

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/239564904>

A Collection of 110 Geometry Theorems and Their Machine Produced Proofs Using Full-Angles y

Article

CITATIONS

9

READS

11,538

3 authors, including:



[Xiao-Shan Gao](#)

Chinese Academy of Sciences

201 PUBLICATIONS 2,602 CITATIONS

[SEE PROFILE](#)

Some of the authors of this publication are also working on these related projects:



Difference Algebra [View project](#)

A Collection of 110 Geometry Theorems and Their Machine Produced Proofs Using Full-Angles[†]

Shang-Ching Chou, Xiao-Shan Gao[‡] and Jing-Zhong Zhang[§]

Department of Computer Science, The Wichita State University
Wichita KS 67260-0083, USA

This is a collection of 110 geometry theorems and their machine proofs produced by a prover using inference rules based on *full-angles*. These examples with difficulties range from moderate to those in the level of International Mathematical Olympiad and those proposed in the recent American Mathematics Monthly.

The prover is implemented using SB-Prolog on a SPARK-10 workstation. Briefly speaking, the prover works as follows.

- S1 The prover first builds a geometry information base (GIB) from the hypotheses by collecting information about parallel lines, perpendicular lines, and circles.
- S2 The prover tries to prove the theorem using backward chaining.
- S3 If the prover fails to find a proof in step S2, it further collects new geometry information related to equal-angles into the GIB and does S2 again.
- S4 If a geometry statement is true, the prover can automatically generate the *machine proof* in TeX typesetting.

Details of the method/prover can be found in the following papers.

1. S. C. Chou, X. S. Gao, & J. Z. Zhang, Automated Generation of Multiple and Shortest Proofs in Geometry with High Level Inference Rules, I. Multiple and Shortest Proofs Generation, CS-WSU-94-2, March, 1994.
2. S. C. Chou, X. S. Gao, & J. Z. Zhang, Automated Generation of Multiple and Shortest Proofs in Geometry with High Level Inference Rules, II. Theorem Proving with Full-angles, CS-WSU-94-3, March, 1994.

[†]The work reported here was supported in part by the NSF Grant CCR-9117870 and Chinese National Science Foundation.

[‡]On leave from Institute of Systems Sciences, Academia Sinica, Beijing 100080.

[§]The permanent address: Chengdu Institute of Computer Application, Academia Sinica, 610015 Chengdu, China

This collection is organized as follows. Section 1 introduces some notations. Section 2 is the collection of the 110 examples. Section 3 includes the time and proof length statistics.

1 Notation Convenience

A geometry statement consists of four parts

(Nm,Pts,Hs,Cs).

1. Nm is the name of the statement, which is used as the identification of the geometry statement.
2. Pts is a list of the points occurring in the statement. The order of the points in the list is used to guide the search in the proving process.
3. Hs is a list of predicates which are the hypotheses of the geometry statement.
4. Cs is a predicate or an equation of full-angles which is the conclusion of the statement.

The following predicates are allowed in the description of geometry statements

coll (collinear)
para (parallel)
perp (perpendicular)
cong (congruent)
eqangle (equal-angle)
midpoint
cyclic
foot
circumcenter
orthocenter
similar (two triangles are similar)
incenter pbisector (perp-bisector).

In all the examples, we use lower case English letter to denote points. This is in accordance with the Prolog rule that capital letters are used as variables.

2 A Collection of 110 Geometry Theorems

2.1 Examples Whose Conclusions Are in the Database

For the following five examples, their proofs are already in the GIB. We thus proved these examples by forward chaining.

Example 1 In triangle ABC , let F the midpoint of the side BC , D and E the feet of the altitudes on AB and AC , respectively. FG is perpendicular to DE at G . Show that G is the midpoint of DE .

Point order: a, b, c, d, e, f, g .

Hypotheses: $\text{foot}(d, c, a, b)$, $\text{foot}(e, b, a, c)$,
 $\text{midpoint}(f, b, c)$, $\text{midpoint}(g, d, e)$.

Conclusion: $\text{perp}(f, g, d, e)$.

The Machine Proof

The result is in the database.

$gf \perp ed$, because $\text{midpoint}(g, d, e)$; and $df = ef$.

$df = ef$, because d, e, b, c are on the circle with f as center.

d, e, b, c are on the circle with f as center, because $\text{midpoint}(f, b, c)$; $bd \perp cd$; $be \perp ce$.

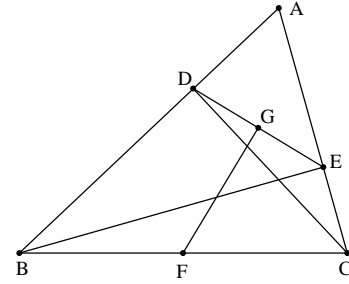


Figure 1

Example 2 Prove that the circumcenter of a triangle is the orthocenter of its medial triangle.

Point order: $a, b, c, a1, b1, c1, o$.

Hypotheses: $\text{midpoint}(a1, b, c)$, $\text{midpoint}(b1, a, c)$,
 $\text{midpoint}(c1, a, b)$, $\text{circumcenter}(o, a, b, c)$.

Conclusion: $\text{perp}(o, a1, b1, c1)$.

The Machine Proof

The result is in the data base.

$oa1 \perp c1b1$, because $oa1 \perp bc$, and $bc \parallel b1c1$.

$bc \parallel b1c1$, because $\text{midpoint}(b1, a, c)$, $\text{midpoint}(c1, a, b)$.

$oa1 \perp bc$, because $\text{midpoint}(a1, b, c)$; $ob = oc$.

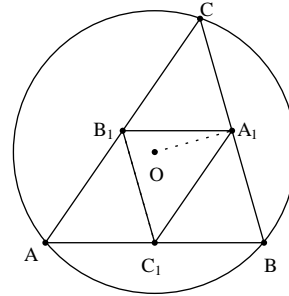


Figure 2

Example 3 Show that the triangle formed by the foot of the altitude to the base of a triangle and the midpoints of the altitudes to the lateral sides is similar to the given triangle; its circumcircle passes through the orthocenter of the given triangle and through the midpoint of its base.

Point order: $a, b, c, d, e, f, h, a1, p, q$.

Hypotheses: $\text{foot}(d, a, b, c)$, $\text{foot}(e, b, a, c)$, $\text{foot}(f, c, a, b)$,
 $\text{coll}(h, a, d)$, $\text{coll}(h, b, e)$, $\text{coll}(h, c, f)$, $\text{midpoint}(a1, b, c)$,
 $\text{midpoint}(p, b, e)$, $\text{midpoint}(q, c, f)$.

Conclusion: $\text{cyclic}(p, q, h, d)$.

The Machine Proof

The conclusion is in the database

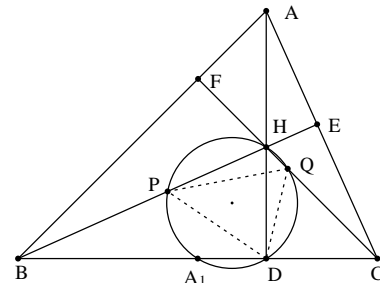


Figure 3

$q, d, a1, h, p$ are cyclic, because $d, a1, h, q$ are cyclic and $d, a1, h, p$ are cyclic.
 $d, a1, h, q$ are cyclic, because $da1 \perp dh$; $qa1 \perp qh$.
 $d, a1, h, p$ are cyclic, because $da1 \perp dh$; $pa1 \perp ph$.
 $qa1 \perp qh$, because $qa1 \parallel ab$; $ab \perp qh$.
 $qa1 \parallel ab$, because $\text{midpoint}(a1, b, c)$; $\text{midpoint}(q, c, f)$.
 $pa1 \perp ph$, because $pa1 \parallel ac$; $ac \perp ph$.
 $pa1 \parallel ac$, because $\text{midpoint}(a1, b, c)$; $\text{midpoint}(p, b, e)$.

Example 4 Show that the anticenter of a cyclic quadrilateral is the orthocenter of the triangle having for vertices the midpoints of the diagonals and the point of intersection of those two lines.

Point order: $a, b, c, d, o, q, s, j, m$.

Hypotheses: $\text{circumcenter}(o, a, b, c, d)$, $\text{coll}(i, a, d)$, $\text{coll}(i, b, c)$,
 $\text{midpoint}(q, b, c)$, $\text{midpoint}(s, a, d)$, $\text{midpoint}(j, s, q)$, $\text{midpoint}(j, o, m)$.

Conclusion: $\text{perp}(s, m, b, c)$.

The Machine Proof

The conclusion is in the database.

$ms \perp cb$, because $sm \parallel oq$; $oq \perp bc$.

$sm \parallel oq$, because $\text{midpoint}(j, o, m); \text{midpoint}(j, s, q)$.

$oq \perp bc$, because midpoint(q, b, c); $ob = oc$.

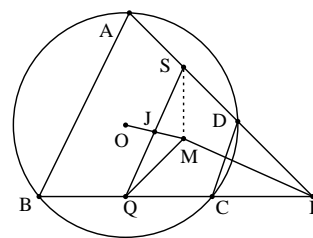


Figure 4

Example 5 *Let H be the orthocenter of triangle ABC . Then the circumcenters of the four triangles ABC , ABH , ACH , and BCH are such that each is the orthocenter of the triangle formed by the remaining three.*

Point order: a, b, c, h, o, p, q, r .

Hypotheses: $\text{orthocenter}(h, a, b, c)$, $\text{circumcenter}(o, a, b, c)$,
 $\text{circumcenter}(p, a, c, h)$, $\text{circumcenter}(q, a, b, h)$.

Conclusion: $[para, p, q, b, c]$.

The Machine Proof

The result is in the database.

$pq \parallel bc$, because $pq \perp ah$, $bc \perp ah$.

$pq \perp ah$, because ah is the axis of $\text{circle}(p, a, c, h)$ and $\text{circle}(q, a, b, h)$.

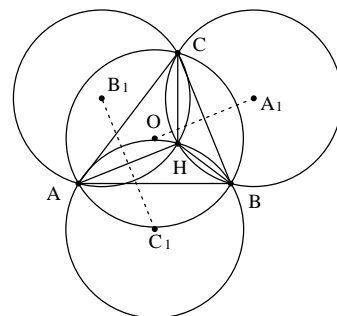


Figure 5

2.2 Examples Proved by Backward Chaining

The proof for the examples in this subsection are obtained by a backward chaining search based on the GIB. Generally speaking, the information in GIB is obvious and does not need further

explanation.

Example 6 (The Orthocenter Theorem) *The three altitudes of a triangle are concurrent.*

Point order: a, b, c, e, f, h .

Hypotheses: $\text{foot}(e, b, a, c)$, $\text{foot}(f, a, b, c)$,
 $\text{coll}(h, a, f)$, $\text{coll}(h, b, e)$.

Conclusion: $\text{perp}(a, b, c, h)$.

The Machine Proof

$$-[hc, ba] + 1$$

(Since a, f, h are collinear; h, c, f, e are cyclic;

$$[hc, ba] = [hc, hf] + [fa, ba] = -[fe, ec] + [fa, ba].)$$

$$= [fe, ec] - [fa, ba] + 1$$

(Since a, c, e are collinear; e, f, a, b are cyclic; $[fe, ec] = [fb, ba]$.)

$$= [fb, ba] - [fa, ba] + 1$$

(Since b, c, f are collinear; $[fb, ba] = [cb, ba]$.)

$$= -[fa, ba] + [cb, ba] + 1$$

(Since $fa \perp bc$; $[fa, ba] = [cb, ba] + 1$.)

$$= 0$$

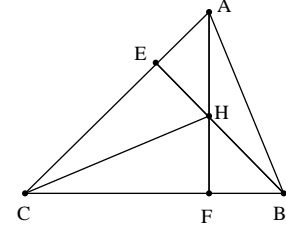


Figure 6

Example 7 (Simson's Theorem) *Let D be a point on the circumcircle of triangle ABC . From D three perpendiculars are drawn to the three sides BC , AC , and AB of triangle ABC . Let E , F , and G be the three feet respectively. Show that E , F and G are collinear.*

Point order: o, a, b, c, d, e, f, g .

Hypotheses: $\text{cong}(o, a, o, b)$, $\text{cong}(o, a, o, c)$, $\text{cong}(o, a, o, d)$,
 $\text{coll}(e, b, c)$, $\text{perp}(e, d, b, c)$, $\text{coll}(f, a, c)$, $\text{perp}(f, d, a, c)$,
 $\text{coll}(g, a, b)$, $\text{perp}(g, d, a, b)$.

Conclusion: $\text{coll}(g, f, e)$.

The Machine Proof

$$[gf, ge]$$

(Since f, g, d, a are cyclic; e, g, d, b are cyclic;

$$[gf, ge] = [gf, gd] + [gd, ge] = [fa, da] - [eb, db].)$$

$$= [fa, da] - [eb, db]$$

(Since b, c, e are collinear; $[eb, db] = -[db, cb]$.)

$$= [fa, da] + [db, cb]$$

(Since a, c, f are collinear; $[fa, da] = -[da, ca]$.)

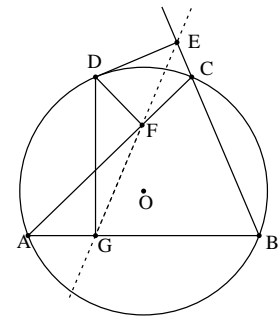


Figure 7

$$= [db, cb] - [da, ca]$$

(Since b, d, c, a are cyclic; $[db, cb] = [da, ca]$.)

$$= 0$$

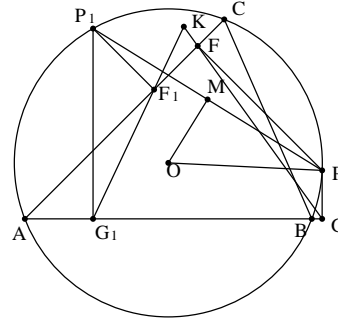


Figure 8

Example 8 (Simson Line, Extension 1) *The angle formed by the Simson lines of two points for the same triangle is measured by half the arc between the two points.*

Point order: $a, b, c, p, p_1, g, g_1, f, f_1, k$.

Hypotheses: $\text{cyclic}(a, b, c, p, p_1)$, $\text{foot}(g, p, a, b)$, $\text{foot}(f, p, a, c)$, $\text{foot}(g_1, p_1, a, b)$, $\text{foot}(f_1, p_1, a, c)$, $\text{coll}(k, f, g)$, $\text{coll}(k, f_1, g_1)$.

Conclusion: $\text{eqangle}(p, a, p_1, f_1, k, f)$.

The Machine Proof

$$-[kf_1, kf] - [p_1a, pa]$$

(Since f_1, g_1, k are collinear; f, g, k are collinear;

$$[kf_1, kf] = [f_1g_1, fg].)$$

$$= -[f_1g_1, fg] - [p_1a, pa]$$

(Since a, f, f_1 are collinear; f_1, g_1, a, p_1 are cyclic;

$$[f_1g_1, fg] = [f_1g_1, f_1a] + [af, fg] = -[fg, fa] + [g_1p_1, p_1a].)$$

$$= [fg, fa] - [g_1p_1, p_1a] - [p_1a, pa]$$

(Since $-[g_1p_1, p_1a] - [p_1a, pa] = -[g_1p_1, pa]$.)

$$= [fg, fa] - [g_1p_1, pa]$$

(Since f, g, a, p are cyclic; $[fg, fa] = [gp, pa]$.)

$$= -[g_1p_1, pa] + [gp, pa]$$

(Since $g_1p_1 \parallel pg$; $[g_1p_1, pa] = [gp, pa]$.)

$$= 0$$

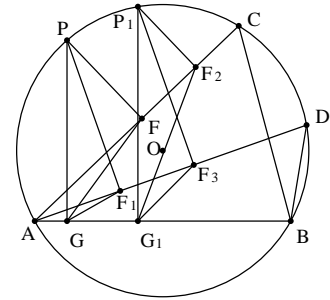


Figure 9

Example 9 (Simson Line, Extension 2) *The angle formed by the Simson lines of the same points with respect to two triangles inscribed in the same circle is the same for all positions of the point on the circle.*

Point order: $a, b, c, d, p, p_1, g, f, f_1, g_1, f_2, f_3$.

Hypotheses: $\text{cyclic}(a, b, c, d, p, p_1)$, $\text{perp}(p, f, a, c)$, $\text{coll}(f, a, c)$,
 $\text{perp}(p, g, a, b)$, $\text{coll}(g, a, b)$, $\text{perp}(p, f_1, a, d)$, $\text{coll}(f_1, a, d)$,
 $\text{perp}(p_1, g_1, a, b)$, $\text{coll}(g_1, a, b)$, $\text{perp}(p_1, f_2, a, c)$, $\text{perp}(p_1, f_3, a, d)$,
 $\text{coll}(f_2, a, c)$, $\text{coll}(f_3, a, d)$.

Conclusion: $\text{eqangle}(f, g, f_1, f_2, g_1, f_3)$.

The Machine Proof

$$\begin{aligned}
& [f_3g_1, f_2g_1] - [f_1g, fg] \\
& \quad (\text{Since } g_1, f_3, f_2, a \text{ are cyclic; } [f_3g_1, f_2g_1] = [f_3a, f_2a].) \\
& = [f_3a, f_2a] - [f_1g, fg] \\
& \quad (\text{Since } a, d, f_3 \text{ are collinear; } [f_3a, f_2a] = -[f_2a, da].) \\
& = -[f_2a, da] - [f_1g, fg] \\
& \quad (\text{Since } a, c, f_2 \text{ are collinear; } [f_2a, da] = -[da, ca].) \\
& = -[f_1g, fg] + [da, ca] \\
& \quad (\text{Since } g, f_1, f, p \text{ are cyclic; } [f_1g, fg] = [f_1p, fp].) \\
& = -[f_1p, fp] + [da, ca] \\
& \quad (\text{Since } f_1p \perp ad \text{ and } fp \perp ac \text{ } [f_1p, fp] = [da, ca].) \\
& = 0
\end{aligned}$$

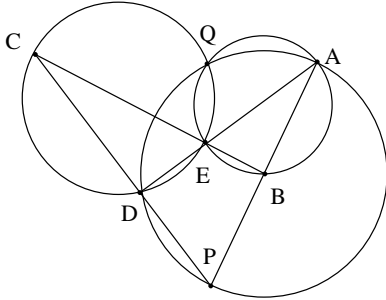


Figure 10

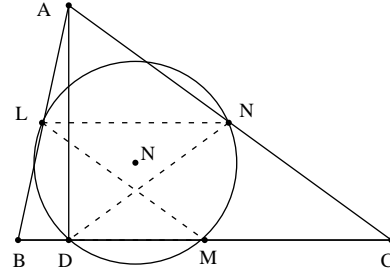


Figure 11

Example 10 (Miquel Point Theorem) *Four lines form four triangles. Show that the circumcircles of the four triangles pass through a common point.*

Point order: a, b, c, d, e, q, p .

Hypotheses: $\text{cyclic}(a, b, e, q)$, $\text{cyclic}(c, d, e, q)$, $\text{coll}(a, b, p)$,
 $\text{coll}(c, d, p)$, $\text{coll}(b, c, e)$, $\text{coll}(a, d, e)$.

Conclusion: $\text{cyclic}(d, a, p, q)$.

The Machine Proof

$$\begin{aligned}
& [pd, pa] - [qd, qa] \\
& \quad (\text{Since } c, d, p \text{ are collinear; } a, b, p \text{ are collinear; } [pd, pa] = [dc, ba].) \\
& = -[qd, qa] + [dc, ba] \\
& \quad (\text{Since } d, q, e, c \text{ are cyclic; } a, q, e, b \text{ are cyclic;}
\end{aligned}$$

$$\begin{aligned}
[qd, qa] &= [qd, qe] + [qe, qa] = -[ec, dc] + [eb, ba]. \\
&= [ec, dc] - [eb, ba] + [dc, ba] \\
&\quad (\text{Since } b, c, e \text{ are collinear; } [ec, dc] - [eb, ba] = -[dc, ba].) \\
&= 0
\end{aligned}$$

Example 11 (Nine Point Circle Theorem) *Let the midpoints of the sides AB, BC , and CA of $\triangle ABC$ be L, M , and N , and AD the altitude on BC . Show that L, M, N , and D are on the same circle.*

Point order: a, b, c, d, l, m, n .

Hypotheses: $\text{midpoint}(m, b, c)$, $\text{midpoint}(n, a, c)$, $\text{midpoint}(l, a, b)$, $\text{foot}(d, a, b, c)$.

Conclusion: $\text{cyclic}(l, d, m, n)$.

The Machine Proof

$$\begin{aligned}
&-[nl, nd] + [ml, md] \\
&\quad (\text{Since } nl \parallel bc, [nl, nd] = -[nd, cb].) \\
&= [nd, cb] + [ml, md] \\
&\quad (\text{Since } d, c, a \text{ have equal distance from point } n, \\
&\quad [nd, cb] = [nd, dc] = [da, ca] + 1.) \\
&= [ml, md] + [da, ca] + 1 \\
&\quad (\text{Since } ml \parallel ca, [ml, md] + [da, ca] = -[md, da].) \\
&= -[md, da] + 1 \\
&\quad (\text{Since } md \perp da, [md, da] = 1.) \\
&= 0
\end{aligned}$$

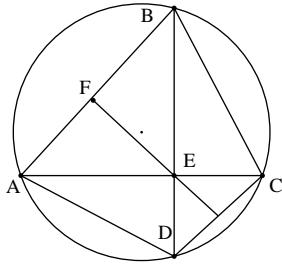


Figure 12

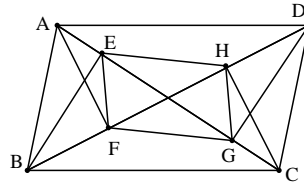


Figure 13

Example 12 (Theorem of Brahmagupta) *In a quadrilateral which is both orthodiagonal and cyclic the perpendicular from the point of intersection of the diagonals to a side bisects the side opposite.*

Point order: a, b, c, d, e, f .

Hypotheses: $\text{cyclic}(a, b, c, d)$, $\text{perp}(a, c, b, d)$, $\text{coll}(e, a, c)$, $\text{coll}(e, b, d)$, $\text{midpoint}(f, a, b)$.

Conclusion: $\text{perp}(e, f, c, d)$.

The Machine Proof

$$[fe, dc] + 1$$

$$\text{(Since } \text{circumcenter}(f, e, a, b); [fe, dc] = [fe, ea] + [ea, dc] = [eb, ba] + [ea, dc] + 1.)$$

$$= [eb, ba] + [ea, dc]$$

$$\text{(Since } b, d, e \text{ are collinear; } [eb, ba] = [db, ba].)$$

$$= [ea, dc] + [db, ba]$$

$$\text{(Since } a, c, e \text{ are collinear; } [ea, dc] = -[dc, ca].)$$

$$= -[dc, ca] + [db, ba]$$

$$\text{(Since } c, d, a, b \text{ are cyclic; } [dc, ca] = [db, ba].)$$

$$= 0$$

Example 13 *Let $ABCD$ be a parallelogram. Then the feet from A, B, C, D to the diagonals of the parallelogram form a parallelogram.*

Point order: a, b, c, d, e, f, h, g .

Hypotheses: $\text{para}(d, a, b, c)$, $\text{para}(d, c, a, b)$, $\text{foot}(e, b, a, c)$,
 $\text{foot}(f, a, b, d)$, $\text{foot}(h, c, b, d)$, $\text{foot}(g, d, a, c)$.

Conclusion: $\text{para}(e, f, h, g)$.

The Machine Proof

$$-[gh, fe]$$

$$\text{(Since } c, e, g \text{ are collinear; } g, h, c, d \text{ are cyclic;}$$

$$[gh, fe] = [gh, gc] + [ce, fe] = [hd, dc] - [fe, ec].)$$

$$= -[hd, dc] + [fe, ec]$$

$$\text{(Since } b, d, h \text{ are collinear; } [hd, dc] = -[dc, db].)$$

$$= [fe, ec] + [dc, db]$$

$$\text{(Since } a, c, e \text{ are collinear; } e, f, a, b \text{ are cyclic; } [fe, ec] = [fb, ba].)$$

$$= [fb, ba] + [dc, db]$$

$$\text{(Since } dc \parallel ab; [dc, db] = -[db, ba].)$$

$$= [fb, ba] - [db, ba]$$

$$\text{(Since } b, d, f \text{ are collinear; } [fb, ba] = [db, ba].)$$

$$= 0$$

$$\begin{aligned}
&= -[kh, fc] + [hg, ig] - 1 \\
&\quad (\text{Since } fc \perp ab \text{ } [kh, fc] = [kh, ba] + 1.) \\
&= -[kh, ba] + [hg, ig] \\
&\quad (\text{Since } a, b, k \text{ are collinear; } a, b, f \text{ are collinear;} \\
&\quad k, h, f, e \text{ are cyclic; } [kh, ba] = [kh, kf] = [he, ef].) \\
&= [hg, ig] - [he, ef] \\
&\quad (\text{Since } c, e, h \text{ are collinear; } h, g, c, f \text{ are cyclic;} \\
&\quad [hg, ig] = [hg, hc] + [ce, ig] = -[ig, ec] + [gf, fc].) \\
&= -[he, ef] - [ig, ec] + [gf, fc] \\
&\quad (\text{Since } c, e, h \text{ are collinear; } -[he, ef] - [ig, ec] = -[ig, ef].) \\
&= -[ig, ef] + [gf, fc] \\
&\quad (\text{Since } a, f, i \text{ are collinear; } i, g, f, d \text{ are cyclic;} \\
&\quad [ig, ef] = [ig, if] + [fa, ef] = [gd, df] - [ef, fa].) \\
&= -[gd, df] + [gf, fc] + [ef, fa] \\
&\quad (\text{Since } b, c, d, g \text{ are collinear; } [gd, df] = -[df, cb].) \\
&= [gf, fc] + [ef, fa] + [df, cb] \\
&\quad (\text{Since } gf \parallel ad; [gf, fc] = [da, fc].) \\
&= [ef, fa] + [df, cb] + [da, fc] \\
&\quad (\text{Since } a, b, f \text{ are collinear; } f, e, b, c \text{ are cyclic; } [ef, fa] = [ec, cb].) \\
&= [ec, cb] + [df, cb] + [da, fc] \\
&\quad (\text{Since } b, c, d \text{ are collinear; } d, f, c, a \text{ are cyclic;} \\
&\quad [df, cb] = [df, dc] = [fa, ca].) \\
&= [ec, cb] + [da, fc] + [fa, ca] \\
&\quad (\text{Since } a, c, e \text{ are collinear; } [ec, cb] + [fa, ca] = [fa, cb].) \\
&= [da, fc] + [fa, cb] \\
&\quad (\text{Since } a, b, f \text{ are collinear; } [fa, cb] = -[cb, ba].) \\
&= [da, fc] - [cb, ba] \\
&\quad (\text{Since } da \perp bc \text{ } fc \perp ab \text{ } [da, fc] = [cb, ba].) \\
&= 0
\end{aligned}$$

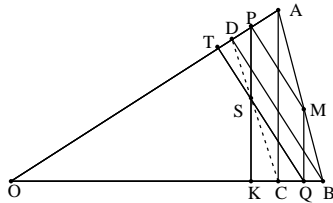


Figure 16

Example 16 *M is a point on side AB of triangle OAB. C and D are the feet of the altitude AC and BD. P and Q are the feet of the perpendiculars from M to OA and OB respectively. T and K are the feet of the perpendiculars from Q and P to OA and OB respectively. S = QP ∩ PK. Show that OS ⊥ PQ.*

Point order: $o, a, b, c, d, m, p, q, k, t, s$.

