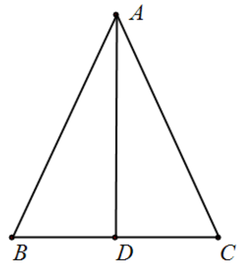
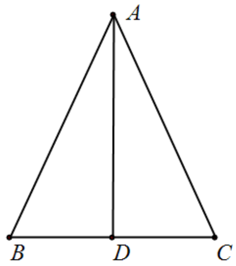
Example 1: As shown in Figure 1 , in $\triangle ABC$, AD is the bisector of $\angle BAC$, *and* AD *is* the midline on the side of BC . Prove: $AB = AC$.



Proof: $\dfrac{\dfrac{C-B}{C-A}}{\dfrac{B-A}{B-C}}+4\dfrac{\dfrac{A-\frac{B+C}{2}}{A-B}}{\dfrac{A-C}{A-\frac{B+C}{2}}}=4$.

Example 2 : As shown in Figure 1 , *in* $\triangle ABC$, D is the midpoint of BC, $\angle ABC = \angle ACB$, to prove: $AD \perp BC$.



Proof: $$\left(\frac{A-\frac{B+C}{2}}{B-C}\right)^2+\frac{\frac{B-A}{B-C}}{\frac{C-B}{C-A}}=\frac{1}{4},$$

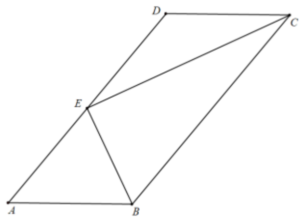
$$\left|\left(\frac{A-\frac{B+C}{2}}{B-C}\right)^2+\frac{\frac{B-A}{B-C}}{\frac{C-B}{C-A}}\right|=\frac{1}{4}\geq\left|\left(\frac{A-\frac{B+C}{2}}{B-C}\right)^2-\frac{\frac{B-A}{B-C}}{\frac{C-B}{C-A}}\right|\geq\left|\frac{\frac{B-A}{B-C}}{\frac{C-B}{C-A}}\right|-\left|\left(\frac{A-\frac{B+C}{2}}{B-C}\right)^2\right|,$$

$\frac{1}{4}\geq\frac{AB\cdot AC}{BC^2}-\frac{AD^2}{BC^2}$, that is $BD^2+AD^2\geq AB\cdot AC$, the equality sign holds if and only if $AB = AC$.

New proposition: In $\triangle ABC$, D is the midpoint of BC, to prove: $BD^2+AD^2\geq AB\cdot AC$, if and only when $AB = AC$ the equality sign holds.

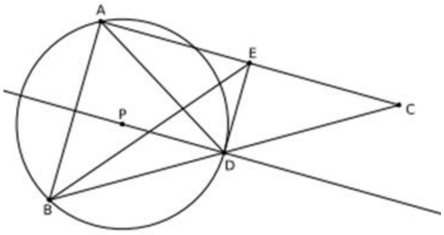
Example 3 : As shown in Figure 1 , in $\triangle ABC$, D is a point on BC. Prove: $AD \perp BC$, $\angle CAD = \angle BAD$, $\angle ABC = \angle ACB$, among these three conditions, if any two are known, the first three.

$$\frac{\frac{C-B}{C-A}\frac{A-C}{A-D}}{\frac{B-A}{B-C}\frac{A-D}{A-B}}\left(\frac{A-D}{B-C}\right)^2 = -1,$$



Example 4 : As shown *in* Figure 1 , in the parallelogram $ABCD$, E is the midpoint of AD, if CE bisects ∠BCD , then EB bisects ∠ABC .

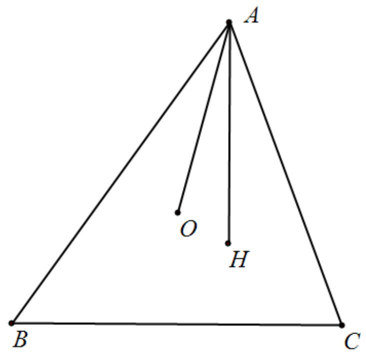
$$\text{prove: } \frac{\dfrac{B-\dfrac{A+C-B+A}{2}}{B-C}}{\dfrac{B-A}{B-\dfrac{A+C-B+A}{2}}}+\frac{\dfrac{C-\dfrac{A+C-B+A}{2}}{B-A}}{\dfrac{C-B}{C-\dfrac{A+C-B+A}{2}}}=2$$



Example 5 : As shown in Figure 1 , $\triangle$in ABC, D is the midpoint of BC, DE is the angle bisector of $\triangle\angle ADC$, *and is also* the tangent of the circumscribed circle of ABD. *Prove:* $\angle BAC = 90°$.

Proof: Suppose $D=0$, $\dfrac{\frac{A}{A-B}}{\frac{B-A}{B}} = \dfrac{\frac{E}{-B}}{\frac{A}{E}}\left(\dfrac{\frac{A}{E}}{\frac{B-A}{B-0}}\right)^2$,

Explanation: $\angle BAC = 90°$ is equivalent to $DA=DB$.



Example 6 : As shown in Figure 10, in $\triangle ABC$, O and H are the circumcenter and orthocenter respectively, to prove: $\angle BAO=\angle CAH$. If $\angle B < \angle C$, then $\angle ACB=\angle ABC+\angle HAO$.

$$\left(\frac{\frac{B-A}{B-C}\frac{A-H}{A-O}}{\frac{C-B}{C-A}}\right)^2=-\left(\frac{A-H}{B-C}\right)^2\frac{\frac{B-A}{B-O}}{\frac{A-O}{A-B}}\frac{\frac{A-C}{A-O}}{\frac{C-O}{C-A}}\frac{\frac{B-O}{B-C}}{\frac{C-B}{C-O}},$$

$$\left(\frac{\frac{A-C}{A-H}}{\frac{A-O}{A-B}}\right)^2=-\left(\frac{B-C}{A-H}\right)^2\frac{\frac{B-A}{B-O}}{\frac{A-O}{A-B}}\frac{\frac{A-C}{A-O}}{\frac{C-O}{C-A}}\frac{\frac{B-O}{B-C}}{\frac{C-B}{C-O}},$$

Make a question, use the combination of known conditions

Another proof: Suppose $O=0$, $H=A+B+C$, $\dfrac{\frac{A-0}{A-B}}{\frac{A-C}{A-(A+B+C)}}=T$, $\dfrac{\frac{A-0}{A-B}}{\frac{B-A}{B-0}}=T_1$, $\dfrac{\frac{C-B}{C-0}}{\frac{B-0}{B-C}}=T_2$, $\dfrac{\frac{C-0}{C-A}}{\frac{A-C}{A-0}}=T_3$, then $T=T_1+T_3-T_1T_2T_3$, the relationship between line segments $AB^2+AC^2=BC^2+AB\cdot AC\cdot\dfrac{AH}{AO}$.

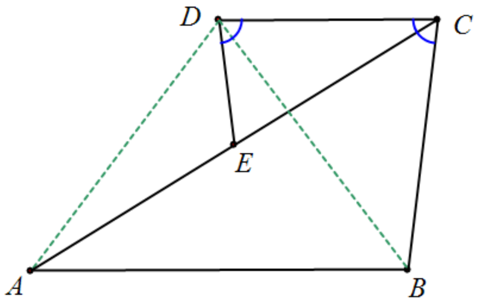
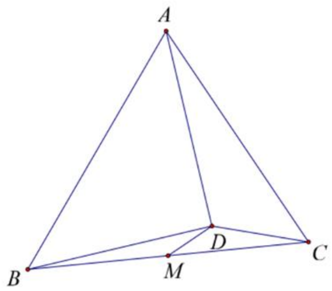


Figure 11

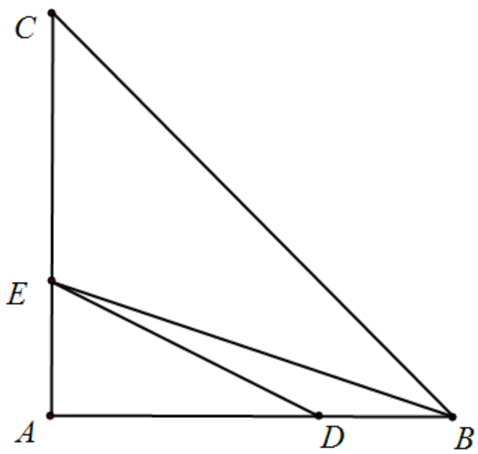
Example 7 : As shown in Figure 11, the trapezoid $ABCD$, AB // DC, and the midpoint of AC is O, to prove: $DA = DB \Leftrightarrow \angle ODC = \angle BCD$.

Proof: Suppose $D = t(A-B)+C$, $\dfrac{\dfrac{A-D}{A-B}}{\dfrac{B-A}{B-D}} = T$, $\dfrac{\dfrac{C-B}{C-D}}{\dfrac{D-C}{D-\dfrac{A+C}{2}}} = T_1$, $T = t(1-t+2tT_1)$.

Example 8 : As shown in Figure 1, it is known that D is a point inside $\triangle ABC$, $\angle DAC = \angle BDM$, M is the midpoint of BC, $\angle ABD = \angle ACD$, verify that $\angle ADB = 90°$.



Proof: Suppose $\left(\frac{D-A}{D-B}\right)^2 = T$, $\frac{\frac{D-\frac{B+C}{2}}{D-B}}{\frac{A-C}{A-D}} = T_1$, $\frac{\frac{B-A}{B-D}}{\frac{C-D}{C-A}} = T_2$, $T + T_2 - 2T_1T_2 - 1 = 0$



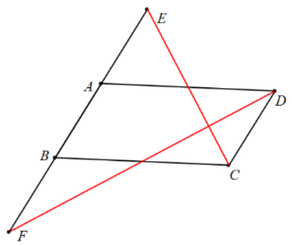
Example 9 : As shown in Figure 1, $\triangle$in ABC, take $BD=$ $AE=$ on $\dfrac{AB}{3}AB$ and AC,

and verify: $AB=AC$, $AB \perp AC$, $\angle ADE = \angle EBC$. Among these three conditions, if any two are known to be true, you can The third is also established.

Proof: Suppose $\dfrac{\dfrac{B-\dfrac{2A+C}{3}}{B-C}}{\dfrac{B-A}{\dfrac{A+2B}{3}-\dfrac{2A+C}{3}}}=t_1$, $\dfrac{\dfrac{C-B}{C-A}}{\dfrac{B-A}{B-C}}=t_2$, $\left(\dfrac{A-C}{A-B}\right)^2=t_3$,

$6t_2-9t_1t_2+t_3=1$.

Example 1 0 : As shown in Figure 1, in the parallelogram $ABCD$, extend AB to E and F in two directions, so that $AE = AB = BF$, connect CE and DF, prove: $AD = 2AB \Leftrightarrow EC \perp FD$.



picture

Proof Suppose $A=0$, $4\dfrac{\dfrac{F-0}{F-D}}{\dfrac{D-F}{D-0}}+\left[\dfrac{-\dfrac{F}{2}-\left(\dfrac{F}{2}+D\right)}{D-F}\right]^2=1$,

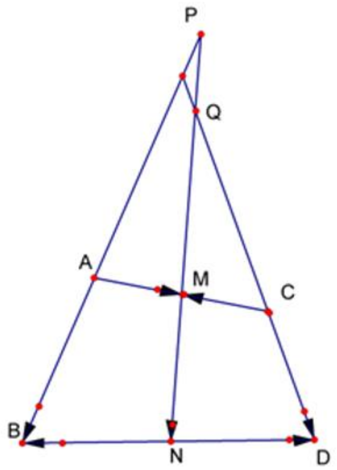
$\dfrac{-4DF}{(D-F)^2}+\dfrac{(D+F)^2}{(D-F)^2}=1$, $(D-F)^2+4DF=(D+F)^2$.

The proof assumes $A=0$ that . $(-B-(B+D))(2B-D)+(4B^2-D^2)=0$

The way to convert it into a general vector is :

$(-\overrightarrow{AB}-(\overrightarrow{AB}+\overrightarrow{AD}))\cdot(2\overrightarrow{AB}-\overrightarrow{AD})=-(4\overrightarrow{AB}^2-\overrightarrow{AD}^2)$.

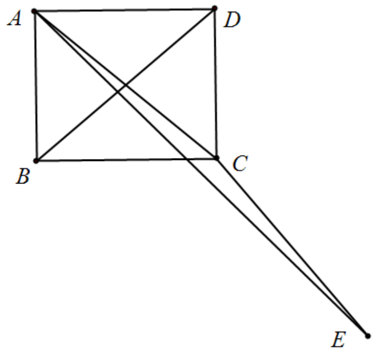
$4\dfrac{\dfrac{2B-0}{2B-D}}{\dfrac{D-2B}{D-0}}+\left(\dfrac{B+D+B}{2B-D}\right)^2=1$, $\dfrac{-8BD}{(2B-D)^2}+\dfrac{(2B+D)^2}{(2B-D)^2}=1$, Example 11: As shown in Figure 1, the line segment $AB = CD$, M and N are the midpoints of the line segments AC and BD respectively. Line MN intersects AB at P and CD at Q respectively. Proof: ∠APM = ∠COM.



set $K = D + B - C$, $4\dfrac{\dfrac{\dfrac{A+C}{2}-\dfrac{D+B}{2}}{C-D}}{\dfrac{A-B}{\dfrac{A+C}{2}-\dfrac{D+B}{2}}} = \dfrac{\dfrac{K-A}{K-B}}{\dfrac{A-B}{A-K}}$

Explanation: This identity can lead to the geometric method: construct parallelogram DCBK, then MN is the median line of △CAK, BA=BK is equivalent to ∠APM = ∠COM.

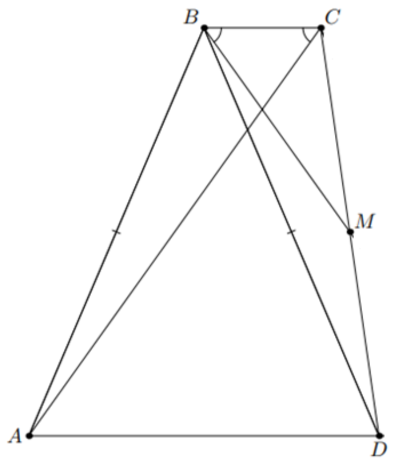
Example 1 2 : As shown in Figure 1, the quadrilateral $ABCD$ is a rectangle , $CE \perp BD$, AE is the bisector of $\angle BAD$, to prove: $CE = BD$.



Proof: Suppose $\dfrac{\dfrac{A-C}{A-E}}{\dfrac{E-A}{E-C}}=T$, $\dfrac{\dfrac{B-C}{A-E}}{\dfrac{A-E}{A-B}}=t_1$, $\dfrac{A+C-B-B}{C-E}=t_2$, $\dfrac{B-C}{A-B}=t_3$,

$$T=\frac{t_1\left(1-t_3{}^2\right)}{t_2t_3}.$$

$$\frac{\dfrac{A-C}{A-E}}{\dfrac{E-A}{E-C}}=\frac{\dfrac{B-C}{A-E}}{\dfrac{A-E}{A-B}}\frac{\dfrac{A-C}{A-B}}{\dfrac{B-A}{B-D}}\frac{E-C}{D-B}\frac{B-A}{B-C}.$$



Example 1 3 : As shown in Figure 1, quadrilateral $ABCD$, $BC \parallel AD$, $BA = BD$, M is the midpoint of CD, prove that $\angle CBM = \angle BCA$.

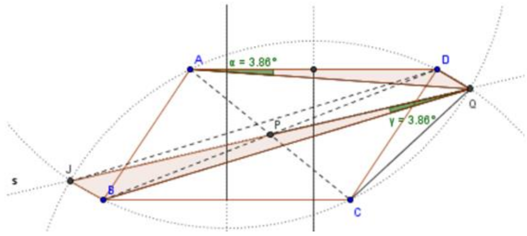
$$\frac{\frac{C-A}{C-B}}{\frac{B-C}{B-\frac{C+D}{2}}}=T,\quad \frac{A-D}{B-C}=T_1,\quad \frac{\frac{A-B}{A-D}}{\frac{D-A}{D-B}}=T_2,\quad 2T-T_1-T_1^2T_2-1=0.$$

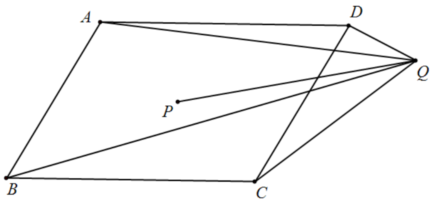
Explanation: Quadrilateral $ABCD$, $BC \parallel AD$, M is the midpoint of CD, prove that $BA = BD \Leftrightarrow \angle CBM = \angle BCA$.

April wrote:

The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

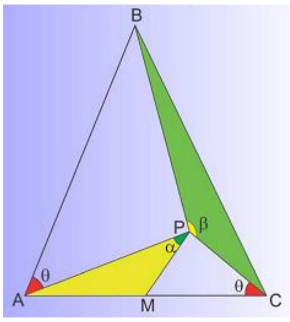
Author: unknown author, Ukraine





Example 1 4 : As shown in Figure 1, the diagonals of the trapezoid $ABCD$ intersect at point P, AD // BC, and point Q is located between BC and AD, satisfying $\angle$AQD = $\angle$ CQB, to prove: $\angle BQP = \angle DAQ$.

Proof: Let $P=0$, $A=tC$, $D=tB$, $\dfrac{\frac{D-A}{Q-A}}{\frac{Q-B}{Q-P}}=T$, $\dfrac{\frac{Q-A}{Q-D}}{\frac{Q-C}{Q-B}}=t_1$, $T=\dfrac{t\left(1-t_1\right)}{\left(1-t\right)t_1}$.

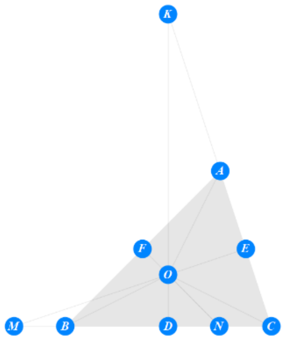


Example 1 5 : As shown in Figure 1, in $\triangle BAC$, $BA = BC$, M is the midpoint *of* AC, if point P satisfies $\angle BAP = \angle ACP$, then $\angle APM$ and $\angle BPC$ are complementary.

Proof: Suppose $M = 0$, $\frac{P-0}{P-A}\frac{P-B}{P+A} = T$, $\frac{\frac{A-B}{A-P}}{\frac{-A-0}{-A-P}} = t_1$, $\frac{\frac{B}{-A}}{\frac{A}{B}} = t_2$,

$$T = -\frac{(1-t_1)^2 + t_2}{4t_1}.$$

Example 16 : As shown in Figure 1, there is a point O on the $\triangle ABC$ plane , D , E , and F are the midpoints of BC , CA , and AB respectively , EO , FO intersect the straight line BC at M , N , and OD intersects the straight line CA at K , if $\angle OBA=\angle OMB$, $\angle OCA=\angle ONB$, to verify $\angle OKA=\angle OAB$, $\angle ODE=\angle OFE$.



$$\frac{\frac{C}{A-C}}{\frac{B-C}{\frac{A+B}{2}}}+\frac{\frac{A}{B-A}}{\frac{C-A}{\frac{B+C}{2}}}+\frac{\frac{B}{A-B}}{\frac{C-B}{\frac{C+A}{2}}}=-\frac{1}{2}.\quad \frac{\frac{\frac{B+C}{2}-0}{\frac{B+C}{2}-\frac{C+A}{2}}}{\frac{\frac{A+B}{2}-\frac{C+A}{2}}{\frac{A+B}{2}-0}}-4\frac{\frac{B}{C-B}}{\frac{A-B}{\frac{C+A}{2}}}=1.$$

$$\frac{\frac{OA}{AB}}{\frac{AC}{OD}}+\frac{\frac{OB}{AB}}{\frac{BC}{OE}}+\frac{\frac{OC}{AC}}{\frac{BC}{OF}}=\frac{1}{2}.$$

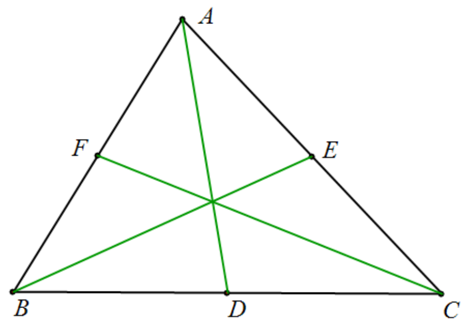
Problem 23 (ISL 1998). Let ABC be a triangle such that $\angle ACB = 2\angle ABC$. Let D be the point of the segment BC such that $CD = 2BD$. The segment AD is extended over the point D to the point E for which $AD = DE$. Prove that $\angle ECD + 180^\circ = 2\angle EBC$.

Example 1 7 : As shown in Figure 1, in $\triangle ABC$, D is the third bisection point of side BC, $2BD = DC$, extend AD to E, so that $AD = DE$, $\angle ACB = 2\angle ABC$, prove: $\angle ECD + 180^\circ = 2\angle EBC$. (ISL 1998)

Proof: Suppose $B=0$, $D=\frac{2B+C}{3}$, $E=2D-A$, $\dfrac{\frac{C-E}{C-B}}{\left(\frac{B-C}{B-E}\right)^2}+\dfrac{\left(\frac{B-A}{B-C}\right)^2}{\frac{C-B}{C-A}}=\frac{4}{27}$.

Explanation: It can be obtained from the identity equation $\angle ECD+180^\circ=2\angle EBC \Leftrightarrow \angle ACB=2\angle ABC$ and the line segment relational expression $-CE\cdot BE^2+CA\cdot BA^2=\frac{4}{27}BC^3$. Note $180^\circ=2\angle EBC-\angle ECD$ that $CE\cdot BE^2$ the previous coefficient is negative.

Example 18 : As shown in Figure 1, $\triangle$in ABC, D, E, and F are the midpoints of BC, CA, and AB respectively. Prove: $\angle DAC=\angle ABE \Leftrightarrow \angle AFC=\angle ADB$. (Second round of the 1995 British Mathematics Competition)



$$\frac{\frac{B-A}{B-\frac{A+C}{2}}}{\frac{A-C}{A-\frac{B+C}{2}}}\left(\frac{\frac{B+C}{2}-B}{\frac{B+C}{2}-A}/\frac{\frac{A+B}{2}-A}{\frac{A+B}{2}-C}+1\right)=2$$

$$\text{or } 2\frac{\frac{A-C}{A-\frac{B+C}{2}}}{\frac{B-A}{B-\frac{A+C}{2}}}-\frac{\frac{C-B}{\frac{B+C}{2}-A}}{\frac{B-A}{\frac{A+B}{2}-C}}=1$$

Explanation: To establish the identity, you can set $A=0$ to simplify the formula, $\frac{\frac{B-A}{B-\frac{A+C}{2}}}{\frac{A-C}{A-\frac{B+C}{2}}}=\frac{B(B+C)}{(2B-C)C}$, $\frac{\frac{B+C}{2}-B}{\frac{B+C}{2}-A}/\frac{\frac{A+B}{2}-A}{\frac{A+B}{2}-C}=-\frac{(B-2C)(B-C)}{B(B+C)}$, and then use observation and try to find the identity.

Example 1 9 : As shown in Figure 1, suppose the convex hexagon $ABCDEF$,

$AB=a$, $BC=b'$, $CD=c$, $DE=a'$, $EF=b$, $FA=c'$, $AD=f'$, $BE=g$, ,

$CF=e$ is inscribed in a circle , then $efg=aa'e+bb'f+cc'g+abc+a'b'c'$.

$$(A-B)(C-D)(E-F)+(B-C)(D-E)(A-F)+(A-B)(F-C)(E-D)$$

$$+(B-C)(A-D)(E-F)+(C-D)(B-E)(A-F)=(A-D)(B-E)(C-F)$$

$$\frac{A-B}{A-D}\frac{C-D}{C-F}\frac{E-F}{B-E}+\frac{B-C}{B-E}\frac{D-E}{A-D}\frac{A-F}{C-F}-\frac{A-B}{B-E}\frac{E-D}{A-D}+\frac{B-C}{B-E}\frac{E-F}{C-F}+\frac{C-D}{A-D}\frac{A-F}{C-F}=1$$

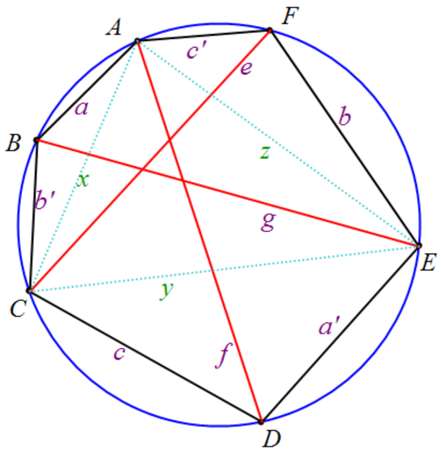
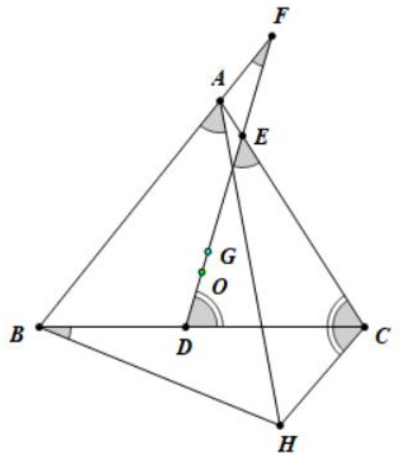
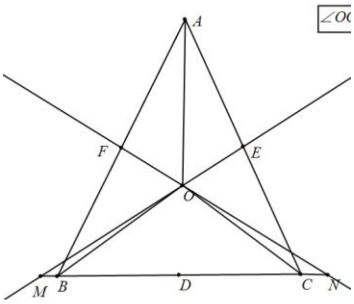


figure 2

Example 20 : As shown in Figure 1, in the quadrilateral $ABHC$, O is any point, G is the center of gravity of $\triangle ABC$, OG intersects the three sides of BC, CA, and AB at D, E, and F *respectively*, if $\angle CBH = \angle OFB$, $\angle BAH = \angle DEC$, to prove: $\angle ODC$ *and* $\angle HCA$ are *complementary*.

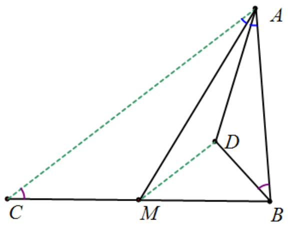


$$\frac{(A-H)\dfrac{A+B+C}{3}}{(A-B)(C-A)}+\frac{(B-H)\dfrac{A+B+C}{3}}{(B-C)(A-B)}+\frac{(C-H)\dfrac{A+B+C}{3}}{(C-A)(B-C)}=0,$$



Example 21: As shown in Figure 1, there is a point O on the $\triangle ABC$ plane, D, E, and F are the midpoints of BC, CA, and AB respectively, and EO, FO intersect BC at M, N. If $\angle OCA = \angle BNO$, $\angle OBA = \angle CMO$, verify $\angle BAO = \angle DAC$.

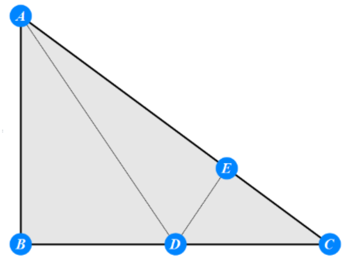
$$\frac{c(\frac{a+b}{2})}{(a-c)(b-c)}+\frac{a(\frac{b+c}{2})}{(b-a)(c-a)}+\frac{b(\frac{c+a}{2})}{(c-b)(a-b)}=-\frac{1}{2}$$



Example 22 : As shown in Figure 1, there is a point D in $\triangle ABC$, M is the midpoint of BC, $\angle CAM = \angle DAB$, $\angle ACB = \angle DBA$, to prove: $DM \parallel AC$.

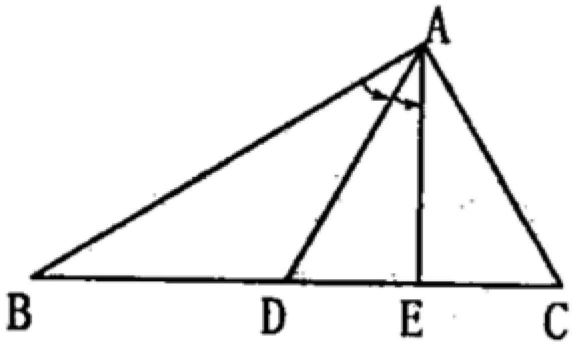
Proof: $2\dfrac{\dfrac{B+C}{2}-D}{C-A}+\dfrac{\dfrac{B-D}{B-A}}{\dfrac{C-A}{C-B}}+2\dfrac{\dfrac{A-\dfrac{B+C}{2}}{A-C}}{\dfrac{A-B}{A-D}}=2$.

Example 23 : As shown in Figure 1, in $\triangle ABC$, D is the midpoint of BC, E is the trisection point of AC, $\angle B=90°$, to prove: $\angle BDA=\angle EDC$.

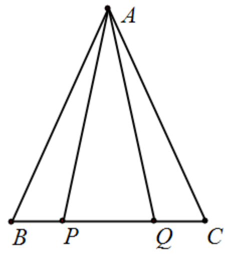


Proof: Suppose $B=0$, $3\frac{\frac{\frac{C}{2}-\frac{2C+A}{3}}{\frac{C}{2}-C}}{\frac{\frac{C}{2}-0}{\frac{C}{2}-A}}+4\frac{A^2}{C^2}=1$.

Example 24 : As shown in Figure 1, in $\triangle ABC$, $BC = 2AC$, D is the midpoint of BC, E is the midpoint of DC, to prove: $\angle BAD = \angle DAE$.

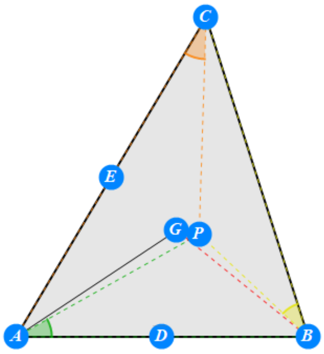


Proof: Let $A=0$, $\frac{1}{2}\frac{\frac{C}{\frac{B+C}{2}}}{\frac{\frac{B+C}{2}}{\frac{B+C}{2}-C}}-\frac{\frac{\frac{B+3C}{4}}{\frac{B+C}{2}}}{\frac{\frac{B+C}{2}}{B}}+1=0$.



Example 25 : As shown *in* Figure 1, let P and Q *be* two fixed points on *line* segment BC , and BP $=$ CQ , A is a moving point outside BC, when point A moves to make $\angle$ BAP = $\angle$CAQ , determine the shape of $\triangle ABC$, and justify your conclusions .

Proof: Suppose $A=0$, $P=tB+(1-t)C$, $Q=B+C-P$, $(1-t)t\dfrac{\dfrac{C-B}{C}}{\dfrac{B}{B-C}}+\dfrac{\dfrac{P}{B}}{\dfrac{C}{Q}}=1$.



Example 26 : As shown in Figure 1, in $\triangle ABC$, G is the center of gravity, point P satisfies $\angle PAB=\angle PBC=\angle PCA$, if the four points A, B, P and G *share a circle, prove that:* C, E, G and P share a circle, A, D, The four points G and E share a circle.

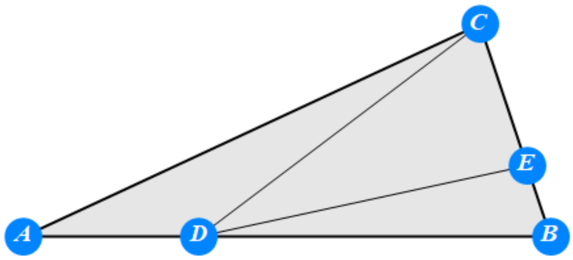
Proof: $$\frac{\dfrac{\dfrac{A+C}{2}-P}{\dfrac{A+C}{2}-\dfrac{A+B+C}{3}}}{\dfrac{C-P}{C-\dfrac{A+B+C}{3}}}=\frac{\dfrac{\dfrac{A+B+C}{3}-P}{\dfrac{A+B+C}{3}-B}}{\dfrac{C-P}{C-A}}+1$$, stating

$\angle PEG=\angle PCG \Leftrightarrow \angle PGB=\angle PAB$,

$$\frac{\dfrac{\dfrac{A+B+C}{3}-\dfrac{A+B}{2}}{\dfrac{A+B+C}{3}-A}}{\dfrac{\dfrac{A+C}{2}-\dfrac{A+B}{2}}{\dfrac{A+C}{2}-A}}+\frac{\dfrac{1}{2}}{\dfrac{\dfrac{A-P}{A-B}}{\dfrac{B-P}{B-C}}-\dfrac{\dfrac{\dfrac{A+B+C}{3}-P}{\dfrac{A+B+C}{3}-B}}{\dfrac{B-P}{B-C}}}=1$$, explain

$\angle AED=\angle AGD \Leftrightarrow \angle PGB=\angle PBC$.

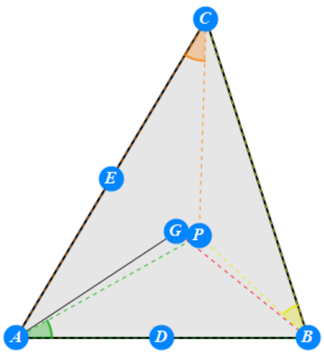
Example 27 : As shown in Figure 1, in $\triangle ABC$, D and E are the three equal points of AB and BC respectively , and $2AD = DB$, $2BE = EC$, if $\angle CDE = \angle CAB$, prove: $\angle EDB = \angle DCA$; $\angle DEC = \angle BCA$; ; $\angle BCD = \angle ABC$ $2\angle DFA = \angle CDF$, where F is the midpoint of BC.



Proof: According to the identity $\frac{3}{2}\frac{\frac{C-\frac{2A+B}{3}}{C-A}}{\frac{\frac{2A+B}{3}-\frac{C+2B}{3}}{\frac{2A+B}{3}-B}}=\frac{\frac{\frac{2A+B}{3}-C}{\frac{2A+B}{3}-\frac{C+2B}{3}}}{\frac{A-C}{A-B}}$ can be obtained $\angle EDB = \angle DCA \Leftrightarrow \angle CDE = \angle CAB$.

According to the identity $\frac{\frac{C-B}{C-A}}{\frac{\frac{C+2B}{3}-\frac{2A+B}{3}}{\frac{C+2B}{3}-C}}=2\left(\frac{\frac{\frac{2A+B}{3}-C}{\frac{2A+B}{3}-\frac{C+2B}{3}}}{\frac{A-C}{A-B}}-1\right)$ can be obtained $\angle DEC = \angle BCA \Leftrightarrow \angle CDE = \angle CAB$.

According to the identity $\frac{\frac{C-B}{C-\frac{2A+B}{3}}}{\frac{B-A}{B-C}}=3\left(1-\frac{\frac{A-C}{A-B}}{\frac{\frac{2A+B}{3}-C}{\frac{2A+B}{3}-\frac{C+2B}{3}}}\right)$ can be obtained $\angle BCD = \angle ABC \Leftrightarrow \angle CDE = \angle CAB$.



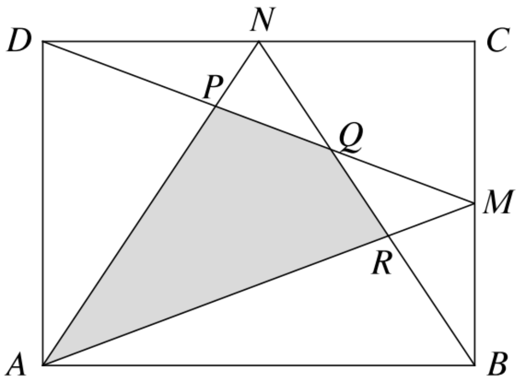
Example 28 : As shown in Figure 9, in $\triangle ABC$, G is the center of gravity, and point P satisfies $\angle PAB=\angle PBC=\angle PCA$. If A, B, P, and G are four points in a circle, prove that: C, E, G, and P are in a circle, and A, D, and The four points G and E share a circle.

Proof: $$\frac{\dfrac{\dfrac{A+C}{2}-P}{\dfrac{A+C}{2}-\dfrac{A+B+C}{3}}}{\dfrac{C-P}{C-\dfrac{A+B+C}{3}}}=\frac{\dfrac{\dfrac{A+B+C}{3}-P}{\dfrac{A+B+C}{3}-B}}{\dfrac{C-P}{C-A}}+1,$$

description $\angle PEG=\angle PCG \Leftrightarrow \angle PGB=\angle PAB$,

$$\frac{\dfrac{\dfrac{A+B+C}{3}-\dfrac{A+B}{2}}{\dfrac{A+B+C}{3}-A}}{\dfrac{\dfrac{A+C}{2}-\dfrac{A+B}{2}}{\dfrac{A+C}{2}-A}}+\frac{\dfrac{1}{2}}{\dfrac{\dfrac{A-P}{A-B}}{\dfrac{B-P}{B-C}}-\dfrac{\dfrac{\dfrac{A+B+C}{3}-P}{\dfrac{A+B+C}{3}-B}}{\dfrac{B-P}{B-C}}}=1,$$

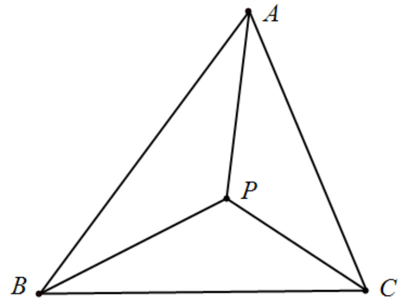
description $\angle AED=\angle AGD \Leftrightarrow \angle PGB=\angle PBC$.



Example 29 : As shown in Figure 1, the parallelogram $ABCD$, M and N are the midpoints of BC and CD respectively , AM intersects BN at R , DM intersects BN at Q , AN intersects DM at P , to prove: A , R , Q , $R4$ The necessary and sufficient condition for point cocircle is $BA \perp BC$.

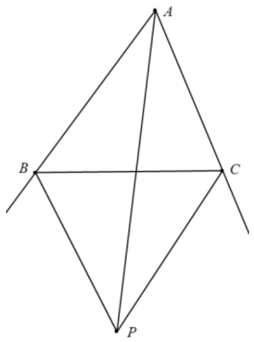
Proof: Suppose $\dfrac{\dfrac{A-\dfrac{A+C-B+C}{2}}{A-\dfrac{B+C}{2}}}{\dfrac{A+C-B-\dfrac{B+C}{2}}{\dfrac{A+C-B+C}{2}-B}}=t$, $\left(\dfrac{B-C}{B-A}\right)^2=s$, $1-4t-4s+ts=0$,

Example 30 : As shown in Figure 4 , in $\triangle ABC$, the angle bisectors of $\angle B$ and $\angle C$ intersect *at* point *P*. *Prove* that AP bisects $\angle A$. (inner theorem)



Proof: $\dfrac{\frac{P-A}{B-A}}{\frac{C-A}{P-A}}+\dfrac{\frac{P-B}{C-B}}{\frac{A-B}{P-B}}+\dfrac{\frac{P-C}{A-C}}{\frac{B-C}{P-C}}=1$.

Explanation: The above formula is equivalent to $\dfrac{(P-A)^2}{(B-A)(C-A)}+\dfrac{(P-B)^2}{(C-B)(A-B)}+\dfrac{(P-C)^2}{(A-C)(B-C)}=1$. If rewritten as $\dfrac{(x-a)^2}{(b-a)(c-a)}+\dfrac{(x-b)^2}{(c-b)(a-b)}+\dfrac{(x-c)^2}{(a-c)(b-c)}=1$, it is a very classic algebraic identity. The classic proof is that the left side of the formula is regarded as a quadratic function about x, *and it is easy to verify that when* $x=$ a , b , c , the left side is equal to 1, so the formula is always true. It is hard to imagine that the complex geometric meaning of such a familiar algebraic identity is an inner theorem. What needs to be emphasized is that while the identity proves the angle relationship, it also proves the side length relationship. According to $\left|\dfrac{\frac{P-A}{B-A}}{\frac{C-A}{P-A}}\right|+\left|\dfrac{\frac{P-B}{C-B}}{\frac{A-B}{P-B}}\right|+\left|\dfrac{\frac{P-C}{A-C}}{\frac{B-C}{P-C}}\right|\geq\left|\dfrac{\frac{P-A}{B-A}}{\frac{C-A}{P-A}}+\dfrac{\frac{P-B}{C-B}}{\frac{A-B}{P-B}}+\dfrac{\frac{P-C}{A-C}}{\frac{B-C}{P-C}}\right|=1$, it can be obtained that the equality sign is established $\dfrac{PA^2}{BA\cdot CA}+\dfrac{PB^2}{CB\cdot AB}+\dfrac{PC^2}{AC\cdot BC}\geq 1$ if and only when P is $\triangle$the incenter.



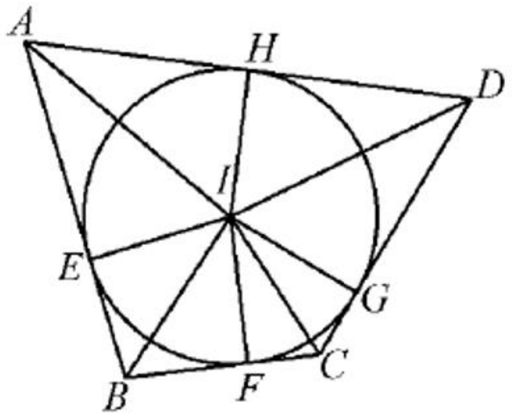
Example 31 : As shown in Figure 5 , in $\triangle ABC$, the bisectors of the exterior angles of ∠B and ∠C intersect *at* point *P. Prove* that *AP* bisects $\angle A$. (side-center theorem)

Proof: $\dfrac{\dfrac{P-A}{B-A}}{\dfrac{C-A}{P-A}}-\dfrac{\dfrac{P-B}{B-A}}{\dfrac{C-B}{P-B}}-\dfrac{\dfrac{P-C}{B-C}}{\dfrac{C-A}{P-C}}=1$.

Explanation: It is not difficult to find that the proof of the inner theorem and the peripheral theorem are the same identity, but a little deformation has been made for the convenience of understanding. This means that the two theorems are somewhat equivalent. However, it is different when it is reflected in the relationship of side lengths, and the above-mentioned identity means that $\dfrac{PA^2}{BA\cdot CA}-\dfrac{PB^2}{CB\cdot AB}-\dfrac{PC^2}{AC\cdot BC}=1$ it is established.

Example 32 : After exploring and obtaining the related properties of the inner triangle, try to generalize the triangle to the quadrilateral.
As shown in the figure , it is known that quadrilateral $ABCD$ is a circumscribed quadrilateral of $\frac{AI^2}{DA\cdot BA}+\frac{BI^2}{AB\cdot CB}+\frac{CI^2}{BC\cdot DC}+\frac{DI^2}{CD\cdot AD}=2$ circle I, then .

We want $\frac{AI^2}{DA\cdot BA}+\frac{BI^2}{AB\cdot CB}+\frac{CI^2}{BC\cdot DC}+\frac{DI^2}{CD\cdot AD}$ to be constant value. Assuming a fixed value, when the quadrilateral $ABCD$ is a square, it is easy to guess that the fixed value is 2 .

