Preservation

If $e : \tau$ & $e \mapsto e'$ then $e' : \tau$

[Goal $\text{let}(e_1; x.e_2) : \tau$

Case:

$$\frac{\cancel{\text{let}(e_1; x.e_2) : \tau_2} \quad e_1 \mapsto e_1'}{\text{let}(e_1; x.e_2) \mapsto \text{let}(e_1'; x.e_2)}$$

$e = \text{let}(e_1; x.e_2)$

$e' = \text{let}(e_1'; x.e_2)$

Above the line $e_1 \mapsto e_1'$ H_1

By inversion on $e : \tau$

Above the line

$\Gamma \vdash e_1 : \tau_1$ * H_2

$\Gamma, x : \tau_1 \vdash e_2 : \tau_2$ * H_3

By IH on H_1 & H_2

$e_1' : \tau_1$

By rule of let

$\text{let}(e_1'; x.e_2) : \tau_2$

<1>

~~Progress for let~~

~~if $e : \tau$ then either e val or there e'.~~

Case : $\dfrac{[e_1 \text{ val}]}{\text{let}(e_1; x.e_2) \mapsto [e_1/x]e_2}$

$e = \text{let}(e_1; x.e_2)$

$e' = [e_1/x]e_2$

Above the line e_1 val * H_1

By inversion on $e : \tau$

Above the line

$\Gamma \vdash e_1 : \tau_1$ * H_2

$\Gamma, x : \tau_1 \vdash e_2 : \tau_2$ * H_3

By substitution lemma on H_3 & H_2 $\rightarrow$ if $\Gamma, x : \tau \vdash e' : \tau'$ & $\Gamma \vdash e : \tau$ then $[e/x]e' : \tau'$

we get

$[e_1/x]e_2 : \tau_2$

//

Progress

If $e : \tau$ then either e val or there exists e' such that $e \mapsto e'$

Case: $\dfrac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash let(e_1; x.e_2) : \tau_2}$

$e = let(e_1; x.e_2)$

$\tau = \tau_2$

Above the line

$\Gamma \vdash e_1 : \tau_1$ * H_1

$\Gamma, x : \tau_1 \vdash e_2 : \tau_2$ * H_2

By IH on H_1 either e_1 is val or there exists e_1' such that $e_1 \mapsto e_1'$

Case 1: $e_1 \mapsto e_1'$

By rule take $e_1' = let(e_1'; x.e_2)$

then $e_1 \mapsto e_1'$

Case 2: e_1 val Apply IH to H_2

Take $e_2' = [e_1/x]e_2$

the above the line $[e_1/x]e_2$ val

e_1 val ▨

$$\frac{[e_1/x]e_2 \Downarrow v_2}{\text{let}(e_1; x.e_2) \Downarrow v_2} \quad \text{lazy}$$

$$\frac{e_1 \Downarrow v_1 \quad [v_1/x]e_2 \Downarrow v_2}{\text{let}(e_1; x.e_2) \Downarrow v_2} \quad \text{eage}$$

Equivalent

if $E \Downarrow v$ then $E \mapsto^* v$

Case : $\dfrac{e_1 \Downarrow v_1 \quad [v_1/x]e_2 \Downarrow v_2}{\text{let}(e_1; x.e_2) \Downarrow v_2}$

$E = \text{let}(e_1; x.e_2)$

$v = v_2$

Above the line

$e_1 \Downarrow v_1 \quad H_1$

$[v_1/x]e_2 \Downarrow v_2 \quad H_2$

By IH on H_1

$e_1 \mapsto^* v_1 \quad \rightarrow H_3$

By IH on H_2

$[v_1/x]e_2 \mapsto^* v_2 \quad \rightarrow H_4$

By transitivity

$[e_1/x]e_2 \mapsto^* v_2 \quad \rightarrow H_5$

By let rule

$\text{let}(e_1; x.e_2) \mapsto [e_1/x]e_2 \quad H_6$

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By H5 & H8 we have 5

let $(e_1 ; x . e_2) \mapsto^* V_2$

If $E \mapsto^* V$ & V val then $E \Downarrow V$

Higer Order

Lazy

Preservation

If $e : \tau$ & $e \mapsto e'$ then $e' : \tau$

Case: $\dfrac{e_1 \mapsto e_1'}{ap(e_1 ; e_2) \mapsto ap(e_1' ; e_2)}$

$e = ap(e_1 ; e_2)$

$e' = ap(e_1' ; e_2)$

above the line $e_1 \mapsto e_1'$ + H_1

By inversion on $e : \tau$

Above the line

$\Pi \vdash e_1 : arr(\tau_2 ; \tau)$ H_2

$\Pi \vdash e_2 : \tau_2$ H_3

By IH on H_2 & H_1

$e_1' : arr(\tau_2 ; \tau)$

By rule of ap

$ap(e_1' ; e_2) : \tau$

<3>

Case: $\left[\dfrac{e_1 \text{ val} \quad e_2 \mapsto e_2'}{ap(e_1; e_2) \mapsto ap(e_1; e_2')}\right]$ lager

$e = ap(e_1; e_2)$

$e' = ap(e_1; e_2')$

Above the line

e_1 val $\ast H_1$

$e_2 \mapsto e_2'$ $\ast H_2$

By inversion on $e : \tau$

Above the line

$e_1 : arr(\tau_2; \tau)$ $\ast H_3$

$e_2 : \tau_2$ $\ast H_4$

By IH on H_4 & H_2:

e_2', τ_2

By ap rule

$ap(e_1; e_2') : \tau$ ▨

Case: $\dfrac{[e_2 \text{ val}]}{ap(\underbrace{lam\{\tau_1\}(x.e_1)}_{e_1}; \underbrace{e_2}_{e_2}) \mapsto [e_2/x]e_1}$

$e = ap(lam\{\tau_1\}(x.e_1))$

$e' = [e_2/x]e_1$

Above the line e_2 val $\ast H_1$

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By inversion of $e : \tau$ 7

$\Gamma \vdash lam\{\tau\}(x.e_1) : arr(\tau_2; \tau)$ * H_2

$\Gamma \vdash e_2 : \tau_2$ * H_3

By inversion on H_2

$\Gamma, x : \tau_2 \vdash e_1 : \tau$ * H_4

By S.L on H_4 & H_3

$\Gamma, x : \tau \vdash e' : \tau'$ & $e : \tau$ then $[e/x]e' : \tau'$

$[e_2/x]e_1 : \tau$

Progress

If $e : \tau$ then eval or there exists e' such that $e \mapsto e'$

Case: $\dfrac{\Gamma, x : \tau_1 \vdash e_1 : \tau_2}{\Gamma \vdash lam\{\tau_1\}(x.e) : arr(\tau_1; \tau_2)}$

$e = lam\{\tau_1\}(x.e)$ or $\lambda(x : \tau)e$

$\tau = arr(\tau_1; \tau_2)$

Above the line

$\Gamma, x : \tau_1 \vdash e_1 : \tau_2$ * H_1

By IH to H_1

either e_1 val or e_1', $e_1 \mapsto e_1'$

By rule : $\dfrac{\text{By rule}}{lam\{\tau_1\}(x.e_1) \text{ val}}$

e_1 val

<5>

Case: $\dfrac{\Gamma \vdash e_1 : arr(\tau_2; \tau_1) \quad \Gamma \vdash e_2 : \tau_2}{ap(e_1; e_2) : \tau_1}$

$e = ap(e_1; e_2)$

$\tau = \tau_1$

Above the line:

$\Gamma \vdash e_1 : arr(\tau_2; \tau_1) \; * H_1$

$\Gamma \vdash e_2 : \tau_2 \; * H_2$

By IH on H_1

Either e_1 val or there exists e_1 such that $e_1 \mapsto e_1'$

Case: $e_1 \mapsto e_1'$

By rule take $e_1' = ap(e_1'; e_2)$

then $e_1 \mapsto e_1'$

Case: e_1 val

By ~~apply~~ apple IH on H_2 ->

case: e_2 val

apply CFL to H_1

WKT

$e_1 = lam\{\tau_1\}(x.e_1)$

By ~~rule~~ take $e_1' = [e_2/x]e_1$

e_2 val

Case $e_2 \mapsto e_2'$

Take $e_2' = ap(e_1; e_2')$

Then e_1 val & $e_2 \mapsto e_2'$

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Evaluation dynamics

$$\frac{}{lam\{\tau\}(x.e) \Downarrow lam\{\tau\}(x.e)}$$

$$\cancel{\frac{e_1 \Downarrow lam\{\tau\}(x.e) \quad ap(lam\{\tau\}(x.e); e_2) \Downarrow v}{ap(e_1; e_2) \Downarrow v}}$$

$$\frac{e_1 \Downarrow lam\{\tau\}(x.e) \quad e_2 \Downarrow v_2 \quad [v_2/x]e \Downarrow v}{ap(e_1; e_2) \Downarrow v}$$

eager

$$\frac{e_1 \Downarrow lam\{\tau\}(x.e) \quad [e_2/x]e \Downarrow v}{ap(e_1; e_2) \Downarrow v}$$

lazy

Theorem 8.1

If $E \Downarrow V$ then $E \mapsto^{*} V$ {left to right}

$$\frac{e_1 \Downarrow lam\{\tau\}(x.e) \quad e_2 \Downarrow v_2 \quad [v_2/x]e \Downarrow v}{ap(e_1; e_2) \Downarrow v}$$

$E = ap(e_1; e_2)$

$V = v$

Above the lin

$e_1 \Downarrow lam\{\tau\}(x.e)$ * H_1 → 1

$e_2 \Downarrow v_2$ * H_2 → 2

$[v_2/x]e \Downarrow v$ * H_3 → 3

By IH on on 1

$e_1 \mapsto^* lam\{\tau\}(x.e_1)$ - (4)

By IH on H2

$e_2 \mapsto^* v_2$ → (5)

By IH on H3

$[v_2/x]e \mapsto^* v$ → (6)

goal

$ap(e_1; e_2) \mapsto^* v$

By congruence lemma (1)

if $e_1 \mapsto^* e_1'$ then $ap(e_1; e_2) \mapsto^* ap(e_1'; e_2)$

on 4

$ap(e_1; e_2) \mapsto^* ap(lam\{\tau\}(x.e); e_2)$ → (7)

By congruence lemma (2)

if e_1 val & $e_2 \mapsto^* e_2'$

then $ap(e_1; e_2) \mapsto ap(e_1; e_2')$

~~$ap(e_1; e_2) \mapsto ap(e_1; v_2)$~~

Apply lemma if $e \Downarrow v$ then v val

on (1)

$lam\{\tau\}(x.e)$ val → (8)

Apply lemma to (2)

v_2 val → (9)

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Apply Congruence lemma 2 on

(5) & (8)

$ap(lam\{\tau_1\}(x.e) ; e_2) \mapsto^* ap(lam\{\tau_1\}(x.e), v_2)$ (10)

By ap rule

$ap(lam\{\tau_1\}(x.e), v_2) \mapsto [v_2/x]e$ (11)

By lemm $E \mapsto E'$ then $E \mapsto^* E'$

$ap(lam\{\tau_1\}(x.e), v_2) \mapsto^* [v_2/x]e$ (12)

Apply transitivity to (7) & (10)

$ap(e_1 ; e_2) \mapsto^* ap(lam\{\tau_1\}(x.e), v_2)$ (13)

Apply transitivity to (13) & (12)

$ap(e_1 ; e_2) \mapsto^* [v_2/x]e$ (14)

Apply transitivity to (14) & (6)

$ap(e_1 ; e_2) \mapsto^* v$

if $E \mapsto^* V$ & v val * Right to left

then $E \Downarrow V$

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case $e \mapsto^* e$

$E = e$ & $V = e$, e val ①

Inversion on e val

$e = lam\{\tau\}(x.e)$

By rule

$\underline{lam\{\tau\}(x.e)} \Downarrow lam\{\tau\}(x.e)$

case $e \mapsto^* e''$

$E = e$, $V = e''$

$e \mapsto e'$ ①, $e' \mapsto^* e''$ ②, e'' val ③

apply IH to ② & ③

$e' \Downarrow e'' \to 4$

by lemma

if $e \mapsto e'$, $e' \Downarrow v$

the $e \Downarrow v$

$\therefore e \Downarrow e''$ ☒

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System 1

Preservation

If $e : \tau$ & $e \mapsto e'$ then $e' : \tau$

Case:

$$\frac{e \mapsto e'}{\mathrm{rec}\{e_0; x, y.e_1\}(e) \mapsto \mathrm{rec}\{e_0; x, y.e_1\}(e')}$$

$e = \mathrm{rec}\{e_0; x, y.e_1\}(e^{=})$

$e' = \mathrm{rec}\{e_0; x, y.e_1\}(e')$

Above the line

$e \mapsto e'$ * H_1

By inversion on $e : \tau$

$\Gamma \vdash e : \mathrm{nat}$ * H_2

$\Gamma \vdash e_0 : \tau$ * H_3

$\Gamma, x : \mathrm{nat}, y : \tau \vdash e_1 : \tau$ * H_4

By IH on H_2 & H_1

$e' : \mathrm{nat}$

By rec rul

$\mathrm{rec}\{e_0; x, y.e_1\}(e') : \tau$

<11>

Case:

$$\frac{}{rec\{e_0; x, y.\ e_1\}(z) \longmapsto e_0}$$

$e = rec\{e_0; x, y.\ e_1\}(z)$

$e' = e_0$

By inversion on $e : \tau$

$\Gamma \vdash z : nat$ * H1

$\Gamma \vdash e_0 : \tau$ * H2

$\Gamma, x : nat, y : \tau \vdash e_1 : \tau$ * H3

Case:

$$\frac{s(e)\ val}{rec\{e_0; x, y.\ e_1\}(s(e)) \longmapsto [e, rec\{e_0; x.y.\ e_1\}(e)/x, y]e_1}$$

$e = rec\{e_0; x, y.\ e_1\}(s(e))$

$e' = [e, rec\{e_0; x, y.\ e_1\}(e)/x, y]e_1$

Above the line

$s(e)\ val$ * H1

By inversion on $e : \tau$

$\Gamma \vdash s(e) : nat$ * H2 By Inver $\Rightarrow$ $e : nat$

$\Gamma \vdash e_0 : \tau$ * H3

<127> $\Gamma, x : nat, y : \tau \vdash e_1 : \tau$ * H4

Apply weakening lem To

H 2

$\Gamma, y : \tau \vdash e : nat$ + H 5

$[e/x]e_1 : \tau$

By S.L on 4 & 5

$y : \tau \vdash [e/x]e_1 : \tau$ — 8

$x : \tau$ $e' :$ τ'

$rec\{e_0, x, y.e_1\}(e) : \tau \rightarrow$ 9

by 2, 3, 4

By S.L on 8 & 9 e : τ

$[e/x][rec\{e_0, x, y.e_1\}(e)/y]e_1$

τ

$\boxtimes$

Case: $\dfrac{e \mapsto e'}{s(e) \mapsto s(e')}$

$e = s(e)$

$e' = s(e')$

Above the line

$e \mapsto e'$ + H 1

By inversion on $e : \tau$

$\Gamma \vdash e : nat$ + H 2

By IH on H2 & H1 <13>

$e' : nat$ By rule $s(e') : nat$

<u>Progress</u>

if $e : \tau$ then either e val or there exists e', $e \mapsto e'$

Case: $\dfrac{}{\Gamma \vdash z : nat}$

$e = z$

$\tau = nat$

by rule z val

Case: $\dfrac{\Gamma \vdash e : nat}{\Gamma \vdash S(e) : nat}$

$e = S(e)$

$\tau = nat$

Above the line

$\Gamma \vdash e : nat$ *H1

By IH on H1 either e val or e', $e \mapsto e'$

Case: $e \mapsto e'$

take $e' = S(e')$

then $e \mapsto e'$

Case: e val

<14> $S(e)$ val by rule.

Case: $\dfrac{\Gamma \vdash e : nat \quad \Gamma \vdash e_0 : \tau \quad \Gamma, x : nat, y : \tau \vdash e_1 : \tau_1}{\Gamma \vdash rec\{e_0; x, y.e_1\} : \tau_1}$

$e = rec\{e_0; x, y.e_1\}(e)$

$\tau = \tau_1$

Above the line

$\Gamma \vdash e : nat \quad H_1$

$\Gamma \vdash e_0 : \tau_1 \quad H_2$

$\Gamma, x : nat, y : \tau \vdash e_1 : \tau_1 \quad H_3$

By IH on H_1 either e val

or there exists e' such that $e \mapsto e'$

Case $e \mapsto e'$

Take $e' = rec\{e_0; x, y.e_1\}(e')$

then $e \mapsto e'$

Case e val

by canonical form lem if $e : nat$ & e val

the $e = z$ or $s(e')$

Subcase $e = z$

Take e' as e_0

$E \mapsto E'$

Subcase $e = s(e')$

take $e' = [e'/x]\{rec\{e_0; x, y.e_1\}(e')\}e_1$

then $e \mapsto e'$

* Big Step rules for rec *

$$\frac{}{z \Downarrow z} \qquad \frac{e \Downarrow v}{s(e) \Downarrow s(v)}$$

$$\frac{e \Downarrow z \qquad e_0 \Downarrow v}{rec\{e_0; x, y \cdot e_1\}(e) \Downarrow v}$$

$$\frac{e \Downarrow s(e) \qquad [e/x][rec\{e_0; x, y \cdot e_1\}e/y]e_1 \Downarrow v}{rec\{e_0; x, y \cdot e_1\}(e) \Downarrow v}$$

$$\frac{e \Downarrow s(e) \qquad rec\{e_0; x, y \cdot e_1\}(e) \Downarrow v_1 \qquad [e/x][v_1/y]e_1 \Downarrow v}{rec\{e_0; x, y \cdot e_1\}(e) \Downarrow v}$$

if $E \Downarrow v$ then $E \mapsto^* v$

case: $\dfrac{e \Downarrow z \qquad e_0 \Downarrow v}{rec\{e_0; x, y \cdot e_1\}(e) \Downarrow v}$

$E = rec\{e_0, x, y \cdot e_1\}$

$v = v$

Above the line

$e \Downarrow z$ * H_1

$e_0 \Downarrow v$ * H_2

By IH on H_1 $e \mapsto^* z$ * H_3

By IH on H_2 $e_0 \mapsto^* v$ * H_4

By congruence lem

$e \mapsto^* e'$ then

$rec\{e_0; x, y.e_1\}(e) \mapsto^* rec\{e_0; x, y.e_1\}(e')$

to (3)

$rec\{e_0; x, y.e_1\}(e) \mapsto^* rec\{e_0; x, y.e_1\}(z)$ (H5)

By rec rule

$rec\{e_0; x, y.e_1\}(z) \mapsto^* e_0$

Blg lem if $E \mapsto E'$ the $E \mapsto^* E'$

$rec\{e_0; x, y.e_1\}(z) \mapsto^* e_0$ (H6)

By transitivity on H_6 & H_4

$rec\{e_0; x, y.e_1\}(z) \mapsto^* V$ (H7)

By transitivity on 5 & 7

$rec\{e_0; x, y.e_1\}(e) \mapsto^* V$.

Case: $\dfrac{e \Downarrow s(e) \quad rec\{e_0; x, y.e_1\}(e) \Downarrow v_1 \quad [v_1/y][e/x]e_1 \Downarrow v}{rec\{e_0; x, y.e_1\}(e) \Downarrow v}$

lary $\dfrac{e \Downarrow s(e) \quad [e/x][rec\{e_0; x, y.e_1\}(e)/y]e_1 \Downarrow v}{rec\{e_0; x, y.e_1\}(e) \Downarrow v}$

$E = rec\{e_0; x, y.e_1\}(e)$

$v = v$

Above the line

$e \Downarrow s(e)$ * H_1

$[e/x][rec\{e_0; x, y.e_1\}(e)/y]e_1 \Downarrow v$ * H_2

By IH on H_1

$e \mapsto^* s(e)$ * H_3

By IH on H_2

$[e/x][rec\{e_0; x, y.e_1\}(e)/y]e_1 \mapsto^* v$ * H_4

By congrence lemm 1

if $E \mapsto^* E'$

then

$rec\{e_0; x, y.e_1\}(e) \mapsto^* rec\{e_0; x.y.e_1\}(e')$

on H_3

$rec\{e_0; x, y.e_1\}(e) \mapsto^* rec\{e_0; x, y.e_1\}(s(e))$

* H_5

By rec rule

$rec\{e_0; x, y.e_1\}s(e) \mapsto$

$[e/x][rec\{e_0; x, y.e_1\}(e)/y]e_1$

By Lemma
$E \mapsto E'$ then $E \mapsto^* E'$

$rec\{e_0; x, y.e_1\}s(e) \mapsto^*$
$[e/x][rec\{e_0; x, y.e_1\}(e)/y]e_1$

By transitivity on H6 & H4 (H6)

$rec\{e_0; x, y.e_1\}s(e) \mapsto^* v$ (H7)

By transitivity on H7 & H5

$rec\{e_0; x, y.e_1\}(e) \mapsto^* v$

Product Type

Preservation:
If $e : \tau$ & $e \mapsto e'$ then $e' : \tau$

Case: $\dfrac{e_1 \mapsto e_1'}{\langle e_1, e_2\rangle \mapsto \langle e_1', e_2\rangle}$

$e = \langle e_1, e_2\rangle$

$e' = \langle e_1', e_2\rangle$

Above the line

$e_1 \mapsto e_1'$ * H1

Apply inversion on $e : L$

$\Gamma \vdash e_1 : \tau_1 \ {}^{*}H_2$

$\Gamma \vdash e_2 : \tau_2 \ {}^{*}H_3$ by IH on H_1 & H_2

$e_1' : \tau_1 \ {}^{*}H_4$

By Inversion on $e : \tau$

or by IH on H_2 & H_1 $\Gamma \vdash e_1 : \tau_1 \times \tau_2 \ {}^{*}H_4$

then By IH on H_4 & H_1, $e_1' : \tau_1$

by Sub

By rule P

$\langle e_1', e_2 \rangle : \tau_1 \times \tau_2$ $\langle e_1', e_2 \rangle : \tau_1 \times \tau_2$

<u>Case</u>:

$$\frac{e \mapsto e'}{fst(e) \mapsto fst(e')}$$

$e = fst(e)$

$e' = fst(e')$

Above the line $e \mapsto e' \ {}^{*}H_1$

By Inversion on $e : \tau$

$\Gamma \vdash e : \tau_1 \times \tau_2 \ {}^{*}H_2$

By inversion on H_2

$\Gamma \vdash e_1 : \tau_1 \ {}^{*}H_3$

$\Gamma \vdash e : \tau_2 \ {}^{*}H_4$

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~~By IH on H_3 & H_1~~

~~$fst(e') : \tau_1$~~

By IH on H_1 & H_2

$$e' : \tau_1 \times \tau_2$$

By rule $\dfrac{\Gamma \vdash e' : prod(\tau_1 ; \tau_2)}{\Gamma \vdash fst(e') : \tau_1}$

$$fst(e') : \tau_1$$

Case: $\dfrac{e \mapsto e'}{snd(e) \mapsto snd(e')}$

$e = snd(e)$

$e' = snd(e')$

Above the line $e \mapsto e'$ * H_1

By Inversion on $e : \tau$

$$e : prod(\tau_1 ; \tau_2) \quad * \; H_2$$

By IH on H_1 & H_2

$$e' : prod(\tau_1 ; \tau_2)$$

by rule

$$snd(e') : \tau_2$$

Case: $\frac{[e_1 \text{ val}] \quad [e_2 \text{ val}]}{\text{fst}\langle e_1, e_2\rangle \mapsto e_1}$

$e = \text{fst}\langle e_1, e_2\rangle$

$e' = e_1$

Preservation

Above the line

$[e_1 \text{ val}] * H_1$

$e_2 \text{ val} * H_2$

$e_1 : \tau_1 \quad e_2 : \tau_2$

$\langle e_1, e_2\rangle : \tau_1 \times \tau_2$

By inversion on $e : \tau$

$\Gamma \vdash \langle e_1, e_2\rangle : \text{prod}(\tau_1, \tau_2) * H_3$

By inversion on H_3

$\Gamma \vdash e_1 : \tau_1$

$\Gamma \vdash e_2 : \tau_2$

Case: $\frac{[e_1 \text{ val}] \quad [e_2 \text{ val}]}{\text{snd}\langle e_1, e_2\rangle \mapsto e_2}$

$e = \text{snd}\langle e_1, e_2\rangle$

$e' = e_2$

Above the line

$e_1 \text{ val} * H_1$

$e_2 \text{ val} * H_2$

By inversion on $e : \tau$

$\Gamma \vdash \langle e_1, e_2\rangle : \text{prod}(\tau_1, \tau_2) * H_3$

By Inversion on H3

$\Gamma \vdash e_1 : \tau_1$

$\Gamma \vdash e_2 : \tau_2$

<u>Progress</u>

If $e : \tau$ then either e val or there exist e', $e \mapsto e'$

Case: $\dfrac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2}$

$e = \langle e_1 e_2 \rangle$, $pair(e_1, e_2)$

$\tau = prod(\tau_1 ; \tau_2)$

Above the line

$\Gamma \vdash e_1 : \tau_1$ * H1

$\Gamma \vdash e_2 : \tau_2$ * H2

By IH on H1 either e1 val or there exist e_1', such that $e_1 \mapsto e_1'$

Case $e_1 \mapsto e_1'$

take $e_1' = pair(e_1', e_2)$

then $e_1 \mapsto e_1'$

Case e1 val

By IH on H2

either e2 val or e_2', $e_2 \mapsto e_2'$

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subclass $e_2 \mapsto e_2'$

take

e_2' as $\langle e_1, e_2' \rangle$

then $e_2 \mapsto e_2'$

e_1 val

subclass e_2 val

we have e_1 val & e_2 val

then $\langle e_1, e_2 \rangle$ val $\Rightarrow$ pair$\langle e_1, e_2 \rangle$ ☒

Case: $\dfrac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash fst(e) : \tau_1}$

$e = fst(e)$

$\tau = \tau_1$

Above the line

$\Gamma \vdash e : prod(\tau_1, \tau_2)$ $\mathcal{H}_1$

By IH on $\mathcal{H}_1$ either e val

or there exists e', $e \mapsto e'$.

