of information before we can proceed. Firstly, the dimension of the Coulomb branch can be read off the quiver directly [25] and is given by

$$\dim(\mathcal{M}_{\mathcal{C}}) = 2 \times \sum_i \mathrm{rank}(G_i), \tag{4.38}$$

[2]For an example of such a procedure, see [8].

where G_i is a gauge group of the quiver. In our case, we have that $\dim(\mathcal{M}_\mathcal{C}) = 2(2+3) = 10$. Secondly, we recall from Chapter 2 that the Hilbert series can always be written as

$$\mathrm{HS}(t) = \frac{Q(t)}{(1-t^\alpha)^{\dim(\mathcal{M}_\mathcal{C})}}. \tag{4.39}$$

And, if $\mathcal{M}_\mathcal{C}$ is hyperkähler, then $Q(t)$ is palindromic [26].

So, for our quiver, we know that the Hilbert series will be of the form

$$\mathrm{HS}(t) = \frac{Q(t)}{(1-t^\alpha)^{10}}. \tag{4.40}$$

We can also get the first terms of the Hilbert series by summing around the origin, i.e.

$$\begin{aligned}\mathrm{HS}(t) &= \sum_{\substack{a_1\le a_2\\ \in[-5,5]}} \sum_{\substack{b_1\le b_2\le b_3\\ \in[-5,5]}} P_{U(2)}P_{U(3)}t^{2\Delta(a_1,\ldots,b_3)}\\ &= 1+14t^2+104t^4+539t^6+2184t^8+7378t^{10}+\mathcal{O}(t^{12}).\end{aligned} \tag{4.41}$$

We now need to guess the correct α value, which can be found by trial and error, and here we have that $\alpha = 2$. And the expression for $Q(t)$ is thus

$$\begin{aligned}Q(t) &= (1-t^2)^{10} \times \mathrm{HS}(t)\\ &= 1+4t^2+9t^4+9t^6+4t^8+t^{10}+\mathcal{O}(t^{12}).\end{aligned} \tag{4.42}$$

Because $Q(t)$ has to be palindromic, we can extract the palindrome, which is the exact form of $Q(t)$ and is no longer an approximation. The exact Hilbert series can then be retrieved by simply dividing by $(1-t^2)^{10}$, and therefore the Hilbert series is

$$\begin{aligned}\mathrm{HS}(t) &= \frac{1+4t^2+9t^4+9t^6+4t^8+t^{10}}{(1-t^2)^{10}}\\ &= \frac{(1+t^2)(1+3t^2+6t^4+3t^6+t^8)}{(1-t^2)^{10}}.\end{aligned} \tag{4.43}$$

We can also employ another trick to get the character expansion form of the Hilbert series. Since the t^2 term gives the dimension of the adjoint representation of the global symmetry, we can guess that the global symmetry is G_2[3]. By comparing each of the coefficients in the unrefined Hilbert series with combinations of the dimensions of representations of G_2, as given in Table 4.1, we can guess the refined Hilbert series:

$$\mathrm{HS}_{\mathrm{unref}}(t) = 1+14t^2+104t^4+539t^6+\mathcal{O}(t^8), \tag{4.44}$$

and therefore

$$\begin{aligned}\mathrm{HS}_{\mathrm{ref}}(t,x) = [0,0]+[0,1]t^2+([2,0]+[0,2])t^4&\\ +([3,0]+[1,2]+[0,3])t^6+\mathcal{O}(t^8).&\end{aligned} \tag{4.45}$$

We can also confirm this by computing the refined Hilbert series in the usual way by

$$\mathrm{HS}(t,z) = \sum_{\substack{a_1\le a_2\\ \in[-5,5]}} \sum_{\substack{b_1\le b_2\le b_3\\ \in[-5,5]}} z_1^{a_1+a_2}z_2^{b_1+b_2+b_3}P_{U(2)}P_{U(3)}t^{2\Delta(a_1,\ldots,b_3)}, \tag{4.46}$$

and after a fugacity mapping, using the Cartan matrix of G_2, we get the same characters.

[3]The fact that it is G_2 can also be seen by considering the discrete quotient of the affine quiver of D_4 and $\mathfrak{S}_3$ [27].

Reps. of G_2	$[0,0]$	$[1,0]$	$[0,1]$	$[2,0]$	$[1,1]$	$[0,2]$	$[3,0]$	$[2,1]$	$[1,2]$	$[0,3]$
Dimension	1	7	14	27	64	77	77	182	189	273

Table 4.1: Dimensions of the first few representations of G_2.

Chapter 5

Conclusion

In this report, we studied $3d$ $\mathcal{N} = 4$ quiver gauge theories, mainly focusing on the Coulomb branch of their moduli spaces. We began by introducing the relevant tools that we frequently relied upon, mainly the Hilbert series and the plethystic programme. Then, we specified the type of theories we were interested in and showed how the methods previously developed applied to the moduli space of supersymmetric theories. Afterwards, we introduced the main way of computing the Hilbert series for the Coulomb branch, the monopole formula, and applied it to a variety of quivers. In particular, we started with simple quivers such as for $U(1)$ with n flavours or $SU(2)$ with n flavours applying principles from Lie algebras. Then we looked at more advanced theories, such as the minimal nilpotent orbit of A_2 and the subregular nilpotent orbit of G_2, where we generalised the method for computing the conformal dimension from Lie algebraic considerations to being able to simply read it off the quiver. For the latter, we have also introduced the perturbative approach, which simplifies the calculation of the Hilbert series and nicely utilises the fact that the Coulomb branch is hyper-Kähler.

For possible extensions, something discussed was studying how the subregular nilpotent orbit of G_2 quiver can be constructed from the D_4 affine quiver via a discrete quotient with $\mathfrak{S}_3$ [27]. Additionally, obvious extensions would be to include more about supersymmetry and string theory; in particular, branes and mirror symmetry have been glossed over.

Appendix A

Lie Algebras and Representation Theory

The pivotal role of Lie algebras and representation theory in modern theoretical physics cannot be overstated, particularly in the context of quiver gauge theories. Although existing literature explores these topics, it often diverges from a directed path to what we need it for. This appendix seeks to provide this targeted and comprehensive discussion, starting with the fundamentals of Lie algebras and extending through to Weyl's character formula. In doing so, we take inspiration from various parts of [28, 29, 30].

A.1 Basics

Definition 9. A **Lie algebra** $\mathfrak{g}$ is a vector space over a field F, in our case this usually is $\mathbb{C}$, endowed with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called a Lie bracket such that the following is satisfied:

1. the operation is bilinear,
2. $[x, x] = 0$,
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (**Jacobi identity**),

for all $x, y, z \in \mathfrak{g}$.

From the first condition and the bilinearity, we can derive the following:

$$\begin{aligned} 0 &= [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] \\ &= [x, y] + [y, x], \end{aligned} \tag{A.1}$$

for all $x, y \in \mathfrak{g}$. So we have that $[x, y] = -[y, x]$, and we say the Lie bracket is **skew-symmetric**.

Example 11. The set of $n \times n$ matrices with complex coefficients called $\mathfrak{gl}(n, \mathbb{C})$, is a Lie algebra where the Lie bracket is defined to be the commutator

$$[A, B] \equiv AB - BA. \tag{A.2}$$

Definition 10. An **abelian Lie algebra** $\mathfrak{a}$ satisfies:

$$[x, y] = 0, \tag{A.3}$$

for all $x, y \in \mathfrak{a}$.

Abelian Lie algebras will become important shortly when we want to distinguish between different types of Lie algebras that have certain related properties.

The dimension of a given Lie algebra $\mathfrak{g}$ is the dimension of $\mathfrak{g}$ considered as a vector space. Hence, we can make the following definition:

Definition 11. The **basis** of a Lie algebra $\mathfrak{g}$ of finite (or countably infinite) dimension d is written as

$$\mathcal{B} = \{e_i \,|\, i = 1, 2, ..., d\} .$$

We refer to the elements e_i as the **generators** of the Lie algebra.

Now, we may look at a specific example of the Lie algebra of the group $SL(2, \mathbb{C})$. Since $SL(2, \mathbb{C}) \subset GL(n, \mathbb{C})$, it inherits the commutator as its Lie bracket.

Example 12. Consider the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. We may choose the following basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.4}$$

From these elements, one can compute the commutation relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H. \tag{A.5}$$

It can easily be seen that the Lie bracket is closed as required, and one can explicitly calculate the other two properties (skew-symmetry and Jacobi identity), showing that this is in fact a Lie algebra. Later, it will also become clear why we have named the last element H instead of, for example, Z.

In the following, we will try to adopt the notation that specific elements are denoted by upper-case letters and unspecified elements by lower-case letters.

Due to the bilinearity of the Lie bracket, we can also define the Lie bracket, and therefore the Lie algebra, by

$$[e_i, e_j] = \sum_{k=1}^{d} c_{ij}{}^k e_k \equiv c_{ij}{}^k e_k, \tag{A.6}$$

where we will generally use the Einstein summation convention of summing over repeated indices. The coefficients $c_{ij}{}^k \in \mathbb{C}$ are called the **structure constants** and one can derive the properties in Definition 9 in terms of these structure constants:

1. $c_{ii}{}^{j} = 0$,

2. $c_{ij}{}^{m}\, c_{mk}{}^{n} + c_{jk}{}^{m}\, c_{mi}{}^{n} + c_{ki}{}^{m}\, c_{mj}{}^{n} = 0$.

Additionally, we say that two Lie algebras are isomorphic if they have the same commutation relations up to a change of basis, $e'_i = a_i{}^j\, e_j$ where $a_i{}^j \in \mathbb{C}$, or more formally:

Definition 12. Two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic if there exists and isomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}'$ such that the algebraic structure is preserved, that is

$$[x, y] \mapsto \varphi\left([x, y]\right) = [\varphi(x), \varphi(y)]$$

for all $x, y \in \mathfrak{g}$. We write $\mathfrak{g} \cong \mathfrak{g}'$.

Example 13. The Lie algebra $\mathfrak{su}(2, \mathbb{R})$ has the 3 generators, known as the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.7}$$

They satisfy the commutation relation

$$[\sigma_i, \sigma_j] = 2i\epsilon^{ijk}\sigma_k. \tag{A.8}$$

Note that here the structure constants are $2i\epsilon^{ijk}$. We can now perform a change of basis by letting $\sigma_i = 2iL_i$, and hence

$$[L_i, L_j] = \epsilon^{ijk}L_k, \tag{A.9}$$

which we realise is commutator relation of angular momentum $\mathfrak{so}(3, \mathbb{R})$. Therefore, we have the isomorphism $\mathfrak{su}(2, \mathbb{R}) \cong \mathfrak{so}(3, \mathbb{R})$.