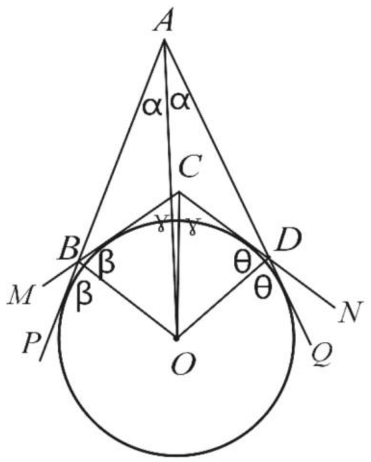


Suppose there is an inscribed circle in the quadrilateral $\frac{1}{z_1}$ *ABCD , and let the circle be the unit circle, the complex forms of the four tangent points are* , $\frac{1}{z_2}$, $\frac{1}{z_3}$, $\frac{1}{z_4}$, then

$a=\frac{2}{z_4+z_1}$, $b=\frac{2}{z_1+z_2}$, $c=\frac{2}{z_2+z_3}$, $d=\frac{2}{z_3+z_4}$, $x=0$, and it can be verified that $\frac{(x-a)^2}{(d-a)(b-a)}+\frac{(x-b)^2}{(a-b)(c-b)}+\frac{(x-c)^2}{(b-c)(d-c)}+\frac{(x-d)^2}{(c-d)(a-d)}=2$ the constant holds true.

Easy to verify $\frac{(x-a)^2}{(d-a)(b-a)}=\frac{(x-c)^2}{(b-c)(d-c)}$, ie $\frac{AI^2}{DA\cdot BA}=\frac{CI^2}{BC\cdot DC}$. the same way $\frac{BI^2}{AB\cdot CB}=\frac{DI^2}{CD\cdot AD}$.

Easy to verify $\frac{(x-a)^2}{(d-a)(b-a)}+\frac{(x-b)^2}{(a-b)(c-b)}=1$, ie $\frac{AI^2}{DA\cdot BA}+\frac{BI^2}{AB\cdot CB}=1$. the same way $\frac{BI^2}{AB\cdot CB}+\frac{CI^2}{BC\cdot DC}=\frac{CI^2}{BC\cdot DC}+\frac{DI^2}{CD\cdot AD}=\frac{DI^2}{CD\cdot AD}+\frac{AI^2}{DA\cdot BA}=1$.



Example 33 : According to the above identity, different interpretations can be made for different graphics, so the following conclusions can be obtained.

As shown in the figure , the tangent lines AP and AQ *of* O *pass through point* A, and *the* tangent *lines* CM and CN of O pass *through* point C. AP and CM intersect at point B , and AQ *and* CN *intersect* at *point* D. *Then* the following identities hold:

$$\frac{AO^2}{DA\cdot AB}=\frac{CO^2}{BC\cdot CD},\ \frac{BO^2}{AB\cdot BC}=\frac{DO^2}{CD\cdot DA},$$

$$\frac{AO^2}{DA\cdot AB}-\frac{BO^2}{AB\cdot BC}=1,\ \frac{BO^2}{AB\cdot BC}-\frac{CO^2}{BC\cdot CD}=-1,$$

$$\frac{CO^2}{BC\cdot CD}-\frac{DO^2}{CD\cdot DA}=1,\ \frac{DO^2}{CD\cdot DA}-\frac{AO^2}{DA\cdot AB}=-1.$$

Example 34 : Suppose I the center of the inscribed circle of is $\triangle ABC$, c_1,c_2,c_3 the three circumscribed circles of I_a,I_b,I_c is , $\triangle ABC$ the centers of the three circumscribed c_1,c_2,c_3 circles of respectively $\triangle ABC$, then

$$\frac{AB\cdot AC}{I_aA^2}+\frac{BA\cdot BC}{I_bB^2}+\frac{CA\cdot CB}{I_cC^2}=1.$$

$$\frac{I_aA^2}{AB\cdot AC}-\frac{I_aB^2}{BA\cdot BC}-\frac{I_aC^2}{CA\cdot CB}=1 .$$

Suppose $A=a^2$, $B=b^2$, $C=c^2$, $I_a=ab-bc+ca$, $I_b=ab+bc-ca$, $I_c=-ab+bc+ca$,

verifiable $\frac{\left(c^2-a^2\right)\left(b^2-a^2\right)}{\left(a^2-I_a\right)^2}+\frac{\left(a^2-b^2\right)\left(c^2-b^2\right)}{\left(b^2-I_b\right)^2}+\frac{\left(b^2-c^2\right)\left(a^2-c^2\right)}{\left(c^2-I_c\right)^2}=1$,

ie $\frac{AB\cdot AC}{I_aA^2}+\frac{BA\cdot BC}{I_bB^2}+\frac{CA\cdot CB}{I_cC^2}=1$.

Verifiable $\frac{\left(a^2-I\right)\left(a^2-I_a\right)}{\left(b^2-a^2\right)\left(c^2-a^2\right)}=1$, ie $\frac{IA\cdot I_aA}{BA\cdot CA}=1$.

Example 34: Convex quadrilateral $ABCD$ with side lengths a, b, c and d is circumscribed on circle O. To prove: $OA\cdot OC+OB\cdot OD=\sqrt{abcd}$.

Proof: Suppose $O=0$, $A=\frac{2}{z_4+z_1}$, $B=\frac{2}{z_1+z_2}$, $C=\frac{2}{z_2+z_3}$, $D=\frac{2}{z_3+z_4}$, then there is an identity

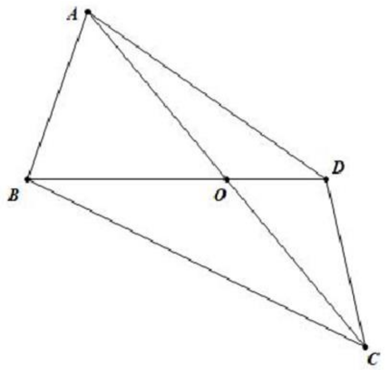
$$\frac{\frac{A-O}{A-B}\frac{C-O}{C-D}}{\frac{A-D}{A-O}\frac{C-B}{C-O}}+\frac{\frac{B-O}{B-C}\frac{D-O}{D-C}}{\frac{B-A}{B-O}\frac{D-A}{D-O}}-2\frac{A-O}{A-B}\frac{B-O}{B-C}\frac{C-O}{C-D}\frac{D-O}{D-A}=1,$$

its geometric meaning $\frac{AO^2}{AB\cdot AD}\frac{CO^2}{CB\cdot CD}+\frac{BO^2}{BC\cdot BA}\frac{DO^2}{DC\cdot DA}+2\frac{AO}{AB}\frac{BO}{BC}\frac{CO}{CD}\frac{DO}{DA}=1$,

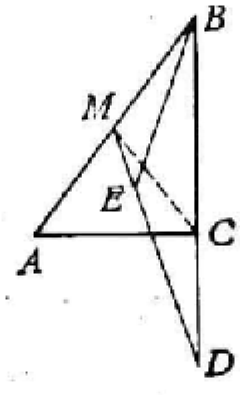
ie $OA\cdot OC+OB\cdot OD=\sqrt{abcd}$.

Example 34 : It is known that BD and AC intersect at the point O, $BO=2OD, AO=OC$, $\angle ABD=2\angle ADB$, to prove:

$\angle BDC=90^{\circ}+\frac{1}{2}\angle CBD$.



Let $O=0$, $C=-A$, $B=-2D$, $\frac{\left(\frac{D-(-A)}{D-0}\right)^2}{\frac{-2D-0}{-2D-(-A)}}+\frac{\frac{-2D-A}{-2D-0}}{\left(\frac{D-0}{D-A}\right)^2}=2$

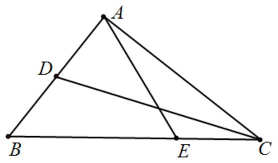


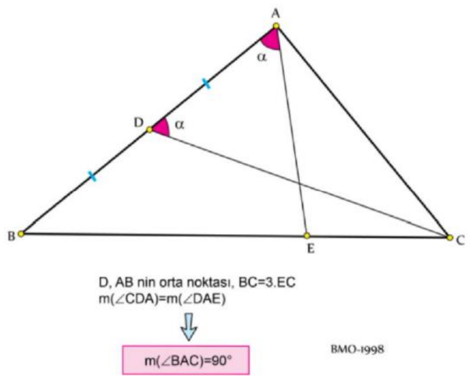
Example 35 : As shown in Figure 1 , the known point M is the midpoint of the hypotenuse AB *of* $Rt\triangle ABC$, extend BC to point D, make 2 $CD = AB$, connect MD , intersect *the* bisector of ∠B at point E , and verify: $BE = ED$.

Proof: Suppose $C=0$, $$\left(\frac{\dfrac{B-0}{B-E}}{\dfrac{D-\dfrac{A+B}{2}}{D-0}}\right)^2 = 2\,\frac{\dfrac{B-0}{B-E}}{\dfrac{B-E}{B-A}}\;\frac{\dfrac{\dfrac{A+B}{2}-0}{\dfrac{A+B}{2}-D}}{\dfrac{D-\dfrac{A+B}{2}}{D-0}}\;\frac{\dfrac{D}{A-B}}{\dfrac{A+B}{B}}$$

[Certificate] Even MC, then $MC = MB$, so $\angle MCB = \angle MBC = 2\angle EBD$.
And $MC = CD$, so $\angle MCB = \angle CMD + \angle D = 2\angle D$;
So $\angle EBD = \angle D$, $BE = ED$. _

Example 36 : As shown in Figure 1 , in $\triangle ABC$, D is the midpoint of AB, E is the third point of BC, and $BE = 2EC$, $\angle ADC = \angle BAE$, find $\angle BAC$. (1998 British Mathematical Olympiad test questions)





$$6\frac{\dfrac{A-\dfrac{B+2C}{3}}{A-B}}{\dfrac{B-A}{\dfrac{A+B}{2}-C}}+4\left(\frac{A-C}{A-B}\right)^2=1,$$

Suppose $A=0$, $$4\left(\frac{C}{B}\right)^2+6\frac{\dfrac{\dfrac{B+2C}{3}}{B}}{\dfrac{B}{\dfrac{B}{2}-C}}=1.$$

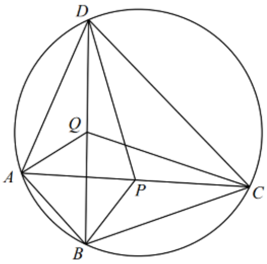
Explanation: According to the identity equation, it can be known $\angle ADC=\angle BAE \Leftrightarrow \angle BAC=90^{\circ}$.

set $$\left[\left(\frac{A+B+2C}{4}-A\right)^2-\left(\frac{A+B+2C}{4}-\frac{A+B}{2}\right)^2\right]-\frac{1}{2}(A-B)(A-C)=0.$$

where the intersection of AE and CD is $\dfrac{A+B+2C}{4}$.

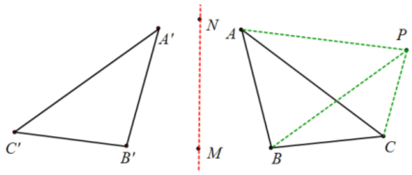
Example 37 : As shown in Figure 1 , P and Q *are the midpoints of* the diagonals AC and BD *of the* inscribed quadrilateral $ABCD$. If ∠BPA = ∠DPA , prove: $\angle AQB = \angle CQB$.

(2011 National High School Mathematics Competition)



Proof: Suppose $\frac{\frac{(B+D)/2-C}{D-B}}{\frac{D-B}{(B+D)/2-A}}=t_1$, $\frac{\frac{(A+C)/2-B}{C-A}}{\frac{C-A}{(A+C)/2-D}}=t_2$, $\frac{\frac{D-A}{D-B}}{\frac{C-A}{C-B}}=t_3$, then

$$-3-4t_1-4t_2+16t_1t_2+16t_3-16t_3^{\ 2}.$$



Example 38 : As shown in the figure, $\triangle ABC$ and $\triangle A_1B_1C_1$ are symmetrical about the line MN, $PA \parallel B_1C_1$, $PB \parallel C_1A_1$, prove $PC \parallel A_1B_1$.

$$\frac{\dfrac{B'-C'}{M-N}}{\dfrac{M-N}{B-C}}\frac{P-A}{B'-C'}+\frac{\dfrac{C'-A'}{M-N}}{\dfrac{M-N}{C-A}}\frac{P-B}{C'-A'}+\frac{\dfrac{A'-B'}{M-N}}{\dfrac{M-N}{A-B}}\frac{P-C}{A'-B'}=0,$$

Problem 1. 設 $\triangle ABC$ 是一個以 A 為頂點的等腰三角形。兩點 P, Q 滿足

$$\measuredangle ABP = \measuredangle BCQ \quad 及 \quad \measuredangle PCA = \measuredangle QBC.$$

證明：A, P, Q 共線。

Example 39 : In $\triangle ABC$, $AB = AC$, two points P and Q satisfy $\angle ABP = \angle BCQ$, $\angle PCA = \angle QBC$, prove: A, P, Q are collinear.

Proof: Suppose $\dfrac{A-P}{P-Q}=t_1, \dfrac{\frac{B-A}{B-C}}{\frac{C-B}{C-A}}=t_2, \dfrac{\frac{B-A}{B-P}}{\frac{C-B}{C-Q}}=t_3, \dfrac{\frac{C-P}{C-A}}{\frac{B-Q}{B-C}}=t_4$,

$-t_2-t_1t_3+t_2t_4+t_1t_2t_4-t_2t_3t_4=0$,

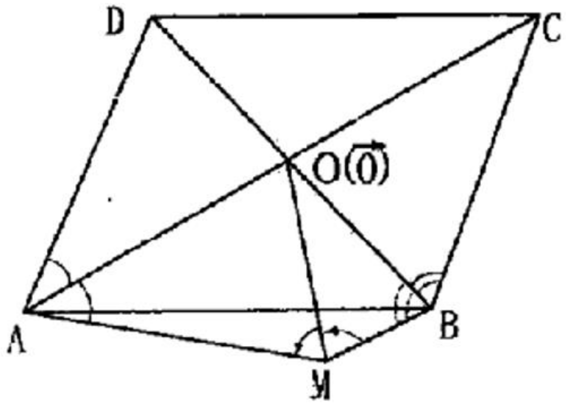
Example 0.1.14 (Reim's). 令 A_1, A_2, B_1, B_2 為共圓四點，C_1 為 A_1B_1 上一點。證明：$\odot(B_1B_2C_1)$，A_2B_2，跟過 C_1 平行於 A_1A_2 的直線共點。

Solution. 設 A_2B_2 與過 C_1 平行於 A_1A_2 的直線交於 C_2。由於 A_1, A_2, B_1, B_2 共圓，(A_1A_2, B_1B_2) 關於 (A_1B_1, A_2B_2) 逆平行。所以由 $A_1A_2 \parallel C_1C_2$ 我們可以得到 (C_1C_2, B_1B_2) 關於 (B_1C_1, B_2C_2) 逆平行，故 B_1, B_2, C_1, C_2 共圓，證畢。

Example 40: It is known that A_1, A_2, B_1, and B_2 share a circle, and C_1 is a point on A_1B_1. Prove: The circumscribed circle of $B_1B_2C_1$, A_2B_2, and C_1 is parallel to the line A_1A_2.

Proof: Suppose $\frac{B_1-C_1}{A_1-B_1}=t_1, \frac{B_2-C_2}{A_2-B_2}=t_2, \frac{\frac{A_2-B_1}{A_2-B_2}}{\frac{A_1-B_1}{A_1-B_2}}=t_3, \frac{\frac{B_1-C_1}{B_1-C_2}}{\frac{B_2-C_1}{B_2-C_2}}=t_4$, $\frac{A_1-A_2}{C_1-C_2}=t_5$,

$-t_4+t_3t_4-t_1t_2t_5+t_1t_2t_4t_5=0$,



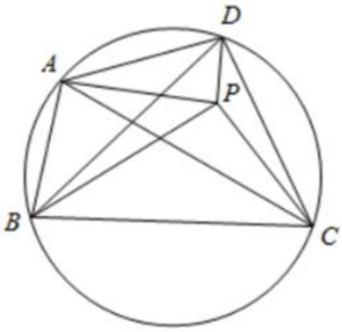
Example 41 : In the parallelogram $ABCD$, the rays AM and AD *are symmetrical* about AC, BM and BC *are symmetrical* about D, and O is the intersection of diagonals AC and BD. Prove: $\angle\ OMA = \angle\ BMO$.

Proof: Suppose $O=0$, $M=xA+yB$, $T=\dfrac{\dfrac{M-0}{M-B}}{\dfrac{M-A}{M-0}}$, $t_1=\dfrac{\dfrac{A-0}{A-M}}{\dfrac{A+B}{A-0}}$, $t_2=\dfrac{\dfrac{B-0}{B+A}}{\dfrac{B-M}{B-0}}$,

$$T=\frac{x+y+1-t_1-t_2+(t_1-t_2)(x-y)}{x+y-1},$$

Explanation: Note $x+y-1\neq 0$ that it is equivalent to that M is not on the straight line AB.

If M *is on* the straight line AB, the conclusion does not hold. Therefore, this question requires an additional condition $AB\ \neq\ BC$ to ensure that M is on the straight line AB.



Example 42 : In *the* convex quadrilateral $ABCD$, the diagonal BD does not bisect any of the opposite angles, the point P *is inside* the quadrilateral $ABCD$, and satisfies $\angle PBC$ = $\angle$DBA and $\angle PDC$ = $\angle$BDA . If A , B , C , D are four points in a circle, prove : AP = CP . (The 45th International Mathematical Olympiad Test Questions)

$$\frac{\frac{A-P}{A-C}}{\frac{C-A}{C-P}}=T\,,\quad \frac{\frac{B-P}{B-C}}{\frac{C-A}{C-D}}=t_1,\quad \frac{\frac{C-B}{C-A}}{\frac{D-C}{D-P}}=t_2\,,\quad \frac{\frac{D-C}{D-B}}{\frac{A-C}{A-B}}=t_3,\quad T-t_1+t_1t_2+t_1t_3-t_2t_3=0,$$

Example 43 : As shown in Figure 7, in $\angle A$ the parallelogram $ABCD$, $\angle B$ the angle bisector of sum intersects at point E, $\angle C$ and $\angle D$ the angle bisector of sum intersects at point G. Prove: $EG // BC$.

Proof: Let $\overrightarrow{AB}$ the unit vector be $\vec{a}$, $\overrightarrow{AD}$ and the unit vector be $\vec{b}$, then

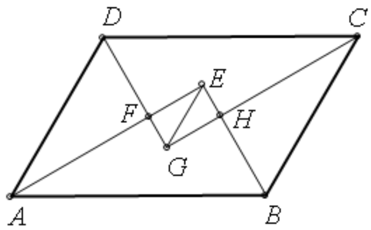
$\overrightarrow{AE}=m(\vec{a}+\vec{b})$, $\overrightarrow{BE}=n(-\vec{a}+\vec{b})$, $\overrightarrow{CG}=-p(\vec{a}+\vec{b})$, $\overrightarrow{DG}=q(\vec{a}-\vec{b})$; from $\overrightarrow{AB}=\overrightarrow{DC}$,

$\overrightarrow{AE}+\overrightarrow{EB}=\overrightarrow{DG}+\overrightarrow{GC}$ that is $m(\vec{a}+\vec{b})-n(-\vec{a}+\vec{b})=q(\vec{a}-\vec{b})+p(\vec{a}+\vec{b})$, $m+n=p+q$,

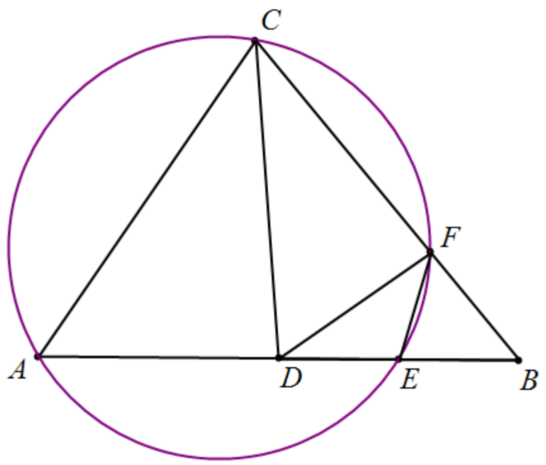
$m-n=p-q$ and get $m=p$, $n=q$. By $\overrightarrow{AE}=\overrightarrow{AB}+\overrightarrow{BE}$, that is

$m(\vec{a}+\vec{b})=k\vec{a}+n(-\vec{a}+\vec{b})$, solved $m=n$. Thus

$\overrightarrow{EG}=\overrightarrow{EB}+\overrightarrow{BC}+\overrightarrow{CG}=-m(-\vec{a}+\vec{b})+\overrightarrow{BC}-m(\vec{a}+\vec{b})=\overrightarrow{BC}-2m\vec{b}$, so $EG // BC$.

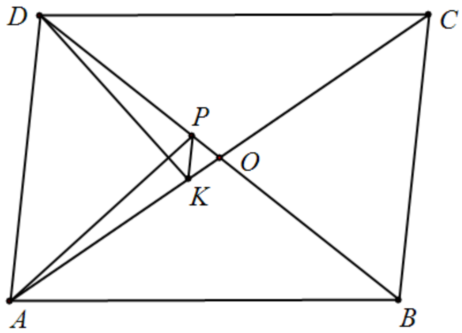


$$2\frac{E-G}{B-C}+\frac{\frac{A-E}{A-B}}{\frac{B-C}{A-E}}+\frac{\frac{B-E}{B-C}}{\frac{B-A}{B-E}}+\frac{\frac{C-G}{B-A}}{\frac{C-B}{C-G}}+\frac{\frac{A+C-B-G}{C-B}}{\frac{A-B}{A+C-B-G}}=2,$$



Example 44 : As shown in the figure, in $\triangle ABC$, D is the midpoint of AB, E is the midpoint of BD, and the circumscribed circle of $\triangle ACE$ intersects CB at F. Prove: $\angle DCA = \angle EFD$.

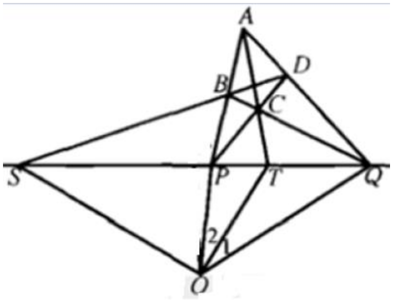
Suppose $C=0$, $F=kB$, $4-4\dfrac{\dfrac{C-\dfrac{A+B}{2}}{C-A}}{\dfrac{F-\dfrac{A+3B}{4}}{F-\dfrac{A+B}{2}}}+(2k-1)\dfrac{\dfrac{C-B}{F-\dfrac{A+3B}{4}}}{\dfrac{A-C}{A-B}}=0$,



Example 45 : As shown in the figure, the diagonals of parallelogram $ABCD$ intersect at O, AP is the angle bisector of $\triangle DAB$, $PK \parallel DA$, to prove: $DK \perp PA$.

Proof: Let $O=0$, $P=sD$, $K=sA$, $\dfrac{\dfrac{A-D}{A-P}}{\dfrac{A-P}{A+D}}=t_1$, $\left(\dfrac{D-K}{A-P}\right)^2=T$, then

$T=1-t_1+s^2t_1$.



Example 46 : As shown in the figure, in the quadrilateral $ABCD$, AB intersects CD at P, AD intersects BC at Q, BD and AC intersect straight line PQ at S and T *respectively, and* the point O *is* outside the straight line PQ. The necessary and sufficient condition for proving $OS \perp OT$ *is* $\angle POT = \angle QOT$. _

Proof: Suppose $O=0$, $D=\dfrac{xA+yB+zC}{x+y+z}$, $P=\dfrac{xA+yB}{x+y}$, $Q=\dfrac{yB+zC}{y+z}$,

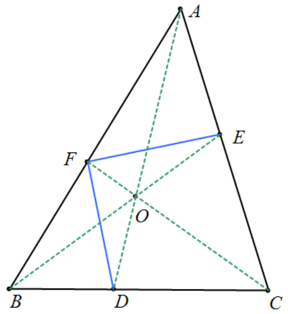
$S=\dfrac{xA+2yB+zC}{x+2y+z}$, $T=\dfrac{xA-zC}{x-z}$,

$$(x+2y+z)^2\left(\frac{S}{T}\right)^2-4(x+y)(y+z)\frac{\frac{P}{T}}{\frac{T}{Q}}=(x-z)^2.$$

O in the condition has a certain degree of activity, some special positions can be selected for compilation.

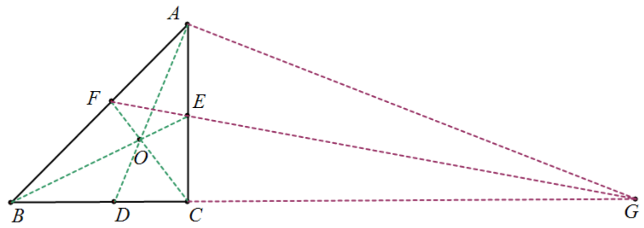
This question is a special case of Example 4 6 .

Example 47 : As shown in the figure, in $\triangle ABC$, AD, BE, CF intersect at a point O, the necessary and sufficient condition to prove $FD \perp FE$ is $\angle BFD = \angle CFD$.



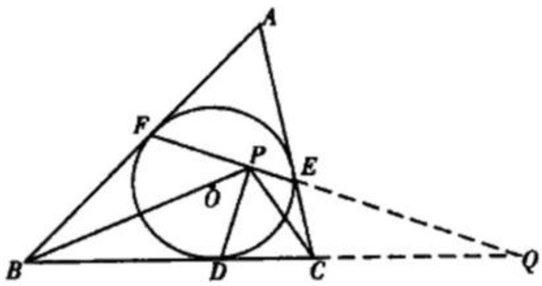
This question is a special case of Example 4 6 .

Example 48 : As shown in the figure, in $\triangle ABC$, AD, BE, CF intersect at a point O, FE intersects BC at G, the necessary and sufficient condition to prove that $AD \perp AG$ is $\angle BAD = \angle CAD$.

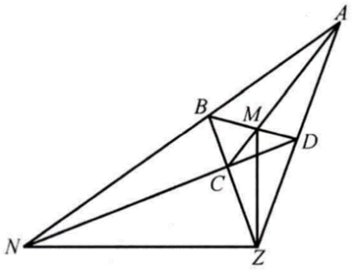


This question is a special case of Example 4 6 .

Example 49 : As shown in the figure, it is known that the inscribed circle of $\triangle ABC$ *cuts* three sides at D , E , F *respectively,* $DP \perp EF$ is in P, prove: $\angle BPD = \angle CPD$.



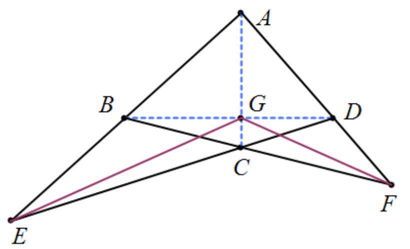
Example 50 : As shown in the figure, in the quadrilateral $ABCD$, AC intersects BD at M, AB intersects CD at N, and AD intersects BC at Z. Prove that the necessary and sufficient condition for $ZM \perp ZN$ is $\angle AZM = \angle BZM$.



Proof: Suppose $D=\frac{xA+yB+zC}{x+y+z}$, $M=\frac{xA+zC}{x+z}$, $Z=\frac{yB+zC}{y+z}$, $N=\frac{xA+yB}{x+y}$,

solve the equation

$k_1\left(\frac{Z-N}{Z-M}\right)^2+k_2\frac{\frac{Z-B}{Z-M}}{\frac{Z-M}{Z-A}}=k_3$, available $(x+y)^2\left(\frac{Z-N}{Z-M}\right)^2-4xy\frac{\frac{Z-B}{Z-M}}{\frac{Z-M}{Z-A}}=(x+z)^2$.



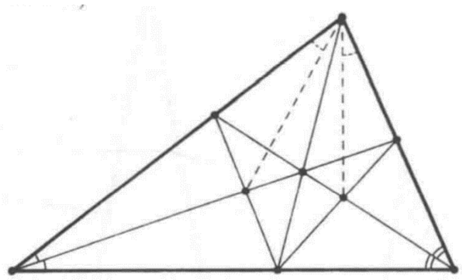
Example 51 : As shown in the figure, in the quadrilateral $ABCD$, AC intersects BD at G, AB intersects CD at E, and AD intersects BC at F. Prove that the necessary and sufficient condition for $AC \perp BD$ is $\angle EGC = \angle FGC$.

Proof: Suppose $D=\dfrac{xA+yB+zC}{x+y+z}$, $G=\dfrac{xA+zC}{x+z}$, $F=\dfrac{yB+zC}{y+z}$, $E=\dfrac{xA+yB}{x+y}$,

solve the equation

$$k_1\left(\frac{B-D}{A-C}\right)^2+k_2\frac{\frac{G-F}{A-C}}{\frac{A-C}{G-E}}=k_3$$, available

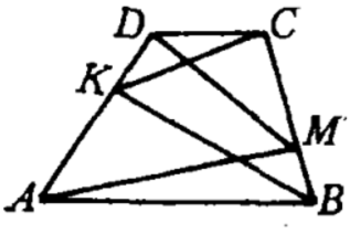
$$y^2(x+y+z)^2\left(\frac{B-D}{A-C}\right)^2-(x+y)(x+z)^2(y+z)\frac{\frac{G-F}{A-C}}{\frac{A-C}{G-E}}=x^2z^2.$$



Example 52 : As shown in the figure, in $\triangle ABC$, I is the center, AD, BE, CF are the angle bisectors, DF intersects BI at P, and DE intersects CI at Q. Prove: $\angle CAQ = \angle PAB$.

Proof: Suppose $I=\frac{aA+bB+cC}{a+b+c}$, then $P=\frac{aA+2bB+cC}{a+2b+c}$, $Q=\frac{aA+bB+2cC}{a+b+2c}$,

$$\frac{\frac{A-P}{A-B}}{\frac{A-C}{A-Q}}+\frac{2(a+2b+c)}{a+b+2c}\frac{\frac{B-P}{B-C}}{\frac{B-A}{B-P}}+\frac{2(a+b+2c)}{a+2b+c}\frac{\frac{C-Q}{C-A}}{\frac{C-B}{C-Q}}=\frac{2a^2+4ab+2b^2+4ac+5bc+2c^2}{(a+2b+c)(a+b+2c)}$$

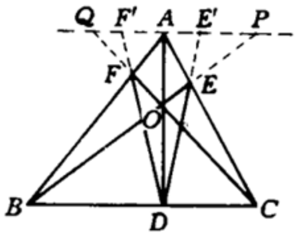


Example 53 : As shown in the figure, the trapezoid $ABCD, AB \parallel CD, AB > CD, K$ and M *are* the points on the waist AD *and* CB respectively , and it is known that $\angle DAM = \angle CBK$, to prove: $\angle DMA = \angle CKB$. (Second Session "Zu Chong's Cup" Junior High School Mathematics Invitational Test Questions)

Proof: Let the intersection of $O=0$ AD and BC, $D=sA$, $C=sB$, $K=kA$, $M=mB$, solve the equation

$$k_1 \frac{\frac{K-C}{K-B}}{\frac{M-A}{M-D}} + k_2 \frac{\frac{A-0}{A-M}}{\frac{B-K}{B-C}} = k_3, \text{ available } \frac{\frac{K-C}{K-B}}{\frac{M-A}{M-D}} - (km-s)\frac{\frac{A-0}{A-M}}{\frac{B-K}{B-C}} = s.$$

Explanation: It is necessary to pay attention to this way of setting the origin, which will make the conclusion easier.



Example 54 : As shown in the figure, it is known that AD is the height of the acute angle $\triangle ABC$, O is any point on AD, connect BO, CO and extend AC, AB to E, F *respectively*, and connect DE, DF. Prove: $\angle EDA = \angle FDA$.

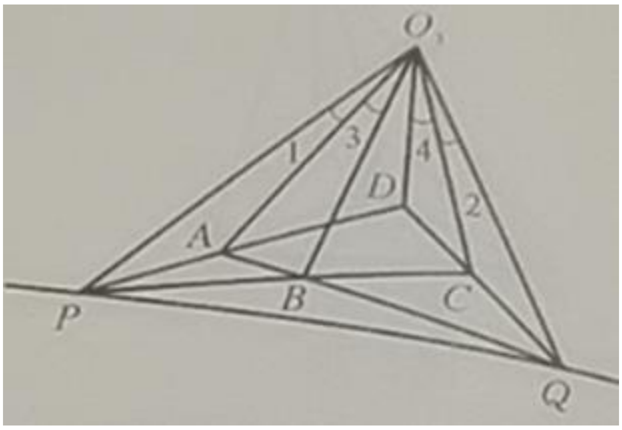
Proof: Suppose $O=\frac{xA+yB+zC}{x+y+z}$, $D=\frac{yB+zC}{y+z}$, $E=\frac{xA+zC}{x+z}$, $F=\frac{xA+yB}{x+y}$,

solve the equation

$$k_1\left(\frac{B-C}{A-D}\right)^2+k_2\frac{\frac{D-F}{D-A}}{\frac{D-A}{D-E}}=k_3$$

, available

$$\frac{y^2z^2}{(y+z)^2}\left(\frac{B-C}{A-D}\right)^2+(x+y)(x+z)\frac{\frac{D-F}{D-A}}{\frac{D-A}{D-E}}=x^2,$$



Example 55 : As shown in the figure, suppose O *is* a point on the quadrilateral $ABCD$ *plane*, AD intersects BC at P, AB intersects DC at Q, the necessary and sufficient condition for ∠1=∠2 is ∠3=∠4.

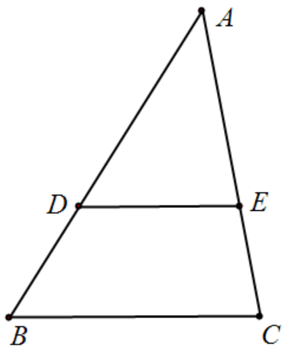
Suppose $O=0$, $D=\frac{xA+yB+zC}{x+y+z}$, $P=\frac{yB+zC}{y+z}$, $Q=\frac{xA+yB}{x+y}$, solve the equation

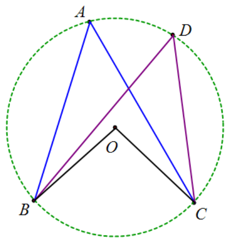
$$k_1\frac{\frac{Q}{C}}{\frac{A}{P}}+k_2\frac{\frac{B}{A}}{\frac{C}{D}}=k_3,\text{Available}(x+y)(y+z)\frac{\frac{Q}{C}}{\frac{A}{P}}-y(x+y+z)\frac{\frac{B}{A}}{\frac{C}{D}}=xz$$

Simple classic case

Example 56 : As shown in Figure 3, $\triangle$in ABC, D and E intersect points on AB and AC respectively. *Prove that* the necessary and sufficient condition for $DE \,//\, BC$ *is* $\angle ADE = \angle ABC$.

$$\frac{B-C}{D-E}=\frac{\frac{B-C}{B-A}}{\frac{D-E}{B-A}}.$$

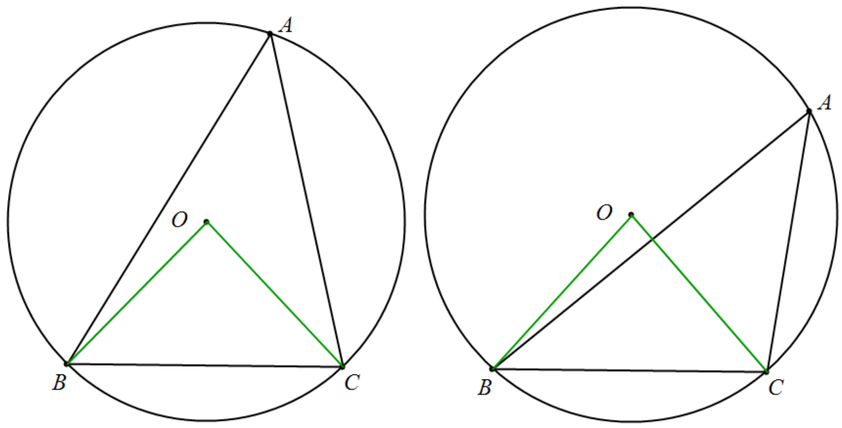




Example 57 : As shown in Figure 3, in $\angle DAC = \angle DBC \Leftrightarrow \angle ADB = \angle ACB$ quadrilateral $ABCD$, .

Proof: $\dfrac{\dfrac{B-C}{A-C}}{\dfrac{B-D}{A-D}} = \dfrac{\dfrac{D-A}{C-A}}{\dfrac{D-B}{C-B}}$.

Explanation: The above identity is obviously established, because the right side of the equation can be regarded as the simple adjustment of the left side. It seems to be a simple algebraic deformation, but it corresponds to a geometric property. Even the angle symbol does not appear in the identity, but the angle relationship is proved, and the original proposition and the inverse proposition are proved together. The premise of reading this proof is to understand the geometric meaning of the complex numbers in the identity. Its detailed representation is $\angle ADB = \angle ACB \Leftrightarrow \dfrac{\dfrac{B-C}{A-C}}{\dfrac{B-D}{A-D}} \in R \Leftrightarrow \dfrac{\dfrac{D-A}{C-A}}{\dfrac{D-B}{C-B}} \in R \Leftrightarrow \angle DAC = \angle DBC$.

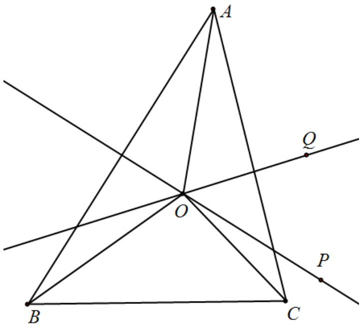


Example 58 : As shown in Figure 3, O is the circumcenter of $2\angle BAC=\angle BOC$ $\triangle ABC$, *to prove:* (Circle center angle theorem).

Proof: $$\frac{\left(\frac{A-C}{A-B}\right)^2}{\frac{O-C}{O-B}}\frac{\frac{C-O}{C-A}}{\frac{A-C}{A-O}}\frac{\frac{B-A}{B-O}}{\frac{A-O}{A-B}}=1.$$

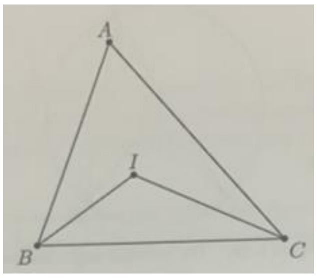
If the three points O, B and C are collinear, it is Thales' theorem.

The known conditions: $\angle OCA=\angle OAC \Leftrightarrow \frac{\frac{C-O}{C-A}}{\frac{A-C}{A-O}}\in R$; $\angle OBA=\angle OAB \Leftrightarrow \frac{\frac{B-A}{B-O}}{\frac{A-O}{A-B}}\in R$; Prove the conclusion $2\angle BAC=\angle BOC \Leftrightarrow \frac{\left(\frac{A-C}{A-B}\right)^2}{\frac{O-C}{O-B}}\in R$. In Euclidean geometry, it is often necessary to give corresponding proofs according to different positions of points. However, using the method of complex identities, the proof can be unified. Moreover, identities are good for understanding inverse propositions. O is the circumcenter of $\triangle ABC$, available $2\angle BAC=\angle BOC$. But not the other way around, because $\frac{\left(\frac{A-C}{A-B}\right)^2}{\frac{O-C}{O-B}}\in R$ they cannot be rolled out separately $\frac{\frac{C-O}{C-A}}{\frac{A-C}{A-O}}\in R, \frac{\frac{B-A}{B-O}}{\frac{A-O}{A-B}}\in R$.



Example 59 : As shown in Figure 3, O is a point inside $\triangle ABC$, *the straight line* OP is the angle bisector of $\angle AOB$, *and the straight line* OQ is the angle bisector of $2\angle QOP = \angle BOC\ \angle AOC$. *Prove:* .

Proof: $\dfrac{\left(\dfrac{O-Q}{O-P}\right)^2}{\dfrac{O-C}{O-B}} \dfrac{\dfrac{P-O}{O-A}}{\dfrac{O-B}{P-O}} \dfrac{\dfrac{O-A}{O-Q}}{\dfrac{O-Q}{O-C}} = 1.$



Example 60 : As shown in Figure 3, I is the heart of $\triangle ABC$, to prove: $\angle BIC = 90^\circ + \frac{1}{2}\angle A$.

Proof: $$\frac{\left(\frac{I-C}{I-B}\right)^2}{\frac{A-C}{A-B}} \frac{\frac{B-I}{B-C}}{\frac{B-A}{B-I}} \frac{\frac{C-B}{C-I}}{\frac{C-I}{C-A}} = -1.$$

Explanation: $\dfrac{\left(\frac{I-C}{I-B}\right)^2}{\frac{A-C}{A-B}} \in R$ Can only describe $2\angle BIC = \angle A$ or

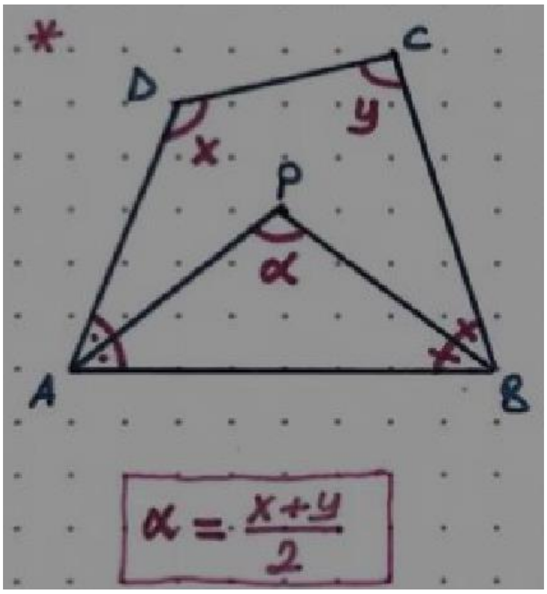
$2\angle BIC - \angle A = 180^\circ$.

Method 1: With the aid of graphs, it is impossible to hold $\angle BIC > \angle A$ since $2\angle BIC = \angle A$.

Method 2: Further clarify the positive and negative of each item, not just judge whether it is a real number. **Among them, known conditions:** $\angle$IBC = $\angle$

ABI ; $\angle$***BCI*** $\Leftrightarrow \dfrac{\frac{B-I}{B-C}}{\frac{B-A}{B-I}} \in R^+$ = $\angle$ICA ; ***so*** $\Leftrightarrow \dfrac{\frac{C-B}{C-I}}{\frac{C-I}{C-A}} \in R^+$, $\dfrac{\left(\frac{I-C}{I-B}\right)^2}{\frac{A-C}{A-B}} \in R^-$ explain

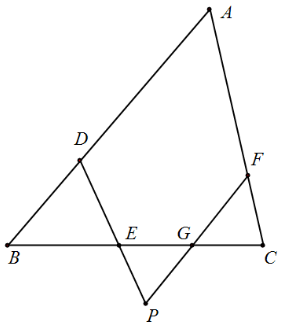
$2\angle BIC - \angle A = 180^\circ$.



