Amplitude Dependence on the Time Period of a Pendulum

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***Abstract*—I demonstrated how the amplitude of a pendulum influences its time period by analysing video recordings and using object detection. I was able to accurately find the factors of the higher order correction terms to the time period equation. But, I found a conflicting value for the initial time period, which seems to be due to a measurement mistake.**

I. Introduction

MANY physical situations can be modelled with the equations of a simple pendulum, for example old pendulum clocks, swings, and wrecking balls. But these equations are only approximations that rely on certain assumptions to be true. One of these assumptions is that the angle of oscillation is small. This, of course, is not always the case and in real world applications we can get large angles. The effect of a large angle is that the time period increases. To be able to make more accurate predictions we need to find correction terms to the usual equations governing the simple pendulum. In this paper, I will try to demonstrate the effect of the amplitude on the time period of oscillation using object detection on videos of a pendulum.

II. Theoretical Background

The equation of the time period of a simple pendulum can be derived by approximating the restoring force on the pendulum bob as

$$F = -mg\sin\theta \approx -mg\theta = -\frac{mg}{l}x, \tag{1}$$

where m is the mass of the bob, g is the acceleration due to gravity, θ is the angle of the pendulum to its equilibrium position, l is the length of the pendulum string, and x is the arc length that the pendulum moves. Solving Equation 1 for x and finding the time period then gives

$$T = 2\pi\sqrt{\frac{l}{g}}, \tag{2}$$

where l and g are the same as in Equation 1. To find the higher order corrections to the time period we can look at the kinetic and potential energy of the pendulum at its initial maximum amplitude and at some later time. Setting the initial and final energy equal to each other we get that

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 + mgl\left(1-\cos\theta\right) = 0 + mgl\left(1-\cos\theta_i\right), \tag{3}$$

where K is the kinetic energy, U is the potential energy, the subscript f denotes the final position and i the initial position. It can be shown that by rearranging and integrating Equation 3, you would get

$$2\pi + \frac{1}{2}\sin^2\left(\frac{\theta_i}{2}\right) + ... = \sqrt{\frac{g}{l}}T, \tag{4}$$

as is described in [1]. Approximating the $\sin^2\left(\frac{\theta_i}{2}\right)$ term as $\left(\frac{\theta_i}{2}\right)^2$ and rearranging for T gives us

$$T \approx 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\theta_i^2}{16}\right) = T_0\left(1+\frac{\theta_i^2}{16}\right), \tag{5}$$

where T_0 is the same as the time period for the simple pendulum. It is important to mention that the approximation here is of a higher order term than the approximation for the simple pendulum.

III. Method

To get accurate values for the time periods, I recorded videos of the pendulum swinging, starting from different amplitudes. In this setup, the camera was attached above the pivot point and filmed top down as can be seen in Figure 1. Here,

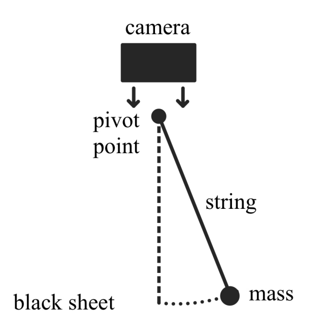


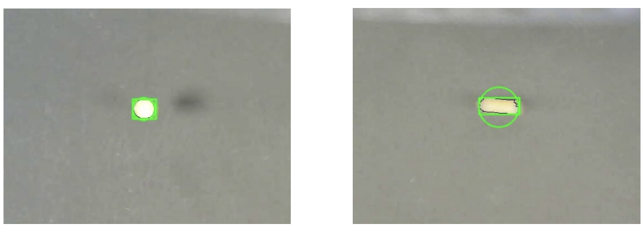
Fig. 1: Schematic of experiment setup. The camera is placed right above the pivot point filming top down. A black sheet is put under the pendulum.

to improve object detection, it was important to ensure that no other moving objects were in the image as well as that the pendulum is above a dark background. Additionally, I placed a ruler in the image and adjusted the string size so that the mass is right above the black sheet. This was initially done to have another way of determining the angle, but I ended up using it to challenge my string length measurement, which I measured to be 44.8 ± 0.05 cm. I used a protractor, that was attached to the pivot point, to measure the angle. Here, the uncertainty is $\pm 0.5°$. I recorded a total of three videos per amplitude for the following amplitudes: $2°, 3°, 6°, 9°, 12°, 15°, 18°$, and

21°. With the additional 2° measurement I wanted to be especially accurate close to 0°.