Example 14. Usually we prefer working with the **complexified Lie algebras**, and in particular for $\mathfrak{su}(2, \mathbb{R})$ we have

$$\mathfrak{su}(2, \mathbb{C}) = \mathfrak{su}(2, \mathbb{R}) + i\mathfrak{su}(2, \mathbb{R}) \tag{A.10}$$

which we will just call $\mathfrak{su}(2)$ from now on. Suppose we define three new generators as

$$H = \sigma_3, \quad X = \frac{1}{2}\left(\sigma_1 + i\sigma_2\right), \quad Y = \frac{1}{2}\left(\sigma_1 - i\sigma_2\right), \tag{A.11}$$

or more explicitly,

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{A.12}$$

The commutation relations can now be written as

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H, \tag{A.13}$$

which, of course, is nothing but $\mathfrak{sl}(2, \mathbb{C})$ as defined in Example 12. We can write $\mathfrak{su}(2) \cong \mathfrak{sl}(2, \mathbb{C})$.

A Lie **subalgebra** $\mathfrak{t}$ is a subspace of $\mathfrak{g}$, which itself is a Lie algebra with respect to the Lie bracket of $\mathfrak{g}$. We can also introduce some more notation, which will turn out to be useful

$$[\mathfrak{g}, \mathfrak{t}] \equiv \mathrm{span}_{\mathbb{C}} \{[x, y] \,|\, x \in \mathfrak{g},\ y \in \mathfrak{t}\}\,. \tag{A.14}$$

Now we can write the subalgebra property of a Lie group $\mathfrak{t} \subseteq \mathfrak{g}$ as

$$[\mathfrak{t}, \mathfrak{t}] \subseteq \mathfrak{t}. \tag{A.15}$$

And if $\mathfrak{t}$ is neither $\mathfrak{g}$ or $\{0\}$, then we say that it is a proper subalgebra; otherwise, it is a trivial subalgebra.

Definition 13. An **ideal** $\mathfrak{i} \subseteq \mathfrak{g}$ is a subspace that satisfies

$$[\mathfrak{i}, \mathfrak{g}] \subseteq \mathfrak{i}.$$

Again, the same distinction for proper and trivial subalgebras applies to the ideals.

Definition 14. If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two commuting Lie algebras, i.e. $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$, then the set of elements of both algebras forms a new Lie algebra $\mathfrak{g}$. We use the following notation:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

and say that $\mathfrak{g}$ is the **direct sum** of $\mathfrak{g}_1$ and $\mathfrak{g}_1$.

The Lie subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ in $\mathfrak{g}$ are ideals with respect to $\mathfrak{g}$. This can be seen from the fact that they commute. In a similar manner, we may also define the semidirect sum:

Definition 15. If a Lie algebra $\mathfrak{g}$ has two subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that $[\mathfrak{g}_1, \mathfrak{g}_2] \subseteq \mathfrak{g}_1$. We write

$$\mathfrak{g} = \mathfrak{g}_1 \oplus_s \mathfrak{g}_2.$$

This is called the **semidirect sum**.

Example 15. Consider the Lie algebra $\mathfrak{so}(4)$ with basis elements $X_1, X_2, X_3, Y_1, Y_2, Y_3$ satisfying

$$\begin{gathered}
[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \\
[Y_1, Y_2] = X_3, \quad [Y_2, Y_3] = X_1, \quad [Y_3, Y_1] = X_2, \\
[X_1, Y_1] = 0, \quad [X_2, Y_2] = 0, \quad [X_3, Y_3] = 0, \\
[X_1, Y_2] = Y_3, \quad [X_1, Y_3] = -Y_2, \\
[X_2, Y_1] = -Y_3, \quad [X_x, Y_3] = Y_1, \\
[X_3, Y_1] = Y_2, \quad [X_3, Y_2] = -Y_1.
\end{gathered} \tag{A.16}$$

If we do a change of basis according to

$$J_i = \frac{X_i + Y_i}{2}, \quad K_i = \frac{X_i - Y_i}{2} \quad (i = 1, 2, 3) \tag{A.17}$$

we get the following relations

$$\begin{gathered}
[J_1, J_2] = J_3, \quad [J_2, J_3] = J_1, \quad [J_3, J_1] = J_2, \\
[K_1, K_2] = K_3, \quad [K_2, K_3] = K_1, \quad [K_3, K_1] = K_2, \\
[J_i, K_j] = 0 \quad (i, j = 1, 2, 3).
\end{gathered} \tag{A.18}$$

It can now readily be seen that $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

Definition 16. We may now make a distinction between different types of Lie algebras:

1. A **simple Lie algebra** does not contain any proper ideals and is not abelian.
2. A **semisimple Lie algebra** is the direct product of simple Lie algebras.
3. A **reductive Lie algebra** is the direct product of simple and abelian Lie algebras.

Some common semisimple Lie algebras include:

$$\mathfrak{sl}(n, \mathbb{C}) \quad n \geq 2, \quad \mathfrak{so}(n, \mathbb{C}) \quad n \geq 3, \quad \mathfrak{sp}(n, \mathbb{C}) \quad n \geq 1. \tag{A.19}$$

Some common reductive Lie algebras are:

$$\mathfrak{gl}(n, \mathbb{C}), \quad \mathfrak{so}(2, \mathbb{C}). \tag{A.20}$$

From a given Lie algebra $\mathfrak{g}$, we can construct other other algebras by considering

$$\begin{aligned}
\mathfrak{g}^{(0)} &= \mathfrak{g} \\
\mathfrak{g}^{(1)} &= [\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \\
\mathfrak{g}^{(2)} &= [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \\
&\dots .
\end{aligned} \tag{A.21}$$

These are called the **derived algebras**.

Definition 17. If for some k, $\mathfrak{g}^{(k)}$ in the series of derived algebras has no elements, we call $\mathfrak{g}$ **solvable**.

We can also consider a different series, called the **lower central series**, defined as

$$\begin{aligned}
\mathfrak{g}^{(0)} &= \mathfrak{g} \\
\mathfrak{g}^{(1)} &= [\mathfrak{g}, \mathfrak{g}^{(0)}] \\
\mathfrak{g}^{(2)} &= [\mathfrak{g}, \mathfrak{g}^{(1)}] \\
&\dots .
\end{aligned} \tag{A.22}$$

Definition 18. If for some k, $\mathfrak{g}^{(k)}$ in the lower central series has no elements we call $\mathfrak{g}$ **nilpotent**.

We will end this section with an explicit example that we will frequently make use of in the following sections:

Example 16. Consider the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ with the following basis

$$
\begin{aligned}
H_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad & H_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
X_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad & X_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad & X_3 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
Y_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad & Y_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad & Y_3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{A.23}
$$

And, just by looking at these elements, we can immediately see that we have two $\mathfrak{sl}(2,\mathbb{C})$ subalgebras given by the elements H_1, X_1, Y_1 and H_2, X_2, Y_2. If we compute the commutation relations, we find the following:

$$
\begin{aligned}
&[H_1, H_2] = 0, \\
\\
&[H_1, X_1] = 2X_1, \qquad && [H_1, Y_1] = -2Y_1, \\
&[H_2, X_1] = -X_1, && [H_2, Y_1] = Y_1, \\
\\
&[H_1, X_2] = -X_1, && [H_1, Y_2] = Y_1, \\
&[H_2, X_2] = 2X_1, && [H_2, Y_2] = -2Y_1, \\
\\
&[H_1, X_3] = X_3, && [H_1, Y_3] = -Y_1, \\
&[H_2, X_3] = X_3, && [H_2, Y_3] = -Y_1.
\end{aligned}
\tag{A.24}
$$

The reason for the very suggestive arrangement and naming will become clear shortly, but first, we need to establish some more theory.

A.2 Cartan Subalgebra

The goal of the following section is to build up all the necessary tools we need to classify finite-dimensional semisimple Lie algebras. For these, there exists an entirely canonical basis, the Cartan-Weyl basis, a feature that for more general Lie algebras (particularly solvable Lie algebras) does not exist. We will hence focus our attention on these and, unless stated otherwise, only consider finite-dimensional semisimple Lie algebras. The first step will be to identify the so-called Cartan subalgebra, but before that, we have to define the adjoint map:

Definition 19. Let $\mathfrak{g}$ be a Lie algebra and $x, y \in \mathfrak{g}$, then we define a linear map

$$
\begin{aligned}
\mathrm{ad}_x : \mathfrak{g} &\to \mathfrak{g} \\
y &\mapsto \mathrm{ad}_x(y) \equiv [x, y]
\end{aligned}
$$

this is called the **adjoint map** or **adjoint representation**.

Although this initially seems unnecessary, it turns out that expressions such as $[x, [x, ...[x, y]]]$ are easier to write as $(\mathrm{ad}_x)^n\, y$. Hence, we will occasionally prefer using this notation over the normal Lie bracket.

Definition 20. The **Cartan subalgebra** $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ satisfies the following conditions:

1. For all $H_1, H_2 \in \mathfrak{h}$, $[H_1, H_2] = 0$ (abelian).

2. If $X \in \mathfrak{g}$ satisfies $[H, X] = 0$ for all $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$ (maximal).

3. For all $H \in \mathfrak{h}$, ad_H is diagonalisable.

In principle, a semisimple Lie algebra can have many different Cartan subalgebras, but it can be shown that they are all related by an automorphism. As a consequence of this, all Cartan subalgebras have the same dimension, and thus, for a Lie algebra $\mathfrak{g}$, the dimension of its Cartan subalgebra is an intrinsic property. This is summarised in the following definition:

Definition 21. The **rank** of a Lie algebra $\mathfrak{g}$ is given by the dimension of its a Cartan subalgebra $\mathfrak{h}$, we write

$$r \equiv \mathrm{rank}\,\mathfrak{g} = \dim \mathfrak{h}. \tag{A.25}$$

Example 17. In Example 16, we considered the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ and it is now clear that we labelled certain elements by H to highlight that they belong to the Cartan subalgebra.

A.3 Roots

The motivation behind studying root systems of Lie algebras comes from the following theorem due to Cartan and Killing:

Theorem 3. *Every Lie algebra has an associated root system which determines the Lie algebra (up to isomorphism).*

For a given Lie algebra $\mathfrak{g}$ with dimension d, one can find the so-called Cartan-Weyl basis

$$\mathcal{B} = \{H_1, ..., H_r, E_1, ..., E_{d-r}\}\,, \tag{A.26}$$

where H is an element of the Cartan subalgebra $\mathfrak{h}$. It is possible to find such a basis due to property 3 in definition 20, with the commutation relations being satisfied:

1. $[H_i, H_j] = 0$.

2. $[H_i, E_\alpha] = \alpha(H_i)E_\alpha = \alpha_i E_\alpha$ where $\alpha_i \in \mathbb{C}$.

3. $[E_\alpha, E_\beta] = N_{\alpha\beta}E_{\alpha+\beta}$ if $\alpha + \beta \neq 0$

We note that α is a function of the elements of the Cartan subalgebra, $\alpha : \mathfrak{h} \to \mathbb{C}$, and hence we say that α lives in the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$. We make the following important definition:

Definition 22. An ordered collection $\alpha = (\alpha_1, \alpha_2, ..., \alpha_r)$ is called a **root** if:

1. For all $i = 1, 2, ..., r$, $\alpha_i \neq 0$.

2. There exists E_α with $E_\alpha \neq 0$ such that $[H_i, E_\alpha] = \alpha_i E_\alpha$.

We call E_α the **root vector** corresponding to the root α.