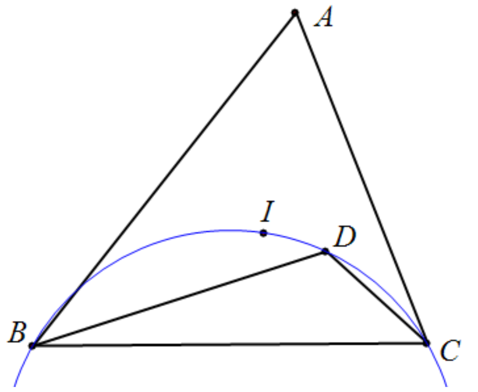
Example 61 : *As* shown in Figure 1, in the quadrilateral $ABCD$, the angle bisectors of ∠A and ∠B intersect *at* point P. *Prove* : $2\angle APB = \angle ADC + \angle BCD$.

$$\frac{\left(\frac{P-B}{P-A}\right)^2}{\frac{D-C}{D-A}\frac{C-B}{C-D}}\frac{\frac{A-P}{A-B}}{\frac{A-D}{A-P}}\frac{\frac{B-A}{B-P}}{\frac{B-P}{B-C}}=1,$$

Explanation: This is the generalization of the conclusion of the above question in the quadrilateral.

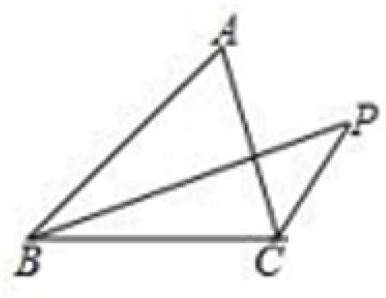


Example 62 : As shown in Figure 1, $\triangle$in ABC, points D and E on BA and BC, and $BD = BE$, points F and G on CA and CB, and $CF = CG$, and DE intersects FG on P, then $\angle DPF + \frac{1}{2}\angle A = 90^\circ$. $\dfrac{\left(\dfrac{G-F}{D-E}\right)^2}{\dfrac{A-C}{A-B}} \dfrac{\dfrac{D-E}{A-B}}{\dfrac{C-B}{E-D}} \dfrac{\dfrac{A-C}{F-G}}{\dfrac{G-F}{B-C}} = -1$,



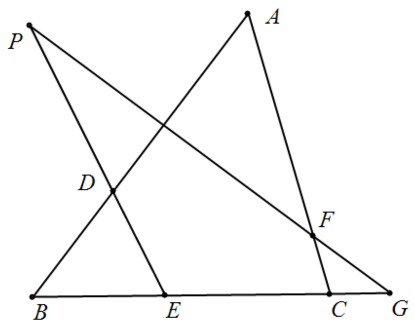
Example 63 : As *shown* in Figure 1, $\triangle$in ABC, I is the inward point, point D satisfies $\angle ABD = \angle BCD$, $\angle DBC = \angle DCA$, to prove: B, C, D, *and* I share a circle.

$$\frac{\frac{B-I}{B-C}\frac{C-B}{C-I}\frac{B-A}{B-D}\frac{C-D}{C-A}}{\frac{B-A}{B-I}\frac{C-I}{C-A}\frac{C-B}{C-D}\frac{B-D}{B-C}}=\left(\frac{\frac{D-C}{D-B}}{\frac{I-C}{I-B}}\right)^2$$



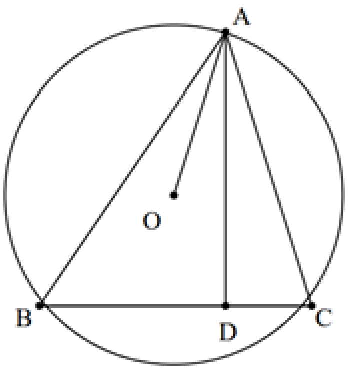
Example 64 : As shown *in* Figure 1, if the point P is *the* intersection of the bisectors of the exterior angles $\angle BPC = \frac{1}{2}\angle A$ of ∠ABC and ∠ACB , then .

$$\frac{\left(\frac{P-C}{P-B}\right)^2}{\frac{A-C}{A-B}} = \frac{\frac{B-A}{B-P}\,\frac{C-P}{B-C}}{\frac{B-P}{B-C}\,\frac{C-A}{C-P}},$$



Example 65 : As shown in Figure 1, $\triangle$in ABC, *there are points* D and E on BA and BC *respectively*, and $BD = BE$, points F and G *on* CA and on the extension line of CB *respectively*, and $CF = CG$, and DE intersects FG at P, then $\angle DPF = \frac{1}{2}\angle A$.

$$\frac{\left(\frac{P-F}{P-D}\right)^2}{\frac{A-C}{A-B}} = \frac{\frac{B-A}{D-P}\,\frac{F-P}{B-C}}{\frac{D-P}{B-C}\,\frac{C-A}{F-P}}$$

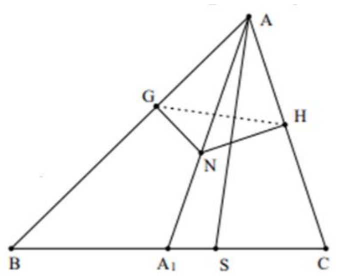


Example 66 : As shown in Figure 3, in $\triangle ABC$, O is the circumcenter, AD is high, $AB > AC$, to prove: $\angle C = \angle B + \angle DAO$.

Proof: $$\frac{\frac{A-D}{A-O}\frac{B-A}{B-C}}{\frac{C-B}{C-A}} = \frac{A-D}{B-C}\left(\frac{B-A}{B-C}\frac{A-C}{A-O}\right).$$

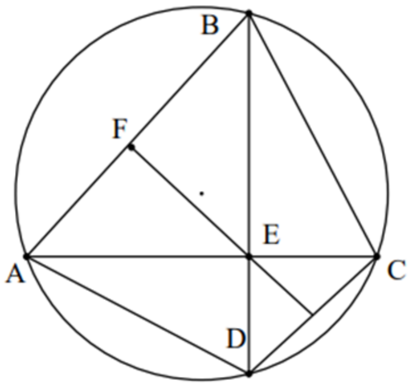
Explain $\frac{B-A}{B-C}\frac{A-C}{A-O}$ that ∠B + ∠CAO = 90 ° . _ See Zhou Gaozhang p 149

Example 67 : As shown in Figure 3, in $\triangle ABC$, D and S are on BC, N is a point on A *and* D, and the feet of N on AB and AC are G and H respectively, $\angle BAD = \angle CAS$, to prove: $AS \perp GH$. _



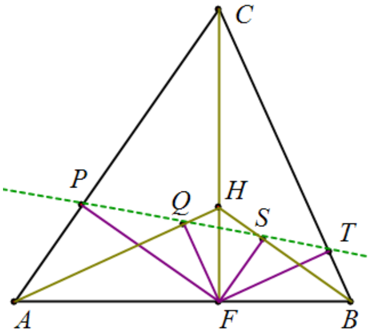
Proof: $\dfrac{H-G}{S-A} = \left(\dfrac{A-B}{A-D} / \dfrac{A-S}{A-C}\right)\left(\dfrac{H-G}{H-A} / \dfrac{N-G}{N-A}\right)\dfrac{H-A}{A-C}\dfrac{A-D}{A-N}\dfrac{N-G}{A-B}$.

Promotion: As shown in the figure, in $\triangle ABC$, D and S are on BC, and the vertical feet of N on AB and AC *are* G and H respectively. To prove: three conditions " N is a point on AD"" $\angle BAD = \angle CAS$ " " $AS \perp GH$ " , if any two are established, the third one is also established.



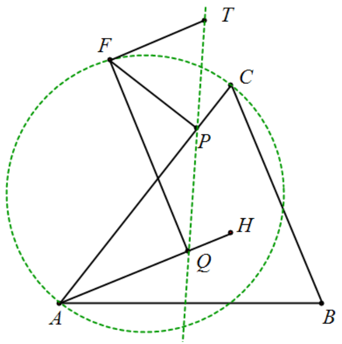
$$\frac{C-D}{F-E}=\frac{B-D}{A-E}\left(\frac{C-D}{C-A}/\frac{B-D}{B-A}\right)\left(\frac{E-A}{E-F}/\frac{A-B}{A-C}\right)$$

Example 68 : As shown in Figure 1, the quadrilateral $ABCD$ is inscribed in the circle , $AC\perp BD$, the diagonals intersect at point E, and F is the midpoint of AB. Prove: $CD\perp FE$.



Example 69 : As shown in Figure 3, in $\triangle ABC$, H is the orthocenter, CF is the height, and the feet of F on AC, AH, and BC are P, Q, and T respectively . Prove: P, Q, and T are collinear.

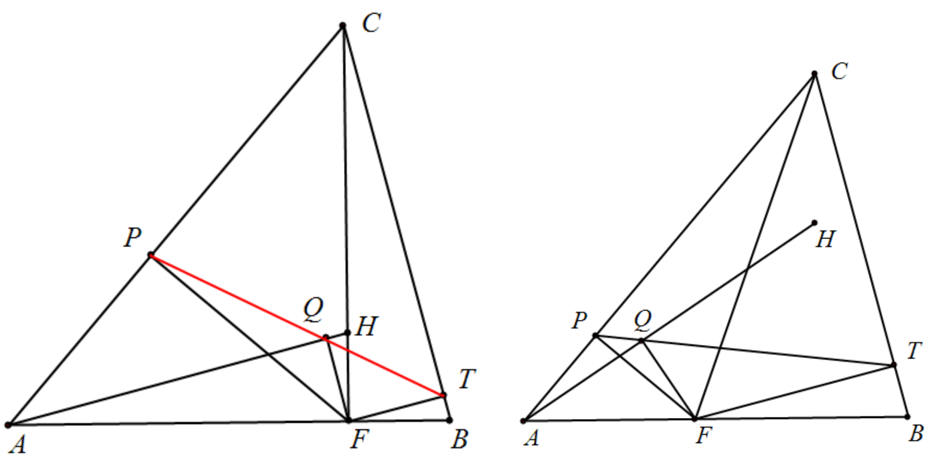
$$\frac{P-Q}{P-T}=\frac{\frac{F-P}{F-C}}{\frac{T-P}{T-C}}\frac{\frac{Q-P}{Q-A}}{\frac{F-P}{F-A}}\left(\frac{Q-A}{T-C}\frac{F-C}{F-A}\right),$$

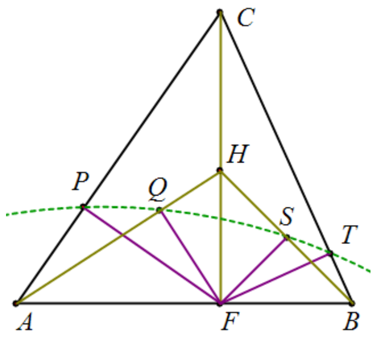


It can be seen from the identity that the three points P, Q, and T are collinear $\Leftrightarrow$ $\boldsymbol{FC \perp FA}$. $\boldsymbol{FC \perp FA}$ **does not require** $\boldsymbol{F}$ **to be on** $\boldsymbol{AB}$**, it only needs to let** F be on the circle whose diameter is AC.

It can be seen from the identity equation that the three points P, Q, T are collinear and the angle between $\Leftrightarrow$ the straight lines CB and HA is equal to the angle between the straight lines AB and FC. **Therefore, a new proposition can be obtained:**

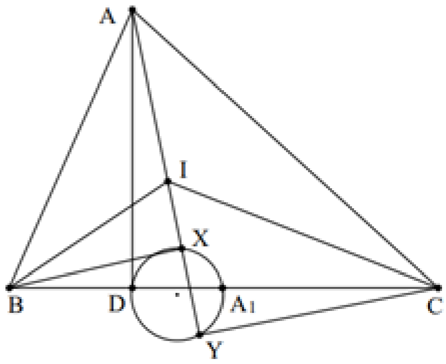
Example 70 : As shown in the figure, there are two points F and H on the $\triangle ABC$ plane , satisfying that the angle between the straight lines CB and HA is equal to the angle between the straight lines AB and FC. The feet of F on AC, AH, BC are P, Q, T respectively . Prove: P, Q, T are collinear.





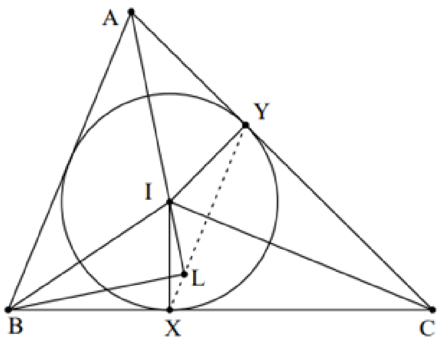
Example 71 : As shown in Figure 3, in $\triangle ABC$, H is a point on high CF, and the feet of F on AC, AH, BH, and BC are P, Q, S, and T respectively. To prove: P, Q, S, and T Four points in a circle.

$$\frac{\frac{P-Q}{P-T}}{\frac{Q-S}{S-T}}\frac{\frac{A-Q}{A-F}}{\frac{P-Q}{P-F}}\frac{\frac{T-P}{T-C}}{\frac{F-P}{F-C}}\frac{\frac{F-T}{F-B}}{\frac{S-T}{S-B}}\frac{\frac{S-Q}{B-S}}{\frac{F-Q}{F-H}}\frac{F-A}{F-C}\frac{F-B}{F-H}\frac{T-C}{T-F}\frac{Q-F}{Q-A}=1.$$



Example 72 : As shown in Figure 3, in △ ABC , I is the center, AD is the height, A *1* is the midpoint *of BC , and the feet of B* and C on *AI are X and Y* respectively . Prove that Y , A_1 , X and D are four points in a circle.

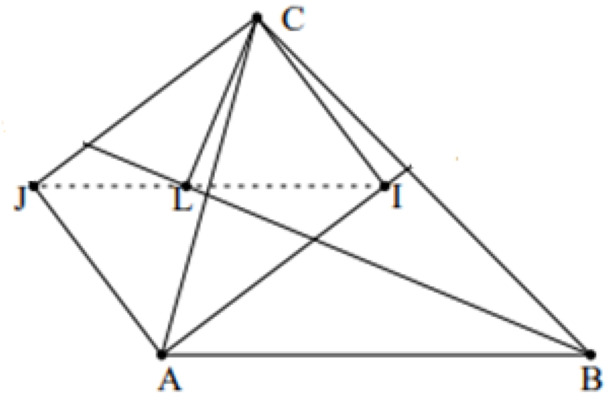
prove: $$\frac{Y-D}{Y-X}\frac{A_1-X}{A_1-D}=\frac{Y-A}{Y-X}\frac{D-C}{D-A_1}\frac{A_1-X}{C-A}\left(\frac{D-Y}{D-C}\frac{A-C}{A-Y}\right)$$



Example 73 : As shown in Figure 3, in $\triangle ABC$, the inscribed circle I cuts BC, AC at X, Y, and $BL \perp AI$ at L. Prove: X, L, Y are collinear.

$$\frac{X-L}{X-Y}=\left(\frac{L-X}{L-I}\frac{B-I}{B-X}\right)\frac{\frac{Y-C}{Y-X}}{\frac{I-C}{I-X}}\left(\frac{\frac{I-C}{I-B}}{\frac{A-C}{A-I}}\frac{B-X}{X-I}\right)\frac{L-I}{A-I}\frac{A-C}{Y-C}.$$

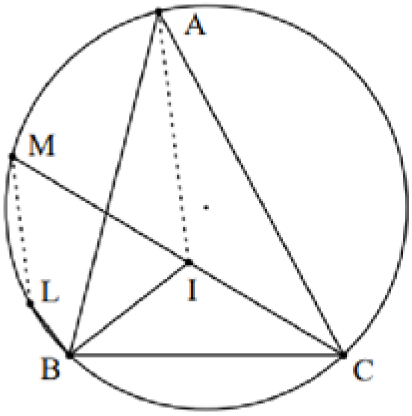
Description: used $\angle BIC=90^{\circ}+\frac{1}{2}\angle A$. See above question.



Example 74 : As shown in Figure 3, in $\triangle ABC$, O is the inner, AI, BL is the angle bisector, AJ is the outer angle bisector, $IA \perp IC$, $LC \perp LB$, $JC \perp JA$, to prove: J, L, I three points collinear.

$$\frac{I-J}{I-L}=\left(\frac{L-O}{L-I}\frac{C-I}{C-O}\right)\left(\frac{J-I}{J-A}\frac{C-A}{C-I}\right)\left(\frac{A-J}{A-C}\frac{O-C}{O-B}\right)\frac{B-O}{L-O}$$

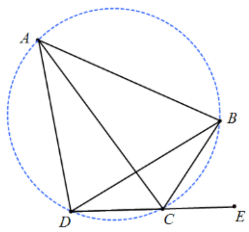
Description: used $\angle BOC=90^{\circ}+\frac{1}{2}\angle A$. See above question.



Example 75 : As shown in Figure 3, in $\triangle ABC$, I is the inner circle, M and L are on the circumscribed circle of $\triangle ABC$, M is on the straight line CI, $BL \perp BI$, to prove: $ML \parallel AI$.

$$\frac{A-I}{L-M}=\left(\frac{L-B}{L-M}\frac{C-M}{C-B}\right)\left(\frac{C-A}{C-I}/\frac{C-M}{C-B}\right)\left(\frac{I-C}{I-B}/\frac{A-C}{A-I}\frac{B-I}{B-L}\right),$$

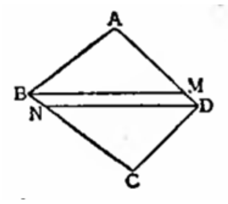
Description: used $\angle BIC=90^{\circ}+\frac{1}{2}\angle A$. See above question.



Example 76 : As shown in Figure 3, the quadrilateral $ABCD$, E is a point on the ray DC, if $DA = DB$, ∠ACD = ∠BCE, to prove: A, B, C, D four points share a circle.

$$\frac{\frac{A-B}{A-D}}{\frac{B-D}{B-A}}\frac{\frac{C-D}{C-A}}{\frac{C-B}{D-C}}=\frac{\frac{A-B}{A-C}}{\frac{D-B}{D-C}}\frac{\frac{A-B}{A-D}}{\frac{C-B}{D-C}},$$

Explanation: This question is very special. The two items on the right side of the equation $\frac{\frac{A-B}{A-C}}{\frac{D-B}{D-C}}\in R$ are $\frac{\frac{A-B}{A-D}}{\frac{C-B}{D-C}}\in R$ the equivalent equations of four points A, B, C, and D co-circling. If the verification conclusion is not established, then the left side of the identity is a real number, and the right side is not a real number, which is a contradiction.



Example 77 : *As shown* in Figure 3, in the quadrilateral $ABCD$, if the diagonals ∠A and ∠C are equal , *then* the bisectors of the *other* pair of diagonals ∠B *and* ∠D are parallel to each other.

$$\left(\frac{N-D}{B-M}\right)^2=\frac{\frac{A-D}{A-B}\frac{B-A}{B-M}\frac{D-N}{D-A}}{\frac{C-B}{C-D}\frac{B-M}{B-C}\frac{D-C}{D-N}}.$$

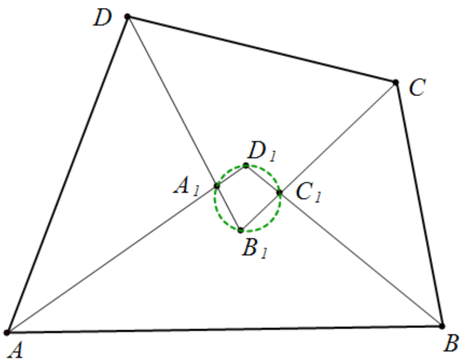
Explanation: The right part of the equation is a positive real number, so it can be judged $\dfrac{N-D}{B-M}$ as a real number.

Example 79 : As shown in Figure 3, there is a *point* P inside $\triangle ABC$, which satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Prove that: B, C, P, *and* I share a circle, and I is the center of $\triangle ABC$.

Proof: $\left(\frac{\frac{P-C}{P-B}}{\frac{I-C}{I-B}}\right)^2 = \frac{\frac{B-I}{B-C}\frac{C-B}{C-I}}{\frac{B-A}{B-I}\frac{C-I}{C-A}}\left(\frac{\frac{B-A}{B-P}\frac{C-P}{C-A}}{\frac{B-P}{B-C}\frac{C-B}{C-P}}\right)$.

Example 80 : As shown in Figure 3, for the quadrilateral $ABCD$, four lines drawn from the four corners intersect to form a quadrilateral $A_1B_1C_1D_1$, satisfying $\angle CDA_1 = \angle DAA_1$, $\angle A_1AB = \angle C_1BC$, $\angle ABC_1 = \angle BCC_1$, $\angle C_1CD = \angle A_1DA$, and verifying that the quadrilateral $A_1B_1C_1D_1$ is a quadrilateral inscribed in a circle.

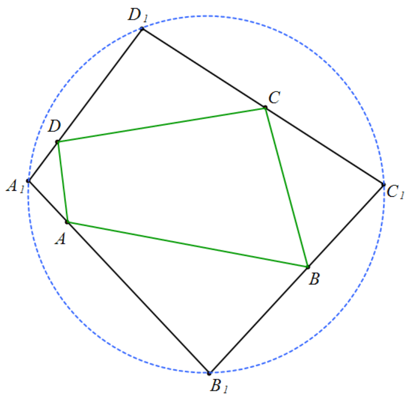
$$\frac{\frac{A-D}{A_1-D_1}\frac{C_1-D_1}{B-C}\frac{C-B}{C_1-B_1}\frac{A_1-B_1}{D-A}}{\frac{D-C}{A_1-B_1}\frac{A_1-D_1}{A-B}\frac{B-A}{C_1-D_1}\frac{C_1-B_1}{C-D}}=\left(\frac{\frac{B_1-A_1}{A_1-D_1}}{\frac{C_1-B_1}{C_1-D_1}}\right)^2$$



Example 81 : As shown in Figure 3, the quadrilateral $ABCD$, the bisectors of the four angles form a quadrilateral $A_1B_1C_1D_1$, verify that the quadrilateral $A_1B_1C_1D_1$ is a quadrilateral inscribed in a circle.

Proof: $\dfrac{\dfrac{A-D}{A_1-D_1}}{\dfrac{A_1-D_1}{A-B}}\dfrac{\dfrac{C_1-D_1}{B-C}}{\dfrac{B-A}{C_1-D_1}}\dfrac{\dfrac{C-B}{C_1-B_1}}{\dfrac{C_1-B_1}{C-D}}\dfrac{\dfrac{A_1-B_1}{D-A}}{\dfrac{D-C}{A_1-B_1}}=\left(\dfrac{\dfrac{B_1-A_1}{A_1-D_1}}{\dfrac{C_1-B_1}{C_1-D_1}}\right)^2$.

Another proof: $\angle A_1D_1C_1+\angle A_1B_1C_1=180^\circ-\dfrac{\angle A}{2}-\dfrac{\angle B}{2}+180^\circ-\dfrac{\angle C}{2}-\dfrac{\angle D}{2}=180^\circ$.



$\angle A_1+\angle C_1+\angle A_1AD+\angle A_1DA+\angle C_1CB+\angle C_1BC$

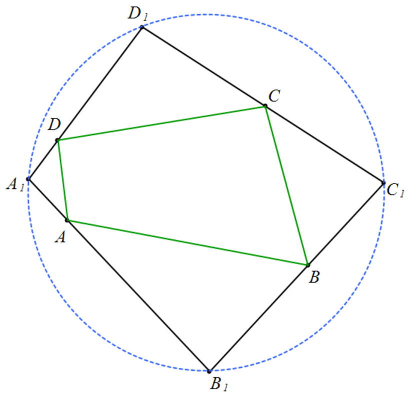
$=\angle B_1+\angle D_1+\angle B_1AB+\angle B_1BA+\angle D_1DC+\angle D_1CD=360^\circ$, easy to get

$\angle A_1+\angle C_1=\angle B_1+\angle D_1=180^\circ$.

Example 82 : As shown in Figure 3, make a quadrilateral circumscribed by quadrilateral $A_1B_1C_1D_1ABCD$, if $\angle A_1AD=\angle B_1AB$, $\angle B_1BA=\angle C_1BC$, $\angle C_1CB=\angle D_1CD$, $\angle D_1DC=\angle A_1DA$, then the quadrilateral $A_1B_1C_1D_1$ is a quadrilateral inscribed in a circle.

$$\frac{\frac{B_1-A_1}{A-D}}{\frac{A-B}{A_1-B_1}}\frac{\frac{B-C}{B_1-C_1}}{\frac{C_1-B_1}{B-A}}\frac{\frac{D_1-C_1}{C-B}}{\frac{C-D}{C_1-D_1}}\frac{\frac{D-A}{D_1-A_1}}{\frac{A_1-D_1}{D-C}}=\left(\frac{\frac{B_1-A_1}{B_1-C_1}}{\frac{A_1-D_1}{D_1-C_1}}\right)^2.$$

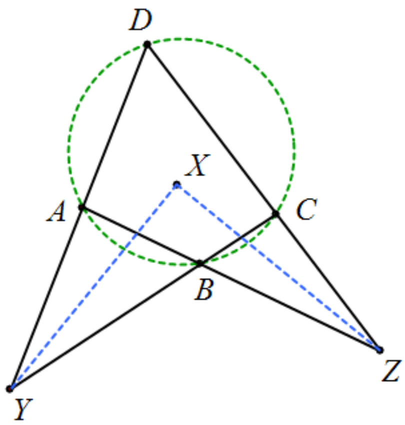
Up and down is an equation. different interpretation



change picture

Example 83 : As shown in Figure 3, construct the circumscribed quadrilateral $A_1B_1C_1D_1$ of the quadrilateral $ABCD$, if $A_1A=A_1D$, $B_1B=B_1A$, $C_1C=C_1B$, $D_1D=D_1C$, then the quadrilateral $ABCD$ is a circular inscribed quadrilateral.

$$\frac{\frac{A-B}{A_1-B_1}}{\frac{C_1-B_1}{B-A}}\frac{\frac{D_1-C_1}{C-B}}{\frac{B-C}{B_1-C_1}}\frac{\frac{C-D}{C_1-D_1}}{\frac{A_1-D_1}{D-C}}\frac{\frac{B_1-A_1}{A-D}}{\frac{D-A}{D_1-A_1}}=\left(\frac{D-C}{D-A}\frac{B-A}{B-C}\right)^2$$



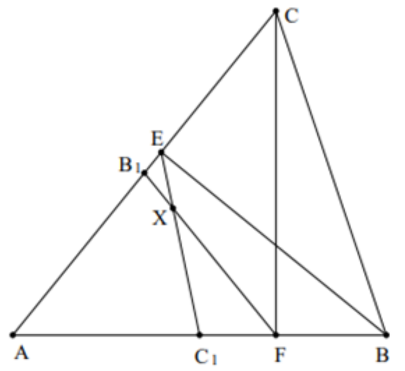
replace and equal try

Example 84 : As shown in Figure 3, the quadrilateral $ABCD$ is inscribed in a circle , straight line DA intersects CB at Y, AB intersects DC at Z, and the angle bisectors of $\angle AYB$ and $\angle$BZC *intersect at point X.* Prove: $XY \perp XZ$.

$$\left(\frac{X-Y}{X-Z}\right)^2=\frac{\frac{X-Y}{C-B}}{\frac{D-A}{X-Y}}\frac{\frac{A-B}{X-Z}}{\frac{X-Z}{D-C}}\left(\frac{A-D}{A-B}\frac{C-B}{C-D}\right).$$

Note that the right side of the equation is a negative real number, so XY and XZ can only be vertical, but not others.

Extension: As shown in the figure, the quadrilateral $ABCD$, straight line DA intersects CB at Y, AB intersects DC at Z, and the angle bisectors of $\angle AYB$ and $\angle$BZC *intersect at point X.* Prove: The necessary and sufficient conditions for $XY \perp XZ$ *are* A, B, The four points C and D *share a circle.*



Example 85 : As shown in Figure 3, in $\triangle ABC$, BE and CF are heights, B_1 and are C_1 the midpoints of FB_1AC and AB respectively , which intersect EC_1 at X.

Prove: $\angle B_1XC_1 = 3\angle A$.

Analysis: 1) $\angle FAC = \angle CFB_1 \Leftrightarrow \frac{A-C}{F-C} / \frac{F-C}{F-B_1} \in R$; 2) $\angle ABE = \angle BEC_1 \Leftrightarrow \frac{E-B}{E-C_1} / \frac{B-A}{B-E} \in R$; 3) $CF \perp AB \Leftrightarrow \left(\frac{C-F}{A-B}\right)^2 \in R$; 4) $EB \perp AC \Leftrightarrow \left(\frac{A-C}{E-B}\right)^2 \in R$;

5) $\angle B_1XC_1 = 3\angle A \Leftrightarrow \left(\frac{A-C}{A-B}\right)^3 / \frac{C_1-E}{B_1-F} \in R$.

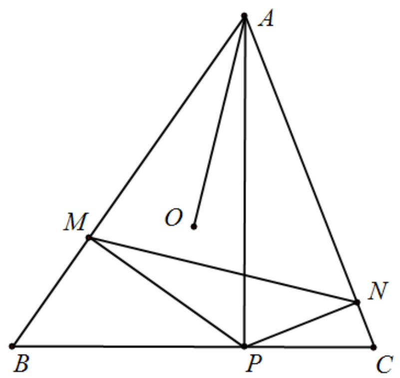
Proof: $\left(\frac{A-C}{A-B}\right)^3 / \frac{C_1-E}{B_1-F} = \left(\frac{C-F}{A-B}\right)^2 \left(\frac{A-C}{E-B}\right)^2 \left(\frac{A-C}{F-C} / \frac{F-C}{F-B_1}\right) \left(\frac{E-B}{E-C_1} / \frac{B-A}{B-E}\right)$.

Since the parallel (including collinear) and perpendicular conclusions involve only two minor terms, it is relatively easy to express them with conditions. However, the angle relationship (including the proof that the triangle is an isosceles triangle) and the four points cocircle involve four small terms, so it is relatively difficult to express it with conditions. Of course this is not absolute.

parallel and perpendicular

Example 86 : As shown in Figure 1, $\triangle$in ABC, O is the circumcenter, BD and CE are high, prove that $AO \perp DE$.

$$\frac{A-O}{D-E}=\frac{\dfrac{A-O}{A-B}}{\dfrac{B-D}{B-C}}\frac{\dfrac{B-D}{B-C}}{\dfrac{E-D}{E-C}}\frac{A-B}{C-E},$$

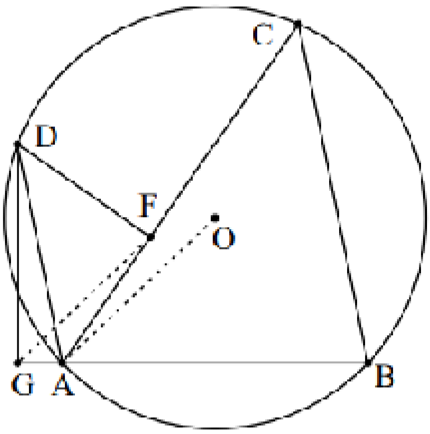


Example 87 : As shown in Figure 3, $\triangle$in ABC, O is the circumcenter, AP is the height, $PM \perp AB$ intersects AB at M, $PN \perp AC$ intersects AC at N, to prove: $AO \perp MN$.

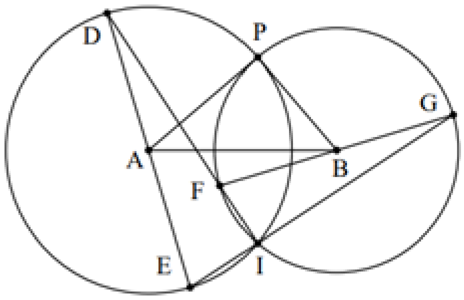
Proof: $$\frac{M-N}{A-O}=\frac{\frac{A-C}{A-P}}{\frac{A-O}{A-M}}\frac{\frac{C-B}{C-A}}{\frac{M-A}{M-N}}\frac{P-A}{B-C}.$$

Explanation: According to $AN\cdot AC=AP^2=AM\cdot AB$ the four points M, B, C and N *are in a circle,* $\angle BCA=\angle AMN$. Or $\angle BCA=\angle NPA=\angle AMN$.

Example 88 : As shown in Figure 3, in $\triangle ABC$, O is the circumcenter, and the parallel line passing through A to BC *intersects the circumscribed circle* of $\triangle ABC$ at point D, and the feet of D on AB and AC *are* G and F respectively . Prove: $GF \parallel AO$.

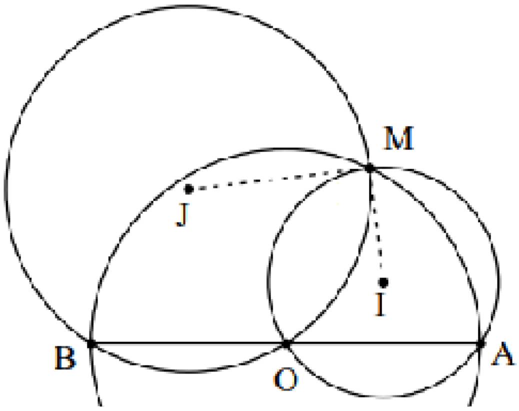


$$\frac{G-F}{O-A}=\frac{B-C}{A-D}\frac{\frac{G-F}{A-B}}{\frac{D-F}{D-A}}\left(\frac{D-F}{A-C}\frac{A-B}{A-O}\frac{C-A}{C-B}\right)$$



Example 89 : As shown in Figure 3, circle A intersects circle B at two points P and I, and $PA \perp PB$, D *is* a point on circle A, D and E *are* symmetrical about A, DI intersects circle B at F, and EI intersects circle B at G, if G is on FB, then $FG \perp DE$.

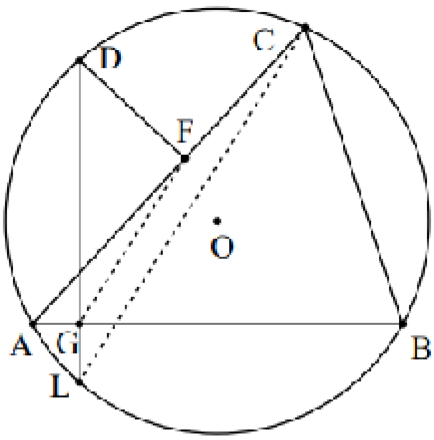
$$\frac{F-G}{E-D}=\frac{G-I}{I-D}\frac{F-G}{B-G}\left(\frac{I-B}{G-I}/\frac{G-I}{G-B}\right)\frac{I-G}{E-I}\frac{A-D}{E-D}\left(\frac{I-A}{I-B}\frac{I-E}{I-D}\right)\left(\frac{D-I}{D-A}/\frac{I-A}{D-I}\right).$$



Example 90 : As shown in Figure 3, O is the midpoint of AB, M is a point on the circle with AB as the diameter, I and J *are* the circumcenters of $\triangle AOM$ and $\triangle BOM$ *respectively, and the proof is:* $MI \perp MJ$.

$$\frac{M-I}{M-J}=\frac{M-A}{M-B}\frac{O-B}{O-A}\left(\frac{B-M}{B-O}\frac{M-O}{M-J}\frac{M-I}{M-O}\frac{A-O}{A-M}\right)$$

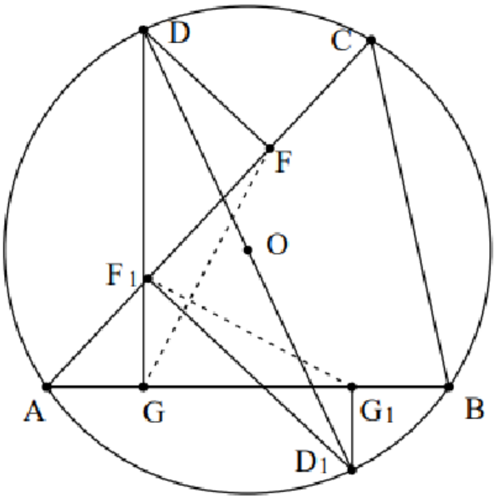
Explanation: It can be generalized by the identity. As shown in the figure, A, B, O are collinear, M is not on AB, I, J *are the circumcentres of* $\triangle AOM$ and $\triangle BOM$ respectively. Prove: $\angle IMJ = \angle AMB$.



Example 91 : As shown in Figure 3, in the quadrilateral $ABCD$ inscribed in the circle O, the feet of D on AC and AB *are* F and G respectively , and DG intersects the circle at L. Prove: FG // CL .

$$\frac{G-F}{L-C}=\left(\frac{G-A}{D-L}\frac{D-F}{A-C}\right)\left(\frac{A-C}{A-D}\frac{L-D}{L-C}\right)\left(\frac{G-F}{G-A}\frac{D-A}{D-F}\right),$$

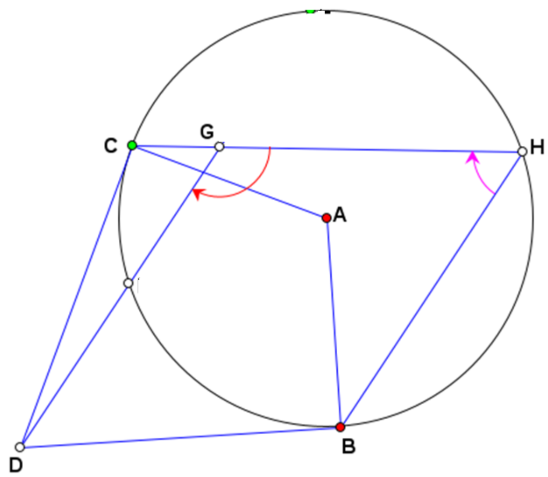
Example 92 : As shown in Figure 3, in $\triangle ABC$, DD_1 is the diameter, the feet of D on AC and AB are F, G, the feet of D_1 on AC and AB are F_1, G_1, to prove: $FG \perp F_1G_1$.



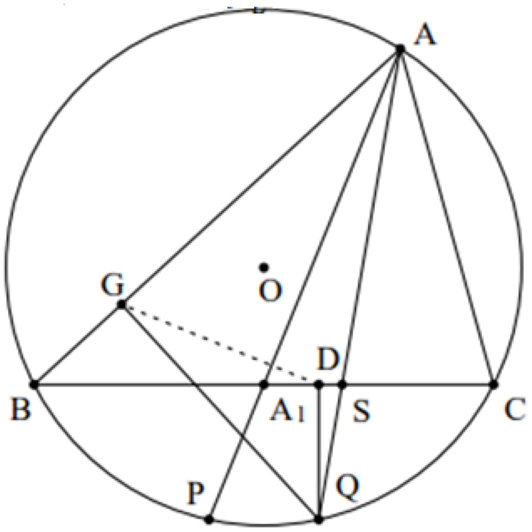
$$\frac{F-G}{F_1-G_1}=\frac{D_1-A}{D-A}\frac{F-D}{D_1-F_1}\frac{A-G}{A-G_1}\left(\frac{F_1-A}{F_1-G_1}\frac{D_1-G_1}{D_1-A}\right)\left(\frac{G-F}{G-A}\frac{D-A}{D-F}\right)\left(\frac{D_1-F_1}{A-F_1}\frac{A-G_1}{D_1-G_1}\right)$$

Generalization: In $\triangle ABC$, D and D_1 are arbitrary points , and the feet of D on AC and AB are F and G, and the feet of D_1 on AC and AB are F_1 and G_1. To prove: FG and F The included angle $_1G_1$ is equal to $\angle DAD_1$.

Example 93 : As shown in Figure 3, in △ BHC , A is the circumcenter, respectively pass through B and C *to draw* the tangent of the circumcircle of △ BHC *, the two circumscribed lines intersect at point* D *, and* the circumcircle of △ ADB *intersects* CH at G. Prove: $DG \parallel BH$.



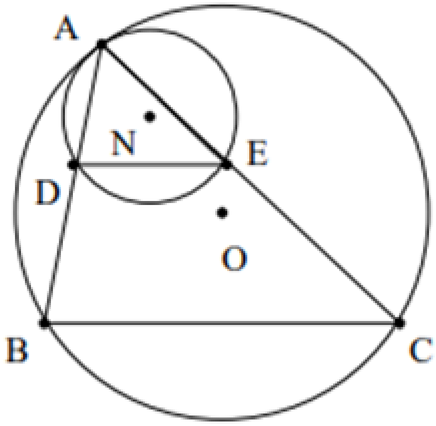
$$\frac{G-D}{H-B}=\frac{C-G}{H-C}\left(\frac{H-C}{H-B}/\frac{A-C}{A-D}\right)\left(\frac{G-A}{G-C}\frac{D-C}{D-A}\right)\left(\frac{G-D}{G-A}/\frac{C-D}{C-A}\right),$$



Example 94 : As shown in Figure 3, in $\triangle ABC$, O is the circumcenter, A_1 is the midpoint of BC, S is on BC, and $\angle BAA_1 = \angle SAC$, AA_1 intersects the circle O at P, AS intersects the circle O at Q, the feet of Q on BA and BC are G and D respectively . To prove: $AP \perp DG$.

$$\frac{A-P}{D-G}=\frac{A-P}{A-A_1}\frac{A-B}{B-G}\frac{A-Q}{A-S}\left(\frac{A-C}{A-Q}/\frac{B-C}{B-Q}\right)\left(\frac{A-S}{A-B}/\frac{A-C}{A-A_1}\right)\left(\frac{B-G}{B-Q}\frac{D-Q}{D-G}\right)\frac{B-C}{D-Q}$$

Example 95 : As shown in Figure 3 , in $\triangle ABC$, D, E are on AB, AC *respectively*, $DE \mathbin{/\mkern-6mu/} BC$, N, O *are* the circumcenters of $\triangle ADE$, $\triangle ABC$ *respectively, to prove:* A, N, O are collinear .

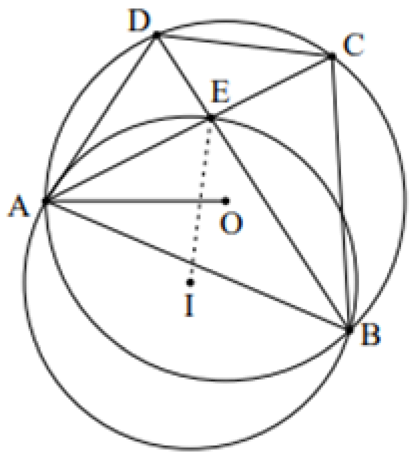


$$\frac{A-N}{A-O}=\frac{B-C}{D-E}\frac{A-E}{A-C}\frac{A-D}{A-B}\left(\frac{A-N}{A-D}\frac{E-D}{E-A}\right)/\left(\frac{A-O}{A-B}\frac{C-B}{C-A}\right),$$

is used: ∠DEA + ∠NAD = 90 ° .

Extension: As shown in the figure, in $\triangle ABC$, D, E are on AB, AC *respectively*, N, O *are* the circumcenters of $\triangle ADE$, $\triangle ABC$ *respectively. Prove:* A, N, O *are* collinear $\Leftrightarrow DE \mathbin{/\mkern-6mu/} BC$.

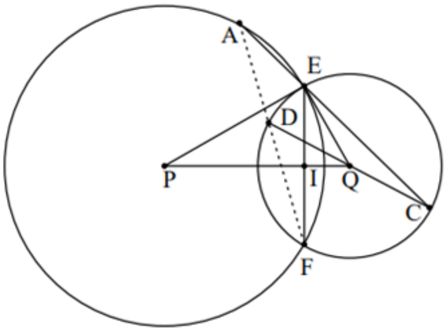
Example 96 : As shown in Figure 3, in the quadrilateral $ABCD$ *inscribed in* the circle O, the diagonals intersect at E, *and* I is the circumcenter of $\triangle ABE$. Prove: $IE \perp DC$.



$$\frac{I-E}{D-C}=\frac{B-E}{B-D}\frac{A-E}{C-A}\left(\frac{B-A}{B-E}\frac{E-I}{A-E}\right)\left(\frac{C-A}{C-D}\frac{B-D}{B-A}\right),$$

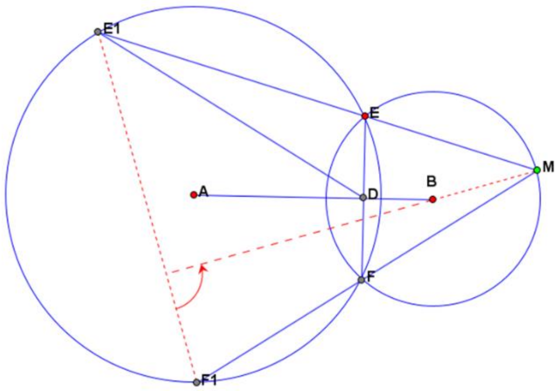
The abbreviation is $\frac{I-E}{D-C}=\left(\frac{B-A}{B-D}\frac{E-I}{C-A}\right)\left(\frac{C-A}{C-D}\frac{B-D}{B-A}\right)$, is the geometric meaning obvious?