Hypotheses: $\text{foot}(c, a, o, b)$, $\text{foot}(d, b, o, a)$, $\text{coll}(m, a, b)$, $\text{foot}(p, m, a, o)$,

$\text{foot}(q, m, o, b), \text{foot}(t, q, o, a), \text{foot}(k, p, o, b), \text{coll}(s, q, t), \text{coll}(s, p, k).$

Conclusion: $\text{perp}(o, s, p, q).$

The Machine Proof

$$[so, qp] + 1$$

$$\begin{aligned} & \text{(Since } q, s, t \text{ are collinear; } s, o, t, k \text{ are cyclic; } [so, qp] = [so, st] + [tq, qp] = -[tk, ko] + [tq, qp].) \\ & = -[tk, ko] + [tq, qp] + 1 \end{aligned}$$

$$\begin{aligned} & \text{(Since } ko \perp ac; [tk, ko] = [tk, ca] + 1.) \\ & = -[tk, ca] + [tq, qp] \end{aligned}$$

$$\begin{aligned} & \text{(Since } ca \perp ob; [tk, ca] = [tk, bo] + 1.) \\ & = -[tk, bo] + [tq, qp] - 1 \end{aligned}$$

$$\begin{aligned} & \text{(Since } d, p, t \text{ are collinear; } t, k, p, q \text{ are cyclic; } [tk, bo] = [tk, tp] + [pd, bo] = [kq, qp] + [pd, bo].) \\ & = [tq, qp] - [kq, qp] - [pd, bo] - 1 \end{aligned}$$

$$\begin{aligned} & \text{(Since } kq \parallel bo; -[kq, qp] - [pd, bo] = [qp, pd].) \\ & = [tq, qp] + [qp, pd] - 1 \end{aligned}$$

$$\begin{aligned} & \text{(Since } qp \parallel qp; [tq, qp] + [qp, pd] = [tq, pd].) \\ & = [tq, pd] - 1 \end{aligned}$$

$$\begin{aligned} & \text{(Since } tq \perp pd; [tq, pd] = 1.) \\ & = 0 \end{aligned}$$

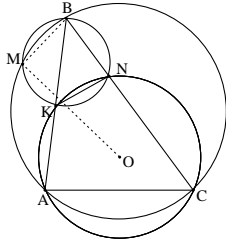


Figure 17

Example 17 *A, C, K, and N are four points on a circle. $B = AK \cap CN$. M is the intersection of the circumcircles of triangles BKN and BAC . Show that K, O, C , and M are cyclic.*

Point order: $a, c, k, n, o, b, m.$

Hypotheses: $\text{circumcenter}(o, a, c, k, n), \text{coll}(b, a, k), \text{coll}(b, c, n), \text{cyclic}(m, a, b, c), \text{cyclic}(m, b, k, n).$

Conclusion: $\text{cyclic}(k, o, c, m).$

The Machine Proof

$$-[mk, mc] + [ok, oc]$$

$$\begin{aligned} & \text{(Since } k, m, b, n \text{ are cyclic; } c, m, b, a \text{ are cyclic;} \\ & [mk, mc] = [mk, mb] + [mb, mc] = -[bn, nk] + [ba, ca].) \end{aligned}$$

$$= [bn, nk] - [ba, ca] + [ok, oc]$$

$$\begin{aligned} & \text{(Since } b, c, n \text{ are collinear; } [bn, nk] = -[nk, nc].) \\ & = -[ba, ca] + [ok, oc] - [nk, nc] \end{aligned}$$

$$\begin{aligned} & \text{(Since } a, b, k \text{ are collinear; } [ba, ca] = [ka, ca].) \\ & = [ok, oc] - [nk, nc] - [ka, ca] \end{aligned}$$

$$\begin{aligned} & \text{(Since } \text{circumcenter}(o, k, a, c); [ok, oc] = [ok, ka] + [ka, oc] = -[oc, ka] + [kc, ca] + 1.) \\ & = -[oc, ka] - [nk, nc] + [kc, ca] - [ka, ca] + 1 \end{aligned}$$

$$\text{(Since } ka \parallel ka; -[oc, ka] - [ka, ca] = -[oc, ca].)$$

$$\begin{aligned}
&= -[oc, ca] + [kc, ca] - [nk, nc] + 1 \\
&\quad (\text{Since } \text{circumcenter}(o, c, a, n); [oc, ca] = [oc, ca] = [nc, na] + 1.) \\
&= -[nc, na] + [kc, ca] - [nk, nc] \\
&\quad (\text{Since } nc \parallel nc; -[nc, na] - [nk, nc] = -[nk, na].) \\
&= -[nk, na] + [kc, ca] \\
&\quad (\text{Since } n, k, a, c \text{ are cyclic; } [nk, na] = [kc, ca].) \\
&= 0
\end{aligned}$$

Example 18 ¹ The same as Example 17. Show that $BM \perp MO$.

Point order: a, c, k, n, o, b, m .

Hypotheses: $\text{circumcenter}(o, a, c, k, n)$, $\text{coll}(b, a, k)$, $\text{coll}(b, c, n)$,
 $\text{cyclic}(k, o, c, m)$, $\text{cyclic}(m, a, b, c)$, $\text{cyclic}(m, b, k, n)$.

Conclusion: $\text{perp}(b, m, m, o)$.

The Machine Proof

$$\begin{aligned}
&[mb, mo] + 1 \\
&\quad (\text{Since } b, m, c, a \text{ are cyclic; } o, m, c, k \text{ are cyclic;} \\
&\quad [mb, mo] = [mb, mc] + [mc, mo] = [ba, ca] - [ok, kc].) \\
&= [ba, ca] - [ok, kc] + 1 \\
&\quad (\text{Since } a, b, k \text{ are collinear; } [ba, ca] = [ka, ca].) \\
&= -[ok, kc] + [ka, ca] + 1 \\
&\quad (\text{Since } \text{circumcenter}(o, k, c, a); [ok, kc] = [ok, kc] = [ka, ca] + 1.) \\
&= 0
\end{aligned}$$

Example 19 Two circles O and Q meet in two points A and B . A line passing through A meets circles O and Q in C and E . A line passing through B meets circles O and Q in D and F . Show that $CD \parallel EF$.

Point order: a, b, c, d, e, f .

Hypotheses: $\text{cyclic}(a, b, c, d)$, $\text{cyclic}(a, b, e, f)$, $\text{coll}(a, c, e)$, $\text{coll}(b, d, f)$.

Conclusion: $\text{para}(c, d, e, f)$.

The Machine Proof

$$\begin{aligned}
&-[fe, dc] \\
&\quad (\text{Since } b, d, f \text{ are collinear; } f, e, b, a \text{ are cyclic;} \\
&\quad [fe, dc] = [fe, fb] + [bd, dc] = [ea, ba] - [dc, db].) \\
&= -[ea, ba] + [dc, db] \\
&\quad (\text{Since } a, c, e \text{ are collinear; } [ea, ba] = [ca, ba].)
\end{aligned}$$

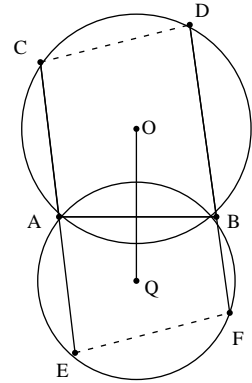


Figure 19

¹This example is from the 1985 International Mathematical Olympiad

$$\begin{aligned}
&= [dc, db] - [ca, ba] \\
&\quad (\text{Since } d, c, b, a \text{ are cyclic; } [dc, db] = [ca, ba].) \\
&= 0
\end{aligned}$$

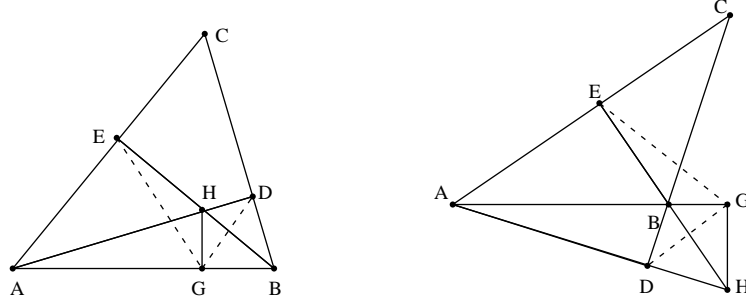


Figure 20

Example 20 In triangle ABC , two altitudes AD and BE meet in H . G is the foot of the perpendicular from point H to AB . Show that $\angle[DG, GH] = \angle[HG, GE]$.

Point order: a, b, c, d, e, h, g .

Hypotheses: $\text{foot}(e, b, a, c)$, $\text{foot}(d, a, b, c)$, $\text{coll}(h, e, b)$, $\text{coll}(h, a, d)$, $\text{foot}(g, h, a, b)$.

Conclusion: $\text{eqangle}(d, g, h, h, g, e)$.

The Machine Proof

$$\begin{aligned}
&-[gh, ge] - [gh, gd] \\
&\quad (\text{Since } gh \perp ab; [gh, ge] = -[ge, ba] + 1.) \\
&= -[gh, gd] + [ge, ba] - 1 \\
&\quad (\text{Since } gh \perp ab; [gh, gd] = -[gd, ba] + 1.) \\
&= [ge, ba] + [gd, ba] \\
&\quad (\text{Since } a, b, g \text{ are collinear; } g, e, a, h \text{ are cyclic;} \\
&\quad [ge, ba] = [ge, ga] = [he, ha].) \\
&= [gd, ba] + [he, ha] \\
&\quad (\text{Since } a, b, g \text{ are collinear; } g, d, b, h \text{ are cyclic;} \\
&\quad [gd, ba] = [gd, gb] = [hd, hb].) \\
&= [he, ha] + [hd, hb] \\
&\quad (\text{Since } b, e, h \text{ are collinear; } a, d, h \text{ are collinear; } [he, ha] = [eb, da].) \\
&= [hd, hb] + [eb, da] \\
&\quad (\text{Since } hb \parallel eb; [hd, hb] + [eb, da] = [hd, da].) \\
&= [hd, da] \\
&\quad (\text{Since } a, d, h \text{ are collinear; } [hd, da] = 0.) \\
&= 0
\end{aligned}$$

Example 21 The circumcenter of triangle ABC is O . AD is the altitude on side BC . Show that $\angle[AO, DA] = \angle[BA, BC] - \angle[BC, CA]$.

Point order: a, b, c, o, d .

Hypotheses: $\text{cong}(o, a, o, b)$, $\text{cong}(o, a, o, c)$, $\text{coll}(d, b, c)$, $\text{perp}(a, d, b, c)$.

Conclusion: $[ao, ad] - [ab, bc] + [bc, ca] = 0$.

The Machine Proof

$$-[da, oa] + [cb, ca] + [cb, ba]$$

(Since $da \perp bc$; $[da, oa] = -[oa, cb] + 1$.)

$$= [oa, cb] + [cb, ca] + [cb, ba] - 1$$

(Since $cb \parallel cb$; $[oa, cb] + [cb, ca] = [oa, ca]$.)

$$= [oa, ca] + [cb, ba] - 1$$

(Since $\text{circumcenter}(o, a, c, b)$; $[oa, ca] = [oa, ac] = -[cb, ba] + 1$.)

$$= 0$$

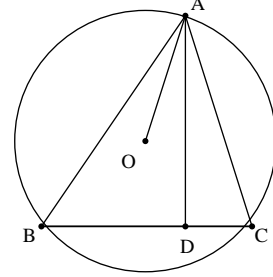


Figure 21

Example 22 In a circle, the lines joining the midpoints of two arcs AB and AC meet line AB and AC at D and E . Show that $AD = AE$.

Point order: a, m, n, o, p, q, d, e .

Hypotheses: $\text{cong}(o, a, o, n)$, $\text{cong}(o, a, o, m)$, $\text{coll}(p, o, n)$, $\text{perp}(p, a, o, n)$, $\text{coll}(q, o, m)$, $\text{perp}(q, a, o, m)$, $\text{coll}(e, a, p)$, $\text{coll}(e, n, m)$, $\text{coll}(d, a, q)$, $\text{coll}(d, n, m)$.

Conclusion: $[ad, ed] + [ae, ed]$.

The Machine Proof

$$-[ed, ea] - [ed, da]$$

(Since d, e, m, n are collinear; a, e, p are collinear; $[ed, ea] = -[pa, nm]$.)

$$= -[ed, da] + [pa, nm]$$

(Since d, e, m, n are collinear; $[ed, da] = -[da, nm]$.)

$$= [da, nm] + [pa, nm]$$

(Since a, d, q are collinear; $[da, nm] = [qa, nm]$.)

$$= [qa, nm] + [pa, nm]$$

(Since $qa \perp mo$; $[qa, nm] = [om, nm] + 1$.)

$$= [pa, nm] + [om, nm] + 1$$

(Since $pa \perp no$; $[pa, nm] = [on, nm] + 1$.)

$$= [on, nm] + [om, nm]$$

(Since $\text{circumcenter}(o, n, m, a)$; $[on, nm] = [on, nm] = [na, ma] + 1$.)

$$= [om, nm] + [na, ma] + 1$$

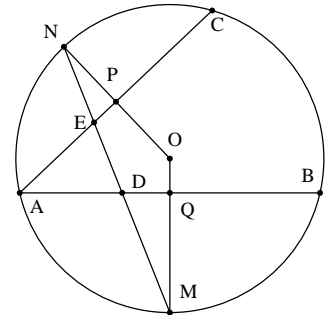


Figure 22

$$\begin{aligned}
& (\text{Since } \text{circumcenter}(o, m, n, a); [om, nm] = [om, mn] = -[na, ma] + 1.) \\
& = 0
\end{aligned}$$

Example 23 From the midpoint C of arc AB of a circle, two secants are drawn meeting line AB at F , G , and the circle at D and E . Show that F , D , E , and G are on the same circle.

Point order: a, c, d, e, o, m, f, g .

Hypotheses: $\text{cong}(o, a, o, c)$, $\text{cong}(o, a, o, d)$, $\text{cong}(o, a, o, e)$,
 $\text{coll}(m, c, o)$, $\text{perp}(m, a, c, o)$, $\text{coll}(f, a, m)$, $\text{coll}(f, c, d)$,
 $\text{coll}(g, a, m)$, $\text{coll}(g, c, e)$.

Conclusion: $[ce, fg] + [cd, de]$.

The Machine Proof

$$\begin{aligned}
& -[gf, ec] - [ed, dc] \\
& \quad (\text{Since } a, f, g, m \text{ are collinear; } [gf, ec] = [ma, ec].) \\
& = -[ma, ec] - [ed, dc] \\
& \quad (\text{Since } ma \perp co; [ma, ec] = [oc, ec] + 1.) \\
& = -[oc, ec] - [ed, dc] - 1 \\
& \quad (\text{Since } \text{circumcenter}(o, c, e, d); [oc, ec] = [oc, ce] = -[ed, dc] + 1.) \\
& = 0
\end{aligned}$$

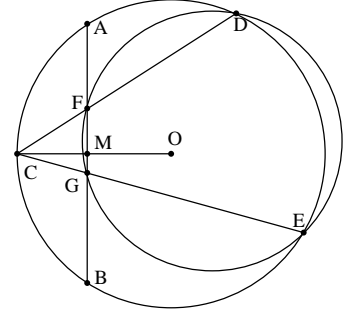


Figure 23

Example 24 Let Q, S and Y be three collinear points and (O, P) be a circle. Circles SPQ and YPQ meet circle (O, P) again at points R and X , respectively. Show that XY and RS meet on the circle (O, P) .

Point order: x, r, p, q, s, y, i .

Hypotheses: $\text{cyclic}(r, p, q, s)$, $\text{coll}(y, q, s)$, $\text{cyclic}(y, p, q, x)$, $\text{coll}(i, x, y)$, $\text{coll}(i, r, s)$.

Conclusion: $[xi, ri] + [rp, xp]$.

The Machine Proof

$$\begin{aligned}
& -[ir, ix] + [pr, px] \\
& \quad (\text{Since } i, r, s \text{ are collinear; } i, x, y \text{ are collinear; } [ir, ix] = -[yx, sr].) \\
& = [yx, sr] + [pr, px] \\
& \quad (\text{Since } q, s, y \text{ are collinear; } y, x, q, p \text{ are cyclic;} \\
& \quad [yx, sr] = [yx, yq] + [qs, sr] = [sq, sr] - [qp, px].) \\
& = [sq, sr] - [qp, px] + [pr, px] \\
& \quad (\text{Since } s, q, r, p \text{ are cyclic; } [sq, sr] = [qp, pr].) \\
& = [qp, pr] - [qp, px] + [pr, px] \\
& \quad (\text{Since } qp \parallel qp; [qp, pr] - [qp, px] = -[pr, px].) \\
& = 0
\end{aligned}$$

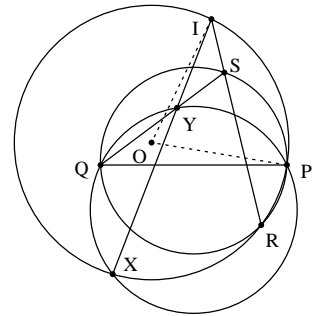


Figure 24

Example 25 *The nine-point circle cuts the sides of the triangle at angles $|B - C|$, $|C - A|$, and $|A - B|$.*

Point order: $a, b, c, f, m, q, p, l, s, n$.

Hypotheses: $\text{foot}(f, c, a, b)$, $\text{midpoint}(m, b, c)$, $\text{midpoint}(q, a, c)$,
 $\text{midpoint}(p, a, b)$, $\text{midpoint}(l, f, p)$, $\text{midpoint}(s, p, q)$, $\text{perp}(n, l, a, b)$, $\text{perp}(n, s, p, q)$.

Conclusion: $[bc, ab] + [ac, ab] + [fn, ln]$.

The Machine Proof

$$\begin{aligned}
& -[nl, nf] + [cb, ba] + [ca, ba] \\
& \quad (\text{Since } nl \parallel cf; [nl, nf] = -[nf, fc].) \\
& = [nf, fc] + [cb, ba] + [ca, ba] \\
& \quad (\text{Since } fc \perp ab; [nf, fc] = [nf, ba] + 1.) \\
& = [nf, ba] + [cb, ba] + [ca, ba] + 1 \\
& \quad (\text{Since circumcenter}(n, f, p, q); [nf, ba] = [nf, fp] = -[pq, qf] + 1.) \\
& = -[pq, qf] + [cb, ba] + [ca, ba] \\
& \quad (\text{Since } pq \parallel cb; -[pq, qf] + [cb, ba] = [qf, ba].) \\
& = [qf, ba] + [ca, ba] \\
& \quad (\text{Since circumcenter}(q, f, a, c); [qf, ba] = [qf, fa] = [fc, ca] + 1.) \\
& = [fc, ca] + [ca, ba] + 1 \\
& \quad (\text{Since } ca \parallel cf; [fc, ca] + [ca, ba] = [fc, ba].) \\
& = [fc, ba] + 1 \\
& \quad (\text{Since } fc \perp ba; [fc, ba] = 1.) \\
& = 0
\end{aligned}$$

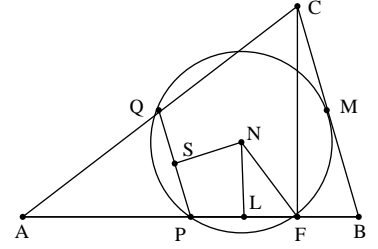


Figure 25

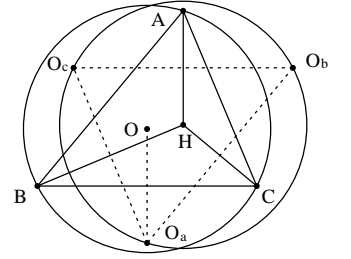


Figure 26

Example 26 *Let H be the orthocenter of triangle ABC . Then the circumcenters of the four triangles ABC , ABH , ACH , and HBC form a triangle congruent to ABC ; the sides of the two triangles are parallel.*

Point order: a, b, c, h, o, p, q, r .

Hypotheses: $\text{orthocenter}(h, a, b, c)$, $\text{circumcenter}(o, a, b, c)$, $\text{circumcenter}(r, b, c, h)$,
 $\text{circumcenter}(p, a, c, h)$, $\text{circumcenter}(q, a, b, h)$.

Conclusion: $\text{para}(p, c, q, b)$.

The Machine Proof

$$\begin{aligned}
& -[qb, pc] \\
& \quad (\text{Since circumcenter}(q, b, a, h); [qb, pc] = [qb, ba] + [ba, pc] = -[pc, ba] + [hb, ha] + 1.) \\
& = [pc, ba] - [hb, ha] - 1 \\
& \quad (\text{Since circumcenter}(p, c, a, h); [pc, ba] = [pc, ca] + [ca, ba] = [hc, ha] + [ca, ba] + 1.) \\
& = [hc, ha] - [hb, ha] + [ca, ba]
\end{aligned}$$

$$\begin{aligned}
& \text{(Since } hc \perp ab; ha \perp bc; [hc, ha] = -[cb, ba].\text{)} \\
& = -[hb, ha] - [cb, ba] + [ca, ba] \\
& \text{(Since } hb \perp ac; ha \perp bc; [hb, ha] = -[cb, ca].\text{)} \\
& = [cb, ca] - [cb, ba] + [ca, ba] \\
& \text{(Since } cb \parallel cb; [cb, ca] - [cb, ba] = -[ca, ba].\text{)} \\
& = 0
\end{aligned}$$

Example 27 Continuing from Example 26, show that the point H is the circumcenter of the triangle $O_aO_bO_c$.

Point order: a, b, c, h, o, p, q, r .

Hypotheses: $\text{orthocenter}(h, a, b, c)$, $\text{circumcenter}(o, a, b, c)$, $\text{circumcenter}(r, b, c, h)$, $\text{circumcenter}(p, a, c, h)$, $\text{circumcenter}(q, a, b, h)$.

Conclusion: $\text{pbisector}(h, p, q)$.

The Machine Proof

$$\begin{aligned}
& -[qp, qh] - [qp, ph] \\
& \text{(Since } qp \parallel bc; [qp, qh] = -[qh, cb].\text{)} \\
& = -[qp, ph] + [qh, cb] \\
& \text{(Since } qp \parallel bc; [qp, ph] = -[ph, cb].\text{)} \\
& = [qh, cb] + [ph, cb] \\
& \text{(Since } \text{circumcenter}(q, h, a, b); [qh, cb] = [qh, ha] + [ha, cb] = [hb, ba] + [ha, cb] + 1.\text{)} \\
& = [ph, cb] + [hb, ba] + [ha, cb] + 1 \\
& \text{(Since } \text{circumcenter}(p, h, a, c); [ph, cb] = [ph, ha] + [ha, cb] = [hc, ca] + [ha, cb] + 1.\text{)} \\
& = [hc, ca] + [hb, ba] + 2[ha, cb] \\
& \text{(Since } hc \perp ab; [hc, ca] = -[ca, ba] + 1.\text{)} \\
& = [hb, ba] + 2[ha, cb] - [ca, ba] + 1 \\
& \text{(Since } hb \perp ac; [hb, ba] = [ca, ba] + 1.\text{)} \\
& = 2[ha, cb] \\
& \text{(Since } ha \perp cb; [ha, cb] = 1.\text{)} \\
& = 0
\end{aligned}$$

Example 28 *The four projections of the foot of the altitude on a side of a triangle upon the other two sides and the other two altitudes are collinear.*

Point order: a, b, c, f, h, p, q, t .

Hypotheses: $\text{foot}(f, c, a, b)$, $\text{coll}(h, c, f)$, $\text{perp}(h, a, b, c)$,
 $\text{perp}(h, b, a, c)$, $\text{foot}(p, f, a, c)$, $\text{foot}(q, f, a, h)$, $\text{foot}(t, f, b, c)$.

Conclusion: $\text{coll}(p, q, t)$.

The Machine Proof

$$-[tp, qp]$$

(Since b, c, t are collinear; t, p, c, f are cyclic; $[tp, qp] = [tp, tc] + [cb, qp] = -[qp, cb] + [pf, fc]$.)

$$= [qp, cb] - [pf, fc]$$

(Since a, h, q are collinear; q, p, a, f are cyclic;

$$[qp, cb] = [qp, qa] + [ah, cb] = [pf, fa] + [ha, cb].)$$

$$= -[pf, fc] + [pf, fa] + [ha, cb]$$

(Since $pf \parallel pf$; $-[pf, fc] + [pf, fa] = [fc, fa]$.)

$$= [ha, cb] + [fc, fa]$$

(Since $ha \perp cb$; $[ha, cb] = 1$.)

$$= [fc, fa] + 1$$

(Since $fc \perp fa$; $[fc, fa] = 1$.)

$$= 0$$

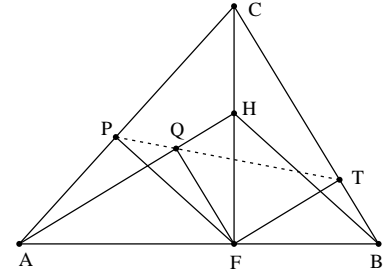


Figure 28

Example 29 *DP, DQ are the perpendiculars from the foot D of the altitude AD of the triangle ABC upon the sides AC, AB. Prove that the points B, C, P, Q are cyclic.*

Point order: a, b, c, d, p, q .

Hypotheses: $\text{foot}(d, a, b, c)$, $\text{foot}(p, d, a, c)$, $\text{foot}(q, d, a, b)$.

Conclusion: $\text{cyclic}(p, b, q, c)$.

The Machine Proof

$$[qp, qb] - [pc, cb]$$

(Since a, b, q are collinear; q, p, a, d are cyclic; $[qp, qb] = [pd, da]$.)

$$= [pd, da] - [pc, cb]$$

(Since $pd \perp ac$; $da \perp bc$; $[pd, da] = -[cb, ca]$.)

$$= -[pc, cb] - [cb, ca]$$

(Since $cb \parallel cb$; $-[pc, cb] - [cb, ca] = -[pc, ca]$.)

$$= -[pc, ca]$$

(Since a, c, p are collinear; $[pc, ca] = 0$.)

$$= 0$$

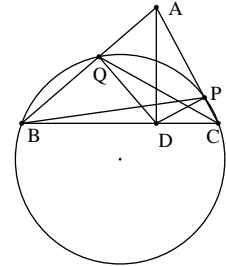


Figure 29

Example 30 *The radii of the circumcircle passing through the vertices of a triangle are perpendicular to the corresponding sides of the orthic triangle.*

Point order: a, b, c, e, f, o .

Hypotheses: $\text{circumcenter}(o, a, b, c)$, $\text{foot}(e, b, a, c)$, $\text{foot}(f, c, a, b)$.

Conclusion: $\text{perp}(o, a, e, f)$.

The Machine Proof

$$[oa, fe] + 1$$

$$\text{(Since } \text{circumcenter}(o, a, b, c); [oa, fe] = [oa, ab] + [ab, fe] = -[fe, ba] - [cb, ca] + 1.)$$

$$= -[fe, ba] - [cb, ca]$$

$$\text{(Since } a, b, f \text{ are collinear; } f, e, b, c \text{ are cyclic; } [fe, ba] = [fe, fb] = [ec, cb].)$$

$$= -[ec, cb] - [cb, ca]$$

$$\text{(Since } cb \parallel cb; -[ec, cb] - [cb, ca] = -[ec, ca].)$$

$$= -[ec, ca]$$

$$\text{(Since } a, c, e \text{ are collinear; } [ec, ca] = 0.)$$

$$= 0$$

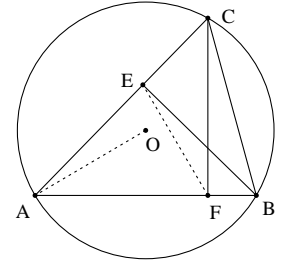


Figure 30

Example 31 *The mediators of the sides AC, AB of the triangle ABC meet the sides AB, AC in P and Q. Prove that the points B, C, P, Q lie on a circle.*

Point order: $a, b, c, o, b1, c1, p, q$.

Hypotheses: $\text{circumcenter}(o, a, b, c)$, $\text{midpoint}(c1, a, b)$, $\text{midpoint}(b1, a, c)$,

$\text{coll}(p, o, c1)$, $\text{coll}(q, o, b1)$, $\text{coll}(p, a, c)$, $\text{coll}(q, a, b)$.

Conclusion: $\text{cyclic}(b, c, p, q)$.