There is some ambiguity in the notation, in the above α_i is defined as the ith component of the root α, but when dealing with many roots it is often convenient to label different roots as α_i. Usually, it should be apparent from the context which version is meant, especially if one is aware that both do exist.

Example 18. In Example 12, we already saw that the commutation relations of $\mathfrak{sl}(2, \mathbb{C})$ and by slightly relabeling the elements we get

$$[H, E_\alpha] = 2E_\alpha, \quad [H, E_\beta] = -2E_\beta, \quad [E_\alpha, E_\beta] = H. \tag{A.27}$$

We can readily identify the roots as

$$\alpha = 2 \quad \text{and} \quad \beta = -2. \tag{A.28}$$

Example 19. A slightly less trivial example is $\mathfrak{sl}(3, \mathbb{C})$, where the commutation relations are as in Example 16. We are now able to appreciate their labelling and arrangement as we can readily read off the roots:

E_α	X_1	X_2	X_3	Y_1	Y_2	Y_3
α	$(2, -1)$	$(-1, 2)$	$(1, 1)$	$(-2, 1)$	$(1, -2)$	$(-1, -1)$

If we consider all elements $x \in \mathfrak{g}$ which satisfy

$$\mathrm{ad}_h(x) = \alpha(h)x, \tag{A.29}$$

where $h \in \mathfrak{h}$ (including those where $\alpha = 0$). Then we can write $\mathfrak{g}$ as the following semidirect sum

$$\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha, \tag{A.30}$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,|\, \mathrm{ad}_h(x) = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$. Or, by separating out the cases where $\alpha \neq 0$ and therefore a root:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \tag{A.31}$$

where $\Phi(\mathfrak{g})$ is the set of all roots.

A.4 The Killing form

Our goal is to define an inner product on the dual space of the Cartan subalgebra $\mathfrak{h}^*$, i.e. for the roots of a Lie algebra. This can be derived from an analogous structure on the normal Cartan subalgebra $\mathfrak{h}$, which itself is obtained from the restriction of an inner product on the whole space of the Lie algebra $\mathfrak{g}$. We define this inner product as follows:

Definition 23. The **Killing form** is map $\kappa(x, y) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined as

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y), \tag{A.32}$$

where tr denotes the trace of a linear map and $\circ$ is the composition of functions.

The following properties follow from the definition:

- The Killing form is bilinear and symmetric. This follows straightforwardly from the properties of the trace.

- It is invariant under $\kappa\left(x, [y, z]\right) = \kappa\left([x, y], z\right)$.

In general, for an arbitrary Lie algebra, the Killing form can be degenerate; there could exist a non-zero $x \in \mathfrak{g}$ such that $\kappa(x, y) = 0$ for all $y \in \mathfrak{g}$. If the Killing form is non-degenerate, there exists a powerful theorem due to Cartan that states:

Theorem 4 (Cartan's Criterion)**.** *A Lie algebra is semisimple if and only if the Killing form is non-degenerate.*

Because of this, the Killing form is, in fact, a proper inner product on the space of a semisimple Lie algebra. We will now develop a valuable representation of the Killing form in terms of the structure constants.

Given a basis $\{e_i \,|\, i = 1, 2, ..., d\}$ of $\mathfrak{g}$, we can write the Killing form in its matrix representation as

$$\kappa_{ij} = \mathrm{tr}\left(\mathrm{ad}_{e_i} \circ \mathrm{ad}_{e_j}\right). \tag{A.33}$$

Now consider $\mathrm{ad}_{e_i} \circ \mathrm{ad}_{e_j}$ acting on a third basis element e_k, we have

$$\begin{aligned} \left(\mathrm{ad}_{e_i} \circ \mathrm{ad}_{e_j}\right)(e_k) &= [e_i, [e_j, e_k]] \\ &= [e_i, {c_{jk}}^l\, e_l] \\ &= {c_{jk}}^l\, {c_{il}}^m\, e_m. \end{aligned} \tag{A.34}$$

And we can see that if we now take the trace of $\mathrm{ad}_{e_i} \circ \mathrm{ad}_{e_j}$ by setting $m = k$ and relabel

$$\kappa_{ij} = {c_{in}}^m\, {c_{jm}}^n. \tag{A.35}$$

This has the nice consequence that we can write Theorem 4 as: A Lie algebra is semisimple if and only if

$$\det\left(\kappa_{ij}\right) \neq 0. \tag{A.36}$$

Or, again, in other words, if there exists an inverse κ^{ij} of the tensor κ_{ij} such that

$$\kappa_{ij}\,\kappa^{jk} = \delta_i^k, \tag{A.37}$$

where δ_i^k is the Kronecker delta.

Example 20. Consider the basis elements of $\mathfrak{sl}(2,\mathbb{C})$ (see Example 12),

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{A.38}$$

Now, we can explicitly compute adjoint maps by first listing all commutation relations:

$$\begin{aligned}
&\mathrm{ad}_H\, H = [H,H] = 0, && \mathrm{ad}_X\, H = [X,H] = -2X, && \mathrm{ad}_Y\, H = [Y,H] = 2Y,\\
&\mathrm{ad}_H\, X = [H,X] = 2X, && \mathrm{ad}_X\, X = [X,X] = 0, && \mathrm{ad}_Y\, X = [Y,X] = -H,\\
&\mathrm{ad}_H\, Y = [H,Y] = -2Y, && \mathrm{ad}_X\, Y = [X,Y] = H, && \mathrm{ad}_Y\, Y = [Y,Y] = 0.
\end{aligned} \tag{A.39}$$

The next step is to collect our elements into a vector $\boldsymbol{\xi} = (H, X, Y)$, and we can identify the adjoint maps as 3×3 matrices mapping $\boldsymbol{\xi}$ to some other arrangement of the basis elements. We have that

$$\mathrm{ad}_H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \mathrm{ad}_X = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}_Y = \begin{pmatrix} 0 & 0 & 2 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.40}$$

This also means that function composition is simple matrix multiplication. And thus, for example,

$$\kappa(H,H) = \mathrm{tr}\left(\mathrm{ad}_H \circ \mathrm{ad}_H\right) = \mathrm{tr}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 8. \tag{A.41}$$

Example 21. An alternative way to compute the Killing form for $\mathfrak{sl}(2,\mathbb{C})$, as compared to Example 20, is using the structure constants. From Eq. A.39 we get that

$$c_{12}{}^2 = 2, \quad c_{13}{}^3 = -2, \quad c_{23}{}^1 = 1, \tag{A.42}$$

where we used the following labelling: $H \sim 1$, $X \sim 2$ and $Y \sim 3$. All other combinations can be derived using the skew-symmetry property or are equal to 0. Now to compute the same quantity as in Example 20,

$$\begin{aligned}
\kappa(H,H) = \kappa_{11} &= c_{1m}{}^n c_{1n}{}^m \\
&= c_{11}{}^1 c_{11}{}^1 + c_{12}{}^1 c_{11}{}^2 + \ldots + c_{13}{}^3 c_{13}{}^3 \\
&= c_{12}{}^2 c_{12}{}^2 + c_{13}{}^2 c_{12}{}^3 + c_{12}{}^3 c_{13}{}^2 + c_{13}{}^3 c_{13}{}^3 \\
&= (2)(2) + (0)(0) + (0)(0) + (-2)(-2) = 8.
\end{aligned} \tag{A.43}$$

Although using the structure constants is arguably the more efficient and straightforward approach, it is useful to get a sense of how both work.

There is a handy trick due to a consequence of Schur's lemma:

Theorem 5. *On a simple Lie algebra, $\mathfrak{g}$ over $\mathbb{C}$, the space of invariant bilinear forms is at most 1-dimensional.*

And since the Killing form is an invariant bilinear form, we have that

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y) = \zeta \mathrm{tr}(xy), \tag{A.44}$$

where $x, y \in \mathfrak{g}$ and ζ is some scalar. If hence suffices to calculate one element using the adjoint maps to fix ζ and the remaining elements can be calculated by simple matrix multiplication of elements in $\mathfrak{g}$.

Example 22. Using the trick we have developed, we can now very easily calculate all elements of κ_{ij} of $\mathfrak{sl}(2, \mathbb{C})$. First of all, we recount that we computed $\kappa_{11} = \kappa(H, H) = 8$ in Example 20. We now consider another invariant and bilinear form, namely $\mathrm{tr}(XY)$ where $X, Y \in \mathfrak{g}$, we get

$$\begin{aligned} \mathrm{tr}(HH) = 2, \quad \mathrm{tr}(XY) = 1, \\ \mathrm{tr}(HX) = \mathrm{tr}(HY) = \mathrm{tr}(XX) = \mathrm{tr}(YY) = 0. \end{aligned} \tag{A.45}$$

Comparing $\kappa(H, H) = 8$ and $\mathrm{tr}(HH) = 2$, we see that $\zeta = 4$, and therefore

$$\kappa_{ij} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}. \tag{A.46}$$

Example 23. Consider the commutation relations for $\mathfrak{sl}(3, \mathbb{C})$ given in Example 16. Instead of calculating all elements of κ_{ij}, an 8×8 matrix, we can restrict ourselves to only Cartan subalgebra elements. So we want to compute a 2×2 matrix κ_{ij} where $I, j = 1, 2$ corresponds to the elements H_1 and H_2. But before that, we need to do the somewhat cumbersome calculation of an element of the whole 8×8 matrix. Consider

$$\begin{aligned} \kappa_{11} &= c_{1m}{}^{n} c_{1n}{}^{m} \\ &= c_{11}{}^{n} c_{1n}{}^{1} + c_{12}{}^{n} c_{1n}{}^{2} + c_{13}{}^{n} c_{1n}{}^{3} + c_{14}{}^{n} c_{1n}{}^{4} \\ &\quad + c_{15}{}^{n} c_{1n}{}^{5} + c_{16}{}^{n} c_{1n}{}^{6} + c_{17}{}^{n} c_{1n}{}^{7} + c_{18}{}^{n} c_{1n}{}^{8}, \end{aligned} \tag{A.47}$$

now, the first term is zero as we have a repeated index, and the term $c_{14}{}^{n}$ is also zero since it corresponds to the commutator of two Cartan subalgebra elements. If we focus on the term $c_{im}{}^{n}$ of the $c_{im}{}^{n} c_{jn}{}^{m}$ pairs where i and m are fixed, we can read off what element appears for their commutator; this fixes n:

$$\begin{aligned} \kappa_{11} &= 0 + c_{12}{}^{2} c_{12}{}^{2} + c_{13}{}^{3} c_{13}{}^{3} + 0 + c_{15}{}^{5} c_{15}{}^{5} + c_{16}{}^{6} c_{16}{}^{6} + c_{17}{}^{7} c_{17}{}^{7} + c_{18}{}^{8} c_{18}{}^{8} \\ &= (2)(2) + (-2)(-2) + (1)(1) + (1)(1) + (1)(1) + (-1)(-1) \\ &= 12. \end{aligned} \tag{A.48}$$

Now we can use the simpler form of $\mathrm{tr}(XY)$, we have that

$$\mathrm{tr}(H_1 H_1) = 2 \quad \text{and} \quad \mathrm{tr}(H_1 H_2) = -1. \tag{A.49}$$

All that remains is to fix the scalar multiple $\zeta = 6$, and hence the restricted matrix is

$$\kappa_{ij} = \begin{pmatrix} 12 & -6 \\ -6 & 12 \end{pmatrix}, \tag{A.50}$$

for $i, j = 1, 2$.