Example 97 : As shown in Figure 3, circle P and circle Q intersect at E and F, there is a point P on circle P, AE intersects circle Q at C, and CD is the diameter of circle Q. Prove that A, D and F are collinear.



$$\frac{D-F}{A-F}=\frac{C-E}{E-A}\left(\frac{F-D}{C-F}\frac{A-E}{A-F}\frac{E-F}{E-P}\right)\left(\frac{C-F}{C-E}/\frac{E-F}{E-P}\right),$$

Example 98 : As shown in Figure 3, circle A intersects circle B at two points EF, *passes through point* M on circle B, draws straight line ME intersects circle A at X, draws straight line MF intersects circle A at Y, and proves $BM \perp XY$.



$$\frac{B-M}{X-Y}=\left(\frac{M-B}{M-X}/\frac{M-Y}{B-D}\right)\left(\frac{X-M}{D-F}/\frac{X-Y}{Y-M}\right)\frac{D-F}{D-B},$$

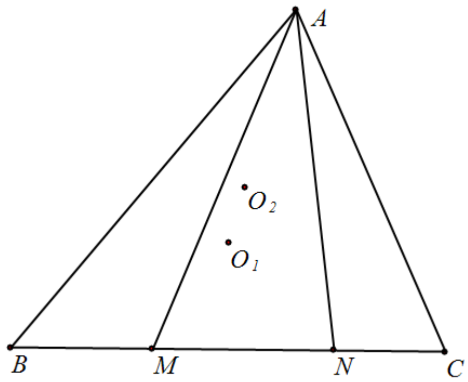
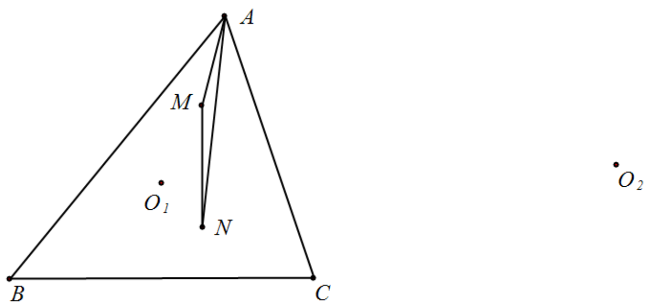


figure 1

Example 99 : As shown in Figure 1 , in the acute angle $\triangle ABC$, $AB > AC$, M, N are two different points on the side of BC , *so that* $\angle$BAM = $\angle$CAN . Let the circumcenters of $\triangle ABC$ and $\triangle$ AMN be O_1 , O_2 respectively , to prove: O_1 , O_2 , A three points are collinear. (Additional test questions for the 2012 National High School Mathematics League)

$$\frac{A-O_1}{A-O_2}=\frac{\frac{A-C}{A-N}}{\frac{A-M}{A-B}}\left(\frac{C-B}{C-A}\frac{A-O_1}{A-B}\right)/\left(\frac{N-M}{N-A}\frac{A-O_2}{A-M}\right)\frac{M-N}{B-C},$$



As shown in Figure 2 , in the acute angle $\triangle ABC$, $AB >$ AC, M , N are two points inside $\triangle$ ABC , so that $\angle$BAM = $\angle$ CAN and $MN \perp BC$. Let the circumcenters of $\triangle ABC$ and $\triangle$ AMN *be* O_1 , O_2 respectively , Prove: $AO_1 \perp AO_2$.

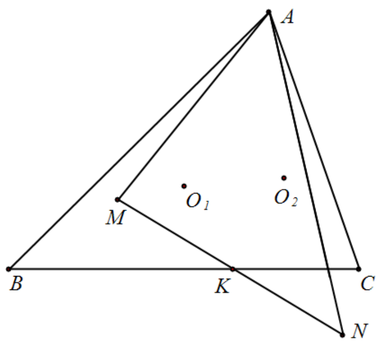
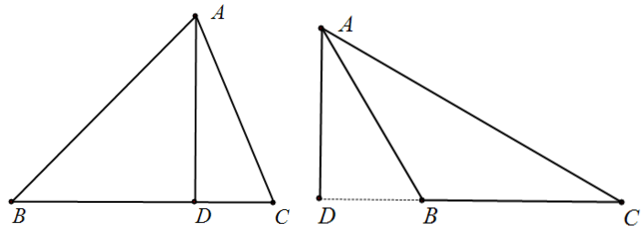


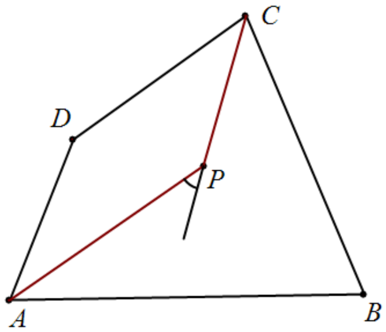
Figure 3

angle relationship

Example 100 : As shown in Figure 4 , in $\triangle ABC$, $AD \perp BC$ is in D, to prove: $BA = BC \Leftrightarrow 2\angle CAD = \angle ABC$.

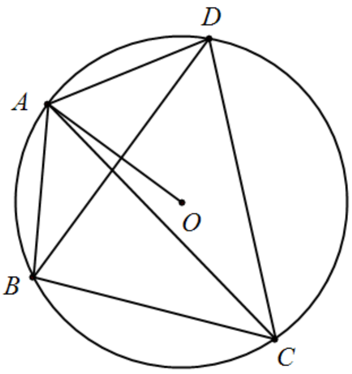


$$\frac{\frac{B-A}{B-C}}{\left(\frac{A-C}{A-D}\right)^2}\frac{\frac{A-C}{A-B}}{\frac{C-B}{C-A}}\left(\frac{B-C}{A-D}\right)^2 = -1,$$



Example 101 : As shown in Figure 4 , quadrilateral $ABCD$, the angle bisectors of $\angle A$ and $\angle C$ intersect at point P. Prove *that the* included angles of straight lines $\frac{\angle D-\angle B}{2}$ CP and PA *are equal*.

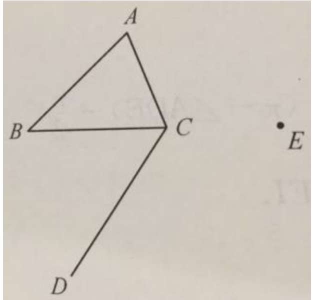
$$\frac{\left(\frac{C-P}{P-A}\right)^2\frac{B-A}{B-C}}{\frac{D-C}{D-A}}\frac{\frac{A-P}{A-B}}{\frac{A-D}{A-P}}\frac{\frac{C-B}{C-P}}{\frac{C-P}{C-D}}=1$$



Example 102 : As shown in Figure 1, the circle O is inscribed in the quadrilateral $ABCD$, $OA \perp BD$, to prove: CA bisects $\angle BCD$.

$$\frac{\frac{C-B}{C-A}}{\frac{C-A}{C-D}}\frac{\frac{O-A}{B-D}}{\frac{A-O}{A-C}\frac{D-C}{D-A}}\frac{\frac{A-C}{A-D}}{\frac{B-C}{B-D}}=1,\quad \left(\frac{B-S}{A-D}\frac{B-D}{B-C}\right)\frac{C-A}{B-S}\frac{\frac{B-C}{B-D}}{\frac{A-C}{A-D}}=-1,$$

Example 103 : As shown in Figure 1, point D is the paracentre of $\triangle ABC$, and the symmetry point of point A with respect to straight line DC is E. Proof: three points B, C, and E are collinear. (2016 Jiangxi Provincial Preliminary Competition)

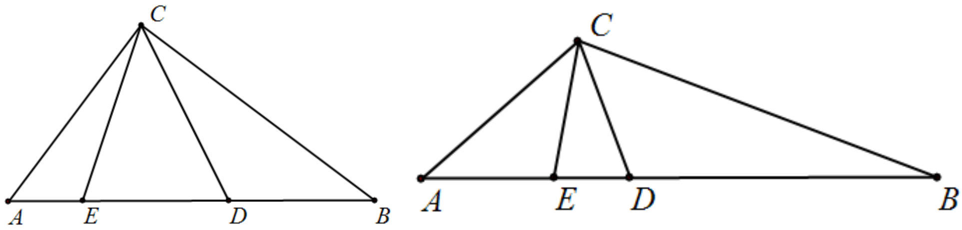


$$\frac{B-C}{C-E}=\frac{\frac{C-B}{C-D}\frac{C-D}{C-A}}{\frac{C-D}{A-C}\frac{C-E}{C-D}},$$

Example 104 : As shown in Figure 1, $\triangle ABC$ and AD are angle bisectors, O, P and Q are the circumcentres of ABC, $\triangle ABD$ and $\triangle ADC$ respectively $\triangle$. Prove: $OP = OQ$.

$$\frac{\frac{P-Q}{P-O}}{\frac{Q-O}{Q-P}}=\left(\frac{A-B}{P-O}\frac{A-C}{O-Q}\right)\left(\frac{P-Q}{A-D}\right)^2\frac{\frac{A-D}{A-B}}{\frac{A-C}{A-D}}$$

Example 105 : As shown in Figure 1, it is known that D and E are two points on the hypotenuse AB *of the right angle* $\triangle ABC$, *and* $AD = AC$, $BE = BC$, find the degree of $\angle ECD$.

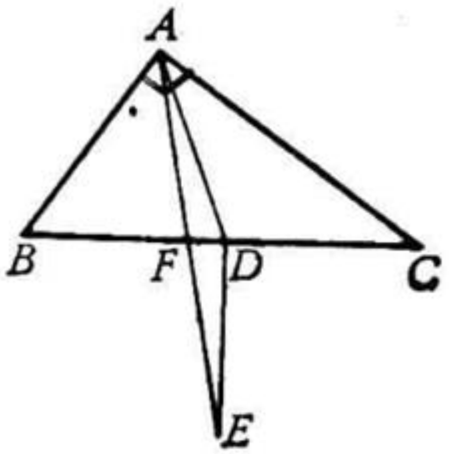


$$\frac{\frac{C-B}{C-A}}{\left(\frac{C-D}{C-E}\right)^2}=-\frac{\frac{E-C}{A-B}}{\frac{C-B}{C-E}}\frac{\frac{B-A}{D-C}}{\frac{C-D}{C-A}}\left(\frac{C-B}{C-A}\right)^2$$, so $\angle ECD = 45°$.

Simplify the above identity $\frac{\frac{A-C}{C-B}}{\left(\frac{C-D}{C-E}\right)^2}=\frac{\frac{E-C}{A-B}}{\frac{C-B}{C-E}}\frac{\frac{B-A}{D-C}}{\frac{C-D}{C-A}}$, and then get the extended proposition:

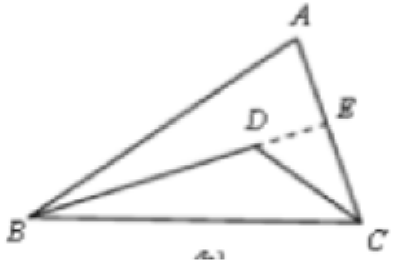
As shown in Figure 1, in $\triangle ABC$, it is known that D and E are two points on AB, *and* $AD = AC$, $BE = BC$, find the complementarity of $\angle ECD$ and $\angle ACB$.

Example 106 : As shown in Figure 1, in the right angle $\triangle ABC$, the bisector of the right angle A *intersects the mid-perpendicular line* DE of BC at point E, then $\angle DAE$ = $\angle DEA$.



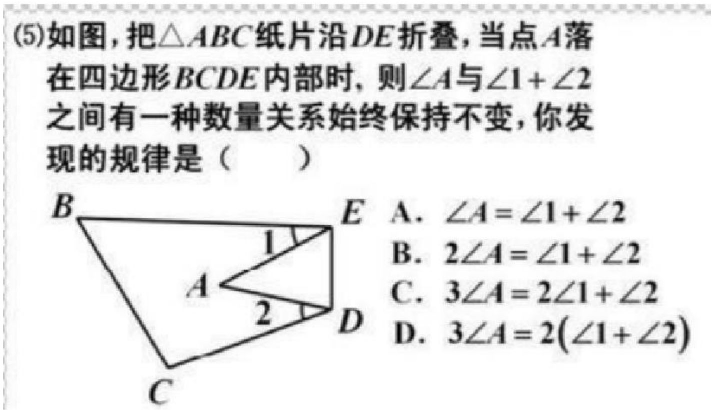
repeat

$$\frac{\frac{A-D}{A-E}}{\frac{E-A}{E-D}}=\frac{\frac{A-C}{A-E}}{\frac{A-E}{A-B}}\frac{\frac{C-B}{C-A}}{\frac{A-C}{A-D}}\left(\frac{A-C}{A-B}\frac{D-E}{B-C}\right),$$



Example 107 : As *shown* in Figure 1 , $\angle BDC = \angle ABD + \angle A + \angle ACD$.

$$\frac{D-C}{D-B}=\frac{B-A}{B-D}\frac{C-D}{C-A}\frac{A-C}{A-B}$$



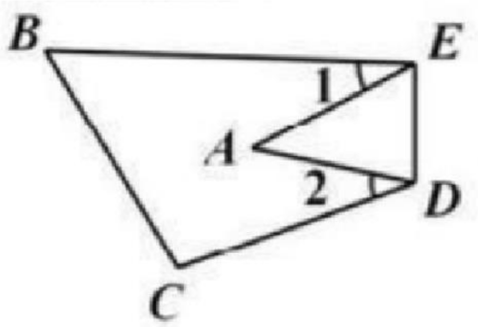
(5)如图，把△ABC纸片沿DE折叠，当点A落在四边形$BCDE$内部时，则$\angle A$与$\angle 1+\angle 2$之间有一种数量关系始终保持不变，你发现的规律是（　　）

A. $\angle A=\angle 1+\angle 2$

B. $2\angle A=\angle 1+\angle 2$

C. $3\angle A=2\angle 1+\angle 2$

D. $3\angle A=2(\angle 1+\angle 2)$



Example 108 : As shown in Figure 1, *fold* the △ ABC sheet along DE, when point A falls inside the quadrilateral $BCDE$, explore the quantitative relationship between ∠1+ ∠2 and ∠A .

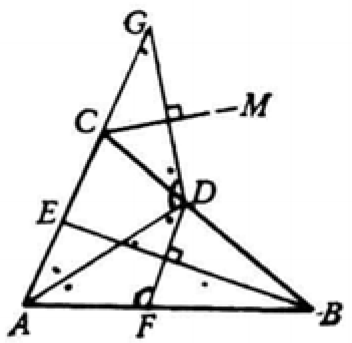
Traditional proof: in △ ADE, ∠A =180° −∠ADE -∠AED, from the properties of folding: ∠1+2∠ADE = *180°* , ∠2+2∠AED = *180°* ,

Then ∠1+∠2=360° −2∠ADE - *2∠AED* = 2(180° −∠ADE - ∠AED *)* = *2∠A* .

$$\frac{E-A}{E-B}\frac{D-C}{D-A}=\frac{D-C}{E-B}\frac{A-E}{A-D}$$

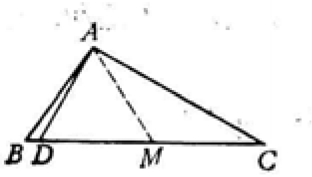
tautological value

Example 109 : As shown in Figure 1, it is known that AD and BE *are* the bisectors of $\angle$A and $\angle$B of $\triangle ABC$, $DF \perp BE$ intersects AB at point F , DG is perpendicular to the *bisector* CM of the exterior angle of $\angle C$, and intersects the extension of AC line at point G. Prove : $\angle$AFD = $\angle$ADG . _



$$\left(\frac{\frac{F-A}{F-D}}{\frac{D-A}{D-G}}\right)^2 = \frac{\frac{A-C}{A-D}}{\frac{A-D}{A-B}} \frac{\frac{C-B}{D-F}}{\frac{F-D}{A-B}} \frac{\frac{G-D}{C-A}}{\frac{B-C}{D-G}} \left(\frac{A-F}{A-B}\right)^2$$

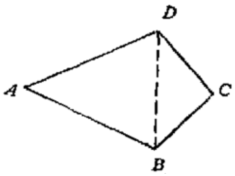
Example 110 : As shown in Figure 1, in △ ABC, ∠B = 2 ∠C , $AD \perp AC$ intersects BC at *point D*. Prove: CD =2 AB .



Explanation: Take the midpoint M *of* CD *, and* change the certificate to $AB = AM$.

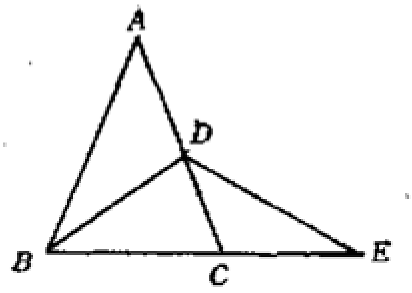
$$\frac{\frac{B-A}{B-C}}{\frac{C-B}{M-A}}\frac{\left(\frac{C-B}{C-A}\right)^2}{\frac{B-A}{B-C}}\frac{\frac{A-C}{A-M}}{\frac{C-B}{C-A}}=1,$$

Example 111 : As shown in Figure 1, $AB = \mathrm{AD}$, $\angle \mathrm{B} = \angle D$. Prove: $CB = CD$.



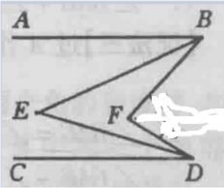
$$\frac{\frac{B-A}{B-D}\frac{D-C}{D-A}\frac{B-D}{B-C}}{\frac{D-B}{D-A}\frac{B-A}{B-C}\frac{D-C}{D-B}}=1,$$

Example 112 : As shown in Figure 1, $\triangle in$ ABC, AB = A C, BD is the bisector of $\angle$ ABC , E is a point on the extension line of side BC, and $BD = DE$. Prove : $CD = CE$.

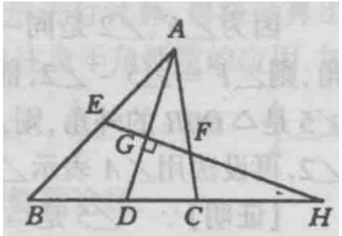


$$\frac{\frac{D-E}{A-C}}{\frac{C-B}{E-D}}=\frac{\frac{C-B}{C-A}}{\frac{B-A}{B-C}}\frac{\frac{B-A}{B-D}}{\frac{B-D}{B-C}}\left(\frac{\frac{B-D}{B-C}}{\frac{C-B}{E-D}}\right)^2$$

Example 113 : As shown in Figure 1, $AB // CD$, BE and DE are the bisectors of $\angle ABF$ and $\angle CDF$ respectively. Prove: $2\angle BED = \angle BFD$.



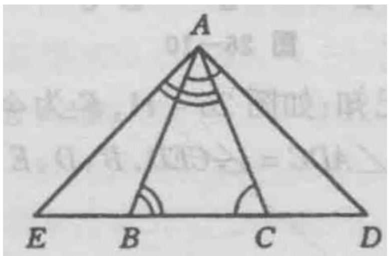
$$\frac{\left(\frac{E-B}{E-D}\right)^2}{\frac{F-B}{F-D}}\frac{\frac{B-F}{B-E}}{\frac{B-E}{B-A}}\frac{\frac{D-E}{D-F}}{\frac{D-C}{D-E}}\frac{C-D}{A-B}=1,$$



Example 114 : As shown in Figure 1, $\triangle$in ABC, AD bisects $\angle BAC$, $EF \perp AD$ meets G, intersects AB at E, intersects AC at F, intersects the extension line of BC at H. Proof: $2\angle H = \angle ACB - \angle B$.

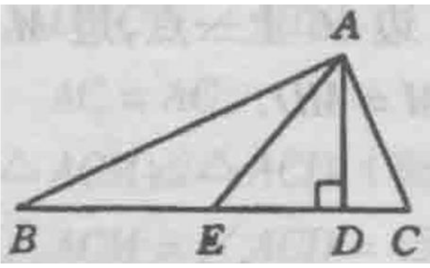
$$\frac{\left(\frac{C-B}{F-E}\right)^2 \frac{B-A}{B-C}}{\frac{C-B}{C-A}} = \frac{\frac{B-A}{E-F}}{\frac{F-E}{C-A}},$$

$$2\angle H = 2\left(90^\circ - \angle ADH\right) = 2\left(90^\circ - \angle B - \frac{1}{2}\angle BAC\right) = 180^\circ - 2\angle B - \angle BAC = \angle ACB - \angle B$$



Example 115 : As shown in Figure 1, in the acute angle $\triangle ABC$, take a point D *on the straight line where BC is located,* so *that* $\angle BAD = \angle ACB$. Then take *another* point *E,* so *that* $\angle CAE = \angle ABC$. Prove: $AD = AE$.

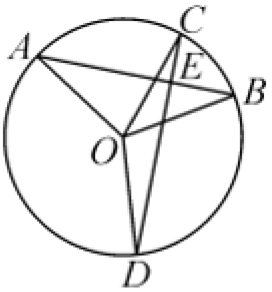
$$\frac{\frac{E-A}{B-C}}{\frac{C-B}{D-A}}\frac{\frac{C-B}{C-A}}{\frac{A-D}{A-B}}\frac{\frac{A-C}{A-E}}{\frac{B-A}{B-C}}=1,$$



Example 116 : As shown in Figure 1, △in ABC , $\angle BAC = 90°$, E is a point on BC, and $BE = AE$. $AD \perp BC$, ∠BAD =3 ∠ CAD. Prove : $AD = ED$.

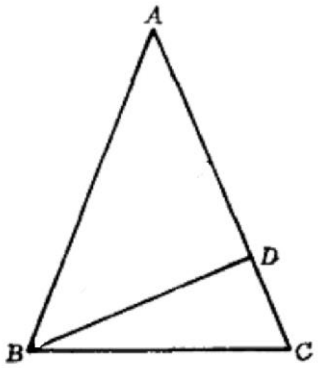
$$\frac{\dfrac{E-A}{B-C}}{\dfrac{A-D}{A-E}} = \left(\frac{\dfrac{A-E}{A-B}}{\dfrac{B-A}{B-C}}\right)^2 \frac{\left(\dfrac{A-C}{A-D}\right)^3}{\dfrac{A-D}{A-B}} \left(\frac{\dfrac{A-D}{C-B}}{\dfrac{A-C}{A-B}}\right)^3 .$$

Example 117 : As shown in Figure 1, in the circle O , the chord $AB \perp CD$, the vertical foot is E, find the degree of $\angle AOD + \angle BOC$.



$$\left(\frac{O-D}{O-A}\frac{O-C}{O-B}\right)\frac{\frac{B-O}{B-A}}{\frac{A-B}{A-O}}\frac{\frac{C-D}{C-O}}{\frac{D-O}{D-C}}\left(\frac{A-B}{C-D}\right)^2=1,$$

Example 118 : As shown in Figure 3, $\triangle$in ABC, $AB = AC, BD \perp AC$. Prove : $2\angle DBC = \angle A$.



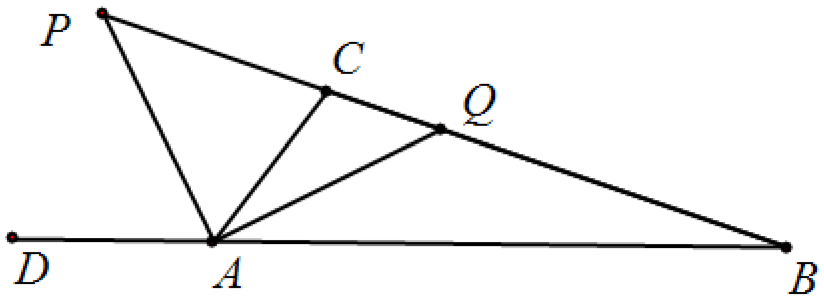
$$\frac{\left(\frac{B-D}{B-C}\right)^2}{\frac{A-C}{A-B}} = -\frac{\frac{B-A}{B-C}}{\frac{C-B}{C-A}}\left(\frac{B-D}{A-C}\right)^2,$$

Explanation: The general method to prove the original question is to make high AE. The identity method can prove three propositions at once.

Example 119 : As shown in Figure 3, $\triangle in$ ABC , $AB =$ AC , if the bisector of $\angle$B intersects AC at P. Prove that $\angle APB = 3 \angle PBC$.

$$\frac{\left(\frac{B-P}{B-C}\right)^3}{\frac{P-B}{P-A}}=\frac{\frac{B-P}{B-C}}{\frac{B-A}{B-P}}\frac{\frac{B-A}{B-C}}{\frac{C-B}{C-A}}\frac{A-P}{A-C}$$

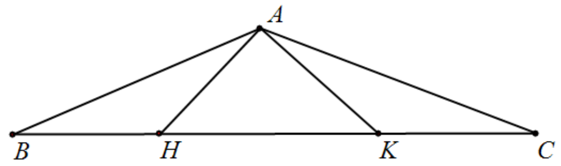
Example 120 : *As shown in Figure 3*, △side *BC* of *ABC* extends to *D, and the* bisector of ∠BAC intersects BC *at* K. *Prove* that ∠*ABD* + ∠ACD =2 ∠AKD .



$$\frac{\frac{B-A}{B-D}\frac{C-A}{C-D}}{\left(\frac{K-A}{K-D}\right)^2}=\frac{\frac{A-C}{A-K}}{\frac{A-K}{A-B}}\frac{(D-K)^2}{(B-D)(C-D)},$$

Example 121 : As shown in Figure 3, $\triangle in\, ABC$, extend BA to D, *and* the bisectors of ∠CAB and ∠CA D *intersect* the straight *line* BC at P and Q. If $AP = AQ$, prove that $\angle ACB$ - ∠ABC =90 °.

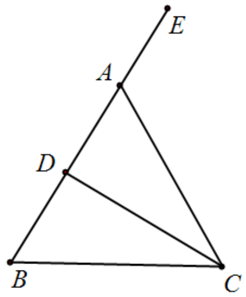
$$\left(\frac{\frac{C-B}{C-A}}{\frac{B-A}{B-C}}\right)^2=-\frac{\frac{A-P}{A-B}}{\frac{A-C}{A-P}}\frac{\frac{Q-A}{A-C}}{\frac{B-A}{Q-A}}\left(\frac{\frac{C-B}{P-A}}{\frac{Q-A}{B-C}}\right)^2,$$



Example 122 : As shown in Figure 3, $\triangle$in ABC, $\angle$BAC is a pure angle , and the perpendicular bisectors of AB and AC intersect BC *at* H and K. Prove that $\angle HAK$ = 2 $\angle$BAC -180° .

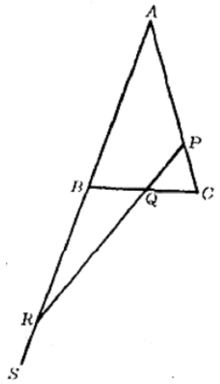
$$\frac{\left(\frac{A-C}{A-B}\right)^2}{\frac{A-K}{A-H}}=-\frac{\frac{A-C}{A-K}\frac{B-C}{B-A}}{\frac{C-B}{C-A}\frac{A-B}{A-H}}$$

Explanation: Note that the right side of the equation is a negative real number.



Example 123 : As shown in Figure 3, $\triangle in\, ABC$, $AB=$ AC , extend BA to E , if the bisector of $\angle$ACB intersects AB at D , prove that $\angle CDE=\frac{3}{4}\angle$CAE .

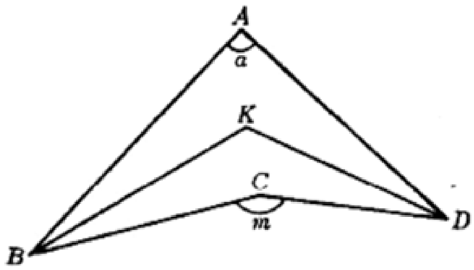
$$\frac{\left(\frac{B-A}{D-C}\right)^4}{\left(\frac{B-A}{A-C}\right)^3}=\left(\frac{\frac{C-B}{C-D}}{\frac{C-D}{C-A}}\right)^2\frac{\frac{B-A}{B-C}}{\frac{C-B}{C-A}},$$



Example 124 : As shown in Figure 3, $AB = AC$, $CP = CQ$, prove that $\angle SRP = 3\angle RPC$.

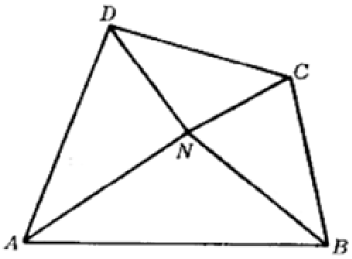
$$\frac{\frac{Q-P}{A-B}}{\left(\frac{A-C}{P-Q}\right)^3}=\frac{\frac{C-B}{C-A}}{\frac{B-A}{B-C}}\left(\frac{\frac{Q-P}{B-C}}{\frac{A-C}{P-Q}}\right)^2,$$

Example 125 : As *shown in Figure* 3 , the bisectors of $\angle ABC$ and $\angle ADC$ intersect at K. Prove : $\angle BAD + \angle BCD = 2\angle BKD$.

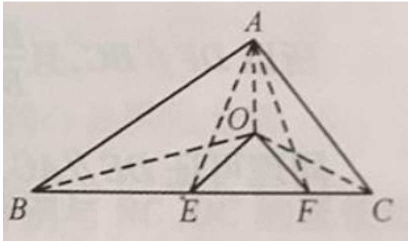


$$\frac{\left(\frac{K-D}{K-B}\right)^2}{\frac{A-D}{A-B}\,\frac{C-D}{C-B}}=\frac{\frac{B-A}{B-K}\,\frac{D-K}{D-A}}{\frac{B-K}{B-C}\,\frac{D-C}{D-K}}$$

Example 126 : As shown in Figure 3, NA and NB are the bisectors of $\angle DAB$ and $\angle CBA$. Prove that $\angle ADC + \angle BCD = 2\angle ANB$.



$$\frac{\left(\frac{N-B}{N-A}\right)^2}{\frac{D-C}{D-A}\frac{C-B}{C-D}}=\frac{\frac{A-D}{A-N}\frac{B-N}{B-C}}{\frac{A-N}{A-B}\frac{B-A}{B-N}}$$

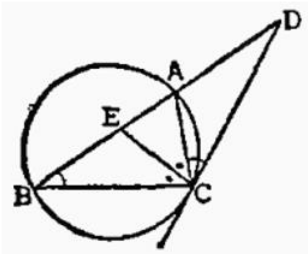


Example 1 27 : As shown in Figure 3, in $\triangle ABC$, 0 is the center , and points E and F are on the large side BC. It is known that $BF = BA, CE = CA.$ Prove: $\angle EOF = \angle B + \angle C$.

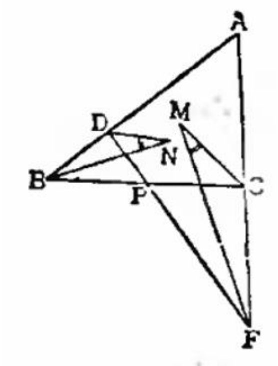
$$\frac{\frac{O-F}{O-E}}{\frac{B-A}{B-C}\frac{C-B}{C-A}}=\frac{\frac{O-F}{O-E}}{\left(\frac{A-F}{A-E}\right)^2}\frac{\frac{A-F}{A-B}}{\frac{F-B}{F-A}}\frac{\frac{A-C}{A-E}}{\frac{E-A}{E-C}}\frac{B-F}{E-C}.$$

Explanation: Note that O is the circumcenter of $\triangle AEF$, $2\angle EAF = \angle EOF$.

Example 128 : As shown in Figure 3, when passing through point C on the circumcircle of $\triangle ABC$, draw a tangent line and intersect the extension line of BA at point D, and the circle with D as the center and DC as the radius intersects AB at point E, then CE bisects $\angle ACB$.

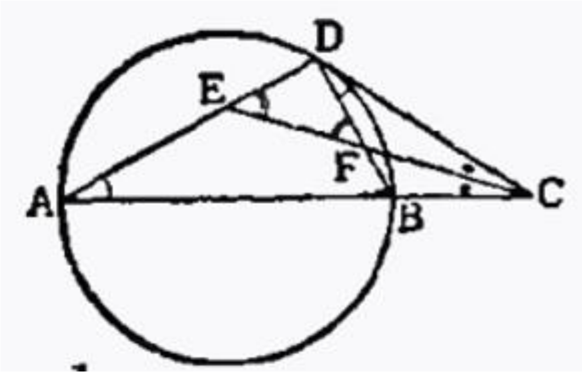


$$\frac{\frac{C-B}{C-E}}{\frac{C-E}{C-A}}=\frac{\frac{C-A}{C-D}}{\frac{B-D}{B-C}}\frac{\frac{E-D}{E-C}}{\frac{C-E}{C-D}}\frac{B-D}{E-D},$$



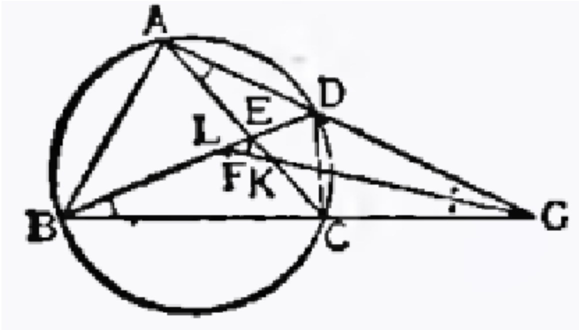
Example 1 29 : As shown in Figure 3, take any point D on the side AB of $\triangle ABC$, take any point F *on the extension line of* AC, and connect it to DF. If *the bisector of* $\angle ADF$ and $\angle ABC$ intersect at N. The bisector of $\angle AFD$ and $\angle ACB$ intersect *at* M, then $\angle BND = \angle CMF$.

$$\left(\frac{\frac{N-B}{N-D}}{\frac{M-C}{M-F}}\right)^2 = \frac{\frac{D-A}{D-N}}{\frac{D-N}{D-F}} \frac{\frac{B-N}{B-C}}{\frac{B-A}{B-N}} \frac{\frac{C-B}{C-M}}{\frac{C-M}{C-A}} \frac{\frac{F-M}{F-A}}{\frac{F-D}{F-M}} \frac{A-B}{A-D} \frac{A-F}{A-C},$$



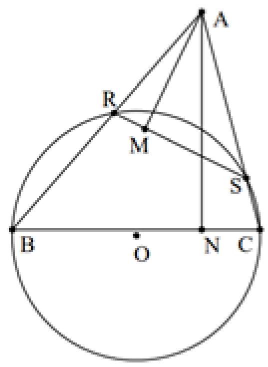
Example 130 : As shown in Figure 3, draw the tangent CD *of the circle* from a point C on the extension line of the diameter AB, *and* the tangent point is D. If the intersection points of the bisectors of AD, BD and ∠ACD are E and F, then DE = DF.

$$\frac{\frac{A-D}{E-C}}{\frac{C-E}{B-D}}=\frac{\frac{C-A}{C-E}}{\frac{C-E}{C-D}}\frac{\frac{A-D}{A-C}}{\frac{D-C}{D-B}},$$



Example 131 : As shown in Figure 3, let the extended lines of sides AD and BC of the inscribed quadrilateral $ABCD$ intersect at point G, AC and BD intersect at point E, and the bisector of AGB passing through E intersect BD and AC at F and K, Prove: $EF = EK$.

$$\frac{\frac{B-D}{K-G}}{\frac{G-K}{C-A}}=\frac{\frac{C-B}{G-K}}{\frac{G-K}{D-A}}\frac{\frac{D-B}{D-A}}{\frac{C-B}{C-A}}.$$



Example 132 : As shown in Figure 3, the quadrilateral $BCSR$ *is inscribed* in a circle , and BR and CS are extended to intersect at point A. The feet of A on BC and RS *are* N and M. Prove: $\angle BAM = \angle CAN$.

$$\frac{A-M}{R-B} / \frac{S-C}{A-N} = \left(\frac{A-N}{B-C}\frac{A-M}{R-S}\right)\left(\frac{R-S}{R-B}\frac{C-B}{C-S}\right).$$

Example 133 : As shown in Figure 8 , two circles intersect at points A and B, draw a straight line passing through A and B respectively and intersect the two circles at C, D, F, F *respectively*. Prove: $CE \parallel DF$.

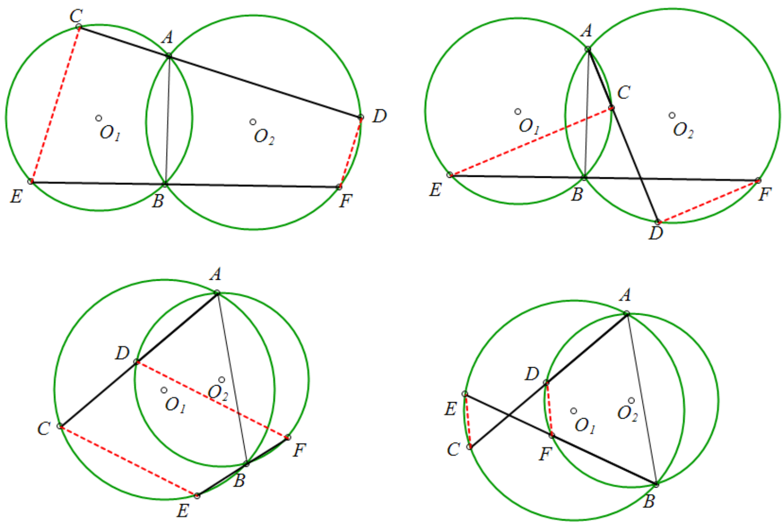


Figure 8 Multiple situations where two intersecting circles produce parallel lines

Proof: $\dfrac{C-E}{D-F}=\dfrac{C-A}{A-D}\dfrac{E-B}{B-F}\dfrac{(B-A)(C-E)}{(B-E)(C-A)}\dfrac{(A-D)(F-B)}{(A-B)(F-D)}$.

Explanation: $\dfrac{C-E}{D-F}\in R \Leftrightarrow CE \parallel DF$, $\dfrac{C-A}{A-D}\in R \Leftrightarrow C$, A, D are collinear,

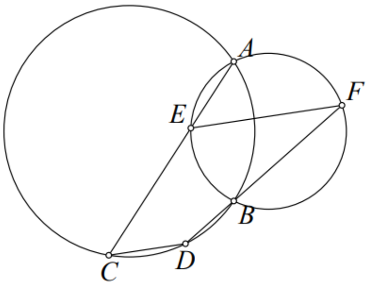
$\dfrac{E-B}{B-F}\in R \Leftrightarrow E$, B, F are three points collinear, $\dfrac{(B-A)(C-E)}{(B-E)(C-A)}\in R \Leftrightarrow A$, C, E, B are four points collinear,

$\dfrac{(A-D)(F-B)}{(A-B)(F-D)}\in R \Leftrightarrow A$, B, F, D share a circle,

The establishment of the above identity is clear at a glance. Based on the identity, it can be known that any four of the five conditions are true, and the remaining one can be deduced to be true. This means that the identity method not only proves the original proposition, but also obtains a new proposition. Paper [24] also established the identity proof on this problem, but it needs to use more complex mathematical knowledge such as conformal geometric algebra, and the amount of calculation is large.

The situation of this question is diverse (Figure 8), and the default solution of traditional geometry is only for the first situation , so the proof is considered simple: $\angle ECA=\angle ABF=180^\circ-\angle ADF$, so $CE//DF$. In fact, $\angle ECA=\angle ABF$ it does not hold in the second case , at this time

$\angle ECA = 180^\circ - \angle ABF$.



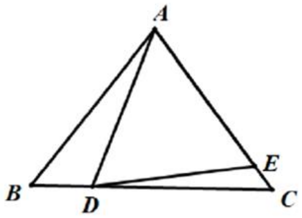
Example 1 33 : As shown in the figure, four points A , B , C , and D *share a circle,* A , B , E , *and* F share a circle, A , E , and C *three points* collide, and B , D , F three points collide , to prove: CD // EF .

$$\frac{\dfrac{E-F}{C-D}}{\dfrac{\frac{F-D}{F-E}}{\frac{A-B}{A-C}}\dfrac{\frac{D-C}{D-F}}{\frac{C-A}{A-B}}}=1$$

,

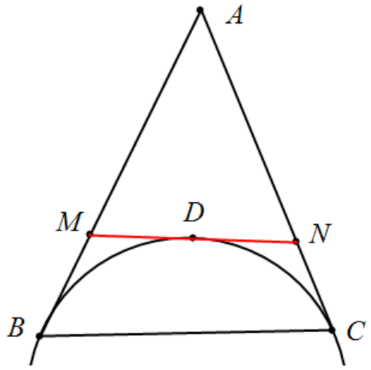
$$\begin{aligned}
\angle(EF,DC) &= \angle(EF,DBF)+\angle(DBF,DC)\\
&= \angle(EF,FB)+\angle(BD,DC)\\
&= \angle(EA,AB)+\angle(BA,AC)\\
&= \angle(EA,AC)=0,
\end{aligned}$$

Example 1 34 : As shown in the figure, in $\triangle ABC$, $AB = AC$, point D is on BC, point E is on AC, and $AD = AE$. Prove: $2\angle EDC = \angle DAB$.

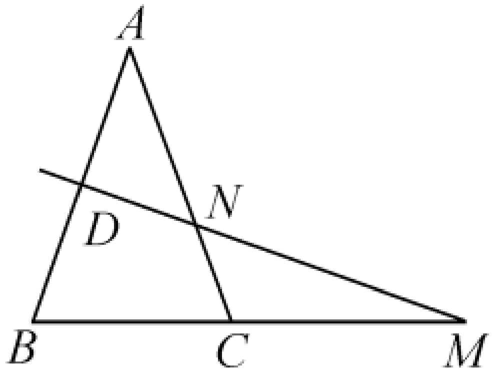


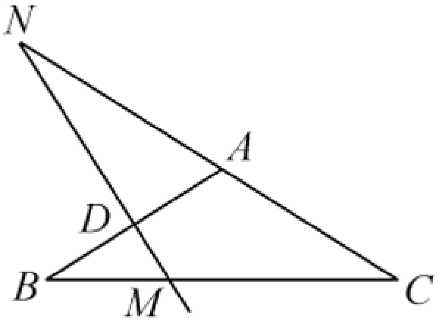
$$\frac{\left(\frac{D-E}{B-C}\right)^2}{\frac{A-D}{A-B}} = \frac{\frac{B-A}{B-C}}{\frac{C-B}{C-A}} \frac{\frac{E-D}{C-A}}{\frac{D-A}{D-E}},$$

Example 1 35 : As shown in the figure, in $\triangle ABC$, D is the incenter, and the tangent line of the circumscribed circle of $\triangle$ *BCD passing through D intersects AB* at M and AC at N. Prove: $AM = AN$.