IV. Results

To analyse the videos and extract the positions of the pendulum bob, I utilised object detection with OpenCV and the K-nearest neighbour background subtraction algorithm[2]. To get the centre of the pendulum, I let OpenCV find a minimum enclosing circle and rectangle, as can be seen in Figure 2. It is important to note the shape of the bob in motion. Due to the cameras low frame rate the bob appears to be stretched and looks almost rectangular. This has an effect on the uncertainty of the position, but because of the applied object detection we can quantify this uncertainty as half the width and height of the enclosing rectangle, for the horizontal and vertical positions respectively. I also had to exclude the first few frames of each videos, as they were reserved for the background subtraction algorithm to learn to distinguish between foreground and background.



(a) Stationary (b) Moving

Fig. 2: Enclosing circle and rectangle around pendulum bob in motion and stationary. The stationary bob fits perfectly into a circle, whereas the moving bob takes a rectangular shape due to the frame rate of the camera[3].

To get the time period of each of the trials, I plotted the horizontal and vertical distance against the time, see Figure 3. What is interesting here is that even though the uncertainty in the position is larger when the bob travels at a faster speed, it does not contribute much when trying to fit a curve to the values. This is because, as well as the uncertainty increasing, the gradient also increases significantly and therefore the y-axis uncertainty is less relevant.

The next part of the analysis was to make a plot of time period versus amplitude, where I am expecting the previously described trend to be visible on. For this I will try to find values for T_0, α, and β in

$$T = T_0 \left(1 + \alpha\theta + \beta\theta^2\right). \tag{6}$$

And I am expecting that $T_0 = 1.343$ s, $\beta = 0$, and $\alpha = \frac{1}{16}$ as discussed earlier. For this I first removed three outliers, as they were more than one standard deviation away from the mean of all the points. I also fitted the curve to the data for the horizontal time periods only, as those were more precise than the points for the vertical periods. But it can be seen that the vertical points still follow the same trend, see Figure 4.

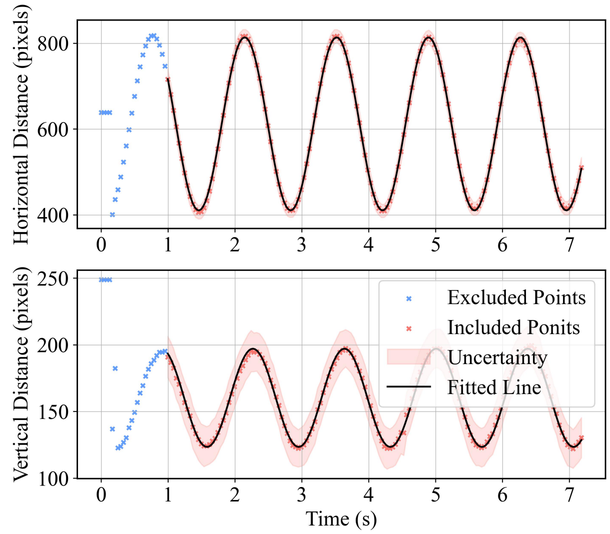


Fig. 3: Graph of first trial for an amplitude of 9° as an example of all the other graphs. The uncertainty in the distances is shown by the red area. And the blue points are excluded, as the background subtraction algorithm requires a few frames to learn[3].