A.5 Root Systems

Before we are able to define an inner product on $\mathfrak{h}^*$, we first need to look at some of the different types of roots that we can have.

Definition 24. We make the following distinctions between roots in Φ:

1. There exists a hyper-plane which divides root space into two disjoint half-spaces $V_\pm$. If $\alpha \in V_+$ we say that $\alpha > 0$ and define the **positive roots** as
$$\Phi_+ = \{\alpha \in \Phi \,|\, \alpha > 0\}. \tag{A.51}$$
If $\alpha \in V_-$ we say that $\alpha < 0$ and define the **negative roots** as
$$\Phi_- = \{\alpha \in \Phi \,|\, \alpha < 0\}. \tag{A.52}$$

2. The **simple** or **fundamental roots** Π are defined as the positive roots that are linearly independent. There are always $\mathrm{rank}(\mathfrak{g})$ fundamental roots.

Example 24. We will now introduce the first root diagram. Consider the roots for $\mathfrak{sl}(3,\mathbb{C})$ as in Example 19, we draw the root diagram as in Figure A.1. Where

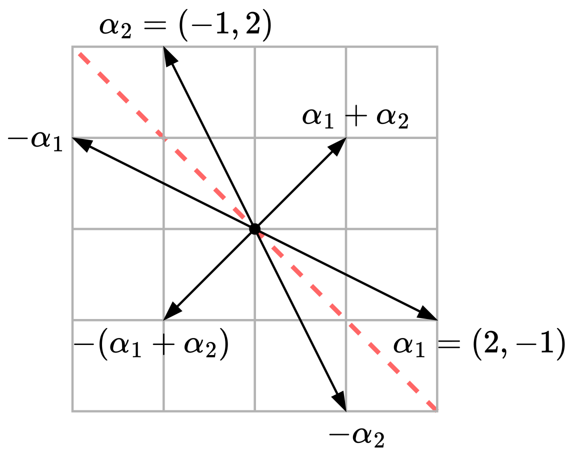


Figure A.1: The root diagram for $\mathfrak{sl}(3,\mathbb{C})$ in the eigenvalue basis. The red line splits the root space into positive and negative roots. Then one can identify $\alpha_1 = (2,-1)$ and $\alpha_2 = (-1,2)$ as the fundamental roots.

we have identified $\alpha_1 = (2,-1)$ and $\alpha_2 = (-1,2)$ as the fundamental roots, since $(1,1) = \alpha_1 + \alpha_2$. Another nice feature is that the fundamental roots are always the ones closest to the hyperplane. The last thing to note is that this diagram was drawn in the so-called eigenvalue basis, but as it turns out, there is another basis that exploits some additional symmetries.

The importance of the fundamental roots is that they form a basis of the dual Cartan subalgebra
$$\mathfrak{h}^* = \mathrm{span}_{\mathbb{C}}(\Pi), \tag{A.53}$$

and additionally, we can consider the restriction to real coefficients only

$$\mathfrak{h}^*_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}(\Pi). \tag{A.54}$$

The reason for introducing the latter will become clear shortly. We may summarise all the relevant spaces in the following way:

$$\Pi \subset \Phi \subset \mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{h}^* \tag{A.55}$$

It turns out that if we restrict the Killing form $\kappa : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ to the Cartan subalgebra $\kappa|_{\mathfrak{h}} : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ it in fact stays non-degenerate. There also exists an isomorphism between $\mathfrak{h}^*$ and $\mathfrak{h}$, which becomes clear from noting that

$$\kappa(H_\alpha, \cdot) : \mathfrak{h} \to \mathbb{C} \quad \text{and} \quad \alpha : \mathfrak{h} \to \mathbb{C}, \tag{A.56}$$

where $H_\alpha \in \mathfrak{h}$ is some fixed element that can be uniquely identified for every $\alpha \in \mathfrak{h}^*$.

Definition 25. The **dual Killing form** on the dual Cartan subalgebra $\kappa^* : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ is given by

$$(\alpha, \beta) = \kappa^*(\alpha, \beta) = \kappa(H_\alpha, H_\beta), \tag{A.57}$$

where $\alpha, \beta \in \mathfrak{h}^*$ and $H_\alpha, H_\beta \in \mathfrak{h}$ are their respective elements as defined above. To simplify the notation, we have also dropped the κ^*.

This rather abstract definition becomes a lot easier to understand if we consider the Killing form for some specified basis and, therefore, can write it as κ_{ij}. The dual Killing form then is nothing but κ^{ij} such that

$$\kappa_{ij}\kappa^{jk} = \delta^k_i. \tag{A.58}$$

Theorem 6. *If we restrict the dual Killing form to $\mathfrak{h}^*_{\mathbb{R}}$, then*

$$\kappa^* : \mathfrak{h}^*_{\mathbb{R}} \times \mathfrak{h}^*_{\mathbb{R}} \to \mathbb{R} \tag{A.59}$$

and

$$(\alpha, \alpha) \geq 0 \quad \textit{and} \quad (\alpha, \alpha) = 0 \Leftrightarrow \alpha = 0. \tag{A.60}$$

We now have a real, positive-definite inner product on $\mathfrak{h}^*_{\mathbb{R}}$ and therefore also on Φ. This allows us to calculate the lengths of roots and the angles between them. But before we look at that, we first introduce a symmetry of the roots:

Definition 26. For any $\alpha \in \Phi$, we define the **Weyl transformation** as

$$\begin{aligned} s_\alpha &: \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}} \\ s_\alpha(\beta) &\equiv \beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}\,\alpha. \end{aligned} \tag{A.61}$$

The corresponding **Weyl group** is defined as

$$\mathcal{W} = \{s_\alpha \,|\, \alpha \in \Phi\}. \tag{A.62}$$

The reason why the Weyl group is so important is that it allows us to reconstruct Φ from the fundamental roots Π. This is captured in the following:

Theorem 7. *1. For all $w \in \mathcal{W}$, there exist $\pi_1, ..., \pi_n \in \Pi$ such that*

$$w = s_{\pi_1} \circ ... \circ s_{\pi_n}, \tag{A.63}$$

where $n \leq rank(\mathfrak{g})$.

2. For all $\alpha \in \Phi$, there exist some $w \in \mathcal{W}$ and $\pi \in \Pi$ such that $\alpha = w(\pi)$.

3. For all $w \in \mathcal{W}$ and $\alpha \in \Phi$, $w(\alpha) \in \Phi$.

Also, even though the Weyl transformation is defined generally for any element in $\mathfrak{h}^*_{\mathbb{R}}$, if we only consider roots in Φ, then we will stay within Φ. Or more formally, for any $\alpha, \beta \in \Pi$

$$s_\alpha(\beta) = \beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}\alpha \in \Phi. \tag{A.64}$$

A neat corollary of this is that if $\alpha \in \Phi$ then we also have that $-\alpha \in \Phi$, this follows simply from considering

$$s_\alpha(\alpha) = \alpha - 2\,\frac{(\alpha, \alpha)}{(\alpha, \alpha)}\,\alpha = -\alpha. \tag{A.65}$$

Furthermore, since any root $\alpha \in \Phi$ can be written in the following way

$$\alpha = \pm \sum_{i=1}^{r} n_i \pi_i, \tag{A.66}$$

where $n \in \mathbb{N}$, the same has to be true for $s_{\pi_i}(\pi_j)$ and therefore this implies that for $i \neq j$

$$-2\frac{(\pi_i, \pi_j)}{(\pi_i, \pi_i)} \in \mathbb{N}. \tag{A.67}$$

Definition 27. The **Cartan matrix** is defined to be the $r \times r$ matrix with entries

$$C_{ij} = 2\frac{(\pi_i, \pi_j)}{(\pi_i, \pi_i)}, \tag{A.68}$$

where $\pi_i, \pi_j \in \Pi$.

From the above, we can see that $C_{ij} \in \{0, -1, -2, ...\}$ if $i \neq j$ and $C_{ii} = 2$. We can also consider the **bond number**

$$n_{ij} = C_{ij}C_{ji} = 4\,\frac{(\pi_i, \pi_j)}{(\pi_i, \pi_i)}\,\frac{(\pi_j, \pi_i)}{(\pi_j, \pi_j)}. \tag{A.69}$$

This can be rewritten as

$$n_{ij} = 4\,\frac{(\pi_i, \pi_j)}{|\pi_i|\,|\pi_j|}\,\frac{(\pi_j, \pi_i)}{|\pi_i|\,|\pi_j|}, \tag{A.70}$$

and we may recognise this as

$$n_{ij} = 4\cos^2(\varphi_{ij}) \tag{A.71}$$

with φ_{ij} being the angle between π_i and π_j. It follows that $0 \leq n_{ij} < 4$ for $i \neq j$, and because the Cartan matrices themselves are restricted to certain values, we find the following possible values:

C_{ij}	C_{ji}	n_{ij}	φ_{ij}
0	0	0	$\pi/2$
-1	-1	1	$\pi/3$ or $2\pi/3$
-1	-2	2	$\pi/4$ or $3\pi/4$
-2	-1		
-1	-3	3	$\pi/6$ or $5\pi/6$
-3	-1		

And finally, if we consider

$$\frac{C_{ji}}{C_{ij}} = \frac{(\pi_i, \pi_i)}{(\pi_j, \pi_j)}, \tag{A.72}$$

where we will assume that $(\pi_i, \pi_i) \geq (\pi_j, \pi_j)$, then one of the following has to hold:

1. $\varphi_{ij} = \pi/2$, and $(\pi_i, \pi_i), (\pi_j, \pi_j)$ are not related.

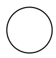
2. $\varphi_{ij} = \pi/3$ or $2\pi/3$, and $(\pi_i, \pi_i) = (\pi_j, \pi_j)$.

3. $\varphi_{ij} = \pi/4$ or $3\pi/4$, and $(\pi_i, \pi_i) = 2(\pi_j, \pi_j)$.

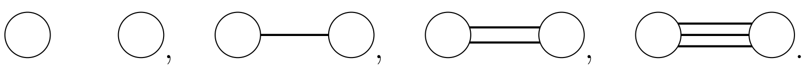
4. $\varphi_{ij} = \pi/6$ or $5\pi/6$, and $(\pi_i, \pi_i) = 3(\pi_j, \pi_j)$.