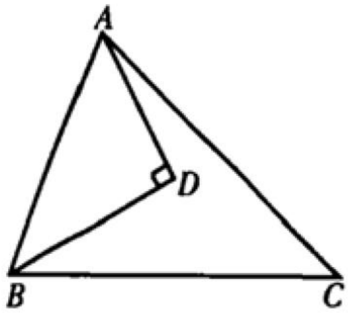
$$\frac{\frac{B-A}{M-N}\frac{B-D}{B-C}\frac{C-B}{C-D}\frac{C-D}{C-A}\frac{M-N}{D-C}}{\frac{N-M}{C-A}\frac{B-A}{B-D}\frac{D-B}{N-M}\frac{C-B}{C-D}\frac{B-D}{B-C}}=1$$





Example 1 36 : As shown in the figure, $\triangle ABC$, $AB=AC$, the perpendicular of AB intersects AB at D, BC at M, and AC at N. Prove that $2\angle BMN=\angle BAC$.

$$\frac{\left(\frac{C-B}{M-N}\right)^2}{\frac{A-C}{A-B}}=-\left(\frac{A-B}{M-N}\right)^2\frac{\frac{C-B}{C-A}}{\frac{B-A}{B-C}},$$



Example 1 37 : As shown in the figure, BD is the bisector of $\angle ABC$, $AD \perp BD$, and the vertical foot is D. Prove: $\angle BAD = \angle DAC + \angle ACB$.

$$\frac{\frac{A-C}{A-D}\frac{C-B}{C-A}}{\frac{A-D}{A-B}} = -\left(\frac{B-D}{A-D}\right)^2 \frac{\frac{B-A}{B-D}}{\frac{B-D}{B-C}},$$

Example 4.5 Let $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ be co-circular points. Let $\mathbf{5}$ be the foot drawn from point $\mathbf{1}$ to line $\mathbf{23}$, and let $\mathbf{6}$ be the foot drawn from point $\mathbf{2}$ to line $\mathbf{14}$. Then $\mathbf{34} \,||\, \mathbf{56}$.

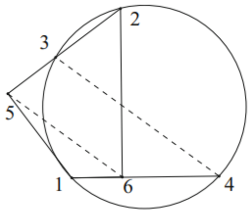


Figure 6 Example 4.5

Geometric construction sequence:

Free points: $\mathbf{1}, \mathbf{2}, \mathbf{3}$.

Semi-free point: $\mathbf{4}$ on circle $\mathbf{123}$ (non-linear construction).

Feet: $\mathbf{5} = \mathrm{Foot}_{\mathbf{1},\mathbf{23}}$, $\mathbf{6} = \mathrm{Foot}_{\mathbf{2},\mathbf{14}}$.

Conclusion: $[\boldsymbol{e}\mathbf{34}\boldsymbol{e}\mathbf{56}] = 0$.

We remove the hypothesis that $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ are co-circular, and compute the conclusion expression, where only the reduced Cayley forms of intersections $\mathbf{5}, \mathbf{6}$ are needed:

$$\begin{aligned}
\mathbf{5} = \;& (\mathbf{2} \wedge \mathbf{3}) \vee_e (\mathbf{1} \wedge \langle e\mathbf{23} \rangle_3^{\sim}) \quad \text{mod } e, \\
\mathbf{6} = \;& (\mathbf{1} \wedge \mathbf{4}) \vee_e (\mathbf{2} \wedge \langle e\mathbf{14} \rangle_3^{\sim}) \quad \text{mod } e. \\
& [e\mathbf{34}e\mathbf{56}] \\
\overset{\mathbf{5},\mathbf{6}}{=} \;& [e\mathbf{34}e\{(\mathbf{2} \wedge \mathbf{3}) \vee_e (\mathbf{1} \wedge \langle e\mathbf{23} \rangle_3^{\sim})\}\{(\mathbf{1} \wedge \mathbf{4}) \vee_e (\mathbf{2} \wedge \langle e\mathbf{14} \rangle_3^{\sim})\}] \\
\overset{\text{expand}}{=} \;& \;(\mathbf{2} \wedge \mathbf{3}) \vee_e (\mathbf{1} \wedge \langle e\mathbf{23} \rangle_3^{\sim}) \vee_e (\mathbf{2} \wedge \langle e\mathbf{14} \rangle_3^{\sim}) \; [e\mathbf{34}e\mathbf{14}] \\
& -(\mathbf{2} \wedge \mathbf{3}) \vee_e (\mathbf{1} \wedge \langle e\mathbf{23} \rangle_3^{\sim}) \vee_e (\mathbf{1} \wedge \mathbf{4}) \; [e\mathbf{34}e\mathbf{2}\langle e\mathbf{14} \rangle_3^{\sim}] \\
\overset{\text{expand}}{=} \;& [e\mathbf{21}\langle e\mathbf{23} \rangle_3^{\sim}][e\mathbf{32}\langle e\mathbf{14} \rangle_3^{\sim}][e\mathbf{34}e\mathbf{14}] + [e\mathbf{231}][e\langle e\mathbf{23} \rangle_3^{\sim}\mathbf{14}][e\mathbf{34}e\mathbf{2}\langle e\mathbf{14} \rangle_3^{\sim}]
\end{aligned}$$

$$\begin{aligned}
\overset{\text{ungrade}}{=} \;& 2^{-2}\{\langle e\mathbf{21}e\mathbf{23} \rangle \langle e\mathbf{32}e\mathbf{14} \rangle [e\mathbf{34}e\mathbf{14}] + [e\mathbf{123}] \langle e\mathbf{23}e\mathbf{14} \rangle \langle e\mathbf{34}e\mathbf{2}e\mathbf{14} \rangle\} \\
\overset{\text{null expand}}{=} \;& \underbrace{-(e \cdot \mathbf{2})(e \cdot \mathbf{4})\langle e\mathbf{23}e\mathbf{14} \rangle}(-\langle e\mathbf{1}\underline{\mathbf{2}}\mathbf{3} \rangle [e\mathbf{1}\underline{\mathbf{4}}\mathbf{3}] + \langle e\mathbf{1}\underline{\mathbf{4}}\mathbf{3} \rangle [e\mathbf{1}\underline{\mathbf{2}}\mathbf{3}]) \\
\overset{\text{contract}}{=} \;& \underbrace{-\langle e\mathbf{1}e\mathbf{3} \rangle}[\mathbf{1234}].
\end{aligned}$$

The last contraction is based on null Cramer's rule:

$$[\mathbf{143}e]\mathbf{123} - [\mathbf{123}e]\mathbf{143} = -[\mathbf{1234}]\mathbf{1}e\mathbf{3}. \tag{45}$$

Homogenization: By

$$\begin{aligned}
e \cdot \mathbf{5} &\overset{\mathbf{5}}{=} e \cdot \mathbf{1}[e\mathbf{23}\langle e\mathbf{23} \rangle_3^{\sim}] = 2(e \cdot \mathbf{1})(e \cdot \mathbf{2})(e \cdot \mathbf{3})(\mathbf{2} \cdot \mathbf{3}), \\
e \cdot \mathbf{6} &\overset{\mathbf{6}}{=} e \cdot \mathbf{2}[e\mathbf{14}\langle e\mathbf{14} \rangle_3^{\sim}] = 2(e \cdot \mathbf{1})(e \cdot \mathbf{2})(e \cdot \mathbf{4})(\mathbf{1} \cdot \mathbf{4}),
\end{aligned}$$

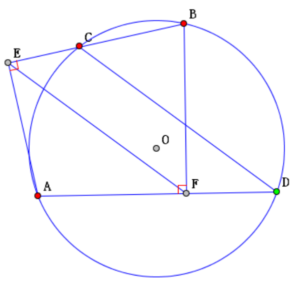
we get

$$\frac{[e\mathbf{34}e\mathbf{56}]}{(e \cdot \mathbf{5})(e \cdot \mathbf{6})} = \frac{\langle e\mathbf{23}e\mathbf{14} \rangle \, [\mathbf{1234}]}{2\,(e \cdot \mathbf{1})(e \cdot \mathbf{2})(\mathbf{1} \cdot \mathbf{4})(\mathbf{2} \cdot \mathbf{3})}, \tag{46}$$

where geometrically,

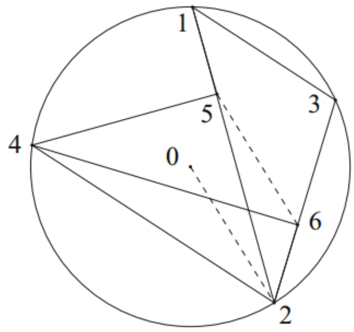
$$\begin{aligned}
\langle e\mathbf{34}e\mathbf{56} \rangle &= 2\, d_{\mathbf{34}} d_{\mathbf{56}} \cos\angle(\mathbf{34}, \mathbf{56}), \\
[e\mathbf{34}e\mathbf{56}] &= 2\, d_{\mathbf{34}} d_{\mathbf{56}} \sin\angle(\mathbf{34}, \mathbf{56}).
\end{aligned} \tag{47}$$

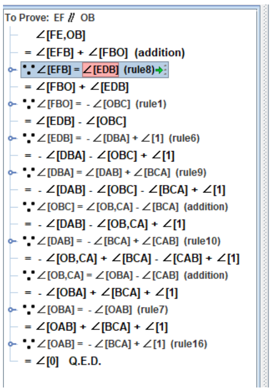
Example 1 38 : As shown in the figure, in the quadrilateral $ADBC$ *inscribed in* the circle O, $AE \perp BC$, $BF \perp AD$, to prove: $EF \,/\!/\, CD$.

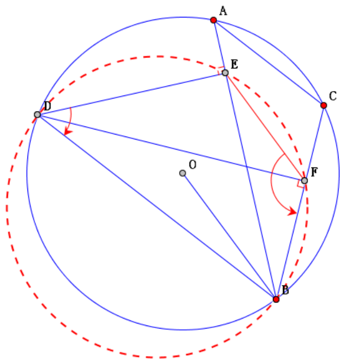


$$\frac{E-F}{D-C} \frac{\frac{F-A}{F-E}}{\frac{B-A}{B-E}} \frac{\frac{B-A}{B-C}}{\frac{D-A}{D-C}} \frac{D-A}{A-F} \frac{B-C}{B-E} = 1 \;,$$

Example 4.6 Let **1**, **2**, **3**, **4** be points on a circle of center **0** such that **13** || **24**. Let **5**, **6** be feet drawn from **4** to lines **12**, **23** respectively. Then **02** || **56**.

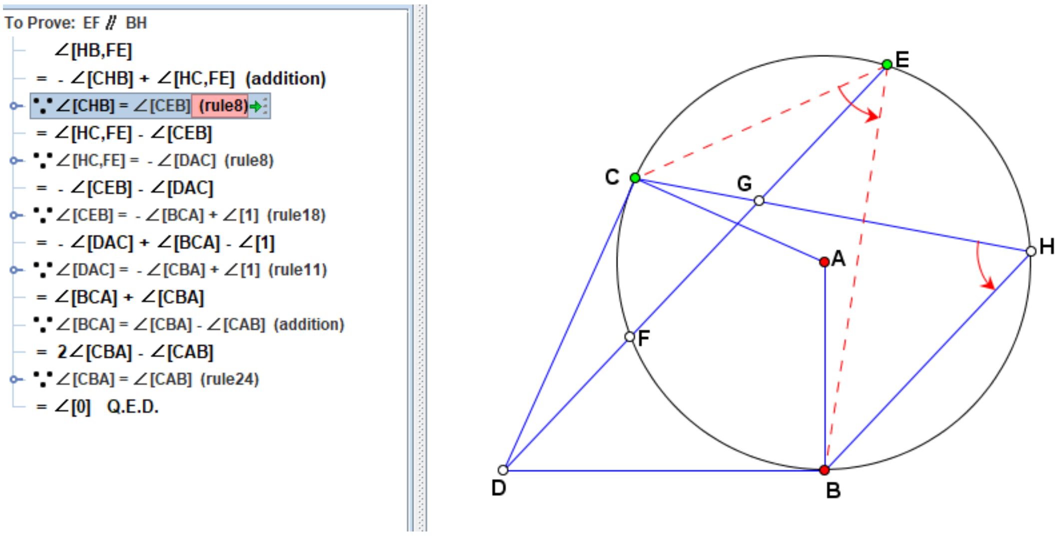






Example 1 39 : As shown in the figure, in the quadrilateral $ADBC$ *inscribed in* the circle O, $AC \,//\, DB$, $DE \perp AC$, $DF \perp BC$, to prove: $OB \,//\, EF$.

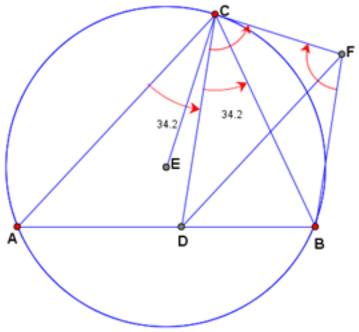
$$\frac{E-F}{O-B}\,\frac{\dfrac{D-E}{D-B}}{\dfrac{F-E}{B-C}}\,\frac{D-B}{A-C}\left(\frac{B-O}{B-A}\,\frac{C-A}{B-C}\right)\frac{A-B}{D-E}=1 \quad ,$$



Example 1 40 : As shown in the figure, it is known that $AB = AC$, take A as the center and AB as the radius to draw a circle, the tangent lines passing through B and C intersect at D, and the straight line DE *passing through* D intersects the circles at E and F, G is in EF point, CG intersects the circle with H, and proves: $HB \,//\, ED$.

$$\frac{H-B}{E-D}\,\frac{\dfrac{E-B}{E-C}}{\dfrac{H-B}{H-C}}\,\frac{\dfrac{E-D}{H-C}}{\dfrac{A-D}{A-C}}\,\frac{\dfrac{A-D}{A-C}}{\dfrac{E-B}{E-C}}=1$$

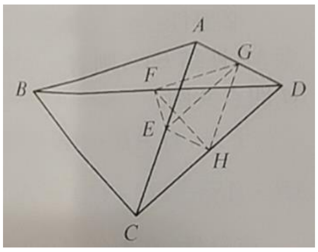
Explanation: Because G is the midpoint of EF, $AG \perp EF$, so D, A, G, and C are in a circle, and ∠CGD = $\angle CAD$.



Example 1 41 : As shown in the figure, in $\triangle ABC$, CD is the angle bisector, E is the circumcenter of $\triangle ABC$, through C, do $CF \perp CE$, through D, do $DF \parallel AC$, and prove: $CD \parallel FB$.

$$\frac{C-D}{F-B}\frac{D-F}{A-C}\frac{\frac{C-B}{C-D}}{\frac{C-D}{C-A}}\frac{\frac{F-B}{F-D}}{\frac{C-B}{C-D}}=1,$$

Explanation: From $\angle FCB = \angle CAB = \angle FDB$, then the four points F, C, D, B *are in a circle,* $\angle BFD = \angle BCD$.



Example 1 42 : As shown in the figure, E , F , G , and H *are the midpoints of* AC , BD , AD , and CD respectively . To prove: ∠ABC = ∠ADC *The* necessary and sufficient condition is that the four points E , F , G , and H share a circle .

$$\frac{\frac{B-A}{B-C}}{\frac{D-C}{D-A}}=\frac{\frac{\frac{C+D}{2}-\frac{A+C}{2}}{\frac{C+D}{2}-\frac{B+D}{2}}}{\frac{\frac{A+D}{2}-\frac{A+C}{2}}{\frac{A+D}{2}-\frac{B+D}{2}}}$$

It is easy to write a new identity equation according to the gourd painting

$$\frac{\frac{B-A}{B-C}}{\frac{D-C}{D-A}}=\frac{\frac{\frac{P+C+D}{3}-\frac{P+A+C}{3}}{\frac{P+C+D}{3}-\frac{P+B+D}{3}}}{\frac{\frac{P+A+D}{3}-\frac{P+A+C}{3}}{\frac{P+A+D}{3}-\frac{P+B+D}{3}}}$$

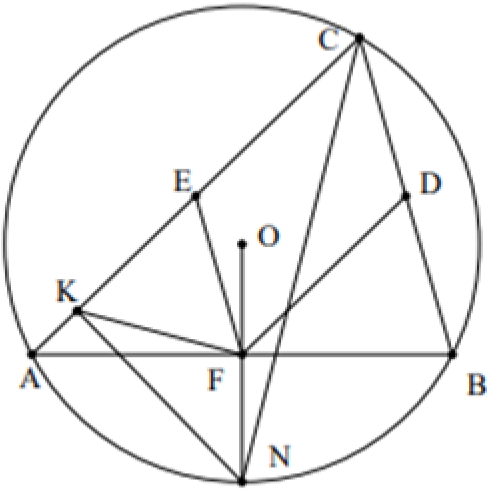
, and its geometric meaning is:

Example 1 43 : As shown in the figure, point P *is* any point on the quadrilateral *ABCD plane, and* E , F , G , H are the centers of gravity of △ PAC , △ PBD , △ PAD , △ PCD *respectively. To* prove : $\angle ABC = \angle ADC$ The condition is that the four points E , F , G , and H share a circle.

Example 1 44 : As shown in Figure 3, it is known that A, B, C and E share a circle, A, B, F and D share a circle, E, F and B share a line, CE intersects DF at K, Prove: K, C, A, D four points share a circle.

$$\frac{\frac{D-F}{C-E}}{\frac{A-D}{C-A}}=\frac{B-F}{E-B}\frac{\frac{C-A}{A-B}}{\frac{E-C}{E-B}}\frac{\frac{D-F}{D-A}}{\frac{F-B}{B-A}}.$$

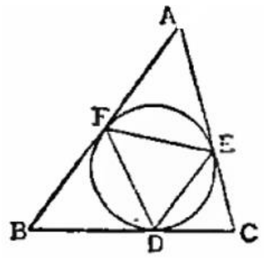
Explanation: If $\angle CAD = 180°$, then point K is at infinity, CE // DF. So it is the same as the previous question. This can be seen more clearly from the identity equation.



Example 1 45 : As shown in Figure 3, in $\triangle ABC$, D, E, and F *are the midpoints of* BC, CA, and AB respectively, N is the midpoint of the inferior arc AB, $NK \perp CA$ is at K, and the proof is: $\angle DFK + \angle KFE = 180°$.

$$\frac{D-F}{F-K} / \frac{F-K}{F-E} = -\frac{F-E}{B-C}\frac{F-D}{A-C}\left(\frac{A-C}{A-K}\right)^2\left(\frac{K-A}{K-F}\frac{N-F}{N-A}\right)^2\left(\frac{C-B}{C-A}\left(\frac{N-A}{N-F}\right)^2\right).$$

Explanation : $\angle BCA + 2\angle \mathrm{ANF} = 180°$ $\Leftrightarrow \frac{C-B}{C-A}\left(\frac{N-A}{N-F}\right)^2 \in R^-$.



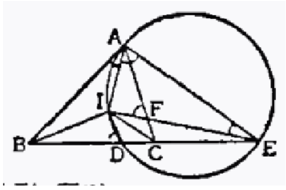
Example 1 46 : As shown in Figure 3, $\triangle$the inscribed circles of ABC *intersect* BC , CA , AB *respectively* at D , E , F , to prove: $2\angle FDE = \angle B + \angle C$.

Proof: $\dfrac{\dfrac{B-A}{B-C}\dfrac{C-B}{C-A}}{\left(\dfrac{D-F}{D-E}\right)^2} = \dfrac{\dfrac{C-B}{D-F}\dfrac{D-E}{B-C}}{\dfrac{F-D}{A-B}\dfrac{A-C}{E-D}}$.

Compared with the traditional proof method :

$$\angle B + \angle C = 180^\circ - 2\angle BDF + 180^\circ - 2\angle EDC$$

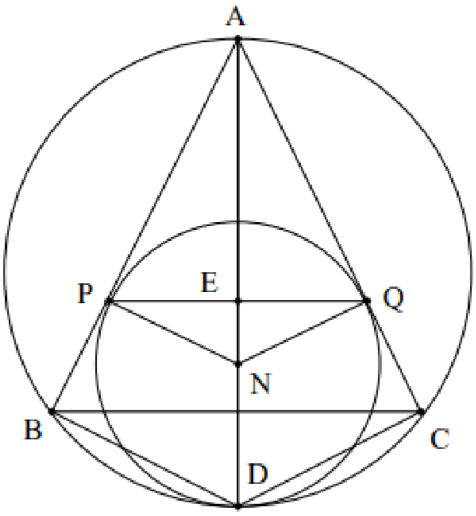
$= 2(180^\circ - \angle BDF - \angle EDC) = 2\angle FDE$, Do you feel that there is something in common?



Example 1 47 : As shown in Figure 3, suppose △a circle is drawn through the center I of ABC and side AB cuts *at point A* , *and* the intersection point with *BC* is *D* and *E* , then *IC* bisects ∠ *DIE* .

$$\frac{\frac{I-E}{I-C}}{\frac{I-C}{I-D}}=\frac{\frac{C-B}{C-I}}{\frac{C-I}{C-A}}\frac{\frac{A-I}{A-B}}{\frac{A-C}{A-I}}\frac{\frac{E-I}{E-A}}{\frac{A-I}{A-B}}\frac{\frac{A-E}{A-I}}{\frac{C-B}{D-I}},$$

Example 148 : As shown in Figure 3, in $\triangle ABC$, $AB = AC$, circle N is inscribed on the circumscribed circle of $\triangle ABC$, and is tangent to AB, AC respectively at P, Q, to prove: the midpoint E *of the line segment* PQ is $\triangle ABC$ *The* center of the inscribed circle. (1978 *IMO* test questions).

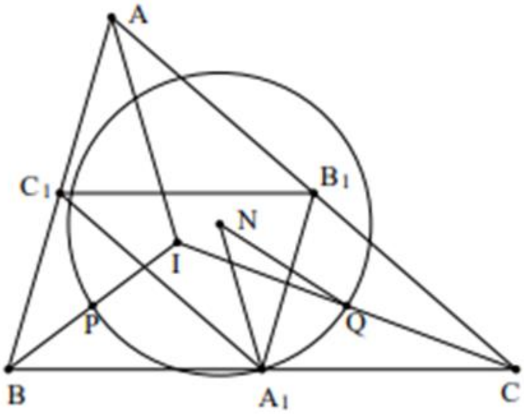


$$\frac{B-A}{B-E}/\frac{B-E}{B-C}=$$

$$\frac{B-A}{P-A}\frac{(B-C)(P-Q)}{(P-E)^2}\left(\frac{P-N}{B-D}\right)^2\left(\frac{D-B}{D-P}/\frac{E-B}{E-P}\right)^2\left(\frac{P-A}{P-Q}/\frac{N-P}{N-A}\right)\left(\frac{N-P}{N-A}/\left(\frac{P-N}{P-D}\right)^2\right)$$

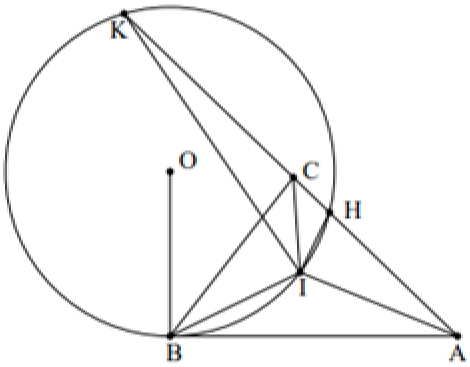
There is a square term in the conclusion of the verification $(B-E)^2$. If only one term is contained in the condition, $B-E$ it is easy to have a chain reaction, because this term will be squared. Incidentally, other parts of the term will also be squared, and these squared terms will either be eliminated by the new squared term, or divided into two, requiring two primary terms to be eliminated. This will result in longer and more complex identities. Therefore, readers should be mentally prepared to encounter such problems.

Example 149 : As shown in Figure 3, in $\triangle ABC$, I is the center, A_1, B_1, C_1 are the midpoints of BC, CA, AB respectively, P, Q *are the midpoints of* IB, IC respectively, and N is $\triangle PA_1$ The circumcenter of Q, to prove: A_1N is the bisector of $\angle C_1A_1B_1$.



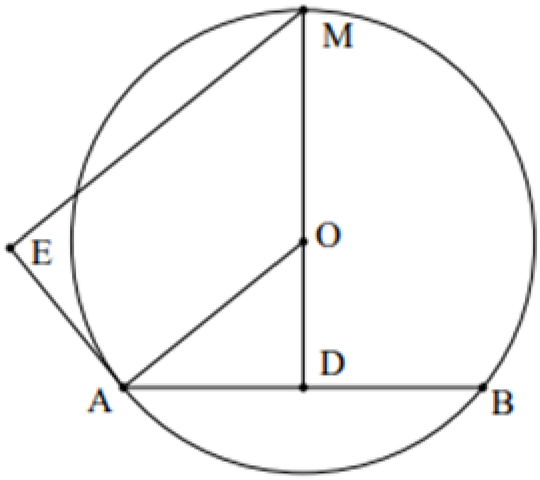
$$\frac{\dfrac{A_1-C_1}{A_1-N}}{\dfrac{A_1-N}{A_1-B_1}}=\frac{A_1-C_1}{C-A}\frac{A_1-B_1}{A-B}\left(\frac{P-Q}{B-C}\right)^2\left(\frac{I-C}{P-A_1}\right)^2\left(\frac{I-B}{Q-A_1}\right)^2\left(\frac{P-A_1}{P-Q}\frac{A_1-Q}{A_1-N}\right)^2\frac{\dfrac{C-B}{C-I}}{\dfrac{C-I}{C-A}}\frac{\dfrac{B-A}{B-I}}{\dfrac{B-I}{B-C}}$$

Example 150 : As shown in Figure 3, in $\triangle ABC$, I is the center, $OB \perp BA$, take O as the center and OB as the radius to draw a circle, intersect AC at two points H and K, and I is on the circle. Prove that: IC is Angle bisector *of* $\angle$HI K.



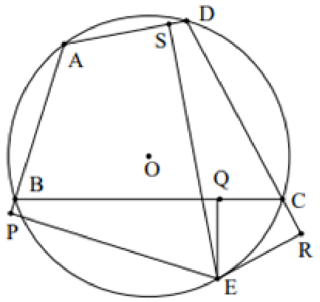
$$\frac{I-K}{I-C}/\frac{I-C}{I-H}=\frac{H-K}{A-C}\left(\frac{C-A}{C-I}/\frac{C-I}{C-B}\right)\left(\frac{B-H}{B-I}/\frac{K-H}{K-I}\right)\left(\frac{H-I}{H-B}/\frac{B-C}{B-I}\right),$$

Example 151 : As shown in Figure 3, in $\triangle MAB$, O is the circumcenter, D is the midpoint of AB, $EM \parallel AO$, $EA \perp AO$, to prove: $MD = ME$.



$$\frac{E-M}{E-D} / \frac{D-E}{D-M} = \left(\frac{E-M}{E-A} \frac{A-B}{D-M}\right)\left(\frac{A-E}{A-M} / \frac{D-E}{D-M}\right)^2 \left(\frac{A-M}{A-B} / \frac{A-E}{A-M}\right),$$

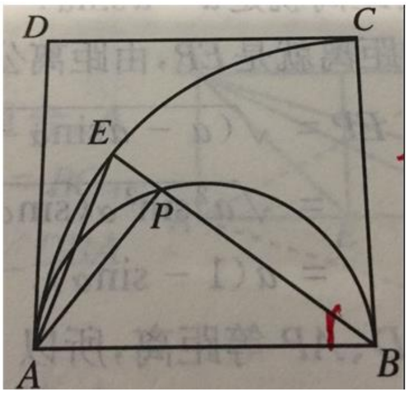
Example 152 : As shown in Figure 3, point E *is on* the circumscribed circle O *of* quadrilateral $ABCD$, and the feet of E on AB , BC , CD , and DA *are* P , Q , R , and S respectively . Prove: $\angle EPS = EQR$.



$$\frac{Q-R}{Q-E}/\frac{P-S}{P-E}=\left(\frac{R-Q}{R-C}\frac{E-C}{E-Q}\right)\left(\frac{S-A}{S-P}\frac{E-P}{E-A}\right)\left(\frac{C-R}{A-S}\frac{E-A}{E-C}\right),$$

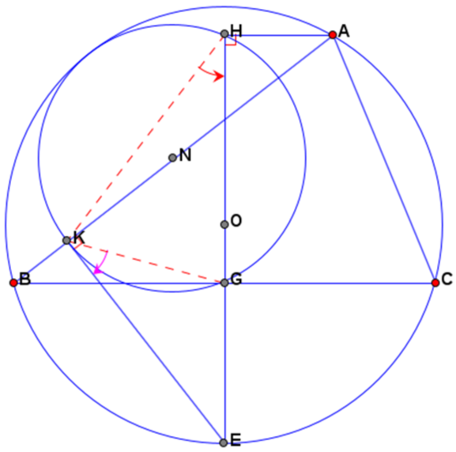
Explanation: This question comes from Zhou Gaozhang, but in fact, the conclusion ∠EPS = EQR only needs four points A , E , C , and D *to be in a circle, and it has nothing to do with the* position of *B.*

Example 153 : As shown in Figure 3, in the square $ABCD$, draw a semicircle within the square with *AB as the diameter , and draw a quarter arc AC within the square* with B as the center diameter . P is a point on the semicircle, extend BP to intersect the arc AC Ding E, even AE. Then AE divides $\angle$ *DAP equally* .



$$\frac{A-D}{A-E}/\frac{A-E}{A-P}=\left(\frac{B-A}{B-P}/\frac{A-D}{A-P}\right)\left(\frac{E-B}{E-A}/\frac{A-E}{A-B}\right)\left(\frac{A-D}{A-B}\right)^2\frac{B-P}{E-B}.$$

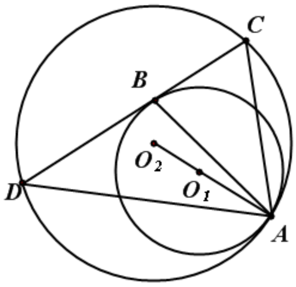
Example 154 : As shown in Figure 3, in $\triangle ABC$, O is the circumcenter, G is the midpoint of BC, OG intersects the circumscribed circle of $\triangle ABC$ on E, $EN \perp AB$ on K, $AH \perp OG$ on H, to prove: $\angle GKE = \angle GHK$.



$$\frac{K-G}{K-E}/\frac{H-G}{H-K}=\left(\frac{A-E}{A-K}/\frac{H-G}{H-K}\right)\left(\frac{K-G}{K-B}\frac{E-B}{E-G}\right)\left(\frac{B-K}{B-G}/\frac{E-K}{E-G}\right)\left(\frac{B-G}{B-E}/\frac{C-E}{C-B}\right)$$

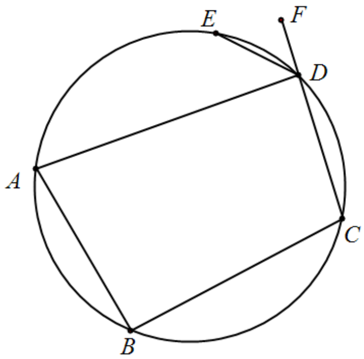
$$\left(\frac{C-E}{C-B}/\frac{A-E}{A-B}\right)\frac{A-K}{B-A}$$

Example 155 : As shown in the figure, two circles are inscribed at a point A, and B the tangent line of the small circle at the point intersects with the larger circle C,D, then AB it will be bisected $\angle CAD$.



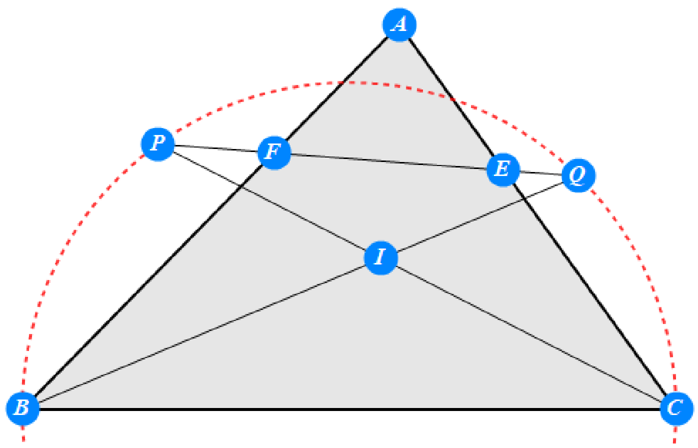
prove: $\frac{\frac{A-B}{A-C}}{\frac{A-D}{A-B}}=\frac{\frac{B-A}{B-O_1}}{\frac{A-O_1}{A-B}}\left(\frac{A-O_2}{A-C}\frac{D-C}{D-A}\frac{O_1-B}{C-D}\right)\frac{A-O_1}{A-O_2}$

Instructions $\frac{A-O_2}{A-C}\frac{D-C}{D-A}$ are used $\angle O_2AC+\angle ADC=90°$.



Example 1 56 : As shown in Figure 3, the quadrilateral $ABCD$ is inscribed in a circle , the angle bisector of $\angle ABC$ intersects the circle at E , and F is on the extension line of CD . Prove that DE is the bisector of $\angle ADF$.

$$\frac{\frac{C-D}{D-E}}{\frac{D-E}{D-A}}=\frac{\frac{D-A}{D-E}}{\frac{B-A}{B-E}}\frac{\frac{A-B}{B-E}}{\frac{B-E}{B-C}}/\left(\frac{E-D}{E-B}\frac{C-B}{C-D}\right),$$



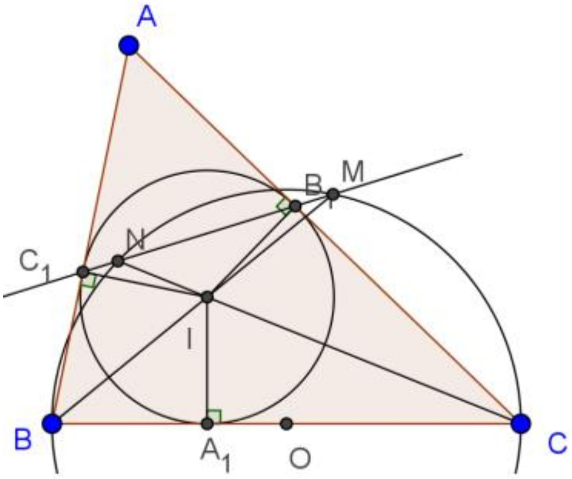
Example 133 : As shown in Figure 9 , in $\triangle ABC$, the inscribed circle I intersects AC and AF at E and F *respectively*, and the straight line EF intersects BI and CI at Q and P *respectively*. Prove that: B, C, P and Q share a circle.

Proof: $$\left(\frac{\frac{E-F}{I-B}}{\frac{C-I}{C-B}}\right)^2=\frac{\frac{E-F}{C-A}}{\frac{B-A}{F-E}}\frac{\frac{B-A}{B-I}}{\frac{B-I}{B-C}}\frac{\frac{C-B}{C-I}}{\frac{C-I}{C-A}}.$$

Explanation: B, C, P, Q four points cocircle are equivalent to $\angle PQB = \angle PCB$. In fact, we only need to slightly change the above formula to get a new identity $$\left(\frac{\frac{F-E}{A-C}}{\frac{B-I}{I-C}}\right)^2=\frac{\frac{E-F}{C-A}}{\frac{B-A}{F-E}}\frac{\frac{B-A}{B-I}}{\frac{B-I}{B-C}}\frac{\frac{C-I}{C-A}}{\frac{C-B}{C-I}},$$ which shows that $\angle QEC = \angle QIC$, and the four points I, C, Q, and E share a circle. It's a way of discovering new propositions. Transform some conditions in the title, including but not limited to: taking the reciprocal, adding and subtracting a real number, etc. After the recombination, if the obtained expression has obvious geometric meaning in the form, it is considered that a new proposition has been obtained.

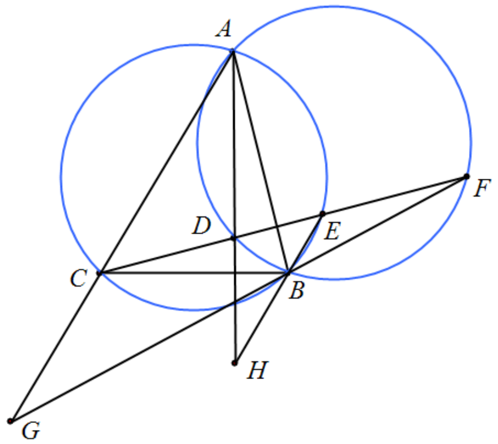
The identities correspond to $\angle PQB = \angle PCB$, $\angle QEC = \angle QIC$.

Automatically discover new questions



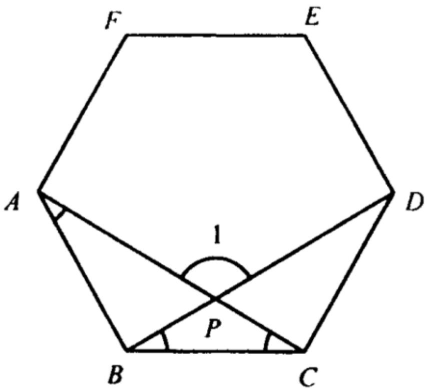
Example 157 : As shown in Figure 3, in △ABC, I is the center, extend BI and CI to intersect the circle with BC as the diameter at M and N, and prove: AI⊥MN.

$$\left(\frac{A-I}{M-N}\right)^2 \frac{\frac{C-N}{C-A}}{\frac{C-B}{C-N}} \left(\frac{\frac{M-N}{M-B}}{\frac{C-N}{C-B}}\right)^2 \frac{\left(\frac{M-B}{I-A}\right)^2}{\frac{C-B}{C-A}} = 1,$$



Example 158 : As shown in Figure 3, there is a point D inside $\triangle ABC$, the straight line CD intersects the circumcircles of $\triangle ABC$ and $\triangle ADB$ at E and F, AC intersects BF at G, and AD intersects EB at H. Prove: $\angle AGB = \angle AHB$.

$$\frac{\frac{H-A}{H-E}}{\frac{G-A}{G-F}} = \frac{\frac{B-A}{H-E}}{\frac{G-A}{C-F}} \frac{\frac{H-A}{C-F}}{\frac{B-A}{G-F}},$$

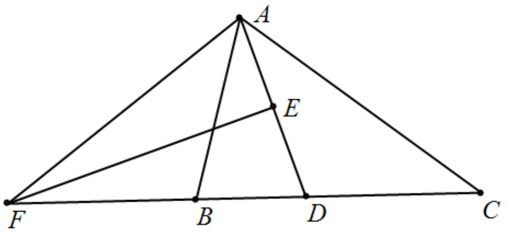


Example 159 : As shown in Figure 3, A, B, C, and D are four adjacent vertices in sequence on a regular polygon. AC intersects BD at P. Prove: $\angle APD=\angle ABC$.

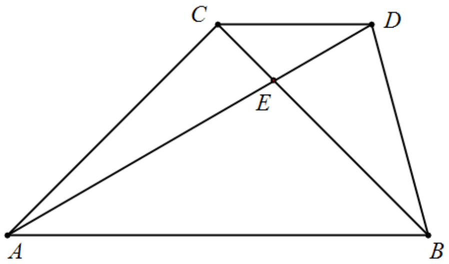
$$\frac{\frac{A-C}{D-B}}{\frac{B-A}{B-C}}=\frac{\frac{A-C}{A-B}}{\frac{C-B}{C-A}}\frac{\frac{C-B}{C-A}}{\frac{B-D}{B-C}}$$

Another proof: $\angle APD=\angle ABP+\angle PAB=\angle ABP+\angle ACB=\angle ABP+\angle PBC=\angle ABC$.

Example 160 : As shown in Figure 1, in $\triangle ABC$, $\angle B = 2\angle C$, the perpendicular line EF of $\angle BAC$ bisector AD intersects the extension line of CB at point F, to prove: $AF = AC$.



$$\frac{\frac{F-A}{B-C}}{\frac{C-B}{C-A}} = \frac{\frac{A-D}{A-B}}{\frac{A-C}{A-D}} \frac{\frac{C-B}{D-A}}{\frac{A-D}{A-F}} \frac{\frac{B-A}{B-C}}{\left(\frac{C-B}{C-A}\right)^2},$$

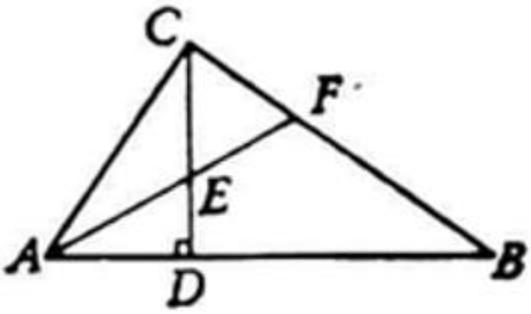


Pass through the right angle vertex *C of the isosceles right angle* $\triangle ABC$ to draw a parallel line to AB, *and take a point D* on it, make $AB = AD$, let AD and BC intersect at point E, *and* verify: $BD = BE$, $2CD = AE$, $2S_{\triangle ABC} = S_{\triangle ABE}$, $2\angle CAD = \angle DAB$.

Example 1 61 : As shown in Figure 1, there is a point **D** in $\triangle$ **ABC**, $\angle$ **CAB**, so that $AD = AB$, $2\angle CAD = \angle DAB$, AD and BC intersect at point E, to prove: The necessary and sufficient condition of **$CA \perp CB$ is** $BD = BE$.

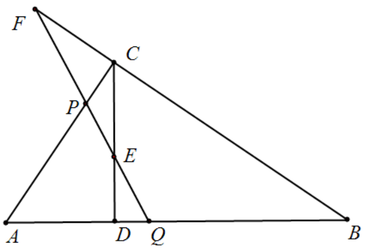
$$\frac{\frac{D-B}{D-A}}{\frac{B-A}{B-D}}\left(\frac{C-B}{C-A}\right)^2\left(\frac{\frac{A-D}{C-B}}{\frac{D-B}{D-A}}\right)^2\frac{\left(\frac{A-C}{A-D}\right)^2}{\frac{A-D}{A-B}}=-1,$$

Example 1 62 : *As shown* in **Figure 1,** it is known that CD is the height on the hypotenuse AB *of* $Rt\triangle ABC$, the bisector of $\angle A$ intersects CD at point E, *and* intersects CB at point F. Prove: $CE = CF$.



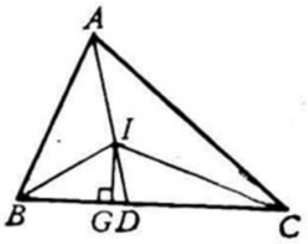
$$\frac{\frac{A-C}{A-F}}{\frac{A-F}{A-B}}\frac{C-B}{C-A}\frac{C-E}{A-B}=\frac{\frac{E-C}{A-F}}{\frac{F-A}{B-C}}.$$

Example 1 63 : As shown in Figure 1, it is known that CD is the height on the hypotenuse AB *of* $Rt\triangle ABC$, take points P and Q on AC and AB *respectively*, so that $AP = AQ$, straight line PQ intersects BC at F and intersects CD at E. Prove : $CE = CF$.



$$\frac{\frac{A-C}{P-Q}}{\frac{Q-P}{A-B}}\frac{C-B}{C-A}\frac{C-E}{A-B}=\frac{\frac{E-C}{P-Q}}{\frac{P-Q}{B-C}}$$

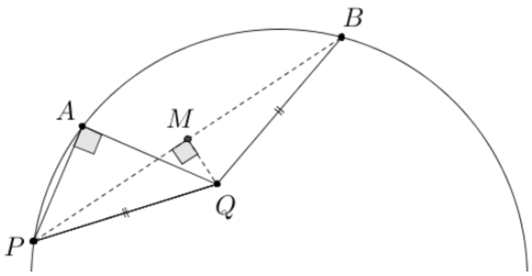
Description: Just replace the straight line AF with PQ.