The Machine Proof

$$[qc, qb] - [pc, pb]$$

$$\text{(Since } qb \perp oc1; [qc, qb] = [qc, c1o] + 1.)$$

$$= [qc, c1o] - [pc, pb] + 1$$

$$\text{(Since } c1o \perp ab; [qc, c1o] = [qc, ba] + 1.)$$

$$= [qc, ba] - [pc, pb]$$

$$\text{(Since } qc = qa \text{ } a, b, q \text{ are collinear; } [qc, ba] = [qc, ca] + [ca, ba] = 2[ca, ba].)$$

$$= -[pc, pb] + 2[ca, ba]$$

$$\text{(Since } pc \perp ob1; [pc, pb] = -[pb, b1o] + 1.)$$

$$= [pb, b1o] + 2[ca, ba] - 1$$

$$\text{(Since } b1o \perp ac; [pb, b1o] = [pb, ca] + 1.)$$

$$= [pb, ca] + 2[ca, ba]$$

$$\text{(Since } pb = pa \text{ } a, c, p \text{ are collinear; } [pb, ca] = [pb, ba] + [ba, ca] = -2[ca, ba].)$$

$$= 0$$

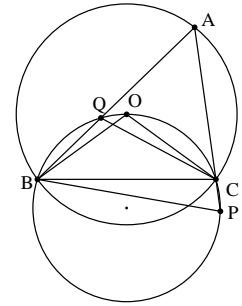


Figure 31

Example 32 *If two lines are antiparallel with respect to an angle, the perpendiculars dropped upon them from the vertex are isogonal in the angle considered.*

Point order: b, c, r, s, o, a, m, n .

Hypotheses: $\text{circumcenter}(o, b, c, r)$, $\text{cong}(o, b, o, s)$, $\text{coll}(a, r, b)$, $\text{coll}(a, s, c)$, $\text{foot}(m, a, r, s)$, $\text{foot}(n, a, b, c)$.

Conclusion: $\text{eqangle}(b, a, m, n, a, c)$.

The Machine Proof

$$\begin{aligned}
& -[na, ac] - [ma, ab] \\
& \quad (\text{Since } na \perp bc; [na, ac] = -[ac, cb] + 1.) \\
& = -[ma, ab] + [ac, cb] - 1 \\
& \quad (\text{Since } ma \perp rs; [ma, ab] = -[ab, sr] + 1.) \\
& = [ac, cb] + [ab, sr] \\
& \quad (\text{Since } a, c, s \text{ are collinear; } [ac, cb] = [sc, cb].) \\
& = [ab, sr] + [sc, cb] \\
& \quad (\text{Since } a, b, r \text{ are collinear; } [ab, sr] = -[sr, rb].) \\
& = -[sr, rb] + [sc, cb] \\
& \quad (\text{Since } r, s, b, c \text{ are cyclic; } [sr, rb] = [sc, cb].) \\
& = 0
\end{aligned}$$

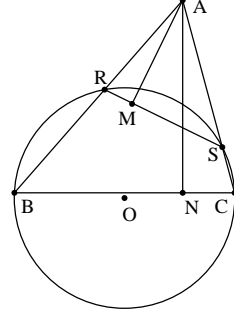


Figure 32

Example 33 *Show that the four perpendiculars to the sides of an angle at four concyclic points form a parallelogram whose opposite vertices lie on isogonal conjugate lines with respect to the given angle.*

Point order: a, b, c, d, i, e, f .

Hypotheses: $\text{cyclic}(a, b, c, d)$, $\text{coll}(i, a, b)$, $\text{coll}(i, c, d)$, $\text{perp}(e, d, d, c)$, $\text{perp}(e, a, a, b)$, $\text{perp}(f, b, b, a)$, $\text{perp}(f, c, c, d)$.

Conclusion: $\text{eqangle}(a, i, e, f, i, c)$.

The Machine Proof

$$\begin{aligned}
& -[fi, ic] - [ei, ia] \\
& \quad (\text{Since } i, f, c, b \text{ are cyclic; } [fi, ic] = [fb, cb].) \\
& = -[fb, cb] - [ei, ia] \\
& \quad (\text{Since } fb \parallel ae; [fb, cb] = [ea, cb].) \\
& = -[ei, ia] - [ea, cb] \\
& \quad (\text{Since } i, e, a, d \text{ are cyclic; } [ei, ia] = [ed, da].) \\
& = -[ed, da] - [ea, cb] \\
& \quad (\text{Since } ed \perp cd; [ed, da] = [dc, da] + 1.)
\end{aligned}$$

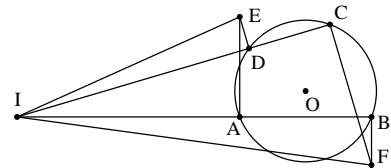


Figure 33

$$\begin{aligned}
&= -[ea, cb] - [dc, da] - 1 \\
&\quad (\text{Since } ea \perp ab; [ea, cb] = -[cb, ba] + 1.) \\
&= -[dc, da] + [cb, ba] \\
&\quad (\text{Since } d, c, a, b \text{ are cyclic; } [dc, da] = [cb, ba].) \\
&= 0
\end{aligned}$$

Example 34 *The perpendicular at the orthocenter H to the altitude HC of the triangle ABC meets the circumcircle of HBC in P . Show that $ABPH$ is a parallelogram.*

Point order: a, b, c, h, o, p .

Hypotheses: $\text{orthocenter}(h, a, b, c)$, $\text{circumcenter}(o, b, c, h)$, $\text{perp}(p, h, h, c)$, $\text{cong}(p, o, o, b)$.

Conclusion: $\text{para}(a, h, b, p)$.

The Machine Proof

$$\begin{aligned}
&-[pb, ha] \\
&\quad (\text{Since } ha \perp bc; [pb, ha] = [pb, cb] + 1.) \\
&= -[pb, cb] - 1 \\
&\quad (\text{Since } ph \perp hc; pb \perp bc; [pb, cb] = [pb, bc] + [bc, cb] = -[cb, cb] + 1.) \\
&= [cb, cb] \\
&\quad (\text{Since } [cb, cb] = 0.) \\
&= 0
\end{aligned}$$

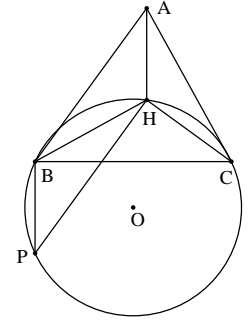


Figure 34

Example 35 *The segment of the altitude extended between the orthocenter and the second point of intersection with the circumcircle is bisected by the corresponding side of the triangle.*

Point order: a, b, c, o, d, e, h, k .

Hypotheses: $\text{circumcenter}(o, a, b, c)$, $\text{foot}(d, c, a, b)$, $\text{foot}(e, b, a, c)$, $\text{coll}(h, b, e)$, $\text{coll}(h, c, d)$, $\text{coll}(k, c, d)$, $\text{cong}(k, o, a, o)$.

Conclusion: $\text{pbisector}(a, h, k)$.

The Machine Proof

$$\begin{aligned}
&-[kh, ka] - [kh, ha] \\
&\quad (\text{Since } kh \perp ab; [kh, ka] = -[ka, ba] + 1.) \\
&= -[kh, ha] + [ka, ba] - 1 \\
&\quad (\text{Since } c, d, h, k \text{ are collinear; } [kh, ha] = -[ha, dc].) \\
&= [ka, ba] + [ha, dc] - 1 \\
&\quad (\text{Since } c, d, k \text{ are collinear; } k, a, c, b \text{ are cyclic;} \\
&\quad [ka, ba] = [ka, kc] + [cd, ba] = [dc, ba] - [cb, ba].) \\
&= [ha, dc] + [dc, ba] - [cb, ba] - 1
\end{aligned}$$

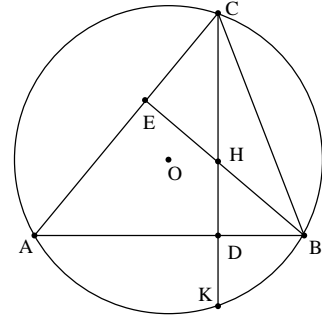


Figure 35

$$\begin{aligned}
& \text{(Since } dc \parallel dc; [ha, dc] + [dc, ba] = [ha, ba].\text{)} \\
& = [ha, ba] - [cb, ba] - 1 \\
& \text{(Since } b, e, h \text{ are collinear; } h, a, e, d \text{ are cyclic;} \\
& \quad [ha, ba] = [ha, he] + [eb, ba] = -[ed, da] + [eb, ba].\text{)} \\
& = -[ed, da] + [eb, ba] - [cb, ba] - 1 \\
& \text{(Since } a, b, d \text{ are collinear; } d, e, b, c \text{ are cyclic; } [ed, da] = [ec, cb].\text{)} \\
& = -[ec, cb] + [eb, ba] - [cb, ba] - 1 \\
& \text{(Since } cb \parallel cb; -[ec, cb] - [cb, ba] = -[ec, ba].\text{)} \\
& = -[ec, ba] + [eb, ba] - 1 \\
& \text{(Since } a, c, e \text{ are collinear; } [ec, ba] = [ca, ba].\text{)} \\
& = [eb, ba] - [ca, ba] - 1 \\
& \text{(Since } eb \perp ac; [eb, ba] = [ca, ba] + 1.\text{)} \\
& = 0
\end{aligned}$$

Example 36 *The circumcircle of the triangle formed by two vertices and the orthocenter of a given triangle is equal to the circumcircle of the given triangle.*

Point order: $a, b, c, o, d, e, h, o1$.

Hypotheses: $\text{circumcenter}(o, a, b, c)$, $\text{foot}(d, c, a, b)$, $\text{foot}(e, b, a, c)$,
 $\text{coll}(h, b, e)$, $\text{coll}(h, c, d)$, $\text{circumcenter}(o1, a, b, h)$.

Conclusion: $\text{pbisector}(a, o, o1)$.

The Machine Proof

$$\begin{aligned}
& -[o1o, o1a] - [o1o, oa] \\
& \text{(Since } o1o \parallel cd; [o1o, o1a] = -[o1a, dc].\text{)} \\
& = -[o1o, oa] + [o1a, dc] \\
& \text{(Since } o1o \parallel cd; [o1o, oa] = [dc, oa].\text{)} \\
& = [o1a, dc] - [dc, oa] \\
& \text{(Since } dc \perp ab; [o1a, dc] = [o1a, ba] + 1.\text{)} \\
& = [o1a, ba] - [dc, oa] + 1 \\
& \text{(Since } \text{circumcenter}(o1, a, b, h); [o1a, ba] = [o1a, ab] = -[hb, ha] + 1.\text{)} \\
& = -[hb, ha] - [dc, oa] \\
& \text{(Since } hb \perp ac; [hb, ha] = -[ha, ca] + 1.\text{)} \\
& = [ha, ca] - [dc, oa] - 1 \\
& \text{(Since } b, e, h \text{ are collinear; } h, a, e, d \text{ are cyclic;} \\
& \quad [ha, ca] = [ha, he] + [eb, ca] = -[ed, da] + [eb, ca].\text{)} \\
& = -[ed, da] + [eb, ca] - [dc, oa] - 1 \\
& \text{(Since } a, b, d \text{ are collinear; } d, e, b, c \text{ are cyclic; } [ed, da] = [ec, cb].\text{)} \\
& = -[ec, cb] + [eb, ca] - [dc, oa] - 1 \\
& \text{(Since } a, c, e \text{ are collinear; } [ec, cb] = -[cb, ca].\text{)}
\end{aligned}$$

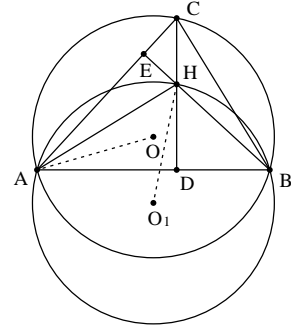


Figure 36

$$\begin{aligned}
&= [eb, ca] - [dc, oa] + [cb, ca] - 1 \\
&\quad (\text{Since } eb \perp ca; [eb, ca] = 1.) \\
&= -[dc, oa] + [cb, ca] \\
&\quad (\text{Since } dc \perp ab; [dc, oa] = -[oa, ba] + 1.) \\
&= [oa, ba] + [cb, ca] - 1 \\
&\quad (\text{Since circumcenter}(o, a, b, c); [oa, ba] = [oa, ab] = -[cb, ca] + 1.) \\
&= 0
\end{aligned}$$

Example 37 A vertex of a triangle is the midpoint of the arc determined on its circumcircle by the two altitudes, produced, issued from the other two vertices.

Point order: $a, b, c, o, h, a1, c1$.

Hypotheses: circumcenter(o, a, b, c), orthocenter(h, a, b, c), coll($a1, a, h$), cong($o, a, o, a1$), coll($c1, c, h$), cong($o, c, o, c1$).

Conclusion: pbisector($b, a1, c1$).

The Machine Proof

$$\begin{aligned}
&-[c1a1, c1b] - [c1a1, a1b] \\
&\quad (\text{Since } c1, a1, b, a \text{ are cyclic; } [c1a1, c1b] = [a1a, ba].) \\
&= -[c1a1, a1b] - [a1a, ba] \\
&\quad (\text{Since } a1, c1, b, a \text{ are cyclic; } [c1a1, a1b] = [c1a, ba].) \\
&= -[c1a, ba] - [a1a, ba] \\
&\quad (\text{Since } c, c1, h \text{ are collinear; } c1, a, c, a1 \text{ are cyclic;} \\
&\quad [c1a, ba] = [c1a, c1c] + [ch, ba] = -[a1c, a1a] + [hc, ba].) \\
&= [a1c, a1a] - [a1a, ba] - [hc, ba] \\
&\quad (\text{Since } a1a \perp bc; [a1c, a1a] = [a1c, cb] + 1.) \\
&= [a1c, cb] - [a1a, ba] - [hc, ba] + 1 \\
&\quad (\text{Since } c, a1, b, a \text{ are cyclic; } [a1c, cb] = [a1a, ba].) \\
&= -[hc, ba] + 1 \\
&\quad (\text{Since } hc \perp ba; [hc, ba] = 1.) \\
&= 0
\end{aligned}$$

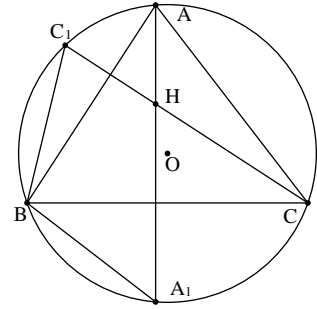


Figure 37

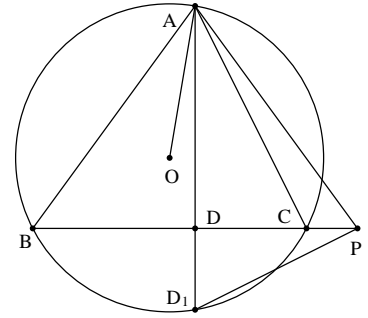


Figure 38

Example 38 If D_1 is the second point of intersection of the altitude ADD_1 of the triangle ABC with the circumcircle, center O , and P is the trace on BC of the perpendicular from D_1 to AC , show that the lines AP , AO make equal angles with the bisector of the angle DAC .

Point order: $a, b, c, o, d, d1, p$.

Hypotheses: circumcenter(o, a, b, c), foot(d, a, b, c), coll($d1, a, d$),

Example 40 Show that a parallel through a tritan-
gent center to a side of a triangle is equal to the sum,
or difference, of the two segments on the other two
sides of the triangle between the two parallel lines
considered.

Point order: a, b, c, i, m .

Hypotheses: $\text{incenter}(i, a, b, c)$, $\text{coll}(m, a, c)$, $\text{para}(i, m, a, b)$.

Conclusion: $\text{pbisector}(m, a, i)$.

The Machine Proof

$$[mi, ia] + [ma, ia]$$

(Since $mi \parallel ab$; $[mi, ia] = -[ia, ba]$.)

$$= [ma, ia] - [ia, ba]$$

(Since a, c, m are collinear; $[ma, ia] = -[ia, ca]$.)

$$= -[ia, ca] - [ia, ba]$$

(Since i is the incenter of triangle a, c, b $[ia, ca] = -[ia, ba]$.)

$$= 0$$

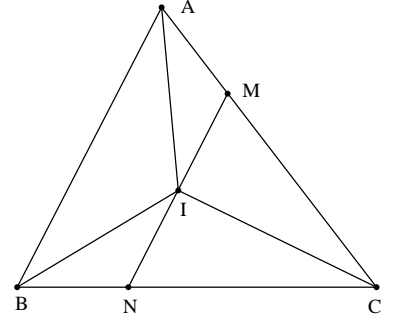


Figure 40

Example 41 The projection of the vertex B of the
triangle ABC upon the internal bisector of the angle
 A lies on the line joining the points of contact of the
incircle with the sides BC and AC . State and prove
an analogous proposition for the external bisectors.

Point order: a, b, c, i, x, y, l .

Hypotheses: $\text{incenter}(i, a, b, c)$, $\text{foot}(x, i, b, c)$, $\text{foot}(y, i, a, c)$, $\text{foot}(l, b, a, i)$.

Conclusion: $\text{coll}(x, y, l)$.

The Machine Proof

$$-[lx, yx]$$

(Since a, i, l are collinear; l, x, i, b are cyclic;

$$[lx, yx] = [lx, li] + [ia, yx] = -[yx, ia] + [xb, ib].)$$

$$= [yx, ia] - [xb, ib]$$

(Since a, c, y are collinear; y, x, c, i are cyclic; $[yx, ia] = [yx, yc] + [ca, ia] = [xi, ic] - [ia, ca]$.)

$$= [xi, ic] - [xb, ib] - [ia, ca]$$

(Since $xi \perp bc$; $[xi, ic] = -[ic, cb] + 1$.)

$$= -[xb, ib] - [ic, cb] - [ia, ca] + 1$$

(Since $xb \parallel cb$; $-[xb, ib] - [ic, cb] = -[ic, ib]$.)

$$= -[ic, ib] - [ia, ca] + 1$$

(Since i is the incenter of triangle a, c, b $[ic, ib] = [ia, ba] + 1$.)

$$= -[ia, ca] - [ia, ba]$$

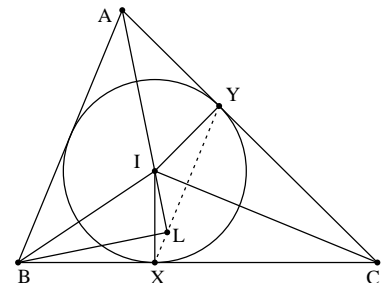


Figure 41

(Since i is the incenter of triangle a, c, b $[ia, ca] = -[ia, ba]$.)
 $= 0$

Example 42 *The midpoint of a side of a triangle, the foot of the altitude on this side, and the projections of the ends of this side upon the internal bisector of the opposite angle are four cyclic points.*

Point order: $a, b, c, i, a1, x, x1, y, d$.

Hypotheses: $\text{incenter}(i, a, b, c)$, $\text{midpoint}(a1, b, c)$, $\text{foot}(x, b, a, i)$,
 $\text{midpoint}(x, b, x1)$, $\text{coll}(x1, a, c)$, $\text{foot}(y, c, a, i)$, $\text{foot}(d, a, b, c)$.

Conclusion: $\text{cyclic}(x, y, d, a1)$.

The Machine Proof

$-[dy, yx] + [da1, xa1]$

(Since $yx \perp bx$; $[dy, yx] = [dy, xb] + 1$.)

$= -[dy, xb] + [da1, xa1] - 1$

(Since $xb \perp ai$; $[dy, xb] = [dy, ia] + 1$.)

$= -[dy, ia] + [da1, xa1]$

(Since b, c, d are collinear; d, y, c, a are cyclic; $[dy, ia] = [dy, dc] + [cb, ia] = [ya, ca] - [ia, cb]$.)

$= [da1, xa1] - [ya, ca] + [ia, cb]$

(Since $ya \parallel ia$; $-[ya, ca] + [ia, cb] = -[cb, ca]$.)

$= [da1, xa1] - [cb, ca]$

(Since $xa1 \parallel ca$; $[da1, xa1] - [cb, ca] = [da1, cb]$.)

$= [da1, cb]$

(Since $da1 \parallel cb$; $[da1, cb] = 0$.)

$= 0$

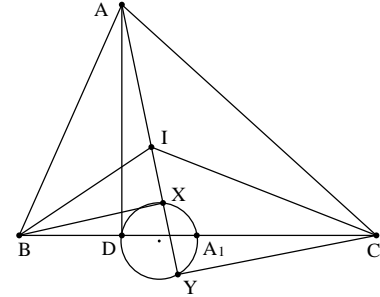


Figure 42

Example 43 *On the hypotenuse AB of right triangle ABC a square ABFE is erected. Let P be the intersection of the diagonals AF and BE of ABFE. Show that $\angle ACP = \angle PCB$.*

Point order: a, b, c, p .

Hypotheses: $\text{perp}(c, a, c, b)$, $\text{perp}(p, a, p, b)$, $\text{cong}(p, a, p, b)$.

Conclusion: $\text{eqangle}(a, c, p, p, c, b)$.

The Machine Proof

$-[pc, cb] - [pc, ca]$

(Since c, p, b, a are cyclic; $[pc, cb] = [pa, ba]$.)

$= -[pc, ca] - [pa, ba]$

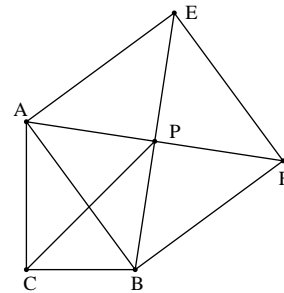


Figure 43

$$\begin{aligned}
& \text{(Since } c, p, a, b \text{ are cyclic; } [pc, ca] = [pb, ba].\text{.)} \\
& = -[pb, ba] - [pa, ba] \\
& \text{(Since } pb = pa \text{ } [pb, ba] = -[pa, ba].\text{.)} \\
& = 0
\end{aligned}$$

Example 44 *In the cyclic quadrilateral $ABCD$ the perpendicular to AB at A meets CD in A_1 , and the perpendicular to CD at C meets AB in C_1 . Show that the line A_1C_1 is parallel to the diagonal BD .*

Point order: a, b, c, d, a_1, c_1 .

Hypotheses: $\text{cyclic}(a, b, c, d)$, $\text{coll}(a_1, c, d)$, $\text{perp}(a_1, a, a, b)$, $\text{coll}(c_1, a, b)$, $\text{perp}(c_1, c, c, d)$.

Conclusion: $\text{para}(a_1, c_1, b, d)$.

The Machine Proof

$$[c_1a_1, db]$$

$$\begin{aligned}
& \text{(Since } a, b, c_1 \text{ are collinear; } c_1, a_1, a, c \text{ are cyclic; } [c_1a_1, db] = [c_1a_1, c_1a] + [ab, db] = \\
& [a_1c, ca] - [db, ba].\text{.)}
\end{aligned}$$

$$= [a_1c, ca] - [db, ba]$$

$$\begin{aligned}
& \text{(Since } a_1, c, d \text{ are collinear; } [a_1c, ca] = [dc, ca].\text{.)} \\
& = [dc, ca] - [db, ba]
\end{aligned}$$

$$\begin{aligned}
& \text{(Since } c, d, a, b \text{ are cyclic; } [dc, ca] = [db, ba].\text{.)} \\
& = 0
\end{aligned}$$

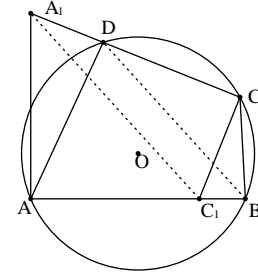


Figure 44

Example 45 Show that the perpendicular from the point of intersection of two opposite sides, produced, of a cyclic quadrilateral upon the line joining the midpoints of the two sides considered passes through the anticenter of the quadrilateral.

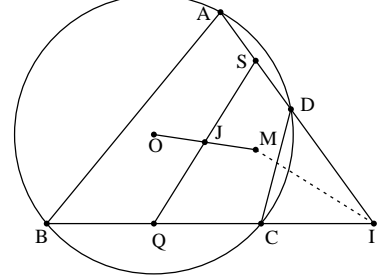


Figure 45

Point order: $a, b, c, d, o, i, q, s, j, m$.

Hypotheses: $\text{circumcenter}(o, a, b, c, d)$, $\text{coll}(i, a, d)$, $\text{coll}(i, b, c)$, $\text{midpoint}(q, b, c)$, $\text{midpoint}(s, a, d)$, $\text{midpoint}(j, s, q)$, $\text{midpoint}(j, o, m)$.

Conclusion: $\text{perp}(i, m, q, s)$.

The Machine Proof

$[mi, sq] + 1$

(Since m is the orthocenter of triangle q, s, i $[mi, sq] = [mi, qs] + [qs, sq] = -[sq, sq] + 1$.)

$= -[sq, sq]$

(Since $[sq, sq] = 0$.)

$= 0$

Example 46 Show that the product of the distances of two opposite sides of a cyclic quadrilateral from a point on the circumcircles is equal to the product of the distances of the other two sides from the same point.

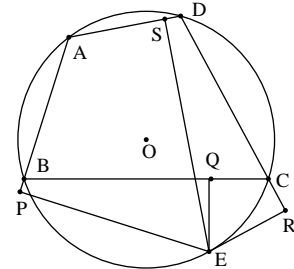


Figure 46

Point order: $a, b, c, d, e, p, q, s, r$.

Hypotheses: $\text{cyclic}(a, b, c, d, e)$, $\text{foot}(p, e, a, b)$, $\text{foot}(q, e, b, c)$, $\text{foot}(r, e, c, d)$, $\text{foot}(s, e, a, d)$.

Conclusion: $\text{eqangle}(e, p, s, e, q, r)$.

The Machine Proof

$[rq, qe] - [sp, pe]$

(Since $qe \perp bc$; $[rq, qe] = [rq, cb] + 1$.)

$= [rq, cb] - [sp, pe] + 1$

(Since c, d, r are collinear; r, q, c, e are cyclic; $[rq, cb] = [rq, rc] + [cd, cb] = [qe, ec] + [dc, cb]$.)

$= -[sp, pe] + [qe, ec] + [dc, cb] + 1$

(Since $pe \perp ab$; $[sp, pe] = [sp, ba] + 1$.)

$= -[sp, ba] + [qe, ec] + [dc, cb]$

(Since a, d, s are collinear; s, p, a, e are cyclic; $[sp, ba] = [sp, sa] + [ad, ba] = [pe, ea] + [da, ba]$.)

$= [qe, ec] - [pe, ea] + [dc, cb] - [da, ba]$

(Since $qe \perp bc$; $[qe, ec] = -[ec, cb] + 1$.)

$$\begin{aligned}
&= -[pe, ea] - [ec, cb] + [dc, cb] - [da, ba] + 1 \\
&\quad (\text{Since } pe \perp ab; [pe, ea] = -[ea, ba] + 1.) \\
&= -[ec, cb] + [ea, ba] + [dc, cb] - [da, ba] \\
&\quad (\text{Since } c, e, b, a \text{ are cyclic; } [ec, cb] = [ea, ba].) \\
&= [dc, cb] - [da, ba] \\
&\quad (\text{Since } c, d, b, a \text{ are cyclic; } [dc, cb] = [da, ba].) \\
&= 0
\end{aligned}$$

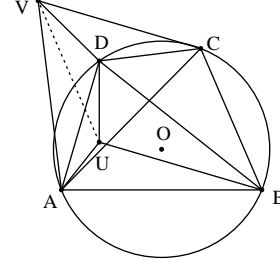


Figure 47

Example 47 A line AD through the vertex A meets the circumcircle of the triangle ABC in D . If U, V are the orthocenters of the triangle ABD, ACD , respectively, prove that UV is equal and parallel to BC .

Point order: a, b, c, d, u, v .

Hypotheses: $\text{cyclic}(a, b, c, d)$, $\text{orthocenter}(u, a, b, d)$, $\text{orthocenter}(v, a, c, d)$.

Conclusion: $\text{cyclic}(u, v, a, d)$.