Case $e \mapsto e'$

Take $e' = fst(e')$

then $e \mapsto e'$

Case e val

By CFL if $e : prod(\tau_1, \tau_2)$

& e val then

$e = pair(e_1, e_2)$ with e_1 val &

e_2 val

by rule e_1 val & e_2 val

$e = \langle e_1, e_2 \rangle$

Subcase ~~fst(e1,e2) ↦ e1~~

taken $e_1'' = e_1$, By rule fst(e1,e2) ↦ e1

$\times$ then $e \mapsto e'$ e1 is value

case : $\dfrac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash snd(e) : \tau_2}$

$e = snd(e)$

$\tau_1 = \tau_2$

Above the line

$\Gamma \vdash e : \tau_1 \times \tau_2$ $\times$ H_1

By IH on H_1 either e val

or there exists e', $e \mapsto e'$

Case: $e \mapsto e'$

Take $e' = snd(e')$

then $e \mapsto e'$

Case: e val

By CFL if $e : prod(\tau_1, \tau_2)$

& e val the $e = Pair(e_1, e_2)$

By rule e_1 val & e_2 val

Subcase : $snd(e_1, e_2) \mapsto e_2$

Take $e' = e_2$

$\times$ the $e \mapsto e'$ snd(e1,e2) ↦ e2, e2 val

Big step rules

$$\frac{}{\text{True} \Downarrow \text{True}}$$

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{\langle e_1, e_2 \rangle \Downarrow \langle v_1, v_2 \rangle}$$

$$\frac{e \Downarrow \langle v_1, v_2 \rangle}{fst(e) \Downarrow v_1} \qquad \frac{e \Downarrow \langle e_1, e_2 \rangle \quad e_1 \Downarrow v}{fst(e) \Downarrow v}$$

$$\frac{e \Downarrow \langle v_1, v_2 \rangle}{snd(e) \Downarrow v_2} \qquad \frac{e \Downarrow \langle e_1, e_2 \rangle \quad e_2 \Downarrow v}{snd(e) \Downarrow v_2}$$

If $E \Downarrow v$ then $E \mapsto^* v$

Case: - $\dfrac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{\langle e_1, e_2 \rangle \Downarrow \langle v_1, v_2 \rangle}$

$E = \langle e_1, e_2 \rangle$

$v = \langle v_1, v_2 \rangle$

Above the line

$e_1 \Downarrow v_1 \; * H_1$

$e_2 \Downarrow v_2 \; * H_2$

By IH on H_1

$e_1 \mapsto^* v_1 \; * H_2$

By IH on H_2

$e_2 \mapsto^* v_2 \; * H_3$

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By congruence rule

$\langle e_1, e_2 \rangle \overset{*}{\mapsto} \langle e_1', e_2 \rangle$

on H_1

$\langle e_1, e_2 \rangle \overset{*}{\mapsto} \langle v_1, e_2 \rangle$ ④ $\langle e_1, e_2 \rangle \mapsto \langle v_1, e_2 \rangle$

by lemma e ⇓ v then v val on H_1

v_1 val ⇒ ⑤ e_1 val E_1

By congruence lemma 2 on

$e_2 \mapsto e_2'$

then

$\langle v_1, e_2 \rangle \overset{*}{\mapsto} \langle v_1, v_2 \rangle$ ⑥

By transitivity on 4 & 6

$\langle e_1, e_2 \rangle \overset{*}{\mapsto} \langle v_1, v_2 \rangle$

<u>Case</u> $\dfrac{e \Downarrow \langle v_1, v_2 \rangle}{fst(e) \Downarrow v_1}$

$E = fst(e)$

$v = v_1$

Above the line

$e \Downarrow \langle v_1, v_2 \rangle$ ^ H_1

By IH on H_1

By inversion on H_1

$e_1 \Downarrow v_1$ ^ H_2

$e_2 \Downarrow v_2$ ^ H_3

by IH on H_2

$e_1 \mapsto^* v_1$

By congr e

$fst(e) \mapsto^* fst(e')$ on ②

$fst(e) \mapsto^* fst(v_1)$ + ④

Apply lemma to 2 $e \Downarrow v$ then v val

v_1 val

$fst(v_1) \mapsto v_1$

By lem a $E \mapsto E'$ & E val

$E \mapsto^* E'$

$fst(v_1) \mapsto^* v_1$ → ⑤

By transitivity on ④ & ⑤

$fst(e) \mapsto^* v_1$

<u>Case</u> $\dfrac{e \Downarrow \langle v_1, v_2 \rangle}{snd(e) \Downarrow v_2}$

$E = snd(e)$

$v = v_2$

Above the line

$e \Downarrow \langle v_1, v_2 \rangle$ + H_1

By Inversion on H_1

$e_1 \Downarrow v_1$ + H_2

<287> $e_2 \Downarrow v_2$ + H_3

By IH on H3

$e_2 \longmapsto^* v_2$

By cong Lemma

$snd(e) \longmapsto^* snd(e')$

on H3

$snd(e) \longmapsto^* snd(v_2)$ *H4

Apply Lemma $e \Downarrow v$ then vval

on H3

v_2 val

By cong lem ②

$snd(v_2) \longmapsto^* v_2$

By lem

$snd(v_2) \longmapsto^* v_2$

By transitivity

$snd(e) \longmapsto^* v_2$ ▨

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Sum types

Preservation

If $e : \tau$ & $e \mapsto e'$ then $e' : \tau$

Case: $\dfrac{e \mapsto e'}{\text{abort}_{\{\tau\}}(e) \mapsto \text{abort}_{\{\tau\}}(e')}$

$e = \text{abort}_{\{\tau\}}(e)$

$e' = \text{abort}_{\{\tau\}}(e')$

Above the line

$e \mapsto e'$ * H_1

By inversion on $e : \tau$

$\Gamma \vdash e : \text{void}$ * H_2

By IH on H_2 & H_2

$e' : \text{void}$

By rule

$\text{abort}_{\{\tau\}}(e') : \tau$ ☐

Case: $\dfrac{e \mapsto e'}{\text{inl}_{\{\tau_1, \tau_2\}}(e) \mapsto \text{inl}_{\{\tau_1, \tau_2\}}(e')}$

$e = \text{inl}_{\{\tau_1, \tau_2\}}(e)$

$e' = \text{inl}_{\{\tau_1, \tau_2\}}(e')$

Above the line

$e \mapsto e'$ * H_1

By inversion on $e : \tau$

$\Gamma \vdash e : \tau_1 \; \# \; H_1$

By IH on H_1 & H_2

$e' : \tau_1$ ☒

By rule

$In[l]\{\tau_1, \tau_2\}(e') : sum(\tau_1 ; \tau_2)$

☒

Case: $\dfrac{e \mapsto e'}{case(e ; x.e_1 ; y.e_2) \mapsto case(e' ; x.e_1 ; y.e_2)}$

$e = case(e ; x.e_1 ; y.e_2)$

$e' = case(e ; x.e_1 ; y.e_2)$

Above the line

$e \mapsto e' \quad H_1$

By inversion on $e : \tau$

$\Gamma \vdash e : sum(\tau_1 ; \tau_2) \; \# \; H_1$

$\Gamma, x : \tau_1 \vdash e_1 : \tau \; \# \; H_2$

$\Gamma, y : \tau_2 \vdash e_2 : \tau \; \# \; H_3$

By IH on H_1, H_2

$e' : sum(\tau_1 ; \tau_2) \; \#$

By rule

$case(e' ; x.e_1 ; y.e_2) : \tau$ ☒

Case : [eval]

$case(inl\{\tau_1, \tau_2\}(e); x.e_1, y.e_2) \longmapsto [e/x]e_1$

$e = case(inl\{\tau_1, \tau_2\}(e); x.e_1, y.e_2)$

$e' = [e/x]e_1$

Above the line eval * H_1

By inversion on $e : \tau$

$\Gamma \vdash e : sum(\tau_1, \tau_2)$ * H_2 ($inl\{\tau_1, \tau_2\}(e)$)

$\Gamma, x : \tau_1, e_1 : \tau$ * H_3

$\Gamma, y : \tau_2, e_2 : \tau$ * H_4

By Inversion on H_2

$\Gamma \vdash e : \tau_1$ * H_5

By SL on H_3 & H_5

$[e/x]e_1 : \tau$ ∎

Preservation of

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Progress

If $e : \tau$ then either e val or there exists e', such that $e \mapsto e'$

Case: $\dfrac{\Gamma \vdash e : void}{\Gamma \vdash abort^{\{\tau\}}(e) : \tau'}$

$e = abort\{\tau'\}(e)$

$\tau = \tau'$

Above the line

$\Gamma \vdash e : void$ * H_1

By IH on H_1 then either e val or there exists e', $e \mapsto e'$

Case: $e \mapsto e'$

Take $e' = abort\{\tau'\}(e')$

then $e \mapsto e'$

Case: e val

e : void & e val

if e : void & e val then false

by CF L

Case: $\dfrac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \cancel{inl(e)} : \tau_1 + \tau_2}$

$in[l]\{\tau_1 ; \tau_2\}(e) : \tau_1 + \tau_2$

$e = in[l]\{\tau_1 ; \tau_2\}(e)$

$\tau : \tau_1 + \tau_2$

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Above the line $\Gamma' \vdash e : L_1 * H_1$ 35

By IH on H_1

Either eval or there exists e', $e \mapsto e'$

Case: $e \mapsto e'$

take $e' : in[l]\{\tau_1 ; \tau_2\}(e')$

then $e \mapsto e'$

Case eval

By rule

$in[l]\{\tau_1 ; \tau_2\}(e)$ ~~eval~~

Case: $\Gamma \vdash e : sum(\tau_1 ; \tau_2)$ $\Gamma, x : \tau_1 \vdash e_1 : \tau'$ $\Gamma, y : \tau_2 \vdash e_2 :$

$case(e ; x \cdot e_1 ; y \cdot e_2) : \tau$

$e = case(e ; x \cdot e_1 ; y \cdot e_2)$

$\tau = \tau'$

Above the line

$\Gamma \vdash e : sum(\tau_1 ; \tau_2) * H_1$

$\Gamma, x : \tau_1 \vdash e_1 : \tau' * H_2$

$\Gamma, y : \tau_2 \vdash e_2 : \tau' * H_3$

By IH on H_1 either eval

or there exists e', $e \mapsto e'$

Case: $e \mapsto e'$

Take $e' : case(e' ; x \cdot e_1 ; y \cdot e_2)$

then $e \mapsto e'$

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Case : e val

By CFL

if $e : sum(\tau_1 ; \tau_2)$ then

there exist (e') such that

$e = inl\ (e')$

or

Subcase $e = inl\}\{\tau_1 ; \tau_2\}(e')$ $inr\ (e')$, e' val

Take $e' = \cancel{inl\{\tau_1 ; \tau_2\}(e')}\ [e'/x]e_1$

then $e \mapsto e'$

Subcase $e = inr\}\{\tau_1 ; \tau_2\}(e')$, e' val

Take $e' = \cancel{inr\{\tau_1 ; \tau_2\}(e')}\ [e'/y]e_2$

then $e \mapsto e'$

Big STEP

$e : sum(\tau_1 ; \tau_2)$ e' s.t. $e = inl(e')$ or e

$$\frac{}{abort\{\tau\}(e) \Downarrow abort\{\tau\}(e)} \checkmark$$

$$\frac{e \Downarrow V}{In[l]\{\tau_1 ; \tau_2\}(e) \Downarrow In[l]\{\tau_1 ; \tau_2\}(V)}$$

$$\frac{e \Downarrow V}{In[r]\{\tau_1 ; \tau_2\}(e) \Downarrow In[r]\{\tau_1 ; \tau_2\}(V)}$$

$$\frac{e \Downarrow inl\}\{\tau_1 ; \tau_2\}(v_1) \quad [v_1/x]e_1 \Downarrow V}{case(e ; x.e_1 ; y.e_2) \Downarrow V}$$

$$\frac{e \Downarrow inr[r]\{\tau_1 ; \tau_2\}(v_2) \quad [v_2/x]e_2 \Downarrow V}{case(e ; x.e_1 ; y.e_2) \Downarrow V}$$

if $E \Downarrow V$ then

Case: $$\frac{e \Downarrow V}{in[l]\{\tau_1;\tau_2\}(e) \Downarrow in[l]\{\tau_1;\tau_2\}(v)}$$

$E = in[l]\{\tau_1;\tau_2\}(e)$

$V = in[l]\{\tau_1;\tau_2\}(v)$

Above the line

$e \Downarrow V \quad * H_1$

By IH on H_1

$e \mapsto^* V \quad * H_2$

By congruence lem

$in[l]\{\tau_1;\tau_2\}(e) \mapsto^* in[l]\{\tau_1;\tau_2\}(e$

on H_1

$in[l]\{\tau_1;\tau_2\}(e) \mapsto^* in[l]\{\tau_1;\tau_2\}(v$

Case: $$\frac{e \Downarrow in[l]\{\tau_1;\tau_2\}(v_2) \quad [v_2/y]e_2 \Downarrow V}{case(e; x.e_1; y.e_2) \Downarrow V}$$

$E = case(e; x.e_1; y.e_2)$

$V = V$

Above the line

$e \Downarrow in[r]\{\tau_1;\tau_2\}(v_2) \quad * H_1$

$[v_2/y]e_2 \Downarrow V \quad * H_2$

By IH on H1

$e \longmapsto^{*} in[\tau]\{\tau_1;\tau_2\}(v_2)$ * H3

By IH on H2

$\{v_2/y\}e_2 \longmapsto^{*} v$ * H3

By congruence lemma

$case(e; x.e_1; y.e_2) \mapsto case(e'; x.e_1; y.e_2)$

on H1

$Case(e; x.e_1; y.e_2) \mapsto$

$Case(in[\tau]\{\tau_1;\tau_2\}(v_2); x.e_1; y.e_2)$

* H4

By rule

$Case(in[\tau]\{\tau_1;\tau_2\}(e); x.e_1; y.e_2)$

$\mapsto [e/y]e_2$

Above eval * H5

By Lemma $e \Downarrow v$ then

$in[\tau]\{\tau_1;\tau_2\}(v_2)$ val

$Case(in[\tau]\{\tau_1;\tau_2\}(v_2); x.e_1; y.e_2)$

$\mapsto [v_2/y]e_2$ * H6

$Case(e; x.e_1; y.e_2) \mapsto^{*} [v_2/y]e_2$

H7

By transitivity on H3 & H4

$Case(e; x.e_1; y.e_2) \mapsto^{*} v$ ∎

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System F

Preservation

If $e : \tau$ & $e \mapsto e'$ then ~~$e \mapsto e'$~~ $e' : \tau$

Case:
$$\frac{e \mapsto e'}{App\{\tau\}(e) \mapsto App\{\tau\}(e')}$$

$e = App\{\tau\}(e)$

$e' = App\{\tau\}(e')$

Above the line

$e \mapsto e'$ + H_1

By inversion on $e : \tau$

$\Delta \Gamma \vdash e : all(t.\tau')$ + H_2

$\Delta \vdash \tau$ type + H_3

By IH on H_2 & H_1

we have $e' : all(t.\tau')$

by rule of APP

$App\{\tau\}(e') : [\tau/t]\tau'$ ∎

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Case:

App $\{\tau\}(\text{Lam}(t.e)) \mapsto [\tau/t]e$

$e = \text{App}\{\tau\}(\text{Lam}(t.e))$

$e' = [\tau/t]e$

By Inversion on $e : \tau$

$\Delta\Gamma \vdash \text{Lam}(t.e) : \text{all}(t.\tau') \quad H_1$

$\Delta \vdash \tau \text{ type} \quad H_3$

By inversion on $e : \text{all}(t.\tau')$ H_1

$\Delta, t \text{ type}\ \Gamma \vdash e : \tau' \quad H_4$

By S.L2 on H_3 & H_4

~~if Δ, t type $\Gamma \vdash [\tau/t]e' :$~~

if Δ, t type $\Gamma \vdash e' : \tau'$ & $\Delta \vdash \tau$ type

then $\Delta[\tau/t]\Gamma \vdash [\tau/t]e' : [\tau/t]\tau'$

So $[\tau/t]e] : [\tau/t]\tau'$

$t \notin \tau'$ then $[\tau/t]\tau' = \tau'$

$[\tau/t]e : \tau'$

Progress

if $e : \tau$ then either eval or there exists e', $e \mapsto e'$

Case: $\dfrac{\Delta, t \text{ type } \Gamma \vdash e : \tau}{\Delta \Gamma \vdash Lam(t.e) : all(t.\tau)}$

$e = Lam(t.e)$

$\tau = all(t.\tau)$ system F

Above the line

$\underbrace{\Delta, t \text{ type } \Gamma \vdash e : \tau}$ + H,

By IH on H either eval or ~~there exists~~ e', $e \mapsto e'$

no IH just by rul

Case: eval

by rul

$\therefore$ $Lam(t.e)$ val.

Case: $e \mapsto e'$

Impossible

<407>

Case: $\Delta' \vdash e : all(t.\tau')$ $\Delta \vdash \tau\ type$ 43

$\Delta\Gamma \vdash App\{\tau\}(e) : [\tau/t]\tau'$

$e = App\{\tau\}(e)$

$\tau = [\tau/t]\tau'$

Progress

Above the line

Suppose

$\Delta\Gamma \vdash e : all(t.\tau')$ *H,

$\Delta \vdash \tau\ type$ *H2

By IH on H, either e val or there exists e', $e \mapsto e'$

Case: $e \mapsto e'$

Take $e' = App\{\tau\}(e')$

then $e \mapsto e'$

Case: e val

By CFL if $e : all(t.\tau')$ & e val then $e = Lam(t.e)$

By ~~rule Lam(t.e) val~~

Take $E' = [\tau/t]e$

the $e \mapsto e'$

$\boxtimes$

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Big step rules

$$\frac{}{Lam\{t \cdot e\} \Downarrow Lam\{t \cdot e\}}$$

$$\frac{e \Downarrow Lam(t \cdot e) \quad [\tau/t]e \Downarrow v}{App\{\tau\}(e) \Downarrow v}$$

$$\frac{e \Downarrow \checkmark}{App\{\tau\}(e) \Downarrow v}$$

If $E \Downarrow v$ then $E \mapsto^* v$

Case : $\dfrac{e \Downarrow Lam(t \cdot e) \quad [\tau/t]e \Downarrow v'}{App\{\tau\}(e) \Downarrow v}$

$E = App\{\tau\}(e)$

$v = v'$

Above the line

$e \Downarrow Lam(t \cdot e)$ * H_1

$[\tau/t]e \Downarrow v'$ * H_2

By IH on H_1

$e \mapsto^* Lam(t \cdot e)$ * H_3

By IH on H_2

$[\tau/t]e \mapsto^* v'$ * H_4

By congruence Lemma

$$App\{\tau\}(e) \mapsto^* App\{\tau\}(e')$$

on H_1

$$App\{\tau\}(e) \mapsto^* App\{\tau\}(Lam(t.e))_{H_2}$$

By lem $e \Downarrow v$ then v val on H_1

$Lam(t.e)$ val

By rule e

$$App\{\tau\}(Lam(t.e)) \mapsto [\tau/t]e \quad H_6$$

By lemma $E \mapsto E'$ & E val

$E \mapsto^* E'$

$$App\{\tau\}(Lam(t.e)) \mapsto^* [\tau/t]e' \quad H_7$$

By transitivity on H_7 & H_4

$$App\{\tau\}(Lam(t.e)) \mapsto^* v' \quad H_8$$

By transitivity on H_5 & H_8

$$App\{\tau\}(e) \mapsto^* v' \quad \blacksquare$$

Regularity

if $\Gamma \vdash E : \tau$ then τ type, Γ valid

Case: $\dfrac{\Delta \vdash \tau_1 \text{ type} \quad \Delta, x : \tau_1 \vdash e : \tau_2}{\Delta \vdash lam\{\tau_1\}(x.e) : arr(\tau_1; \tau_2)}$

$E = lam\{\tau_1\}(x.e)$

$\tau = arr(\tau_1; \tau_2)$

Above the line

$\Delta \vdash \tau_1 \text{ type} \quad * H_1$

$\Delta, x : \tau_1 \vdash e : \tau_2 \quad * H_2$

~~Γ valid * H_3~~ — By strengthening on H_2

By IH on H_2 & ~~H_3~~ $\Delta, e : \tau_2 - H_2$

τ_2 type $* H_3$

Using 1 & 3

we have $arr(\tau_1; \tau_2)$ type.

Case: $\dfrac{\Delta \vdash e_1 : arr(\tau_2; \tau) \quad \Delta \vdash e_2 : \tau_2}{\Delta \vdash ap(e_1; e_2) : \tau}$

$E = ap(e_1; e_2)$

$\tau = \tau'$

Above the line

$\Delta \vdash e_1 : arr(\tau_2; \tau') \quad * H_1$

<44> $\Delta \vdash e_2 : \tau_2 \quad * H_2$

By inversion on H_1

$\Delta \vdash \tau_2$ type $* H_1$

$\Delta \Gamma, x : \tau_2 \vdash e : \tau' * H_2$ By strengthening on H_2 $\Delta \Gamma, e : \tau'$ — ③

By IH on ③ then τ' type $\Delta \Gamma e : \tau'$ type

<u>Case:</u> $\dfrac{\Delta, t \text{ type } \Gamma \vdash e : \tau}{\Delta \Gamma \vdash Lam(t \cdot e) : all(t \cdot \tau)}$

Regularity

$E = Lam(t \cdot e)$

$\tau = all(t \cdot \tau')$

Above the line

$\Delta, t \text{ type } \Gamma \vdash e : \tau * H_1$

By IH on H_1

$\Delta, t \text{ type} \vdash \tau$ type $* H_3$

By rule ✓

$all(t \cdot \tau)$ type

Cass: $\dfrac{\Delta \Gamma \vdash e : all(t \tau') \quad \Delta \vdash \tau \text{ type}}{\Delta \Gamma \vdash App\{\tau\}(e) : [\tau/t]\tau'}$ ~~strong~~ ~~rule~~

$E = App\{\tau\}(e)$

$\tau = [\tau/t]\tau'$

Above the line

$\Delta \Gamma \vdash e : all(t \tau') * H_1$

$\Delta \vdash \tau$ type $* H_2$

By Inversion on H_1

$\Delta, t\ type\ \Gamma \vdash e : \tau^*$ * H_3

Apply IH on H_3

$\Delta, t\ type\ \Gamma \vdash \tau'\ type$ * H_4

By SL(1) on H_4 & H_2

we have $[\tau / t]\tau'\ type$

System FE

<u>Preservation</u>

If $e : \tau$ & $e \mapsto e'$ then $e' : \tau$

Case:
$$\frac{e \mapsto e'}{pack\{t.\tau\}\{\rho\}(e) \mapsto pack\{t.\tau\}\{\rho\}(e')}$$

$e = pack\{t.\tau\}\{\rho\}(e)$

$e' = pack\{t.\tau\}\{\rho\}(e')$

Above the line

$e \mapsto e'$ * H_1

By inversion on $e : \tau$

$\Delta \vdash \rho\ type$ * H_1

<46> $\Delta, t\ type \vdash \tau\ type$ * H_2

By Inversion on H_1

$\Delta, t\ type\ \Gamma \vdash e : \tau^* \quad * H_3$

Apply IH on H_3

$\Delta, t\ type\ \Gamma \vdash \tau'\ type \quad * H_4$

By SL(1) on H_4 & H_2

we have $[\tau / t]\tau'\ type$ ▨

System FE

<u>Preservation</u>

If $e : \tau$ & $e \mapsto e'$ then $e' : \tau$

Case:
$$\frac{e \mapsto e'}{pack\{t.\tau\}\{\rho\}(e) \mapsto pack\{t.\tau\}\{\rho\}(e')}$$

$e = pack\{t.\tau\}\{\rho\}(e)$

$e' = pack\{t.\tau\}\{\rho\}(e')$

Above the line

$e \mapsto e' \quad * H_1$

By inversion on $e : \tau$

$\Delta \vdash \rho\ type \quad * H_1$

$\Delta, t\ type \vdash \tau\ type \quad * H_2$

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$\Delta\Gamma \vdash e : [\rho/t]\tau$ * H_3

By IH on H_3 & H_1

$e' : [\rho/t]\tau$

By rule

pack $\{t.\tau\}\{\rho\}(e') : \text{some}(t.\tau)$

$\blacksquare$

Case: $\dfrac{e_1 \mapsto e_1'}{\text{open}\{t.\tau\}\{\rho\}(e_1; t, x.e_2) \mapsto \text{open}\{t.\tau\}\{\rho\}(e_1'; t, x.e_2)}$

$e = \text{open}\{t.\tau\}\{\rho\}(e_1; t, x.e_2)$

$e' = \text{open}\{t.\tau\}\{\rho\}(e_1'; t, x.e_2)$

Above the line.