Example 1 64 : As shown in Figure 1, the known point I is $\triangle ABC$ the center of , and the extension line of ***AI intersects*** with D ***BC***, $IG \perp BC, G$ which is the vertical foot. Prove: $\angle BID = \angle CIG$.

$$-\left(\frac{I-G}{B-C}\right)^2 \frac{\dfrac{I-D}{A-B}\,\dfrac{B-A}{B-I}\,\dfrac{C-B}{C-I}}{\dfrac{A-C}{I-D}\,\dfrac{B-I}{B-C}\,\dfrac{C-I}{C-A}} = \left(\frac{\dfrac{I-D}{I-B}}{\dfrac{I-C}{I-G}}\right)^2,$$

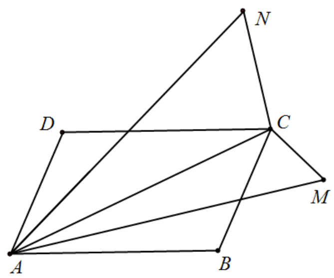
Proof: $\angle CIG = 90^{\circ} - \frac{1}{2}\angle ACB = \frac{1}{2}\left(180^{\circ} - \angle ACB\right) = \frac{1}{2}(\angle CAB + \angle ABC) = \angle BID$.

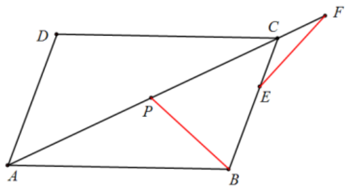


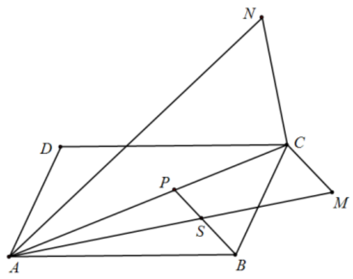
1. In the figure below, the points P, A, B lie on a circle. The point Q lies inside the circle such that $\angle PAQ = 90^\circ$ and $PQ = BQ$. Prove that the value of $\angle AQB - \angle PQA$ is equal to the arc AB.

Example 1 65 : As shown in Figure 1, three points P, A, and B are on a circle, Q is inside the circle, and $AP \perp AQ$, $QP = QB$, prove: $\angle AQB - \angle PQA$ is equal to the central angle subtended by the arc AB. (2015 Iran Mathematics Contest Questions)

$$\left(\frac{A-P}{A-Q}\right)^2 \frac{\dfrac{B-Q}{B-P}}{\dfrac{P-B}{P-Q}} \frac{\dfrac{Q-A}{Q-B}}{\dfrac{Q-P}{Q-A}\left(\dfrac{P-A}{P-B}\right)^2} = -1$$



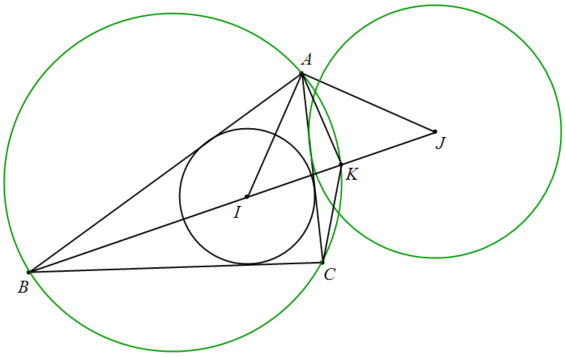




Example 1 66 : As shown in Figure 1, in the parallelogram $ABCD$, M and N *are* the circumcentres of ABC and $\triangle ADC$ respectively . Prove: $\triangle CM \perp AN$, $\angle AMC = \angle ANC$.

$$\left(\frac{C-M}{A-N}\right)^2=-\frac{\frac{C-M}{D-A}\frac{A-D}{A-N}}{\frac{A-C}{C-M}\frac{A-N}{A-C}},\ \left(\frac{\frac{N-C}{N-A}}{\frac{M-A}{M-C}}\right)^2=\frac{\frac{A-C}{A-M}\frac{C-M}{D-A}\frac{A-D}{A-N}\frac{C-N}{A-C}}{\frac{A-M}{A-B}\frac{A-C}{C-M}\frac{A-N}{A-C}\frac{B-A}{C-N}}.$$

Example 1 67 : As shown in Figure 1, the distance from the intersection point of the bisector of an interior angle of a triangle and its circumscribed circle to the other two vertices and the distance between the center and the center of the triangle are equal. As shown in the figure, in $\triangle ABC$, I and J are the inner and outer centers respectively, and BI intersects the circumscribed circle of $\triangle ABC$ at K, then $KA = KC = KI = KJ$.

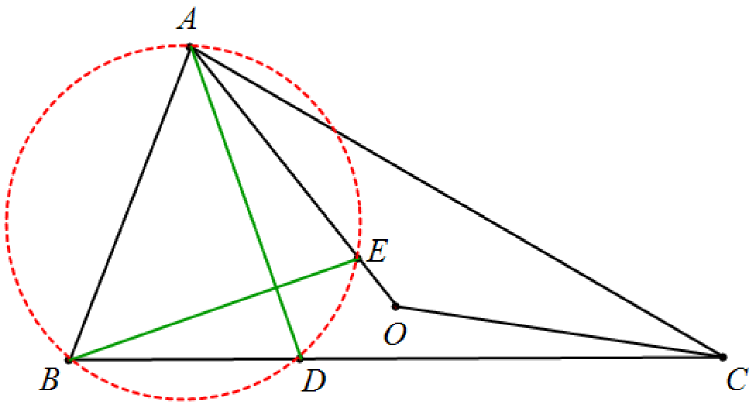


$$\frac{\frac{A-K}{A-I}}{\frac{I-A}{B-K}} = \frac{\frac{B-K}{B-C}}{\frac{B-A}{B-K}} \frac{\frac{A-C}{A-I}}{\frac{A-I}{A-B}} \frac{\frac{C-B}{C-A}}{\frac{K-B}{K-A}}.$$

Explanation: This identity states *that* $KA = KI$. At the same time, replace I *with* J, which proves that $KA = KJ$.

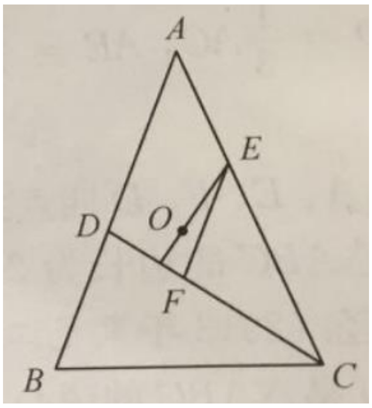
Variant: As shown in the figure, $\triangle$in ABC, the angle bisector of $\angle$ABC intersects the circumscribed circle $\triangle$*of* ABC at K, take points M and N *on* AB and AC *respectively*, so that $AM = AN$, and the straight line MN intersects BK and AK at P and Q, to prove: $KP = KQ$.

four points in circle



Example 1 68 : As shown in Figure 1, $\triangle$in ABC, O is the circumcenter, AD is the angle bisector, $BE \perp AD$ intersects AO at E, and prove that: A, B, D, and E share a circle.

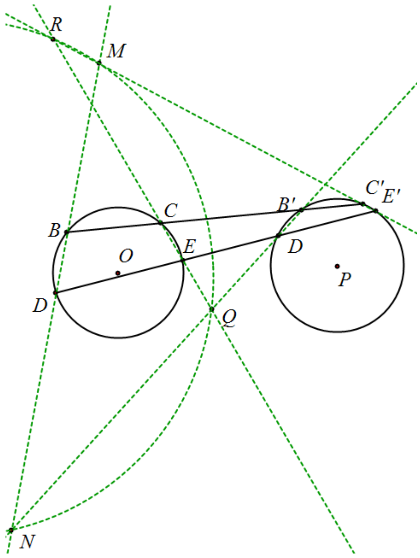
$$\frac{\frac{E-B}{O-A}}{\frac{C-B}{D-A}}=\frac{A-D}{A-B}\frac{A-C}{A-O}\frac{\frac{E-B}{D-A}}{\frac{A-C}{A-B}\frac{C-B}{C-O}}\frac{A-C}{A-D}\frac{C-O}{C-A},$$



Example 1 69 : As shown in Figure 1, in the acute angle $\triangle ABC$, the bisector of $\angle ACB$ *intersects* AB at point *D*, passes through $\triangle$the circumcenter *O of ABC*, *draws a perpendicular line from CD*, *intersects* AC *at* point *E*, *and* passes through point *E* to draw parallel to *AB* The line intersects *CD* at point *F*. Prove: *C*, *E*, *O*, *F* *are* four points in a circle. (2010 Fujian Provincial Preliminaries)

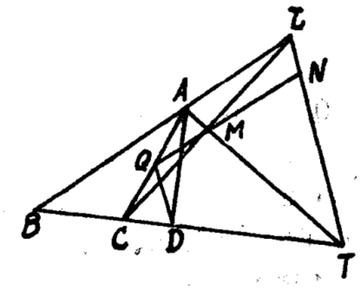
$$\frac{\dfrac{F-E}{D-C}}{\dfrac{O-E}{O-C}}=\frac{\dfrac{C-B}{C-D}}{\dfrac{C-D}{C-A}}\frac{E-F}{A-B}\left(\frac{D-C}{O-E}\frac{C-O}{C-A}\right)\frac{B-A}{B-C}。$$

Example 1 70 : As shown in Figure 1, let BC and DE be the chords of *the circle* O, extend BC and DE to intersect the known circle P at B', C', D', E ', *and* then extend BD, CE, $B'D$' , C' E' intersect at M, N and R, Q, then the four points M, R, Q, N share a circle.

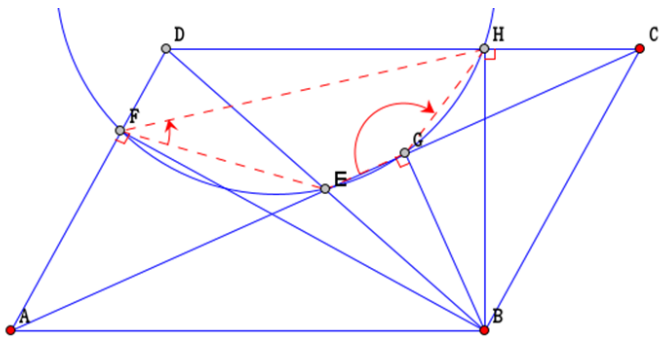


$$\frac{\dfrac{B-D}{B_1-D_1}}{\dfrac{C_1-E_1}{C-E}}=\frac{\dfrac{D-B}{D-E}}{\dfrac{C-B}{E-C}}\frac{\dfrac{C-B}{E_1-C_1}}{\dfrac{D_1-B_1}{D-E}},$$

Example 1 71 : As shown in Figure 1, $\triangle$in ABC, the bisector of the exterior angle of $\angle$A intersects the extension line of BC at T, and $CM \perp AT$ *is* drawn from C at M, *and* intersects *the* extension line of BA *at L,* taking AC Point Q, QM intersect LT at N, then A leads $AD \perp BC$ to D, then D, Q, N, T are four points in a circle.



$$\frac{\frac{D-Q}{Q-N}}{\frac{C-B}{T-L}} = \frac{\frac{C-A}{B-C}}{\frac{C-B}{D-Q}} \frac{\frac{C-B}{C-A}}{\frac{B-A}{L-T}} \frac{B-A}{Q-N},$$



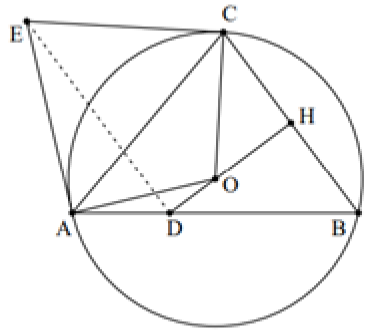
Example 172 : As shown in Figure 3, in the parallelogram $ABCD$, AC intersects BD at E, and the feet of B on AD, AC, and CD are respectively F, G, and H. Prove that: E, F, G, and H are four points in a circle .

$$\frac{\frac{F-H}{F-E}}{\frac{G-H}{G-E}}=\frac{\frac{B-H}{B-C}}{\frac{G-H}{G-C}}\frac{\frac{H-F}{H-D}}{\frac{B-F}{B-D}}\frac{\frac{B-F}{B-E}}{\frac{F-E}{F-B}}\left(\frac{B-C}{B-F}\frac{D-H}{B-H}\right)\frac{G-E}{C-G}\frac{B-E}{B-D}.$$

multiple choice questions

This question is not proved by geometry experts , so the question is automatically generated

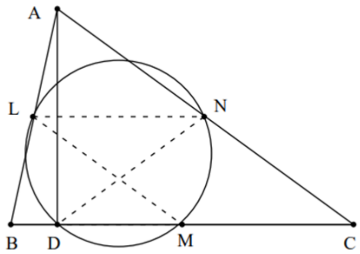
Mix the conditions together to multiply and divide, as short as possible, and finally see if there is any geometric meaning.



Example 173 : As shown in Figure 3, in $\triangle ABC$, O is the circumcenter, $EA \perp AO$, $EC \perp CO$, the point O is on AB, and $DO \perp BC$, to prove: E, D, O, and C are all circles.

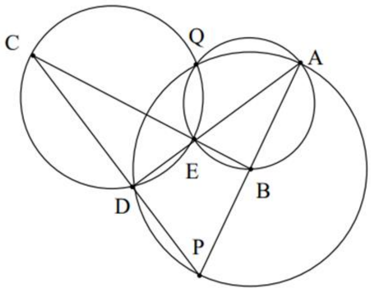
$$\frac{E-O}{E-A}\frac{D-A}{D-O}=\left(\frac{C-A}{C-E}\frac{O-E}{O-A}\right)\left(\frac{B-C}{D-O}\frac{B-A}{B-C}\frac{C-O}{C-A}\right)\left(\frac{O-A}{E-A}\frac{E-C}{C-O}\right)\frac{D-A}{B-A}.$$

Example 174 : As shown in Figure 3, in $\triangle ABC$, AD is high, and M, N, L *are the midpoints of* BC, CA, AB respectively . Prove that D, M, N, L share a circle.

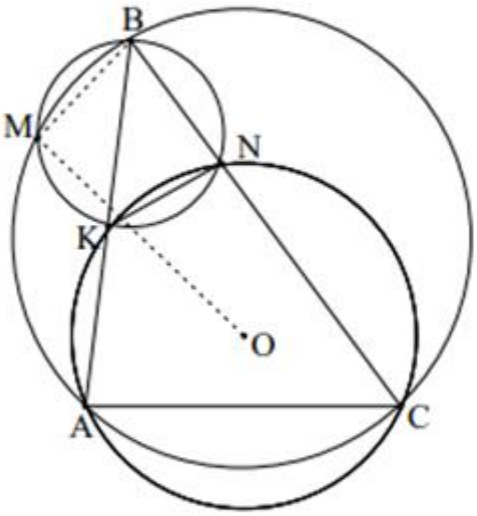


$$\frac{N-D}{N-L}\frac{M-L}{M-D}=\frac{B-C}{N-L}\frac{M-L}{C-A}\left(\frac{N-D}{B-C}\frac{C-A}{M-D}\right),$$

Example 175 : As shown in Figure 3, the straight lines AB and CD intersect at P , AD and BC intersect at E , and the circumscribed circles of $\triangle ABE$ and $\triangle CDE$ intersect at point Q. Prove that the four points A , Q , D , P *share a circle.* (Mick's theorem)



$$\frac{P-D}{P-A}\frac{Q-A}{Q-D}=\frac{P-D}{D-C}\frac{B-A}{P-A}\frac{E-C}{B-E}\left(\frac{Q-E}{Q-D}\frac{C-D}{C-E}\right)\left(\frac{Q-A}{Q-E}\frac{B-E}{B-A}\right),$$



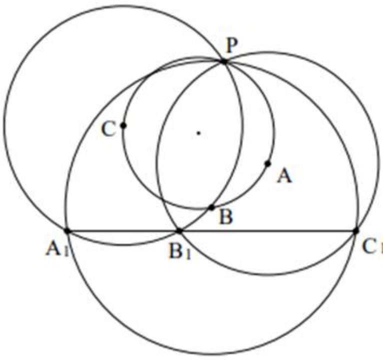
Example 1 76 : As shown in Figure 3, A, C, K, N share a circle, O is its center, AK intersects CN at B, and the circumscribed circles of $\triangle ACB$ and $\triangle KNB$ intersect at M. Prove: M, K, O, C are four points in a circle, $MB \perp MO$. (1985 International Olympic Examination Questions)

$$\frac{M-K}{M-C}\frac{O-C}{O-K}=\left(\frac{M-B}{M-C}/\frac{A-B}{A-C}\right)\left(\frac{M-K}{M-B}\frac{N-B}{N-K}\right)\frac{N-C}{N-B}\frac{A-B}{K-A}\left(\frac{K-C}{K-O}\frac{A-K}{A-C}\right)$$

$$\left(\frac{N-A}{N-C}\frac{C-O}{C-A}\right)\left(\frac{N-K}{N-A}/\frac{C-K}{C-A}\right),$$

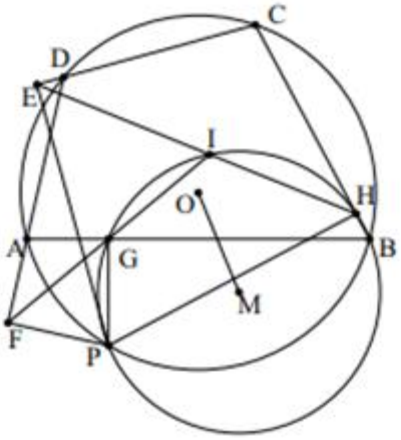
$$\frac{M-O}{M-B}=\frac{\dfrac{A-B}{A-C}}{\dfrac{M-B}{M-C}}\frac{\dfrac{K-C}{K-O}}{\dfrac{M-C}{M-O}}\frac{A-K}{A-B}\left(\frac{K-O}{K-C}\frac{A-C}{A-K}\right)。$$

Example 177 : As shown in Figure 3, it is known that A_1, B_1, and C_1 are collinear , A is the circumcenter of $\triangle PB_1C_1$, B *is* the circumcenter of $\triangle PA_1C_1$, and C is $\triangle PA_1B$ *The circumcenter of* $_1$, to prove: P, C, B, A four points share a circle.



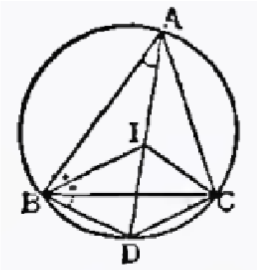
$$\frac{C-P}{C-A}\frac{B-A}{B-P}=\frac{A_1-C_1}{A_1-B_1}\frac{A-B}{P-C_1}\frac{P-B_1}{A-C}\left(\frac{P-C}{P-B_1}\frac{A_1-B_1}{A_1-P}\right)\left(\frac{A_1-P}{A_1-C_1}\frac{P-C_1}{P-B}\right),$$

Example 178 : As shown in Figure 3, P is a point on the circumscribed circle O *of the quadrilateral* $ABCD$ *, and the feet of* P on CD , DA , AB , and BC *are* E , F , G , and H respectively . EH intersects FG and I. Prove: G , P , H , I are four points



in a circle.

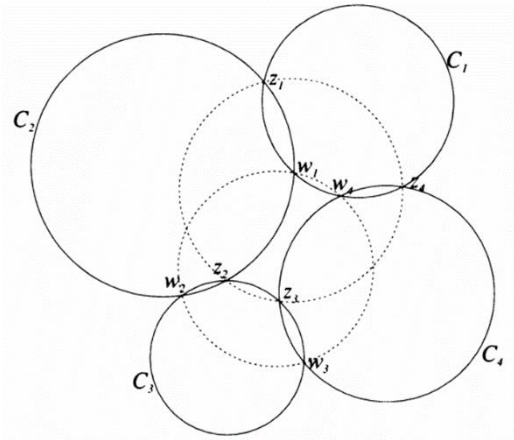
$$\frac{I-H}{I-G}\frac{P-G}{P-H}=\left(\frac{H-E}{H-C}/\frac{P-E}{P-C}\right)\left(\frac{P-F}{P-A}/\frac{G-F}{G-A}\right)\left(\frac{C-D}{C-P}\frac{A-P}{A-D}\right)\left(\frac{C-H}{H-P}\frac{G-P}{A-G}\right)\frac{I-H}{H-E}\frac{F-G}{G-I}\frac{A-D}{F-P}\frac{E-P}{C-D}$$



Example 179 : As shown in Figure 3, if $\triangle$the center of *ABC is I and the circumcenter of* $\triangle BCI$ is *D*, then the three points A, I, and D are collinear, and the four points A, B, C, and D *are in the same circle* .

$$\left(\frac{A-I}{I-D}\right)^2=\frac{\frac{B-I}{B-D}}{\frac{I-D}{I-B}}\frac{\frac{C-D}{C-B}}{\frac{B-C}{B-D}}\frac{\frac{I-C}{I-D}}{\frac{C-D}{C-I}}\frac{\frac{A-I}{A-B}}{\frac{A-C}{A-I}}\frac{\frac{B-A}{B-I}}{\frac{B-I}{B-C}}\frac{\frac{C-B}{C-I}}{\frac{C-I}{C-A}}.$$

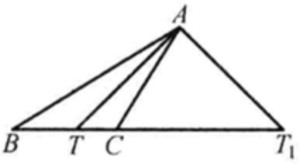
$$\frac{\frac{C-D}{C-B}}{\frac{A-D}{A-B}}=\frac{\frac{B-I}{B-D}}{\frac{I-D}{I-B}}\frac{\frac{B-A}{B-I}}{\frac{B-I}{B-C}}\frac{\frac{C-D}{C-B}}{\frac{B-C}{B-D}}\frac{I-D}{A-D},$$



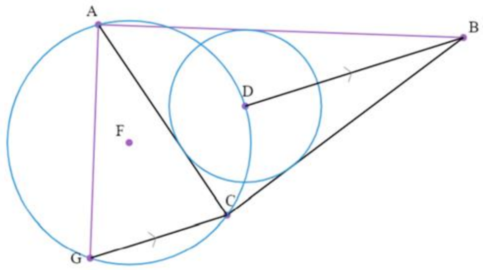
Example 180 : As shown in Figure 3, there are four circles on the plane C_1, C_2, C_3, C_4. C_1 Intersecting C_2 at Z_1 sum W_1, C_2 intersecting C_3 at Z_2 sum W_2, C_3 intersecting C_4 at Z_3 sum W_3, C_4 intersecting C_1 at Z_4 sum W_4, to prove Z_1 that, Z_2, Z_3, Z_4 the necessary and sufficient condition for the four points to be in the same circle is that W_1, W_2, W_3, W_4 the four points are in the same circle.

$$\frac{\frac{Z_1-Z_2}{W_2-Z_2}\frac{Z_2-Z_3}{W_3-Z_3}\frac{Z_3-Z_4}{W_4-Z_4}\frac{Z_4-Z_1}{W_1-Z_1}}{\frac{Z_1-W_1}{W_2-W_1}\frac{Z_2-W_2}{W_3-W_2}\frac{Z_3-W_3}{W_4-W_3}\frac{Z_4-W_4}{W_1-W_4}}=\frac{\frac{Z_1-Z_2}{Z_3-Z_2}\frac{W_1-W_2}{W_3-W_2}}{\frac{Z_1-Z_4}{Z_3-Z_4}\frac{W_1-W_4}{W_3-W_4}}.$$

Example 181 : As shown in Figure 1, in $\triangle ABC$, $\angle ACB - \angle B = 90°$, *the bisectors of the inner and outer angles of* $\angle BAC$ intersect BC *and* its extension line at T, T_1, and prove : $AT = AT_1$.

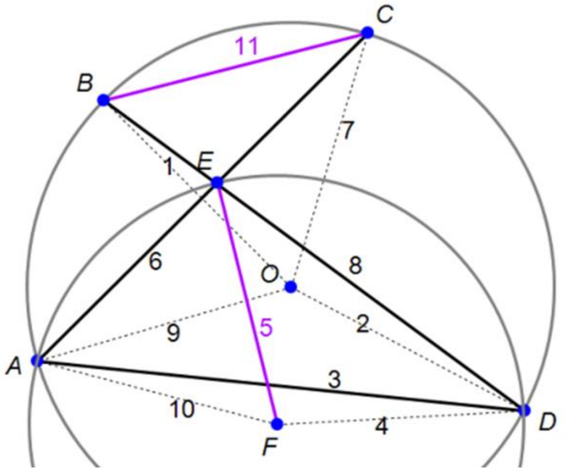


$$\left(\frac{\frac{T-A}{B-C}}{\frac{C-B}{T_1-A}}\right)^2 \left(\frac{\frac{C-B}{C-A}}{\frac{B-A}{B-C}}\right)^2 \frac{\frac{A-C}{A-T}}{\frac{A-T}{A-B}} \frac{\frac{B-A}{A-T_1}}{\frac{A-T_1}{A-C}} = -1,$$



Example 1 82 : As shown in Figure 1, D is the center of $\triangle ABC$, G is a point on the circumcircle of $\triangle ADC$, and $CG \parallel BD$, prove that $AB \perp AG$.

$$\frac{A-B}{A-G}\frac{\frac{B-D}{B-A}}{\frac{D-C}{D-A}}\left(\frac{G-A}{G-C}\frac{D-C}{D-A}\right)\frac{G-C}{D-B}=-1.$$



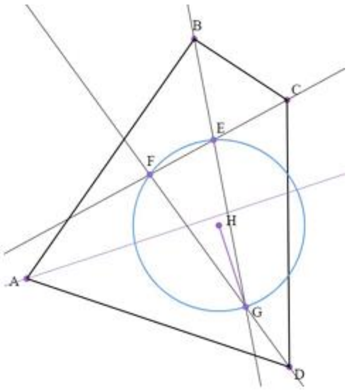
Example 1 83 : As shown in Figure 1, the quadrilateral $ABCD$ is inscribed in the circle O, E is the intersection point of the diagonals, F is the circumcenter of $\triangle AED$, prove: $BC \perp EF$.

$$\frac{B-C}{E-F}=\frac{\frac{B-C}{B-D}}{\frac{A-C}{A-D}}\left(\frac{B-D}{E-F}\frac{A-C}{A-D}\right),$$

Explanation : $\frac{B-D}{E-F}\frac{A-C}{A-D}$ *It* is a pure imaginary number that uses ∠DEF + ∠CAD = 90 ° .

$$\left(\frac{B-C}{E-F}\right)^2\left(\frac{\frac{E-F}{B-D}}{\frac{A-C}{A-D}}\right)^2\frac{\frac{C-O}{C-B}}{\frac{B-C}{B-O}}\frac{\frac{A-C}{A-O}}{\frac{C-O}{C-A}}\frac{\frac{B-D}{B-O}}{\frac{D-O}{D-B}}\frac{\frac{A-O}{A-D}}{\frac{D-A}{D-O}}=1,$$

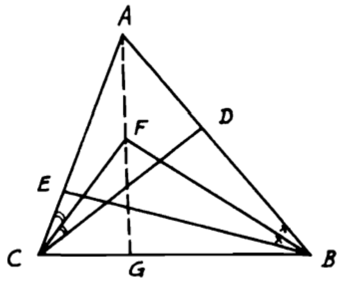
Example 1 84 : As shown in Figure 1, AK, BE, CF, DG *are* the bisectors of the four interior angles of quadrilateral $ABCD$ respectively , and H is △the circumcenter of EFG. *Prove:* $AK \perp HG$.



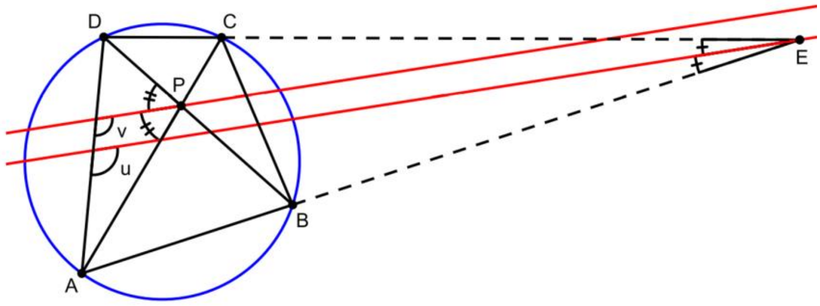
$$\left(\frac{A-K}{H-G}\right)^2\left(\frac{F-E}{F-G}\frac{G-H}{G-E}\right)^2\frac{\frac{A-B}{A-K}\frac{E-G}{B-A}\frac{C-B}{E-F}\frac{G-F}{D-C}}{\frac{A-K}{A-D}\frac{B-C}{E-G}\frac{E-F}{C-D}\frac{D-A}{G-F}}=1,$$

1.2. *ABC is a triangle. D and E are any two points on AB and AC. The bisectors of the angles ABE and ACD meet in F. Show that* $\angle BDC + \angle BEC = 2\angle BFC$.

Example 1 85 : As shown in Figure 1, in $\triangle ABC$, D and E are points on AB and AC respectively , and the angle bisectors of $\angle ABE$ and $\angle ACD$ intersect at F. Prove: $\angle BDC + \angle BEC = 2\angle BFC$.



$$\frac{\frac{D-B}{D-C}\frac{E-B}{E-C}}{\left(\frac{F-B}{F-C}\right)^2}=\frac{\frac{C-F}{C-D}\frac{B-E}{B-F}}{\frac{C-E}{C-F}\frac{B-F}{B-D}},$$



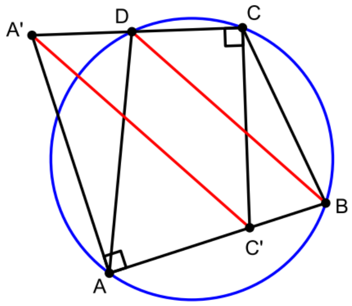
Example 1 86 : As shown in Figure 1, quadrilateral $ABCD$, AC intersects BD at P, *and* AB intersects DC at E. Prove that the necessary and sufficient condition for the four points A, B, C, and D *to be in* a circle is the angle bisector of $\angle APD$ and $\angle AED$ parallel.

$$\left(\frac{L_2}{L_1}\right)^2 \frac{\frac{A-C}{A-B}}{\frac{D-C}{D-B}} \frac{\frac{L_1}{B-D}}{\frac{C-A}{L_1}} \frac{\frac{B-A}{L_2}}{\frac{L_2}{C-D}} = 1,$$

$$u = \pi - \angle A - \frac{\pi - \angle A - \angle D}{2} = \frac{\pi - \angle A + \angle D}{2},$$

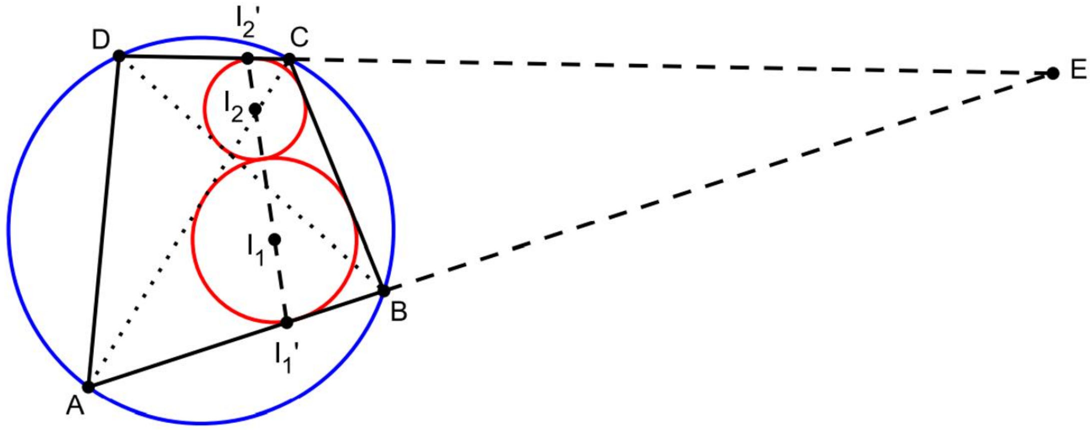
$$v = \frac{\pi - \angle DAC - \angle BDA}{2} + \angle BDA = \frac{\pi - \angle DAC + \angle BDA}{2},$$

$$u = v \Leftrightarrow -\angle A + \angle D = -\angle DAC + \angle BDA.$$



Example 1 87 : As shown in Figure 1, quadrilateral $ABCD$, $AA'\perp AB$ intersects CD at A', $CC'\perp CD$ intersects AB at C', prove: The necessary and sufficient condition for four points A, B, C and D to be cocircles is $A'C'//BD$.

$$\frac{A'-C'}{D-B}\frac{\frac{A-C}{A-B}}{\frac{D-C}{D-B}}\frac{\frac{A'-C}{A'-C'}}{\frac{A-C}{A-C'}}\frac{D-C}{A'-C}\frac{A-B}{A-C'}=1$$

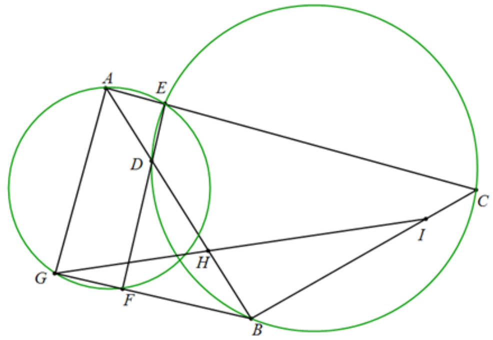


Example 1 88 : As shown in Figure 1, quadrilateral $ABCD$, AB intersects DC at E, I_1, I_2 are respectively the inner centers of △ I_1I_2ABC and △ DBC, intersects AB at I_1', intersects at $I_2'DC$, to prove: A, B, C, D four points cocircle The necessary and sufficient condition is $EI_1' = EI_2'$.

$$\left(\frac{\frac{B-I_1}{B-I_2}}{\frac{C-I_1}{C-I_2}}\right)^2 \frac{\frac{A-C}{A-B}}{\frac{D-C}{D-B}} \frac{\frac{C-B}{C-I_2}}{\frac{C-I_2}{C-D}} \frac{\frac{B-A}{B-I_1}}{\frac{B-I_1}{B-C}} \frac{\frac{C-I_1}{C-A}}{\frac{C-B}{C-I_1}} \frac{\frac{B-I_2}{B-C}}{\frac{B-D}{B-I_2}} = 1,$$

$$\frac{\frac{D-C}{I_2-I_1}}{\frac{I_1-I_2}{A-B}} \frac{\frac{I_1-I_2}{I_1-C}}{\frac{B-I_2}{B-C}} \frac{\frac{A-C}{A-B}}{\frac{D-C}{D-B}} \frac{\frac{C-B}{C-I_1}}{\frac{I_2-B}{I_2-I_1}} \frac{\frac{C-I_1}{C-A}}{\frac{C-B}{C-I_1}} \frac{\frac{B-I_2}{B-C}}{\frac{B-D}{B-I_2}} = 1,$$

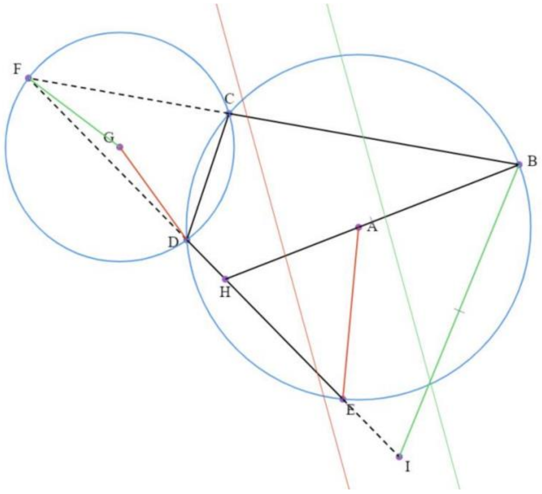
Explanation: To prove $EI_1' = EI_2'$, it is necessary to introduce a straight line $I_1'I_2'$, which is inconvenient to eliminate. It needs to be divided into two steps, first prove that the four points B, I_1, I_2, and C are in a circle, and then prove it $EI_1' = EI_2'$. But because of this, it will cause trouble for the push back. The following proof may be more appropriate.



Example 189 : As shown in Figure 1 , △in ABC , take point B *on* AB , △the circumscribed circle of BCD intersects AC *at* E , take point F *on the extension line of* ED , the circumscribed circle of BE and $\triangle AEF$ intersects at G , and at AB *Take point* H above , make $AG = AH$, extend GH to intersect BC at I , and prove that $BG = BI$.

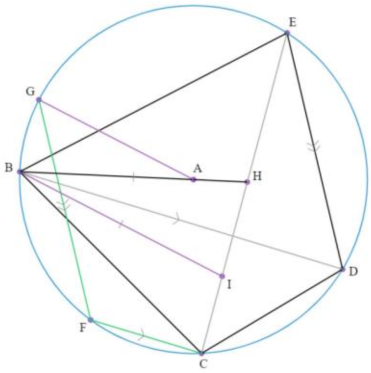
$$\frac{\frac{G-I}{G-B}}{\frac{C-B}{I-G}}=\frac{\frac{A-C}{E-F}}{\frac{B-C}{A-B}}\frac{\frac{F-E}{G-B}}{\frac{A-C}{A-G}}\frac{\frac{I-G}{B-A}}{\frac{G-A}{G-I}},$$

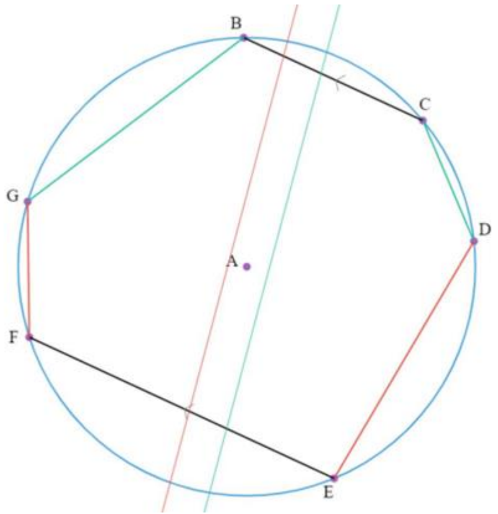
D and H do not appear in the identity , and these two points are virtually eliminated. If H is on the extension line *of* BA , is the conclusion still valid?



Example 1 90 : As shown in Figure 1 , circle A is inscribed with quadrilateral $BCDE$, F is the intersection point of BC and ED, G is the circumcenter of $\triangle CDF$, extend BA to intersect DE at H, let DG and EA *be symmetrical* about the straight line l_1, on the extension line of DE Take point I, GF and IB *are symmetrical* about the straight line l_2, and prove it $BI = BH \Leftrightarrow l_1 // l_2$.

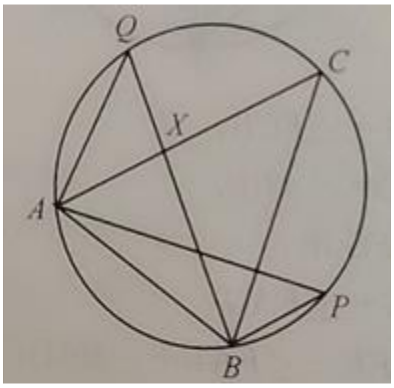
$$\frac{\frac{E-F}{D-G}\frac{H-B}{F-E}\frac{L}{I-B}\frac{D-G}{L}\frac{B-E}{B-H}}{\frac{F-G}{F-E}\frac{E-F}{I-B}\frac{G-F}{L}\frac{L}{E-A}\frac{E-A}{E-B}}\left(\frac{F-G}{F-E}\frac{E-F}{E-B}\right)^2=-1$$





Example 1 91 : As shown in Figure 1 , the circle inscribes the polygon $BCDEFG$, where FG and ED *are symmetrical* about the straight line l_1, BG and DC *are symmetrical* about the straight line l_2, then $EF//BC \Leftrightarrow l_1//l_2$.

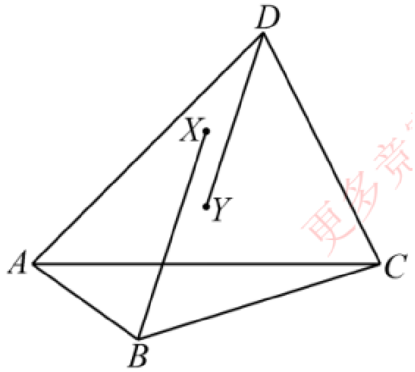
Proof: If known $l_1//l_2$, then by $\dfrac{\dfrac{B-C}{B-E}}{\dfrac{D-C}{E-D}}\dfrac{\dfrac{G-F}{B-G}}{\dfrac{E-F}{E-B}}\dfrac{\dfrac{L}{E-D}}{\dfrac{F-G}{L}}\dfrac{\dfrac{B-G}{L}}{\dfrac{L}{C-D}}\dfrac{E-F}{C-B}=1$, can be obtained $EF//BC$. If it is known $EF//BC$, then $\dfrac{\dfrac{B-C}{B-E}}{\dfrac{D-C}{E-D}}\dfrac{\dfrac{G-F}{B-G}}{\dfrac{E-F}{E-B}}\dfrac{\dfrac{l_1}{E-D}}{\dfrac{F-G}{l_1}}\dfrac{\dfrac{B-G}{l_2}}{\dfrac{l_2}{C-D}}\dfrac{E-F}{C-B}$ it is a positive real number because it $\left(\dfrac{l_1}{l_2}\right)^2$ is a positive real number, and it can be obtained $EF//BC$.



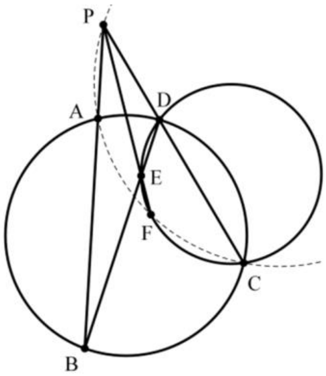
Example 1 92 : As shown in Figure 1 , *in* $\triangle ABC$, the high line $\triangle$passing through point A *intersects* the circumscribed circle *of* ABC at point P, X is a point on the line segment AC , and BX intersects the circle at Q. Proof: The necessary and sufficient condition for $BX= CX$ *is* PQ is the diameter of the circumscribed circle. (2003 British Mathematics Contest Questions)

$$\frac{\frac{\frac{B-C}{B-P}}{\frac{A-C}{A-P}}\frac{\frac{B-Q}{B-C}}{\frac{C-B}{C-A}}\frac{C-B}{A-P}\frac{B-P}{Q-B}}{}=1,$$

Example 1 93 : As shown in Figure 1 , *in* the convex quadrilateral $ABCD$, $\angle BAD = \angle DCB$, X and Y *are the circumcenters of* $\triangle ABC$ and $\triangle ACD$ *respectively* . Prove: $BX // DY$.

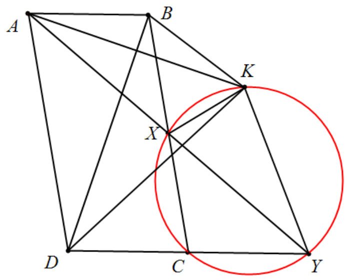


$$\frac{D-Y}{B-X}=\left(\frac{D-Y}{D-A}\frac{C-A}{C-D}\right)\left(\frac{B-A}{B-X}\frac{C-B}{C-A}\right)\frac{\dfrac{A-D}{A-B}}{\dfrac{C-B}{C-D}},$$



Example 194 : As shown in Figure 1 , A , B , C and D share a circle, BA intersects CD at P , point E extends PE on line segment BD , and intersects the circumscribed circle of $\triangle CDE$ at F. Prove: P , F , A , C are four points in a circle.