The Machine Proof

$$\begin{aligned}
&[vd, ud] - [va, ua] \\
&\quad (\text{Since } vd \perp ac; ud \perp ab; [vd, ud] = [ca, ba].) \\
&= -[va, ua] + [ca, ba] \\
&\quad (\text{Since } va \perp cd; ua \perp bd; [va, ua] = [dc, db].) \\
&= -[dc, db] + [ca, ba] \\
&\quad (\text{Since } d, c, b, a \text{ are cyclic; } [dc, db] = [ca, ba].) \\
&= 0
\end{aligned}$$

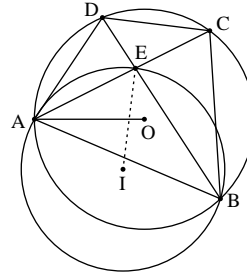


Figure 48

Example 48 Let E be the intersection of the two diagonals AC and BD of cyclic quadrilateral $ABCD$. Let I be the center of circumcircle of ABE . Show the $IE \perp DC$.

Point order: a, b, c, d, e, i .

Hypotheses: $\text{cyclic}(a, b, c, d)$, $\text{coll}(e, a, c)$, $\text{coll}(e, b, d)$, $\text{circumcenter}(i, a, b, e)$.

Conclusion: $\text{perp}(i, e, c, d)$.

The Machine Proof

$$\begin{aligned}
&[ie, dc] + 1 \\
&\quad (\text{Since } \text{circumcenter}(i, e, a, b); [ie, dc] = [ie, ea] + [ea, dc] = [eb, ba] + [ea, dc] + 1.) \\
&= [eb, ba] + [ea, dc] \\
&\quad (\text{Since } b, d, e \text{ are collinear; } [eb, ba] = [db, ba].) \\
&= [ea, dc] + [db, ba] \\
&\quad (\text{Since } a, c, e \text{ are collinear; } [ea, dc] = -[dc, ca].)
\end{aligned}$$

$$\begin{aligned}
&= -[dc, ca] + [db, ba] \\
&\quad (\text{Since } c, d, a, b \text{ are cyclic; } [dc, ca] = [db, ba].) \\
&= 0
\end{aligned}$$

Example 49 *In an orthodiagonal quadrilateral the midpoints of the sides lie on a circle having for center the centroid of the quadrilateral.*

Point order: $a, b, c, d, p, q, s, r, o$.

Hypotheses: $\text{perp}(a, c, b, d)$, $\text{midpoint}(p, a, b)$, $\text{midpoint}(q, b, c)$, $\text{midpoint}(s, a, d)$, $\text{midpoint}(r, c, d)$, $\text{coll}(o, p, r)$, $\text{coll}(o, q, s)$.

Conclusion: $\text{pbisector}(o, s, r)$.

The Machine Proof

$$\begin{aligned}
&[or, rs] + [os, rs] \\
&\quad (\text{Since } o, p, r \text{ are collinear; } [or, rs] = -[rs, rp].) \\
&= [os, rs] - [rs, rp] \\
&\quad (\text{Since } o, q, s \text{ are collinear; } [os, rs] = -[rs, sq].) \\
&= -[rs, rp] - [rs, sq] \\
&\quad (\text{Since } rs \parallel ac; [rs, rp] = -[rp, ca].) \\
&= -[rs, sq] + [rp, ca] \\
&\quad (\text{Since } rs \parallel ac; [rs, sq] = -[sq, ca].) \\
&= [rp, ca] + [sq, ca] \\
&\quad (\text{Since } r, p, q, s \text{ are cyclic; } rq \perp ac; \\
&\quad [rp, ca] = [rp, rq] + [rq, ca] = -[sq, sp] - [ca, ca] + 1.) \\
&= -[sq, sp] + [sq, ca] - [ca, ca] + 1 \\
&\quad (\text{Since } sq \parallel sq; -[sq, sp] + [sq, ca] = [sp, ca].) \\
&= [sp, ca] - [ca, ca] + 1 \\
&\quad (\text{Since } sp \perp ca; [sp, ca] = 1.) \\
&= -[ca, ca] \\
&\quad (\text{Since } [ca, ca] = 0.) \\
&= 0
\end{aligned}$$

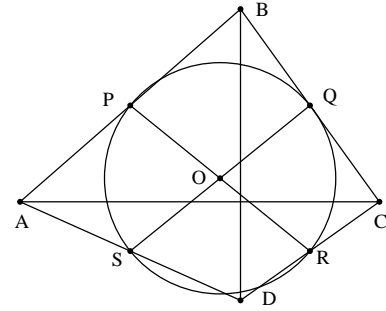


Figure 49

Example 50 *If an orthodiagonal quadrilateral is cyclic, the anticenter coincides with the point of intersection of its diagonals.*

Point order: a, b, c, d, o, m, p, r .

Hypotheses: $\text{circumcenter}(o, a, b, c, d)$, $\text{perp}(a, c, b, d)$, $\text{coll}(m, a, c)$, $\text{coll}(m, b, d)$, $\text{midpoint}(p, a, b)$, $\text{midpoint}(r, c, d)$.

Conclusion: $\text{para}(o, p, r, m)$.

The Machine Proof

$-[rm, po]$

(Since $po \perp ab$; $[rm, po] = [rm, ba] + 1$.)

$= -[rm, ba] - 1$

(Since $\text{circumcenter}(r, m, c, d)$; $[rm, ba] = [rm, mc] + [mc, ba] = [md, dc] + [mc, ba] + 1$.)

$= -[md, dc] - [mc, ba]$

(Since b, d, m are collinear; $[md, dc] = -[dc, db]$.)

$= -[mc, ba] + [dc, db]$

(Since a, c, m are collinear; $[mc, ba] = [ca, ba]$.)

$= [dc, db] - [ca, ba]$

(Since $db \perp ac$; $[dc, db] = [dc, ca] + 1$.)

$= [dc, ca] - [ca, ba] + 1$

(Since c, d, a, b are cyclic; $[dc, ca] = [db, ba]$.)

$= [db, ba] - [ca, ba] + 1$

(Since $db \perp ac$; $[db, ba] = [ca, ba] + 1$.)

$= 0$

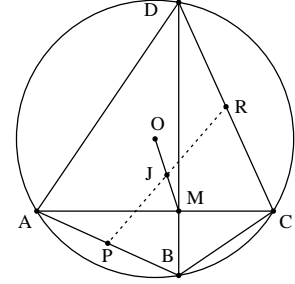


Figure 50

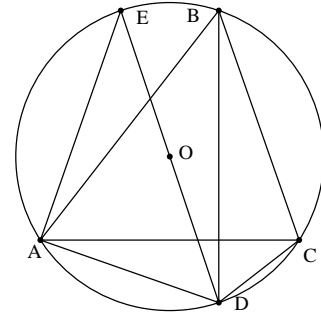


Figure 51

Example 51 *If the diagonals of a cyclic quadrilateral ABCD are orthogonal, and E is the diametric opposite of D on its circumcircle, show that $AE = CB$.*

Point order: a, b, c, d, o, e .

Hypotheses: $\text{circumcenter}(o, a, b, c, d, e)$, $\text{perp}(a, c, b, d)$, $\text{coll}(e, d, o)$.

Conclusion: $\text{para}(b, e, a, c)$.

The Machine Proof

$[eb, ca]$

(Since $\text{circumcenter}(o, e, b, a)$; $[eb, ca] = [eb, oa] + [ba, ca] = [ea, oa] - [ca, ba] + 1$.)

$= [ea, oa] - [ca, ba] + 1$

(Since d, e, o are collinear; e, a, d, b are cyclic;

$[ea, oa] = [ea, ed] + [do, oa] = [od, oa] - [db, ba]$.)

$= [od, oa] - [db, ba] - [ca, ba] + 1$

(Since $\text{circumcenter}(o, d, a, b)$; $[od, oa] = [od, da] + [da, oa] = -[oa, da] + [db, ba] + 1$.)

$$\begin{aligned}
&= -[oa, da] - [ca, ba] \\
&\quad (\text{Since circumcenter}(o, a, d, b); [oa, da] = [oa, ad] = -[db, ba] + 1.) \\
&= [db, ba] - [ca, ba] - 1 \\
&\quad (\text{Since } db \perp ac; [db, ba] = [ca, ba] + 1.) \\
&= 0
\end{aligned}$$

Example 52 Let D be a point on the side CB of a right triangle ABC such that the circle (O) with diameter CD touches the hypotenuse AB at E . Let $F = AC \cap DE$. Show that $AF = AE$.

Point order: c, d, e, o, a, f .

Hypotheses: $\text{perp}(c, e, d, e)$, $\text{midpoint}(o, c, d)$, $\text{perp}(a, c, c, d)$,
 $\text{perp}(a, e, e, o)$, $\text{coll}(f, a, c)$, $\text{coll}(f, d, e)$.

Conclusion: $\text{pbisector}(a, e, f)$.

The Machine Proof

$$\begin{aligned}
&[fa, fe] - [fe, ae] \\
&\quad (\text{Since } a, c, f \text{ are collinear; } d, e, f \text{ are collinear; } [fa, fe] = [ac, ed].) \\
&= -[fe, ae] + [ac, ed] \\
&\quad (\text{Since } d, e, f \text{ are collinear; } [fe, ae] = -[ae, ed].) \\
&= [ae, ed] + [ac, ed] \\
&\quad (\text{Since } ae \perp eo; [ae, ed] = [oe, ed] + 1.) \\
&= [ac, ed] + [oe, ed] + 1 \\
&\quad (\text{Since } ac \perp cd; [ac, ed] = -[ed, dc] + 1.) \\
&= [oe, ed] - [ed, dc] \\
&\quad (\text{Since } oe = od \text{ } c, d, o \text{ are collinear; } [oe, ed] = [oe, ed] + [ed, ed] = -[ed, ed] + [ed, dc].) \\
&= -[ed, ed] \\
&\quad (\text{Since } [ed, ed] = 0.) \\
&= 0
\end{aligned}$$

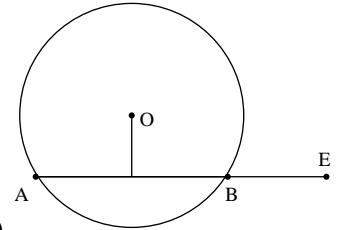


Figure 52

Example 53 Let C be a point on a chord AB of circle O . Let D and E be the intersections of perpendicular of OC through C with the two tangents of the circle at A and B , respectively. Show that $CE = CD$.

Point order: o, a, b, c, e, d .

Hypotheses: $\text{cong}(o, a, o, b)$, $\text{coll}(c, a, b)$, $\text{perp}(e, b, b, o)$,
 $\text{perp}(e, c, c, o)$, $\text{perp}(d, a, a, o)$, $\text{coll}(e, c, d)$.

Conclusion: $\text{pbisector}(o, e, d)$.

The Machine Proof

$$-[de, do] - [de, eo]$$

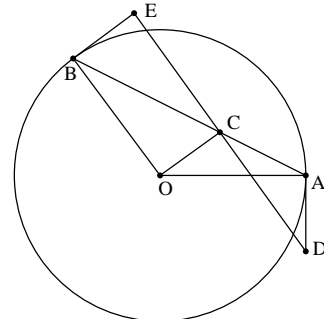


Figure 53

$$\begin{aligned}
& \text{(Since } de \perp oc; [de, do] = -[do, co] + 1.) \\
& = -[de, eo] + [do, co] - 1 \\
& \text{(Since } c, d, e \text{ are collinear; } [de, eo] = [ec, eo].) \\
& = [do, co] - [ec, eo] - 1 \\
& \text{(Since } c, d, e \text{ are collinear; } d, o, c, a \text{ are cyclic; } [do, co] = [do, dc] + [ce, co] = [ec, co] - [ca, ao].) \\
& = -[ec, eo] + [ec, co] - [ca, ao] - 1 \\
& \text{(Since } ec \parallel eo; -[ec, eo] + [ec, co] = [eo, co].) \\
& = [eo, co] - [ca, ao] - 1 \\
& \text{(Since } e, o, b, c \text{ are cyclic; } eb \perp bo; \\
& \quad [eo, co] = [eo, eb] + [eb, co] = -[cb, co] - [co, bo] + 1.) \\
& = -[cb, co] - [ca, ao] - [co, bo] \\
& \text{(Since } co \parallel eo; -[cb, co] - [co, bo] = -[cb, bo].) \\
& = -[cb, bo] - [ca, ao] \\
& \text{(Since } a, b, c \text{ are collinear; } [cb, bo] = [ba, bo].) \\
& = -[ca, ao] - [ba, bo] \\
& \text{(Since } a, b, c \text{ are collinear; } [ca, ao] = [ba, ao].) \\
& = -[ba, bo] - [ba, ao] \\
& \text{(Since } ob = oa \text{ } [ba, bo] = -[ba, ao].) \\
& = 0
\end{aligned}$$

Example 54 Let G be a point on the circle (O) with diameter BC , A be the midpoint of the arc BG . $AD \perp BC$. $E = AD \cap BG$ and $F = AC \cap BG$. Show that $AE = BE (= EF)$.

Point order: $a, b, c, g, o, m, d, e, f$.

Hypotheses: $\text{perp}(b, a, a, c)$, $\text{midpoint}(o, b, c)$, $\text{foot}(d, a, b, c)$,
 $\text{foot}(m, b, a, o)$, $\text{coll}(e, b, m)$, $\text{coll}(e, a, d)$, $\text{coll}(f, a, c)$, $\text{coll}(f, b, m)$.

Conclusion: $\text{pbisector}(e, a, b)$.

The Machine Proof

$$\begin{aligned}
& [eb, ba] + [ea, ba] \\
& \text{(Since } b, e, m \text{ are collinear; } [eb, ba] = [mb, ba].) \\
& = [ea, ba] + [mb, ba] \\
& \text{(Since } a, d, e \text{ are collinear; } [ea, ba] = [da, ba].) \\
& = [da, ba] + [mb, ba] \\
& \text{(Since } da \perp bc; [da, ba] = [cb, ba] + 1.) \\
& = [mb, ba] + [cb, ba] + 1 \\
& \text{(Since } mb \perp ao; [mb, ba] = [oa, ba] + 1.) \\
& = [oa, ba] + [cb, ba] \\
& \text{(Since circumcenter}(o, a, b, c); [oa, ba] = [oa, ab] = -[cb, ca] + 1.)
\end{aligned}$$

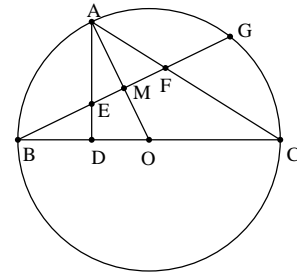


Figure 54

$$\begin{aligned}
&= -[cb, ca] + [cb, ba] + 1 \\
&\quad \text{(Since } cb \parallel cb; -[cb, ca] + [cb, ba] = [ca, ba].\text{)} \\
&= [ca, ba] + 1 \\
&\quad \text{(Since } ca \perp ba; [ca, ba] = 1.\text{)} \\
&= 0
\end{aligned}$$

Example 55 *The same as the preceding example. Show that $EA = EM$.*

Point order: $a, b, c, g, o, m, d, e, f$.

Hypotheses: $\text{perp}(b, a, a, c)$, $\text{midpoint}(o, b, c)$, $\text{foot}(d, a, b, c)$,
 $\text{foot}(m, b, a, o)$, $\text{coll}(e, b, m)$, $\text{coll}(e, a, d)$, $\text{coll}(f, a, c)$, $\text{coll}(f, b, m)$.

Conclusion: $\text{pbisector}(e, f, a)$.

The Machine Proof

$$\begin{aligned}
&[fe, fa] - [fa, ea] \\
&\quad \text{(Since } b, e, f, m \text{ are collinear; } a, c, f \text{ are collinear; } [fe, fa] = [mb, ca].\text{)} \\
&= -[fa, ea] + [mb, ca] \\
&\quad \text{(Since } a, c, f \text{ are collinear; } [fa, ea] = -[ea, ca].\text{)} \\
&= [ea, ca] + [mb, ca] \\
&\quad \text{(Since } a, d, e \text{ are collinear; } [ea, ca] = [da, ca].\text{)} \\
&= [da, ca] + [mb, ca] \\
&\quad \text{(Since } da \perp bc; ca \perp ab; [da, ca] = [cb, ba].\text{)} \\
&= [mb, ca] + [cb, ba] \\
&\quad \text{(Since } mb \perp ao; ca \perp ab; [mb, ca] = [oa, ba].\text{)} \\
&= [oa, ba] + [cb, ba] \\
&\quad \text{(Since circumcenter}(o, a, b, c); [oa, ba] = [oa, ab] = -[cb, ca] + 1.\text{)} \\
&= -[cb, ca] + [cb, ba] + 1 \\
&\quad \text{(Since } cb \parallel cb; -[cb, ca] + [cb, ba] = [ca, ba].\text{)} \\
&= [ca, ba] + 1 \\
&\quad \text{(Since } ca \perp ba; [ca, ba] = 1.\text{)} \\
&= 0
\end{aligned}$$

Example 56 Let M be the midpoint of the arc AB of circle (O) , D be the midpoint of AB . The perpendicular through M is drawn to the tangent of the circle at A meeting that tangent at E . Show $ME = MD$.

Point order: a, b, m, o, d, e .

Hypotheses: $\text{cong}(m, a, m, b)$, $\text{cong}(o, a, o, m)$, $\text{cong}(o, a, o, b)$, $\text{midpoint}(d, a, b)$, $\text{coll}(m, o, d)$, $\text{perp}(e, a, a, o)$, $\text{para}(m, e, a, o)$.

Conclusion: $\text{pbisector}(m, e, d)$.

The Machine Proof

$$\begin{aligned}
& -[ed, em] - [ed, dm] \\
& \quad (\text{Since } em \parallel ao; [ed, em] = [ed, oa].) \\
& = -[ed, dm] - [ed, oa] \\
& \quad (\text{Since } d, e, m, a \text{ are cyclic; } [ed, dm] = [ea, ma].) \\
& = -[ed, oa] - [ea, ma] \\
& \quad (\text{Since } e, d, a, m \text{ are cyclic; } ea \perp ao; [ed, oa] = [ed, ea] + [ea, oa] = [dm, ma] - [oa, oa] + 1.) \\
& = -[ea, ma] - [dm, ma] + [oa, oa] - 1 \\
& \quad (\text{Since } ea \perp ao; [ea, ma] = [oa, ma] + 1.) \\
& = -[dm, ma] + [oa, oa] - [oa, ma] \\
& \quad (\text{Since } oa \parallel oa; [oa, oa] - [oa, ma] = -[oa, ma].) \\
& = -[dm, ma] - [oa, ma] \\
& \quad (\text{Since } dm \perp ab; [dm, ma] = -[ma, ba] + 1.) \\
& = -[oa, ma] + [ma, ba] - 1 \\
& \quad (\text{Since circumcenter}(o, a, m, b); [oa, ma] = [oa, am] = -[mb, ba] + 1.) \\
& = [mb, ba] + [ma, ba] \\
& \quad (\text{Since } mb = ma \text{ } [mb, ba] = -[ma, ba].) \\
& = 0
\end{aligned}$$

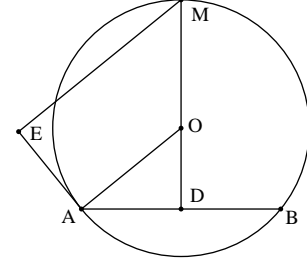


Figure 56

Example 57 The circle with the altitude AD of triangle ABC as a diameter meets AB and AC at E and F , respectively. Show that B, C, E and F are on the same circle.

Point order: a, b, c, d, o, e, f .

Hypotheses: $\text{foot}(d, a, b, c)$, $\text{midpoint}(o, a, d)$, $\text{coll}(e, a, b)$, $\text{cong}(o, a, o, e)$, $\text{coll}(f, a, c)$, $\text{cong}(o, a, o, f)$.

Conclusion: $\text{cyclic}(b, c, e, f)$.

The Machine Proof

$$[fe, eb] - [fc, cb]$$

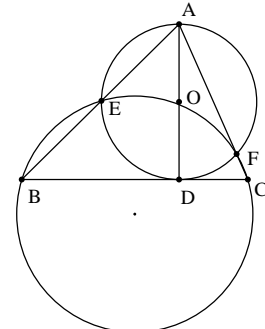


Figure 57

$$\begin{aligned}
& \text{(Since } a, b, e \text{ are collinear; } e, f, a, d \text{ are cyclic; } [fe, eb] = [fd, da].\text{)} \\
& = [fd, da] - [fc, cb] \\
& \text{(Since } fd \perp ac; da \perp bc; [fd, da] = -[cb, ca].\text{)} \\
& = -[fc, cb] - [cb, ca] \\
& \text{(Since } cb \parallel cb; -[fc, cb] - [cb, ca] = -[fc, ca].\text{)} \\
& = -[fc, ca] \\
& \text{(Since } a, c, f \text{ are collinear; } [fc, ca] = 0.\text{)} \\
& = 0
\end{aligned}$$

Example 58 Let A, B, C, D be four points on circle (O) . $E = CD \cap AB$. CB meets the line passing through E and parallel to AD at F . GF is tangent to circle (O) at G . Show that $FG = FE$.

Point order: a, b, c, d, e, f .

Hypotheses: $\text{cyclic}(a, b, c, d)$, $\text{coll}(e, a, b)$, $\text{coll}(e, c, d)$, $\text{coll}(f, b, c)$, $\text{para}(f, e, a, d)$.

Conclusion: $\text{eqangle}(f, e, b, e, c, b)$.

The Machine Proof

$$\begin{aligned}
& [fe, eb] - [ec, cb] \\
& \text{(Since } fe \parallel ad; [fe, eb] = -[eb, da].\text{)} \\
& = -[ec, cb] - [eb, da] \\
& \text{(Since } c, d, e \text{ are collinear; } [ec, cb] = [dc, cb].\text{)} \\
& = -[eb, da] - [dc, cb] \\
& \text{(Since } a, b, e \text{ are collinear; } [eb, da] = -[da, ba].\text{)} \\
& = -[dc, cb] + [da, ba] \\
& \text{(Since } c, d, b, a \text{ are cyclic; } [dc, cb] = [da, ba].\text{)} \\
& = 0
\end{aligned}$$

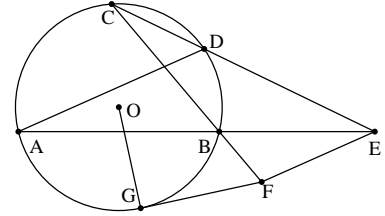


Figure 58

Example 59 The bisector of triangle ABC at vertex C bisects the arc AB of the circumcircle of triangle ABC .

Point order: a, b, c, o, m, n .

Hypotheses: $\text{circumcenter}(o, a, b, c)$, $\text{midpoint}(m, a, b)$, $\text{coll}(n, m, o)$, $\text{cong}(o, a, o, n)$.

Conclusion: $\text{eqangle}(a, c, n, n, c, b)$.

The Machine Proof

$$\begin{aligned}
& -[nc, cb] - [nc, ca] \\
& \text{(Since } c, n, b, a \text{ are cyclic; } [nc, cb] = [na, ba].\text{)} \\
& = -[nc, ca] - [na, ba] \\
& \text{(Since } c, n, a, b \text{ are cyclic; } [nc, ca] = [nb, ba].\text{)}
\end{aligned}$$

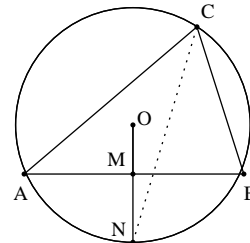


Figure 59

$$\begin{aligned}
&= -[nb, ba] - [na, ba] \\
&\quad (\text{Since } nb = na \text{ } [nb, ba] = -[na, ba].) \\
&= 0
\end{aligned}$$

Example 60 Let N be the traces of the internal bisectors of the triangle ABC on the circumscribed circle (O). Show that the Simson line of N is the external bisector of the medial triangle of ABC .

Point order: $a, b, c, o, d, e, f, n, k$.

Hypotheses: $\text{circumcenter}(o, a, b, c)$, $\text{midpoint}(d, b, c)$, $\text{midpoint}(e, a, c)$, $\text{midpoint}(f, a, b)$, $\text{coll}(n, o, f)$, $\text{cong}(o, a, o, n)$, $\text{foot}(k, n, a, c)$.

Conclusion: $\text{eqangle}(e, f, k, k, f, d)$.

The Machine Proof

$$\begin{aligned}
&-[kf, fe] - [kf, fd] \\
&\quad (\text{Since } fe \parallel bc; [kf, fe] = [kf, cb].) \\
&= -[kf, fd] - [kf, cb] \\
&\quad (\text{Since } fd \parallel ac; [kf, fd] = [kf, ca].) \\
&= -[kf, cb] - [kf, ca] \\
&\quad (\text{Since } a, e, k \text{ are collinear; } k, f, a, n \text{ are cyclic;} \\
&\quad [kf, cb] = [kf, ka] + [ae, cb] = [nf, na] + [ea, cb].) \\
&= -[kf, ca] - [nf, na] - [ea, cb] \\
&\quad (\text{Since } a, c, k \text{ are collinear; } k, f, a, n \text{ are cyclic; } [kf, ca] = [kf, ka] = [nf, na].) \\
&= -2[nf, na] - [ea, cb] \\
&\quad (\text{Since } nf \perp ab; [nf, na] = -[na, ba] + 1.) \\
&= 2[na, ba] - [ea, cb] \\
&\quad (\text{Since } \text{circumcenter}(o, n, a, b); no \perp ab; [na, ba] = -[fo, oa] + 2[fo, ba].) \\
&= -[fo, oa] + 2[fo, ba] - [ea, cb] \\
&\quad (\text{Since } fo \perp ab; [fo, oa] = -[oa, ba] + 1.) \\
&= 2[fo, ba] - [ea, cb] + [oa, ba] - 1 \\
&\quad (\text{Since } fo \perp ba; [fo, ba] = 1.) \\
&= -[ea, cb] + [oa, ba] - 1 \\
&\quad (\text{Since } a, c, e \text{ are collinear; } [ea, cb] = -[cb, ca].) \\
&= [oa, ba] + [cb, ca] - 1 \\
&\quad (\text{Since } \text{circumcenter}(o, a, b, c); [oa, ba] = [oa, ab] = -[cb, ca] + 1.) \\
&= 0
\end{aligned}$$

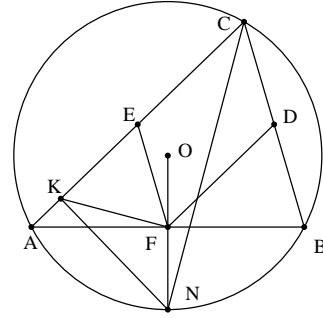


Figure 60

Example 61 Let A and B be the two common points of two circles (O) and (O_1) . Through B a line is drawn meeting the circles at C and D respectively. Show $AC/AD = OA/O_1A$.

We need only to show that triangles AOO_1 and ACD are similar.

Point order: a, b, c, o, x, d, o_1 .

Hypotheses: $\text{circumcenter}(o, a, b, c)$, $\text{midpoint}(x, a, b)$, $\text{cong}(o_1, a, o_1, b)$,

$\text{coll}(o_1, x, o)$, $\text{coll}(d, b, c)$, $\text{cong}(d, o_1, o_1, a)$.

Conclusion: $\text{eqangle}(a, o, o_1, a, c, d)$.

The Machine Proof

$$-[o_1o, oa] + [dc, ca]$$

(Since o, o_1, x are collinear; $[o_1o, oa] = [xo, oa]$.)

$$= [dc, ca] - [xo, oa]$$

(Since b, c, d are collinear; $[dc, ca] = [cb, ca]$.)

$$= -[xo, oa] + [cb, ca]$$

(Since $xo \perp ab$; $[xo, oa] = -[oa, ba] + 1$.)

$$= [oa, ba] + [cb, ca] - 1$$

(Since $\text{circumcenter}(o, a, b, c)$; $[oa, ba] = [oa, ab] = -[cb, ca] + 1$.)

$$= 0$$

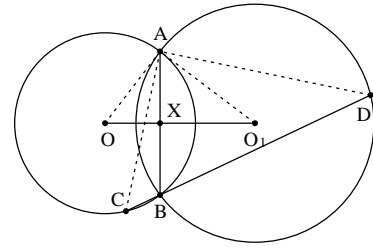


Figure 61

Example 62 If three chords drawn through a point of a circle are taken for diameters of three circles, these circles intersect, in pairs, in three new points, which are collinear.

Point order: d, a, b, c, e, f, g .