A.6 Dynkin Diagrams and Root Diagrams

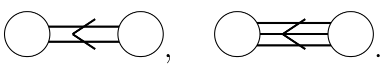
We have now built all the necessary tools to fully classify all semisimple Lie algebras. We start with listing the rules needed to create a so-called **Dynkin diagram**:

1. For every fundamental root $\pi \in \Pi$, we will draw a node: 

2. If $\pi_i, \pi_j \in \Pi$, then draw n_{ij} lines between them:



3. If $n_{ij} \geq 2$, then according to our conditions above, one root is larger than the other. We draw a $>$ sign onto the connecting lines to indicate which one is larger:



Theorem 8 (Cartan, Killing)**.** *Any finite-dimensional simple Lie algebra over* $\mathbb{C}$ *is entirely defined by its set of fundamental roots* Π*, and they only come in one of the forms shown in Figure A.2 and Figure A.3.*

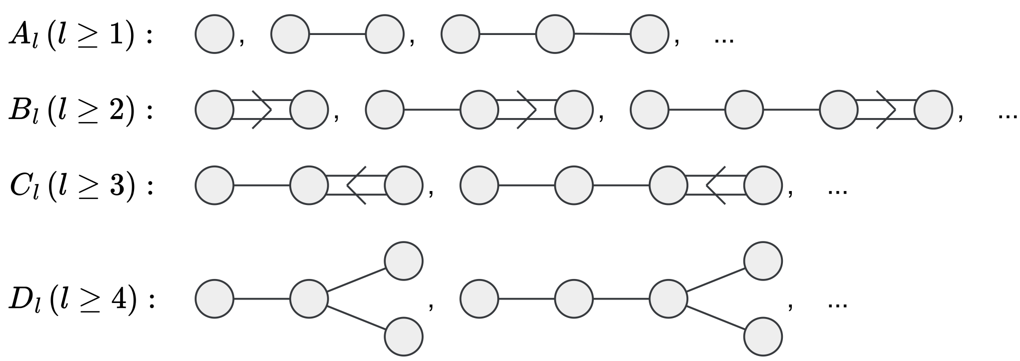


Figure A.2: The four classical Lie algebra sequences.

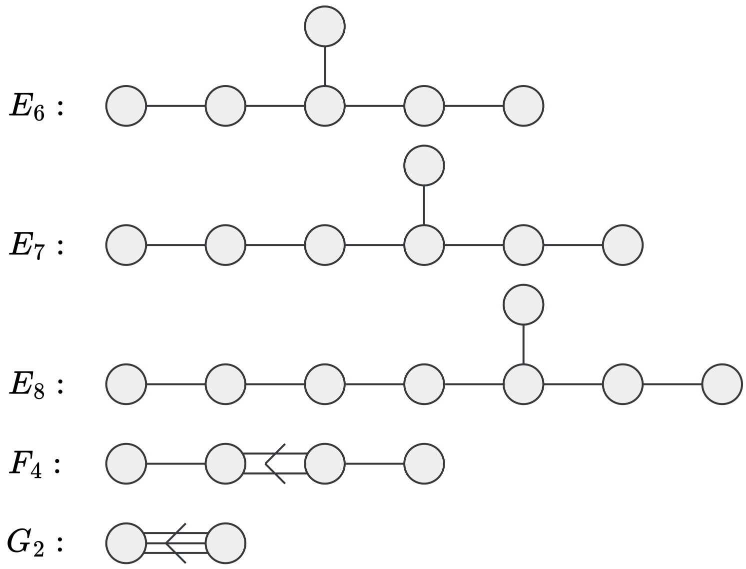
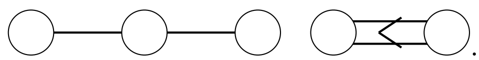


Figure A.3: The five exceptional Lie algebras.

These are all simple Lie algebras that exist, and by using disconnected graphs, we can create direct sums of simple Lie algebra, i.e. semisimple Lie algebras. Consider, for example,



We can identify this as $A_3 \oplus B_2$. We also have the following isomorphisms:

$$A_l \cong \mathfrak{sl}(l+1,\mathbb{C}), \quad B_l \cong \mathfrak{so}(2l+1,\mathbb{C}), \quad C_l \cong \mathfrak{sp}(2l,\mathbb{C}), \quad D_l \cong \mathfrak{so}(2l,\mathbb{C}). \tag{A.73}$$

Root diagrams are depictions as in Example 24 showing all the roots in Φ in some basis. We will continue by drawing all examples of rank 1 and 2, i.e. those that live in 2 dimensions.

Example 25 (Rank 1)**.** Here there exists one fundamental root, say α, then due to the Weyl symmetry $-\alpha$ is also in Φ and there are no other roots possible. This is the Lie algebra A_1 and can be drawn as in Figure A.4.

Figure A.4: The rank 1 root system of A_1.

Example 26 (Rank 2)**.** We now have four unique root systems that are allowed by the rules that we have established. They are called $A_1 \times A_1$, A_2, B_2 and G_2, and are as shown in Figure A.5. All other rank 2 root systems are isomorphic to one of these.

- $A_1 \times A_1$ corresponds to the case where $\varphi_{ij} = \pi/2$ and the lengths of the roots are unrelated.

- A_2 corresponds to the case where $\varphi_{ij} = \pi/3$ and all roots have the same length.

- For B_2 the lengths differ by a factor of $\sqrt{2}$ and the angle between roots is $\varphi_{ij} = \pi/4$.

- Finally, for G_2 the lengths differ by a factor of $\sqrt{3}$ and the angles are $\varphi_{ij} = \pi/6$.

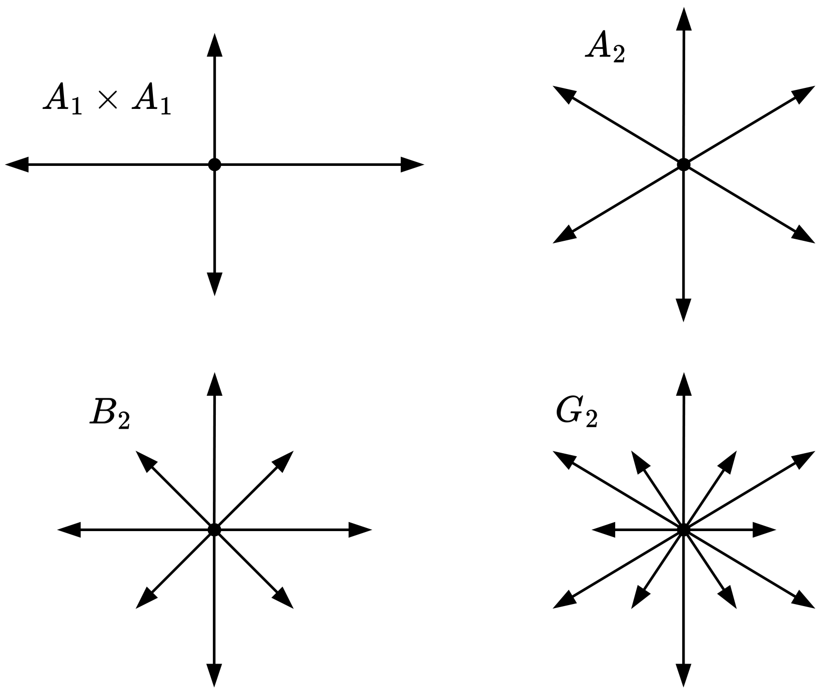
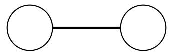


Figure A.5: The four rank 2 root systems.

We will end this section by constructing the complete Lie algebra A_2 from only its Dynkin diagram.

Example 27. Consider the Dynkin diagram of A_2:



It immediately follows that we have two fundamental roots $\Pi = \{\pi_1, \pi_2\}$ and that the bond number between the fundamental roots is $n_{12} = C_{12}C_{21} = 1$. Hence they are of the same length and the angle between the them is $\varphi_{12} = \pi/3$ or $2\pi/3$, and if we look at the restriction on C_{ij} we can determine which of the two angles we actually have. Note that

$$C_{12} = 2\,\frac{(\pi_1, \pi_2)}{(\pi_1, \pi_1)} < 0 \tag{A.74}$$

and since $(\pi_1, \pi_1) > 0$, this implies that $(\pi_1, \pi_2) < 0$. So the projection of π_1 onto π_2 is negative, and therefore, the angle between them is $\varphi_{12} = 2\pi/3$, this is shown by the red arrows in Figure A.6 and this holds true for all fundamental roots in other Lie algebras. Now, by repeated action of the Weyl group, we can determine all other roots in Φ; in particular, we find

$$s_{\pi_1}(\pi_2) = \pi_2 - 2\,\frac{(\pi_1, \pi_2)}{(\pi_1, \pi_1)}\,\pi_1 = \pi_1 + \pi_2. \tag{A.75}$$

This is shown by the dotted line, again, in Figure A.6. Considering $s_{\pi_1}(\pi_1)$, $s_{\pi_2}(\pi_2)$ and $s_{\pi_1+\pi_2}(\pi_1 + \pi_2)$, we of course get the negative version of each root. All other Weyl group elements only permute between these roots. We have hence created the complete root diagram, and now we can derive the Lie algebra relations.

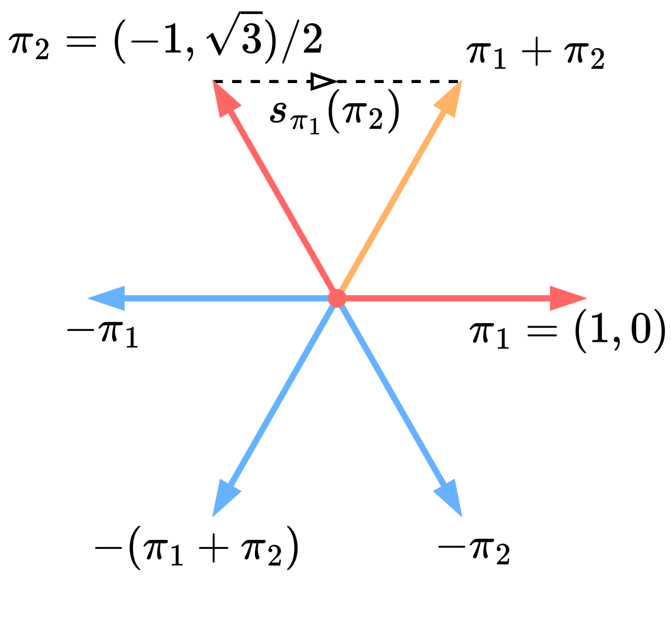


Figure A.6: The A_2 diagram construction.