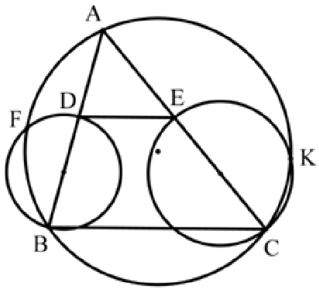
$$\frac{\frac{B-P}{A-C}}{\frac{F-P}{F-C}}=\frac{\frac{B-D}{P-C}}{\frac{F-P}{F-C}}\frac{\frac{B-P}{B-D}}{\frac{C-A}{C-P}},$$



Example 1 95 : As shown in Figure 1 , parallelogram $ABCD$, the angle bisector of $\angle A$ intersects BC *at* X , DC intersects *at* Y , and K and A are symmetrical about BD. Prove : C, X, Y, K *are* four points in a circle .

$$\left(\frac{\frac{D-A}{X-K}}{\frac{B-A}{Y-K}}\right)^2 \frac{\frac{A-B}{A-D}}{\frac{K-D}{K-B}} \frac{\frac{K-X}{K-B}}{\frac{D-A}{X-K}} \frac{\frac{B-A}{Y-K}}{\frac{K-Y}{K-D}} = 1.$$

Example 196 : As shown in Figure 1 , in $\triangle ABC$, if $DE // BC$ intersects AB and AC at points D and E *respectively,* the second intersection point between the circle whose diameter is BD and CE and the circumscribed circle of $\triangle ABC$ *is respectively Prove* for F and K : D, E, K, F are all circles. (Wanxiren's proposition)

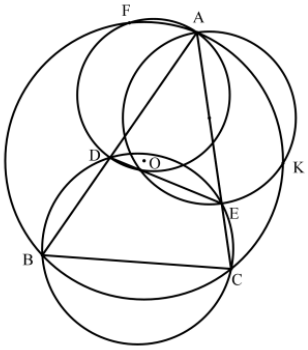


$$\frac{\frac{F-D}{F-K}\frac{F-K}{F-B}\frac{K-C}{K-E}}{\frac{D-E}{E-K}\frac{C-K}{B-C}\frac{F-D}{F-B}}\frac{D-E}{B-C}=1,$$

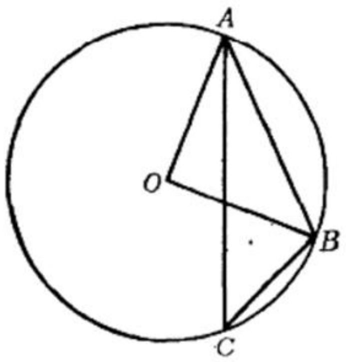
Note: This question has nothing to do with A. D, E, K, F four points share a circle $\Leftrightarrow DE // BC$. $\angle DFB=\angle CKE=90^{\circ}$ restrictions that can be removed .

90°

Example 197 : As shown in Figure 1 , *in* △ ABC , points D and E *are on* sides AB and AC respectively , so that ∠ADE = ∠ACB , the second intersection point between the circle whose diameter is AD *and* AE and the circumscribed circle of △ ABC *They are* F and K respectively . To prove: D , E , K and F share a circle.

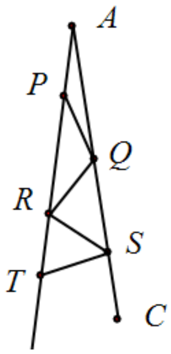


$$\frac{\frac{F-K}{F-D}\frac{B-A}{D-E}\frac{K-E}{K-A}\frac{A-K}{K-C}\frac{F-A}{F-K}}{\frac{E-K}{D-E}\frac{C-B}{C-A}\frac{F-A}{F-D}\frac{B-A}{B-C}\frac{C-A}{C-K}}=1$$



Example 1 98 : As shown in Figure 1 , O is $\triangle$the circumcenter of ABC, if $\angle ACB = \angle OAB$, then $AO \perp OB$.

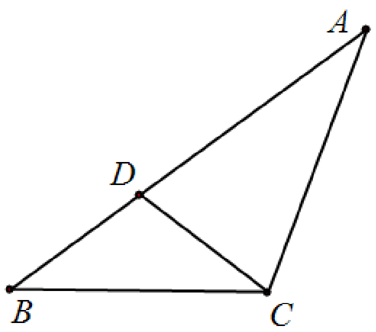
$$\left(\frac{O-B}{O-A}\right)^2 \frac{\frac{C-A}{C-B}\frac{C-O}{C-A}\frac{C-A}{C-B}\frac{B-C}{B-O}}{\frac{B-O}{B-A}\frac{A-C}{A-O}\frac{A-B}{A-O}\frac{C-O}{C-B}} = -1,$$



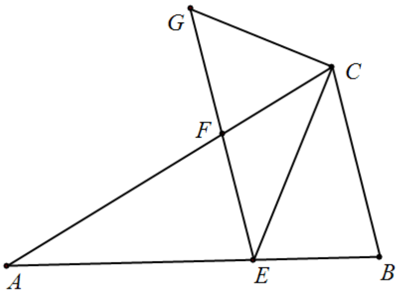
Example 199 : As shown in Figure 1 , it is known that $APRT$ and $AQSC$ are both straight lines , and $AP = PQ = QR = RS = ST$. Prove: $\angle CST = 5\angle AQP$.

$$\frac{\left(\frac{Q-P}{C-A}\right)^5}{\frac{A-C}{S-T}} = \left(\frac{\frac{Q-P}{C-A}}{\frac{A-C}{A-P}}\right)^4 \frac{\frac{P-Q}{A-P}}{\frac{A-P}{R-Q}} \frac{\frac{A-C}{Q-R}}{\frac{S-R}{C-A}} \frac{\frac{R-S}{A-P}}{\frac{A-P}{T-S}},$$

Example 200 : As shown in Figure 1 , $\triangle$in ABC, $\angle C = 3\angle B$, intercept $AD = AC$ on AB, and prove : $CD = DB$.



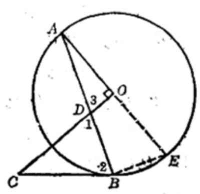
$$\left(\frac{\frac{C-B}{C-D}}{\frac{B-A}{B-C}}\right)^2 \frac{\left(\frac{B-A}{B-C}\right)^3}{\frac{C-B}{C-A}} \frac{\frac{C-D}{C-A}}{\frac{B-A}{D-C}} = 1,$$



Example 1 77 : As shown in Figure 1 , *in* △ABC, *the* bisector of ∠C intersects AB at E, *the* parallel line drawn from E to BC *intersects* AC at F, *and the* bisector of the exterior angle of ∠C intersects at G, then E $F = FG$.

Proof: Suppose $A=0$, $E=kB$, $F=kC$, $G=2F-E$, $\dfrac{\frac{C-G}{B-C}}{\frac{C-A}{C-G}}+\dfrac{\frac{C-E}{C-A}}{\frac{C-B}{C-E}}=4k-4k^2$.

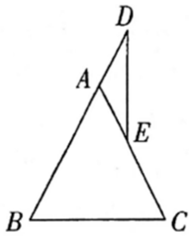
C is the origin



Example 178 : As shown in Figure 1 , AB is the chord of circle O , OC is perpendicular to OA , intersects AB at D, *and* intersects the tangent of B *at* C , then $BC = CD$.

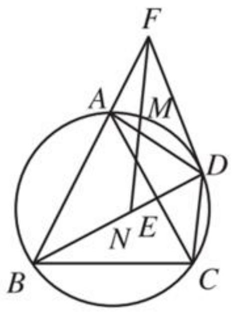
$$\frac{\frac{A-B}{O-C}}{\frac{B-C}{B-A}}=\frac{\frac{B-A}{B-O}}{\frac{A-O}{A-B}}\frac{\frac{A-O}{O-C}}{\frac{B-C}{B-O}},$$

Example 1 79 : As shown in Figure 1 , in $\triangle ABC$, $AB = AC$, D is a point on the extension line of BA, E is on AC, and $AD = AE$, to prove: $DE \perp BC$.

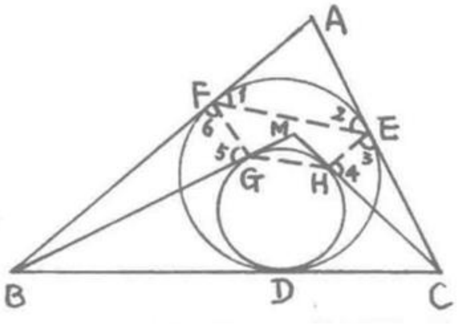


$$\frac{\dfrac{C-B}{C-A}}{\dfrac{B-A}{B-C}}\,\frac{\dfrac{C-A}{E-D}}{\dfrac{D-E}{A-B}}\left(\frac{E-D}{B-C}\right)^2=-1.$$

Example 1 80 : As shown in Figure 1 , *the diagonals* AC and BD of the inscribed quadrilateral $ABCD$ intersect at point E, and $AC \perp BD$, $AB = AC$, pass through point D and make $DF \perp BD$, intersect the extension line of BA at point F, the bisector of $\angle BFD$ intersects AD and BD at *points* M *and* N *respectively*. To prove : $\angle BAD = 3\angle DAC$.



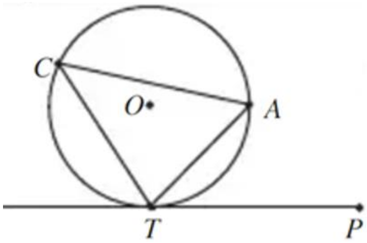
$$\frac{\left(\frac{A-D}{A-C}\right)^3}{\frac{A-D}{A-B}}=-\left(\frac{D-B}{C-A}\right)^2\left(\frac{\frac{A-D}{A-C}}{\frac{B-D}{B-C}}\right)^2\frac{\frac{B-A}{B-C}}{\frac{C-B}{C-A}},$$



Example 182 : As shown in Figure 1 , let M *be* a point inside △ABC △, the inscribed circle of ABC and side BC , the tangent points of CA, AB *are* D, E, F *respectively*, the △inscribed circle of MBC and sides BC , CM , the tangent points of MB are D , H , G , and prove that the four points E , F , G , and H *share a circle.*

$$\left(\frac{\frac{F-E}{F-G}}{\frac{H-E}{G-H}}\right)^2 \frac{\frac{B-A}{F-E}}{\frac{E-F}{C-A}} \frac{\frac{B-M}{G-H}}{\frac{H-G}{C-M}} \frac{\frac{F-G}{A-B}}{\frac{M-B}{G-F}} \frac{\frac{H-E}{M-C}}{\frac{A-C}{E-H}} = 1,$$

Explanation: This question looks complicated, but you only need to convert $AF = AE$, $MG = MH$, $BF = BD = BG$, $CD = CE = CH$, and these line segment relationships into angle representations.

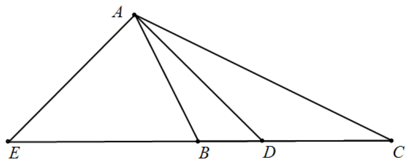


Example 1 83 : As shown in Figure 1 , the theorem of chord cutting angle : the degree of chord cutting angle is equal to half of the angle of the center of the arc subtended by it, and equal to the circumference angle of the arc subtended by it. TA is a chord on the circle O , TP is the tangent, C is a point on the circle, and is on the arc opposite to the chord $\angle PTA = \angle ACT$ AT .

Proof 1: $\angle PTA = 90^{\circ} - \angle ATO = \frac{1}{2}\left(180^{\circ} - \angle ATO - \angle OAT\right) = \frac{1}{2}(\angle AOT) = \angle ACT$.

Proof 2 : $$\left(\frac{\frac{T-A}{C-P}}{\frac{C-A}{C-T}}\right)^2 = -\left(\frac{T-O}{T-P}\right)^2 \frac{\frac{A-T}{A-O}}{\frac{T-O}{T-A}} \frac{\frac{O-A}{O-T}}{\left(\frac{C-A}{C-T}\right)^2}$$

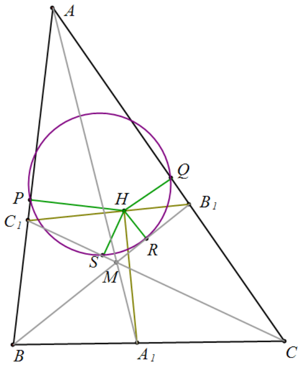
It can be found that the ideas of the two proofs are exactly the same, but the writing is different. The difference seems to be that Proof 1 is more concise. So what is the advantage of Proof 2? The identity formula shows: four conditions $\angle PTA = \angle ACT$, $TO \perp TP$, $OA = OT$, $2\angle ACT = \angle AOT$, if any three are known, it can be determined that the fourth one is also true.



Example 184 : As shown in Figure 1 , in $\triangle ABC$, D and E are respectively on CB and its extension line, $AD = AE$, $\angle BAD = \angle CAD$, $AD \perp AE$, to prove: $\angle B - \angle C = \frac{\pi}{2}$.

Proof: $A=0$, $\frac{\frac{B}{B-C}}{\frac{C-B}{C}} = \frac{\frac{C}{D}}{\frac{D}{B}} \frac{\frac{E}{B-C}}{\frac{C-B}{D}} \frac{D}{E}$.

In $\triangle ABC$, D and E are respectively on CB and its extension line. Prove: $AD = AE$, $\angle BAD = \angle CAD$, $\angle B - \angle C = \angle EAD$, if any two of these three conditions are known, we can get The third was established.

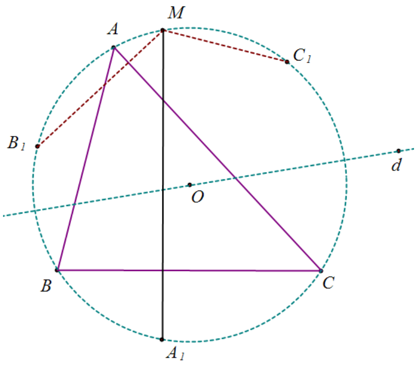


Example 186 : As shown in Figure 1 *, there is a point* M in $\triangle ABC$, AM, BM, and CM intersect opposite sides at A_1, B_1, C_1, and A_1H is perpendicular to B_1C_1 at H, passing through H to AB and AC, BB_1, CC_1 as vertical line segments HP, HQ, HR, HS, to prove: P, Q, R, S are four points that share a circle.

$$\frac{\frac{P-Q}{P-S}}{\frac{R-Q}{S-R}}=-\left(\frac{B_1-C_1}{H-A_1}\right)^2\frac{\frac{P-Q}{P-H}}{\frac{A-C}{A-H}}\frac{\frac{P-H}{P-S}}{\frac{C_1-B_1}{C_1-C}}\frac{\frac{C-A}{B_1-C_1}}{\frac{R-Q}{R-H}}\frac{\frac{M-H}{C-C_1}}{\frac{R-H}{R-S}}\frac{\frac{H-A_1}{H-M}}{\frac{H-A}{A_1-H}},$$

$\dfrac{\frac{H-A_1}{H-M}}{\frac{H-A}{A_1-H}}\in R$ Equivalent to $\angle AHA_1$ and $\angle MHA_1$ complementary to, equivalent to

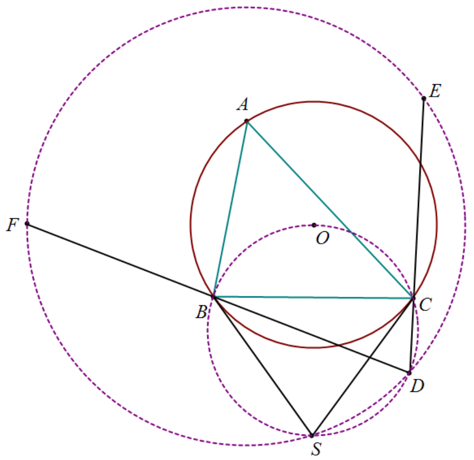
$\angle AHC_1=\angle MHC_1$, needs to be supplemented to prove this conclusion.



Example 187 : As shown in Figure 1 , *in* △ ABC, the straight line d *passes through* the circumcenter O *of* △ ABC, the symmetric points of A, B, and C with respect to d *are* A_1, B_1, C_1, and the straight line passing through A_1 is perpendicular to BC. The circle is in M, prove that $MB_1 \perp AC$, $MC_1 \perp AB$.

Proof: The following only proves $MB_1 \perp AC$.

Proof: $$\frac{B_1-M}{A-C}\frac{\frac{C_1-A_1}{C_1-B_1}}{\frac{C-B}{C-A}}\frac{\frac{M-A_1}{M-B_1}}{\frac{C_1-A_1}{C_1-B_1}}\frac{B-C}{M-A_1}=-1.$$

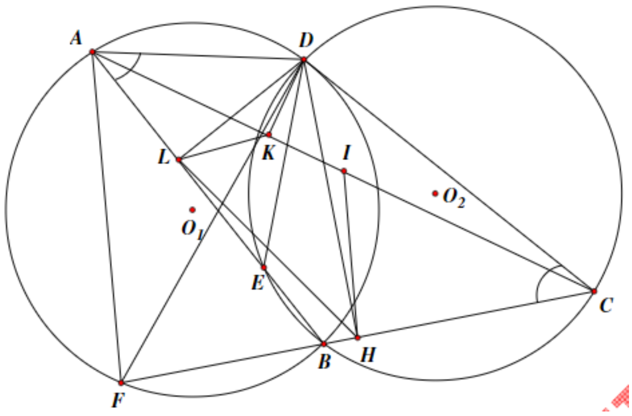


Example 188 : As shown in Figure 1 , SB and SC *are* two tangents of the circumscribed circle O *of* $\triangle ABC$, B and E are symmetrical about AC , C and F are symmetrical about AB , *and* BF intersects CE at D. Prove: B , C , D , The four points *of* S are in a circle.

Proof: $$\frac{\dfrac{B-S}{B-F}}{\dfrac{C-S}{C-E}}=\frac{\dfrac{B-A}{B-C}\dfrac{C-B}{C-A}\left(\dfrac{A-C}{A-B}\right)^2}{\dfrac{B-F}{B-A}\dfrac{C-A}{C-E}\dfrac{S-C}{B-S}},$$

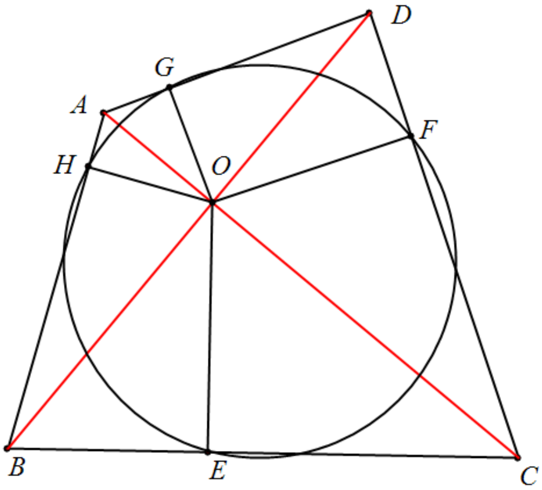
Explanation: It is also possible to study the four points of F , E , D , and S sharing a circle, and the five points of B , C , D , S , and O sharing a circle.

Example 189 : As shown in Figure 1 , the quadrilateral $ABCD$, $\angle DAB=\angle BCD$, draw perpendicular segments DL, DH, DK through D to AB, BC, CA, and I is the midpoint of AC. Prove: K, L, H, I 4 points in a circle.



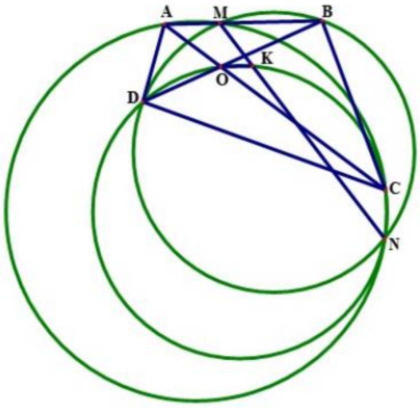
Proof: $$\frac{\frac{K-H}{K-L}}{\frac{I-H}{I-L}}=\frac{\frac{B-A}{A-D}}{\frac{K-L}{K-D}}\frac{\frac{C-D}{B-C}}{\frac{K-D}{K-H}}\frac{\frac{A-D}{A-B}}{\frac{F-D}{B-C}}\frac{\frac{F-A}{C-E}}{\frac{D-C}{D-F}}\frac{A-F}{I-H}\frac{L-I}{E-C}.$$

Explanation: Assuming that AB intersects $\odot O_2$ with E, it is easy to prove that $\triangle DAE$ is an isosceles triangle. Let BC intersect $\odot O_1$ at F, it is easy to prove that $\triangle DFC$ is an isosceles triangle. Then $LK \parallel EC$, $IH \parallel AF$, $\triangle DAE \backsim \triangle DFC$, $\triangle DAF \backsim \triangle DEC$



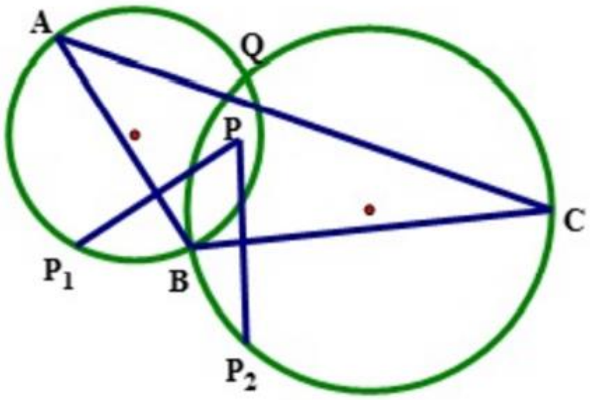
Example 1 90 : As shown in Figure 1 , the quadrilateral $ABCD$, AC intersects BD at O, and the feet of O on the four sides of BC, CD, DA, and AB are E, F, G, and H. Prove: The necessary and sufficient condition for $AC \perp BD$ is E, F, G, H are four points in a circle.

Proof: $$\frac{\frac{G-F}{G-H}}{\frac{E-F}{E-H}}=\left(\frac{A-C}{B-D}\right)^2\frac{\frac{G-O}{G-H}}{\frac{A-C}{A-B}}\frac{\frac{G-F}{G-O}}{\frac{D-C}{D-B}}\frac{\frac{E-H}{E-O}}{\frac{B-A}{B-D}}\frac{\frac{E-O}{E-F}}{\frac{C-A}{C-D}}.$$



Example 1 91 : As shown in Figure 1 , the quadrilateral $ABCD$, AC intersects BD at O, M is a point on AB, the circumcircle of $\triangle$ ACM *intersects* with the circumcircle of $\triangle$ BDM *at* N, to prove: B, O, C, N four points circle; if MN intersects the circumscribed circle of $\triangle$ BOC on K, then AB // OK.

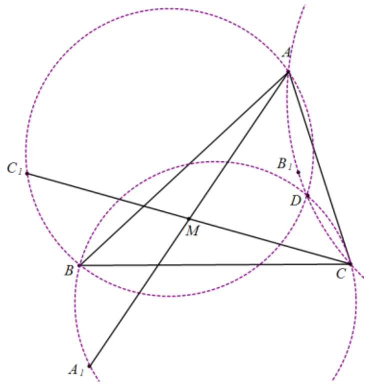
Proof: $\dfrac{\dfrac{A-C}{B-D}}{\dfrac{N-C}{D-N}}\dfrac{\dfrac{A-M}{A-C}}{\dfrac{N-M}{N-C}}\dfrac{\dfrac{B-D}{B-M}}{\dfrac{N-D}{N-M}}\dfrac{M-B}{A-M}=1$, $\dfrac{O-K}{A-B}\dfrac{\dfrac{N-M}{N-C}}{\dfrac{O-K}{A-C}}\dfrac{\dfrac{A-B}{A-C}}{\dfrac{N-M}{N-C}}=1$.



Example 1 92 : As shown in Figure 1 , point P on the $\triangle$ ABC *plane* , P , P_1 is symmetric about AB , P , is symmetric P_2 about BC , P , is symmetric P_3 about CA , prove that the circumscribed circles of $\triangle ABP_1$, $\triangle BCP_2$, $\triangle CAP_3$ intersect at one point.

Proof: Assuming that the circumscribed circle of $\triangle ABP_1$, $\triangle BCP_2$ intersects at Q , it is only necessary to prove *that* $\angle$AQC = $\angle APC$.

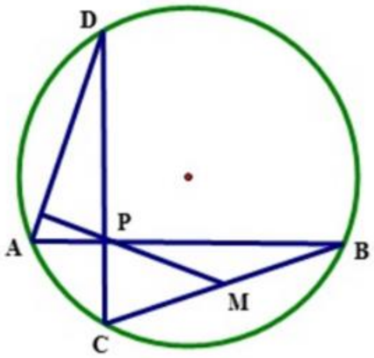
Proof: $\dfrac{\frac{Q-C}{Q-A}}{\frac{P-A}{P-C}}=\dfrac{\frac{Q-B}{Q-A}}{\frac{B-P_1}{P_1-A}}\dfrac{\frac{P-B}{P-A}}{\frac{P_1-A}{P_1-B}}\dfrac{\frac{Q-C}{Q-B}}{\frac{P_2-C}{B-P_2}}\dfrac{\frac{P-C}{P-B}}{\frac{P_2-B}{P_2-C}}$.



Example 1 93 : As shown in Figure 1 , *point* M in $\triangle ABC$, extend AM to AA_1, make $AM=A_1M$, extend BM to BB_1, make $BM=B_1M$, extend CM to CC_1, make $CM=C_1M$, to prove: the circumscribed circles of $\triangle ABC_1$, $\triangle A_1BC$, $\triangle AB_1C$ intersect at point D.

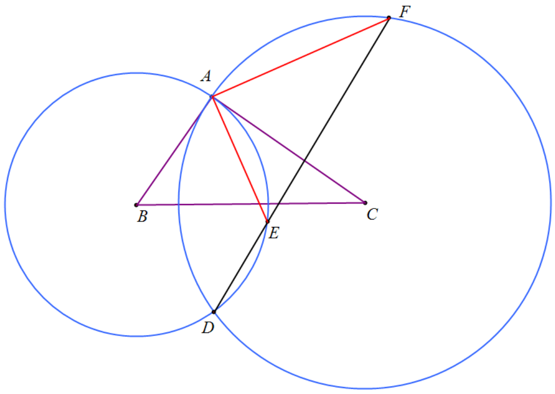
Proof: Assume that the circumscribed circles of $\triangle ABC_1$ and $\triangle A_1BC$ intersect at point D, and then prove that D *is* on the circumscribed circle of $\triangle AB_1C$.

Proof: $$\frac{\frac{D-A}{D-C}}{\frac{B_1-A}{B_1-C}}=\frac{A_1-B}{B_1-A}\frac{C_1-A}{A_1-C}\frac{B_1-C}{C_1-B}\frac{\frac{B-D}{D-C}}{\frac{A_1-B}{A_1-C}}\frac{\frac{D-A}{B-D}}{\frac{C_1-A}{C_1-B}},$$



Example 1 94 : As shown in Figure 1 , it is known that $AB \perp CD$, AB intersects CD at P, and M is the midpoint of BC. Prove: The necessary and sufficient condition for $PM \perp AD$ is that A, C, B, and D share a circle.

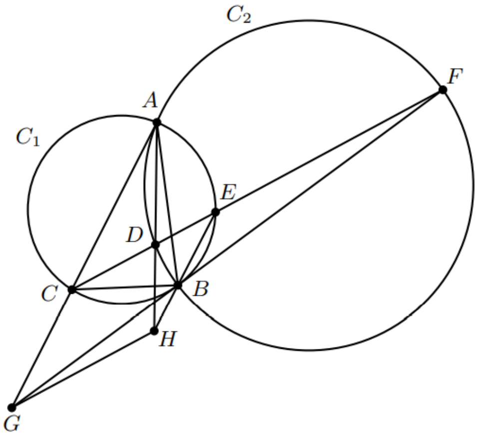
Proof: $\frac{P-M}{D-A}=\frac{\frac{P-M}{P-C}}{\frac{C-D}{C-B}}\frac{\frac{C-D}{C-B}}{\frac{A-D}{A-B}}\frac{C-P}{A-B}$,



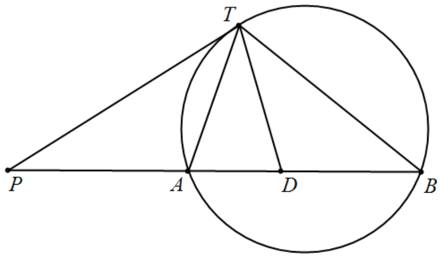
Example 195 : As shown in Figure 1 , $\triangle ABC$, $AB \perp AC$, take B, C as the center, BA, CA as the radius to draw a circle, intersection point A, D, there is E *on* circle B, extend DE to intersect DE at F, verify : $AE \perp AF$.

Proof: $\dfrac{A-E}{A-F}=\dfrac{\dfrac{D-F}{D-C}\dfrac{A-E}{A-D}\dfrac{F-C}{F-A}\dfrac{A-D}{A-B}}{\dfrac{F-C}{F-D}\dfrac{D-F}{D-C}\dfrac{A-F}{A-C}\dfrac{F-D}{F-A}}\dfrac{A-B}{A-C}$, Example 196 : As shown in Figure 1 , two circles intersect at points A and B, and a straight line intersects two circles at four points D, D, E, and F. G is the intersection of AC and BF, and H is the intersection of AD and BE. Prove: GH // CF.

Proof: As a $\dfrac{\frac{H-A}{H-E}}{\frac{G-A}{G-F}}=\dfrac{\frac{B-A}{H-E}}{\frac{G-A}{C-F}}\dfrac{\frac{H-A}{C-F}}{\frac{B-A}{G-F}}$ result $\angle AGB=\angle AHB$, the four points $\angle ACF=\angle ABE=\angle AGH$ A, B, H, and G *share a circle*, .

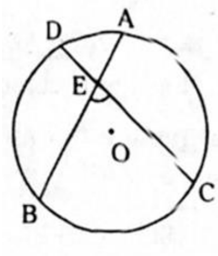


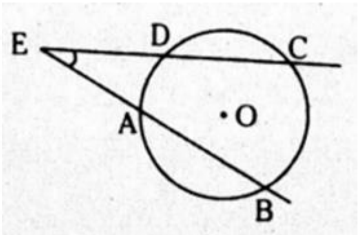
Explanation: Since the *GH line is added last and does not appear in any conditions, it is impossible to obtain terms like* $G-F$ by conditional simplification, so it is difficult to solve the identity directly in one step, and can only be handled around the corner .



Example 1 97 : As shown in Figure 1 , in △ ABT, draw a tangent to the circumscribed circle of △ ABT *through* T, *intersect* AB at P, and TD bisect ∠ATB .
Prove: $PD = PT$.

Proof: $\dfrac{\dfrac{B-P}{D-T}}{\dfrac{T-D}{T-P}} = \dfrac{\dfrac{T-B}{T-D}}{\dfrac{T-D}{T-A}} \dfrac{\dfrac{B-P}{B-T}}{\dfrac{T-A}{T-P}}$,



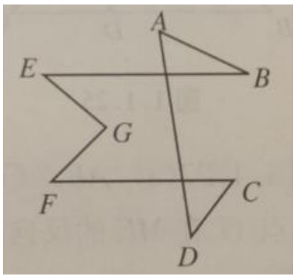


Example 1 98 : *As* shown in Figure 1 , ***there are*** A , B , C , and D on circle O , and AB intersects CD with E. Prove: $2\angle CEB = \angle COB + \angle DOA$.

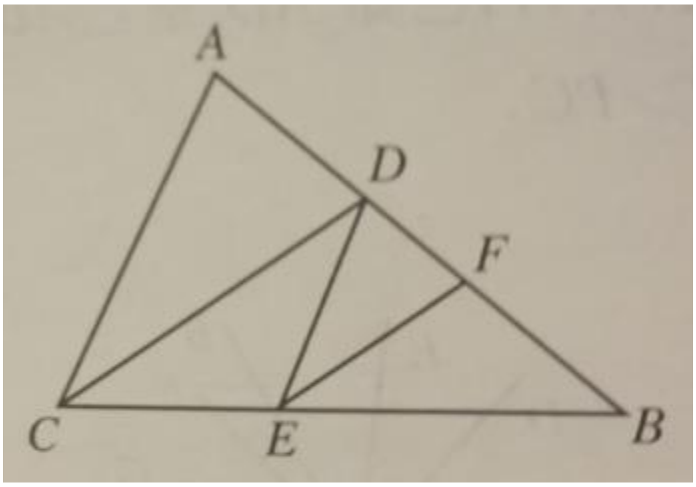
Proof: $$\frac{\left(\frac{D-C}{A-B}\right)^2}{\frac{O-C}{O-B}\frac{O-D}{O-A}}=\frac{\frac{D-C}{D-O}\frac{A-O}{A-B}}{\frac{C-O}{C-D}\frac{B-A}{B-O}},$$

Explanation: The identities use directed angles. If a conventional angle is used, as shown in Figure 1, $2\angle CEB = \angle COB + \angle DOA$. As shown *in* Figure 2 , $2\angle CEB = \angle COB - \angle DOA$.

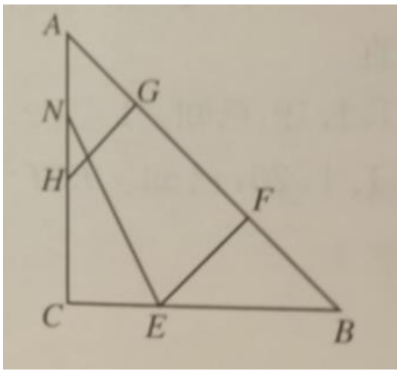
Example 1 99 : As shown in Figure 1 , it is known that $\angle EGF = \angle BEG + \angle CFG$. Proof: $\angle A + \angle B + \angle C + \angle D = 180°$.



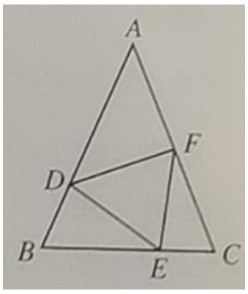
Proof: $\dfrac{\dfrac{E-B}{E-G}\dfrac{F-G}{F-C}}{\dfrac{G-F}{G-E}} = -\dfrac{A-B}{A-D}\dfrac{B-E}{B-A}\dfrac{C-D}{C-F}\dfrac{D-A}{D-C}$, Example 200 : As shown in Figure 1 , it is known that CD bisects $\angle ACB, AC // DE, CD // EF$. Prove: EF bisects $\angle DEB$.



Proof: $\dfrac{\dfrac{E-D}{E-F}}{\dfrac{E-F}{C-B}}=\dfrac{\dfrac{C-A}{C-D}}{\dfrac{C-D}{C-B}}\dfrac{E-D}{C-A}\left(\dfrac{C-D}{E-F}\right)^2$, Example 2 0 1: As shown in Figure 1 , in $\triangle ABC$, $\angle ACB=90°$, straight line EF intersects sides CB and AB *at points* E and F *respectively,* straight line HG intersects sides AC and AB *at points* H and G *respectively,* And $HG // EF$. *To* prove: ∠CEF − ∠AHG =90° .



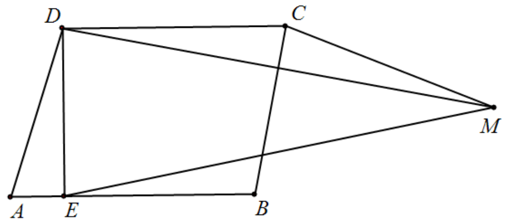
Proof: $\dfrac{\dfrac{C-B}{E-F}}{\dfrac{C-A}{H-G}}=\dfrac{H-G}{E-F}\dfrac{C-B}{C-A}$,



Example 202 : As shown in Figure 1 , in $\triangle ABC$, $AB = AC$, D, E, F are on AB, BC, CA respectively , and $DE = EF = FD$. To prove: $2\angle DEB = \angle ADF + \angle CFE$.

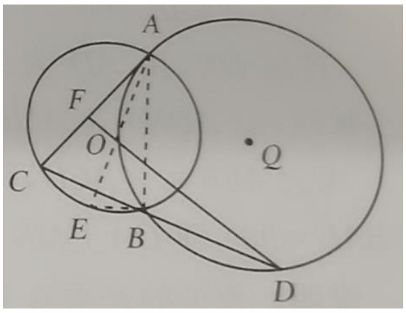
Proof: $$\frac{\left(\frac{C-B}{E-D}\right)^2}{\frac{B-A}{D-F}\frac{A-C}{F-E}} = \frac{\frac{C-B}{C-A}\frac{D-F}{D-E}}{\frac{B-A}{B-C}\frac{E-D}{E-F}},$$

Example 203 : As shown in Figure 1 , in the parallelogram $ABCD$, $DE \perp AB$ at point E, $MD = ME$, $MC = CD$. Prove: $\angle EMC = 3\angle BEM$.



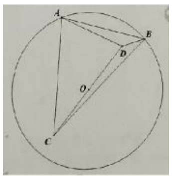
Proof: $$\frac{\frac{M-E}{M-C}}{\left(\frac{E-M}{A-B}\right)^3}=\left(\frac{A-B}{D-E}\right)^4\frac{\frac{M-D}{M-C}}{\frac{A-B}{D-M}}\left(\frac{\frac{E-D}{E-M}}{\frac{D-M}{D-E}}\right)^2,$$

$AB \parallel DC$ is used in the identity , and there is no need for the quadrilateral $ABCD$ to be a parallelogram.



Example 204 : As shown in Figure 1 , it is known that circle O and circle Q intersect at points A and B, circle Q passes through point O, C is a point on the superior arc AB *of* circle O, *and* the extension line of CB intersects circle Q at point D. Prove: $DO \perp AC$.

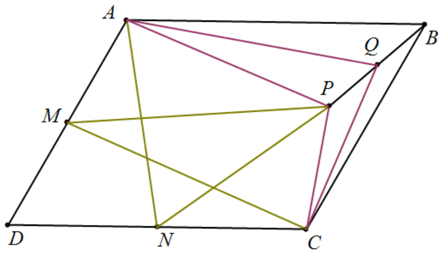
Proof: $\frac{A-C}{D-O}=\frac{\frac{C-A}{C-B}}{\frac{B-C}{B-A}}\frac{\frac{D-B}{D-O}}{\frac{A-B}{A-O}}\frac{C-B}{D-B}\frac{B-C}{A-O}$, Example 205 : As shown in Figure 1 , A and B are two points on O O, and O is the midpoint of CD. *Prove* : *The* necessary and sufficient condition for ∠BAD + ∠BAC =90 ° is ∠ABD + $\angle ABC$ =90° . (Huang Libing proposition)



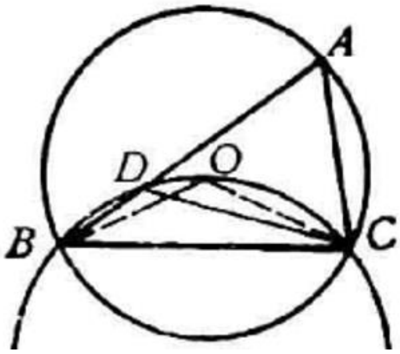
Proof: $\frac{A-D}{A-B}\frac{A-C}{A-B}-\frac{B-D}{B-A}\frac{B-C}{B-A}=2\frac{\frac{A+B}{2}-\frac{C+D}{2}}{A-B}$,

ABCD is parallelogram and P a point inside, such that the midpoint of AD is equidistant from P and C, and the midpoint of CD is equidistant from P and A. Let Q be the midpoint of PB. Show that ∠PAQ = ∠PCQ.

Example 206 : As shown in Figure 1 , P is a point inside the parallelogram $ABCD$, M and N *are the midpoints of* DA and DC respectively , $MP = MC$, $NA = NP$, proof: $\angle PAQ = \angle PCQ$.

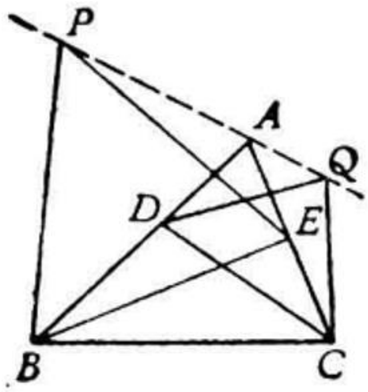


prove: $$\frac{\dfrac{A-\frac{P+B}{2}}{A-P}}{\dfrac{C-P}{C-\frac{P+B}{2}}}=\frac{\frac{A+C-B+C}{2}-\frac{A+P}{2}}{P-A}\frac{\frac{A+C-B+A}{2}-\frac{C+P}{2}}{P-C}$$



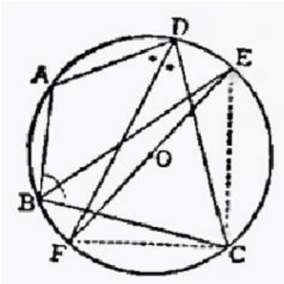
Example 207 : As shown in Figure 1 , O *is* the circumcenter of $\triangle ABC$, *and* D is a point on AB. If B, C, O, and D share a circle, prove: $DA = DC$.

prove: $$\frac{\frac{A-C}{A-B}}{\frac{C-D}{C-A}}=\frac{\frac{O-C}{O-B}}{\frac{D-C}{A-B}}\frac{\frac{A-C}{A-O}}{\frac{C-O}{C-A}}\frac{\frac{A-O}{A-B}}{\frac{B-A}{B-O}}$$



Example 208 : As shown in Figure 1 , D and E *are* the points on the sides AB and AC respectively $\triangle ABC$. Make isosceles $\triangle BEP$ and isosceles $\triangle CDQ$ to make the vertices $\angle P=\angle Q=\angle A$. Prove that the three points P, A and Q are collinear .

prove: $\dfrac{P-A}{A-Q}=\dfrac{\dfrac{E-B}{E-P}}{\dfrac{A-B}{A-P}}\dfrac{\dfrac{C-D}{C-Q}}{\dfrac{E-B}{E-P}}\dfrac{\dfrac{Q-C}{Q-A}}{\dfrac{D-C}{A-B}}$



Example 209 : As shown in Figure 1 , *if the intersections of the* bisectors of the diagonals B and D *of* the inscribed quadrilateral $AECD$ and the circumscribed circle are E and F *respectively*, then EF is the diameter of the circle.

prove: $-\left(\frac{C-E}{C-F}\right)^2=\frac{\frac{B-E}{B-C}\frac{D-C}{D-F}\frac{B-A}{B-C}}{\frac{B-A}{B-E}\frac{D-F}{D-A}\frac{A-D}{D-C}}\left(\frac{\frac{E-C}{E-F}}{\frac{D-C}{D-F}}\right)^2\left(\frac{\frac{B-C}{B-E}}{\frac{F-C}{F-E}}\right)^2$

Another proof: $\angle CEF=\angle CDF=\frac{1}{2}\angle ADC$, $\angle CFE=\angle CBE=\frac{1}{2}\angle ABC$,

$\angle CEF+\angle CFE=\frac{1}{2}(\angle ADC+\angle ABC)=90^\circ$, thus $\angle ECF=90^\circ$.