$e_1 \mapsto e_1'$ H_0

By inversion on $e : \tau$

then

$\Delta\Gamma \vdash e_1 : \text{some}(t.\tau)$ * H_1

$\Delta, t \text{ type } \Gamma, x : \tau \vdash e_2 : \tau_2$ * H_2

$\Delta \vdash \tau_2 \text{ type}$ * H_3

By IH on H_1 & H_0

then $e_1' : \text{some}(t.\tau')$

By rule: $\text{open}\{t.\tau\}\{\rho\}(e_1'; t, x.e_2) :$

$\Delta : 2$ $\blacksquare$

Case: [eval]

open{t.τ}{ρ}(pack{t.τ}{ρ}(e); t,x.e₂)

Preservation

$\longmapsto [\rho, e/t, x]e_2$

$e = \text{open}\{t.\tau\}\{\rho\}(\text{pack}\{t.\tau\}\{\rho\}(e); t, x.e_2)$

$e' = [\rho, e/t, x]e_2$

Above the line

eval ∗ H1

By inversion on $e : \tau$

$\Delta \Gamma \vdash \text{pack}\{t.\tau\}\{\rho\}(e) : \text{some}(t.\tau)$ ∗ H2

$\Delta, t \text{ type}\ \Gamma, x : \tau \vdash e_2 : \tau_2$ ∗ H3

$\Delta \vdash \tau_2 \text{ type}$ ∗ H3

$x : \tau_1 \vdash e_1$

not context

By Inversion on H2

$\Delta \vdash \rho \text{ type}$ ∗ H4

$\Delta, t \text{ type} \vdash \tau \text{ type}$ ∗ H5

$\Delta \Gamma \vdash e : [\rho/t]\tau$ ∗ H6

By S.L on H3 & H6 (S.L)

$[\rho, e/t, x]e_2 : \tau_2$

$x : [\rho/t]\tau_1 \vdash [\rho/t]e_2 : [\rho/t]\tau_2$

But by 3) the $x : [\rho/t]\tau_1 \vdash [\rho/t]e_2 : \tau_2$

S.L. 3 on 4 & 6

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Progress

- If $e : \tau$ then either e val or there exists e', $e \mapsto e'$

Case: $\dfrac{\Delta \vdash \rho \text{ type} \quad \Delta, t \text{ type} \vdash \tau \text{ type} \quad \Delta; \Gamma \vdash e : [\rho/t]\tau}{\Delta; \Gamma \vdash \text{pack}\{t.\tau\}\{\rho\}(e) : \text{some}(t.\tau)}$

$e = \text{pack}\{t.\tau\}\{\rho\}(e)$

$\tau = \text{some}(t.\tau)$

Above the line

$\Delta \vdash \rho \text{ type} \quad * \quad H_1$

$\Delta, t \text{ type} \vdash \tau \text{ type} \quad * \quad H_2$

$\Delta; \Gamma \vdash e : [\rho/t]\tau \quad * \quad H_3$

By IH on H_3 either e val or there exists e val or e', $e \mapsto e'$

Case: $e \mapsto e'$

Take $e' = \text{pack}\{t.\tau\}\{\rho\}(e')$

then $e \mapsto e'$

Case: e val

By rule $\text{pack}\{t.\tau\}\{\rho\}(e)$ val

$\blacksquare$

$t \notin \Gamma_2$

Case: $\Delta \Gamma \vdash e_1 : \text{some}(t.\tau)$ $\Delta, t\ \text{type}\ \Gamma, x:\tau \vdash e_2 : \tau_2$ $\Delta \vdash$

$\Delta \Gamma \vdash \text{open}\{t.\tau\}\{p\}(e_1; t, x.e_2) : \tau_2$

$e = \text{open}\{t.\tau\}\{p\}(e_1; t, x.e_2)$

$\tau = \tau_2$

Above the line

$\Delta \Gamma \vdash e_1 : \text{some}(t.\tau)$ $\mathcal{H}_1$

$\Delta, t\ \text{type}\ \Gamma, x:\tau \vdash e_2 : \tau_2$ $\mathcal{H}_2$

$\Delta \vdash \tau_2\ \text{type}$ $\mathcal{H}_3$

By IH on $\mathcal{H}_1$

Either e_1 val or there e_1' such that $e_1', e_1 \mapsto e_1'$

Case: $e_1 \mapsto e_1'$

Take: $e_1' = \text{open}\{t.\tau\}\{p\}(e_1'; t, x.e_2)$

Then $e_1 \mapsto e_1'$

Case: e_1 val

By CFL $e_1 : \text{some}(t.\tau)$

& e_1 val

$e_1 = \text{pack}\{t.\tau\}\{p\}(e)$

take $e_1' = [p, e/t, x]e_2$

Calypse the $e_1 \mapsto e_1'$

Big Step

$$\frac{}{\text{pack}\{t.\tau\}\{\rho\}(e) \Downarrow \text{pack}\{t.\tau\}\{\rho\}(e)}$$

$$\frac{e \Downarrow \text{pack}\{t.\tau\}\{\rho\}(e) \quad [\rho/t]e_2 \Downarrow}{\text{open}\{t.\tau\}\{\rho\}(e_1; t.x.e_2) \Downarrow V}$$

$$\frac{e \Downarrow V}{\text{pack}\{t.\tau\}\{\rho\}(e) \Downarrow V}$$

$$\frac{e \Downarrow \text{pack}\{t.\tau\}\{\rho\}(e) \quad [\rho/t][e/x]e_2 \Downarrow}{\text{open}\{t.\tau\}\{\rho\}(e_1; t.x.e_2) \Downarrow V}$$

If $E \Downarrow v$ then $E \mapsto^* V$

Case $\dfrac{e \Downarrow V'}{\text{pack}\{t.\tau\}\{\rho\}(e) \Downarrow V}$

$E = \text{pack}\{t.\tau\}\{\rho\}(e)$

$V = V'$

Above the line

$e \Downarrow V'$ IH

Apply IH to H_1

$e \mapsto^* v' \quad * H_2$

By congruence lemma -

$\text{pack}\{t.\tau\}\{\rho\}(e) \mapsto^* \text{pack}\{t.\tau\}\{\rho\}(e)$

on H_1

$\text{pack}\{t.\tau\}\{\rho\}(e) \mapsto^* \text{pack}\{t.\tau\}\{\rho\}$

Regularity

If $e : \tau$ then τ type

Case: $\dfrac{\Delta \vdash \rho \text{ type} \quad \Delta, t \text{ type} \vdash \tau \text{ type} \quad \Delta \Gamma \vdash e : [\rho/t]\tau}{\text{pack}\{t.\tau\}\{\rho\}(e) : \text{some}(t.\tau)}$

$e = \text{pack}\{t.\tau\}\{\rho\}(e)$

$\tau = \text{some}(t.\tau)$

Above the line

$\Delta \vdash \rho \text{ type} \quad * H_1$

$\Delta, t \text{ type} \vdash \tau \text{ type} \quad * H_2$

$\Delta \Gamma \vdash e : [\rho/t]\tau \quad * H_3$

By rule $\dfrac{\Delta, t \text{ type} \vdash \tau \text{ type}}{\text{some}(t.\tau) \text{ type}}$

some(t.τ) type

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$\boxtimes$

$\Delta\Gamma \vdash e_1 : some(t.\tau)$ $\Delta, t\ type\ \Gamma, x:\tau \vdash e_2 : \tau_2$ $\Delta \vdash \tau_2\ type$

$open\{t.\tau\}\{p\}(e_1; t, x.e_2) : \tau_2$

$e = open\{t.\tau\}\{e\}(e_1; t, x.e_2)$

$\tau = \tau_2$

Regularity

Above the line

$\Delta\Gamma \vdash e_1 : some(t.\tau)$ * H_1

$\Delta, t\ type\ \Gamma, x:\tau \vdash e_2 : \tau$ * H_2

proved $\boxed{\Delta \vdash \tau_2\ type}$ * H_3

Transition Lemma H_2

$\Gamma \vdash e_2 : \tau_2$ — (H2)

Union H_3

$e_2 : \tau_2$

τ_2 type

ETT System PCF

20-30 | 10-20 | 10-15

Preservation

If $e : \tau$ & $e \mapsto e'$ then $e' : \tau$

Case: $\dfrac{e \mapsto e'}{ifz\{e_0; x.e_1\}(e) \mapsto ifz\{e_0; x}$

$e = ifz\{e_0; x.e_1\}(e)$

$e' = ifz\{e_0; x.e_1\}(e')$

Above the line

$e \mapsto e'$ * H_1

By inversion on $e : \tau$

$\Gamma \vdash e : nat$ * H_2 & $\Gamma, x : nat, e : \tau$ *

$\Gamma \vdash e_0 : \tau$ * H_3

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Apply IH to H2 & H1

$e' : nat$

by rule

$ifz\{e_0; x.e_1\}(e') : \tau$

Case

$$\frac{}{ifz\{e_0; x.e_1\}(z) \mapsto e_0}$$

~~by~~ $e = ifz\{e_0; x.e_1\}(z)$

$e' = e_0$

By inversion on $e : \tau$

$\Gamma \vdash e : nat$ * H1

$\Gamma \vdash e_0 : \tau$ * H2

$\Gamma, x : nat, e_1 : \tau$ * H3

Case:

$$\frac{s(e)\ val}{ifz\{e_0; x.e_1\}\, s(e) \mapsto [e/x]e_1}$$

$e = ifz\{e_0; x.e_1\}\, s(e)$

$e' = [e/x]e_1$

Above the line

$s(e)\ val$ *

$\Gamma \vdash e : nat * H_1$ by invers $e : nat$ H_1'

$\Gamma \vdash e_0 : \tau * H_2$

$\Gamma, x : nat \vdash e_1 : \tau * H_3$

Apply S.L. on H_1'' & H_3

$[e/x]e_1 : \tau$ ▨

<u>Case</u>:

$$\frac{}{fix\{\tau\}(x.e) \mapsto [fix\{\tau\}(x.e)/x]e}$$

$e = fix\{\tau\}(x.e)$

$e' = [fix\{\tau\}(x.e)/x]e$

By inversion on $e : \tau$

$\Gamma, x : \tau \vdash e : \tau * H_1$

~~By inversion on~~

By rule

$\Gamma \vdash fix\{\tau\}(x.e) : \tau * H_2$

By S.L on H_1 & H_2

$[fix\{\tau\}(x.e)/x]e : \tau$ ▨

If $e : \tau$ then either eval or there exists e', $e \mapsto e'$

Case: $\dfrac{\Gamma \vdash e : nat \quad \Gamma \vdash e_0 : \tau' \quad \Gamma, x : nat \vdash e_1 : \tau'}{\Gamma \vdash ifz\{e_0; x.e_1\}(e) : \tau'}$

$e = ifz\{e_0; x.e_1\}(e)$

$\tau : \tau'$

Above the line

$\Gamma \vdash e : nat \ast H_1$

$\Gamma \vdash e_0 : \tau' \ast H_2$

$\Gamma, x : nat \vdash e_1 : \tau' \ast H_3$

By IH on H_1 either eval or there exists e' ; $e \mapsto e'$

Case: $e \mapsto e'$

Take e' : $ifz\{e_0; x.e_1\}(e')$

then $e \mapsto e'$

Case: eval

By CFL $e : nat$ the

$e = z$ or $e = s(e)$

Subcase: $e = z$

Take $e' = e_0$

then $e \mapsto e'$

Subcase $e = s(e)$

take $e' = [e/x]e_1$

then $e \mapsto e'$

Case: $\dfrac{\Gamma, x:\tau' \vdash e : \tau'}{\Gamma \vdash fix\{\tau\}(x.e) : \tau'}$

$e = fix\{\tau\}(x.e)$

$\tau = \tau'$

Above the line

$\Gamma, x:\tau' \vdash e : \tau'$ *H1

NOT By IH on H1 either e val

or there exists e', $e \mapsto e'$

Case: $e \mapsto e'$

Take $e' = [fix\{\tau\}(x.e)/x]e$

then $e \mapsto e'$

Case: e val

Impossible ✗

Big Step rules

$$z \Downarrow z$$

$$\frac{e \Downarrow v}{s(e) \Downarrow v}$$

$$\overline{lam\{\tau\}(x.e) \Downarrow lam\{\tau\}(x.e)}$$

lazy
$$\frac{e_1 \Downarrow lam\{\tau\}(x.e) \quad [e_2/x]e \Downarrow v}{ap(e_1; e_2) \Downarrow v}$$

Eager
$$\frac{e_1 \Downarrow lam\{\tau\}(x.e \quad e_2 \Downarrow v_1 \quad [v_1/x]e \Downarrow v}{ap(e_1; e_2) \Downarrow v}$$

$$\frac{e \Downarrow z \quad e_0 \Downarrow v}{ifz\{e_0; x.e_1\}(e) \Downarrow v}$$

$$\frac{e \Downarrow s(e) \quad [e/x]e_1 \Downarrow v}{ifz\{e_0; x.e_1\}(e) \Downarrow v}$$

$$\frac{[fix\{\tau\}(x.e)/x]e \Downarrow v}{fix\{\tau\}(x.e) \Downarrow v}$$

If $E \Downarrow v$ then $E \mapsto^* v$

Case: $\dfrac{e \Downarrow z \quad e_0 \Downarrow v'}{\text{if } z\{e_0; x.e_1\}(e) \Downarrow v'}$

$E = \text{if } z\{e_0; x.e_1\}(e)$

$v = v'$

Above the line

$e \Downarrow z$: H_1

$e_0 \Downarrow v'$: H_2

By IH on H_1

$e \mapsto^* z$: H_3

By IH on H_2

$e_0 \mapsto^* v'$: H_4

By congruence lemm $E' \mapsto^* E''$

then $\text{if } z\{e_0; x.e_1\}(e) \mapsto^* \text{if } z\{e_0; x.e_1\}(e')$

on 3

$\text{if } z\{e_0; x.e_1\}(e) \mapsto^* \text{if } z\{e_0; x.e_1\}(z)$ H_5

By <u>if z rule</u>

$\text{if } z\{e_0; x.e_1\}(z) \mapsto e_0$ $\rightarrow H_{51}$

By lemma $E \Downarrow v$ the v val 59

$e_0 \Downarrow v'$ the v val $\rightarrow H_6$

By lemm if $E \mapsto E'$ & e' val $E \mapsto E'$

$\text{if } z\{e_0; x.e_1\}(z) \mapsto^* e_0$: H_7

By transitivity on H_7 & H_4

$if\, z\{e_0; x.e_1\}(z) \xrightarrow{+} V' \rightarrow H_8$

By transitivity on H_8 & H_3

$if\, z\{e_0; x.e_1\}(e) \longmapsto V'$ ▨

<u>Case</u>:

$$\frac{e \Downarrow s(e) \quad [e/x]e_1 \Downarrow V'}{if\, z\{e_0; x.e_1\}(e) \Downarrow V'}$$

$E = if\, z\{e_0; x.e_1\}(e)$

$V = V'$

Above the line

$e \Downarrow s(e)$ * H_1

$[e/x]e_1 \Downarrow V'$ * H_2

By IH on H_1

$e \longmapsto^* s(e)$ * H_3

By IH on H_2

$[e/x]e_1 \longmapsto^* V'$ * H_4

By congruence lemma

$if\, z\{e_0; x.e_1\}(e) \longmapsto^* if\, z\{e_0; x.e_1\}(e')$

on H_3

$if\,z\,\{e_0 ; x.e_1\}(e) \longmapsto^{*} if\,z\,\{e_0 ; x.e_1\}(s(e))$ H5

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By <u>if z rule</u>

$if\,z\,\{e_0 ; x.e_1\}(s(e)) \longmapsto [e/x]e_1$ H6

By lemma $E \Downarrow v$ then v val

v' val

By lemma

$if\,z\{e_0 ; x.e_1\}(s(e)) \longmapsto^{*} [e/x]e_1$ H7

By transitivity on H7 & H4

$if\,z\{e_0 ; x.e_1\}(s(e)) \longmapsto^{*} v'$ H8

By transitivity on H5 & H8

$if\,z\{e_0 ; x.e_1\}(e) \longmapsto^{*} v'$

<u>Case</u> $\dfrac{[fix\{\tau\}(x.e)/x]e \Downarrow v'}{fix\{\tau\}(x.e) \Downarrow v'}$

$E = fix\{\tau\}(x.e)$

$v = v'$

Above the line

$[fix\{\tau\}(x.e)/x]e \Downarrow v$ H1

By IH on H1

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$[fix\{\tau\}(x.e)/x]e \longmapsto^{*} v$ H2

By fix rule

$\text{fix}\{\tau\}(x.e) \longmapsto [\text{fix}\{\tau\}(x.e)/x]e$ H_5

By Lemma E.1 v the val on H,

v val $\rightarrow H_4$

By conf lemma on H_3 & H_4

$\text{fix}\{\tau\}(x.e) \mapsto [\text{fix}\{\tau\}(x.e)/x]e$ H_5

By transitivity on H_3 & H_5

n ∈ Nat ::= z | s(n) $\text{fix}\{\tau\}(x.e) \mapsto^* v'$ ▨

Typed version of syntax char

Syntax chart for E

$n \in \text{Nat} ::= z \mid s(n)$

$s \in \text{Str} ::= \text{null} \mid \text{cons}(n; s)$

$\tau \in \text{Typ} ::= \text{num} \mid \text{str}$

$e \in \text{Exp} ::= x \mid \text{num}(n) \mid \text{str}(s) \mid$
$\text{plus}(e_1; e_2) \mid \text{times}(e_1; e_2) \mid$
$\text{cat}(e_1; e_2) \mid \text{len}(e_1) \mid$
$\text{let}(e_1; x.e_2)$

$$\left.\begin{array}{l} nat : type \\ str : type \\ typ : type \\ Exp : Type \end{array}\right\} \text{syntactic categories}$$

$z : nat$

$s(n) : nat \to nat$

$null : str$

$cons(n; s) : nat \to str \to str$

$num\ t : typ$

$str\ t : typ$

$num : nat \to Exp$

$str : str \to exp$

$plus : exp \to exp \to exp$

$times : exp \to exp \to exp$

cat : "

$len : exp \to exp$

$let : exp \to (exp \to exp) \to exp$

<u>Jugements</u>

$e : \tau$ $\quad$ $type_of : exp \to typ \to o$

$e_1 \mapsto e_2$ $\quad$ $step : exp \to exp \to o$

$e\ NV$ $\quad$ $eval :$ "

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash let(e_1 ; x . e_2) : \tau_2}$$

$$\frac{\cdot \vdash num(3) : num \quad \dfrac{\dfrac{}{x : num \vdash x : num} \quad \dfrac{}{y : num \vdash x : num}}{x : num \vdash plus(x ; x) : num}}{\cdot \vdash let(num(3) ; x . plus(x ; x)) : num}$$

18) $(P \vee Q) \to R = (P \to R) \wedge (Q \wedge R)$

Guideline for construction of elements

7. → Produce if $X, \to, \forall$ → (Not $\vee, \exists$) ↳ dont produce right ay

↳ use if $X, \vee, \exists$ ↳ (not $\to, \forall$)

lam $(x . e)$:

$x : (P + Q) \to R$

$e = \langle e_1 , e_2 \rangle$ pair (e1,e2) $e_1 : P \to R$, $e_2 : Q \to R$

$e_1 = lam(p . f_1^{\to R})$

$f_1 = ap\ x ; \quad ; Q\}(p)) : P$

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$e_2 = lam\ (q \cdot f_2)$

$f_2 = ap(x ; inr\{P ; Q\}(q)))$

$(P \to R) \wedge (Q \wedge R) \to (P \vee R) \to R$

~~lam(x · e)~~

$lam\{P \to R \times Q \to R\}(x \cdot e)$.

$lam\{P + Q\}(s \cdot e_1)$ | $e : (P+Q) \to R$, $e : R$

$e_1 = case(s ; p \cdot f_1 ; q \cdot f_2)$

$p : P$

$q : Q$

$f_1 = ap(fst(x) ; p) : R$

$f_2 = ap(snd(x) ; q) : R$

7). $[(\exists t \cdot P(t)) \mapsto A] \to \forall t\ P(t) \to A$

$lam\{(\exists t \cdot P(t)) \mapsto A\}(x \cdot e)$ $\quad e : \forall t\ P(t) \to A$

$e = \Lambda t \cdot e_1$ $\quad e_1 : P(t) \to A$

$e_1 = lam\{P(t)\}(p \cdot a) : A$

$a = ap(f ; pack\{t \cdot P(t)\}\{t\}(p))$

$(t) \to A\} \to [(\exists t\ P(t) \to A]$ 18

lam $\{\forall t\ P(t) \to A\}(x \cdot e)$

lam $\{\exists t\ P(t)\}(u \cdot f)$ $\quad e : \exists t\ P(t) \to A$

$f : A$

θ = open $(e; t, u.\ ap(x[t]; u))$

	const	elim
void	none	abort
unit	$\langle\rangle$	none
sum	inl, inr	case
prod	$\langle\rangle$	fst, snd
$\exists$	pack	open
$\forall$	Λ	$[\ ]$
$\to$	λ	$[\ ]$

1) $T \wedge P = P$

$T \times P = P$

lam $\{T \times P\}(x \cdot e) \cdot$ snd(x) $\quad e : P$

2) $P = T \times P$

lam $\{P\}(x \cdot e) \cdot$ pair$(e_1; e_2)$ $\quad e : T \times P$

e_1 : unit

e_2 : snd(p) x

$\lambda f : P \cdot \langle$unit, $x \rangle$

3) $T \vee P = T$ $\quad e$

lam $\{T + P\}(x \cdot e) \cdot$ ~~inl~~ true

4) $T \to T + P$ $\quad e : T + P$

lam $\{T\}(x \cdot e)$: inl$\{T, P\}\{x\}$.

λf : unit. inl(f)

because there is no variable of type P in context

5) $F \wedge P = F$

void $\times$ P $\rightarrow$ void

lam{void $\times$P}(x.e). fst(x)

6) $F \rightarrow F \times P$.

lam{void}(x.e) pair(e_1; e_2) $\quad e : F \times P$

e_1 : x

e_2 : abort{P}(x)

7) $F \vee P = P$

void + P = P

lam{void+P}(x.e). $\quad e : P$

case(x; void.x.abort{P}(x); b.b)

8) $P \rightarrow F + P$

lam{P}(x.e). inr{F, P}(x) $\quad e : F+P$

$\lambda f : P$. inl{abort{P}(void)} ↑ inr(f)

9) $(T \rightarrow P) \rightarrow P$

lam{T → P}(x.e). ap(x; <>);

$\lambda f : T \rightarrow P$. f(<>)

10) $P \to (F \to P)$ $\quad e : F \to P$

$lam\{P\}(x \cdot e) \cdot lam\{F\}(f \cdot e_1) \cdot x$ $\quad e_1 : F$

11) $P \to T = T$

$e : T$

$lam\{P \to T\}(x \cdot e) \cdot \langle\rangle$

12) $T \to (P \to T)$

$e : P \to T$

$e_1 : T$

$lam\{T\}(x \cdot e) \cdot lam\{P\}(y \cdot e_1) \cdot \langle\rangle$ or x

13) $F \to P = T$

$void \to P = unit$ $\quad e \cdot un$

$lam\{void \to P\}(x \cdot e) \cdot \langle\rangle$

14) $T \to (F \to P)$

$lam\{T\}(x \cdot e) \cdot lam\{void\}(y \cdot e_1)$ $\quad e : F \to P$

$e_1 : P$

~~abort~~

$abort\{P\}(y)$.

15) $P \wedge P \to P$

λf:

$lam\{P \times P\}(x \cdot e) \cdot fst(x)$

16) $P \to P \times P$

$lam\{P\}(x \cdot e) \cdot pair(x, x)$ $\quad e : P \times P$

$\lambda f : P \cdot \langle f, f \rangle$

17) $P \vee P = P$

$lam\{P+P\}(x \cdot e) \cdot case(x; \vec{a}^{P} \cdot a$
$\vec{b}^{P} : b)$

18) $P \to P + P$

$lam\{P\}(x \cdot e) \cdot inl\{P; P\}(x)$

19) $P \to P = T$

$lam\{P \to P\}(x \cdot \vec{e}^{T}) \cdot \langle\rangle$

20) $T \to (P \to P)$

$lam\{T\}(x \cdot \vec{e}^{P \to T}) \cdot lam\{P\}(y \cdot \vec{e}_1^{P})$
$\cdot y$

21) $P \wedge Q = Q \wedge P$

$P \times Q = Q \times P$

$lam\{P \times Q\}(x \cdot \vec{e}^{Q \times P}) \cdot pair(snd(x); fst(x))$

$\lambda p : P \times Q \cdot \langle snd(p), fst(p) \rangle$

22) $P \vee Q = Q \vee P$

$lam\{P+Q\}(x \cdot \vec{e}^{Q+P}) \cdot case(x \cdot \vec{a}^{P} \cdot e_1 \; \vec{b}^{Q} \cdot e_2)$

$e_1 : inr\{P, Q\}(a^{x})$

$e_2 : inl\{P, Q\}(b^{x})$

23) $P \wedge (Q \wedge R) = (P \wedge Q) \wedge R$

$e : (P \times Q) \times R$

$lam\{P \times (Q \times R)\}(x.e)$.

$pair(pair(fst(x); \overset{?}{fst(snd(x))}); snd(snd(x)))$

$\lambda p : P \times (Q \times R).\ \langle \langle fst(p), fst(snd(p)) \rangle, snd(snd(p)) \rangle$

24) $(P \wedge Q) \wedge R \rightarrow P \wedge (Q \wedge R)$

$P \times (Q \times R)$

$lam\{(P \times Q) \times R\}(x.e)$.

$pair(fst(fst(x)); pair(snd(fst(x)); snd(x)))$

$\lambda p : (P \times Q) \times R.\ \langle fst(fst(p)), \langle snd(fst(p)), snd(p) \rangle \rangle$

25) $P \vee (Q \vee R) = (P \vee Q) \vee R$

$lam\{P + (Q + R)\}.(x.e)$.