Hypotheses: $\text{cyclic}(a, b, c, d)$, $\text{perp}(e, a, e, d)$, $\text{perp}(e, b, e, d)$,

$\text{perp}(f, a, f, d)$, $\text{perp}(f, c, f, d)$, $\text{perp}(g, c, g, d)$, $\text{perp}(g, b, g, d)$.

Conclusion: $\text{coll}(e, f, g)$.

The Machine Proof

$$-[ge, fe]$$

(Since b, c, g are collinear; g, e, b, d are cyclic;

$$[ge, fe] = [ge, gb] + [bc, fe] = -[fe, cb] + [ed, bd].)$$

$$= [fe, cb] - [ed, bd]$$

(Since a, c, f are collinear; f, e, a, d are cyclic;

$$[fe, cb] = [fe, fa] + [ac, cb] = [ed, ad] - [cb, ca].)$$

$$= -[ed, bd] + [ed, ad] - [cb, ca]$$

(Since $ed \parallel ed$; $-[ed, bd] + [ed, ad] = [bd, ad]$.)

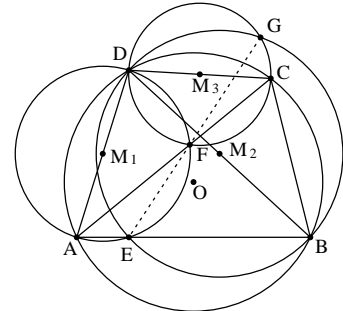


Figure 62

$$\begin{aligned}
&= -[cb, ca] + [bd, ad] \\
&\quad (\text{Since } c, b, a, d \text{ are cyclic; } [cb, ca] = [bd, ad].) \\
&= 0
\end{aligned}$$

Example 63 *If three circles having a point in common intersect in pairs in three collinear points, their common point is concyclic with their centers.*

Point order: $p, a1, b1, c1, a, b, c$.

Hypotheses: $\text{circumcenter}(a, p, b1, c1)$, $\text{circumcenter}(b, p, a1, c1)$, $\text{circumcenter}(c, p, b1, a1)$, $\text{coll}(a1, b1, c1)$.

Conclusion: $\text{cyclic}(p, a, b, c)$.

The Machine Proof

$$\begin{aligned}
&[ca, cp] - [ba, bp] \\
&\quad (\text{Since } ca \perp pb1; [ca, cp] = -[cp, b1p] + 1.) \\
&= -[cp, b1p] - [ba, bp] + 1 \\
&\quad (\text{Since } \text{circumcenter}(c, p, b1, a1); [cp, b1p] = [cp, pb1] = -[b1a1, a1p] + 1.) \\
&= -[ba, bp] + [b1a1, a1p] \\
&\quad (\text{Since } ba \perp pc1; [ba, bp] = -[bp, c1p] + 1.) \\
&= [bp, c1p] + [b1a1, a1p] - 1 \\
&\quad (\text{Since } \text{circumcenter}(b, p, c1, a1); [bp, c1p] = [bp, pc1] = -[c1a1, a1p] + 1.) \\
&= -[c1a1, a1p] + [b1a1, a1p] \\
&\quad (\text{Since } a1, b1, c1 \text{ are collinear; } [c1a1, a1p] = [b1a1, a1p].) \\
&= 0
\end{aligned}$$

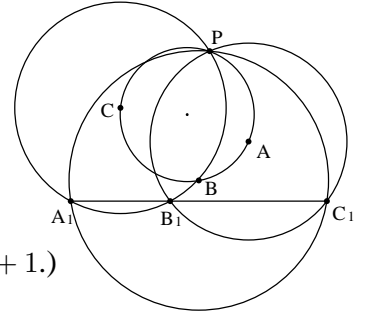


Figure 63

Example 64 *Given two circles (A), (B) intersecting in E, F, show that the chord E_1F_1 determined in (A) by the lines MEE_1 , MFF_1 joining E, F to any point M of (B) is perpendicular to MB.*

Point order: $e, f, m, b, d, a, e1, f1$.

Hypotheses: $\text{circumcenter}(b, e, f, m)$, $\text{midpoint}(d, e, f)$, $\text{coll}(a, d, b)$, $\text{coll}(e1, m, e)$, $\text{cong}(e1, a, e, a)$, $\text{coll}(f1, m, f)$, $\text{cong}(f1, a, e, a)$.

Conclusion: $\text{perp}(e1, f1, m, b)$.

The Machine Proof

$$\begin{aligned}
&[f1e1, bm] + 1 \\
&\quad (\text{Since } f, f1, m \text{ are collinear; } f1, e1, f, e \text{ are cyclic;} \\
&\quad [f1e1, bm] = [f1e1, f1f] + [fm, bm] = [e1e, fe] - [bm, mf].) \\
&= [e1e, fe] - [bm, mf] + 1 \\
&\quad (\text{Since } e, e1, m \text{ are collinear; } [e1e, fe] = [me, fe].) \\
&= -[bm, mf] + [me, fe] + 1
\end{aligned}$$

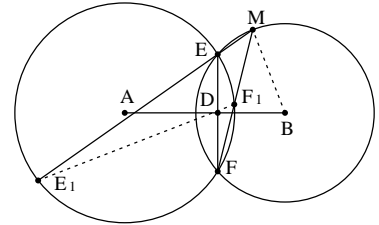


Figure 64

$$\begin{aligned}
& (\text{Since } \text{circumcenter}(b, m, f, e); [bm, mf] = [bm, mf] = [me, fe] + 1.) \\
& = 0
\end{aligned}$$

Example 65 Let D and E be two points on sides AB and AC of triangle ABC such that $DE \parallel BC$. Show that the circumcircles of triangle ABC and ADE are tangent.

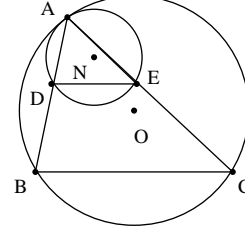


Figure 65

Point order: a, b, c, o, d, e, n .

Hypotheses: $\text{circumcenter}(o, a, b, c)$, $\text{coll}(d, a, b)$, $\text{coll}(e, a, c)$, $\text{para}(d, e, b, c)$, $\text{circumcenter}(n, a, d, e)$.

Conclusion: $\text{coll}(a, n, o)$.

The Machine Proof

$[na, oa]$

$$\begin{aligned}
& (\text{Since } \text{circumcenter}(n, a, d, e); [na, oa] = [na, ad] + [ad, oa] = -[ed, ea] + [da, oa] + 1.) \\
& = -[ed, ea] + [da, oa] + 1 \\
& (\text{Since } ed \parallel bc; [ed, ea] = -[ea, cb].) \\
& = [ea, cb] + [da, oa] + 1 \\
& (\text{Since } a, c, e \text{ are collinear; } [ea, cb] = -[cb, ca].) \\
& = [da, oa] - [cb, ca] + 1 \\
& (\text{Since } a, b, d \text{ are collinear; } [da, oa] = -[oa, ba].) \\
& = -[oa, ba] - [cb, ca] + 1 \\
& (\text{Since } \text{circumcenter}(o, a, b, c); [oa, ba] = [oa, ab] = -[cb, ca] + 1.) \\
& = 0
\end{aligned}$$

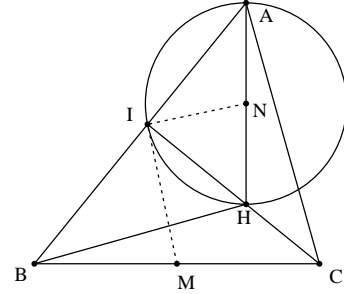


Figure 66

Example 66 Show that in a triangle ABC the circles on AH and BC as diameters are orthogonal.

Point order: a, b, c, i, h, n, m .

Hypotheses: $\text{foot}(i, c, a, b)$, $\text{coll}(h, c, i)$, $\text{perp}(h, a, b, c)$, $\text{midpoint}(m, c, b)$, $\text{midpoint}(n, a, h)$.

Conclusion: $\text{perp}(m, i, n, i)$.

The Machine Proof

$[mi, ni] + 1$

$$\begin{aligned}
& (\text{Since } \text{circumcenter}(m, i, b, c); [mi, ni] = [mi, ib] + [ib, ni] = -[ni, ib] + [ic, cb] + 1.) \\
& = -[ni, ib] + [ic, cb] \\
& (\text{Since } \text{circumcenter}(n, i, a, h); [ni, ib] = [ni, ia] = [hi, ha] + 1.) \\
& = -[hi, ha] + [ic, cb] - 1 \\
& (\text{Since } hi \parallel ic; -[hi, ha] + [ic, cb] = [ha, cb].) \\
& = [ha, cb] - 1
\end{aligned}$$

$$\begin{aligned}
& (\text{Since } ha \perp cb; [ha, cb] = 1.) \\
& = 0
\end{aligned}$$

Example 67 *The circle IBC is orthogonal to the circle on $I_b I_c$ as diameter.*

Point order: $a, b, c, i, o, b1, c1, m$.

Hypotheses: $\text{incenter}(i, a, b, c)$, $\text{circumcenter}(o, b, c, i)$, $\text{coll}(b1, b, i)$, $\text{perp}(b1, c, c, i)$, $\text{coll}(c1, c, i)$, $\text{perp}(c1, b, b, i)$, $\text{midpoint}(m, b1, c1)$.

Conclusion: $\text{perp}(m, b, o, b)$.

The Machine Proof

$$[mb, ob] + 1$$

$$\begin{aligned}
& (\text{Since } \text{circumcenter}(m, b, c, b1); [mb, ob] = [mb, bc] + [bc, ob] = -[b1c, b1b] - [ob, cb] + 1.) \\
& = -[b1c, b1b] - [ob, cb]
\end{aligned}$$

$$\begin{aligned}
& (\text{Since } b1c \perp ci; [b1c, b1b] = -[b1b, ic] + 1.) \\
& = [b1b, ic] - [ob, cb] - 1
\end{aligned}$$

$$\begin{aligned}
& (\text{Since } b, b1, i \text{ are collinear; } [b1b, ic] = -[ic, ib].) \\
& = -[ob, cb] - [ic, ib] - 1
\end{aligned}$$

$$\begin{aligned}
& (\text{Since } \text{circumcenter}(o, b, c, i); [ob, cb] = [ob, bc] = -[ic, ib] + 1.) \\
& = 0
\end{aligned}$$

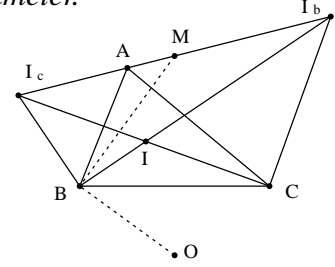


Figure 67

Example 68 *Show that given two perpendicular diameters of two orthogonal circles, the lines joining an end of one of these diameters to the ends of the other pass through the points common to the two circles.*

Point order: $p, d, i, a, e, x, b, g, f$.

Hypotheses: $\text{circumcenter}(a, p, d, i, e)$, $\text{midpoint}(a, d, e)$, $\text{midpoint}(x, p, i)$, $\text{coll}(b, a, x)$, $\text{perp}(b, p, a, p)$, $\text{coll}(e, i, g)$, $\text{cong}(g, b, p, b)$, $\text{midpoint}(b, g, f)$.

Conclusion: $\text{perp}(f, g, d, e)$.

The Machine Proof

$$[fg, ed] + 1$$

$$\begin{aligned}
& (\text{Since } b, f, g \text{ are collinear; } [fg, ed] = [gb, ed].) \\
& = [gb, ed] + 1
\end{aligned}$$

$$\begin{aligned}
& (\text{Since } e, g, i \text{ are collinear; } bg = bi \text{ } [gb, ed] = [gb, gi] + [ie, ed] = -[bi, ei] + [ei, ed].) \\
& = -[bi, ei] + [ei, ed] + 1
\end{aligned}$$

$$\begin{aligned}
& (\text{Since } bi \perp ia; ei \perp di; [bi, ei] = [ai, id].) \\
& = [ei, ed] - [ai, id] + 1
\end{aligned}$$

$$\begin{aligned}
& (\text{Since } ei \perp di; [ei, ed] = -[ed, id] + 1.) \\
& = -[ed, id] - [ai, id]
\end{aligned}$$

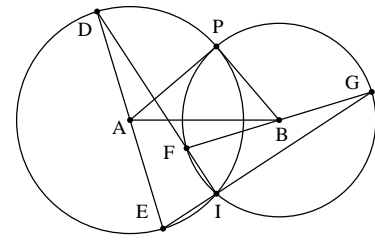


Figure 68

$$\begin{aligned}
& \text{(Since } a, d, e \text{ are collinear; } [ed, id] = [ad, id].\text{)} \\
& = -[ai, id] - [ad, id] \\
& \text{(Since } \text{circumcenter}(a, i, d, p); [ai, id] = [ai, id] = [ip, dp] + 1.\text{)} \\
& = -[ad, id] - [ip, dp] - 1 \\
& \text{(Since } \text{circumcenter}(a, d, i, p); [ad, id] = [ad, di] = -[ip, dp] + 1.\text{)} \\
& = 0
\end{aligned}$$

Example 69 Show that if AB is a diameter and M any point of a circle, center O , the two circles AMO , BMO are orthogonal.

Point order: a, b, m, o, i, j .

Hypotheses: $\text{perp}(a, m, b, m)$, $\text{midpoint}(o, a, b)$,
 $\text{circumcenter}(i, o, a, m)$, $\text{circumcenter}(j, o, b, m)$.

Conclusion: $\text{perp}(i, m, m, j)$.

The Machine Proof

$$\begin{aligned}
& -[jm, im] + 1 \\
& \text{(Since } \text{circumcenter}(j, m, o, b); [jm, im] = [jm, mo] + [mo, im] = -[im, om] - [ob, mb] + 1.\text{)} \\
& = [im, om] + [ob, mb] \\
& \text{(Since } \text{circumcenter}(i, m, o, a); [im, om] = [im, mo] = -[oa, ma] + 1.\text{)} \\
& = [ob, mb] - [oa, ma] + 1 \\
& \text{(Since } ob \parallel oa; [ob, mb] - [oa, ma] = -[mb, ma].\text{)} \\
& = -[mb, ma] + 1 \\
& \text{(Since } mb \perp ma; [mb, ma] = 1.\text{)} \\
& = 0
\end{aligned}$$

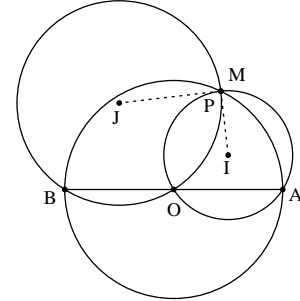


Figure 69

Example 70 If the line joining the ends A, B of a diameter AB of a given circle (O) to a given point P meets (O) again in A_1, B_1 , show that the circle PA_1B_1 is orthogonal to (O) .

Point order: $a, b, a1, o, b1, p, o1$.

Hypotheses: $\text{circumcenter}(o, a, b, a1, b1)$, $\text{midpoint}(o, a, b)$,
 $\text{coll}(p, a, a1)$, $\text{coll}(p, b, b1)$, $\text{circumcenter}(o1, p, a1, b1)$.

Conclusion: $\text{perp}(o1, a1, o, a1)$.

The Machine Proof

$[o1a1, oa1] + 1$

(Since $\text{circumcenter}(o1, a1, p, b1)$; $[o1a1, oa1] = [o1a1, a1p] + [a1p, oa1] = -[pb1, b1a1] + [pa1, oa1] + 1$.)

$= -[pb1, b1a1] + [pa1, oa1]$

(Since $b, b1, p$ are collinear; $[pb1, b1a1] = -[b1a1, b1b]$.)

$= [pa1, oa1] + [b1a1, b1b]$

(Since $a, a1, p$ are collinear; $[pa1, oa1] = -[oa1, a1a]$.)

$= [b1a1, b1b] - [oa1, a1a]$

(Since $b1, a1, b, a$ are cyclic; $[b1a1, b1b] = [a1a, ba]$.)

$= -[oa1, a1a] + [a1a, ba]$

(Since $oa1 = oa$; a, b, o are collinear; $[oa1, a1a] = [oa1, a1a] + [a1a, a1a] = -[a1a, a1a] + [a1a, ba]$.)

$= [a1a, a1a]$

(Since $[a1a, a1a] = 0$.)

$= 0$

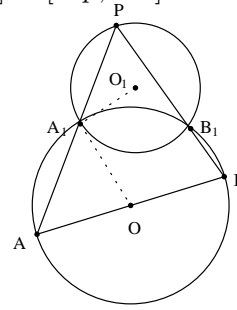


Figure 70

Example 71 Let D be a point on the circumcircle of triangle ABC . If line DA is parallel to BC , show that the Simson line $D(ABC)$ is parallel to the circumradius OA .

Point order: a, b, c, d, o, f, g .

Hypotheses: $\text{circumcenter}(o, a, b, c, d)$, $\text{para}(d, a, b, c)$, $\text{foot}(g, d, a, b)$, $\text{foot}(f, d, a, c)$.

Conclusion: $\text{para}(g, f, o, a)$.

The Machine Proof

$[gf, oa]$

(Since a, b, g are collinear; g, f, a, d are cyclic;

$[gf, oa] = [gf, ga] + [ab, oa] = [fd, da] - [oa, ba]$.)

$= [fd, da] - [oa, ba]$

(Since $da \parallel bc$; $[fd, da] = [fd, cb]$.)

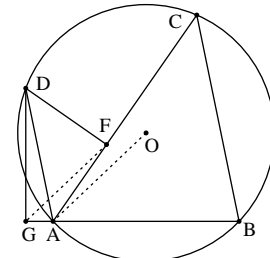


Figure 71

$$\begin{aligned}
&= [fd, cb] - [oa, ba] \\
&\quad (\text{Since } fd \perp ac; [fd, cb] = -[cb, ca] + 1.) \\
&= -[oa, ba] - [cb, ca] + 1 \\
&\quad (\text{Since circumcenter}(o, a, b, c); [oa, ba] = [oa, ab] = -[cb, ca] + 1.) \\
&= 0
\end{aligned}$$

Example 72 If E, F, G are the feet of the perpendiculars from a point D of the circumcircle of a triangle ABC upon its sides BC, CA, AB , prove that the triangle DFG, DBC are similar.

Point order: a, b, c, d, o, f, g .

Hypotheses: $\text{circumcenter}(o, a, b, c, d)$, $\text{foot}(g, d, a, b)$, $\text{foot}(f, d, a, c)$.

Conclusion: $\text{similar}(d, f, g, d, c, b)$.

The Machine Proof

$$\begin{aligned}
&-[gd, fd] - [dc, db] \\
&\quad (\text{Since } gd \perp ab; fd \perp ac; [gd, fd] = -[ca, ba].) \\
&= -[dc, db] + [ca, ba] \\
&\quad (\text{Since } d, c, b, a \text{ are cyclic; } [dc, db] = [ca, ba].) \\
&= 0
\end{aligned}$$

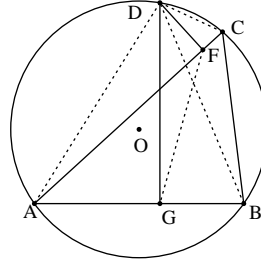


Figure 72

Example 73 If the perpendicular from a point D of the circumcircle (O) of a triangle ABC to the sides BC, CA, AB meet (O) again in the points N, M, L , the three lines AN, BM, CL are parallel to the simson of D for ABC .

Point order: a, b, c, d, o, f, g, l .

Hypotheses: $\text{circumcenter}(o, a, b, c, d)$, $\text{foot}(g, d, a, b)$, $\text{foot}(f, d, a, c)$,

$\text{coll}(l, d, g)$, $\text{cong}(l, o, o, a)$.

Conclusion: $\text{para}(c, l, f, g)$.

The Machine Proof

$$\begin{aligned}
&[lc, gf] \\
&\quad (\text{Since } d, g, l \text{ are collinear; } l, c, d, a \text{ are cyclic;} \\
&\quad [lc, gf] = [lc, ld] + [dg, gf] = -[gf, gd] - [da, ca].) \\
&= -[gf, gd] - [da, ca] \\
&\quad (\text{Since } gd \perp ab; [gf, gd] = [gf, ba] + 1.) \\
&= -[gf, ba] - [da, ca] - 1 \\
&\quad (\text{Since } a, b, g \text{ are collinear; } g, f, a, d \text{ are cyclic;} \\
&\quad [gf, ba] = [gf, ga] = [fd, da].) \\
&= -[fd, da] - [da, ca] - 1
\end{aligned}$$

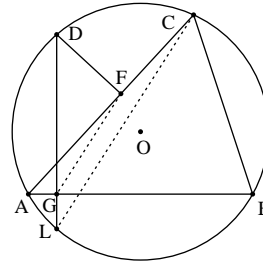


Figure 73

$$\begin{aligned}
& \text{(Since } da \parallel da; -[fd, da] - [da, ca] = -[fd, ca].\text{)} \\
& = -[fd, ca] - 1 \\
& \text{(Since } fd \perp ca; [fd, ca] = 1.\text{)} \\
& = 0
\end{aligned}$$

Example 74 Show that the simson of the point where an altitude cuts the circumcircle again passes through the foot of the altitude and is antiparallel to the corresponding side of the triangle with respect to the other two sides.

Point order: a, b, c, o, g, d, e, f .

Hypotheses: $\text{circumcenter}(o, a, b, c)$, $\text{foot}(g, c, a, b)$, $\text{coll}(d, c, g)$, $\text{cong}(d, o, o, a)$, $\text{foot}(e, d, a, c)$, $\text{foot}(f, d, b, c)$.

Conclusion: $\text{cyclic}(a, b, e, f)$.

The Machine Proof

$$[fe, ea] - [fb, ba]$$

$$\begin{aligned}
& \text{(Since } a, c, e \text{ are collinear; } e, f, c, d \text{ are cyclic; } [fe, ea] = [fd, dc].\text{)} \\
& = [fd, dc] - [fb, ba]
\end{aligned}$$

$$\text{(Since } fd \perp bc; dc \perp ab; [fd, dc] = [cb, ba].\text{)}$$

$$= -[fb, ba] + [cb, ba]$$

$$\text{(Since } b, c, f \text{ are collinear; } [fb, ba] = [cb, ba].\text{)}$$

$$= 0$$

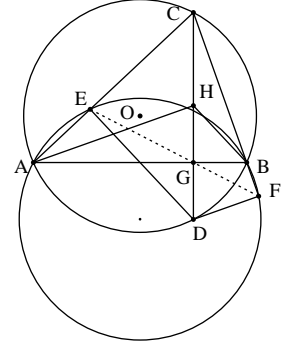


Figure 74

Example 75 Let A and B be the intersections of two circles O_1 and O_2 . Through A a secant is drawn meeting the two circles at C and D , respectively. Show that angle CBD is equal to the angle formed by lines O_1C and O_2D .

Point order: $a, b, c, o1, o, o2, d, i$.

Hypotheses: $\text{circumcenter}(o1, a, b, c)$, $\text{midpoint}(o, a, b)$, $\text{coll}(o1, o2, o)$, $\text{coll}(d, a, c)$, $\text{cong}(o2, a, o2, d)$, $\text{coll}(o1, c, i)$, $\text{coll}(o2, d, i)$.

Conclusion: $\text{eqangle}(c, b, d, o1, i, o2)$.

The Machine Proof

$$[io2, io1] - [db, cb]$$

$$\begin{aligned}
& \text{(Since } d, i, o2 \text{ are collinear; } c, i, o1 \text{ are collinear; } [io2, io1] = [do2, o1c].\text{)} \\
& = [do2, o1c] - [db, cb]
\end{aligned}$$

$$\begin{aligned}
& \text{(Since } a, c, d \text{ are collinear; } o2d = o2a \text{ } [do2, o1c] = [do2, da] + [ac, o1c] = -[o2a, ca] - [o1c, ca].\text{)} \\
& = -[db, cb] - [o2a, ca] - [o1c, ca]
\end{aligned}$$

$$\text{(Since circumcenter}(o2, d, b, a); [db, cb] = [db, o2a] + [ba, cb] = [da, o2a] - [cb, ba] + 1.\text{)}$$

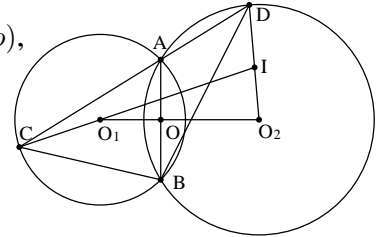


Figure 75

$$\begin{aligned}
&= -[da, o2a] - [o2a, ca] - [o1c, ca] + [cb, ba] - 1 \\
&\quad (\text{Since } o2a \parallel o2a; -[da, o2a] - [o2a, ca] = -[da, ca].) \\
&= -[da, ca] - [o1c, ca] + [cb, ba] - 1 \\
&\quad (\text{Since } da \parallel ca; -[da, ca] - [o1c, ca] = -[o1c, ca].) \\
&= -[o1c, ca] + [cb, ba] - 1 \\
&\quad (\text{Since circumcenter}(o1, c, a, b); [o1c, ca] = [o1c, ca] = [cb, ba] + 1.) \\
&= 0
\end{aligned}$$

Example 76 Let E be a point on the circumcircle of equilateral triangle ABC . $D = BC \cap AE$. Show that $BE \cdot CE = ED \cdot EA$.

We need only to show that triangles EDC and EBA are similar.

Point order: b, c, a, o, e, d .

Hypotheses: $\text{cong}(a, b, a, c)$, $\text{cong}(b, c, b, a)$, $\text{pbisector}(o, a, b)$, $\text{pbisector}(o, c, b)$, $\text{pbisector}(o, e, b)$, $\text{coll}(a, e, d)$, $\text{coll}(b, c, d)$.

Conclusion: $\text{similar}(e, d, c, e, b, a)$.

The Machine Proof

$$\begin{aligned}
&[de, ec] + [ea, eb] \\
&\quad (\text{Since } a, d, e \text{ are collinear; } [de, ec] = [ea, ec].) \\
&= [ea, ec] + [ea, eb] \\
&\quad (\text{Since } e, a, c, b \text{ are cyclic; } [ea, ec] = [ab, cb].) \\
&= [ea, eb] + [ab, cb] \\
&\quad (\text{Since } e, a, b, c \text{ are cyclic; } [ea, eb] = [ac, cb].) \\
&= [ac, cb] + [ab, cb] \\
&\quad (\text{Since } ac = ab \text{ } [ac, cb] = -[ab, cb].) \\
&= 0
\end{aligned}$$

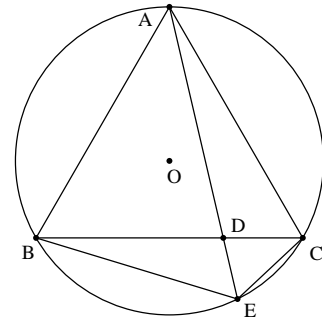


Figure 76

Example 77 A circle touching AB at B and passing through the incenter I (i.e., the center I of the inscribed circle) of the triangle ABC meets AC in H, K . Prove that IC bisects the angle HIK .

Point order: b, a, c, i, o, h, k .

Hypotheses: $\text{incenter}(i, a, b, c)$, $\text{perp}(b, o, a, b)$, $\text{pbisector}(o, b, i)$, $\text{pbisector}(o, b, h)$, $\text{coll}(a, c, h)$, $\text{coll}(a, c, k)$, $\text{pbisector}(o, b, k)$.

Conclusion: $\text{eqangle}(h, i, c, i, k)$.