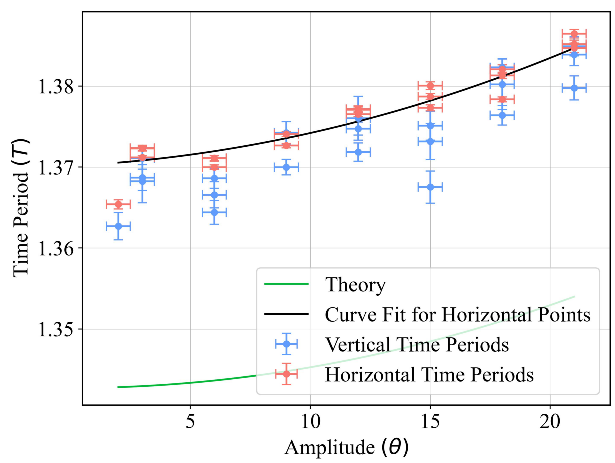


Fig. 4: Graph of time period versus amplitude. Here the curve fit is for only the horizontal time periods, which are more precise. A clear discrepancy between theoretical curve and fitted curve can be seen[3].

I get the following values for the constants $T_0 = 1.370 \pm 0.001$ s, $\alpha = 0.6 \times 10^{-2} \pm 1 \times 10^{-2}$, and $\beta = 6.3 \times 10^{-2} \pm 2.8 \times 10^{-2}$. What is most obvious from Figure 4 is that my data seems to follow the predicted trend, yet completely differs in the value of T_0.

Another thing to mention is that even though my value for β seems to be spot on, to 2 significant figures, the uncertainty is relatively high with about 44%. Similarly, the uncertainty for α is 170%. But to agree with a theoretical value of zero, it ideally needs an uncertainty of 100% to include zero.

A possible explanation for the difference in T_0 is that I incorrectly measured the length of the string. To get a theoretical value of $T_0 = 1.370$ s, the string would have needed to have a length of $l = 0.466$ m. This is a difference of 1.8 cm to the original value and seems not very likely. Yet, if we use the geometric setup in Figure 5, we can get the string length, l, in terms of the horizontal distance between two maximum amplitudes, x, and the maximum amplitude, θ. But we have to approximate the arc length, s, as x. We get the following equation

$$l = \frac{s}{2\theta} \approx \frac{x}{2\theta}. \tag{7}$$

It is easy to see from Equation 7 that the approximation $s \approx x$ will slightly underestimate l, because $s > x$. As I had a ruler in the videos, I was able to determine that 100 pixels ≈ 3.299 cm. As the ruler was slightly lower than the mass of the pendulum this is overestimating x, and therefore l. This is demonstrated in Figure 6. Furthermore, I used the videos of the largest amplitude as the percentage uncertainty in θ will be a lot less. Putting all this together, we get that $x = 34.5 \pm 0.66$ cm and therefore $l = 47.1 \pm 1.6$ cm. This is further indicating that I incorrectly measured the length, if the overestimation from the pixel to metre conversion is reasonably low.

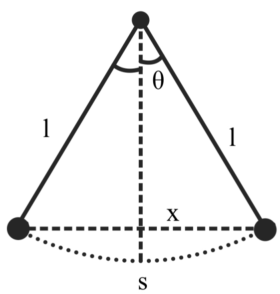


Fig. 5: Schematic of setup used to determine the length, l, of the pendulum string.

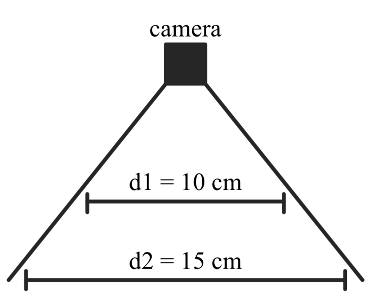


Fig. 6: Schematic showing that, even though d_1 and d_2 have different lengths, they will have the same distance in pixels on a camera. The diagonal lines visualise the point of view of the camera. Therefore, using a conversion acquired from d_2 will overestimate a value measured on the same level as d_1. Diagram not to scale.

V. Conclusion

In this paper, I demonstrated that the time period of a pendulum depends on the amplitude and was able to replicate the theoretically predicted trend, yet I found a discrepancy between the predicted value for T_0 and the measured value. It is inconclusive whether this was due to a wrong string length measurement, as further analysis indicated, or another not yet considered factor.

Acknowledgment

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