$case(x; \overset{P}{a}.\ e_1 \quad \overset{Q+R}{b}.\ e_2)$

$e_1 = inl(a)$

$e_2 = case(b; \overset{Q}{u}.\ inl(u) \quad \overset{R}{v}.\ inr(v))$

26) $(P + Q) + R \rightarrow P + (Q + R)$

$P + (Q + R)$

$lam\{(P+Q) + R\}(x.e)$. $case(x; \overset{P+Q}{a}.\ e_1 \quad \overset{R}{b}.\ e_2)$

$e_1 : case(a; \overset{P}{u}.\ inl(u) \quad \overset{Q}{v}.\ inr\{P, Q+R\}(inl(v)))$

$e_2 : inr(inr(b))$

27) $P \times (Q + R) \to (P \times Q) + (P \times R)$

$e : (P \times Q) + (P \times R)$

$lam\{P \times (Q+R)\}(x \cdot e) \cdot case(snd(x); u \cdot e_1 \; v \cdot e_2)$

$e_1 = inl(pair(fst(x); \cancel{inl(snd(x))}\; u))$

$e_2 = inr(pair(fst(x); \cancel{inr(snd(x))}\; v))$

28) $(P \times Q) + (P \times R) \to P \times (Q + R)$

$lam\{(P \times Q) + (P \times R)\}(x \cdot e) \cdot pair(e_1, e_2)$

$e_1 = case(x; a \cdot fst(a) \; b \cdot fst(b))$

$e_2 = case(x; u \cdot inl(snd(u)) \; v \cdot inr(snd(v)))$

or

$- case(x; a \cdot \langle fst(a), inl(snd(a)) \rangle \; b \cdot \langle fsta; inr(snd(b)) \rangle)$

$P \times (Q+R)$

29) $P + (Q \times R) \to (P + Q) \times (P + R)$

$lam\{P + (Q \times R)\}(x \cdot e) \cdot \underline{pair}(e_1; e_2)$

$e_1 = case(x; a \cdot inl(a) \; b \cdot inr(fst(b)))$

$e_2 = case(x; u \cdot inl(u), v \cdot inr$

30) $(P+Q)\times(Q+R) \to P+(Q\times R)$

$P+(Q\times R)$

$lam\{(P+Q)\times(P+R)\}(x.e)$

$case(fst(x); \overset{P}{a}.inl(a)\ \overset{Q}{b}.$

$P+Q\times Q$

$e_2 = case(snd(x); \overset{P}{u}; inl(u)\ \overset{Q}{v}\ \langle b, v\rangle$

37) $(P\to Q)\wedge(P\to R) = P\to(Q\wedge R)$

$e: P\to(Q\times R)$

$lam\{(P\to Q)\times(P\to R)\}(x.e)$

$lam\{P\}(y.\overset{Q\times R}{e_1}).pair(\overset{Q}{e_2}, \overset{R}{e_3})$

e_2: $ap(fst(x); \overset{y}{\cancel{fst(x)}})$

e_3: $ap(snd(x); \overset{y}{\cancel{snd(x)}})$

$\lambda f: (P\to Q)\times(P\to R).\lambda x.P.\langle \overset{Q}{fst(f)(x)}, \overset{R}{snd(f)(x)}\rangle$

32) $(P \vee Q) \to R = (P \to R) \times (Q$

$(P \to R) \times (Q \to R)$

$lam\{(P+Q) \to R\}(x \cdot e) = \langle e_1 , e_2 \rangle$

$e_1 = P \to R$

$e_2 = Q \to R$

$e_1 : lam\{P\}(y \cdot f).ap(x; inl\{P, Q\}(y))$

$e_2 : lam\{Q\}(w \cdot f_1).ap(x; inr\{P, Q\}(w))$

$\lambda f : (P+Q) \to R . \langle e_1 , e_2 \rangle$

$e_1 = \lambda p : P . f(inl(p))$

$e_2 = \lambda q : Q . f(inr(q))$

33) $(P \to R) \times (Q \to R) \to ((P+Q) \to R)$

$lam\{(P \to R) \times (Q \to R)\}(x \cdot e) = lam\{P+Q\}(y \cdot e_1)$

$case(y; u \cdot ap(fst(x); u); v : ap(snd(x); v))$

34) $(P \to (Q \to R) = ((P \wedge Q) \to R)$

$lam\{P \to (Q \to R)\}(x \cdot e) = lam\{P \times Q\}(y \cdot e_1)$

$e = lam\{P \times Q\}(y \cdot e_1)$

$e_1 = ap(x; ap(fst(y)); snd(y))$

$ap(ap(x; fst(y)); snd(y))$

35) $((P \wedge Q) \to R) \to (P \to (Q \to R))$ 76

$lam\{(P \wedge Q) \to R\}(x \cdot e)$ $\Big|_{e = lam\{P\}(y \cdot e_1)}$ $(P \times Q) \to R$ $P \to (Q \to R)$

$e_1 = lam\{Q\}(v \cdot e_2)$

$e_2 = ap(x; pair(y, v))$

Implication

1. $((P \to Q) + (P \to R)) \to (P \to (Q + R))$

$lam\{(P \to Q) + (P \to R)\}(x \cdot e)$ $P \to (Q + R)$

$lam\{P\}(p \cdot e_2)$ $Q + R$. $case(x; u \cdot f_1, v \cdot f_2)$ $P \to Q$ $P \to Q$

$f_1 = (inl\{Q, R\}(x); p)$

$f_2 = (inr\{Q, R\}(x); p)$

2. $((P \to R) + (Q \to R)) \to ((P \times Q) \to R)$

$lam\{(P \to R) + (Q \to R)\}(x \cdot e)$

$e = lam\{P \times Q\}(p \cdot e_2)$ $P \times Q$ R $\Big|_{e_2 =}$ $case(x; u \cdot f_1, v \cdot f_2)$ $P \to R$ $Q \to R$

$f_1 = ap(fst(p); u)$ $\times$ $ap(v; fst(p))$

$f_2 = ap(snd(p); v)$ $\times$ $ap(v; snd(p))$

Quantifier Equalities

1) $\forall t \cdot \forall u \cdot s(t,u) \Rightarrow \forall u \cdot \forall t \; s(t,u)$

$lam\{\forall t \cdot \forall u \cdot s(t,u)\}(x \cdot e)$.

$e = \underline{Lam}(u \cdot \overset{e = \forall t \, s(t,u)}{e_2}) \; \{ e_2 = Lam(t \cdot \overset{e_T(s(t,u)}{e_1})$

?

~~App{t,u}(x)~~

~~ap(App{t}(f), App{u}(f))~~

~~App{t,u}(x)~~

$App\{u\}(App\{t\}(x))$ $\quad e_1 = \cancel{App}\{t,u\}(x)$

$\lambda f : \forall t \cdot \forall u \cdot s(t,u)$.

$\Lambda t \cdot \Lambda u \cdot f[t][u]$

2) $\forall t \cdot P(t) \wedge Q(t) = (\forall t \cdot P(t)) \wedge (\forall t \cdot Q(t))$

$\lambda f : \forall t \cdot P(t) \wedge Q(t)$.

$\langle \Lambda t \cdot fst f(t), \Lambda t \cdot snd(f) \rangle$

$lam\{\forall t \cdot P(t) \times Q(t)\}(x \cdot e)$.

$pair(Lam(t \cdot \overset{P(t)}{e_1}) \cdot App\{t\}(fst(x)),$

$Lam(t \cdot \overset{Q(t)}{e_2}) \cdot snd(x))$.

3) $(\forall t \cdot P(t) \to \forall t \cdot Q(t)) \to \forall t \cdot P(t) \to Q(t)$

$\lambda f : \forall t \cdot P(t) \to \forall t \cdot Q(t) .\ \Lambda t .\ \lambda x : P(t) .$
$f(t)(x)$

4) $(\forall t \cdot P(t) \to Q(t)) \to ((\forall t \cdot P(t)) \to (\forall t \cdot Q(t))$

$\lambda f : \forall t \cdot P(t) \to Q(t)$
$\lambda x : \forall t \cdot P(t) .\ \Lambda t (f[t])(x[t])$

$lam\{\forall t \cdot P(t) \to Q(t)\}(x \cdot e)$
$lam\{\forall t \cdot P(t)\}(y \cdot e_1) .$
$Lam(t \cdot e_2) .$
$ap(App\{t\}(x) ; App\{t\}(y)) .$

5) $(\forall t \cdot P(t)) \wedge (\forall t \cdot Q(t)) \to \forall t\, P(t) \wedge Q(t)$

$\lambda f : (\forall t \cdot P(t)) \wedge (\forall t\, Q(t)) .\ \Lambda t\ App\{t\}(fst(f))$
$\langle fst(f)[t] , snd(f)[t] \rangle$
$P(t)$ $Q(t)$

$lam\{(\forall t \cdot P(t)) \times (\forall t\, Q(t))\}(x \cdot e) ,$
$e = Lam\{t \cdot e_2\}$ e_1 $pair(App\{t\}(fst(f)) ;$
$App\{t\}(snd(f)))$

6) $((\forall t.P(t)) \vee (\forall t.Q(t)) \to (\forall t.P(t) \vee Q(t))$

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$\lambda f.((\forall t.P(t)) + (\forall t.Q(t)))$

~~case(f; x. Δt y. Δt)~~ $\forall t\,P(t)$ $\forall t\,Q(t)$

$\Delta t.\ case(f;\ x.\ inl(x[t]) \quad y.\ inr(y[t]))$ $\forall t.P(t)$ $\forall t(Qt)$

? App{t}(inl(x))

7) $(\forall t.A \to P(t)) \to A \to \forall t.P(t)$

$\lambda f.\forall t.A \to P(t).\ \lambda x.A.\ \Delta t.$

$(f[t])(x)$

$lam\{\forall t.A \to P(t)\}(\cancel{lam}.\ e)$

$e := lam\{A\}(y.\ e_1)$ $\to \forall t.P(t)$

$lam(t.\ e_2)$ $\to P(t)$

?

$e_2 := ap(App\{t\}(x); y)$

---

8) $(A \to \forall t.P(t)) \to (\forall t.A \to P(t))$

$\lambda f.A \to \forall t.P(t).\ \lambda x.\forall t.A.$

$\Delta t.\ \lambda x.A.\ (f\ x)[t]$

$lam\{A \to \forall t.P(t)\}(x.\ e)$ $\to \forall t.A \to P(t)$

$e := lam(t.\ e_1)$ $\to A \to P(t)$ $\{e_1 := lam\{A\}(y.\ e_2)$ $\to P(t)$

?

$App\{t\}(ap(f; x))$

x y

1) $\exists t.\ \exists u.\ S(t,u) \Rightarrow \exists u.\ \exists t.\ S(t,u)$

$\lambda f : \exists t.\ \exists u.\ S(t,u).$

$\text{open}(f;\ t,\ x^{\exists u.\ S(t,u)}.\ \text{open}(x;\ u,\ y^{S(t,u)}.$

$\text{pack}\{u,\ \exists t.\ S(t,u)\}\{u\}$

$(\text{pack}\{t,\ S(t,u)\}\{t\}(y$

2) $\exists t.\ P(t) \vee Q(t) \Rightarrow (\exists t.\ P(t)) \vee (\exists t.\ Q(t))$

$\lambda f : \exists t.\ P(t) + Q(t).\ \text{open}(f;\ t,\ u^{P(t) \vee Q(t)}.$

$\text{case}(u;\ x^{P(t)}.\ e_1,\ y^{Q(t)}.\ e_2)$

$e_1 = \text{inl}(\text{pack}\{t,\ P(t)\}\{t\}(x))$

$e_2 = \text{inr}(\text{pack}\{t,\ Q(t)\}\{t\}(y))$

3) $(\exists t.\ P(t)) + (\exists t.\ Q(t)) \rightarrow \exists t.\ P(t) + Q(t)$

$\lambda f : (\exists t.\ P(t)) + (\exists t.\ Q(t)).\ \text{case}(f;\ x^{\exists t.\ P(t)}.\ e_1,\ y^{\exists t.\ Q(t)}.\ e_2)$

$e_1 = \text{open}(\cancel{x};\ t,\ u^{P(t)}.\ \text{pack}\{t,\ P(t) + Q(t)\}\{t\}(\text{in}$

$e_2 = \text{open}(y;\ t,\ v^{Q(t)}.\ \text{pack}\{t,\ Q(t)\}\{t\}$

$(\text{inr}$

4) $\exists t. A \wedge P(t) = A \wedge \exists t. P(t)$

$\lambda f. \exists t. A \times P(t). \langle e_1, \quad \rangle$

$e_1 = \text{pack}\{t, A \times P(t)\}\{t\}(\overset{A}{\text{fst}}(f))(t)$ $\quad \exists t. P(t)$

$e_2 = \text{pack}\{t, P(t)\}\{t\}(\text{snd} f)$

$\text{open}(f; t, \overset{A \times P(t)}{x}. \langle \text{fst}(x) \quad e_1 \rangle$

$A \quad , \quad P(t)$

$e_1 = \text{pack}\{t, P(t)\}\{t\}(\text{snd}(x))$

5) $A \wedge \exists t. P(t) \to \exists t. A \wedge P(t)$

$\lambda f: A \times \exists t. P(t). \text{open}(\text{snd}(f); t. \overset{P(t)}{u}.$

$\text{pack}\{t. A \times P(t)\}\{t\}(\langle \text{fst}(f), u \rangle)$

7) $(\exists t. P(t)) \to A = \forall t. P(t) \to A$

$\lambda f: (\exists t. P(t) \to A). \Lambda t. \lambda x. P(t).$

$\cancel{\text{open}(f; t, \overset{P(t) \to A}{u}. (f[t])\, x)}$

no open becau $\exists$ is bound to the no fun

$(u)) \; f(\text{pack}\{t. P(t)\}\{t\})(x)$

8) $\forall t. P(t) \to A = (\exists t. P(t)) \to A$

$\lambda f: (\forall t. P(t) \to A). \lambda x. \exists t P(t)$

$\text{Open}(x; t. \overset{P(t)}{u}, \cancel{\text{pack}} f[t] u)$

$) (\exists t \cdot P(t) \wedge Q(t)) \to (\exists t \cdot P(t)$ 82

$\wedge (\exists t \cdot Q(t)))$

$\lambda f : \exists t \cdot P(t) \times Q(t) . \langle \underset{\exists t \cdot P(t)}{\underline{e_1}} , \underset{\exists t \cdot Q(t)}{\underline{e_2}} \rangle$

$e_1 = open(f ; t, u^{P(t) \times Q(t)} . pack\{t \cdot P(t)\}\{t\} fst(u))$

$e_2 = open(f ; t, v \cdot pack\{t \cdot Q(t)\}\{t\} snd(v))$

or

$\lambda f : \exists t \cdot P(t) \times Q(t) . Open(f ; t, u^{P(t) \times Q(t)} .$
$\langle e_1 , e_2 \rangle$

$e_1 = pack\{t , P(t)\}\{t\}(fst(u))$

$e_2 = pack\{t , Q(t)\}\{t\}(snd(u))$

10) $(\forall t \cdot P(t) \to Q(t)) \to ((\exists t \cdot P(t)) \to (\exists t \cdot Q(t)))$

$\lambda f : (\forall t \cdot P(t) \to Q(t) . \lambda x . \exists t \cdot P(t) .$
$open(x ; t, u^{P(t)} . pack\{t , Q(t)\}\{t\}(f[t])(u))$

11) $(\exists t \cdot P(t) \to Q(t)) \to ((\forall t \cdot P(t) \to (\exists t \cdot Q(t)))$

$\lambda f : (\exists t \cdot P(t) \to Q(t)) . \lambda x . \forall t \cdot P(t) .$
$Open(f ; t, u^{P(t) \to Q(t)} . pack\{t , Q(t)\}(t)(u(x[t])))$

12) $(\exists t.\forall u.s(t,u)) \to (\forall u.\exists t.s(t,u))$

$\lambda f:\exists t.\forall u:s(t,u).\ \Lambda u.\ open(f;t,x^{\forall u.s(t,u)}.$

$pack\{t, s(t,u)\}\{t\}(x[u])$

or

$\lambda f:\exists t.\forall u.s(t,u).\ open(f;t,x^{\forall u s(t,u)}.$

$\Lambda u.\ pack\{t, s(t,u)\}\{t\}(x[u])$

Defining typing rule for case e.

$$\frac{\Gamma \vdash e:\tau_1+\tau_2 \quad \Gamma, x:\tau_1 \vdash e_1:\tau \quad \Gamma, y:\tau_2 : e_2:\tau}{\Gamma \vdash case(e, x.e_1, y.e_2) : \tau}$$

dy

$$\frac{\Gamma \vdash e:\tau_1}{\Gamma \vdash inl(e) : \tau_1+\tau_2}$$

constr

Inductive proofs.

even: $\dfrac{}{zeven}$

$$\text{rec}\{e_0 ; x, y . e_1\}(e)$$

τ $\quad$ τ $\quad$ nat

$$f\ 0 = e_0$$

$$f\ s(n) = g_1(\overset{x}{n}, \overset{y}{f\ n})$$

predec $\quad$ recurcall

$$\text{fact}\ 0 = 1 \qquad 0!$$

$$\text{fact}\ s(n) = \underbrace{s(n)}_{x} * \underbrace{\text{fact}\ n}_{y}$$

$$\text{fact} = \lambda n : \text{nat}.\ \text{rec}\{s(z) ; x, y . s(x) * y\}(n)$$

$$\text{fact} = \text{fix}\{\text{nat} \to \text{nat}\}(f . \lambda n : \text{nat} .$$

$$\text{if}\ z = \{s(z) ; \overset{pr.}{x} . s(x) * f x\}(n)$$

Recursive ons

$2 + 3 = 5$

1) plus n z = n

plus n s(m) = s(plus n m)

plus = λn:nat λm:nat . rec{n; x, y . s(m

plus = fix{nat → nat → nat}(p . λn:nat λm:nat

rec{e0; x, y . e1} ifz{n; x . s(p n x)} m

or

fix{nat → nat → nat}(p . λn:nat

λm: nat . ifz{m; n; x: s(p n x)}

what the recursion is on

2) Times n z = z

Times n s(m) = plus (Times n m) n

Times = λn:nat λm:nat . rec{z; x, y .

plus y n

Times = fix{nat → nat → nat}(T; λn:nat, λm:nat.

ifz(m; z : x . P(T n m) n))

3) pred z : z

pred $s(n)$ = n

pred = λn:nat. rec$\{z; x, y . x\}(n)$

pred = fix (nat → nat)$\{\lambda n$:nat.
ifz$\{n; z, x . x\}$

4) double z = z

double $s(n)$ = $s(s(\text{double } n))$

double = λn:nat. rec$\{z; x, y . s(s(y))\}(n)$

double = fix$\{nat \to nat\}(D . \lambda n$:nat. ifz$(n;$
$z; x . s(s(D x)))$

induction always on the second argument

5) expo n z = $s(z)$

exp n $s(m)$ = Times (exp n m) n

exp = λn:nat λm:nat. rec
$\{s(z); x, y . \text{Times } y\ n\}(m)$

exp = fix$\{nat \to nat \to nat\}.(T, E . \lambda n$:nat.
λm:nat. ifz$(m; s(z), x . T(E\ n\ x)\ n)$

$leq\ z\ m = s(z)$; in $0 < n$ 0

$leq\ s(n)\ z = z$ $\quad 0 < 1 = 1$ $\quad 1 < 0 =$

$leq = \lambda n{:}nat\ \lambda m{:}nat.\ rec\{s(z); \overset{n}{x}, \overset{leq\ n}{y}.\ s(x) = leq\ (n)\ m\}.\ rec\{z; \overset{m}{p}.\ q.\ y(p)$

$s(n) = leq\ (n\ m)$

$z = s(z)$

$leq\ s(n)\ z = z$

$leq\ (n)\ y(m)$

$leq\ (n\ m)$

↳ not using it

$leq = fix\{nat \to nat \to nat\}(L.\ \lambda n{:}nat.\ ifz\ (n; s(z), \overset{n}{x}.\ \lambda m{:}nat.\ ifz\ (m; z; \overset{n}{y}.\ (L\ x)\ y)).$

8) $fact\ z = z$ ↓

$fact\ s(n) = Times\ s(n)\ (fact\ n)$

$fact = \lambda n{:}nat.\ rec\{z; \overset{n}{x}, y.$

$fact = fix\{nat \to nat\}(T, F.\ \lambda n{:}nat.\ ifz\ (n;$
$z, x.\ T\ s(x)\ (F\ x))$

$= a$

ifz $s(n)$ a b $= b$

ifz z a $b = a$

$\lambda n{:}nat\ \lambda a{:}\tau\ \lambda b{:}\tau\ rec\{a; x, y.\ b\}$

8) ~~fact~~ y no fac.

$5 - 3 = 2$

$0 - m = 0$

$S(n) - m$

9) sub z m $= z$

sub $S(n)$ z $= S(n)$

sub $S(n)$ $S(m)$ $=$ sub n m

$0!$

$S(n)$

$= 3!$

$S(n)$.

$5 - 3$

$= 2$

$4 - 3$

$f(sub\ n\ m)$

sub $= \lambda n{:}nat\ \lambda m{:}nat.\ rec\{z; x, y.\ y\}$

$rec\{S(n), p, q.\ y(p)\}m\}(n)$

sub $= fix\{nat \to nat \to nat\}.(S.\ \lambda n{:}nat$

$ifz\{n; z, x.\ \lambda m{:}nat.\ ifz\{m; S(n), y$

$S\ x\ y))$

$8 - 3$

$7 - 5$

$= 2$

10) Tsub n $z = n$

Tsub n $S(m) =$ pred(Tsub n m)

$10 - 7$

$= 3$

$3 - 2$

$2 - 1$

$4 - 3$

$= 1$

$5 - 3$

$= 2$

Tsub $= \lambda n{:}nat\ \lambda m{:}nat.\ rec\{n; x, y.\ pred(y)\}(m)$

Tsub $= fix\{nat \to nat \to nat\}.(P, T.\ \lambda n{:}nat, \lambda m{:}nat.$

$ifz(m; n, x.\ P(T\ n\ x))$

~~fib z = z~~

~~fib s(n) =~~

fib z n = n

fib s(m) n =

plus s(n,y)

2

n n

1 < 2

1, 1, 2

2 + 2

= 4

1 + 2

s(plus

n y)

11) less than =

~~lt z n = s(z)~~

~~lt s(n) z = z~~

5 >3 lt

lt z z = z

lt z s(n) = s(z)

lt s(n1) z = z

lt s(n1) s(n2) = lt n1 n2

lt = λn : nat. rec{ z ; x, y. rec{ s(z) ; p, q. rec{ z ; u, v.

= λn : nat. rec{ λm : nat. rec{ z ; x, y. s(z) }(m) }(n)

lt n1 n2 = λn2 : nat. λ p, q. rec{ z ; u, v. p(u) } n2 }n1

lt = fix{nat → nat → nat}(λ. λn1 : nat λm : nat

ifz(ifz(z ; x. s(z))(m) ; }(n),

x, ifz{z ; y. L^x y}(n)

$lt\ a\ b = 0$ $\quad$ $if\ (b - a) = 0$

$lt\ a\ b = 1$ $\quad$ $if\ (b - a) = 1$

$lt = \lambda a{:}nat\ \lambda b{:}nat.\ rec\{z; x, y.$
$if\ z(Tsub\ (b\ a); z; s(z))\}\ b$

Define for Tsub to 0

$lt = fix\{\cancel{nat \to nat \to nat}\}$
$\lambda a{:}nat\ \lambda b{:}nat.\ if\ z(Tsub\ (a, b); z; s(z))$

12) GT:

$gt\ \ 0\ 0 = 0$
$gt\ \ s(m)\ 0 = s(z)$
$gt\ \ 0\ s(n) = 0$
$gt\ \ s(n_2)\ s(n_1) = gt\ n_2\ n_1$

$gt = \lambda n_1{:}nat.\ rec\{\lambda m{:}nat.\ rec\{z; x.\ y.\ s(z)\}(m).$

$\lambda n_2{:}nat.\ p, q.\ rec\{z; u.\ v.$
$\{n_2\}$

$gt = fix\{nat \to nat \to nat\}.\ (a.\ \lambda m{:}nat,\ \lambda n_1{:}nat.$
$if\ z(if\ z(m; 0; c.\ s(z)); X$
$\lambda n_2{:}nat.\ if\ z(n_2; 0, y.\ a\ y\ X)$

gt b a = 1 s(z)

gt b a = 0 z

gt : λa:nat λb:nat . rec{z ; x, y.
if z (Tsub (b a); s(0). x. 0) }b

λa:nat . λb:nat . if z (Tsub (b,a); s(z); z)

13) mod n m

mod = λn:nat λm: nat .
rec{ z ; x, y. if z (Tsub (m, y)); 0; x. s(y))}(n)

mod s(m) n =
s (Tsub (m n))
(0-n) = 0
(s(m)-n) = s(m-n)

14) Quot n m

λn: nat λm: nat . rec{0; x, y.
if z (mod n m; s(y); p. y)}(n)

15) fib z = z
fib s(z) = s(z)
fib s(n) = fib (n) + fib(n-1)

fib := λn:nat . rec{z ; x, y. rec{s(z); p, q. plus (y(x), y(z))}}

Imp

$T \in Typ ::= \quad | \text{nat list}$

$e \in Exp ::= n \,..\, | \text{nill} | \text{cons}(e_1; e_2) |$

$\text{lrec}\{e_0; x, l, y \cdot e_1\}(\text{nill}) \mapsto e_0$ $\quad \text{lrec}\{e_0; x, l, y \cdot e_1\}(e)$

$\text{lrec}\{e_0, x, y e_1; e_2\}(\text{cons}(n_1; l_2))$

$\text{lrec}\{e_0 ; x, l, y \cdot e_1\}(\text{nill}) \to e_0$

$[n_1/x]\,[l_2/l]$

$\text{lrec}\{e_0 ; x, l, y \cdot e_1\}(\text{cons}(n_1, l_2)) \mapsto$

$[n_1/x]\,[l_2/l]\,[\text{lrec}\{e_0; x, l, y\, e_1\}(l)/y]\, e_1$

---

fib :=

$\text{fix}(\text{nat} \to \text{nat}).(p, F, \lambda n : \text{nat} \cdot \text{ifz}(x; z, x.$
$\text{ifz}(x; s(z).y.\, p(F(x)).F(y))$

Uniqty of typing

if $E : \tau$ &

if $E : \tau$ & $E : \tau'$ then
$\tau = \tau'$

Determ = $E \mapsto E'$ & $E \mapsto E''$ then
$E' = E''$

to other system

1) $\text{Unit} = \forall t.\, t \to t$

$\text{Unit} \to \forall t.\, t \to t$

$\lambda f : \text{unit}.\ \lambda x.\ \forall t.\, t.\ x[f]$ $\quad x[t]$

$\forall t\ t \to t \to \text{unit}$

$\lambda f : \forall t\ t.\ \lambda x.\ t.\ \langle\rangle$

2) $\tau_1 \times \tau_2 \to \forall t.\, (\tau_1 \to \tau_2 \to t) \to t$

$\lambda p : \tau_1 \times \tau_2.\ \lambda f : \forall t.\, (\tau_1 \to \tau_2 \to t).\ f[p](\text{fst}\, p)(\text{snd}\, p)$