The Machine Proof

$$\begin{aligned}
&[ki, ic] + [hi, ic] \\
&\quad (\text{Since } a, h, k \text{ are collinear; } k, i, h, b \text{ are cyclic;}
\end{aligned}$$

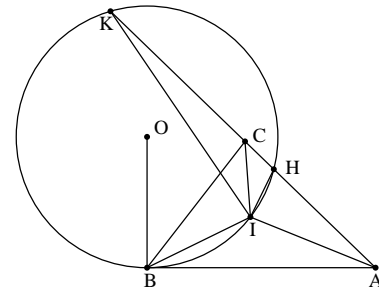


Figure 77

$$\begin{aligned}
& [ki, ic] = [ki, kh] + [ha, ic] = [ha, ic] - [hb, ib].) \\
& = [hi, ic] + [ha, ic] - [hb, ib] \\
& \quad (\text{Since circumcenter}(o, h, i, b); [hi, ic] = [hi, ob] + [ib, ic] = [hb, ob] - [ic, ib] + 1.) \\
& = [ha, ic] + [hb, ob] - [hb, ib] - [ic, ib] + 1 \\
& \quad (\text{Since } hb \parallel hb; [hb, ob] - [hb, ib] = -[ob, ib].) \\
& = [ha, ic] - [ob, ib] - [ic, ib] + 1 \\
& \quad (\text{Since } a, c, h \text{ are collinear; } [ha, ic] = -[ic, ca].) \\
& = -[ob, ib] - [ic, ib] - [ic, ca] + 1 \\
& \quad (\text{Since } ob \perp ba; [ob, ib] = -[ib, ab] + 1.) \\
& = -[ic, ib] - [ic, ca] + [ib, ab] \\
& \quad (\text{Since } i \text{ is the incenter of triangle } a, c, b [ic, ib] = [ia, ab] + 1.) \\
& = -[ic, ca] - [ia, ab] + [ib, ab] - 1 \\
& \quad (\text{Since } i \text{ is the incenter of triangle } c, a, b [ic, ca] = -[ic, cb].) \\
& = [ic, cb] - [ia, ab] + [ib, ab] - 1 \\
& \quad (\text{Since } i \text{ is the incenter of triangle } c, a, b [ic, cb] = [ia, ab] - [ib, ab] + 1.) \\
& = 0
\end{aligned}$$

Example 78 *ABC is triangle inscribed in a circle; DE is the diameter bisecting BC at G; from E a perpendicular EK is drawn to one of the sides, and the perpendicular from the vertex A on DE meets DE in H. Show that EK touches the circle GHK.*

Point order: $a, b, c, o, g, e, k, h, n$.

Hypotheses: $\text{circumcenter}(o, b, a, c)$, $\text{midpoint}(g, b, c)$, $\text{coll}(g, o, e)$,

$\text{pbisector}(o, b, e)$, $\text{perp}(k, e, a, b)$, $\text{coll}(k, a, b)$, $\text{perp}(a, h, o, g)$, $\text{coll}(h, o, g)$, $\text{circumcenter}(n, g, h, k)$.

Conclusion: $\text{perp}(e, k, k, n)$.

The Machine Proof

$$\begin{aligned}
& -[nk, ke] + 1 \\
& \quad (\text{Since } ke \perp ab; [nk, ke] = [nk, ba] + 1.) \\
& = -[nk, ba] \\
& \quad (\text{Since circumcenter}(n, k, g, h); [nk, ba] = [nk, kg] + [kg, ba] = [hk, hg] + [kg, ba] + 1.) \\
& = -[hk, hg] - [kg, ba] - 1 \\
& \quad (\text{Since } hg \perp bc; [hk, hg] = [hk, cb] + 1.) \\
& = -[hk, cb] - [kg, ba] \\
& \quad (\text{Since } e, g, h \text{ are collinear; } h, k, e, a \text{ are cyclic;} \\
& \quad [hk, cb] = [hk, he] + [eg, cb] = [ka, ea] + [eg, cb].) \\
& = -[kg, ba] - [ka, ea] - [eg, cb] \\
& \quad (\text{Since } a, b, k \text{ are collinear; } k, g, b, e \text{ are cyclic;} \\
& \quad [kg, ba] = [kg, kb] = [eg, eb].)
\end{aligned}$$

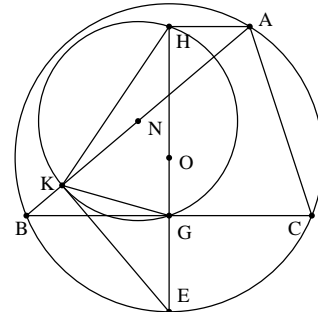


Figure 78

$$\begin{aligned}
&= -[ka, ea] - [eg, eb] - [eg, cb] \\
&\quad (\text{Since } a, b, k \text{ are collinear; } [ka, ea] = -[ea, ba].) \\
&= -[eg, eb] - [eg, cb] + [ea, ba] \\
&\quad (\text{Since } eg \perp bc; [eg, eb] = -[eb, cb] + 1.) \\
&= -[eg, cb] + [eb, cb] + [ea, ba] - 1 \\
&\quad (\text{Since } eg \perp cb; [eg, cb] = 1.) \\
&= [eb, cb] + [ea, ba] \\
&\quad (\text{Since } b, e, c, a \text{ are cyclic; } [eb, cb] = [ea, ca].) \\
&= [ea, ca] + [ea, ba] \\
&\quad (\text{Since circumcenter}(o, a, e, c, b); oc \perp eb; [ea, ca] = -[ea, ba].) \\
&= 0
\end{aligned}$$

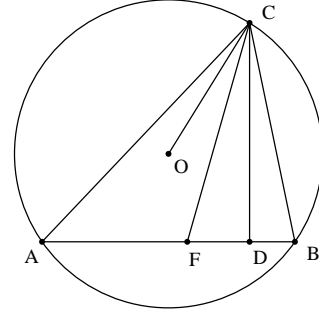


Figure 79

Example 79 *The angle between the circumdiameter and the altitude issued from the same vertex of a triangle is bisected by the bisector of angle of the triangle at the vertex considered.*

Point order: a, b, c, o, d, f .

Hypotheses: $\text{pbisector}(o, a, b)$, $\text{pbisector}(o, a, c)$, $\text{coll}(a, b, d)$,

$\text{perp}(a, b, c, d)$, $\text{coll}(f, a, b)$, $\text{eqangle}(a, c, f, f, c, b)$.

Conclusion: $\text{eqangle}(d, c, f, f, c, o)$.

The Machine Proof

$$\begin{aligned}
&-[fc, dc] - [fc, oc] \\
&\quad (\text{Since } \angle[cf, cd] = \angle[ac, cf]; [fc, dc] = [fc, bc] + [bc, dc] = -[fc, ca] - [dc, cb].) \\
&= -[fc, oc] + [fc, ca] + [dc, cb] \\
&\quad (\text{Since } fc \parallel fc; -[fc, oc] + [fc, ca] = [oc, ca].) \\
&= [dc, cb] + [oc, ca] \\
&\quad (\text{Since } dc \perp ab; [dc, cb] = -[cb, ba] + 1.) \\
&= [oc, ca] - [cb, ba] + 1 \\
&\quad (\text{Since circumcenter}(o, c, a, b); [oc, ca] = [oc, ca] = [cb, ba] + 1.) \\
&= 0
\end{aligned}$$

Example 80 *Let U and U_1 be the intersections of the bisectors of angle A of a triangle ABC with side BC . If the tangent at A to the circumcircle meets BC in T , we have $TA \equiv TU \equiv TU_1$.*

Point order: a, b, c, o, u, t .

Hypotheses: $\text{pbisector}(o, b, c)$, $\text{pbisector}(o, b, a)$,

$\text{eqangle}(b, a, u, u, a, c)$, $\text{coll}(b, u, c)$, $\text{perp}(o, a, a, t)$, $\text{coll}(b, c, t)$.

Conclusion: $\text{pbisector}(t, a, u)$.

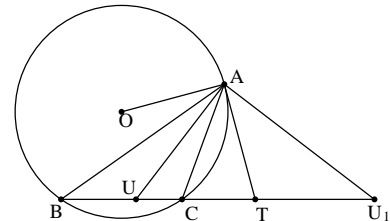


Figure 80

The Machine Proof

$$[tu, ua] + [ta, ua]$$

(Since b, c, t, u are collinear; $[tu, ua] = -[ua, cb]$.)

$$= [ta, ua] - [ua, cb]$$

(Since $ta \perp ao$; $[ta, ua] = -[ua, oa] + 1$.)

$$= -[ua, oa] - [ua, cb] + 1$$

(Since $\angle[au, ao] = \angle[ab, au]$; $[ua, oa] = [ua, ac] + [ac, oa] = -[ua, ba] - [oa, ca]$.)

$$= -[ua, cb] + [ua, ba] + [oa, ca] + 1$$

(Since $ua \parallel ua$; $-[ua, cb] + [ua, ba] = [cb, ba]$.)

$$= [oa, ca] + [cb, ba] + 1$$

(Since $\text{circumcenter}(o, a, c, b)$; $[oa, ca] = [oa, ac] = -[cb, ba] + 1$.)

$$= 0$$

Example 81 *If the two bisectors of the angle A of the triangle ABC are equal, and the circle having BC for diameter cuts the sides AB, AC in the points P, Q , show that $CP \equiv CQ$.*

Point order: u, v, a, b, c, o, p, q .

Hypotheses: $\text{perp}(a, u, a, v)$, $\text{cong}(a, u, a, v)$, $\text{coll}(u, v, b)$,
 $\text{eqangle}(c, a, u, u, a, b)$, $\text{coll}(u, v, c)$, $\text{midpoint}(o, b, c)$, $\text{coll}(a, b, p)$,
 $\text{coll}(a, c, q)$, $\text{pbisector}(o, b, q)$, $\text{pbisector}(o, b, p)$.

Conclusion: $\text{pbisector}(c, p, q)$.

The Machine Proof

$$-[qp, qc] - [qp, pc]$$

(Since q, p, c, b are cyclic; $[qp, qc] = [pb, cb]$.)

$$= -[qp, pc] - [pb, cb]$$

(Since $pc \perp ab$; $[qp, pc] = [qp, ba] + 1$.)

$$= -[qp, ba] - [pb, cb] - 1$$

(Since a, c, q are collinear; q, p, c, b are cyclic; $[qp, ba] = [qp, qc] + [ca, ba] = [pb, cb] + [ca, ba]$.)

$$= -2[pb, cb] - [ca, ba] - 1$$

(Since a, b, p are collinear; $[pb, cb] = -[cb, ba]$.)

$$= 2[cb, ba] - [ca, ba] - 1$$

(Since b, c, u, v are collinear; $[cb, ba] = -[ba, vu]$.)

$$= -[ca, ba] - 2[ba, vu] - 1$$

(Since $\angle[ac, ab] = \angle[ua, ab]$; $[ca, ba] = [ca, ua] + [ua, ba] = -2[ba, au]$.)

$$= 2[ba, au] - 2[ba, vu] - 1$$

(Since $ba \parallel ba$; $2[ba, au] - 2[ba, vu] = -2[au, vu]$.)

$$= -2[au, vu] - 1$$

(Since $au = av$ $ua \perp av$; $[au, vu] =_1 423772$.)

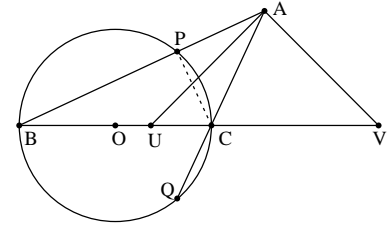


Figure 81

$$= 0$$

Example 82 *The feet of the four perpendiculars dropped from a vertex of a triangle upon the four bisectors of the other two angles are collinear.*

Point order: a, b, c, o, i, e, j, l .

Hypotheses: $\text{incenter}(o, a, b, c)$, $\text{foot}(i, c, a, o)$, $\text{perp}(e, a, o, a)$,
 $\text{foot}(j, c, a, e)$, $\text{foot}(l, c, b, o)$.

Conclusion: $\text{coll}(i, j, l)$.

The Machine Proof

$$-[li, ji]$$

(Since b, l, o are collinear; l, i, o, c are cyclic;

$$[li, ji] = [li, lo] + [ob, ji] = -[ji, ob] + [ic, oc].)$$

$$= [ji, ob] - [ic, oc]$$

(Since a, e, j are collinear; j, i, a, c are cyclic;

$$[ji, ob] = [ji, ja] + [ae, ob] = [ea, ob] + [ic, ca].)$$

$$= [ea, ob] - [ic, oc] + [ic, ca]$$

(Since $ea \parallel ic$; $[ea, ob] - [ic, oc] = [oc, ob]$.)

$$= [ic, ca] + [oc, ob]$$

(Since $ic \perp ao$; $[ic, ca] = [oa, ca] + 1$.)

$$= [oc, ob] + [oa, ca] + 1$$

(Since o is the incenter of triangle a, c, b $[oc, ob] = [oa, ba] + 1$.)

$$= [oa, ca] + [oa, ba]$$

(Since o is the incenter of triangle a, c, b $[oa, ca] = -[oa, ba]$.)

$$= 0$$

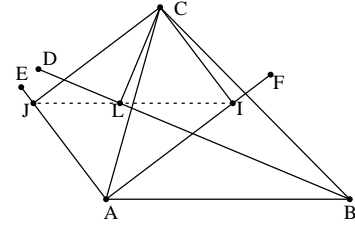


Figure 82

Example 83 *The angle which a side of a triangle makes with the corresponding side of the orthic triangle is equal to the difference of the angles of the given triangle adjacent to the side considered.*

Point order: a, b, c, d, e, x .

Hypotheses: $\text{coll}(b, c, d)$, $\text{perp}(b, c, a, d)$, $\text{coll}(a, c, e)$,
 $\text{perp}(a, c, b, e)$, $\text{coll}(a, b, x)$, $\text{coll}(d, e, x)$.

Conclusion: $[dx, xa] - [ca, ab] - [cb, ba]$.

The Machine Proof

$$[xd, xa] - [cb, ba] - [ca, ba]$$

(Since $xa \parallel ba$; $[xd, xa] - [cb, ba] = [xd, cb]$.)

$$= [xd, cb] - [ca, ba]$$

(Since d, e, x are collinear; $[xd, cb] = [ed, cb]$.)

$$= [ed, cb] - [ca, ba]$$

(Since a, c, e are collinear; e, d, a, b are cyclic;
 $[ed, cb] = [ed, ea] + [ac, cb] = [db, ba] - [cb, ca]$.)

$$= [db, ba] - [cb, ca] - [ca, ba]$$

(Since $db \parallel cb$; $[db, ba] - [cb, ca] = [ca, ba]$.)

$$= 0$$

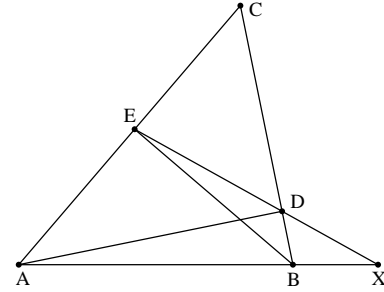


Figure 83

Example 84 *The tangent to the nine-point circle at the midpoint of a side of the given triangle is antiparallel to this side with respect to the two other sides of the triangle.*

Point order: $b, c, a, a_1, b_1, c_1, n, k, j$.

Hypotheses: $\text{midpoint}(a_1, b, c)$, $\text{midpoint}(c_1, b, a)$, $\text{midpoint}(b_1, a, c)$,
 $\text{pbisector}(n, a_1, b_1)$, $\text{pbisector}(n, a_1, c_1)$, $\text{perp}(a_1, k, a_1, n)$, $\text{coll}(a, c, k)$,
 $\text{coll}(k, a_1, j)$, $\text{coll}(a, b, j)$.

Conclusion: $\text{cyclic}(k, j, b, c)$.

The Machine Proof

$$[jk, jb] - [kc, cb]$$

(Since a_1, j, k are collinear; a, b, j are collinear; $[jk, jb] = [ka_1, ab]$.)

$$= [ka_1, ab] - [kc, cb]$$

(Since $ka_1 \perp a_1 n$; $[ka_1, ab] = [na_1, ab] + 1$.)

$$= -[kc, cb] + [na_1, ab] + 1$$

(Since a, c, k are collinear; $[kc, cb] = [ac, cb]$.)

$$= [na_1, ab] - [ac, cb] + 1$$

(Since $\text{circumcenter}(n, a_1, c_1, b_1)$);

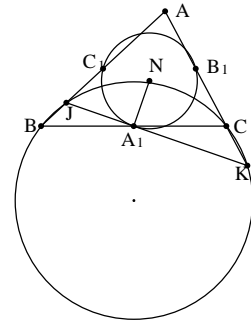


Figure 84

$$\begin{aligned}
[na_1, ab] &= [na_1, a_1c_1] + [a_1c_1, ab] = -[c_1b_1, b_1a_1] + [c_1a_1, ab] + 1.) \\
&= -[c_1b_1, b_1a_1] + [c_1a_1, ab] - [ac, cb] \\
&\quad (\text{Since } c_1a_1 \parallel ac; [c_1a_1, ab] - [ac, cb] = -[ab, cb].) \\
&= -[c_1b_1, b_1a_1] - [ab, cb] \\
&\quad (\text{Since } b_1a_1 \parallel ab; -[c_1b_1, b_1a_1] - [ab, cb] = -[c_1b_1, cb].) \\
&= -[c_1b_1, cb] \\
&\quad (\text{Since } c_1b_1 \parallel cb; [c_1b_1, cb] = 0.) \\
&= 0
\end{aligned}$$

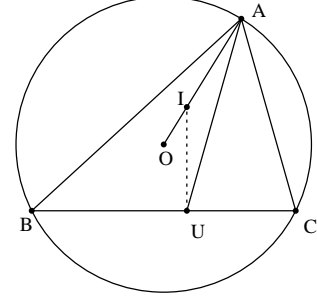


Figure 85

Example 85 Show that the mediator of the bisector AU of the triangle ABC , the perpendicular to BC at U , and the circumdiameter of ABC passing through A are concurrent.

Point order: a, b, c, o, u, i .

Hypotheses: $\text{pbisector}(o, a, b)$, $\text{pbisector}(o, b, c)$, $\text{eqangle}(b, a, u, u, a, c)$, $\text{coll}(b, c, u)$, $\text{pbisector}(i, a, u)$, $\text{coll}(i, o, a)$.

Conclusion: $\text{perp}(i, u, b, c)$.

The Machine Proof

$$\begin{aligned}
&[iu, cb] + 1 \\
&\quad (\text{Since } iu = ia \text{ } a, i, o \text{ are collinear; } [iu, cb] = [iu, ua] + [ua, cb] = [ua, oa] + [ua, cb].) \\
&= [ua, oa] + [ua, cb] + 1 \\
&\quad (\text{Since } \angle[au, ao] = \angle[ab, au]; [ua, oa] = [ua, ac] + [ac, oa] = -[ua, ba] - [oa, ca].) \\
&= [ua, cb] - [ua, ba] - [oa, ca] + 1 \\
&\quad (\text{Since } ua \parallel ua; [ua, cb] - [ua, ba] = -[cb, ba].) \\
&= -[oa, ca] - [cb, ba] + 1 \\
&\quad (\text{Since circumcenter}(o, a, c, b); [oa, ca] = [oa, ac] = -[cb, ba] + 1.) \\
&= 0
\end{aligned}$$

Example 86 The internal bisector of the angle B of the triangle ABC meets the sides B_1C_1 , B_1A_1 of the medial triangle in the points A_2 , C_2 . Prove that AA_2 , CC_2 are perpendicular to the bisector, and that $B_1A_2 = B_1C_2$. Similarly for the external bisector.

Point order: $a, b, c, a_1, b_1, c_1, a_2, c_2$.

Hypotheses: $\text{midpoint}(a, b_1, c)$, $\text{midpoint}(a, c_1, b)$, $\text{midpoint}(b, a_1, c)$, $\text{eqangle}(a, b, a_2, a_2, b, c)$, $\text{coll}(b_1, c_1, a_2)$, $\text{coll}(b, a_2, c_2)$, $\text{coll}(a_1, b_1, c_2)$.

Conclusion: $\text{pbisector}(b_1, a_2, c_2)$.

The Machine Proof

$$-[c_2a_2, c_2b_1] - [c_2a_2, a_2b_1]$$

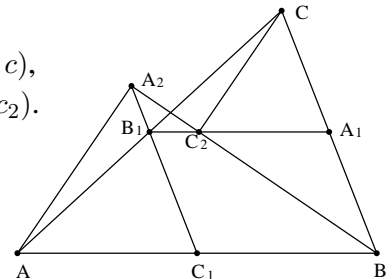


Figure 86

$$\begin{aligned}
& \text{(Since } a_2, b, c_2 \text{ are collinear; } a_1, b_1, c_2 \text{ are collinear; } [c_2 a_2, c_2 b_1] = [a_2 b, b_1 a_1].\text{.)} \\
& = -[c_2 a_2, a_2 b_1] - [a_2 b, b_1 a_1] \\
& \text{(Since } a_2, b, c_2 \text{ are collinear; } [c_2 a_2, a_2 b_1] = -[a_2 b_1, a_2 b].\text{.)} \\
& = [a_2 b_1, a_2 b] - [a_2 b, b_1 a_1] \\
& \text{(Since } a_2 b_1 \parallel bc; [a_2 b_1, a_2 b] = -[a_2 b, cb].\text{.)} \\
& = -[a_2 b, b_1 a_1] - [a_2 b, cb] \\
& \text{(Since } \angle[ba_2, a_1 b_1] = \angle[ab, ba_2]; [a_2 b, b_1 a_1] = [a_2 b, bc] + [bc, b_1 a_1] = -[a_2 b, ba] - [b_1 a_1, cb].\text{.)} \\
& = -[a_2 b, cb] + [a_2 b, ba] + [b_1 a_1, cb] \\
& \text{(Since } a_2 b \parallel a_2 b; -[a_2 b, cb] + [a_2 b, ba] = [cb, ba].\text{.)} \\
& = [b_1 a_1, cb] + [cb, ba] \\
& \text{(Since } cb \parallel cb; [b_1 a_1, cb] + [cb, ba] = [b_1 a_1, ba].\text{.)} \\
& = [b_1 a_1, ba] \\
& \text{(Since } b_1 a_1 \parallel ba; [b_1 a_1, ba] = 0.\text{.)} \\
& = 0
\end{aligned}$$

Example 87 With the usual notation, show that the angle formed by the lines C_1E , B_1F is equal to $3A$, or its supplement.

Point order: $a, b, c, e, f, c_1, b_1, x$.

Hypotheses: $\text{coll}(a, b, f)$, $\text{coll}(a, c, e)$, $\text{perp}(a, b, c, f)$,

$\text{perp}(a, c, b, e)$, $\text{midpoint}(c_1, a, b)$, $\text{midpoint}(b_1, a, c)$, $\text{coll}(x, c_1, e)$, $\text{coll}(x, b_1, f)$.

Conclusion: $3[ba, ac] - [b_1x, xc_1]$.

The Machine Proof

$$\begin{aligned}
& -[xb_1, xc_1] - 3[ca, ba] \\
& \text{(Since } b_1, f, x \text{ are collinear; } c_1, e, x \text{ are collinear;} \\
& \quad [xb_1, xc_1] = [b_1f, c_1e].\text{.)} \\
& = -[b_1f, c_1e] - 3[ca, ba] \\
& \text{(Since circumcenter}(b_1, f, a, c); [b_1f, c_1e] = [b_1f, fa] + [fa, c_1e] = -[c_1e, fa] + [fc, ca] + 1.\text{.)} \\
& = [c_1e, fa] - [fc, ca] - 3[ca, ba] - 1 \\
& \text{(Since circumcenter}(c_1, e, a, b); [c_1e, fa] = [c_1e, ea] + [ea, fa] = -[fa, ea] + [eb, ba] + 1.\text{.)} \\
& = -[fc, ca] - [fa, ea] + [eb, ba] - 3[ca, ba] \\
& \text{(Since } fc \perp ab; [fc, ca] = -[ca, ba] + 1.\text{.)} \\
& = -[fa, ea] + [eb, ba] - 2[ca, ba] - 1 \\
& \text{(Since } a, b, f \text{ are collinear; } [fa, ea] = -[ea, ba].\text{.)} \\
& = [eb, ba] + [ea, ba] - 2[ca, ba] - 1 \\
& \text{(Since } eb \perp ac; [eb, ba] = [ca, ba] + 1.\text{.)} \\
& = [ea, ba] - [ca, ba] \\
& \text{(Since } a, c, e \text{ are collinear; } [ea, ba] = [ca, ba].\text{.)} \\
& = 0
\end{aligned}$$

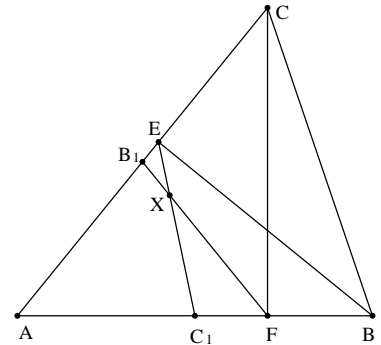


Figure 87

Example 88 Show that the symmetrics, P , Q of a given point L with respect to the sides Ox , Oy of a given angle, and the points $P_1 = (LQ, Ox)$, $Q_1 = (LP, Oy)$ lie on a circle passing through O .

Point order: $o, x, y, l, a, b, p, q, p_1, q_1, i$.

Hypotheses: $\text{perp}(l, a, o, x)$, $\text{coll}(a, o, x)$, $\text{perp}(l, b, o, y)$,
 $\text{coll}(o, b, y)$, $\text{midpoint}(a, l, p)$, $\text{midpoint}(b, l, q)$, $\text{coll}(p_1, l, q)$,
 $\text{coll}(o, x, p_1)$, $\text{coll}(q_1, l, p)$, $\text{coll}(q_1, o, y)$.

Conclusion: $\text{cyclic}(o, p, q, p_1)$.

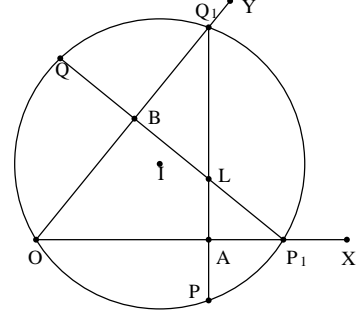


Figure 88

The Machine Proof

$$\begin{aligned}
& [p_1p, p_1o] - [qp, qo] \\
& \quad (\text{Since } p_1o \perp la; [p_1p, p_1o] = [p_1p, al] + 1.) \\
& = [p_1p, al] - [qp, qo] + 1 \\
& \quad (\text{Since } al \perp ox; [p_1p, al] = [p_1p, xo] + 1.) \\
& = [p_1p, xo] - [qp, qo] \\
& \quad (\text{Since } p_1p = p_1l \text{ } b, l, p_1 \text{ are collinear; } [p_1p, xo] = [p_1p, pl] + [pl, xo] = [pl, bl] + [pl, xo].) \\
& = -[qp, qo] + [pl, bl] + [pl, xo] \\
& \quad (\text{Since } qp \parallel ab; [qp, qo] = -[qo, ba].) \\
& = [qo, ba] + [pl, bl] + [pl, xo] \\
& \quad (\text{Since } b, l, q \text{ are collinear; } oq = ol \text{ } [qo, ba] = [qo, ql] + [lb, ba] = -[ba, bl] + [bl, lo].) \\
& = [pl, bl] + [pl, xo] - [ba, bl] + [bl, lo] \\
& \quad (\text{Since } a, l, p \text{ are collinear; } [pl, bl] = -[bl, al].) \\
& = [pl, xo] - [ba, bl] - [bl, al] + [bl, lo] \\
& \quad (\text{Since } bl \parallel bl; -[ba, bl] - [bl, al] = -[ba, al].) \\
& = [pl, xo] - [ba, al] + [bl, lo] \\
& \quad (\text{Since } pl \perp xo; [pl, xo] = 1.) \\
& = -[ba, al] + [bl, lo] + 1 \\
& \quad (\text{Since } a, b, l, o \text{ are cyclic; } [ba, al] = [bo, lo].) \\
& = [bl, lo] - [bo, lo] + 1 \\
& \quad (\text{Since } b, o, y \text{ are collinear; } [bo, lo] = -[lo, yo].) \\
& = [bl, lo] + [lo, yo] + 1 \\
& \quad (\text{Since } bl \perp oy; [bl, lo] = -[lo, yo] + 1.) \\
& = 0
\end{aligned}$$

Example 89 Show that the nine-point center of the triangle IBC lies on the internal bisector of the angle A_1 of the complementary triangle $A_1B_1C_1$ of the given triangle ABC .

Point order: $b, c, a, i, a_1, b_1, c_1, p, q, n$.