USAJMO 2012

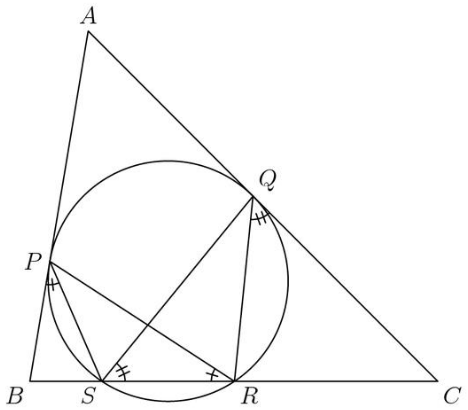
www.artofproblemsolving.com/community/c3975

by BOGTRO, tc1729, rrusczyk

Day 1 April 24th

1 Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic (in other words, these four points lie on a circle).

Example 210 : As shown in Figure 1 , given $\triangle ABC$, P and Q *are points* on sides AB and AC *respectively* , and $AP = AQ$; S and R are two different points on side BC , and point S is located between B and R , ∠BPS = ∠PRS , ∠CQR = ∠QSR . _ _ Proof: The four points P , Q , R , S share a circle. (2012 USA Mathematical Olympiad *USAJMO* test questions)



$$\frac{\frac{P-S}{P-B}}{\frac{C-B}{R-P}}\frac{\frac{S-Q}{B-C}}{\frac{Q-C}{Q-R}}\frac{\frac{Q-C}{Q-P}}{\frac{P-Q}{P-B}}\left(\frac{P-Q}{P-S}/\frac{R-Q}{B-C}\right)\left(\frac{C-B}{R-P}/\frac{Q-S}{Q-P}\right)=1$$

This question is a bit confusing to understand. It is easy to understand if it is said another way: As shown in the figure, given $\triangle ABC$, P and Q *are* points on sides AB and AC respectively , S and R are two different points on side BC , *and point* S *is* located between B and R , ∠BPS = $\angle PRS$, ∠CQR = ∠QSR . _ _ If P , Q , R , S share a circle , prove that $AP = AQ$.

Full－ angle method

Example 81 *If the two bisectors of the angle A of the triangle ABC are equal, and the circle having BC for diameter cuts the sides AB, AC in the points P, Q, show that $CP \equiv CQ$.*

Point order: u, v, a, b, c, o, p, q.

Hypotheses: perp(a, u, a, v), cong(a, u, a, v), coll(u, v, b), eqangle(c, a, u, u, a, b), coll(u, v, c), midpoint(o, b, c), coll(a, b, p), coll(a, c, q), pbisector(o, b, q), pbisector(o, b, p).

Conclusion: pbisector(c, p, q).

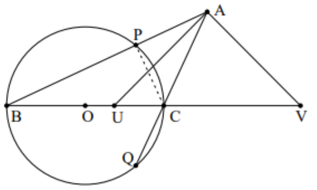


Figure 81

The Machine Proof

$-[qp, qc] - [qp, pc]$

(Since q, p, c, b are cyclic; $[qp, qc] = [pb, cb]$.)

$= -[qp, pc] - [pb, cb]$

(Since $pc \perp ab$; $[qp, pc] = [qp, ba] + 1$.)

$= -[qp, ba] - [pb, cb] - 1$

(Since a, c, q are collinear; q, p, c, b are cyclic; $[qp, ba] = [qp, qc] + [ca, ba] = [pb, cb] + [ca, ba]$.)

$= -2[pb, cb] - [ca, ba] - 1$

(Since a, b, p are collinear; $[pb, cb] = -[cb, ba]$.)

$= 2[cb, ba] - [ca, ba] - 1$

(Since b, c, u, v are collinear; $[cb, ba] = -[ba, vu]$.)

$= -[ca, ba] - 2[ba, vu] - 1$

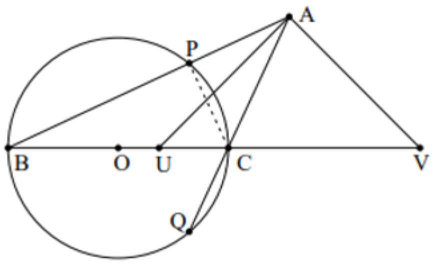
(Since $\angle[ac, ab] = \angle[ua, ab]$; $[ca, ba] = [ca, ua] + [ua, ba] = -2[ba, au]$.)

$= 2[ba, au] - 2[ba, vu] - 1$

(Since $ba \parallel ba$; $2[ba, au] - 2[ba, vu] = -2[au, vu]$.)

$= -2[au, vu] - 1$

(Since $au = av$ $ua \perp av$; $[au, vu] =_1 423772$.)



Example 2 11 : As shown in the figure, in $\triangle ABC$, AU and AV are the bisectors of the inner and outer angles of $\angle A$ respectively. A circle with BC as *the* diameter intersects AB and AC at P and Q *respectively*. If $AU = AV$, then $CP = CQ$.

$$\frac{\frac{P-C}{P-Q}}{\frac{Q-P}{C-A}} = \frac{\frac{B-A}{B-C}}{\frac{Q-P}{C-A}} \frac{\frac{P-C}{P-Q}}{\frac{B-C}{B-Q}} \frac{\frac{C-B}{V-A}}{\frac{U-A}{B-C}} \frac{\frac{A-U}{A-B}}{\frac{A-C}{A-U}} \frac{A-V}{A-U} \frac{A-C}{B-Q}$$

Example 78 *ABC is triangle inscribed in a circle; DE is the diameter bisecting BC at G; from E a perpendicular EK is drawn to one of the sides, and the perpendicular from the vertex A on DE meets DE in H. Show that EK touches the circle GHK.*

Point order: $a, b, c, o, g, e, k, h, n$.

Hypotheses: circumcenter(o, b, a, c), midpoint(g, b, c), coll(g, o, e), pbisector(o, b, e), perp(k, e, a, b), coll(k, a, b), perp(a, h, o, g), coll(h, o, g), circumcenter(n, g, h, k).

Conclusion: perp(e, k, k, n).

The Machine Proof

$-[nk, ke] + 1$

(Since $ke \perp ab$; $[nk, ke] = [nk, ba] + 1$.)

$= -[nk, ba]$

(Since circumcenter(n,k, g, h); $[nk, ba] = [nk, kg] + [kg, ba] = [hk, hg] + [kg, ba] + 1$.)

$= -[hk, hg] - [kg, ba] - 1$

(Since $hg \perp bc$; $[hk, hg] = [hk, cb] + 1$.)

$= -[hk, cb] - [kg, ba]$

(Since e, g, h are collinear; h, k, e, a are cyclic; $[hk, cb] = [hk, he] + [eg, cb] = [ka, ea] + [eg, cb]$.)

$= -[kg, ba] - [ka, ea] - [eg, cb]$

(Since a, b, k are collinear; k, g, b, e are cyclic; $[kg, ba] = [kg, kb] = [eg, eb]$.)

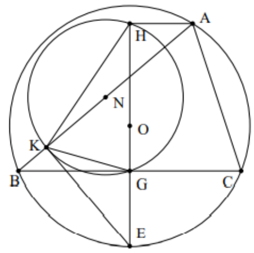


Figure 78

$= -[ka, ea] - [eg, eb] - [eg, cb]$

(Since a, b, k are collinear; $[ka, ea] = -[ea, ba]$.)

$= -[eg, eb] - [eg, cb] + [ea, ba]$

(Since $eg \perp bc$; $[eg, eb] = -[eb, cb] + 1$.)

$= -[eg, cb] + [eb, cb] + [ea, ba] - 1$

(Since $eg \perp cb$; $[eg, cb] = 1$.)

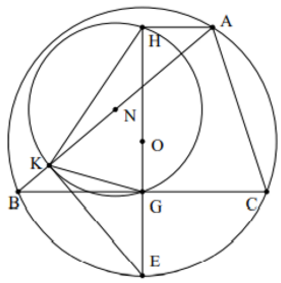
$= [eb, cb] + [ea, ba]$

(Since b, e, c, a are cyclic; $[eb, cb] = [ea, ca]$.)

$= [ea, ca] + [ea, ba]$

(Since circumcenter(o,a, e, c, b); $oc \perp eb$; $[ea, ca] = -[ea, ba]$.)

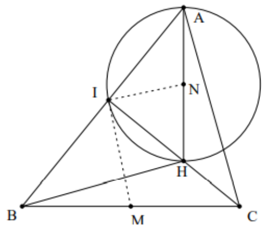
$= 0$



Example 2 12 : As shown in the figure, in $\triangle ABC$, O is the circumcenter, E *is* the midpoint of the inferior arc BC , G is the midpoint of BC , $EK \perp AB$ is at K , $AH \perp GE$ is at H , and N is the circumcenter of $\triangle GHK$, to prove: E is on AB .

$$\frac{N-K}{B-K}=\left(\frac{H-G}{H-K}\frac{K-N}{K-G}\right)\frac{\left(\frac{E-G}{G-H}\frac{A-K}{A-B}\right)}{\left(\frac{A-E}{A-B}\frac{E-B}{E-H}\right)}\frac{\frac{A-E}{A-K}\frac{K-G}{K-B}}{\frac{H-E}{H-K}\frac{E-G}{E-B}}$$

Example 66 *Show that in a triangle ABC the circles on AH and BC as diameters are orthogonal.*



Point order: a, b, c, i, h, n, m.

Hypotheses: foot(i, c, a, b), coll(h, c, i), perp(h, a, b, c), midpoint(m, c, b), midpoint(n, a, h).

Conclusion: perp(m, i, n, i).

The Machine Proof

$[mi, ni] + 1$

(Since circumcenter(m,i, b, c); $[mi, ni] = [mi, ib] + [ib, ni] = -[ni, ib] + [ic, cb] + 1$.)

$= -[ni, ib] + [ic, cb]$

(Since circumcenter(n,i, a, h); $[ni, ib] = [ni, ia] = [hi, ha] + 1$.)

$= -[hi, ha] + [ic, cb] - 1$

(Since $hi \parallel ic$; $-[hi, ha] + [ic, cb] = [ha, cb]$.)

$= [ha, cb] - 1$ (Since $ha \perp cb$; $[ha, cb] = 1$.)

$= 0$

Example 2 13 : As shown in the figure, in $\triangle ABC$, H is the orthocenter, CI is the height, M and N *are the midpoints of* BC and AH respectively , to prove: $IM \perp IN$.

$$\frac{I-M}{I-N}=\frac{\left(\frac{I-M}{A-B}\right)^2}{\frac{M-I}{B-C}}\frac{\left(\frac{B-A}{I-N}\right)^2}{\frac{A-H}{N-I}}\frac{A-H}{B-C}$$

Example 58 *Let A, B, C, D be four points on circle* (O). $E = CD \cap AB$. *CB meets the line passing through E and parallel to AD at F. GF is tangent to circle* (O) *at G. Show that* $FG = FE$.

Point order: a, b, c, d, e, f.

Hypotheses: cyclic(a, b, c, d), coll(e, a, b), coll(e, c, d), coll(f, b, c), para(f, e, a, d).

Conclusion: eqangle(f, e, b, e, c, b).

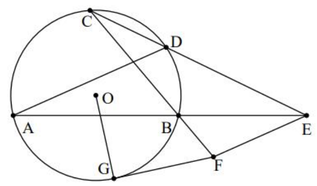


Figure 58

The Machine Proof

$[fe, eb] - [ec, cb]$

(Since $fe \parallel ad$; $[fe, eb] = -[eb, da]$.)

$= -[ec, cb] - [eb, da]$

(Since c, d, e are collinear; $[ec, cb] = [dc, cb]$.)

$= -[eb, da] - [dc, cb]$

(Since a, b, e are collinear; $[eb, da] = -[da, ba]$.)

$= -[dc, cb] + [da, ba]$

(Since c, d, b, a are cyclic; $[dc, cb] = [da, ba]$.)

$= 0$

Example 2 14 : As shown in the figure, four points A, B, C and D *are on the circle* O, AB intersects CD at E, make $EF \mathbin{//} AD$ intersect CB at F, and draw the tangent line FG *of* circle O *through* F, to prove: $FG = FE$.

$$\frac{\frac{C-E}{C-F}}{\frac{E-F}{E-B}} = \frac{C-B}{C-F}\frac{A-D}{F-E}\frac{C-E}{C-D}\frac{B-E}{A-B}\frac{\frac{C-D}{C-B}}{\frac{A-D}{A-B}}$$

Explanation: To change the certificate $FG^2 = FE^2 = FB \cdot FC$, it means to prove that $\triangle CFE \backsim \triangle EFB$, that is, to prove that $\angle FEB = \angle ECB$.

Example 64 *Given two circles* (A), (B) *intersecting in* E, F, *show that the chord* E_1F_1 *determined in* (A) *by the lines* MEE_1, MFF_1 *joining* E, F *to any point* M *of* (B) *is perpendicular to* MB.

Point order: $e, f, m, b, d, a, e1, f1$.

Hypotheses: circumcenter(b, e, f, m), midpoint(d, e, f), coll(a, d, b), coll$(e1, m, e)$, cong$(e1, a, e, a)$, coll$(f1, m, f)$, cong$(f1, a, e, a)$.

Conclusion: perp$(e1, f1, m, b)$.

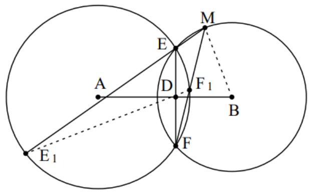


Figure 64

The Machine Proof

$[f1e1, bm] + 1$

(Since f, $f1$, m are collinear; $f1$, $e1$, f, e are cyclic; $[f1e1, bm] = [f1e1, f1f] + [fm, bm] = [e1e, fe] - [bm, mf]$.)

$= [e1e, fe] - [bm, mf] + 1$

(Since e, $e1$, m are collinear; $[e1e, fe] = [me, fe]$.)

$= -[bm, mf] + [me, fe] + 1$ (Since circumcenter(b,m, f, e); $[bm, mf] = [bm, mf] = [me, fe] + 1$.)

$= 0$

Example 2 15 : As shown in the figure, circle A and circle B intersect at points E and F, *and there is a point* M on circle B , extend ME and MF to intersect circle A at E_1 and F_1, and prove $E_1F_1 \perp MB$.

$$\frac{M-B}{E_1-F_1}=\frac{\frac{F_1-F}{F_1-E_1}}{\frac{E-F}{E-E_1}}\left(\frac{E-F}{E-E_1}\frac{M-B}{F-F_1}\right)$$

Example 23 *From the midpoint C of arc AB of a circle, two secants are drawn meeting line AB at F, G, and the circle at D and E. Show that F, D, E, and G are on the same circle.*

Point order: a, c, d, e, o, m, f, g.

Hypotheses: cong(o, a, o, c), cong(o, a, o, d), cong(o, a, o, e), coll(m, c, o), perp(m, a, c, o), coll(f, a, m), coll(f, c, d), coll(g, a, m), coll(g, c, e).

Conclusion: $[ce, fg] + [cd, de]$.

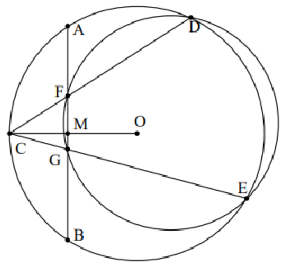


Figure 23

The Machine Proof

$-[gf, ec] - [ed, dc]$

(Since a, f, g, m are collinear; $[gf, ec] = [ma, ec]$.)

$= -[ma, ec] - [ed, dc]$

(Since $ma \perp co$; $[ma, ec] = [oc, ec] + 1$.)

$= -[oc, ec] - [ed, dc] - 1$

(Since circumcenter(o,c, e, d); $[oc, ec] = [oc, ce] = -[ed, dc] + 1$.)

$= 0$

Example 2 16 : As shown in the figure, there are five points A, B, C, D, E on circle O, $CO \perp AB$, CD, CE intersect AB at F, G, to prove: F, G, E, D share a circle .

$$\frac{\frac{D-E}{D-C}}{\frac{E-C}{G-F}}=\frac{F-G}{C-O}\left(\frac{C-O}{C-E}\frac{D-E}{D-C}\right)$$

Example 67 *The circle IBC is orthogonal to the circle on I_bI_c as diameter.*

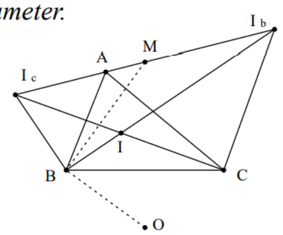


Figure 67

Point order: $a, b, c, i, o, b1, c1, m$.

Hypotheses: incenter(i, a, b, c), circumcenter(o, b, c, i), coll$(b1, b, i)$, perp$(b1, c, c, i)$, coll$(c1, c, i)$, perp$(c1, b, b, i)$, midpoint$(m, b1, c1)$.

Conclusion: perp(m, b, o, b).

The Machine Proof

$[mb, ob] + 1$

(Since circumcenter(m,b, c, $b1$); $[mb, ob] = [mb, bc] + [bc, ob] = -[b1c, b1b] - [ob, cb] + 1$.)

$= -[b1c, b1b] - [ob, cb]$

(Since $b1c \perp ci$; $[b1c, b1b] = -[b1b, ic] + 1$.)

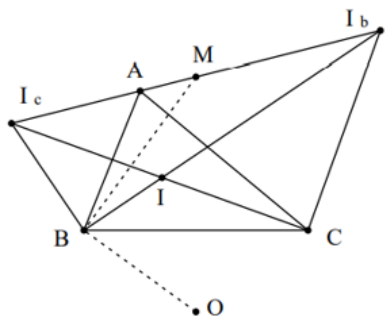
$= [b1b, ic] - [ob, cb] - 1$

(Since b, $b1$, i are collinear; $[b1b, ic] = -[ic, ib]$.)

$= -[ob, cb] - [ic, ib] - 1$

(Since circumcenter(o,b, c, i); $[ob, cb] = [ob, bc] = -[ic, ib] + 1$.)

$= 0$



Example 2 17 : As shown in the figure, in △ ABC , I is the inner, I_b , are I_c the circumcenters of B and C respectively , M is I_bI_c the midpoint, O *is* the circumcenter of △ BCI *, prove* $BM \perp BO$.

$$\frac{B-M}{B-O}=-\left(\frac{B-M}{B-C}\frac{I_b-C}{I_b-B}\right)\left(\frac{B-C}{B-O}\frac{B-I_b}{I-C}\right)\frac{I-C}{I_b-C}$$

Example 84 *The tangent to the nine-point circle at the midpoint of a side of the given triangle is antiparallel to this side with respect to the two other sides of the triangle.*

Point order: $b, c, a, a_1, b_1, c_1, n, k, j$.

Hypotheses: midpoint(a_1, b, c), midpoint(c_1, b, a), midpoint(b_1, a, c), pbisector(n, a_1, b_1), pbisector(n, a_1, c_1), perp(a_1, k, a_1, n), coll(a, c, k), coll(k, a_1, j), coll(a, b, j).

Conclusion: cyclic(k, j, b, c).

The Machine Proof

$[jk, jb] - [kc, cb]$

(Since a_1, j, k are collinear; a, b, j are collinear; $[jk, jb] = [ka_1, ab]$.)

$= [ka_1, ab] - [kc, cb]$

(Since $ka_1 \perp a_1n$; $[ka_1, ab] = [na_1, ab] + 1$.)

$= -[kc, cb] + [na_1, ab] + 1$

(Since a, c, k are collinear; $[kc, cb] = [ac, cb]$.)

$= [na_1, ab] - [ac, cb] + 1$

(Since circumcenter(n, a_1, c_1, b_1);

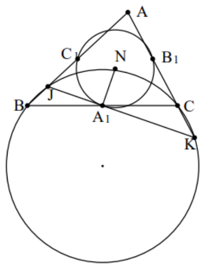


Figure 84

$[na_1, ab] = [na_1, a_1c_1] + [a_1c_1, ab] = -[c_1b_1, b_1a_1] + [c_1a_1, ab] + 1$.)

$= -[c_1b_1, b_1a_1] + [c_1a_1, ab] - [ac, cb]$

(Since $c_1a_1 \parallel ac$; $[c_1a_1, ab] - [ac, cb] = -[ab, cb]$.)

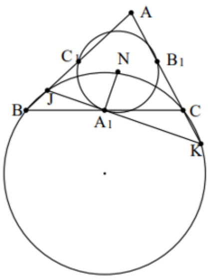
$= -[c_1b_1, b_1a_1] - [ab, cb]$

(Since $b_1a_1 \parallel ab$; $-[c_1b_1, b_1a_1] - [ab, cb] = -[c_1b_1, cb]$.)

$= -[c_1b_1, cb]$

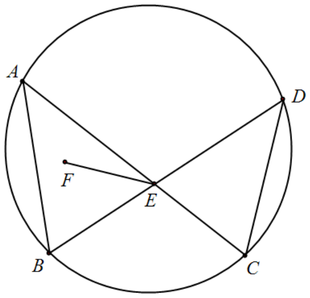
(Since $c_1b_1 \parallel cb$; $[c_1b_1, cb] = 0$.)

$= 0$



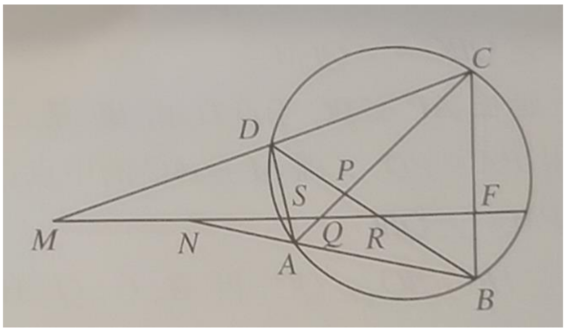
Example 2 18 : As shown in the figure, in $\triangle ABC$, A_1, B_1, and C_1 are the midpoints of BC, CA, and AB respectively, and N is the circumcenter of $\triangle A_1B_1C_1$, and pass through A_1 to make $JK \perp A_1N$, intersect AB on J, intersect AC on K, and prove: J, B, K, and C are all circles.

prove: $$\frac{\frac{J-K}{A-B}}{\frac{A-C}{C-B}}\left(\frac{A_1-C_1}{A_1-N}\frac{B_1-A_1}{B_1-C_1}\right)=\frac{J-K}{A_1-N}\frac{B-C}{B_1-C_1}\frac{A_1-B_1}{A-B}\frac{A_1-C_1}{A-C}$$



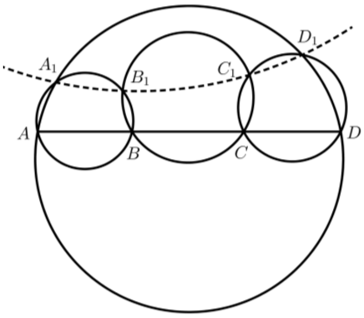
Example 2 19 : As shown in the figure, A , B , C and D share a circle, AC intersects BD at E , and F is the circumcenter of $\triangle ABE$. Prove: $EF \perp CD$.

$$-\left(\frac{C-D}{E-F}\right)^2=\frac{\left(\frac{A-C}{A-B}\right)^2}{\frac{F-E}{F-B}}\left(\frac{\frac{B-A}{B-D}}{\frac{C-A}{C-D}}\right)^2\frac{\frac{D-B}{E-F}}{\frac{B-F}{B-D}},$$



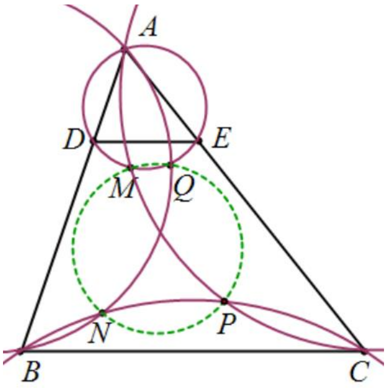
Example 2 20 : As shown in the figure, the quadrilateral $ABCD$ is inscribed in a circle , and the straight line MN forms equal angles with AD and BC . Prove that this straight line forms equal angles with another set of sides AB and CD , and forms equal angles with the two diagonals AC and BD also form equal angles.

$$\frac{\frac{A-B}{M-N}}{\frac{M-N}{D-C}}=\frac{A-D}{\frac{M-N}{\frac{N-M}{B-C}}}\frac{\frac{B-A}{A-D}}{\frac{C-B}{C-D}},\quad \frac{\frac{A-C}{M-N}}{\frac{N-M}{B-D}}=\frac{\frac{A-D}{M-N}}{\frac{N-M}{B-C}}\frac{\frac{D-B}{D-A}}{\frac{C-B}{C-A}},$$



Example 2 21 : As shown in the figure, AD is a chord on a circle, B and C are on AD, four points A, B, B_1, A_1 share a circle, B, C, B_1, C_1 share a circle, C, D, C_1, C_1 share a circle, and A , D , D 1 , A 1 share a circle. Prove: A_1, B_1, C_1, D_1 share a circle.

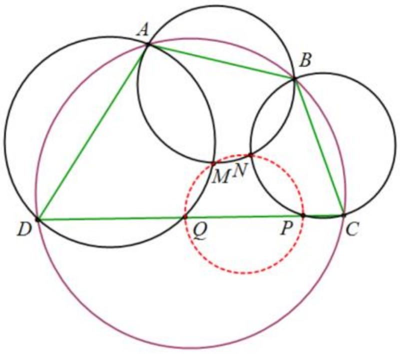
$$\frac{\frac{B_1-A_1}{B_1-C_1}}{\frac{A_1-D_1}{D_1-C_1}}=\frac{\frac{A_1-B_1}{B_1-B}}{\frac{A-A_1}{A-D}}\frac{\frac{C-C_1}{A-D}}{\frac{B_1-C_1}{B_1-B}}\frac{\frac{C_1-D_1}{D_1-D}}{\frac{C-C_1}{A-D}}\frac{\frac{D-D_1}{A-D}}{\frac{A_1-D_1}{A_1-A}}$$ 。



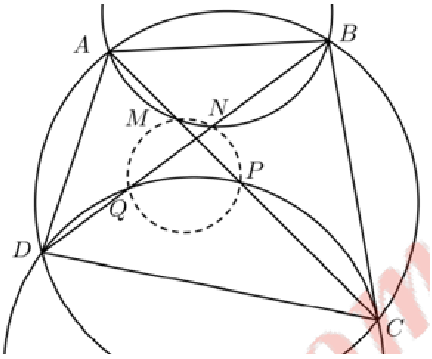
Example 2 22 : As shown in the figure, in $\triangle ABC$, D and E are on AB and AC, $DE \parallel BC$, five points A, D, M, Q and E share a circle, and four points A, B, N and Q share a circle. B, C, P, N four points share a circle, A, M, P, C four points share a circle, prove: M, N, P, Q four points share a circle.

$$\frac{\frac{M-Q}{M-P}}{\frac{N-Q}{N-P}}=\frac{\frac{M-A}{M-P}}{\frac{C-A}{P-C}}\frac{\frac{D-A}{D-Q}}{\frac{M-A}{M-Q}}\frac{\frac{N-P}{N-B}}{\frac{C-P}{B-C}}\frac{\frac{N-B}{N-Q}}{\frac{B-A}{A-Q}}\frac{\frac{Q-D}{Q-A}}{\frac{E-D}{E-A}}\frac{A-B}{A-D}\frac{A-C}{A-E}\frac{D-E}{B-C},$$

Example 2 23 : As shown in the figure, the four points A, B, C, and D share a circle, the four points A, D, Q, and M share a circle, the four points P, C, B, and N share a circle, and A, M, N, and B The four points are in a circle, to prove: M, N, P, Q are in a circle.



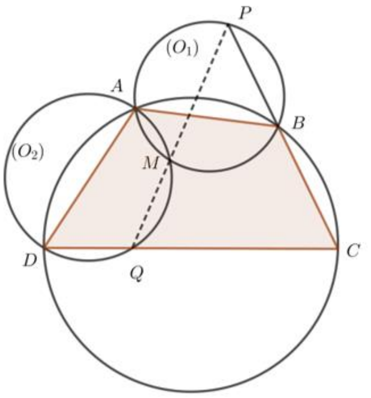
$$\frac{\frac{M-N}{M-Q}\frac{B-C}{B-N}\frac{N-B}{M-N}\frac{Q-M}{D-C}\frac{A-B}{A-D}}{\frac{P-N}{D-C}\frac{C-D}{P-N}\frac{A-B}{A-M}\frac{A-M}{A-D}\frac{C-B}{D-C}}=1,$$



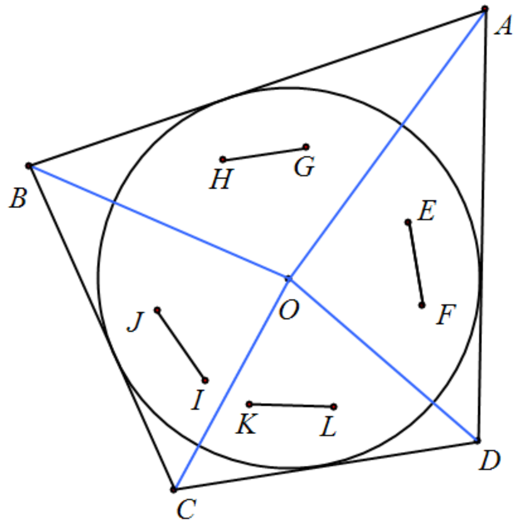
Example 2 24 : As shown in the figure, quadrilateral ABCD *has points* M and P on AC, points N and Q *on* BD, four points A, B, M and N share a circle, D, C, P and Q share a circle, A, B, C, D four points share a circle, and prove that: P, Q, M, N share a circle.

$$\frac{\frac{M-N}{A-C}}{\frac{D-B}{Q-P}}=\frac{\frac{D-B}{D-C}}{\frac{A-B}{A-C}}\frac{\frac{Q-P}{B-D}}{\frac{C-A}{D-C}}\frac{\frac{A-B}{A-C}}{\frac{D-B}{M-N}}$$

Example 2 25 : As shown in the figure, four points A, B, C, and D *share a circle;* A, D, Q, and M share a circle; A, M, B, and P *share* a circle; C, D, and Q share a circle. Collinear, P, B, C three points are collinear, prove that P, M, Q three points are collinear.



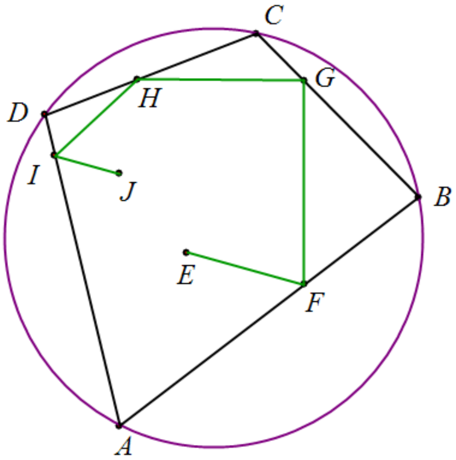
$$\frac{M-Q}{P-M}=\frac{\frac{B-C}{P-M}\frac{D-A}{D-C}\frac{A-B}{A-D}}{\frac{A-B}{A-M}\frac{A-M}{M-Q}\frac{C-B}{D-C}},$$



Example 2 26 : As shown in the figure, the quadrilateral ABCD , the angle bisectors of the four corners intersect at point O, the known straight lines EF and GH are symmetrical about AO, the straight lines GH and IJ are symmetrical about BO, and the straight lines IJ and LK are symmetrical about CO. Prove: Lines LK and EF are symmetrical about DO.

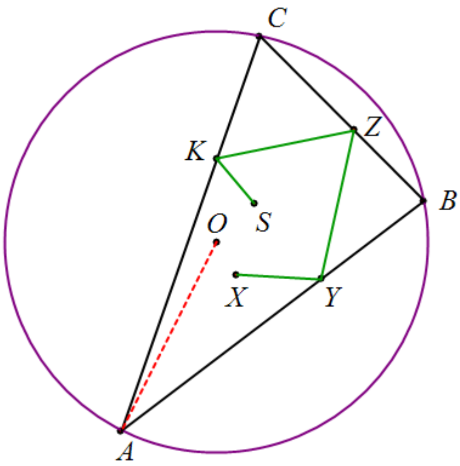
$$\frac{\frac{A-D}{A-O}\frac{B-O}{B-C}\frac{C-B}{C-O}\frac{D-O}{D-A}\frac{A-O}{G-H}\frac{G-H}{B-O}\frac{C-O}{K-L}\frac{K-L}{D-O}}{\frac{A-O}{A-B}\frac{B-A}{B-O}\frac{C-O}{C-D}\frac{D-C}{D-O}\frac{E-F}{A-O}\frac{B-O}{I-J}\frac{I-J}{C-O}\frac{D-O}{E-F}}=1$$

geometric dual



Example 2 27 : As shown in the figure, the circle O is inscribed with the quadrilateral ABCD . It is known that the straight lines EF and GF are symmetrical about AB, the straight lines GF and HG are symmetrical about BC, the straight lines HG and HI are symmetrical about CD, and the straight lines HI and JI are symmetrical about DA. Prove that: IJ//EF.

$$\frac{I-J}{E-F}=\frac{\frac{A-O}{A-B}\frac{C-B}{C-O}\frac{C-O}{C-D}\frac{A-D}{A-O}\frac{B-A}{F-E}\frac{G-H}{B-C}\frac{D-C}{H-G}\frac{I-J}{D-A}}{\frac{B-A}{B-O}\frac{B-O}{B-C}\frac{D-C}{D-O}\frac{D-O}{D-A}\frac{F-G}{A-B}\frac{C-B}{G-F}\frac{H-I}{C-D}\frac{A-D}{I-H}}$$

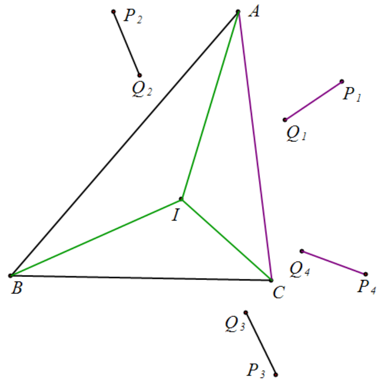


remove circle

Example 2 28 : As shown in the figure, in $\triangle ABC$, O is the circumcenter, the straight lines XY and ZY are symmetrical about AB, *the straight lines* ZY and ZK are symmetrical about BC, *and the straight lines* KZ and KS *are symmetrical* about CA. Prove that the straight line KS *is* symmetrical about AO *and* XY parallel.

$$\frac{\dfrac{A-O}{X-Y}}{\dfrac{S-K}{A-O}}=\frac{\dfrac{A-O}{A-B}}{\dfrac{B-A}{B-O}}\frac{\dfrac{C-B}{C-O}}{\dfrac{B-O}{B-C}}\frac{\dfrac{C-O}{C-A}}{\dfrac{A-C}{A-O}}\frac{\dfrac{B-A}{Y-X}}{\dfrac{Y-Z}{A-B}}\frac{\dfrac{Z-K}{B-C}}{\dfrac{C-B}{Z-Y}}\frac{\dfrac{A-C}{K-Z}}{\dfrac{K-S}{C-A}},$$

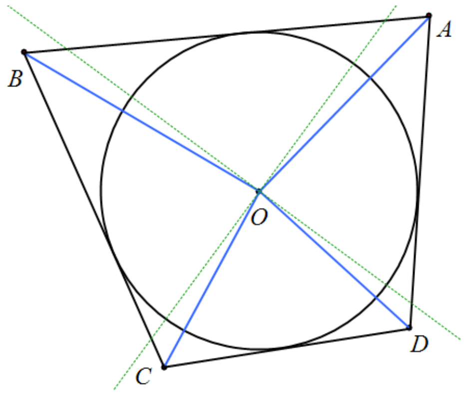
Try changing it to an inscribed circle



Example 2 29 : As shown in the figure, I is the center of $\triangle ABC$, the straight lines P 1 Q 1 and P 2 Q 2 are symmetrical about AI, the straight lines P 2 Q 2 and P 3 Q 3 are symmetrical about BI, and the straight lines P 3 Q 3 and P 4 Q 4 Symmetric about CI, verify: the symmetric straight line P 4 Q 4 about AC is parallel to P 1 Q 1.

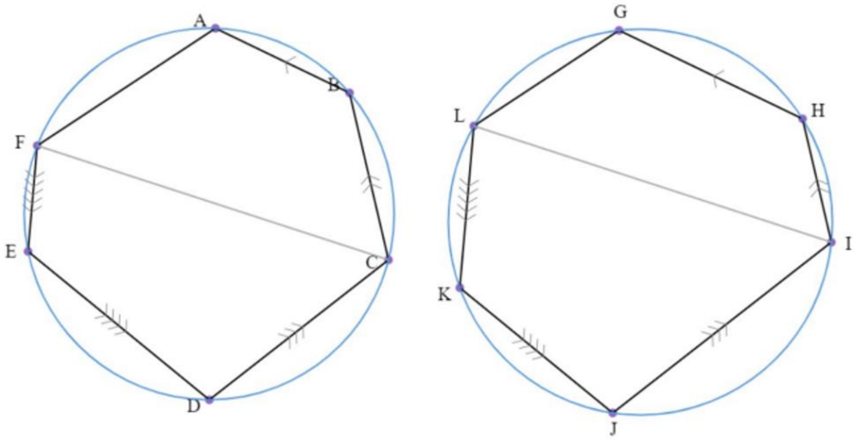
$$\frac{\frac{A-C}{A-I}\ \frac{B-I}{B-C}\ \frac{C-B}{C-I}\ \frac{A-I}{P_2-Q_2}\ \frac{P_2-Q_2}{B-I}\ \frac{C-I}{P_4-Q_4}\ \frac{P_1-Q_1}{A-C}}{\frac{A-I}{A-B}\ \frac{B-A}{B-I}\ \frac{C-I}{C-A}\ \frac{P_1-Q_1}{A-I}\ \frac{B-I}{P_3-Q_3}\ \frac{P_3-Q_3}{C-I}\ \frac{C-A}{P_4-Q_4}}=1,$$

circle inscribed angle bisector vertical



Example 2 30 : As shown in the figure, the quadrilateral *ABCD is* circumscribed on the circle *O*. Prove that the angle bisector *OX of* $\angle AOC$ *is perpendicular to* the angle bisector *OY of* $\angle$BOD .

$$\left(\frac{O-Y}{O-X}\right)^4=\left(\frac{\frac{A-D}{A-O}\frac{B-O}{B-C}\frac{C-B}{C-O}\frac{D-O}{D-A}}{\frac{A-O}{A-B}\frac{B-A}{B-O}\frac{C-O}{C-D}\frac{D-C}{D-O}}\right)\left(\frac{\frac{O-A}{O-X}\frac{O-Y}{O-B}}{\frac{O-X}{O-C}\frac{O-D}{O-Y}}\right)^2,$$



Example 2 31 : As shown in the figure, the hexagon $ABCDEF$, $A_1B_1C_1D_1E_1F_1$ is inscribed by two circles . It is known that $AB // A_1B_1$, $BC // B_1C_1$, $CD // C_1D_1$, $DE // D_1E_1$, $EF // E_1F_1$, to prove: $AF // A_1F_1$.

$$\frac{A-F}{A_1-F_1}=\frac{A-B}{A_1-B_1}\frac{B_1-C_1}{B-C}\frac{C_1-D_1}{C-D}\frac{D-E}{D_1-E_1}\frac{E-F}{E_1-F_1}\frac{\frac{A_1-B_1}{A_1-F_1}}{\frac{C_1-B_1}{F_1-C_1}}\frac{\frac{C-B}{F-C}}{\frac{A-B}{A-F}}\frac{\frac{D_1-E_1}{D_1-C_1}}{\frac{F_1-C_1}{F_1-E_1}}\frac{\frac{F-C}{F-E}}{\frac{D-E}{D-C}},$$

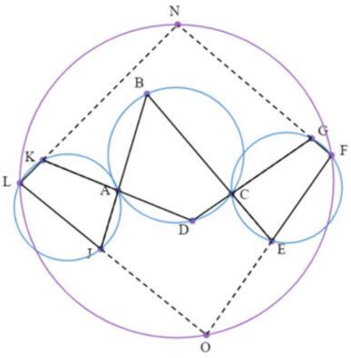
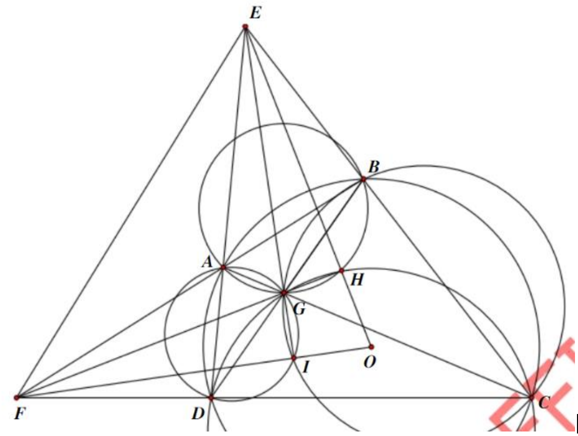


Figure 14: ABCD is a cyclic quadrilateral. G lies on DC extended, E lies on BC extended, J lies on BA extended and K lies on DA extended. CGFE and AKLJ are cyclic quadrilaterals. N is the intersection of LK and FG, O is the intersection of EF and JL. LNFO is a cyclic quadrilateral.

Example 2 32 : As shown in the figure, the quadrilateral $ABCD$ is inscribed in a circle , G is on the extension line of DC , E is on the extension line of BC , J is on the extension line of BA , and K is on the extension line *of* DA . $CGFE$ and $AKLJ$ are quadrilaterals inscribed in a circle. N is the intersection of LK and FG , O is the intersection of EF and JL . Prove: $LNFO$ is a quadrilateral inscribed in a circle.

$$\frac{\frac{A-O}{X-Y}}{\frac{S-K}{A-O}}=\frac{\frac{A-O}{A-B}}{\frac{B-A}{B-O}}\frac{\frac{C-B}{C-O}}{\frac{B-O}{B-C}}\frac{\frac{C-O}{C-A}}{\frac{A-C}{A-O}}\frac{\frac{B-A}{Y-X}}{\frac{Y-Z}{A-B}}\frac{\frac{Z-K}{B-C}}{\frac{C-B}{Z-Y}}\frac{\frac{A-C}{K-Z}}{\frac{K-S}{C-A}}.$$



Inscribed quadrilateral

Example 2 33 : As shown in the figure, the circle O inscribes the quadrilateral $ABCD$, AD intersects BC at E , AB intersects CD at F , AC intersects BD at G , the circumcircle of △ ABG and the circumcircle of △ CDG intersect at G , H , △ The circumscribed circle of ADG and the circumscribed circle of △ BCG intersect at G , I , and prove that: F , I , O are collinear; O *is* the orthocenter of △ EFG .

Proof: First prove that A , B , I , O are four points in a circle.

$$\frac{\frac{I-A}{I-B}}{\frac{O-A}{O-B}}\frac{\frac{D-A}{D-G}}{\frac{I-A}{I-G}}\frac{\frac{I-B}{I-G}}{\frac{C-B}{C-G}}\frac{C-A}{C-G}\frac{D-G}{D-B}\frac{\frac{O-A}{O-B}}{\frac{C-A}{C-B}\frac{D-A}{D-B}}=1,$$

Similarly, four points D , C , I , O *share a circle.* According to the circular power theorem, F *is on* the root axis of the circumcircle of △ ABI and △ DCI *, so the three points* F , I , O are collinear. Similarly, the three points E , G , and I are collinear. The three points E , H and O are collinear.

$$\frac{I-O}{I-G}=-\frac{\frac{I-O}{I-C}}{\frac{D-O}{D-C}}\frac{\frac{I-C}{I-G}}{\frac{B-C}{G-B}}\frac{B-D}{B-G}\left(\frac{B-C}{B-D}\frac{D-O}{D-C}\right),$$ so $IO \perp IG$, similarly $HO \perp HG$,

so O is the orthocenter of △ EFG .