Example 4.5 Let $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ be co-circular points. Let $\mathbf{5}$ be the foot drawn from point $\mathbf{1}$ to line $\mathbf{23}$, and let $\mathbf{6}$ be the foot drawn from point $\mathbf{2}$ to line $\mathbf{14}$. Then $\mathbf{34} \,||\, \mathbf{56}$.

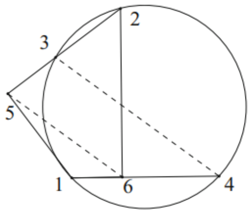


Figure 6 Example 4.5

Geometric construction sequence:

Free points: $\mathbf{1}, \mathbf{2}, \mathbf{3}$.

Semi-free point: $\mathbf{4}$ on circle $\mathbf{123}$ (non-linear construction).

Feet: $\mathbf{5} = \text{Foot}_{\mathbf{1},\mathbf{23}}$, $\mathbf{6} = \text{Foot}_{\mathbf{2},\mathbf{14}}$.

Conclusion: $[\boldsymbol{e}\mathbf{34}\boldsymbol{e}\mathbf{56}] = 0$.

We remove the hypothesis that $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ are co-circular, and compute the conclusion expression, where only the reduced Cayley forms of intersections $\mathbf{5}, \mathbf{6}$ are needed:

$$\begin{aligned}
\mathbf{5} =& \quad (\mathbf{2}\wedge\mathbf{3}) \vee_e (\mathbf{1}\wedge\langle e\mathbf{23}\rangle_3^{\sim}) \quad \text{mod } e,\\
\mathbf{6} =& \quad (\mathbf{1}\wedge\mathbf{4}) \vee_e (\mathbf{2}\wedge\langle e\mathbf{14}\rangle_3^{\sim}) \quad \text{mod } e.\\
& [e\mathbf{34}e\mathbf{56}]\\
\overset{\mathbf{5},\mathbf{6}}{=}& [e\mathbf{34}e\{(\mathbf{2}\wedge\mathbf{3}) \vee_e (\mathbf{1}\wedge\langle e\mathbf{23}\rangle_3^{\sim})\}\{(\mathbf{1}\wedge\mathbf{4}) \vee_e (\mathbf{2}\wedge\langle e\mathbf{14}\rangle_3^{\sim})\}]\\
\overset{\text{expand}}{=}& \ (\mathbf{2}\wedge\mathbf{3}) \vee_e (\mathbf{1}\wedge\langle e\mathbf{23}\rangle_3^{\sim}) \vee_e (\mathbf{2}\wedge\langle e\mathbf{14}\rangle_3^{\sim})\ [e\mathbf{34}e\mathbf{14}]\\
& -(\mathbf{2}\wedge\mathbf{3}) \vee_e (\mathbf{1}\wedge\langle e\mathbf{23}\rangle_3^{\sim}) \vee_e (\mathbf{1}\wedge\mathbf{4})\ [e\mathbf{34}e\mathbf{2}\langle e\mathbf{14}\rangle_3^{\sim}]\\
\overset{\text{expand}}{=}& [e\mathbf{21}\langle e\mathbf{23}\rangle_3^{\sim}][e\mathbf{32}\langle e\mathbf{14}\rangle_3^{\sim}][e\mathbf{34}e\mathbf{14}] + [e\mathbf{231}][e\langle e\mathbf{23}\rangle_3^{\sim}\mathbf{14}][e\mathbf{34}e\mathbf{2}\langle e\mathbf{14}\rangle_3^{\sim}]
\end{aligned}$$

$$\begin{aligned}
\overset{\text{ungrade}}{=}& \quad 2^{-2}\{\langle e\mathbf{21}e\mathbf{23}\rangle\langle e\mathbf{32}e\mathbf{14}\rangle[e\mathbf{34}e\mathbf{14}] + [e\mathbf{123}]\langle e\mathbf{23}e\mathbf{14}\rangle\langle e\mathbf{34}e\mathbf{2}e\mathbf{14}\rangle\}\\
\overset{\text{null expand}}{=}& \quad \underbrace{-(e\cdot\mathbf{2})(e\cdot\mathbf{4})\langle e\mathbf{23}e\mathbf{14}\rangle}(-\langle e\mathbf{1}\underline{\mathbf{2}}\mathbf{3}\rangle[e\mathbf{1}\underline{\mathbf{4}}\mathbf{3}] + \langle e\mathbf{1}\underline{\mathbf{4}}\mathbf{3}\rangle[e\mathbf{1}\underline{\mathbf{2}}\mathbf{3}])\\
\overset{\text{contract}}{=}& \quad \underbrace{-\langle e\mathbf{1}e\mathbf{3}\rangle}[\mathbf{1234}].
\end{aligned}$$

The last contraction is based on null Cramer's rule:

$$[\mathbf{143}e]\mathbf{123} - [\mathbf{123}e]\mathbf{143} = -[\mathbf{1234}]\mathbf{1}e\mathbf{3}. \tag{45}$$

Homogenization: By

$$\begin{aligned}
e\cdot\mathbf{5} &\overset{\mathbf{5}}{=} e\cdot\mathbf{1}[e\mathbf{23}\langle e\mathbf{23}\rangle_3^{\sim}] = 2(e\cdot\mathbf{1})(e\cdot\mathbf{2})(e\cdot\mathbf{3})(\mathbf{2}\cdot\mathbf{3}),\\
e\cdot\mathbf{6} &\overset{\mathbf{6}}{=} e\cdot\mathbf{2}[e\mathbf{14}\langle e\mathbf{14}\rangle_3^{\sim}] = 2(e\cdot\mathbf{1})(e\cdot\mathbf{2})(e\cdot\mathbf{4})(\mathbf{1}\cdot\mathbf{4}),
\end{aligned}$$

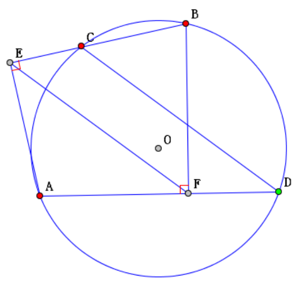
we get

$$\frac{[e\mathbf{34}e\mathbf{56}]}{(e\cdot\mathbf{5})(e\cdot\mathbf{6})} = \frac{\langle e\mathbf{23}e\mathbf{14}\rangle\,[\mathbf{1234}]}{2\,(e\cdot\mathbf{1})(e\cdot\mathbf{2})(\mathbf{1}\cdot\mathbf{4})(\mathbf{2}\cdot\mathbf{3})}, \tag{46}$$

where geometrically,

$$\begin{aligned}
\langle e\mathbf{34}e\mathbf{56}\rangle &= 2\,d_{\mathbf{34}}d_{\mathbf{56}}\cos\angle(\mathbf{34},\mathbf{56}),\\
[e\mathbf{34}e\mathbf{56}] &= 2\,d_{\mathbf{34}}d_{\mathbf{56}}\sin\angle(\mathbf{34},\mathbf{56}).
\end{aligned} \tag{47}$$

Example 1 38 : As shown in the figure, in the quadrilateral $ADBC$ *inscribed in* the circle O , $AE \perp BC$, $BF \perp AD$, to prove: $EF \,/\!/\, CD$.



$$\frac{E-F}{D-C}\,\frac{\dfrac{F-A}{F-E}}{\dfrac{B-A}{B-E}}\,\frac{\dfrac{B-A}{B-C}}{\dfrac{D-A}{D-C}}\,\frac{D-A}{A-F}\,\frac{B-C}{B-E}=1 \quad ,$$