Hypotheses: $\text{incenter}(i, a, b, c)$, $\text{midpoint}(p, b, i)$, $\text{midpoint}(q, c, i)$, $\text{pbisector}(n, a_1, p)$, $\text{pbisector}(n, a_1, q)$, $\text{midpoint}(a_1, c, b)$, $\text{midpoint}(b_1, c, a)$, $\text{midpoint}(c_1, a, b)$.

Conclusion: $\text{eqangle}(b_1, a_1, n, n, a_1, c_1)$.

The Machine Proof

$$\begin{aligned}
& -[na_1, c_1a_1] - [na_1, b_1a_1] \\
& \quad (\text{Since } c_1a_1 \parallel ca; [na_1, c_1a_1] = [na_1, ac].) \\
& = -[na_1, b_1a_1] - [na_1, ac] \\
& \quad (\text{Since } b_1a_1 \parallel ba; [na_1, b_1a_1] = [na_1, ab].) \\
& = -[na_1, ac] - [na_1, ab] \\
& \quad (\text{Since } \text{circumcenter}(n, a_1, q, p); [na_1, ac] = [na_1, a_1q] + [a_1q, ac] = -[qp, pa_1] + [qa_1, ac] + 1.) \\
& = -[na_1, ab] + [qp, pa_1] - [qa_1, ac] - 1 \\
& \quad (\text{Since } \text{circumcenter}(n, a_1, q, p); [na_1, ab] = [na_1, a_1q] + [a_1q, ab] = -[qp, pa_1] + [qa_1, ab] + 1.) \\
& = 2[qp, pa_1] - [qa_1, ac] - [qa_1, ab] \\
& \quad (\text{Since } qp \parallel bc; pa_1 \parallel ci; [qp, pa_1] = -[ic, cb].) \\
& = -[qa_1, ac] - [qa_1, ab] - 2[ic, cb] \\
& \quad (\text{Since } qa_1 \parallel bi; [qa_1, ac] = [ib, ac].) \\
& = -[qa_1, ab] - 2[ic, cb] - [ib, ac] \\
& \quad (\text{Since } qa_1 \parallel bi; [qa_1, ab] = [ib, ab].) \\
& = -2[ic, cb] - [ib, ac] - [ib, ab] \\
& \quad (\text{Since } i \text{ is the incenter of triangle } b, a, c \text{ } [ib, ac] = -2[ic, cb] + [ib, cb].) \\
& = -[ib, ab] - [ib, cb] \\
& \quad (\text{Since } i \text{ is the incenter of triangle } b, a, c \text{ } [ib, ab] = -[ib, cb].) \\
& = 0
\end{aligned}$$

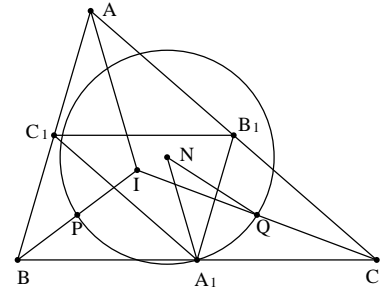


Figure 89

Example 90 If the perpendiculars dropped from a point D of the circumcircle (O) of the triangle ABC upon the sides, meet (O) in the points A_1, B_1, C_1 , show that the two triangle $ABC, A_1B_1C_1$ are congruent and symmetrical with respect to an axis.

Point order: $a, b, c, d, o, f, g, c_1, b_1$.

Hypotheses: $\text{pbisector}(o, a, b)$, $\text{pbisector}(o, a, c)$, $\text{pbisector}(o, a, d)$,
 $\text{coll}(a, b, g)$, $\text{coll}(a, c, f)$, $\text{perp}(a, b, d, g)$, $\text{perp}(a, c, d, f)$,
 $\text{coll}(d, g, c_1)$, $\text{pbisector}(o, a, c_1)$, $\text{coll}(d, f, b_1)$, $\text{pbisector}(o, a, b_1)$.

Conclusion: $\text{para}(c_1, c, b_1, b)$.

The Machine Proof

$$\begin{aligned}
 & -[b_1b, c_1c] \\
 & \quad (\text{Since } b_1, d, f \text{ are collinear; } b_1, b, d, c_1 \text{ are cyclic;} \\
 & \quad [b_1b, c_1c] = [b_1b, b_1d] + [df, c_1c] = -[c_1d, c_1b] - [c_1c, fd].) \\
 & = [c_1d, c_1b] + [c_1c, fd] \\
 & \quad (\text{Since } c_1d \perp ab; [c_1d, c_1b] = -[c_1b, ba] + 1.) \\
 & = [c_1c, fd] - [c_1b, ba] + 1 \\
 & \quad (\text{Since } fd \perp ac; [c_1c, fd] = [c_1c, ca] + 1.) \\
 & = [c_1c, ca] - [c_1b, ba] \\
 & \quad (\text{Since } c, c_1, a, b \text{ are cyclic; } [c_1c, ca] = [c_1b, ba].) \\
 & = 0
 \end{aligned}$$

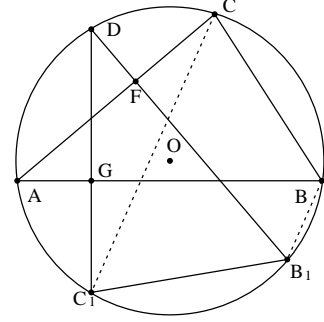


Figure 90

Example 91 The perpendiculars through the points P, P_1 to the Simson lines $P_1(ABC), P(ABC)$, respectively, intersect on the circumcircle of the triangle ABC .

Point order: $a, b, c, o, p, p_1, g, g_1, f, f_1, k$.

Hypotheses: $\text{pbisector}(o, a, b)$, $\text{pbisector}(o, a, c)$, $\text{pbisector}(o, a, p)$, $\text{pbisector}(o, a, p_1)$,
 $\text{perp}(p, g, a, b)$, $\text{coll}(g, a, b)$, $\text{perp}(p, f, a, c)$, $\text{coll}(a, c, f)$, $\text{perp}(p_1, g_1, a, b)$,
 $\text{coll}(g_1, a, b)$, $\text{perp}(p_1, f_1, a, c)$, $\text{coll}(a, c, f_1)$, $\text{perp}(k, p, f_1, g_1)$, $\text{perp}(k, p_1, f, g)$.

Conclusion: $\text{cyclic}(p, p_1, a, k)$.

The Machine Proof

$$\begin{aligned}
 & [kp_1, kp] - [p_1a, pa] \\
 & \quad (\text{Since } kp_1 \perp gf; kp \perp g_1f_1; [kp_1, kp] = -[f_1g_1, fg].) \\
 & = -[f_1g_1, fg] - [p_1a, pa] \\
 & \quad (\text{Since } a, c, f_1 \text{ are collinear; } f_1, g_1, a, p_1 \text{ are cyclic;} \\
 & \quad [f_1g_1, fg] = [f_1g_1, f_1a] + [ac, fg] = -[fg, ca] + [g_1p_1, p_1a].) \\
 & = [fg, ca] - [g_1p_1, p_1a] - [p_1a, pa] \\
 & \quad (\text{Since } p_1a \parallel p_1a; -[g_1p_1, p_1a] - [p_1a, pa] = -[g_1p_1, pa].)
 \end{aligned}$$

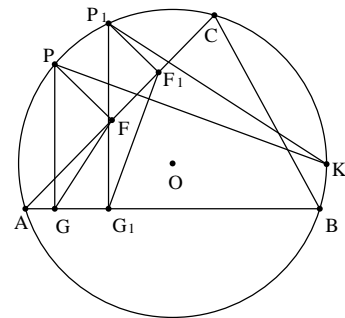


Figure 91

$$\begin{aligned}
&= [fg, ca] - [g_1p_1, pa] \\
&\quad (\text{Since } a, c, f \text{ are collinear; } f, g, a, p \text{ are cyclic;} \\
&\quad [fg, ca] = [fg, fa] = [gp, pa].) \\
&= -[g_1p_1, pa] + [gp, pa] \\
&\quad (\text{Since } g_1p_1 \parallel pg; [g_1p_1, pa] = [gp, pa].) \\
&= 0
\end{aligned}$$

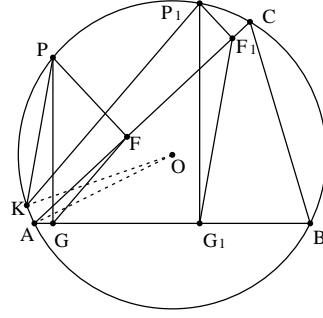


Figure 92

Example 92 The parallels through the points P, P_1 to the Simson lines $P_1(ABC), P(ABC)$, respectively, intersect on the circumcircle of the triangle ABC .

Point order: $a, b, c, o, p, p_1, g, g_1, f, f_1, k$.

Hypotheses: $\text{pbisector}(o, a, b), \text{pbisector}(o, a, c), \text{pbisector}(o, a, p), \text{pbisector}(o, a, p_1),$
 $\text{perp}(p, g, a, b), \text{coll}(g, a, b), \text{perp}(p, f, a, c), \text{coll}(a, c, f), \text{perp}(p_1, g_1, a, b),$
 $\text{coll}(g_1, a, b), \text{perp}(p_1, f_1, a, c), \text{coll}(a, c, f_1), \text{para}(k, p, f_1, g_1), \text{para}(k, p_1, f, g).$

Conclusion: $\text{cyclic}(p, p_1, a, k).$

The Machine Proof

$$\begin{aligned}
&[kp_1, kp] - [p_1a, pa] \\
&\quad (\text{Since } kp_1 \parallel gf; kp \parallel g_1f_1; [kp_1, kp] = -[f_1g_1, fg].) \\
&= -[f_1g_1, fg] - [p_1a, pa] \\
&\quad (\text{Since } a, c, f_1 \text{ are collinear; } f_1, g_1, a, p_1 \text{ are cyclic;} \\
&\quad [f_1g_1, fg] = [f_1g_1, f_1a] + [ac, fg] = -[fg, ca] + [g_1p_1, p_1a].) \\
&= [fg, ca] - [g_1p_1, p_1a] - [p_1a, pa] \\
&\quad (\text{Since } p_1a \parallel p_1a; -[g_1p_1, p_1a] - [p_1a, pa] = -[g_1p_1, pa].) \\
&= [fg, ca] - [g_1p_1, pa] \\
&\quad (\text{Since } a, c, f \text{ are collinear; } f, g, a, p \text{ are cyclic; } [fg, ca] = [fg, fa] = [gp, pa].) \\
&= -[g_1p_1, pa] + [gp, pa] \\
&\quad (\text{Since } g_1p_1 \parallel pg; [g_1p_1, pa] = [gp, pa].) \\
&= 0
\end{aligned}$$

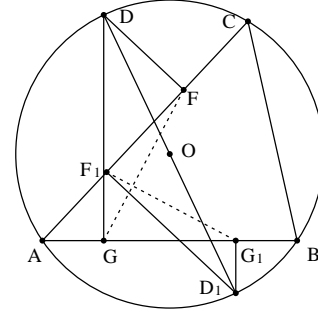


Figure 93

Example 93 The Simson lines of two diametrically opposite points are perpendicular.

Point order: $a, b, c, d, o, d_1, f, g, f_1, g_1, k, a_1, b_1, c_1, n$.

Hypotheses: $\text{circumcenter}(o, a, b, c, d, d_1), \text{coll}(a, b, g),$
 $\text{coll}(a, c, f), \text{perp}(a, b, d, g), \text{perp}(a, c, d, f), \text{midpoint}(o, d, d_1), \text{coll}(a, b, g_1), \text{coll}(a, c, f_1),$
 $\text{perp}(a, b, d_1, g_1), \text{perp}(a, c, d_1, f_1), \text{coll}(g, f, k), \text{coll}(g_1, f_1, k), \text{midpoint}(a_1, c, b), \text{midpoint}(b_1, a, c),$
 $\text{midpoint}(c_1, a, b), \text{pbisector}(n, c_1, a_1), \text{pbisector}(n, c_1, b_1), \text{pbisector}(n, c_1, k).$

Conclusion: $\text{perp}(f, g, f_1, g_1).$

The Machine Proof

$$-[g_1f_1, gf] + 1$$

$$\begin{aligned}
& \text{(Since } a, b, g_1 \text{ are collinear; } g_1, f_1, a, d_1 \text{ are cyclic;} \\
& [g_1 f_1, gf] = [g_1 f_1, g_1 a] + [ab, gf] = [f_1 d_1, d_1 a] - [gf, ba].) \\
& = -[f_1 d_1, d_1 a] + [gf, ba] + 1 \\
& \text{(Since } f_1 d_1 \parallel df; [f_1 d_1, d_1 a] = [fd, d_1 a].) \\
& = [gf, ba] - [fd, d_1 a] + 1 \\
& \text{(Since } a, b, g \text{ are collinear; } g, f, a, d \text{ are cyclic; } [gf, ba] = [gf, ga] = [fd, da].) \\
& = -[fd, d_1 a] + [fd, da] + 1 \\
& \text{(Since } fd \parallel fd; -[fd, d_1 a] + [fd, da] = [d_1 a, da].) \\
& = [d_1 a, da] + 1 \\
& \text{(Since } d_1 a \perp da; [d_1 a, da] = 1.) \\
& = 0
\end{aligned}$$

Example 94 *The four Simson lines of a point of a circle for the four triangles determined by the vertices of a quadrilateral inscribed in that circle, admit the point considered for their Miquel point.*

Point order: $a, b, c, d, p, o, e, f, g, h, i$.

Hypotheses: $\text{cyclic}(a, b, c, d, p)$, $\text{perp}(p, g, a, b)$, $\text{perp}(p, h, b, c)$, $\text{perp}(p, e, c, d)$, $\text{perp}(p, f, d, a)$, $\text{coll}(g, a, b)$, $\text{coll}(h, b, c)$, $\text{coll}(e, c, d)$, $\text{coll}(f, d, a)$, $\text{coll}(i, g, f)$, $\text{coll}(i, e, h)$.

Conclusion: $\text{cyclic}(g, h, p, i)$.

The Machine Proof

$$\begin{aligned}
& [ih, ig] - [hp, gp] \\
& \text{(Since } e, h, i \text{ are collinear; } f, g, i \text{ are collinear; } [ih, ig] = [he, gf].) \\
& = [he, gf] - [hp, gp] \\
& \text{(Since } b, c, h \text{ are collinear; } h, e, c, p \text{ are cyclic;} \\
& [he, gf] = [he, hc] + [cb, gf] = -[gf, cb] + [ep, pc].) \\
& = -[hp, gp] - [gf, cb] + [ep, pc] \\
& \text{(Since } hp \perp bc; gp \perp ab; [hp, gp] = [cb, ba].) \\
& = -[gf, cb] + [ep, pc] - [cb, ba] \\
& \text{(Since } cb \parallel cb; -[gf, cb] - [cb, ba] = -[gf, ba].) \\
& = -[gf, ba] + [ep, pc] \\
& \text{(Since } a, b, g \text{ are collinear; } g, f, a, p \text{ are cyclic;} \\
& [gf, ba] = [gf, ga] = [fp, pa].) \\
& = -[fp, pa] + [ep, pc] \\
& \text{(Since } fp \perp ad; [fp, pa] = -[pa, da] + 1.) \\
& = [ep, pc] + [pa, da] - 1 \\
& \text{(Since } ep \perp cd; [ep, pc] = -[pc, dc] + 1.) \\
& = -[pc, dc] + [pa, da] \\
& \text{(Since } c, p, d, a \text{ are cyclic; } [pc, dc] = [pa, da].)
\end{aligned}$$

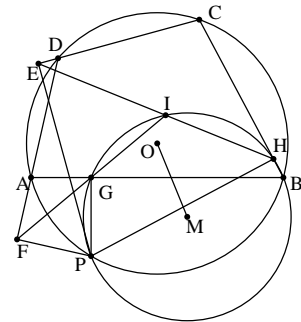


Figure 94

= 0

Example 95 *The traces, on the circumcircle of a triangle, of a median and the corresponding symmedian determine a line parallel to the side of the triangle opposite the vertex considered.*

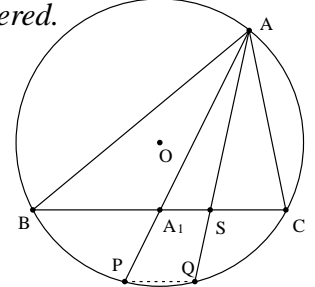


Figure 95

Point order: a, b, c, o, a_1, s, q, p .

Hypotheses: $\text{circumcenter}(o, a, b, c, p, q)$, $\text{midpoint}(a_1, b, c)$, $\text{coll}(a, a_1, p)$, $\text{eqangle}(b, a, s, a_1, a, c)$, $\text{coll}(b, s, c)$, $\text{coll}(a, s, q)$.

Conclusion: $\text{para}(p, q, b, c)$.

The Machine Proof

$[pq, cb]$

(Since a, a_1, p are collinear; p, q, a, b are cyclic; $[pq, cb] = [pq, pa] + [aa_1, cb] = [qb, ba] + [a_1a, cb]$.)

$= [qb, ba] + [a_1a, cb]$

(Since a, q, s are collinear; q, b, a, c are cyclic; $[qb, ba] = [qb, qa] + [as, ba] = [sa, ba] + [cb, ca]$.)

$= [sa, ba] + [a_1a, cb] + [cb, ca]$

(Since $cb \parallel cb$; $[a_1a, cb] + [cb, ca] = [a_1a, ca]$.)

$= [sa, ba] + [a_1a, ca]$

(Since $\angle[ab, as] = \angle[ac, aa_1]$; $[sa, ba] = -[a_1a, ca]$.)

= 0

Example 96 *If from a point on the symmedian perpendiculars are drawn to the including sides of the triangle, the line joining the feet of these perpendiculars is perpendicular to the corresponding median of the triangle.*

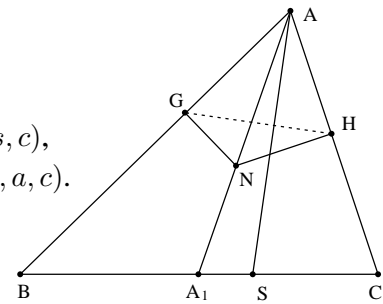


Figure 96

Point order: b, c, a, a_1, s, n, g, h .

Hypotheses: $\text{midpoint}(a_1, b, c)$, $\text{eqangle}(b, a, s, a_1, a, c)$, $\text{coll}(b, s, c)$, $\text{coll}(n, a, a_1)$, $\text{perp}(n, g, a, b)$, $\text{coll}(a, g, b)$, $\text{perp}(n, h, a, c)$, $\text{coll}(h, a, c)$.

Conclusion: $\text{perp}(g, h, a, s)$.

The Machine Proof

$[hg, sa] + 1$

(Since a, c, h are collinear; h, g, a, n are cyclic; $[hg, sa] = [hg, ha] + [ac, sa] = [gn, na] - [sa, ac]$.)

$= [gn, na] - [sa, ac] + 1$

(Since $gn \perp ba$; $[gn, na] = -[na, ab] + 1$.)

$= -[na, ab] - [sa, ac]$

(Since a, a_1, n are collinear; $[na, ab] = [a_1a, ab]$.)

$= -[sa, ac] - [a_1a, ab]$

$$\begin{aligned}
& \text{(Since } \angle[ba, as] = \angle[aa_1, ca] ; [sa, ac] = [sa, ba] + [ba, ac] = -[a_1a, ac] - [ac, ab].) \\
& = [a_1a, ac] - [a_1a, ab] + [ac, ab] \\
& \text{(Since } a_1a \parallel a_1a; [a_1a, ac] - [a_1a, ab] = -[ac, ab].) \\
& = 0
\end{aligned}$$

Example 97 *The median and the symmedian of a triangle ABC issued from A meet the circumcircle in P, Q. Show that the Simson lines of P, Q are respectively perpendicular to AP, AQ.*

Point order: $a, b, c, o, a_1, s, q, p, d, g$.

Hypotheses: $\text{circumcenter}(o, a, b, c, p, q)$, $\text{midpoint}(a_1, b, c)$, $\text{coll}(a, a_1, p)$, $\text{eqangle}(b, a, s, a_1, a, c)$, $\text{coll}(b, s, c)$, $\text{coll}(a, s, q)$, $\text{coll}(d, b, c)$, $\text{perp}(q, d, b, c)$, $\text{coll}(a, b, g)$, $\text{perp}(q, g, a, b)$.

Conclusion: $\text{perp}(d, g, a, p)$.

The Machine Proof

$$[gd, pa] + 1$$

$$\text{(Since } a, b, g \text{ are collinear; } g, d, b, q \text{ are cyclic; } [gd, pa] = [gd, gb] + [ba, pa] = [dq, qb] - [pa, ba].)$$

$$= [dq, qb] - [pa, ba] + 1$$

$$\text{(Since } dq \parallel oa_1; [dq, qb] = -[qb, a_1o].)$$

$$= -[pa, ba] - [qb, a_1o] + 1$$

$$\text{(Since } a, a_1, p \text{ are collinear; } [pa, ba] = [a_1a, ba].)$$

$$= -[qb, a_1o] - [a_1a, ba] + 1$$

$$\text{(Since } a_1o \perp bc; [qb, a_1o] = [qb, cb] + 1.)$$

$$= -[qb, cb] - [a_1a, ba]$$

$$\text{(Since } b, q, c, a \text{ are cyclic; } [qb, cb] = [qa, ca].)$$

$$= -[qa, ca] - [a_1a, ba]$$

$$\text{(Since } a, q, s \text{ are collinear; } [qa, ca] = [sa, ca].)$$

$$= -[sa, ca] - [a_1a, ba]$$

$$\text{(Since } \angle[ab, as] = \angle[aa_1, ac] ; [sa, ca] = [sa, ab] + [ab, ca] = -[a_1a, ca] - [ca, ba].)$$

$$= [a_1a, ca] - [a_1a, ba] + [ca, ba]$$

$$\text{(Since } a_1a \parallel a_1a; [a_1a, ca] - [a_1a, ba] = -[ca, ba].)$$

$$= 0$$

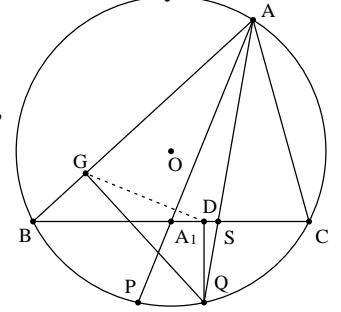


Figure 97

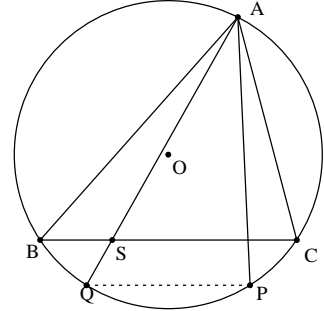


Figure 98

Example 98 *The line joining the traces, on the circumcircle of a triangle, of two isogonal lines of an angle of the triangle is parallel to the side opposite the vertex considered.*

Point order: a, b, c, o, p, s, q .

Hypotheses: $\text{pbisector}(o, p, b)$, $\text{pbisector}(o, c, b)$, $\text{pbisector}(o, a, b)$,

$\text{eqangle}(b, a, s, p, a, c), \text{coll}(b, s, c), \text{coll}(a, s, q), \text{pbisector}(o, q, b).$

Conclusion: $\text{para}(p, q, b, c).$

The Machine Proof

$[qp, cb]$

(Since a, q, s are collinear; q, p, a, c are cyclic; $[qp, cb] = [qp, qa] + [as, cb] = [sa, cb] + [pc, ca].$)

$= [sa, cb] + [pc, ca]$

(Since $\angle[ab, as] = \angle[ap, ac]$; $[sa, cb] = [sa, ab] + [ab, cb] = -[pa, ca] - [cb, ba].$)

$= [pc, ca] - [pa, ca] - [cb, ba]$

(Since c, p, a, b are cyclic; $[pc, ca] = [pb, ba].$)

$= [pb, ba] - [pa, ca] - [cb, ba]$

(Since $\text{circumcenter}(o, p, b, a)$; $[pb, ba] = [pb, oa] + [ba, ba] = [pa, oa] - [ba, ba] + 1.$)

$= [pa, oa] - [pa, ca] - [cb, ba] - [ba, ba] + 1$

(Since $pa \parallel oa$; $[pa, oa] - [pa, ca] = -[oa, ca].$)

$= -[oa, ca] - [cb, ba] - [ba, ba] + 1$

(Since $\text{circumcenter}(o, a, c, b)$; $[oa, ca] = [oa, ac] = -[cb, ba] + 1.$)

$= -[ba, ba]$

(Since $[ba, ba] = 0.$)

$= 0$

Example 99 *The line joining the two projections of a given point upon the sides of an angle is perpendicular to the isogonal conjugate of the line joining the given point to the vertex of the angle.*

Point order: $a, b, c, m, n, q, p.$

Hypotheses: $\text{eqangle}(b, a, m, n, a, c), \text{perp}(m, q, a, b), \text{coll}(q, a, b),$

$\text{perp}(m, p, a, c), \text{coll}(p, a, c).$

Conclusion: $\text{perp}(n, a, p, q).$

The Machine Proof

$-[pq, na] + 1$

(Since a, c, p are collinear; p, q, a, m are cyclic; $[pq, na] = [pq, pa] + [ac, na] = [qm, ma] - [na, ca].$)

$= -[qm, ma] + [na, ca] + 1$

(Since $qm \perp ab$; $[qm, ma] = -[ma, ba] + 1.$)

$= [na, ca] + [ma, ba]$

(Since $\angle[an, ac] = \angle[ab, am]$; $[na, ca] = -[ma, ba].$)

$= 0$

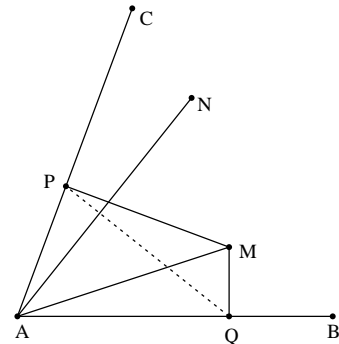


Figure 99

Example 100 *ABC is an equilateral triangle. Produce AB to D such that $BD = 2AB$. F is the foot of the perpendicular line from D to BC. Show that $AC \perp AF$.*

Point order: a, b, c, m, d, f .

Hypotheses: $\text{cong}(a, c, c, b)$, $\text{cong}(c, b, b, a)$,
 $\text{midpoint}(b, a, m)$, $\text{midpoint}(m, b, d)$, $\text{foot}(f, d, b, c)$.

Conclusion: $\text{perp}(c, a, a, f)$.

The Machine Proof

$$-[fa, ca] + 1$$

(Since b, c, f are collinear; f, a, c, m are cyclic;

$$[fa, ca] = [fa, fc] + [cb, ca] = -[mc, ma] + [cb, ca].)$$

$$= [mc, ma] - [cb, ca] + 1$$

(Since $mc \perp ac$; $[mc, ma] = -[ma, ca] + 1$.)

$$= -[ma, ca] - [cb, ca]$$

(Since a, b, m are collinear; $[ma, ca] = -[ca, ba]$.)

$$= -[cb, ca] + [ca, ba]$$

(Since $\angle[ac, bc] = \angle[ab, ac]$; $[cb, ca] = [cb, ac] + [ac, ca] = -[ca, ca] + [ca, ba]$.)

$$= [ca, ca]$$

(Since $[ca, ca] = 0$.)

$$= 0$$

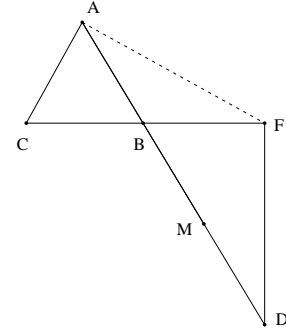


Figure 100

Example 101 *The two tangents to the circumcircle of ABC at A and C meet at E. The mediator of BC meet AB at D. Show that A, O, O, and D are cyclic.*

Point order: a, b, c, o, h, d, e .

Hypotheses: $\text{circumcenter}(o, a, b, c)$, $\text{midpoint}(h, c, b)$,
 $\text{coll}(d, a, b)$, $\text{coll}(d, o, h)$, $\text{perp}(e, a, a, o)$, $\text{perp}(e, c, c, o)$.