3) $\forall t.\, (\tau_1 \to \tau_2 \to t) \to t \to \tau_1 \times \tau_2$

$\lambda f : \forall t.\, (\tau_1 \to \tau_2 \to t).\ \lambda x.\ t.\ \langle$

τ_1, τ_2

4) pair $\langle e_1, e_2 \rangle = \Lambda t.\ \lambda f : (\tau_1 \to \tau_2 \to t).\ f(e_1)(e$

5) void ... $\Lambda \tau$

6) sum $(e_1 + e_2)$ $\cancel{\Lambda t\ \lambda f : (\tau_1 \to t}$

$\Lambda t.\ \lambda x : \tau_1 \to t.\ \lambda y : \tau_2 \to t.\ x.\ e_1$

$y\ e_2$

nat $\Lambda t.\ \lambda f.\ t.\ \lambda g.\ t \to t.\ g(f)$

τ

natlist τ. $\Lambda t. \lambda x:t\ \lambda f:(nat \to t \to t).\ x$

$\exists t.p(t)$. $\Lambda u.\lambda f.(\forall t.\ p(u) \to t).\ f[u](p(u))$

3) void $\to \forall t.t$

$\lambda v:void.\ \Lambda t.\ abort\{t\}(v)$

$\forall t.t \to void$

$\lambda f:\forall t.t.\ f[void]$

4) $\tau_1 + \tau_2 \to \forall t.((\tau_1 \to t) \to (\tau_2 \to t) \to t)$

$\lambda f:\tau_1+\tau_2 \to \Lambda t.\ \lambda x:\tau_1 \to t\ \lambda y:\tau_2 \to t.$

$case(f;\ \overset{\tau_1}{x}.\ f(x)\ \overset{\tau_2}{y}.\ f(y))$

5) $nat \to \forall t\ (t \to (t \to t) \to t)$

6) $natlist \to \forall t\ (t \to nat \to t \to t) \to t)$

7) $\exists t.p(t) \to \forall t.((\forall u.p(u) \to t) \to t)$

$fib(n) = fst(fib(n))$

$fib(0) = \langle 0, 1 \rangle$ or $\langle 1, 1 \rangle$

$fib(n) = \langle x, y \rangle \to y$

$fib(n+1) = \langle y, x+y \rangle$

$\lambda n{:}nat.\ fst(rec\{\langle 1,1\rangle; x, y^{(x,y)}.$
$\langle snd(y), plus\ fst(y)\ snd(y)\rangle\}_n$

$\boxed{\text{Finality}}$ Eval & $E \mapsto E'$ then false

~~Intro~~ Induction proofs

Natural numbers $n ::= z \mid s(n)$

Number list $l ::= nil \mid cons(n; l)$

Number trees $t ::= empty \mid node(n; t_1; t_2$

1)

$\dfrac{\dfrac{}{even\ z}}{z\ even}$ $\quad \dfrac{even\ n}{even\ s(s(n))}$ $\quad \dfrac{n\ even}{s(s(n))\ even}$

$\dfrac{}{odd\ s(z)}$ $\quad \dfrac{odd\ n}{odd\ s(s(n))}$ $\quad \dfrac{n\ odd}{s(s(n))\ odd}$

$\cancel{z\ even}$ $\quad \dfrac{}{even(z)}$ $\quad \cancel{even\ s(s(n))}$ $\cancel{even\ s(n)}$ $\dfrac{even\ n}{even\ s(s(n))}$

$$\frac{}{z \text{ even}} \quad (1) \qquad \frac{n \text{ even}}{s(s(n)) \text{ even}} \quad (2)$$

$$\frac{}{s(z) \text{ odd}} \quad (3) \qquad \frac{n \text{ odd}}{s(s(n)) \text{ odd}} \quad (4)$$

2) $\forall n$, even $n \rightarrow$ odd $s(n)$

By induction on even n

Case 1: even z = odd $s(z)$

By rule (3) $s(z)$ odd

Case 2: even $s(s(n))$ = odd $s(s(s(n)))$

Above the line

even n * H_1

By IH on H_1

odd $s(n)$, H_2

By rule (2)

odd $s(n)$ = odd $s(s(s(n)))$

2b) $\forall n$ odd $n \rightarrow$ even $s(n)$

By induction on odd n

Case 1) odd $s(z)$

By rule (1) $s(s(z))$ even

case 2) odd $s(s(n))$

Above the line

odd n * H_1

By IH on H_1

even $s(n)$ * H_2

By rule 4

even $s(n) = s(s(s(n)))$ even $\square$

3) $\forall n$ even $n \vee$ odd n

By induction on n

$n = z$

case 1; $\forall z$ even $z \vee$ odd z

by rule 1

even z $\therefore$ even z + odd z

Case 2; $n = s(n)$

~~case 2 - even By IH on n~~

~~By rule 2 $s(s(n)$ even~~

~~Above the line n even * H_1~~

~~Apply lemma 2 to H_1~~

~~odd $s(n)$ * H_2~~

By IH on n'

case 2.1. even n', either even n' $\vee$ odd n'

So by lemma 2 even n' = odd $(s(n'))$

So even $s(n') \vee$ odd $s(n')$

Case 2.2 , odd n'

So by lemm 3 odd n' = even $(s n')$

even $s(n')$ then even $s(n') \vee$ odd $s(n')$

5).

$z + m = m$ ①

$s(n) + m = s(n+m)$ ② $3 + 2 = 5$

$z \times m = z$ ③ $6 \times 2 = 12$

$s(n) \times m = (m \times n) + m$ ④

$m + (m \times n)$

6) $\forall n \forall m \forall l \; n + (m + l) = (n + m) + l$

By induction on n

Case 1 : $n = z$

$z + (m + l) = (z + m) + l$

$= m + l$ by rule ①

$=$

$= m + l$ by rul ①

$= (z + m) + l$

Case 2 = $n = s(n')$

∴ There should be atleast one IH

$s(n) + (m + l)$

$= s(n + (m + l))$

by IH

$= s((n + m) + l)$

$= s(n + m) + l$

$= (s(n) + m) + l$

6) $\forall n \; n+Z = n$ & $\forall n \forall m \; n+S(m) = S(n+m)$

By induction on n.

$n = Z$

Case 1

$\forall n \quad Z + Z = Z$

by rule 1

(case 1.1

~~Case 2: $n = S(n')$~~

$S(n') + Z = S(n')$

by rule 1

$n+Z = n$ & $n + S(m) = S(n+m)$ is true for all Z

Case 2 $\quad n = S(n')$

$S(n')+Z = S(n')$

by rule 1

Case 2.2

$S(n') + S(m)$

$= S(n' + S(m))$ by rule 2

$=$ by IH

$= S(S(n' + m))$

by rule 2

$S(S(n') + m$ ∎

7) $\forall n \forall m \quad n + m = m + n$

by induction on n

Case 1 ; $n = z$

$z + m = m$ by rule ①

$= m$ by rule ②

$m + z$

Case 2 : $n = s(n')$

$s(n') + m$

$= s(n' + m)$

~~by lem~~ ~~n~~ by IH

$= s(m + n')$

by lemma 6

$= m + s(n')$

$n + s(m)$

$= s(n+m)$

7) $n \cdot (m \cdot l) = (n \cdot m) \cdot l$

by induction on n

either $n = z$

Case 1. $n = z$

$z \cdot (m \cdot l) = \cancel{m \cdot l}\ z$

$= z$ rule 2

$= z \cdot l$ = rule 3

$(z \cdot m) \cdot l$

Case 2 = $n = s(n')$

goal

$s(n') \cdot (m \cdot l)$

$= m \cdot l + n' \cdot (m \cdot l)$

by IH

$= m \cdot l + (n' \cdot m) \cdot l$

$= m \cdot l + (n' \cdot m) \cdot l$ distrib of mult

$= (m + n' \cdot m) \cdot l$ by rule

$= (s(n') \cdot m) \cdot l$

Distributivity over addition

$(m + n) \cdot l = m \cdot l + n \cdot l$

by induction on m

Case 1 $m = z$

$(z + n) \cdot l$

$= n \cdot l$ by rule 1

$= z + n \cdot l$ by rule 3

$= z \cdot l + n \cdot l$

Case 2: $m = s(m')$

$(s(m') + n) \cdot l$

$= (s(m' + n)) \cdot l$ rule 2 Sum

$= l + (m' + n) \cdot l$ rule 4

$= l + (m' \cdot l + n \cdot l)$ by IH asso. of plus

$= (l + (m' \cdot l)) + n \cdot l$ by a

by rule of mult

$n \cdot m = m \cdot n$

By induction on n

Case 1 = $n = z$

$z \cdot m = \cancel{m}\, z$ rule 3

$= z$

$m \cdot z$

Case 2 = $n = s(n')$

$s(n') \cdot m = m + n' \cdot m$ $\quad m \cdot s(n')$

by $\cancel{d}$ 1

$= m + m \cdot n'$

$= m + m \cdot n'$ by $\times$

$= m \cdot s(n')$ by multiplication rule

8) <u>length</u>

$|nill| = z$ ①

$|cons(n; l)| = |l| + 1$ ②

or $s|l|$

<u>concat</u> $l_1 \cdot l_2$

$nill \cdot l_2 = l_2$ ③

$cons(n; l_1) \cdot l_2 = cons(n; l_1 \cdot l_2)$ ④

rev(l)

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rev(nill) = nill ⑤

rev(cons(n; l)) = rev(l) + cons(n; nill) ⑥

a) $|l_1 \cdot l_2| = |l_1| + |l_2|$

By induction on l_1

Case: l_1 = nill

nill $\cdot\, l_2 = l_2$ by ③ → z + any = any

$= z + |l_2|$

$|\text{nill}| + |l_2|$

Case: $l_1 = \text{cons}(n; l_1)$

$|\text{cons}(n; l_1) \cdot l_2|$

$= |\text{cons}(n; l_1 \cdot l_2)|$ by 4

$= s\,|l_1 \cdot l_2|$ by ②

$= s(|l_1| + |l_2|)$ by IH

by rule 6

$= s|l_1| + |l_2|$ by ②

$|\text{cons}(n; l_1)| + |l_2|$

(9b) $|rev(l)| = |l|$

by induction on l

Case $l = nill$

$|rev(nill)| = |nill|$ 5

Case 2 $l = cons(n; l)$

$|rev(cons(n; l))|$

$= |rev(l) \cdot cons(n; nill)|$ 6

by IH

$= |l| \cdot cons(n; nill)|$

$= |l| \cdot S|nill|$

by lem 9a

$= |l| + S|nil|$

$= S(|l| + Z)$

$= S(|l|)$ by 2

$|cons(n; l)|$

10) Size.

$size(empty) = Z$

$size(node; t_1; t_2) = S(size(t_1) + size(t_2))$ 1

or

$size(t_1) + size(t_2) + 1$ 2

Height

Hight (empty) = empty ③

Height (node; t_1, t_2) =

$$s(\max(H(t_1), H(t_2)))$$

④

11) left to right reversal of a tree

LTR(empty) = empty ⑤

LTR(node (n; t_1, t_2))

= node(n; LTR(t_2) ; LTR(t_1))

12) $\forall t$, size (LTR (t)) = size (t)

by induction on t

<u>case 1</u>: t = empty

size (LTR(empty))

= size (empty)

<u>case 2</u>

size (LTR (node (n; t_1; t_2))

= size (node(n; LTR(t_2); LTR(t_1))

by lemma size(t_2) + size(t_1) = size(t_1) + size(t_2)

= s(size(LTR(t_2)) + size(LTR(t_1)))

By IH

= s(size(t_2) + size(t_1))

= size(node(n; t_1; t_2))

13) $rev(l_1 \cdot l_2) = rev(l_2) \cdot rev(l_1)$

By induction on l_1

Case 1) : $l_1 = nil$

$rev(l_2) = rev(l_2)$ by rule 3 of l_1

$= rev(l_2) \cdot rev(nill)$

Case 2) $l_1 = cons(n; l_1)$

$rev(cons(n; l_1) \cdot l_2)$

$= rev(cons(n; l_1 \cdot l_2))$ by rule 4 of conc

$= rev(l_1 \cdot l_2) + cons(n; nill)$ by rule 6.

by IH

$= rev(l_2) \cdot rev(l_1)$

$= rev(l_2) \cdot rev(l_1) + cons(n; nil$

$= rev(l_2) \cdot rev(cons(n; l_1))$

by rule 6 -> rev

13 b $rev(rev(l)) = l$

By induction on l

case 1 : $l = nill$

$rev(rev(nil))$

$= rev(nil)$ by rules of rev

$= nill$ by rules of rev

$\square$

case 2: $l = cons(n; l)$

$rev(rev(cons(n; l)))$

$= rev(rev(l) \cdot cons(n; nill))$

$= rev(rev(l)) \cdot rev(cons(n; nill))$ by rule 6 of rev

by I H

$= l \cdot cons(n; nill)$ by I H

$= cons(n; l)$; // by rule

$= cons(n; l)$

14) Height (LTR(t)) = Height (t)

By induction on t

Case 1) t = empty.

Height(LTR(empty))

= Height (empty)
by subs

Case 2: 2) Height (LTR(node(n; t_1; t_2)))

= Height (node(n; LTR(t_2); LTR(t_1)))

= S (max (Height (LTR(t_2)), H(LTR(t_1)))

By IH

= S(max (H(t_2), H(t_1)))

By lemma H(t_2), H(t_1)
= H(t_1), H2

= S (max (H(t_1), H(t_2))

Height (node(n; t_1; t_2))

15)

Preorder: R L Rig

Pre (empty) = ~~empty~~ nill ①

Pre(node(n; t_1; t_2)) = ②

cons(n; preorder(t_1)) · pre(t_2)

Inorder L Ro Ri

In order (empty) = ~~empty~~ nill ③

Inorder(n; t_1; t_2) =

Inorder(t_1) · cons(n; Inorder(t_2))

Post ord L Ri Ro

Post (empty) = ~~empty~~ nill

Post(n; t_1; t_2)

= Post(t_1) · Post(t_2) · cons(n; nill)

16) reverse(inorder(t)) = inorde(LTR(t))

By induction on t

Case 1 t = empty

reverse(inorder(empty))

= reverse(~~empty~~ nilf) by ③

= inorder(empty) empty nill

case2 : $t = node(n; t_1; t_2)$

$reverse(inorder(node(n; t_1; t_2)))$

$= rev(in(t_1) \cdot cons(n; in(t_2)))$
by in rule

$= rev(cons(n; in(t_2) \cdot in(t_1)))$
by concat

$\Rightarrow$ ~~$rev(in(t_2)) \cdot rev(in(t_1))$~~

$= rev(in(t_2) \cdot in(t_1)) \cdot cons(n; nil)$
by lem

$= rev(in(t_2)) \cdot rev(in(t_1)) \cdot cons(n; nil)$
by IH

$= in(LTR(t_2)) \cdot in(LTR(t_1)) \cdot cons(n; nil)$

$in(LTR(t_2)) \cdot (cons(n; nil) \cdot in(LTR($

~~$Inorder(LTR(node(n; t_1; t_2)))$~~

$In(node(n; LTR(t_2), LTR(t_1)))$
by in ow

$In(LTR(node(n; t_1; t_2)))$

$\boxtimes$

7) $Size(t) = |Pre(t)|$

By Induction on t

Case 1 $t = empty$

$Size(empty) = Z$ by rule

$|nil| \rightarrow$ by null

$|Pre(empty)|$

Case 2 : $t = node(n; t_1; t_2)$

$Size(node(n; t_1; t_2))$

$= S(Size(t_1) + Size(t_2))$

$=$ by IH

$S(|Pre(t_1)| + |Pre(t_2)|)$

by rule 6

$= S|Pre(t_1)| + S(|Pre(t_2)|)$

$= \cancel{(|Pre(t_1)|+}$ by def. rule

$|l_1 \cdot l_2| = |l_1| + |l_2|$

$= |cons(n; \cancel{Pre}\ t_1)| + |pre\ t_2|$

$= |cons(n; pre(t_1)) \cdot pre(t_2)|$

$|Pre(node(n; t_1; t_2))|$

$Size(t) = |Post(t)|$

Case 1 $Size(empty) = Z$

$= |nil|$ by rule 1

$|Post(empty)|$

Case 2 : $t = node(n; t_1; t_2)$

$size(node(n; t_1; t_2))$

$= s(size(t_1) + size(t_2))$

by IH

$= s(|post(t_1)| + |post(t_2)|)$

$= |post(t_1)| + s(|post(t_2)|)$

by lem 9

$= |post(t_1)| + |cons(n; post(t_2))|$

$|l_1 \cdot l_2| = |l_1 \cdot l_2|$

$= |Post(t_1) \cdot cons(n; Post(t_2))|$

by r

by post rule $= |Post(t_1) \cdot Post(t_2) \cdot cons(n; nil)|$

$|post(node(n; t_1; t_2))|$

#

Proof of non existence of void types

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$\frac{}{\text{unit nonempty}}$ → ①

$\frac{}{\text{void empty}}$ → ②

$\frac{\tau_1 \text{ empty}}{\tau_1 \times \tau_2 \text{ empty}}$ → ③

$\frac{\tau_2 \text{ empty}}{\tau_1 \times \tau_2 \text{ empty}}$ 4

$\frac{\tau_1 \text{ nonempty} \quad \tau_2 \text{ nonempty}}{\tau_1 \times \tau_2 \text{ nonempty}}$ ⑤

$\frac{\tau_1 \text{ empty} \quad \tau_2 \text{ empty}}{\tau_1 + \tau_2 \text{ empty}}$ ⑥

$\frac{\tau_1 \text{ nonempty}}{\tau_1 + \tau_2 \text{ nonempty}}$ → ⑦

$\frac{\tau_2 \text{ nonempty}}{\tau_1 + \tau_2 \text{ nonempty}}$ ⑧

$\frac{\tau_1 \text{ nonempty} \quad \tau_2 \text{ empty}}{\tau_1 \to \tau_2 \text{ empty}}$ ⑨

$\frac{\tau_1 \text{ empty}}{\tau_1 \to \tau_2 \text{ nonempty}}$ → ⑩

$\frac{\tau_2 \text{ nonempty}}{\tau_1 \to \tau_2 \text{ nonempty}}$ → ⑪

is essential, τ empty & $\Gamma \vdash e : \tau$ then "false".

→ By induction on $\Gamma \vdash e : \tau$

Case

$\Gamma, x : \tau \vdash x : \tau$

$e = x$, $\tau = \tau$

τ empty $* H_1$

$x : \tau \in \Gamma$ ∴ τ nonempty H_2

false

→ Case $\langle\rangle : unit$

$e = \langle\rangle$

$\tau = unit$

τ empty $* H_1$

by rule ① unit nonempty

↳ false

→ Case: $\langle e_1, e_2 \rangle : \tau_1 \times \tau_2$

$e = \langle e_1, e_2 \rangle$

$\tau = \tau_1 \times \tau_2$

τ empty $* H_1$

$\Gamma \vdash e_1 : \tau_1 \quad * H_2$

$\Gamma \vdash e_2 : \tau_2 \quad * H_3$

$\tau_1 \times \tau_2$ empty H_1

by Inversion on H_1

τ_1 empty $\& H_4$

on H.2

By IH
false //

b) τ_2 empty + H5

by IH on false H_3 & H5

Case: $\dfrac{\Pi \vdash e : \tau_1 \times \tau_2}{\Pi \vdash fst(e) : \tau_1}$

$e = fst(e)$

$\tau_1 = \tau$

τ_1 empty * H_1

Above the line

$\Pi \vdash e : \tau_1 \times \tau_2$ * H_2

by rule

$\tau_1 \times \tau_2$ empty * H_3

by IH on H_2 & H_3 false

Case $\dfrac{\Pi \vdash e : void}{\Pi \vdash abort\{\tau\}(e) : \tau}$

$e = abort\{\tau\}(e)$

$\tau = \tau$

τ empty * H_1

$\Pi \vdash e : void$ * H_2

by rule void empty * H_3

by IH on H_2 & H_3

false.

Case $\dfrac{\Gamma \vdash e : \tau_1}{\text{in}[\ell]\{\tau_1;\tau_2\}(e) : \tau_1 + \tau_2}$

$e = \text{in}[\ell]\{\tau_1;\tau_2\}(e)$

$\tau = \tau_1 + \tau_2$

$\tau_1 + \tau_2$ empty * H1

$\Gamma \vdash e : \tau_1$ * H2

by inversion on H1

τ_1 empty * H3

τ_2 empty * H4

by IH on H2 & H3 <u>false</u>

Case: $\dfrac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma, x:\tau_1 \vdash e : \tau \quad \Gamma, y:\tau_2 : e : \tau}{\Gamma \vdash \text{case}(e; x.e_1; y.e_2) : \tau}$

$e = \text{case}(e; x.e_1; y.e_2)$

$\tau = \tau$

τ empty * H1

$\Gamma \vdash e : \tau_1 + \tau_2$ * H2

$\Gamma, x:\tau_1 \vdash e : \tau$ * H3

$\Gamma, y:\tau_2 : e : \tau$ * H4

Apply <u>lemma</u> on $\tau = \tau_1 + \tau_2$

Case 1

$\tau_1 + \tau_2$ empty H4

by IH on H2 & H4

false

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Case 2

$\tau_1 + \tau_2$ nonempty (5)

by inversion

Case 2.1

τ_1 nonempty (6)

$\Gamma, x : \tau_1$ is existential

since τ_1 is nonempty

by IH false

Case 2.2

τ_2 nonempty (7)

$\Gamma, y : \tau_2$ is existential

since τ_2 is nonempty

by IH false

Case: $\dfrac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\text{lam}\{\tau_1\}(x.e) : \tau_1 \to \tau_2}$

$e = \text{lam}\{\tau_1\}(x.e)$

$\tau = \tau_1 \to \tau_2$

$\tau_1 \to \tau_2$ * empty * H1

$\Gamma, x : \tau_1 \vdash e_2 : \tau_2$ * H2

by inversion on H1

τ_1 nonempty * H3

τ_2 empty * H4

By IH on H2 & H4 > false

$\text{let}([2+x]; z \cdot (z+2+x))$

$= [2+x]/y \cdot \text{let}(y; z \cdot (z+y))$

$= [a/x]\,\text{let}(e_1; x \cdot e_2)$

$\text{let}([a/x]e_1; x \cdot e_2)$

same variable operator

$[a/x]\,\text{let}(e_1; y \cdot e_2)$

$\text{let}([a/x]e_1; [a/x]e_2)$

$[2+x]/y\ \text{let}(y; x \cdot x+y)$

$= \text{let}([2+x]/y \cdot y; x \cdot [2+x]/y \cdot x+y$

substituted both s.

$= \text{let}([2+x]; x \cdot x(x+2+x))$

x $\quad (2+x)/2 \cdot (x+2)$

$2+x$ $\quad 2x+2y+x+x^2$ / 2

$(2x+x^2)+(2+x^2)$

Lemma

type τ either τ empty or τ nonempty

Induction on τ

Case unit

unit nonempty by rule

case Void void empty by rule e

case $\tau_1 \times \tau_2$ apply IH

either $\tau_1 \times \tau_2$ empty

above the line τ_1 : empty

by rule $\tau_1 \times \tau_2$ empty

$\tau_1 \times \tau_2$ empty

τ_2 : empty

$fix\{\tau\}(R.\ ifz(e_0; [P(e)/x]e_1))$

$rec\{e_0; x.y.e_1\}(e)$

Equivalency

① Rec ≡ ifz

$ifz(n; e_0; x.e_1) \equiv rec\{e_0; x, y.e_1\}(e)$

$rec\{e_0; x, y.e_1\}(e)$

$\equiv fix\{\tau\}(R.\ ifz(e_0; [\overset{pred}{P(e)}/x], [\overset{re}{R/y}]e_1))(e)$

$rec(e_0; x, y.e_1)(n) \equiv$

$= fix\{\tau\}(R.\ ifz(e_0; [n/x][R/y]e_1))(n)$

8) snoc (l, n)

cat $l_1 \cdot$ nill $= l_1$ ——> (a)

cat $l_1 \cdot$ (snoc l_2, n) = snoc(cat(l_1 l_2), n) ——> (b)

$(l_1 \cdot l_2) \cdot l_3 = l_1 \cdot (l_2 \cdot l_3)$

By induction on l_3

Case 1. l_3 = nill

$(l_1 \cdot l_2) \cdot$ nill

$= (l_1 \cdot l_2)$ by a

$= l_1 \cdot (l_2 \cdot$ nill$)$

Case 2 l_3 = (snoc l_3, n)

$(l_1 \cdot l_2) \cdot$ (snoc l_3, n)

$l_1 \cdot$ snoc (l_2 snoc)

$=$ snoc(cat $(l_1 \cdot l_2) \cdot l_3, n$) by 5

$=$ snoc$(l_1 \cdot (l_2 \cdot l_3), n)$ by rule

$= l_1 \cdot$ snoc$(l_2 \cdot l_3, n)$

$= l_1 \cdot (l_2 \cdot$ snoc(l_3, n) by rul

$= l_1 \cdot (l_2 \cdot$ (snoc l_3, n))

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(2) let = λ

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau}{\text{let}(e_1 ; x . e_2) : \tau}$$

$$\equiv \text{ap}(\underbrace{\text{lam}\{\tau\}(x . e_2)}_{E_1}, \underbrace{e_1}_{E_2})$$

$$\frac{\dfrac{x : \tau_1 \vdash e_2 : \tau}{\text{lam}\{\tau\}(x . e_2) : \tau_1 \to \tau} \quad \dfrac{}{e_1 : \tau}}{\text{ap}(\text{lam}\{\tau\}(x . e_2), e_1)}$$

---

fact = $= s(z)$

fact $s(n) = s(n) \cdot \text{fact}(n)$

$\text{fact}(n) = \lambda n : \text{nat} . \text{rec}\{z ; x , y . s(x) \times y\} n$

$\equiv f(2)$

$= \lambda n : \text{nat} . \text{rec}\{s(z) ; x , y . s(x) + y\}(n)$

$\mapsto \text{rec}\{s(z) ; x , y . s(x) \cdot y\}(2)$

$\mapsto [1/x](\text{rec}\{s(z) ; x , y . s(x) \cdot y\}(1)/y)$

$(s(x) \cdot y$

$\mapsto s(1) * \text{rec}\{s(z) ; x , y . s(x) \cdot y\}(1)$

$\mapsto s(1) * [0/x](\text{rec}\{s(z) ; x , y . s(x) \cdot y\}(0)/y)$

$(s(x) \cdot y)$

$\mapsto s(1) * s(0) * (\text{rec}\{s(z) ; x , y . s(x) \cdot y\}(0))$

$\mapsto s(1) * s(0) * s(z)$

$= 2 * 1 * 1$

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$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x:\tau_1 \vdash e_2 : \tau_2 \quad \Gamma, x:\tau_1, y:\tau_2 \vdash e_3 : \tau_3}{\Gamma \vdash \text{let}\ (e_1, x.e_2, x, y.e_3) : \tau_3}$$

$(a \to b) \to a \to b$

$\lambda f : (a \to b) . \lambda x . a . f x$

2) $(a \to (b \to c) \to (a \to b) \to (a \to c)$

$\lambda f : (a \to (b \to c)) . \lambda x : (a \to b)\ \lambda u : a . f (x a)$

3) $(a \to (b \to c)) \to (b \to (a \to c))$

$\lambda f : (a \to (b \to c)), \lambda x : b . \lambda u : a . (f u) x$

4) $(a \to b) \to (a \to a \to b)$

$\lambda f : (a \to b) \to \lambda x : a\ \lambda y : a\ f y$

5) $(a \to b) \to ((a \to b) \to b) \to b)$

$\lambda f : (a \to b) . \lambda x : (a \to b) \to b . x f$

$\exists u \cdot s(t,u) \rightarrow \forall u \cdot s(u,t)$

$f(a)$.