Since we have two fundamental roots, that means that the Cartan subalgebra has dimension two, and because we have 6 roots, there are 6 non-Cartan subalgebra basis elements in $\mathfrak{g}$. We can write

$$\dim(\mathfrak{g}) = |\Pi| + |\Phi|, \tag{A.76}$$

where $|\Pi| = \dim(\mathfrak{h})$. We can associate each root with a corresponding root vector E_{π_1}, which we will label by its associated root. Also, we take the Cartan subalgebra as $\mathfrak{h} = \{H_1, H_2\}$ with the first element H_1 corresponding to the first element of a root. We can essentially read off the following commutation relations:

$$
\begin{aligned}
&[H_1, E_{\pi_1}] = E_{\pi_1}, && [H_1, E_{-\pi_1}] = -E_{-\pi_1}, \\
&[H_2, E_{\pi_1}] = 0, && [H_2, E_{-\pi_1}] = 0, \\
\\
&[H_1, E_{\pi_2}] = -E_{\pi_2}/2, && [H_1, E_{-\pi_2}] = E_{-\pi_2}/2, \\
&[H_2, E_{\pi_2}] = \sqrt{3}E_{\pi_2}/2, && [H_2, E_{-\pi_2}] = -\sqrt{3}E_{-\pi_2}/2, \\
\\
&[H_1, E_{\pi_1+\pi_2}] = E_{\pi_1+\pi_2}/2, && [H_1, E_{-(\pi_1+\pi_2)}] = -E_{\pi_1+\pi_2}/2, \\
&[H_2, E_{\pi_1+\pi_2}] = \sqrt{3}E_{\pi_1+\pi_2}/2, && [H_2, E_{-(\pi_1+\pi_2)}] = -\sqrt{3}E_{\pi_1+\pi_2}/2.
\end{aligned}
\tag{A.77}
$$

And also trivially have that $[H_1, H_2] = 0$. The last thing we need to consider is the Lie bracket $[E_\alpha, E_\beta]$ for some $\alpha, \beta \in \Phi$. Consider the Jacobi identity for E_α, E_β and some $h \in \mathfrak{h}$,

$$
\begin{aligned}
[h, [E_\alpha, E_\beta]] &= -[E_\beta, [h, E_\alpha]] - [E_\alpha, [E_\beta, h]] \\
&= -\alpha(h)[E_\beta, E_\alpha] + \beta(h)[E_\alpha, E_\beta] \\
&= (\alpha + \beta)(h)[E_\alpha, E_\beta].
\end{aligned}
\tag{A.78}
$$

This means that if $(\alpha + \beta) \in \Phi$, then $[E_\alpha, E_\beta] = E_\gamma$. In general, we have

$$
[E_\alpha, E_\beta] \begin{cases} = E_\gamma, & \text{if } \alpha + \beta \in \Phi \\ = 0, & \text{if } \alpha + \beta \notin \Phi \text{ and } \alpha + \beta \neq 0 \\ \in \mathfrak{h}, & \text{if } \alpha + \beta = 0 \end{cases} .
\tag{A.79}
$$

For example, this means that

$$
[E_{\pi_1}, E_{\pi_2}] = E_{\pi_1+\pi_2}, \quad [E_{\pi_1}, E_{\pi_1+\pi_2}] = 0, \quad .... \tag{A.80}
$$

We have now found all Lie algebra relations for A_2, and hence, the Lie algebra is fully defined by its Dynkin diagram.

A.7 Weights

Definition 28. Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be a representation of a Lie algebra $\mathfrak{g}$ on a vector space V and let $\lambda : \mathfrak{h} \to \mathbb{C}$ be a linear functional. λ is called a **weight** of ρ if

$$
\rho(H)v = \lambda(H)v, \tag{A.81}
$$

where $v \neq 0$ and $H \in \mathfrak{h}$.

Definition 29. The **weight space** of V corresponding to a weight λ is the subspace V_λ defined by

$$
V_\lambda = \{v \in V \mid \rho(H)v = \lambda(H)v \text{ for all } H \in \mathfrak{h}\}. \tag{A.82}
$$

In the special case that ρ is the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, we recognise that the weights are nothing more than the roots (and 0 which we excluded for roots),

$$\mathrm{ad}_H(E) = [H, E] = \alpha(H)E. \tag{A.83}$$

This leads to the important result that for a representation ρ of $\mathfrak{g}$ with a weight vector v corresponding to a weight λ and a root vector E_α corresponding to a root α:

$$\begin{aligned} \rho(H)\rho(E)v &= \rho(HE_\alpha)v = \rho(E_\alpha H + \alpha(H)E_\alpha)v \\ &= \rho(E_\alpha)\rho(H)v + \alpha(H)\rho(E_\alpha)v \\ &= (\lambda + \alpha)(H)\rho(E_\alpha)v. \end{aligned} \tag{A.84}$$

This means that either $\rho(E_\alpha)v = 0$ or that $\rho(E_\alpha)v$ is a new weight vector with weight $\lambda + \alpha$.

Definition 30. The **coroot** $\alpha^\vee$ to a root α is defined as

$$\alpha^\vee = 2\,\frac{\alpha}{(\alpha, \alpha)}. \tag{A.85}$$

One may notice the similarity to the Cartan matrix, and in fact, this definition allows us to simplify the definition of the Cartan matrix to read

$$C_{ij} = (\pi_i^\vee, \pi_j), \tag{A.86}$$

where $\pi_i, \pi_j \in \Pi$.

If we now compute an analogous quantity:

Definition 31. An element λ is an **integral element** if for all roots $\alpha \in \Phi$ the quantity

$$(\alpha^\vee, \lambda) = 2\frac{(\alpha, \lambda)}{(\alpha, \alpha)} \in \mathbb{Z}. \tag{A.87}$$

Definition 32. An element λ is **dominant** (relative to a set of positive roots Φ_+) if

$$(\alpha, \lambda) \geq 0, \tag{A.88}$$

for all $\alpha \in \Phi_+$.

The basis of weight space is the dual to the basis $\mathcal{B}$ of root space. The elements of $\mathcal{B}^*$ are called the fundamental weights and are defined as follows:

Definition 33. The **fundamental weights** $\omega_1, ..., \omega_r$ form a basis of weight space (which is dual to the basis of root space) according to

$$\omega_i(\pi_j^\vee) = 2\frac{(\omega_i, \pi_j)}{(\pi_j, \pi_j)} = \delta_{ij}, \tag{A.89}$$

where $\pi_j \in \Pi$.

We are now able to relate the fundamental weights and roots to the Cartan matrix. We find that

$$\pi_i = \omega_j\, C_{ji} \quad \text{and} \quad \omega_i = \pi_j\, (C^{-1})_{ji}. \tag{A.90}$$

Additionally, we can write any dominant integral weight λ as

$$\lambda = \sum_i n_i \omega_i, \tag{A.91}$$

where $n_i \in \mathbb{N}$.

Example 28. For A_2, we can either solve the simultaneous equations we get from Eq. A.89 to get the fundamental weights or calculate them via the Cartan matrix. We choose the latter. The Cartan matrix and its inverse are

$$C_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad (C^{-1})_{ij} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{A.92}$$

Therefore, using the fundamental roots calculated in Example 27,

$$\begin{aligned} \omega_1 &= \pi_1\, (C^{-1})_{11} + \pi_2\, (C^{-1})_{21} \\ &= \frac{1}{2}\, (1, 1/\sqrt{3}), \\ \omega_2 &= \pi_1\, (C^{-1})_{12} + \pi_2\, (C^{-1})_{22} \\ &= (1, 1/\sqrt{3}). \end{aligned} \tag{A.93}$$

If we draw this in a root diagram and highlight the dominant integral weights, we get Figure A.7.

To develop the notion of a highest weight, we need to introduce the concept of **partial ordering** of weights.

Definition 34. Consider two elements λ and μ, we say that λ is larger than μ (relative to a set of positive roots Φ_+) if $\lambda - \mu$ can be written as

$$\lambda - \mu = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_r\alpha_r, \tag{A.94}$$

where $\alpha_i \in \Phi_+$ and c_i are non-negative real numbers. We denote this by $\lambda \succeq \mu$.

From this, it follows that if some λ is dominant, then $\lambda \succeq 0$.

Theorem 9.
1. *Every irreducible, finite-dimensional representation of $\mathfrak{g}$ has a highest weight.*
2. *The highest weight is always a dominant integral element.*
3. *Two irreducible representations with the same highest weight are isomorphic.*
4. *Every dominant integral element corresponds to the highest weight of an irreducible representation.*

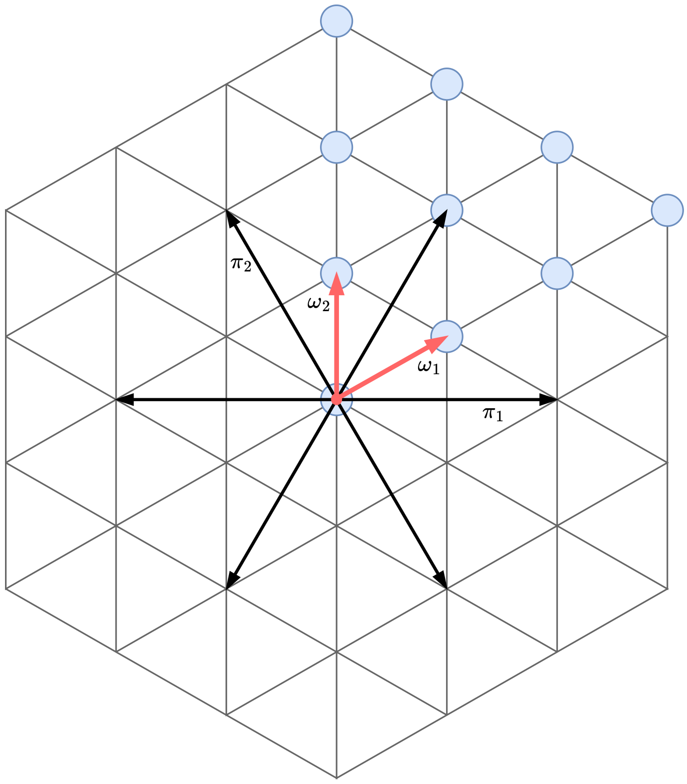


Figure A.7: The weight space of A_2. The black arrows are the roots Φ, the red arrows are the fundamental weights, and the blue circles highlight all dominant integral weights in this weight space.

A useful tool to analyse certain representations is their character; for a Lie algebra $\mathfrak{g}$ with rank r, the character is given by

$$\chi(V)(x_1, ..., x_r) = \sum_{w\in\Gamma} \dim(V_w) \prod_{i=1}^{r} x_i^{n_i}, \tag{A.95}$$

where $w \in \Gamma$ refers to a weight for a representation, $\dim(V_w)$ is the dimension of the corresponding weight space, and n_i is the so called Dynkin label. Since any weight can be written as

$$w = \sum_{i=1}^{r} n_i \omega_i, \tag{A.96}$$

where $\omega_i \in \Pi$ and here $n_i \in \mathbb{Z}$. One should be careful as there are different restrictions on the values of n_i for different types of weights. For example, in Eq. **??**, we defined a dominant integral weight similarly but had $n_i \in \mathbb{N}$. Also, we often refer to a certain representation by its highest weight and write its Dynkin labels in the following form $[n_1, ..., n_r]_G$, where the n_i's refer to exclusively the highest weight. Again, the context of the notation is important to know what it means.

Example 29. Suppose we have the adjoint representation of $A_2 \cong \mathfrak{su}(3)$ with highest weight $[1, 1]$, see Figure A.8.