Conclusion: $\text{cyclic}(a, o, e, d)$.

The Machine Proof

$$-[eo, ea] + [do, da]$$

(Since $ea \perp ao$; $[eo, ea] = [eo, oa] + 1$.)

$$= -[eo, oa] + [do, da] - 1$$

(Since o, e, a, c are cyclic; $[eo, oa] = [ec, ca]$.)

$$= -[ec, ca] + [do, da] - 1$$

(Since $ec \perp co$; $[ec, ca] = [oc, ca] + 1$.)

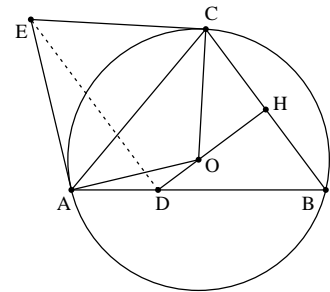


Figure 101

$$\begin{aligned}
&= [do, da] - [oc, ca] \\
&\quad (\text{Since } d, h, o \text{ are collinear; } a, b, d \text{ are collinear; } [do, da] = [ho, ba].) \\
&= [ho, ba] - [oc, ca] \\
&\quad (\text{Since } ho \perp bc; [ho, ba] = [cb, ba] + 1.) \\
&= -[oc, ca] + [cb, ba] + 1 \\
&\quad (\text{Since circumcenter}(o, c, a, b); [oc, ca] = [oc, ca] = [cb, ba] + 1.) \\
&= 0
\end{aligned}$$

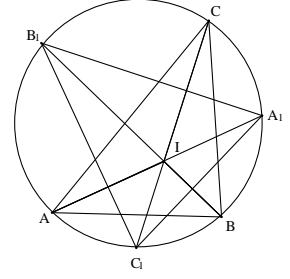


Figure 102

Example 102² Let ABC be inscribed in a circle and let A_1 , B_1 , and C_1 be the midpoints of the arc BC , CA , and AB respectively. Show that the incenter of triangle ABC is the orthocenter of triangle $A_1B_1C_1$.

Point order: a, b, c, a_1, b_1, c_1 .

Hypotheses: $\text{cyclic}(a, b, c, a_1, b_1, c_1)$, $\text{cong}(a_1, b, a_1, c)$, $\text{cong}(b_1, a, b_1, c)$, $\text{cong}(c_1, a, c_1, b)$.

Conclusion: $\text{perp}(a, a_1, b_1, c_1)$.

The Machine Proof

$$\begin{aligned}
&-[c_1b_1, a_1a] + 1 \\
&\quad ([c_1b_1, a_1a] = [c_1b_1, c_1a] + [ac_1, a_1a] = [c_1a, a_1a] + [b_1a_1, a_1a], \text{ because } \text{cyclic}(c_1, b_1, a, a_1) \\
&).) \\
&= -[c_1a, a_1a] - [b_1a_1, a_1a] + 1 \\
&\quad ([c_1a, a_1a] = [c_1a, ba] - [a_1a, ba].) \\
&= -[c_1a, ba] - [b_1a_1, a_1a] + [a_1a, ba] + 1 \\
&\quad ([c_1a, ba] = [b_1a, ca] + [a_1a, ba] + 1, \text{ because } \text{cyclic}(a, b, c, a_1, b_1, c_1), a_1b = a_1c, b_1a = b_1c, \\
&c_1a = c_1b.) \\
&= -[b_1a_1, a_1a] - [b_1a, ca] \\
&\quad ([b_1a_1, a_1a] = -[b_1a, ca], \text{ because } \text{cyclic}(a_1, b_1, a, c, a), b_1a = b_1c.) \\
&= 0
\end{aligned}$$

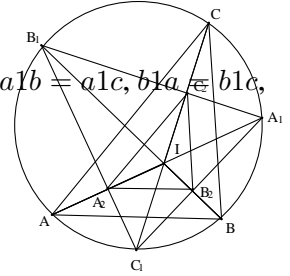


Figure 103

Example 103³ Continue from the preceding example. Show that the pedal triangle of triangle $A_1B_1C_1$ is homothetic to triangle ABC .

Point order: $a, b, c, a_1, b_1, c_1, a_2, b_2, c_2$.

Hypotheses: $\text{cyclic}(a, b, c, a_1, b_1, c_1)$, $\text{cong}(a_1, b, a_1, c)$, $\text{cong}(b_1, a, b_1, c)$, $\text{cong}(c_1, a, c_1, b)$, $\text{foot}(a_2, a_1, b_1, c_1)$, $\text{foot}(b_2, b_1, a_1, c_1)$, $\text{foot}(c_2, c_1, a_1, b_1)$.

Conclusion: $\text{para}(a, b, a_2, b_2)$.

²This example is a problem proposed in American Mathematics Monthly, 1993 (Problem 10317)

³This example is a problem proposed in American Mathematics Monthly, 1993 (Problem 10317)

The Machine Proof

$$\begin{aligned} & -[b2a2, ba] \\ & \quad ([b2a2, ba] = [b2a2, b2a1] + [a1c1, ba] = [a2b1, b1a1] + [c1a1, ba], \\ & \quad \text{because collinear}(a1, b2, c1), \text{cyclic}(b2, a2, a1, b1).) \\ & = -[a2b1, b1a1] - [c1a1, ba] \\ & \quad ([a2b1, b1a1] = [c1b1, b1a1], \text{because collinear}(a2, b1, c1).) \\ & = -[c1b1, b1a1] - [c1a1, ba] \\ & \quad ([c1b1, b1a1] = [c1b1, b1b] - [b1a1, b1b].) \\ & = -[c1b1, b1b] - [c1a1, ba] + [b1a1, b1b] \\ & \quad ([c1b1, b1b] = -[c1b, ba], \text{because cyclic}(b1, c1, b, a, b), c1b = c1a.) \\ & = -[c1a1, ba] + [c1b, ba] + [b1a1, b1b] \\ & \quad ([c1a1, ba] = [c1a1, c1b] + [bc1, ba] = [c1b, ba] + [b1a1, b1b], \text{because cyclic}(c1, a1, b, b1).) \\ & = 0 \end{aligned}$$

2.3 Examples Proved by Combining Forward Chaining with Backward Chaining

The examples in this subsection are the most difficult problems solved by our prover. Their proof need the database containing information about equal-angles. Generally speaking, the information in GIB is not obvious and need further explanation.

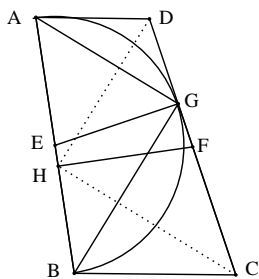


Figure 104

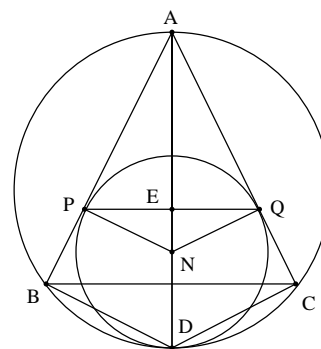


Figure 105

Example 104 ⁴ In quadrilateral $ABCD$, $BC \parallel AD$ and the circle with AB as its diameter is tangent to CD . Show that the circle with CD as its diameter is tangent to AB .

Point order: a, b, c, d, e, g, f, h .

Hypotheses: $\text{para}(a, d, b, c)$, $\text{midpoint}(e, a, b)$, $\text{foot}(g, e, c, d)$,
 $\text{perp}(a, g, b, g)$, $\text{midpoint}(f, c, d)$, $\text{foot}(h, f, a, b)$.

Conclusion: $\text{perp}(c, h, h, d)$.

The Machine Proof

$$-[hd, hc] + 1$$

(Since a, b, h are collinear; h, d, a, g are cyclic; h, c, b, g are cyclic;

$$[hd, hc] = [hd, ha] + [hb, hc] = [gd, ga] - [gc, gb].)$$

$$= -[gd, ga] + [gc, gb] + 1$$

(Since c, d, g are collinear; $-[gd, ga] + [gc, gb] = -[gb, ga]$.)

$$= -[gb, ga] + 1$$

(Since $gb \perp ga$, $[gb, ga] = 1$.)

$$= 0$$

The following information are in the GIB.

Points a, d, g, h are cyclic, because $\angle[ad, cd] = \angle[ab, gh]$.

This equation if further derived from the following two equations:

$\angle[ad, cd] = \angle[ef, cd]$, because $EF \parallel AD$;

$\angle[ab, gh] = \angle[ef, cd]$, because g, e, f, h are cyclic.

Points b, c, g, h are cyclic, because $\angle[bc, cd] = \angle[ab, gh]$.

This equation if further derived from the following two equations:

$\angle[bc, cd] = \angle[ef, cd]$, because $EF \parallel BC$;

$\angle[ab, gh] = \angle[ef, cd]$, because g, e, f, h are cyclic.

Example 105 ⁵ In triangle ABC , $AB = AC$. A circle is tangent to the circumcircle of triangle ABC and is tangent to AB , AC at P and Q . Show that the midpoint of the PQ is the incenter of triangle ABC .

⁴This example is from the 1984 International Mathematical Olympiad

Point order: p, q, a, n, d, b, c, e .

Hypotheses: $\text{cyclic}(a, b, c, d)$, $\text{cong}(a, p, a, q)$, $\text{cong}(a, b, a, c)$,
 $\text{cong}(d, b, d, c)$, $\text{coll}(n, a, d)$, $\text{foot}(p, n, a, b)$, $\text{foot}(q, n, a, c)$,
 $\text{circumcenter}(n, d, p, q)$, $\text{coll}(e, p, q)$, $\text{coll}(e, a, d)$.

Conclusion: $\text{eqangle}(a, b, e, e, b, c)$.

The Machine Proof

$$\begin{aligned}
& -[eb, cb] - [eb, ba] \\
& \quad (\text{Since } cb \parallel pq; [eb, cb] = [eb, qp].) \\
& = -[eb, ba] - [eb, qp] \\
& \quad (\text{Since } ba \perp pn; [eb, ba] = [eb, np] + 1.) \\
& = -[eb, np] - [eb, qp] - 1 \\
& \quad (\text{Since } np \perp pa; [eb, np] = [eb, ap] + 1.) \\
& = -[eb, ap] - [eb, qp] \\
& \quad (\text{Since } e, p, q \text{ are collinear; } e, b, p, d \text{ are cyclic;} \\
& \quad [eb, ap] = [eb, ep] + [pq, ap] = [bd, dp] - [ap, qp].) \\
& = -[eb, qp] - [bd, dp] + [ap, qp] \\
& \quad (\text{Since } e, p, q \text{ are collinear; } e, b, p, d \text{ are cyclic;} \\
& \quad [eb, qp] = [eb, ep] = [bd, dp].) \\
& = -2[bd, dp] + [ap, qp] \\
& \quad (\text{Since } bd \parallel pn; [bd, dp] = -[dp, np].) \\
& = 2[dp, np] + [ap, qp] \\
& \quad (\text{Since circumcenter}(n, d, p, q); dn \perp pq; [dp, np] = [na, np].) \\
& = [na, np] + [ap, qp] \\
& \quad (\text{Since } na \perp pq; np \perp pa; [na, np] = -[ap, qp].) \\
& = 0
\end{aligned}$$

The GIB contains the following circles.

p, a, n, q are cyclic, because $pa \perp pn$; $qa \perp qn$.
 b, c, p, q are cyclic, because $\angle[pa, pq] = \angle[bc, qa]$.
 b, d, e, p are cyclic, because $\angle[pa, pq] = \angle[bc, ca] = \angle[db, ad]$.
 c, d, e, q are cyclic, because $\angle[ad, dc] = \angle[ab, bc] = \angle[pq, qa]$.

⁵This problem is from the 1978 International Mathematical Olympiad

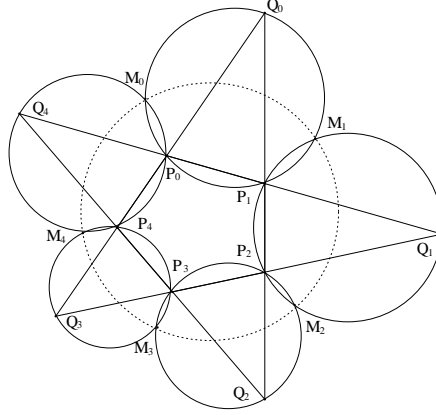


Figure 106

Example 106 (The Five Circle Theorem) As in Figure 9, $P_0P_1P_2P_3P_4$ is a pentagon. $Q_i = P_{i-1}P_i \cap P_{i+1}P_{i+2}$, $M_i = \text{circle}(Q_{i-1}P_{i-1}P_i) \cap \text{circle}(Q_iP_iP_{i+1})$ (the subscripts are understood to be mod 4). Show that points M_0, M_1, M_2, M_3, M_4 are cyclic.

Point order: $p_0, p_1, p_2, p_3, p_4, q_0, q_1, q_2, q_3, q_4, m_0, m_1, m_2, m_3, m_4$.

Hypotheses: $\text{coll}(q_0, p_1, p_2, q_2)$, $\text{coll}(q_1, p_2, p_3, q_3)$, $\text{coll}(q_2, p_3, p_4, q_4)$,
 $\text{coll}(q_3, p_4, p_0, q_0)$, $\text{coll}(q_4, p_0, p_1, q_1)$, $\text{cyclic}(m_0, m_1, q_0, p_0, p_1)$, $\text{cyclic}(m_0, m_4, q_4, p_0, p_4)$,
 $\text{cyclic}(m_3, m_4, q_3, p_3, p_4)$, $\text{cyclic}(m_3, m_2, q_2, p_2, p_3)$, $\text{cyclic}(m_1, m_2, q_1, p_1, p_2)$.

Conclusion: $\text{cyclic}(m_0, m_1, m_2, m_4)$.

The Machine Proof

$$[m_4m_1, m_4m_0] - [m_2m_1, m_2m_0]$$

(Since m_1, m_4, p_0, q_1 are cyclic; m_0, m_4, p_0, p_4 are cyclic;

$$[m_4m_1, m_4m_0] = [m_4m_1, m_4p_0] + [m_4p_0, m_4m_0] = [m_1q_1, q_1p_0] - [m_0p_4, p_4p_0].)$$

$$= -[m_2m_1, m_2m_0] + [m_1q_1, q_1p_0] - [m_0p_4, p_4p_0]$$

(Since m_1, m_2, p_1, p_2 are cyclic; m_0, m_2, p_1, q_4 are cyclic;

$$[m_2m_1, m_2m_0] = [m_2m_1, m_2p_1] + [m_2p_1, m_2m_0] = [m_1p_2, p_2p_1] - [m_0q_4, q_4p_1].)$$

$$= [m_1q_1, q_1p_0] - [m_1p_2, p_2p_1] + [m_0q_4, q_4p_1] - [m_0p_4, p_4p_0]$$

(Since p_0, p_1, q_1 are collinear; q_1, m_1, p_1, p_2 are cyclic;

$$[m_1q_1, q_1p_0] = [q_1m_1, q_1p_1] = [m_1p_2, p_2p_1].)$$

$$= [m_0q_4, q_4p_1] - [m_0p_4, p_4p_0]$$

(Since p_0, p_1, q_4 are collinear; q_4, m_0, p_0, p_4 are cyclic;

$$[m_0q_4, q_4p_1] = [q_4m_0, q_4p_0] = [m_0p_4, p_4p_0].)$$

$$= 0$$

The following information are in the GIB.

$\text{cyclic}(m_3, m_0, p_4, q_0, q_2)$
 $\text{cyclic}(m_1, m_4, p_0, q_1, q_3)$
 $\text{cyclic}(m_1, m_3, p_2, q_0, q_3)$
 $\text{cyclic}(m_2, m_4, p_3, q_1, q_4)$

$\text{cyclic}(m_2, m_0, p_1, q_2, q_4)$

The first of the five circles is the combination of the following two circles

$\text{cyclic}(m_3, p_4, q_0, q_2)$,

since $\angle[p_4m_3, q_0p_4] = \angle[q_2m_3, q_0q_2]$. Both the above two angles equal to $\angle[m_3p_3, p_2p_3]$, since $\text{cyclic}(p_3, p_4, q_3, m_3)$ and $\text{cyclic}(p_2, p_3, q_2, m_3)$.

$\text{cyclic}(m_0, p_4, q_0, q_2)$,

since $\angle[q_0m_0, q_0q_2] = \angle[p_4m_0, q_2p_4]$. Both the above two angles equal to $\angle[m_0p_0, p_0p_1]$, since $\text{cyclic}(p_0, p_4, q_4, m_0)$ and $\text{cyclic}(p_0, p_1, q_0, m_0)$.

Example 107 A line AD through the vertex A meets the circumcircle of the triangle ABC in D . If U, V are the orthocenters of the triangle ABD, ACD , respectively, prove that UV is equal and parallel to BC .

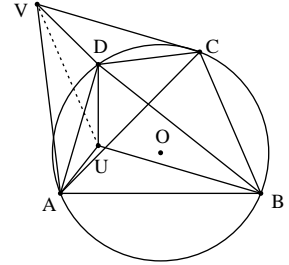


Figure 107

Point order: a, b, c, d, u, v .

Hypotheses: $\text{cyclic}(a, b, c, d)$, $\text{orthocenter}(u, a, b, d)$, $\text{orthocenter}(v, a, c, d)$.

Conclusion: $\text{para}(u, v, b, c)$.

The Machine Proof

$[vu, cb]$

(Since v, u, a, d are cyclic; $va \perp cd$; $[vu, cb] = [vu, va] + [va, cb] = [ud, da] + [dc, cb] + 1$.)

$= [ud, da] + [dc, cb] + 1$

(Since $ud \perp ab$; $[ud, da] = -[da, ba] + 1$.)

$= [dc, cb] - [da, ba]$

(Since c, d, b, a are cyclic; $[dc, cb] = [da, ba]$.)

$= 0$

Example 108 If P is any point on a semicircle, diameter AB , and BC, CD are two equal arcs, then if $E = CA \cap PB$, $F = AD \cap PC$, prove that AD is perpendicular to EF .

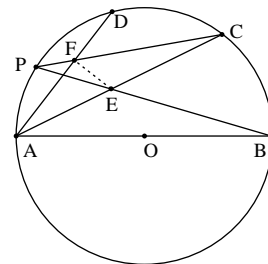


Figure 108

Point order: a, b, c, p, o, d, e, f .

Hypotheses: $\text{midpoint}(o, a, b)$, $\text{perp}(a, c, c, b)$, $\text{perp}(a, p, p, b)$,

$\text{cong}(o, a, o, d)$, $\text{cong}(c, d, c, b)$, $\text{coll}(e, a, c)$, $\text{coll}(e, p, b)$,

$\text{coll}(f, a, d)$, $\text{coll}(f, p, c)$.

Conclusion: $\text{perp}(e, f, a, d)$.

The Machine Proof

$[fe, da] + 1$

(Since $da \parallel co$; $[fe, da] = [fe, oc]$.)

$= [fe, oc] + 1$

$$\begin{aligned}
& \text{(Since } a, d, f \text{ are collinear; } f, e, a, p \text{ are cyclic;} \\
& [fe, oc] = [fe, fa] + [ad, oc] = [ep, pa] + [da, oc].) \\
& = [ep, pa] + [da, oc] + 1 \\
& \text{(Since } ep \perp pa; [ep, pa] = 1.) \\
& = [da, oc] \\
& \text{(Since } da \parallel oc; [da, oc] = 0.) \\
& = 0
\end{aligned}$$

Example 109 *The centers of the four circles (pqr) , (qrs) , (rsp) , (spq) and the Miquel point lie on the same circle.*

Point order: $a, b, d, c, e, f, m, o_1, o_2, o_3, o_4$.

Hypotheses: $\text{coll}(a, b, c)$, $\text{coll}(c, d, e)$, $\text{coll}(e, f, b)$,
 $\text{coll}(d, f, a)$, $\text{cyclic}(m, a, b, f)$, $\text{cyclic}(m, a, c, d)$,
 $\text{pbisector}(o_1, a, b)$, $\text{pbisector}(o_1, a, f)$, $\text{pbisector}(o_2, b, c)$,
 $\text{pbisector}(o_2, b, e)$, $\text{pbisector}(o_3, d, f)$, $\text{pbisector}(o_3, d, e)$,
 $\text{pbisector}(o_4, a, c)$, $\text{pbisector}(o_4, a, d)$.

Conclusion: $\text{cyclic}(o_1, o_2, o_3, o_4)$.

The Machine Proof

$$\begin{aligned}
& [o_4o_2, o_4o_1] - [o_3o_2, o_3o_1] \\
& \text{(Since } o_4o_2 \perp cm; [o_4o_2, o_4o_1] = -[o_4o_1, mc] + 1.) \\
& = -[o_4o_1, mc] - [o_3o_2, o_3o_1] + 1 \\
& \text{(Since } o_4o_1 \perp am; [o_4o_1, mc] = -[mc, ma] + 1.) \\
& = -[o_3o_2, o_3o_1] + [mc, ma] \\
& \text{(Since } o_3o_2 \perp em; [o_3o_2, o_3o_1] = -[o_3o_1, me] + 1.) \\
& = [o_3o_1, me] + [mc, ma] - 1 \\
& \text{(Since } o_3o_1 \perp fm; [o_3o_1, me] = [mf, me] + 1.) \\
& = [mf, me] + [mc, ma] \\
& \text{(Since } m, f, e, d \text{ are cyclic; } [mf, me] = [fd, ed].) \\
& = [mc, ma] + [fd, ed] \\
& \text{(Since } m, c, a, d \text{ are cyclic; } [mc, ma] = [cd, da].) \\
& = [fd, ed] + [cd, da] \\
& \text{(Since } ed \parallel cd; [fd, ed] + [cd, da] = [fd, da].) \\
& = [fd, da] \\
& \text{(Since } a, d, f \text{ are collinear; } [fd, da] = 0.) \\
& = 0
\end{aligned}$$

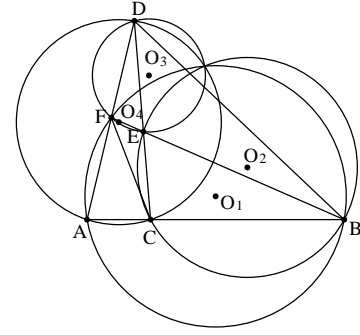


Figure 109

Example 110 *Continue from Example 109. The centers of the four circles (pqr) , (qrs) , (rsp) , (spq) and the Miquel point lie on the same circle.*

Point order: $a, b, d, c, e, f, m, o_1, o_2, o_3, o_4$.

Hypotheses: $\text{coll}(a, b, c)$, $\text{coll}(c, d, e)$, $\text{coll}(e, f, b)$,
 $\text{coll}(d, f, a)$, $\text{cyclic}(m, a, b, f)$, $\text{cyclic}(m, a, c, d)$, $\text{pbisector}(o_1, a, b)$,
 $\text{pbisector}(o_1, a, f)$, $\text{pbisector}(o_2, b, c)$, $\text{pbisector}(o_2, b, e)$, $\text{pbisector}(o_3, d, f)$,
 $\text{pbisector}(o_3, d, e)$, $\text{pbisector}(o_4, a, c)$, $\text{pbisector}(o_4, a, d)$.

Conclusion: $\text{cyclic}(o_1, o_2, o_3, m)$.

The Machine Proof

$$\begin{aligned}
& -[o_3o_2, o_3o_1] + [o_2m, o_1m] \\
& \quad (\text{Since } o_3o_2 \perp em; [o_3o_2, o_3o_1] = -[o_3o_1, me] + 1.) \\
& = [o_3o_1, me] + [o_2m, o_1m] - 1 \\
& \quad (\text{Since } o_3o_1 \perp fm; [o_3o_1, me] = [mf, me] + 1.) \\
& = [o_2m, o_1m] + [mf, me] \\
& \quad (\text{Since circumcenter}(o_2, m, e, b); \\
& \quad [o_2m, o_1m] = [o_2m, me] + [me, o_1m] = -[o_1m, me] + [mb, eb] + 1.) \\
& = -[o_1m, me] + [mf, me] + [mb, eb] + 1 \\
& \quad (\text{Since circumcenter}(o_1, m, a, b); \\
& \quad [o_1m, me] = [o_1m, ma] + [ma, me] = -[me, ma] + [mb, ba] + 1.) \\
& = [mf, me] + [me, ma] + [mb, eb] - [mb, ba] \\
& \quad (\text{Since } me \parallel me; [mf, me] + [me, ma] = [mf, ma].) \\
& = [mf, ma] + [mb, eb] - [mb, ba] \\
& \quad (\text{Since } mb \parallel mb; [mb, eb] - [mb, ba] = -[eb, ba].) \\
& = [mf, ma] - [eb, ba] \\
& \quad (\text{Since } m, f, a, b \text{ are cyclic; } [mf, ma] = [fb, ba].) \\
& = [fb, ba] - [eb, ba] \\
& \quad (\text{Since } b, e, f \text{ are collinear; } [fb, ba] = [eb, ba].) \\
& = 0
\end{aligned}$$

3 Time and Proof Length Statistics

Exs	St	Mt	Time	Exs	St	Mt	Time	Exs	St	Mt	Time
1	2	2	0.89	38	6	4	0.735	75	7	5	0.922
2	2	2	1.016	39	6	3	8.530	76	5	2	0.719
3	2	2	0.916	40	4	2	0.532	77	9	5	1.218
4	2	2	0.86	41	7	4	1.25	78	11	4	1.562
5	2	1	0.453	42	7	3	2.531	79	5	3	0.703
6	5	3	0.64	43	4	2	0.422	80	6	4	0.656
7	5	2	0.969	44	4	2	0.5	81	9	3	1.249
8	7	3	1.11	45	3	2	0.937	82	7	3	1.359
9	7	2	1.562	46	9	5	1.453	83	5	3	0.672
10	4	3	0.5	47	4	2	0.593	84	8	3	2.858
11	5	3	1.0	48	5	2	0.453	85	5	4	0.641
12	5	2	0.516	49	9	4	1.703	86	8	3	1.656
13	6	2	1.297	50	8	3	1.0	87	9	4	1.313
14	6	3	1.375	51	6	4	2.172	88	11	4	2.063
15	14	3	2.75	52	7	3	1.359	89	10	4	2.421
16	8	4	2.469	53	10	4	1.156	90	5	3	1.031
17	10	5	1.531	54	8	3	1.156	91	7	3	1.453
18	4	3	0.531	55	9	3	1.282	92	7	3	1.484
19	4	2	0.375	56	11	4	1.187	93	6	3	5.076
20	7	3	1.078	57	5	2	0.75	94	9	3	1.406
21	4	4	0.391	58	5	2	0.484	95	5	3	0.734
22	8	3	0.937	59	4	2	0.781	96	6	3	0.828
23	4	3	0.657	60	11	4	2.32	97	9	3	1.437
24	5	3	0.515	61	5	3	0.593	98	8	5	1.032
25	8	4	2.75	62	5	3	1.265	100	6	3	2.219
26	6	3	0.937	63	7	3	0.672	101	7	3	0.984
27	8	4	1.063	64	4	3	0.734	99	4	3	0.562
28	6	3	1.172	65	6	3	0.532	102	7	4	5.99
29	5	2	0.593	66	5	3	0.593	103	7	3	22.95
30	5	2	0.5	67	5	3	1.015	104	6	3	3.889
31	7	3	1.25	68	8	3	1.828	105	12	3	9.515
32	6	3	0.703	69	5	3	0.641	106	7	4	21.436
33	7	3	0.844	70	7	2	0.968	107	4	3	3.421
34	8	4	0.765	71	5	3	0.844	108	5	3	6.342
35	10	4	1.11	72	3	2	0.422	109	9	3	6.797
36	12	4	1.266	73	6	3	0.703	110	9	4	6.875
37	7	4	1.016	74	5	2	1.109				

Table 1. Statistics for the Examples in This paper

We include four indexes for each theorem.

1. Exs indicates the example number.
2. St indicates the proof length.

3. Mt indicates the length of the maximal algebraic expressions in the proof.
4. Time indicates the cpu time used to obtain the first proof using the depth-first search strategy.