$f(a) \quad \exists u \cdot s(t,u) \rightarrow \forall u \cdot s(u,t)$

$\vdash \lambda s: \exists t \cdot P(t) \rightarrow \exists(t) \cdot \lambda f: \exists t \cdot P(t) \cdot open(s, a, x)$

using fix

fix can sort of be

$fix : (\tau \to \tau) \to \tau$

A function $f : A \to A$ has a fixed point a if $f(a) = a$.

$fix\{\tau\}(x.e) \mapsto [fix\{\tau\}(x.e)/x]e$

$e : x \to e(x)$

$e\,(fix\{\tau\}(x.e))$

fact $0 = 1$

fact $(sn) = (sn) * fact\ n$ } functional equations

$\forall f f' : nat \to nat$ if both f & f' satisfy the equation for fact, then $f = f'$

Claim $\forall n\ f(n) = f'(n)$

By induction on n

Base $n = 0$

$f\ 0 = 1 = f'0$

Inductive Assume $f(n) = f'(n)$

Goal: $f(sn) = f'(sn)$

$f(sn) = (sn) * fn$

$= (sn) * f'n$ by IH

$= f'(sn)$

$(nat \to \underline{nat}) \to (nat \to nat)$

x

fact = fix $\{nat \to nat\}$ $(f. \lambda x{:}nat.\ if\ z\ (x; 1; x'.\ S(x').f\ x'))$

fact $\mapsto$ [fact / f] $\lambda x{:}nat.\ if\ z$

$= \lambda x{:}nat.\ if\ z\ (x; 1; x'.\ S(x').fact\ x')$

fib 0 = 0

fib 1 = 1

fib = fix $\{nat \to nat\}$ $(f.\ \lambda n{:}nat.$

$if\ z\ (n; 0; x.\ if\ z\ (x; 1; y.\ f x + f y))$

[1, 2, 3] = cons 1 (cons 2 (cons 3 nil))

<u>append function</u>

app

induction on first argument

app nil l_2 = l_2

app (cons n l_1) l_2 = cons n (app l_1 l_2)

not going to work

if we do an induction on second

app l_1 nil = l_1

app l_1 (cons n l_2) = ~~cons~~ (app l_1 l_2)

app (cons n l_1) l_2

cons n (app l_1 l_2)

why not max $(h(t_1), h(t_2))$

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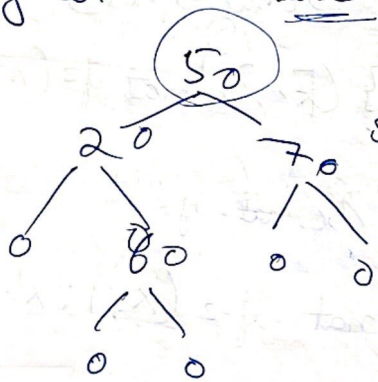

$s(\max(h(t_1), h(t_2)))$

Rules

$$\frac{}{\Pi, x:\tau \vdash x:\tau}$$

$$\frac{}{\Pi \vdash str[s] : str}$$

$$\frac{}{\Pi \vdash num[n] : num}$$

$$\frac{\Pi \vdash e_1 : num \quad \Pi \vdash e_2 : num}{\Pi \vdash plus(e_1; e_2) : num}$$

$$\frac{\Pi \vdash e_1 : num \quad \Pi \vdash e_2 : num}{\Pi \vdash times(e_1; e_2) : num}$$

$$\frac{\Pi \vdash e_1 : str \quad \Pi \vdash e_2 : str}{\Pi \vdash cat(e_1; e_2) : str}$$

$$\frac{\Pi \vdash e : str}{\Pi \vdash len(e) : num}$$

$$\frac{}{\Pi \vdash \cancel{let(e_1; e_2)}}$$

$$\frac{\Pi \vdash e_1 : \tau_1 \quad \Pi, x:\tau_1 \vdash e_2 : \tau_2}{\Pi \vdash let(e_1; x.e_2) : \tau_2}$$

$$\frac{}{num[n] \ val}$$

$$\frac{}{str[s] \ val}$$

$$\frac{n_1 + n_2 \text{ is } n \text{ nat}}{plus(num[n_1]; num[n_2]) \mapsto num[n]}$$

$$\frac{e_1 \mapsto e_1'}{plus(e_1; e_2) \mapsto plus(e_1'; e_2)}$$

$$\frac{e_1 \ val \quad e_2 \mapsto e_2'}{plus(e_1; e_2) \mapsto plus(e_1; e_2')}$$

$$\frac{s^1 \wedge s^2 = s \quad s \wedge s}{cat(str[s_1]; str[s_2]) \mapsto str[s]}$$

$$\frac{e_1 \mapsto e_1'}{cat(e_1; e_2) \mapsto cat(e_1'; e_2)}$$

$$\frac{e_1\ val \quad e_2 \mapsto e_2'}{cat(e_1; e_2) \mapsto cat(e_1; e_2')}$$

$$\frac{e_1 \mapsto e_1'}{let(e_1; x.e_2) \mapsto let(e_1'; x.e_2)}$$

$$\frac{e_1\ val}{let(e_1; x.e_2) \mapsto [e_1/x]e_2}$$

$$\frac{e_1 \Downarrow v_1 \quad [v_1/x]e_2 \Downarrow v}{let(e_1; x.e_2) \Downarrow v}$$

Higher Order

$$\frac{\Gamma, x:\tau_1 \vdash e : \tau_2}{\Gamma \vdash lam\{\tau\}(x.e) : arr(\tau_1; \tau_2)}$$

$$\frac{\Gamma \vdash e_1 : arr(\tau_2; \tau) \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash ap(e_1; e_2) : \tau}$$

$$\frac{lam\{\tau\}(x.e)\ val}{}$$

$$\frac{e_1 \mapsto e_1'}{ap(e_1; e_2) \mapsto ap(e_1'; e_2)}$$

$$\frac{e_1\ val \quad e_2 \mapsto e_2'}{ap(e_1; e_2) \mapsto ap(e_1; e_2')}$$

$$\frac{e_2\ val}{ap(lam\{\tau\}(x.e_1); e_2) \mapsto [e_2/x]e_1}$$

$$\frac{}{lam\{\tau\}(x.e) \Downarrow lam\{\tau\}(x.e)}$$

$$\frac{e_1 \Downarrow lam\{\tau\}(x.e) \quad [e_2/x]e \Downarrow v}{ap(e_1; e_2) \Downarrow v}$$

System T

$$\frac{}{\Gamma, x:\tau \vdash x : \tau}$$

$$\frac{\Gamma \vdash e : nat}{\Gamma \vdash s(e) : nat}$$

$$\frac{}{\Gamma \vdash z : nat}$$

$$\frac{\Gamma \vdash e_0 : nat \quad \Gamma \vdash e_0 : \tau}{\Gamma \vdash rec\{e_0; x.y.e_1\}(e) : \tau}$$

$$\frac{\Gamma \vdash e : nat \quad \Gamma \vdash e_1 : \tau}{}$$

$$\frac{\Gamma \vdash x : \tau_1 \quad \Gamma \vdash e : \tau_2}{\Gamma \vdash lam\{\tau\}(x.e) : arr(\tau_1;\tau_2)}$$

$$\frac{\Gamma \vdash e_1 : arr(\tau_2;\tau) \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash ap(e_1;e_2) : \tau}$$

$$\frac{}{z\ val}$$

$$\frac{\{e\ val\}}{s(e)\ val}$$

$$lam\{\tau\}(x.e)\ val$$

$$\left[\frac{e \mapsto e'}{s(e) \mapsto s(e')}\right]$$

$$\frac{e \mapsto e'}{rec\{e_0;x,y.e_1\}(e) \mapsto rec\{e_0;x.y.e_1\}(e')}$$

$$rec\{e_0;x,y.e_1\}(z) \mapsto e_0$$

$$\frac{s(e)\ val}{rec\{e_0;x,y.e_1\}(s(e)) \mapsto [e/x][rec\{e_0;x,y.e_1\}(e)/y]e_1}$$

$$\frac{e \Downarrow z}{s(e) \Downarrow v}$$

$$\frac{e \Downarrow z \quad e_0 \Downarrow v}{rec\{e_0;x,y.e_1\}(e) \Downarrow v}$$

$$\frac{e \Downarrow s(e) \quad [e, rec\{e_0;x,y.e_1\}(e)/y]e_1 \Downarrow v}{rec\{e_0;x,y.e_1\}(e) \Downarrow v}$$

Product type

$$\frac{}{\Gamma \vdash \langle\rangle : unit}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1;e_2\rangle : prod(\tau_1;\tau_2)}$$

$$\frac{\Gamma \vdash e : prod(\tau_1;\tau_2)}{\Gamma \vdash fst(e) : \tau_1}$$

$$\frac{\Gamma \vdash e : prod(\tau_1;\tau_2)}{\Gamma \vdash snd(e) : \tau_2}$$

$$\frac{}{\langle\rangle\ val}$$

$$\frac{\{e_1\ val\} \quad \{e_2\ val\}}{\langle e_1;e_2\rangle\ val}$$

$$\frac{e_1 \mapsto e_1'}{\langle e_1;e_2\rangle \mapsto \langle e_1';e_2\rangle}$$

$$\frac{[e \text{ val}] \quad e_2 \mapsto e_2'}{\langle e_1, e_2 \rangle \mapsto \langle e_1, e_2' \rangle}$$

$$\frac{e \mapsto e'}{\text{fst}(e) \mapsto \text{fst}(e')}$$

$$\frac{e \mapsto e'}{\text{snd}(e) \mapsto \text{snd}(e')}$$

$$\frac{[e_1 \text{ val}] \quad [e_2 \text{ val}]}{\text{fst}(\langle e_1, e_2 \rangle) \mapsto e_1}$$

$$\frac{[e_1 \text{ val}] \quad [e_2 \text{ val}]}{\text{snd}(\langle e_1, e_2 \rangle) \mapsto e_2}$$

$$\frac{}{\langle\rangle \Downarrow \langle\rangle}$$

$$\frac{e_1 \Downarrow v \quad e_2 \Downarrow v}{\langle e_1, e_2 \rangle \Downarrow v}$$

$$\frac{e \Downarrow \langle v_1, v_2 \rangle}{\text{fst}(e) \Downarrow v_1}$$

$$\frac{e \Downarrow \langle v_1, v_2 \rangle}{\text{snd}(e) \Downarrow v_2}$$

Sum Ty

$$\frac{\Gamma \vdash e : \text{void}}{\Gamma \vdash \text{abort}\{\tau\}(e) : \tau}$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \text{inl}[\tau_2]\{\tau_1, \tau_2\}(e) : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \text{inr}[\tau_1]\{\tau_1, \tau_2\}(e) : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma, x : \tau_1 \vdash e_1 : \tau \quad \Gamma, y : \tau_2 \vdash e_2 : \tau}{\Gamma \vdash \text{case}(e; x.e_1; y.e_2) : \tau}$$

$$\frac{e \mapsto e'}{\text{abort}(e) \mapsto \text{abort}(e')}$$

$$\frac{e \text{ val}}{\text{inl}[\tau_1]\{\tau_1, \tau_2\}(e) \text{ val}}$$

$$\frac{[e \text{ val}]}{\text{inr}[\tau_1]\{\tau_1, \tau_2\}(e) \text{ val}}$$

$$\frac{e \mapsto e'}{\text{inl}[\tau_1]\{\tau_1, \tau_2\}(e) \mapsto \text{inl}\{\tau_1, \tau_2\}(e')}$$

$$\frac{e \mapsto e'}{\text{inr}[\tau_1]\{\tau_1, \tau_2\}(e) \mapsto \text{inr}\{\tau_1, \tau_2\}(e')}$$

$$\frac{e \mapsto e'}{\text{case}(e; x.e_1; y.e_2) \mapsto \text{case}(e'; x.e_1; y.e_2)}$$

$$\frac{[e \text{ val}]}{\text{case}(\text{inl}[\tau_2]\{\tau_1, \tau_2\}(e); x.e_1; y.e_2) \mapsto [e/x]e_1}$$

$$\frac{e \Downarrow v}{\text{abort}\{\tau\}(e) \Downarrow v}$$

$$\frac{e \Downarrow v}{\text{in}[l]\{\tau_1;\tau_2\}(e) \Downarrow \text{in}[l]\{\tau_1;\tau_2\}(v)}$$

$$\frac{e \Downarrow v}{\text{in}[r]\{\tau_1;\tau_2\}(e) \Downarrow \text{in}[r]\{\tau_1;\tau_2\}(v)}$$

$$\frac{e_0 \Downarrow \text{in}[l]\{\tau_1;\tau_2\}(e) \quad [e/x]e_1 \Downarrow v}{\text{case}(e;x.e_1;y.e_1) \Downarrow v}$$

$$\frac{e_0 \Downarrow \text{in}[r]\{\tau_1;\tau_2\}(e) \quad [e/y]e_2 \Downarrow v}{\text{case}(e;x.e_1;y.e_2) \Downarrow v}$$

System F

$$\frac{\Delta, t\ \text{type} \vdash \tau\ \text{type}}{\Delta \vdash \tau_1\ \text{type} \quad \Delta \vdash \tau_2\ \text{type}}$$

$$\frac{\Delta \vdash \text{arr}(\tau_1;\tau_2)\ \text{type}}{\Delta, t\ \text{type} \vdash \tau\ \text{type}}$$

$$\frac{\Delta \vdash \text{all}(t.\tau)\ \text{type}}{\Delta, \Gamma, x:\tau \vdash x:\tau}$$

$$\frac{\Delta, \Gamma, x:\tau_1 \vdash e:\tau_2}{\Delta, \Gamma \vdash \text{lam}\{\tau\}(x.e):\text{arr}(\tau_1;\tau_2)}$$

$$\frac{\Delta, \Gamma \vdash e_1:\text{arr}(\tau_2;\tau) \quad \Delta, \Gamma \vdash e_2:\tau_2}{\Delta, \Gamma \vdash \text{ap}(e_1;e_2):\tau}$$

$$\frac{\Delta, t, y:\tau_1 \vdash e:\tau}{\Delta, \Gamma \vdash \text{Lam}(t.e):\text{all}(t.\tau)}$$

$$\frac{\Delta, \Gamma \vdash e:\text{all}(t.\tau) \quad \Delta \vdash \tau_1\ \text{type}}{\Delta, \Gamma \vdash \text{App}\{\tau\}(e):[\tau_1/t]\tau}$$

$$\text{Lam}(t.e)\ \text{val}$$

$$\frac{e \mapsto e'}{\text{App}\{\tau\}(e) \mapsto \text{App}\{\tau\}(e')}$$

$$\text{App}\{\tau\}(\text{Lam}(t.e)) \mapsto [\tau/t]e$$

$$\text{Lam}(t.e) \Downarrow \text{Lam}(t.e)$$

$$\frac{e \Downarrow v}{\text{App}\{\tau\}(e) \Downarrow v}$$

$$\frac{e \Downarrow \text{Lam}(t.e) \quad [\tau/t]e \Downarrow v}{\text{App}\{\tau\}(e) \Downarrow v}$$

$$\frac{\Delta \vdash \{\rho\} \text{ type} \quad \Delta, t \text{ type}, \Gamma \vdash \tau \text{ type} \quad \Delta; \Gamma \vdash e : [\rho/t]\tau}{\Delta; \Gamma \vdash \text{pack}\{t.\tau\}\{\rho\}(e) : \text{some}(t.\tau)}$$

$$\frac{\Delta; \Gamma \vdash e_1 : \text{some}(t.\tau) \quad \Delta, t \text{ type}; x : \tau \vdash e_2 : \tau_2 \quad \Delta \vdash \tau_2 \text{ type}}{\Delta; \Gamma \vdash \text{open}\{t.\tau\}\{\rho\}(e_1; t, x.e_2) : \tau_2}$$

$$\frac{[e \text{ val}]}{\text{pack}\{t.\tau\}\{\rho\}(e) \text{ val}}$$

$$\frac{e \mapsto e'}{\text{pack}\{t.\tau\}\{\rho\}(e) \mapsto \cancel{\text{pack}\{t.\tau\}\{\rho\}(e')}}$$

$$\frac{e_1 \mapsto e_1'}{\text{open}\{t.\tau\}\{\rho\}(e_1; t, x.e_2) \mapsto \text{open}\{t.\tau\}\{\rho\}(e_1'; t, x.e_2)}$$

$$\frac{e_1 \text{ val}}{\text{open}\{t.\tau\}\{\rho\}(\text{pack}\{t.\tau\}\{\rho\}(e_1); t, x.e_2) \mapsto \cancel{[e_1/t][\text{pack}\{t.\tau\}\{\rho\}(e_1)/x]} \; [\rho, e_1/t, x]e_2}$$

$$\frac{e \Downarrow v}{\text{pack}\{t.\tau\}\{\rho\}(e) \Downarrow v}$$

$$\frac{e \Downarrow \text{pack}\{t.\tau\}\{\rho\}(e_1) \quad [\rho, e_1/t, x]e_2 \Downarrow v}{\text{open}\{t.\tau\}\{\rho\}(e; t, x.e_2) \Downarrow v}$$

$$\frac{}{\Gamma, x : \tau \vdash x : \tau}$$

$$\frac{\Gamma \vdash z : nat \qquad \Gamma \vdash e : nat}{\Gamma \vdash s(e) : nat}$$

$$\frac{\Gamma \vdash e : nat \qquad \Gamma \vdash e_0 : \tau \qquad \Gamma, x : nat \vdash e_1 : \tau}{\Gamma \vdash ifz\{e_0; x.\, e_1\}(e) : \tau}$$

$$\frac{\Gamma, x : \tau \vdash e : \tau}{\Gamma \vdash fix\{\tau\}(x.e) : \tau}$$

$$\frac{}{z \text{ val}}$$

$$\frac{e \text{ val}}{s(e) \text{ val}}$$

$$\cancel{\frac{\Gamma \vdash z : nat \qquad \Gamma \vdash e : nat}{\Gamma \vdash s(e) : nat}}$$

$$\frac{e \mapsto e'}{s(e) \mapsto s(e')}$$

$$\frac{e \mapsto e'}{ifz\{e_0; x.\, e_1\}(e) \mapsto ifz\{e_0; x.\, e_1\}(e')}$$

$$\frac{}{ifz\{e_0; x.\, e_1\}(z) \mapsto e_0}$$

$$\frac{s(e) \text{ val}}{ifz\{e_0; x.\, e_1\}(s(e)) \mapsto [e/x]e_1}$$

$$\frac{}{fix\{\tau\}(x.e) \mapsto [fix\{\tau\}(x.e)/x]e}$$

$$\frac{}{z \Downarrow z}$$

$$\frac{e \Downarrow v}{s(e) \Downarrow v}$$

$$\frac{e \Downarrow z \qquad e_0 \Downarrow v}{ifz\{e_0; x.\, e_1\}(e) \Downarrow v}$$

$$\frac{e \Downarrow s(e) \qquad [e/x]e_1 \Downarrow v}{ifz\{e_0; x.\, e_1\}(e) \Downarrow v}$$

$$\frac{[fix\{\tau\}(x.e)/x]e \Downarrow v}{fix\{\tau\}(x.e) \Downarrow v}$$

SL

if $\Gamma, x : \tau \vdash e' : \tau'$ & $e : \tau$ then $[e/x]e' : \tau'$

weakening

if $\Gamma \vdash e' : \tau'$ then $\Gamma, x : \tau \vdash e' : \tau$

Determ

$e \mapsto e'$ & $e \mapsto e''$ then e' & e'' are alpha equivalent

CFL

if $e : \mathrm{arr}(\tau_1, \tau_2)$ & eval then $e = \lambda(x : \tau)\, e$ or $\mathrm{lam}\{\tau\}(x.e)$

If $e : \tau$ & eval

1) if $\tau = \mathrm{nat}$ then $e = s(s(s(z)))$

S.L

1) if Δ, t type $\vdash \tau'$ type & $\Delta \vdash \tau$ type then $[\tau/t]\tau'$ type

2) if Δ, t type, $\Gamma \vdash e' : \tau'$ & $\Delta \vdash \tau$ type then $\Delta, [\tau/t]\Gamma \vdash [\tau/t]e' : [\tau/t]\tau'$

3) if Δ, Γ then $\Delta, x : \tau \vdash e' : \tau'$, $\Delta \vdash e : \tau$

if $\tau = all(t \cdot \tau')$ - $e = Lam(t \cdot e)$

if $e = som(t \cdot \tau)$ & eval

$e = pack\{t \cdot \tau\}\{p\}(e)$

Abelle

Syntax chart for Highe Or

~~Fig~~

$\tau \in Typ ::= arr(\tau_1; \tau_2)$ $\tau_1 \to \tau_2$

$e \in Exp ::= lam\{\tau\}(x \cdot e)$ $\lambda(x:\tau)e$

$ap(e_1; e_2)$ $e_1(e_2)$

kind Typ type

kind Exp type

let(e1; x.exp) : τ2 / τ1

$arr(\tau_1; \tau_2)$: typ → typ → type

$lam\{\tau\}(x \cdot e)$: typ → (exp → exp) → exp

$ap(e_1; e_2)$: exp → exp → exp

$$\frac{\Pi, x:\tau_1 \vdash e : \tau_2}{\Pi \vdash lam\{\tau\}(x.e) : arr(\tau_1; \tau_2)}$$

&(let E1 E2) T2 :- & E1 T1, pi x\ &
x T1 => & (E2 x) T2.

~~&(lam T)(x~~

& (lam T E) (arr T1 T2) :-
pi x \ & x T1
=> & (E x) T2

$\Gamma \vdash ap(e_1; e_2) : \tau$

$\mathcal{J}(ap\ E_1\ E_2)\ T :- \mathcal{J}\ E_1\ (arr\ T_2\ T),$
$\mathcal{J}\ E_2\ T_2$

System T

kind nat typ.
kind typ typ.
kind exp typ.