Suppose one knows all the weights and their multiplicities. In that case, we can readily compute the character of the representation by assigning a fugacity x_1 and

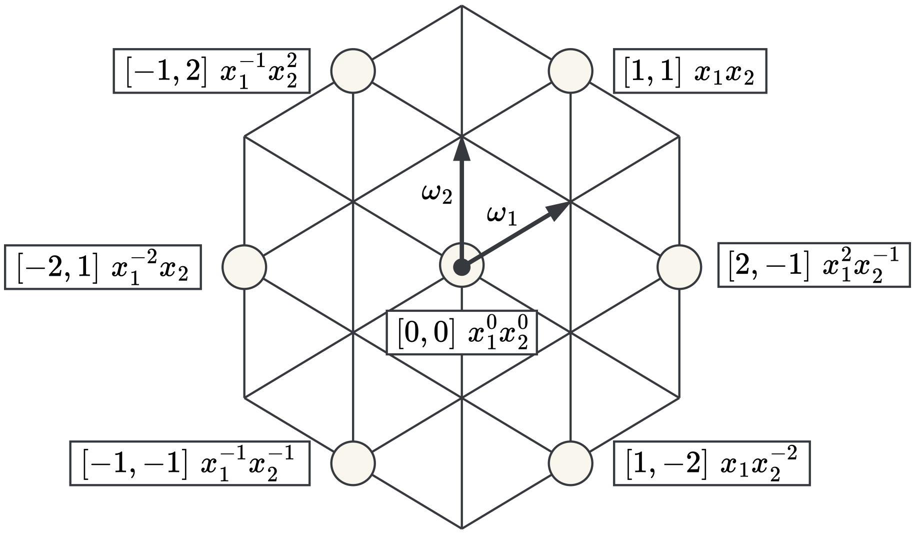


Figure A.8:

x_2, corresponding to the Cartan subalgebra elements, to each weight in the representation. Then, we use the Dynkin labels as exponents. The full character expression, then, is all of the terms added together

$$\chi([1,1]) = 1 + x_1x_2 + x_1^2x_2^{-1} + x_1^1x_2^{-2} + x_1^{-1}x_2^{-1} + x_1^{-2}x_2^1 + x_1^{-1}x_2^2. \tag{A.97}$$

We also have the following useful results by Weyl:

Theorem 10 (Weyl Character Formula)**.** *The character of an irreducible representation with highest weight μ is*

$$\chi(V_\mu) = \frac{\sum_{s\in\mathcal{W}} det(s) \times e(\mu+\rho)}{\sum_{s\in\mathcal{W}} det(s) \times e(\rho)}, \tag{A.98}$$

where $\mathcal{W}$ is the Weyl group, $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ and $det(s)$ is positive or negative depending on whether the element is a rotation or reflection.

Theorem 11 (Weyl Dimension Formula)**.** *The dimension of a weight space V_λ corresponding to a weight λ is given by*

$$dim(V_\lambda) = \frac{\prod_{\alpha\in\Phi_+}(\lambda+\rho,\alpha)}{\prod_{\alpha\in\Phi_+}(\rho,\alpha)} \tag{A.99}$$

Example 30. We can apply the Weyl dimension formula to $\mathfrak{su}(3)$ and since the root system is of rank 2, a representation is described by a highest weight $[m_1, m_2]$. The dimension of this representation is then given by

$$\dim(V_{[m_1,m_2]}) = \frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2). \tag{A.100}$$

Appendix B

Supersymmetry

The purpose of this appendix is to summarise some of the main results in supersymmetry. We will mostly follow [18], with occasionally supplementing from [6, 16].

B.1 SuperPoincaré Algebra

We recall the Poincaré algebra:

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \\ [M^{\mu\nu}, P^\rho] &= i(\eta^{\nu\rho} P^\mu - \eta^{\mu\rho} P^\nu) \\ [M^{\mu\nu}, M^{\rho\nu}] &= i\left(M^{\mu\sigma}\eta^{\nu\rho} - M^{\nu\sigma}\eta^{\mu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma}\right), \end{aligned} \tag{B.1}$$

where P^μ are the generators of translations and $M^{\mu\nu}$ are the generators of Lorentz transformations.

By splitting M into parts corresponding to rotations and boosts, one can show that the Lorentz algebra $\mathfrak{so}(1,3)$ (as defined by the last line of B.1) may be decomposed as $\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. And hence we may use the labels of representations for $SU(2)$ to label representations of $SO(1,3)$. The simplest representations are thus

$$\begin{aligned} (0,0):&\quad \text{Scalar}, \\ (\tfrac{1}{2},0):&\quad \text{Left-handed Weyl Spinor}, \\ (0,\tfrac{1}{2}):&\quad \text{Right-handed Weyl Spinor}, \\ (\tfrac{1}{2},\tfrac{1}{2}):&\quad \text{Vector}. \end{aligned} \tag{B.2}$$

We now extend the Poincaré algebra by adding the so-called supercharges Q_α and $\bar{Q}_{\dot{\alpha}}$, which are a left-handed and right-handed Weyl spinor respectively. The **superPoincaré** algebra has all the same relations as the Poincaré algebra but also following relations involving the supercharges:

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= 2(\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu \\ [M^{\mu\nu}, Q_\alpha] &= (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta \\ [M^{\mu\nu}, \bar{Q}_{\dot{\alpha}}] &= (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}_{\dot{\beta}} \\ \{Q_\alpha, \bar{Q}_\beta\} &= \{Q_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = [P^\mu, Q_\alpha] = 0. \end{aligned} \tag{B.3}$$

All internal symmetries must commute with the supercharges, except for the **R-symmetry** which is an internal $U(1)$ symmetry which acts as

$$Q_\alpha \to e^{i\gamma} Q_\alpha \quad \text{and} \quad \bar{Q}_{\dot{\alpha}} \to e^{-i\gamma} \bar{Q}_{\dot{\alpha}}. \tag{B.4}$$

Instead of adding just a single set of supercharges we can add multiple, we denote the number of sets of supercharges by $\mathcal{N}$ and hence can write Q^A_α and $\bar{Q}^A_{\dot{\alpha}}$ where $A = 1, ..., \mathcal{N}$. If $\mathcal{N} > 1$ we have additional (anti-)commutation relations

$$\begin{aligned}
\{Q^A_\alpha, \bar{Q}^B_{\dot{\alpha}}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^{AB}, \\
\{Q^A_\alpha, Q^B_\beta\} &= \epsilon_{\alpha\beta} Z^{AB}, \\
\{\bar{Q}^A_{\dot{\alpha}}, \bar{Q}^B_{\dot{\beta}}\} &= \epsilon_{\dot{\alpha}\dot{\beta}} (Z^\dagger)^{AB},
\end{aligned} \tag{B.5}$$

where Z^{AB} is a central charge that commute with all other elements of the algebra. Another feature of $\mathcal{N} > 1$ theories is that the R-symmetry can take on more interesting forms, for example, in $\mathcal{N} = 2$ the R-symmetry is $U(2)_R \cong U(1)_R \times SU(2)_R$.

B.2 Representations of the SuperPoincaré Algebra

B.2.1 Representations of the Poincaré Algebra

Representations of the Poincaré algebra can be constructed by considering the Casimir operators

$$C_1 = P_\mu P^\mu \quad \text{and} \quad C_2 = W_\mu W^\mu, \tag{B.6}$$

where $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$ is the Pauli-Lubański vector. Hence we can label the states by the eigenvalues of C_1 and C_2.

We will generally only study theories without mass parameter, and hence we want to construct massless representations. But we note that for a massless particle $C_1 = C_2 = 0$ in its rest frame, to remedy this we boost into a frame where $P^\mu = (E, 0, 0, E)$. In this new frame we have that $W^\mu = M_{12} P^\mu$, and the constant of proportionality between W and P is determined by the eigenvalue of the $U(1)$ rotation in the (x^1, x^2)-plane. The eigenvalue is the helicity $h \in \frac{1}{2}\mathbb{Z}^+$. There is another caveat to this, due to CPT symmetry for any state labeled with h there needs to be an additional state with helicity $-h$. We summarise that massless states come in the following form

$$|p_\mu; h\rangle \quad \text{and} \quad |p_\mu; -h\rangle . \tag{B.7}$$

B.2.2 $4d$ $\mathcal{N} = 1$ Representations

In the superPoincaré algebra our C_2 Casimir from the Poincaré algebra is not longer a Casimir operator as it does not commute with the supercharges. One way to get around this problem is to construct a new Casimir for the superPoincaré algebra, but as it turns out there is an easier method. We can in fact use a representation of the Poincaré algebra and act on it with something resembling raising and lowering operators to build up full representations of the superPoincaré algebra.

For our massless particles in the frame $p_\mu = (E, 0, 0, E)$, the supersymmetry algebra becomes

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = 2E(1+\sigma^3)_{\alpha\dot{\alpha}} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{B.8}$$

And we see that $\{Q_1, \bar{Q}_1\} = 4E$ with all other combinations being equal to 0. Now consider the operators

$$a = \frac{1}{2\sqrt{E}} Q_1 \quad \text{and} \quad a^\dagger = \frac{1}{2\sqrt{E}} \bar{Q}_1, \tag{B.9}$$

which satisfy the following relations we are familiar with from lowering and raising operators

$$\{a, a^\dagger\} = 1 \quad \text{and} \quad \{a, a\} = \{a^\dagger, a^\dagger\} = 0, \tag{B.10}$$

as well as

$$[M^{12}, a] = -\frac{1}{2} a \quad \text{and} \quad [M^{12}, a^\dagger] = \frac{1}{2} a^\dagger. \tag{B.11}$$

If we act with M^{12} and a on some state $|p_\mu; h\rangle$

$$\begin{aligned} M^{12} a |p_\mu; h\rangle &= ([M^{12}, a] + a M^{12}) |p_\mu; h\rangle \\ &= (-\frac{1}{2} a + a M^{12}) |p_\mu; h\rangle \\ &= (h - \frac{1}{2}) a |p_\mu; h\rangle . \end{aligned} \tag{B.12}$$

So, the state $a|p_\mu; h\rangle$ has helicity $h - \frac{1}{2}$ and similarly $a^\dagger |p_\mu; h\rangle$ has helicity $h + \frac{1}{2}$.

Suppose that we have some vacuum state $|\Omega\rangle$, then

$$\begin{aligned} a|\Omega\rangle &\equiv 0 \\ |\Omega\rangle &= |p_\mu; h\rangle \\ a^\dagger |\Omega\rangle &= |p_\mu; h + \tfrac{1}{2}\rangle \\ a^\dagger a^\dagger |\Omega\rangle &= 0, \end{aligned} \tag{B.13}$$

where the last line follows from Eq. B.10.

We are now able to list the representations of $4d$ $\mathcal{N} = 1$ supersymmetry, they are also called the **supermultiplets**:

Chiral Multiplet - If we start with a Poincaré representation of helicity 0, we get the states

State	Multiplicity
$\lvert P, -\frac{1}{2}\rangle$	1
$\lvert P, 0\rangle$	2
$\lvert P, \frac{1}{2}\rangle$	1

The physical content of the chiral multiplet is one complex scalar φ and one Weyl spinor ψ. Some examples in the minimal supersymmetric standard model (MSSM) would be: (squark, quark), (higgs, higgsino) and (slepton, lepton)

Vector Multiplet - Starting with a representation of helicity $\frac{1}{2}$, we have

State	Multiplicity
$\|P,-1\rangle$	1
$\|P,-\frac{1}{2}\rangle$	1
$\|P,\frac{1}{2}\rangle$	1
$\|P,1\rangle$	1

For this we have one Weyl spinor χ and one vector boson A_μ. In the MSSM we have, for example: (gluino, gluon), (photino, photon) and (Wino, W).

