Example 84 *The tangent to the nine-point circle at the midpoint of a side of the given triangle is antiparallel to this side with respect to the two other sides of the triangle.*

Point order: $b, c, a, a_1, b_1, c_1, n, k, j$.

Hypotheses: midpoint(a_1, b, c), midpoint(c_1, b, a), midpoint(b_1, a, c), pbisector(n, a_1, b_1), pbisector(n, a_1, c_1), perp(a_1, k, a_1, n), coll(a, c, k), coll(k, a_1, j), coll(a, b, j).

Conclusion: cyclic(k, j, b, c).

The Machine Proof

$[jk, jb] - [kc, cb]$

(Since a_1, j, k are collinear; a, b, j are collinear; $[jk, jb] = [ka_1, ab]$.)

$= [ka_1, ab] - [kc, cb]$

(Since $ka_1 \perp a_1n$; $[ka_1, ab] = [na_1, ab] + 1$.)

$= -[kc, cb] + [na_1, ab] + 1$

(Since a, c, k are collinear; $[kc, cb] = [ac, cb]$.)

$= [na_1, ab] - [ac, cb] + 1$

(Since circumcenter(n, a_1, c_1, b_1);

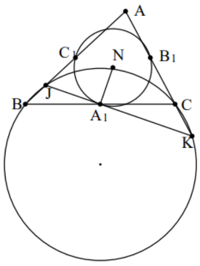


Figure 84

$[na_1, ab] = [na_1, a_1c_1] + [a_1c_1, ab] = -[c_1b_1, b_1a_1] + [c_1a_1, ab] + 1$.)

$= -[c_1b_1, b_1a_1] + [c_1a_1, ab] - [ac, cb]$

(Since $c_1a_1 \parallel ac$; $[c_1a_1, ab] - [ac, cb] = -[ab, cb]$.)

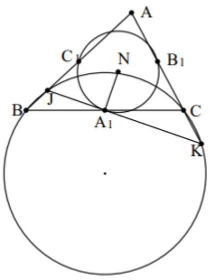
$= -[c_1b_1, b_1a_1] - [ab, cb]$

(Since $b_1a_1 \parallel ab$; $-[c_1b_1, b_1a_1] - [ab, cb] = -[c_1b_1, cb]$.)

$= -[c_1b_1, cb]$

(Since $c_1b_1 \parallel cb$; $[c_1b_1, cb] = 0$.)

$= 0$



Example 2 18 : As shown in the figure, in $\triangle ABC$, A_1, B_1, and C_1 are the midpoints of BC, CA, and AB respectively, and N is the circumcenter of $\triangle A_1B_1C_1$, and pass through A_1 to make $JK \perp A_1N$, intersect AB on J, intersect AC on K, and prove: J, B, K, and C are all circles.

prove: $\dfrac{\frac{J-K}{A-B}}{\frac{A-C}{C-B}}\left(\dfrac{A_1-C_1}{A_1-N}\dfrac{B_1-A_1}{B_1-C_1}\right)=\dfrac{J-K}{A_1-N}\dfrac{B-C}{B_1-C_1}\dfrac{A_1-B_1}{A-B}\dfrac{A_1-C_1}{A-C}$