$arr(\tau_1; \tau_2) : typ \to typ \to typ$

$\dfrac{\rho\ typ\ \Delta, t, t_y, x \mapsto}{\rho\ \tau\ \{\ \}\ (e)}$

$x : exp$

$z : nat$

$s(e) : \cancel{exp \to exp}\ nat \to nat$

$rec\{e_0; x, y - e_1\}(e) : exp \to (exp \to exp \to exp) \to exp \to exp$

$$\frac{\Gamma \vdash e : nat \quad \Gamma \vdash e_0 : \tau \quad \Gamma, x : nat, y : \tau \vdash e_1 : \tau}{\Gamma \vdash rec\{e_0; x \cdot y - e_1\}(e) : \tau}$$

$\mathcal{J}(rec\ E_0\ E_1\ E)\ T :- \mathcal{J}\ E\ nat,$
$\mathcal{J}\ E_0\ T,\ pi\ x/\ \mathcal{J}\ x\ nat\ pi\ y/$
$\mathcal{J}\ y\ T \Rightarrow \mathcal{J}(E_1\ x\ y)\ T$

$pi\ x/\ pi\ y/$
$\mathcal{J}\ x\ na$
$= \mathcal{J}(\ \ T\ x)$
$\Rightarrow \mathcal{J}\ E_1\ T$

Tautologies

1) $(((P \to Q) \to Q) \to Q) \to (P \to Q)$

$\lambda f : (((P \to Q) \to Q) \to Q) . \lambda x : P .$

$\lambda h : (((P \to Q) \to Q) \to Q)$

$\lambda g : (P \to Q) \to Q$ $\quad \lambda x : P$

$\lambda g : P \to Q . \; g(x)$

$\lambda g : P \to Q . \; h\, g\, f\, x$

2) $(P \to Q) \to (((P \to Q) \to Q) \to Q)$

$\lambda f : P \to Q . \; \lambda x : (P \to Q) \to Q . \; f(x)$

3) $(((P \vee (P \to R)) \to R) \to R$

$\lambda f : ((P \vee (P \to R)) \to R)$

4) $R \to (((P \vee (P \to R)) \to R)$

$\lambda f : R . \; \lambda x : P + (P \to R) . \; f$

5) $(\forall t . (\exists u . s(t,u)) \to \forall u . s(u,t))$

$\to ((\exists t . \exists u . s(t,u)) \to$

$\lambda p : \forall t . (\exists u . s(t,u)) \to \forall u . s(u,t)$

$\lambda q : \exists t . \exists u . s(t,u) .$

$\Lambda t .$

$\Lambda b . \; open(q ; t, x^{\exists u . s(t,u)} \to p[t](x)[b])[a]$

$p[t](x)[b]$

$\forall u . s(a,t)$

$s(b)t$

$\forall u . s(b,u)$

$p[b](pack\{u . s(b,u)\}\{t\}$

Actually

$a\ z \cdot m = s(m)$

$a\ s(n) \cdot z = a\ n\ s(z)$

$a\ s(n) \cdot s(m) = a\ n\ (a\ s(n)\ m)$

$a = \lambda n{:}nat.\ rec\{\lambda m{:}nat.\ rec\{s(m);\ x, y = s(z)\}(m);\ p, q.\ y\ q\}(n)$

$a\ z\ m = s(m)$

$a\ s(n)\ z = a\ n\ s(z)$

universal variable cannot be the same as the $\exists$ variable

1) $T \wedge P \to P$

$\lambda p{:}T \times P.\ snd\ p$

$dom\{T \times P\}(x.e).\ snd(x)$

2) $P \to T \times P$

$\lambda f{:}P.\ \langle \langle\rangle, f \rangle$

$dom\{P\}(x.e)_{e=pair(e_1, e_2)}$

$e_1 = \langle\rangle$

$e_2 = x$

2a) $a)\ T \vee P = T$

$\lambda_s{:}T + P.\ \langle\rangle$

b) $T \to T + P$

$\lambda f{:}T.\ inl\{T, P\}(f)$

		const	dest
unit	$\langle\rangle$		no
void	nonl		abort
$\times$	$\langle\rangle$		fst, snd
$+$	inr, inl		case
$\forall$	Λ		[]
$\exists$	pack		open
$\to$	λ		()

3a) $F \times P \to F$

$\lambda p{:}F \times P.\ fst(p)$

$dom\{F \times P\}(x.e).\ fst(p)$

3b) $F \to F \times P$

$\lambda f{:}F.\ \langle f$ [illegible]

4a) $F + P \to P$

$\lambda s : F + P \,.\, \text{case}\,(s ; x . \text{abort}\{P\}(x) ; y . y)\ P$

4b) $F \to F + P$

$\lambda f : F \,.\, \text{inl}\{F, P\}(f)$

5a) $T \to P = P$

$\lambda f : T \to P \,.\, f(\langle\rangle)$

5b) $P \to (T \to P)$

$\lambda f : P \,.\, \lambda x . \text{unit} . f$

6a) $P \to T \to T$

$\lambda f : P \to T \,.\, \langle\rangle$

6b) $T \to (P \to T)$

$\lambda f : T \,.\, \lambda x . P . f$

7a) $(F \to P) \to T$

$\lambda f : F \to P \,.\, \langle\rangle$

7b) $T \to (F \to P)$

$\lambda f : \text{unit} \,.\, \lambda x . \text{void} . \text{abort}\{P\}(x)$

$\lambda f : P \times P \cdot fst(f)$

8) b) $P \to P \times P$

$\lambda y : P \cdot \langle y , y \rangle$

9) a) $P + P \to P$

$\lambda s : P + P \cdot case\{s ; \overset{P}{x} \cdot x ; \overset{P}{y} \cdot y\}$

) b) $P \to P + P$

$\lambda f : P \cdot inl\{P, P\}(f)$

10) a) $(P \to P) \to T$

$\lambda f : P \to P \cdot \langle \rangle$

10) b) $T \to P \to P$

$\lambda f : unit \cdot \lambda x : P \cdot x$

11 a) $P + Q \to Q + P$

$\lambda s : P + Q \cdot case(s \cdot \overset{P}{x} \cdot inr(x) \; \overset{Q}{y} \cdot inl(y))$

$Q + P$

commutativity

) $P \times Q = Q \times P$

$\lambda p : P \times Q \cdot \langle \underset{Q}{snd(p)} , \underset{P}{fst(p)} \rangle$

3) a) $P \times (Q \times R) = (P \times Q) \times R$

$\lambda f : P \times (Q \times R) \cdot \langle \underset{P \times Q}{\langle fst(f), fst(snd(f)) \rangle} , \underset{R}{snd(snd(f))} \rangle$

13 b) $\lambda p : (P \times Q) \times R \cdot \langle fst(fst(p)), \langle snd(fst(p)), snd(p) \rangle \rangle$ $Q \times R)$ 140

14) a) $P + (Q + R) = (P + Q) + R$

$\lambda s : P + (Q + R) \cdot case(s ; \overset{P}{x} \cdot inl(inl(x)), \overset{Q+R}{y} \cdot case(y ; \overset{Q}{u} \cdot inl(inr(u)), \overset{R}{v} \cdot inr(v)))$

$(P+Q)+R$

14) b) $(P + Q) + R \xrightarrow{\text{associativity}} P + (Q + R)$

$\lambda s : (P + Q) + R \cdot case(s ; \overset{P+Q}{x} \cdot e_1, \overset{R}{y} \cdot inr(inr(y)))$

$e_1 = case(x ; \overset{P}{u} \cdot inl(u), \overset{Q}{v} \cdot inr(inl(v)))$

15) a) $P \times (Q + R) = (P \times Q) + (P \times R)$ distrib

$\lambda f : P \times (Q + R) \cdot case(snd(f) \cdot \overset{Q}{x} \cdot e_1, \overset{R}{y} \cdot e_2)$

$e_1 = inl\{P, Q\}\langle fst(f), x \rangle$

$e_2 = inr\{P, R\}\langle fst(f), y \rangle$

15) b) $(P \times Q) + (P \times R) \rightarrow P \times (Q + R)$

$\lambda s : (P \times Q) + (P \times R) \cdot \langle e_1, e_2 \rangle$ $Q+R$

$e_1 = case(s ; \overset{P \times Q}{x} \cdot fst(x), \overset{P \times R}{y} \cdot fst(y))$

$e_2 : case(s ; \overset{P \times Q}{u} \cdot inl(snd(u)), \overset{P \times R}{v} \cdot inr(snd(v)))$

16a)

$\lambda s : P + (Q \times R) . \langle e_1 , e_2 \rangle$

$e_1 = case(s ; \overset{P}{x} . inl(x) , \overset{(Q\times R)}{y} . inr(fst(y)))$

$e_2 = case(s ; \overset{P}{u} . inl(u) , \overset{Q\times R}{v} . inr(snd(v)))$

$? \ (P+Q) \times (P+R) \to P + (Q \times R)$

$\lambda p : (P+Q) \times (P+R) . case(fst(p) ; \overset{P}{x} . inl(x) ; \overset{Q}{y} . inr\{P + Q\times R\} \langle y , \ldots \rangle$

$case(snd(p) . \overset{P}{u} . inl(u) ; \overset{R}{v} . \overset{Q}{\ldots} , \ldots = R$

$\lambda p : (P+Q) \times (P+R) . case(fst(p) . \overset{P}{x} . inl(x)$
$\overset{Q}{y} . case(snd(p) . \overset{P}{u} . inl(u) \overset{QR}{v} . inr\{P+(Q\times R)\}$
$\langle y , v \rangle)$

$P + (Q \times R)$

17) a) $(P \to Q) \times (P \to R) = P \to (Q \times R)$

$\lambda p . (P \to Q) \times (P \to R) . \lambda x . P . \langle \underset{Q}{fst(p)(x)} , \underset{R}{snd p(x)} \rangle$

b) $P \to (Q \times R) \to (P \to Q) \times (P \to R)$

$\lambda f : P \to (Q \times R) . \langle \underset{P \to Q}{\lambda x . P . (f(x)) fst(f)} , \underset{P \to R}{\lambda y . P (f(y))} \rangle$

18 a)

$((P+Q) \to R) \to (P \to R) \times (Q \to R)$

$\lambda f : (P+Q) \to R .$

$\cancel{f(\text{case}(f, x^P.}$

$\cancel{\text{inl}(\lambda x.P.}$ $\cancel{y^Q}$

$\langle \quad e_1 \quad , \quad e_2 \quad \rangle$

$e_1 = \lambda x : P . f(\text{inl}\{P,Q\} x)$

$e_2 = \lambda y : Q . f(\text{inr}\{P,Q\}(y))$

18) b) $(P \to R) \times (Q \to R) \to (P+Q) \to R$

$\lambda p : (P \to R) \times (Q \to R) . \lambda x : P+Q .$

$\text{case}(x ; x^P . \text{fst}(p)(x) \quad y^Q . \text{snd}(p) y)$

19) a) $P \to (Q \to R) \to ((P \times Q) \to R)$

$\lambda f : P \to (Q \to R) . \lambda x : P \times Q .$

$(f(\text{fst}(x)))\text{snd}(x)$

19) b) $(P \times Q) \to R . P \to (Q \to R)$

$\lambda f : (P \times Q) \to R . \lambda x . P , \lambda y . Q$

$f(\langle x, y \rangle)$

Inductive proof

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1)

$f : (((a \to b) \to b) \to b), x : a \vdash f(\lambda g : a \to b\, g(x)) :$

$f : (((a \to b) \to b) \to b \cdot \lambda x : a \cdot f(\lambda g : a \to b \cdot g(x)) : a \to b$

$\vdash \lambda f : (((a \to b) \to b) \to b) \cdot \lambda x : a.$

$f(\lambda g : a \to b \cdot g(x)) :$

$(((a \to b) \to b) \to b)$

$\to (a \to b)$

lam{(a→b)→b}n(apb(x;

2)

$f : (a \to b) \vdash g : ((a \to b) \to b) \cdot g(f) : b.$

$f : (a \to b) . \lambda g : ((a \to b) \to b) \cdot g(f) \cdot ((a \to b) \to b) \to b$

$\vdash \lambda f : (a \to b) \cdot \lambda g : ((a \to b) \to b) \cdot g(f) :$

$(a \to b) \to$

$(((a \to b) \to b)$

$\to b)$

1. $((P \to Q) \vee (P \to R)) \to (P \to (Q \vee R))$

$\lambda s : ((P \to Q) + (P \to R)) . \lambda k \cdot P.$

$\cancel{\text{case}\ (s ; x \cdot inl[x(k)], y \cdot inr[y(k)])}$

$\text{case}\ (s ; \overset{P \to Q}{x} \cdot inl[x(k)], \overset{P \to R}{y} \cdot inr[y(k)]).$

2. $((P \to R) + (Q \to R)) \to ((P \times Q) \to R)$

$\lambda s : ((P \to R) + (Q \to R)) . \lambda x : P \times Q.$

$\text{case}(s ; \overset{P \to R}{u} \cdot u[fst(x)], \overset{Q \to R}{v} \cdot v[snd(x)])$

3. $a \to (b \to a)$

$\lambda f : a \cdot \lambda x : b \cdot f$

4. $a \to (b \to b)$

$\lambda f : a . \lambda x : b \cdot x$

5. $(a \to b) \to (a \to b)$

$\lambda f : a \to b \cdot \lambda x : a \cdot f(x)$

6. $(a \to (b \to c)) \to ((a \to b) \to (a \to c))$

$\lambda f : a \to (b \to c) . \lambda x : a \to b . \lambda y \cdot a \cdot f(y)[x(y)]$

7) $(a \to (b \to c)) \to (b \to (a \to c))$ 145

$\lambda f : a \to (b \to c) : \lambda x \cdot b \cdot \lambda y \cdot a$

$f(y)(x)$

8) $(a \to b) \to (a \to a \to b)$

$\lambda f : a \to b \cdot \lambda x \cdot a \; \lambda x' \cdot a \cdot f(x)$

9) $(a \to b) \to (((a \to b) \to b) \to b)$

$\lambda f : a \to b \cdot \lambda g : (a \to b) \to b \cdot g(f)$

10) $(((a \to b) \to b) \to b) \to (a \to b)$

$\lambda f : (((a \to b) \to b) \to b) \cdot \lambda x \cdot a \cdot$

$f(\lambda g : a \to b \cdot g(x))$

<u>Quantifier Equalities</u>

1. $\forall t \cdot \forall u \cdot S(t, u) = \forall u \cdot \forall t \cdot S(t, u)$

$\lambda f : \forall t \cdot \forall u \cdot S(t, u) \cdot \Lambda b \cdot \Lambda a \cdot$

$f[a][b] \cdot$

$\forall u \cdot \forall t \cdot S(t, u)$

$\text{lam}\{\forall t \cdot \forall u \cdot S(t, u)\}(x \cdot e^{\forall u \cdot \forall t})$

$e = \text{Lam}(y \cdot e'')$

$e' = \text{Lam}(t \cdot e_2)$

$e_2 \cdot \text{App}\{u\}(\text{App}\{t\}(x))$

a) $\forall t \cdot P(t) \times Q(t) \Rightarrow (\forall t \cdot P(t)) \times (\forall t \cdot Q(t))$

$\lambda f : \forall t \cdot P(t) \times Q(t) \cdot \langle \Lambda a \cdot fst(f)[a], \Lambda a' \cdot snd(f)[a'] \rangle$

$\forall t \cdot P(t) \qquad \forall t \cdot Q(t)$

$lam\{\forall t \cdot P(t) \times Q(t)\}(x \cdot e)$

$e = Pair(e_1, e_2)$

$e_1 = Lam(t \cdot e_1')$

$e_1' = App\{t\}(fst(x))$

$e_2 : Lam(t \cdot e_2')$

$e_2' \; snd(x)$

2b) $(\forall t \cdot P(t)) \times (\forall t \cdot Q(t)) \rightarrow \forall t \cdot P(t) \times Q(t)$

$\lambda p : (\forall t \cdot P(t)) \times (\forall t \cdot Q(t)) \cdot \Lambda t \cdot \langle e_1, e_2 \rangle$

$P(t), Q(t)$

$e_1 = fst(p)[t]$

$e_2 = snd(p)[t]$

$lam\{(\forall t \cdot P(t)) \times (\forall t \cdot Q(t))\}(x \cdot e)$

$e = Lam(t, e')$

$e' = pair(App\{t\}(fst(p)), App\{t\}(snd(p)))$

3a) $(\forall t \cdot A \to P(t)) \to A \to \forall t \cdot P(t)$ (47)

$\lambda f : \forall t \cdot A \to P(t) \,.\, \lambda x : A \,.\, \Lambda t \,.\, (f[t])(x)$

$\mathrm{lam}\{\forall t \cdot A \to P(t)\}(x \cdot e)$

$e = \mathrm{lam}\{A\}(y \cdot e')$

$e' = \mathrm{Lam}(t \cdot e_2)$

$e_2 = \mathrm{ap}(\mathrm{App}\{t\}(x) ; y)$

3b) $(A \to \forall t \cdot P(t)) \to \forall t \cdot A \to P(t)$

$\lambda f : A \to \forall t \cdot P(t) \,.\, \Lambda t \,.\, \lambda x \cdot A \,.$
$f(x)[t]$

$\mathrm{lam}\{A \to \forall t \cdot P(t)\}(x \cdot e)$

$e = \mathrm{Lam}(t ; e')$

$e' = \mathrm{lam}\{A\}(y \cdot e_2)$

$e_2 = \mathrm{App}\{t\}(\mathrm{ap}(x ; y))$

4) $((\forall t \cdot P(t)) + (\forall t \cdot Q(t))) \to (\forall t \cdot P(t) + Q(t))$

$\lambda s : (\forall t \cdot P(t)) + (\forall t \cdot Q(t)) \,.\, \Lambda t \,.$

$\mathrm{case}(s ; x : \overset{\forall t \cdot P(t)}{\mathrm{inl}[s[t]]} ; y : \overset{\forall t \cdot Q(t)}{\mathrm{inr}[s[t]]})$

$\mathrm{App}\{t\}(\mathrm{inl}(s))$

5) $(\forall t \cdot P(t) \to Q(t)) \to ((\forall t \cdot P(t)) \to (\forall t \cdot Q(t)))$

$\lambda f : \forall t \cdot P(t) \to Q(t) \;\; \lambda x : \forall t \; P(t) \,.$

$\Lambda t \,.\, f(x)$ or $(f[t])x[t]$

1) $\exists t \cdot \exists u \cdot S(t,u) = \exists u \cdot \exists t \cdot S(t,u)$

$\lambda f : \exists t \cdot \exists u \cdot S(t,u) .\ \mathrm{open}(f; t; x^{\exists u \cdot S(t,u)}.$

$\mathrm{open}(x; u \cdot y^{S(t,u)} \cdot \mathrm{pack}\{u, \exists t \cdot S(t,u)\}\{u\}$

$(\mathrm{pack}\{t \cdot S(t,u)\}\{t\}(y)))$ (e)

2) $\exists t \cdot P(t) + Q(t) = (\exists t \cdot P(t)) + (\exists t \cdot Q(t))$

$\lambda f : \exists t \cdot P(t) + Q(t) .$

$\mathrm{open}(f; t, x^{P(t)+Q(t)} \cdot \mathrm{case}(x; u^{P(t)} \cdot e_1, v^{Q(t)} \cdot e_2)$

$e_1 = (\mathrm{pack}\{t, P(t)\}\{t\} u)$ inl

$e_2 = (\mathrm{pack}\{t \cdot Q(t)\}\{t\} v)$ inr

2b) $(\exists t \cdot P(t)) + (\exists t \cdot Q(t)) \to \exists t \cdot P(t) + Q(t)$

$\lambda f : (\exists t \cdot P(t) + \exists t \cdot Q(t)) .\ \mathrm{Case}(f; x^{\exists t \cdot P(t)} \cdot e_1, y^{\exists t \cdot Q(t)} \cdot e_2)$

$e_1 = \mathrm{open}(x; t \cdot u^{P(t)} \cdot (\mathrm{pack}\{t \cdot P(t)\}\{t\}(u))$ inl

$e_2 = \mathrm{open}(y; t \cdot v^{Q(t)} \cdot (\mathrm{pack}\{t \cdot Q(t)\}\{t\}(v))$ inr

$\lambda f : \exists t \cdot A \times P(t) . \langle \qquad \rangle$

$e_1 = \cancel{open}\{f ; t \cdot u^{A \times P(t)} \cdot fst(u)\}^{A}$ $\quad \exists t \cdot P(t)$

$e_2 = open\{f ; t \cdot v^{A \times P(t)} \cdot pack\{t \cdot P(t)\}\{t\}$

$open(f ; t \cdot u^{A \times P(t)} : \langle fst(u) , e_2 \rangle snd(v))$

$e_2 = pack\{t \cdot P(t)\}\{t\}(snd\, u)$

) $A \times \exists t \cdot P(t) \rightarrow \exists t \cdot A \times P(t)$

$\lambda f : A \times \exists t \cdot P(t) . open(snd(f) ; t \cdot x^{P(t)} \cdot$

$pack\{t \cdot A \times P(t)\}\{t\}(\langle fst(f), x \rangle)$

?

4) $(\exists t \cdot P(t)) \rightarrow A \quad \rightarrow \forall t \cdot P(t) \rightarrow A$

$\lambda f : (\exists t \cdot P(t)) \rightarrow A . \lambda x : P(t) .$

no open.

$\cancel{open(f ; t \cdot u^{P(t) \rightarrow A} \cdot u(x))}$

$f(pack\{t \cdot P(t)\}\{t\}(x))$

5) $(\forall t \cdot P(t) \rightarrow A) \rightarrow (\exists t \cdot P(t)) \rightarrow A$

$\lambda f : \forall t \cdot P(t) \rightarrow A . \lambda x : \exists t \cdot P(t) .$

$open(x ; t \cdot u^{P(t)} : f(t) \cdot u)$

6) $\exists t \cdot P(t) \times Q(t) \rightarrow ((\exists t \cdot P(t)) \times (\exists t \cdot Q(t)))$

$\lambda f : \exists t \cdot P(t) \times Q(t) . \langle e_1 , e_2$

$e_1 = open(f ; t \cdot u^{P(t) \times Q(t)} \cdot pack\{t \cdot P(t)\}\{t\} fst(u))^{\exists t \cdot P(t)}$

$e_2 = open(f ; t \cdot v^{P(t) \times Q(t)} \cdot pac\{t \cdot Q(t)\}\{t\} snd(v))$

7) $(\forall t \cdot P(t) \to Q(t)) \to ((\exists t \cdot P(t)) \to (\exists t \cdot Q(t))$

$\lambda f : \forall t \cdot P(t) \to Q(t) \,.\, \lambda x \cdot \exists t \cdot P(t) \,.$

$\text{open}(x ; t \cdot \overset{P(t)}{u} \,.\, \text{pack}\, \{t \cdot Q(t)\}\{t$

or

$f[t](\lambda x \cdot P(t) \cdot x)$

8) $(\exists t \cdot P(t) \to Q(t)) \to ((\forall t \cdot P(t)) \to (\exists t \cdot Q(t))$

$\lambda f : \exists t \cdot P(t) \to Q(t) \,.\, \lambda x \cdot \forall t \cdot P(t) \,.$

$\text{open}(f ; t \cdot \overset{P(t) \to Q(t)}{u} \,.\, \cancel{x[t](u)}$

$\text{pack}\, \{t \cdot Q(t)\}\{t\}(u[t](x))$

a) $\exists t \cdot \forall u \cdot S(t,u) \to (\forall u \cdot \exists t \cdot S(t,u))$

$\lambda f : \exists t \cdot \forall u \cdot S(t,u) \,.\, \lambda u \,.\, \text{open}(f ; t, \overset{\forall u S(t,u)}{x} \,.$

$\text{pack}\, \{t \cdot S(t,u)\}\{t\}\, x[u] \,.$

$$\frac{\Gamma, x:\tau \vdash x:\tau}{\Gamma \vdash e_1 : num \quad \Gamma \vdash e_2 : num}$$
$$\Gamma \vdash plus(e_1;e_2) : num$$

$$\frac{\Gamma \vdash e_1 : num \quad \Gamma \vdash e_2 : num}{\Gamma \vdash times(e_1;e_2) : num}$$

$$\frac{\Gamma \vdash e_1 : str \quad \Gamma \vdash e_2 : str}{\Gamma \vdash cat(e_1;e_2) : str}$$

$$\frac{\Gamma \vdash e : str}{\Gamma \vdash ln(e) : num}$$

$$\frac{\Gamma \vdash e_0 : \tau_1 \quad \Gamma, x:\tau_1 \vdash e_1 : \tau}{\Gamma \vdash let(e_0;x.e_1) : \tau}$$

$$\frac{e_1 \mapsto e_1'}{plus(e_1;e_2) \mapsto plus(e_1';e_2)}$$

$$\frac{e_1\ val \quad e_2 \mapsto e_2'}{plus(e_1;e_2) \mapsto plus(e_1;e_2')}$$

$$\frac{}{num[n]\ val}$$

$$\frac{}{str[s]\ val}$$

$$\frac{n_1 + n_2 \text{ is } n \quad n\ val}{plus(num[n_1];num[n_2]) \mapsto num[n]}$$

$$\frac{s_1 \wedge s_2 = s}{cat(str[s_1];str[s_2]) \mapsto str[s]}$$

$$\frac{e_0 \mapsto e_0'}{let(e_0;x.e_1) \mapsto let(e_0';x.e_1)}$$

$$\frac{e_0\ val}{let(e_0;x.e_1) \mapsto [e_0/x]e_1}$$

$$\frac{e \Downarrow v}{let(e_0;x.e_1) \Downarrow v}$$

$$\frac{e_0 \Downarrow v_0 \quad [v_0/x]e_1 \Downarrow v}{let(e_0;x.e_1) \Downarrow v}$$

$$\frac{}{num[n] \Downarrow num[n]}$$

$$\frac{}{str[s] \Downarrow str[s]}$$

$$\frac{e_1 \Downarrow num[n_1] \quad e_2 \Downarrow num[n_2] \quad n_1+n_2 \text{ is } n}{plus(e_1;e_2) \Downarrow num[n]}$$

$$\frac{e_1 \Downarrow \mathrm{str}[s_1] \quad e_2 \Downarrow \mathrm{str}[s_2] \quad s_1 \$ s_2 = s_{\mathrm{nat}}}{\mathrm{cat}(e_1; e_2) \Downarrow \mathrm{str}[s]}$$

$$\frac{e \Downarrow \mathrm{str}[s] \quad |s| = n \quad n \geq 0}{\mathrm{len}(e) \Downarrow \mathrm{num}[n]}$$

Higher order

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \mathrm{lam}\{\tau\}(x.e) : \mathrm{arr}(\tau_1; \tau_2)}$$

$$\frac{\Gamma \vdash e_1 : \mathrm{arr}(\tau_2; \tau) \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \mathrm{ap}(e_1; e_2) : \tau}$$

$$\mathrm{lam}\{\tau\}(x.e)\ \mathrm{val}$$

$$\frac{e_1 \mapsto e_1'}{\mathrm{ap}(e_1; e_2) \mapsto \mathrm{ap}(e_1'; e_2)}$$

$$\frac{e_1\ \mathrm{val} \quad e_2 \mapsto e_2'}{\mathrm{ap}(e_1; e_2) \mapsto \mathrm{ap}(e_1; e_2')}$$

$$\frac{[e_2\ \mathrm{val}]}{\mathrm{ap}(\mathrm{lam}\{\tau\}(x.e); e_2) \mapsto [e_2/x]e}$$

$$\frac{\mathrm{lam}\{\tau\}(x.e) \Downarrow \mathrm{lam}\{\tau\}(x.e) \quad e_2 \Downarrow v_1 \quad [v_1/x]e \Downarrow v}{\mathrm{ap}(e_1; e_2) \Downarrow v}$$

$$\frac{e_1 \Downarrow \mathrm{lam}\{\tau\}(x.e) \quad [e_2/x]e \Downarrow v}{\mathrm{ap}(e_1; e_2) \Downarrow v}$$