B.2.3 $4d$ $\mathcal{N}=2$ Representations

This formalism is easily extended to $\mathcal{N}>1$, and we can act on a state with different raising operators until one of them repeats, in general we have the following states

$$\begin{aligned} &|\Omega\rangle \\ a_1^\dagger &|\Omega\rangle \\ a_1^\dagger a_2^\dagger &|\Omega\rangle \\ &... \\ a_1^\dagger ... a_{\mathcal{N}}^\dagger &|\Omega\rangle. \end{aligned} \tag{B.14}$$

And if $\mathcal{N}=2$, the highest possible state is $a_1^\dagger a_2^\dagger|\Omega\rangle$. We repeat the same procedure as above to get:

Hypermultiplet - By starting with $|p_\mu; h=-\frac{1}{2}\rangle$

State	Multiplicity
$\|P,-\frac{1}{2}\rangle$	2
$\|P,0\rangle$	4
$\|P,\frac{1}{2}\rangle$	2

It consists of two chiral multiplets, and therefore of two complex scalars $\varphi_{1,2}$ and two Weyl spinors $\psi_{1,2}$.

Vector multiplet - We start with $|p_\mu; h=0\rangle$ and by applying the raising operators plus the CPT conjugates we have

State	Multiplicity
$\vert P,-1\rangle$	1
$\vert P,-\frac{1}{2}\rangle$	2
$\vert P,0\rangle$	2
$\vert P,\frac{1}{2}\rangle$	2
$\vert P,1\rangle$	1

This consists of one $\mathcal{N}=1$ vector multiplet and one $\mathcal{N}=1$ chiral multiplet. Hence we have one complex scalar φ, two Weyl spinors $\psi_{1,2}$ and one vector boson A_μ.

B.3 Supersymmetric Field Theories

We now want to construct supersymmetric field theories, i.e. field theories which are invariant under supersymmetry. To do this we extend our usual Minkowski space to include two Grassmann-valued spinors θ_α and $\bar{\theta}^{\dot{\alpha}}$. The resulting, so-called, **superspace** is now parameterised by the coordinates $(x^\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}})$.

We can now Taylor expand a **scalar superfield** into its component fields

$$\begin{aligned} S(x,\theta,\bar{\theta}) = \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}N(x) + (\theta\sigma^\mu\bar{\theta})V_\mu(x) \\ +(\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\rho(x) + (\theta\theta\bar{\theta}\bar{\theta})D(x), \end{aligned} \tag{B.15}$$

where ϕ, M, N and D are scalar fields, $\psi_\alpha, \rho_\alpha, \bar{\chi}_{\dot{\alpha}}$ and $\bar{\lambda}_{\dot{\alpha}}$ are Weyl spinor fields, and V_μ is a vector field. This includes more fields than expected as compared to the representation theory of the previous section, and indeed it turns out that $S(x,\theta,\bar{\theta}$ is not irreducible.

By the defining differential operators which behave analogously to the supercharges, i.e. they form a superPoincaré algebra of operators,

$$\begin{aligned} \mathcal{Q}_\alpha &= -i\frac{\partial}{\partial\theta^\alpha} - (\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\frac{\partial}{\partial x^\mu} \\ \bar{\mathcal{Q}}_{\dot{\alpha}} &= +i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\frac{\partial}{\partial x^\mu}. \end{aligned} \tag{B.16}$$

If we can now find some other operator, say $\mathcal{D}_\alpha$, that commutes with $\mathcal{Q}_\alpha$, we know that it will preserve supersymmetry. Two such operators are

$$\begin{aligned} \mathcal{D}_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\frac{\partial}{\partial x^\mu} \\ \bar{\mathcal{D}}_{\dot{\alpha}} &= \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\frac{\partial}{\partial x^\mu}, \end{aligned} \tag{B.17}$$

and therefore $\mathcal{D}S$ and $\bar{\mathcal{D}}S$ are both valid superfields and we can impose that, for example, $\bar{\mathcal{D}}S = 0$.

We define a **chiral superfield** Φ as just such a field where $\bar{\mathcal{D}}\Phi = 0$, and it can be expanded to give

$$\begin{aligned} \Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta^2F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) \\ -i\frac{1}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}\theta^2\bar{\theta}^2\partial_\mu\partial^\mu\phi(x), \end{aligned} \tag{B.18}$$

where $\theta^2 = \theta_\alpha \theta^\alpha$, $\bar{\theta}^2 = \bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}}$, and $\partial_\mu = \frac{\partial}{\partial x^\mu}$. The chiral superfield hence consists of a complex scalar field $\phi(x)$, a Weyl spinor field $\psi_\alpha(x)$, and the auxiliary field $F(x)$. We can also define the **anti-chiral superfield** $\bar{\Phi}$ by $\mathcal{D}\bar{\Phi} = 0$, which is just the complex conjugate of the chiral superfield.

Lastly, we can define the **vector superfield** V by the constraint that $V^\dagger = V$, and it expands as

$$\begin{aligned} V(x,\theta,\bar{\theta}) = C(x) + \theta\chi(x) + \bar{\theta}\bar{\chi}(x) + i\theta^2 M(x) - i\bar{\theta}^2 M^\dagger(x) + \theta\sigma^\mu\bar{\theta}A_\mu(x) \\ +\theta^2\bar{\theta}\left(\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right) + \bar{\theta}^2\theta\left(\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) \\ +\frac{1}{2}\theta^2\bar{\theta}^2\left(D(x) - \frac{1}{2}\partial_\mu\partial^\mu C(x)\right), \end{aligned} \tag{B.19}$$

where C and D are real scalar fields, M is a complex scalar, χ_α and λ_α are Weyl spinors, and, of course, A_μ is a vector field. Here we again have an auxiliary field, D.

Bibliography

[1] E. Noether, *Invariante variationsprobleme*, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* **1918** (1918) 235.

[2] S. Coleman and J. Mandula, *All possible symmetries of the s matrix*, *Phys. Rev.* **159** (1967) 1251.

[3] R. Haag, J.T. Łopuszański and M. Sohnius, *All possible generators of supersymmetries of the s-matrix*, *Nuclear Physics B* **88** (1975) 257.

[4] D. Balin and A. Love, *Supersymmetric Gauge Field Theory and String Theory*, Taylor & Francis Group (1994).

[5] W. Fulton and J. Harris, *Representation Theory: A First Course*, Graduate Texts in Mathematics, Springer New York, NY (1991), 10.1007/978-1-4612-0979-9.

[6] K. Gledhill, *Global Symmetry and Poisson Brackets for Moduli Spaces of Supersymmetric Quiver Gauge Theories with 8 Supercharges*, Ph.D. thesis, Imperial College London, 2023.

[7] G. Kumaran, *Discrete Gauging, Folding, and hyper-Kähler Quotients of Coulomb Branches of* 3d $\mathcal{N}=4$ *Quiver Gauge Theories*, MSc thesis, Imperial College London, 2022.

[8] K. Keeratikarn, *Monopole Formula and Ungauging Schemes of Coulomb branches of* 3d $\mathcal{N}=4$ *Gauge Theories*, MSc thesis, Imperial College London, 2022.

[9] T. Datsev, *Nilpotent Orbit Coulomb Branches*, MSc thesis, Imperial College London, 2022.

[10] J. Bao, 3d $\mathcal{N}=4$ *Coulomb Branches and Quiver Gauge Theories in* 6d, MSc thesis, Imperial College London, 2022.

[11] S. Benvenuti, B. Feng, A. Hanany and Y.-H. He, *Counting bps operators in gauge theories: quivers, syzygies and plethystics*, *Journal of High Energy*

Physics **2007** (2007) 050–050.

[12] B. Feng, A. Hanany and Y.-H. He, *Counting gauge invariants: the plethystic program*, *Journal of High Energy Physics* **2007** (2007) 090–090.

[13] A. Hanany and R. Kalveks, *Highest weight generating functions for hilbert series*, *Journal of High Energy Physics* **2014** (2014) .

[14] D. Miketa, *Algebraic construction of Coulomb branches in* 3d $\mathcal{N} = 4$ *quiver gauge theories*, Ph.D. thesis, Imperial College London, 2020.

[15] Z. Zhong, *Quiver gauge theories in* 3d*,* 5d *and* 6d, MSc thesis, Imperial College London, 2020.

[16] F. Carta, *Moduli Spaces of* $\mathcal{N} = 4$*,* $d = 3$ *Quiver Gauge Theories and Mirror Symmetry*, Ph.D. thesis, University of Padua, 2014.

[17] D. Skinner, *Supersymmetry*, University of Cambridge Part III Lecture Notes (2019).

[18] D. Tong, *Supersymmetric Field Theory*, University of Cambridge Part III Lecture Notes (2023).

[19] K. Intriligator and N. Seiberg, *Mirror symmetry in three dimensional gauge theories*, *Physics Letters B* **387** (1996) 513–519.

[20] A. Hanany and E. Witten, *Type iib superstrings, bps monopoles, and three-dimensional gauge dynamics*, *Nuclear Physics B* **492** (1997) 152–190.

[21] P. Safronov, *Hyperkäler Manifolds*, MIT Notes of Talbot Workshop (2011).

[22] S. Cremonesi, A. Hanany and A. Zaffaroni, *Monopole operators and hilbert series of coulomb branches of 3d* $\mathcal{N} = 4$ *gauge theories*, *Journal of High Energy Physics* **2014** (2014) .

[23] D. Gaiotto and E. Witten, *s-duality of boundary conditions in* $\backslash = 4$ *super yang-mills theory*, *Advances in Theoretical and Mathematical Physics* **13** (2009) 721–896.

[24] G. Cheng, A. Hanany, Y. Li and Y. Zhao, *Coulomb branch for a-type balanced quivers in 3d* $\mathcal{N} = 4$ *gauge theories*, 2017.

[25] A. Hanany and R. Kalveks, *Quiver theories for moduli spaces of classical group nilpotent orbits*, *Journal of High Energy Physics* **2016** (2016) .

[26] J. Gray, Y.-H. He, A. Hanany, N. Mekareeya and V. Jejjala, *Sqcd: a geometric aperçu*, *Journal of High Energy Physics* **2008** (2008) 099–099.

[27] A. Hanany, G. Kumaran, C. Li, D. Liu and M. Sperling, *Actions on the quiver – discrete quotients on the coulomb branch*, 2023.

[28] J. Fuchs and C. Schweigert, *Symmetries, Lie Algebras and Representations*, Cambridge Monographs on Mathematical Physics, Cambridge University Press (2003).

[29] B.C. Hall, *Lie Groups, Lie Algebras, and Representations*, Graduate Texts in Mathematics, Springer Cham (2015), 10.1007/978-3-319-13467-3.

[30] F. Iachello, *Lie Algebras and Applications*, Lecture Notes in Physics, Springer Berlin Heidelberg (2015), 10.1007/978-3-662-44494-8.