System F

$$\Gamma \vdash x : \tau \quad \vdash x : \tau$$

$$\Gamma \vdash z : \mathrm{nat}$$

$$\frac{\Gamma \vdash e : \mathrm{nat}}{\Gamma \vdash s(e) : \mathrm{nat}}$$

$$\frac{\Gamma \vdash e : \mathrm{nat} \quad \Gamma \vdash e_0 : \tau \quad \Gamma, x : \mathrm{nat}, y : \tau \vdash e_1 : \tau}{\Gamma \vdash \mathrm{rec}\{e_0; x, y.e_1\}(e) : \tau}$$

$$z\ \mathrm{val}$$

$$\frac{e\ \mathrm{val}}{s(e)\ \mathrm{val}}$$

$$\frac{e \mapsto e'}{s(e) \mapsto s(e')}$$

$$\frac{e \mapsto e'}{rec\{e_0; x, y.e_1\}(e) \mapsto rec\{e_0; x, y.e_1\}(e')}$$

$$\frac{s(e)\ val}{rec\{e_0; x, y.e_1\}(e) \mapsto [e, rec\{e_0; x, y.e_1\}(e)/x, y]e_1}$$

$$\frac{s(e) \Downarrow \quad [e, rec\{e_0; x, y.e_1\}(e)/x, y]e_1 \Downarrow V}{rec\{e_0; x, y.e_1\}(e) \Downarrow V}$$

$$\frac{e \Downarrow z \quad e_0 \Downarrow V}{rec\{e_0; x, y.e_1\}z \mapsto e_0}$$

$$\frac{s(e) \Downarrow \quad e \Downarrow s(e) \quad [e, rec\{e_0; x, y.e_1\}(e)/x, y]e_1 \Downarrow V}{rec\{e_0; x, y.e_1\}(e) \Downarrow V}$$

$$\frac{\Gamma \vdash fst(e) : prod(e_1; e_2)}{\Gamma \vdash snd(e) : \tau_2}$$

$$\frac{\langle\rangle\ val}{}$$

$$\frac{[e_1\ val]\ [e_2\ val]}{\langle e_1, e_2\rangle\ val}$$

$$\frac{e_1 \mapsto e_1'}{\langle e_1, e_2\rangle \mapsto \langle e_1', e_2\rangle}$$

$$\frac{e \mapsto e'}{fst(e) \mapsto fst(e')}$$

$$\frac{e \mapsto e'}{snd(e) \mapsto snd(e')}$$

$$\frac{[e_1\ val]\ [e_2\ val]}{fst\langle e_1; e_2\rangle \mapsto e_1}$$

$$\frac{[e_1\ val]\ [e_2\ val]}{snd\langle e_1; e_2\rangle \mapsto e_2}$$

Product Rule

$$\frac{}{\Gamma \vdash \langle\rangle : unit}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1; e_2\rangle : \tau_1 \times \tau_2}$$

$$\frac{e \Downarrow \langle v_1, v_2\rangle}{fst(e) \Downarrow v_1}$$

$$\frac{e \Downarrow \langle v_1, v_2\rangle}{snd(e) \Downarrow v_2}$$

Sum types

$$\frac{\Gamma \vdash e : void}{\Gamma \vdash abort\{\tau\}(e) : \tau}$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash inl\{\tau_1;\tau_2\}(e) : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash inr\{\tau_1;\tau_2\}(e) : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e_0 : \tau_1 + \tau_2 \quad \Gamma, x : \tau_1 \vdash e_1 : \tau \quad \Gamma, y : \tau_2 \vdash e_2 : \tau}{\Gamma \vdash case(e_0; x.e_1; y.e_2) : \tau}$$

$$\frac{e \mapsto e'}{abort\{\tau\}(e) \mapsto abort\{\tau\}(e')}$$

$$\frac{[e\ val]}{inl\{\tau_1;\tau_2\}(e)\ val}$$

$$\frac{[e\ val]}{inr\{\tau_1;\tau_2\}(e)\ val}$$

$$\frac{e \mapsto e'}{inl\{\tau_1;\tau_2\}(e) \mapsto inl\{\tau_1;\tau_2\}(e')}$$

$$\frac{e \mapsto e'}{inr\{\tau_1;\tau_2\}(e) \mapsto inr\{\tau_1;\tau_2\}(e')}$$

$$\frac{e_0 \mapsto e_0'}{case(e_0; x.e_1; y.e_2) \mapsto case(e_0'; x.e_1; y.e_2)}$$

$$\frac{[e_0\ val]}{case(inl\{\tau_1;\tau_2\}(e_0); x.e_1; y.e_2) \mapsto [e_0/x]e_1}$$

$$\frac{[e_0\ val]}{case(inr\{\tau_1;\tau_2\}(e_0); x.e_1; y.e_2) \mapsto [e_0/y]e_2}$$

$$\frac{e \Downarrow v_1}{inl\{\tau_1;\tau_2\}(e) \Downarrow inl\{\tau_1;\tau_2\}(v_1)}$$

$$\frac{e_0 \Downarrow inl\{\tau_1;\tau_2\}(e) \quad [e_1 \Downarrow v]}{case(e_0; x.e_1; y.e_2) \Downarrow v}$$

$$\frac{e \Downarrow v}{inr\{\tau_1;\tau_2\}(e) \Downarrow inr\{\tau_1;\tau_2\}(v)}$$

$$\frac{e \Downarrow v}{abort\{\tau\}(e) \Downarrow v}$$

$$\frac{\Delta, t\ \text{type} \vdash \tau_1\ \text{type}}{\Delta \vdash \tau_1\ \text{type}} \qquad \frac{\Delta \vdash \tau_1\ \text{type} \quad \Delta \vdash \tau_2\ \text{type}}{\Delta \vdash \text{arr}(\tau_1; \tau_2)\ \text{type}}$$

$$\frac{\Delta, t\ \text{type} \vdash \tau\ \text{type}}{\Delta \vdash \text{all}(t.\tau)\ \text{type}}$$

$$\frac{}{\Delta\Gamma, x:\tau \vdash x:\tau}$$

$$\frac{\Delta\Gamma \vdash \cancel{\text{type}}\ e:\tau}{\Delta\Gamma \vdash \text{lam}(t.e) : \text{all}(t.\tau)}$$

$$\frac{\Delta\Gamma \vdash e : \text{all}(t.\tau) \quad \Delta\Gamma \vdash \tau\ \text{type}}{\Delta\Gamma \vdash \text{App}\{\tau\}(e) : \tau}$$

$$\frac{}{\text{lam}(t.e)\ \text{val}}$$

$$\frac{e \mapsto e'}{\text{App}\{\tau\}(e) \mapsto \text{App}\{\tau\}(e')}$$

$$\frac{}{\text{App}\{\tau\}(\text{lam}(t.e)) \mapsto [\tau/t]e}$$

$$\frac{}{(t.e) \Downarrow \text{lam}(t.e)\ \text{val}}$$

$$\frac{e \Downarrow \text{lam}(t.e) \quad [\tau/t]e \Downarrow V}{\text{App}\{\tau\}(e) \Downarrow V}$$

$$\frac{e \Downarrow V}{\text{App}\{\tau\}(e) \Downarrow V}$$

$$\frac{\text{Abs}(\tau\ \text{type}\ e : P[\tau]^{\tau})}{\Delta \vdash \tau\ \text{type} \quad \Delta, t\ \text{type} \vdash \tau\ \text{type}}$$

$$\Delta\Gamma \vdash \text{pack}\{t.\tau\}\{e\}(e) : \text{some}(t.\tau)$$

$$\frac{\Delta\Gamma \vdash e_1 : \text{some}(t.\tau) \quad \Delta\Gamma \vdash \text{open}\{t.\tau\}\{e\}(e_1; t, x.e_2) : \tau_2 \quad \Delta, t\ \text{type}, x:\tau \vdash e_2 : \tau_2 \quad \tau_2\ \text{type}}$$

$$\frac{\text{pack}\{t.\tau\}\{e\}(e)\ \text{val} \quad e \mapsto e'}{\text{pack}\{t.\tau\}\{e\}(e) \mapsto \text{pack}\{t.\tau\}\{e\}(e')}$$

$$\frac{e \mapsto e}{\text{open}\{t.\tau\}\{e\}(e_1; t, x.e_1) \mapsto e'}$$

$$\frac{e \text{ val}}{\text{open}(t.\tau)\{\rho\}(\text{pack}\{t.\tau\}\{\rho\}(e); t, x.e_1) \mapsto [\rho, e/t, x]e_1}$$

$$\frac{\text{pack}\{t.\tau\}\{\rho\}(e) \Downarrow \text{pack}\{t.\tau\}\{\rho\}(e) \qquad e \Downarrow V}{\text{pack}\{t.\tau\}\{\rho\}(e) \Downarrow V}$$

$$\frac{e_1 \Downarrow \text{pack}\{t.\tau\}\{\rho\}(e) \qquad [\rho, e/t, x]e_2 \Downarrow V}{\text{open}\{t.\tau\}\{\rho\}(e_1; t, x.e_2) \Downarrow V}$$

PCF

$$\frac{}{\Gamma, x:\tau \vdash x:\tau}$$

$$\frac{}{\Gamma \vdash z : \text{nat}}$$

$$\frac{\Gamma \vdash e : \text{nat}}{\Gamma \vdash s(e) : \text{nat}}$$

$$\frac{\Gamma, x:\tau_1 \vdash e:\tau_2}{\Gamma \vdash \text{lam}\{\tau_1\}(x.e) : \text{parr}(\tau_1;\tau_2)}$$

$$\frac{\Gamma \vdash e_1 : \text{parr}(\tau_2;\tau_1) \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \text{ap}(e_1; e_2) : \tau_1}$$

$$\frac{\Gamma, x:\tau \vdash e:\tau}{\Gamma \vdash \text{fix}\{\tau\}(x.e) : \tau}$$

$$\frac{e \mapsto e'}{\text{ifz}\{e_0; x.e_1\}(e) \mapsto \text{ifz}\{e_0; x.e_1\}(e')}$$

$$\frac{\Gamma \vdash e : \text{nat} \qquad \Gamma \vdash e_0 : \tau \qquad \Gamma, x:\text{nat} \vdash e_1 : \tau}{\Gamma \vdash \text{ifz}\{e_0; x, e_1\}(e) : \tau}$$

$$\frac{\text{ifz}\{e_0; x.e_1\}(z) \mapsto e_0 \quad [e_0 \text{ val}]}{\text{ifz}\{e_0; x.e_1\}(s(e)) \mapsto [e_0/x]e_1}$$

$$\frac{\Gamma, x : \tau \vdash x : \tau}{\Gamma \vdash num[n] : nat}$$

$$\frac{\Gamma \vdash str[s] : nat \qquad \Gamma \vdash e_1 : num \qquad \Gamma \vdash e_2 : num}{\Gamma \vdash plus(e_1; e_2) : num}$$

$$\frac{\Gamma \vdash e_1 : str \qquad \Gamma \vdash e_2 : str}{\Gamma \vdash cat(e_1; e_2) : str}$$

$$\frac{\Gamma \vdash e : str}{\Gamma \vdash len(e) : num}$$

$$\frac{\Gamma \vdash e_0 : \tau_1 \qquad \Gamma, x : \tau_1 \vdash e_1 : \tau}{\Gamma \vdash let(e_0; x.e_1) : \tau}$$

$$\frac{e_1 \Downarrow num[n_1] \qquad e_2 \Downarrow num[n_2] \qquad n_1 + n_2 = n}{plus(e_1; e_2) \Downarrow num[n]}$$

$$\frac{e_1 \Downarrow str[s_1] \qquad e_2 \Downarrow str[s_2] \qquad s_1 \hat{\ } s_2 = s}{cat(e_1; e_2) \Downarrow str[s]}$$

$$\frac{e \Downarrow str[s] \qquad |s| = n}{len(e) \Downarrow num[n]}$$

$$\frac{e_0 \Downarrow v \qquad [e_0/x]e_1 \Downarrow v}{let(e_0; x.e_1) \Downarrow v}$$

$$\frac{e_1 \mapsto e_1'}{plus(e_1; e_2) \mapsto plus(e_1'; e_2)}$$

$$\frac{e_1 \ val \qquad e_2 \mapsto e_2'}{plus(e_1; e_2) \mapsto plus(e_1; e_2')}$$

$$\frac{e \mapsto e'}{len(e) \mapsto len(e')}$$

$$\frac{e_0 \mapsto e_0'}{let(e_0; x.e_1) \mapsto let(e_0'; x.e_1)}$$

$$\frac{e_0 \ val}{let(e_0; x.e_1) \mapsto [e_0/x]e_1}$$

$$\frac{len(e) \ val}{len(e) \ val}$$

$$\frac{e_1 \mapsto e_1' \qquad e_2 \mapsto e_2'}{plus(e_1; e_2) \mapsto plus(e_1'; e_2')}$$

$$\frac{e_1 \mapsto num[n_1] \qquad e_2 \mapsto num[n_2] \qquad n_1 + n_2 = n}{plus(num[n_1]; num[n_2]) \mapsto num[n]}$$

$$\frac{e_1 \mapsto str[s_1] \qquad e_2 \mapsto str[s_2] \qquad s_1 \hat{\ } s_2 = s}{cat(str[s_1]; str[s_2]) \mapsto str[s]}$$

$$\frac{num[n] \ val}{num[n] \ val}$$

$$\frac{str[s] \ val \qquad |s| = n}{len(str[s]) \mapsto num[n]}$$

$$\frac{\Gamma, x : \tau \vdash x : \tau}{}$$

$$\frac{}{\Gamma \vdash num[n] : num}$$

$$\frac{}{\Gamma \vdash str[s] : str}$$

$$\frac{\Gamma \vdash e_1 : num \quad \Gamma \vdash e_2 : num}{\Gamma \vdash plus(e_1; e_2) : num}$$

$$\frac{\Gamma \vdash e_1 : num \quad \Gamma \vdash e_2 : num}{\Gamma \vdash times(e_1; e_2) : num}$$

$$\frac{\Gamma \vdash e_1 : str \quad \Gamma \vdash e_2 : str}{\Gamma \vdash cat(e_1; e_2) : str}$$

$$\frac{\Gamma \vdash e : str}{\Gamma \vdash len(e) : num}$$

$$\frac{\Gamma \vdash e_0 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_1 : \tau}{\Gamma \vdash let(e_0; x.e_1) : \tau}$$

$$\frac{}{num[n]\ val}$$

$$\frac{}{str[s]\ val}$$

$$\frac{n_1 + n_2 \text{ in nat}}{plus(num[n_1]; num[n_2]) \mapsto num[n]}$$

$$\frac{s_1 \hat{\ } s_2 \text{ is } s\ s'}{cat(str[s_1]; str[s_2]) \mapsto str[s]}$$

$$\frac{e_1 \mapsto e_1'}{plus(e_1; e_2) \mapsto plus(e_1'; e_2)}$$

$$\frac{e_1\ val \quad e_2 \mapsto e_2'}{plus(e_1; e_2) \mapsto plus(e_1; e_2')}$$

$$\frac{e_0 \mapsto e_0'}{let(e_0; x.e_1) \mapsto let(e_0'; x.e_1)}$$

$$\frac{e_0\ val}{let(e_0; x.e_1) \mapsto [e_0/x]e_1}$$

$$\frac{}{num[n] \Downarrow num[n]}$$

$$\frac{}{str[s] \Downarrow str[s]}$$

$$\frac{e_1 \Downarrow num[n_1] \quad e_2 \Downarrow num[n_2] \quad n_1 + n_2 \text{ in nat}}{plus(e_1; e_2) \Downarrow num[n_1]}$$

$$\frac{e \Downarrow str[s] \quad |s| = n \text{ nat}}{len(e) \Downarrow num[n]}$$

Higher

$$\frac{\Gamma, x:\tau_1 \vdash e:\tau_2}{\Gamma \vdash lam\{\tau_1\}(x.e) : arr(\tau_1;\tau_2)}$$

$$\frac{\Gamma \vdash e_1 : arr(\tau_2;\tau) \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash ap(e_1;e_2) : \tau}$$

$$\frac{}{lam\{\tau\}(x.e)\ val}$$

$$\frac{e_1 \mapsto e_1'}{ap(e_1;e_2) \mapsto ap(e_1';e_2)}$$

$$\frac{e_1\ val \quad e_2 \mapsto e_2'}{ap(e_1;e_2) \mapsto ap(e_1;e_2')}$$

$$\frac{\{e_2\ val\}}{ap(lam\{\tau\}(x.e);e_2) \mapsto [e_2/x]e}$$

$$\frac{}{lam\{\tau\}(x.e) \Downarrow lam\{\tau\}(x.e)}$$

$$\frac{e_1 \Downarrow lam\{\tau\}(x.e_1) \quad [e_2/x]e_1 \Downarrow v}{ap(e_1;e_2) \Downarrow v}$$

$\lambda(x:\tau)e \quad e_1$

System T

$$\frac{}{\Gamma, x:\tau \vdash x:\tau}$$

$$\frac{\Gamma \vdash z : nat}{}$$

$$\frac{\Gamma \vdash e : nat}{\Gamma \vdash s(e) : nat}$$

$$\frac{\Gamma \vdash e : nat \quad \Gamma \vdash e_0 : \tau \quad \Gamma, x:nat, y:\tau \vdash e_1 : \tau}{\Gamma \vdash rec\{e_0;x,y.e_1\}(e) : \tau}$$

$$\frac{}{z\ val}$$

$$\frac{\{e\ val\}}{s(e)\ val}$$

$$\frac{e \mapsto e'}{s(e) \mapsto s(e')}$$

$$\frac{e \mapsto e'}{rec\{e_0;x,y.e_1\}(e) \mapsto rec\{e_0;x,y.e_1\}(e')}$$

$$\frac{}{rec\{e_0;x,y.e_1\}(z) \mapsto e_0}$$

$$\frac{e(e)\ val}{rec\{e_0;x,y.e_1\}(s(e)) \mapsto [e/x][rec\{e_0;x,y.e_1\}(e)/y]e_1}$$

$$\frac{e \Downarrow v}{s(e) \Downarrow v}$$

$$\frac{e \Downarrow z \quad e_0 \Downarrow v}{rec\{e_0; x, y.\, e_1\}(e) \Downarrow v}$$

$$\frac{e \Downarrow s(e) \quad [e/x]rec\{e_0; x, y.\, e_1\}(e_0)/y \Downarrow v}{rec\{e_0; x, y.\, e_1\}(e) \Downarrow v}$$

$rec\, e\{z \to e_0 \mid s(x) \text{ with } y \to e_1\}$

$rec\, e\{z \to e_0 \mid s(x) \text{ with } y \to e_1\}$

product syntax

$$\frac{}{\Gamma \vdash \langle\rangle : unit}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1, e_2 \rangle : prod(\tau_1; \tau_2)}$$

$$\frac{\Gamma \vdash e : prod(\tau_1; \tau_2)}{\Gamma \vdash fst(e) : \tau_1}$$

$$\frac{\Gamma \vdash e : prod(\tau_1; \tau_2)}{\Gamma \vdash snd(e) : \tau_2}$$

$$\frac{\{e_1\, val\} \quad \{e_2\, val\}}{\langle e_1; e_2 \rangle\, val} \quad \langle\rangle\, val$$

$$\frac{e_1 \mapsto e_1'}{\langle e_1; e_2 \rangle \mapsto \langle e_1'; e_2 \rangle}$$

$$\frac{e_1\, val \quad e_2 \mapsto e_2'}{\langle e_1; e_2 \rangle \mapsto \langle e_1; e_2' \rangle}$$

$$\frac{e \mapsto e'}{fst(e) \mapsto fst(e')}$$

$$\frac{e \mapsto e'}{snd(e) \mapsto snd(e')}$$

$$\frac{[e_1\, val] \quad [e_2\, val]}{fst\langle e_1; e_2 \rangle \mapsto e_1}$$

$$\frac{[e_1\, val] \quad [e_2\, val]}{snd\langle e_1; e_2 \rangle \mapsto e_2}$$

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{\langle e_1; e_2 \rangle \Downarrow v, \text{ where } v = \langle v_1, v_2 \rangle}$$

$$\frac{e \Downarrow \langle v_1, v_2 \rangle}{fst(e) \Downarrow v_1}$$

$$\frac{e \Downarrow \langle v_1, v_2 \rangle}{snd(e) \Downarrow v_2}$$

$$\frac{e_1 \Downarrow v \quad e_2 \Downarrow v}{\langle e_1; e_2 \rangle \Downarrow v}$$

$$\frac{}{\langle\rangle \Downarrow \langle\rangle}$$

$$\frac{e \Downarrow \langle v_1, v_2 \rangle}{fst(e) \Downarrow v_1}$$

$$\frac{e \Downarrow \langle v_1, v_2 \rangle}{snd(e) \Downarrow v_2}$$

$$\frac{\Gamma \vdash e : \text{void}}{\Gamma \vdash \text{abort}\{\tau\}(e) : \tau}$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \text{inl}\{\tau_1;\tau_2\}(e) : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \text{inr}\{\tau_1;\tau_2\}(e) : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e_0 : \tau_1 + \tau_2 \quad \Gamma, x : \tau_1 \vdash e_1 : \tau \quad \Gamma, y : \tau \vdash e_2 : \tau}{\Gamma \vdash \text{case}(e_0; x.e_1; y.e_2) : \tau}$$

$$\frac{e \Downarrow v}{\text{inr}\{\tau_1;\tau_2\}(e) \Downarrow \text{inr}\{\tau_1;\tau_2\}(v)}$$

$$\frac{e \mapsto e'}{\text{abort}\{\tau\}(e) \mapsto \text{abort}\{\tau\}(e')}$$

$$\frac{e \Downarrow v}{\text{inl}\{\tau_1;\tau_2\}(e) \Downarrow \text{inl}\{\tau_1;\tau_2\}(v)}$$

$$\frac{e \text{ val}}{\text{inl}\{\tau_1;\tau_2\}(e) \text{ val}}$$

$$\frac{e_0 \Downarrow \text{inr}\{\tau_1;\tau_2\}(e) \quad [e_0/x]e_1 \Downarrow v}{\text{case}(e_0; x.e_1; y.e_2) \Downarrow v}$$

$$\frac{e \Downarrow v}{\text{abort}\{\tau\}(e) \Downarrow v}$$

$$\frac{e \text{ val}}{\text{inr}\{\tau_1;\tau_2\}(e) \text{ val}}$$

$$\frac{e \mapsto e'}{\text{inl}\{\tau_1;\tau_2\}(e) \mapsto \text{inl}\{\tau_1;\tau_2\}(e')}$$

$$\frac{e \mapsto e'}{\text{inr}\{\tau_1;\tau_2\}(e) \mapsto \text{inr}\{\tau_1;\tau_2\}(e')}$$

$$\frac{e_0 \mapsto e_0'}{\text{case}(e_0; x.e_1; y.e_2) \mapsto \text{case}(e_0'; x.e_1; y.e_2)}$$

$$\frac{[e_0 \text{ val}]}{\text{case}(\text{inl}\{\tau_1;\tau_2\}(e_0); x.e_1; y.e_2) \mapsto [e_0/x]e_1}$$

System F

$$\frac{\Delta \vdash \tau_1 \text{ type} \quad \Delta \vdash \tau_2 \text{ type}}{\Delta \vdash arr(\tau_1; \tau_2) \text{ type}}$$

$$\frac{\Delta, t \text{ type} \vdash \tau \text{ type}}{\Delta \vdash all(t.\tau) \text{ type}}$$

$$\frac{}{\Delta, t \text{ type} \vdash t \text{ type}}$$

$$\frac{\Delta, \tau_1 \text{ type} \quad \Delta, x : \tau_1 \vdash e : \tau_2}{\Delta \vdash lam\{\tau\}(x.e) : arr(\tau_1; \tau_2)}$$

$$\frac{\Delta \vdash e_1 : arr(\tau_2; \tau) \quad \Delta \vdash e_2 : \tau_2}{\Delta \vdash ap(e_1; e_2) : \tau}$$

$$\frac{\Delta, t \text{ type} \vdash e : \tau}{\Delta \vdash Lam(t.e) : all(t.\tau)}$$

$$\frac{\Delta \vdash e : all(t.\tau') \quad \Delta \vdash \tau \text{ type}}{\Delta \vdash App\{\tau\}(e) : [\tau/t]\tau'}$$

$$App\{\tau\}(Lam(t.e)) \mapsto [\tau/t]e$$

$$\frac{}{Lam(t.e) \text{ val}}$$

$$\frac{e \mapsto e'}{App\{\tau\}(e) \mapsto App\{\tau\}(e')}$$

$$\frac{}{Lam(t.e) \Downarrow Lam(t.e)}$$

$$\frac{e \Downarrow Lam(t.e) \quad [\tau/t]e \Downarrow v}{App\{t\}(e) \Downarrow v}$$

$$\frac{e \Downarrow \quad \Downarrow}{App\{t\}(e) \Downarrow v}$$

System Ab

$$\frac{\Delta \vdash \rho \text{ type} \quad \Delta, y : \tau, t \text{ type} \vdash e : \tau}{\Delta \vdash pack\{t.\tau\}\{\rho\}(e) : some(t.\tau)}$$

$$\frac{\Delta, t \text{ type}, x : \tau \vdash e_1 : \tau_2}{\Delta \vdash open\{t.\tau\}\{\rho\}(e_0; t.x.e_1) : \tau_2}$$

$$\frac{e \text{ val}}{pack\{t.\tau\}\{\rho\}(e) \text{ val}}$$

$$\frac{e \mapsto e'}{pack\{t.\tau\}\{\rho\}(e) \mapsto pack\{t.\tau\}\{\rho\}(e')}$$

$$\frac{e \mapsto e_0}{\text{open}\{t.\tau\}\{\rho\}(e_0; t.x.e_1) \mapsto \text{open}\{t.\tau\}\{\rho\}(e_0'; t.x.e_1)}$$

$$\frac{\Gamma \vdash e_1 : \text{pair}(\tau_1;\tau_2) \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash \text{ap}(e_1; e_2) : \tau}$$

$$\frac{\Gamma, x : \tau \vdash e : \tau}{\Gamma \vdash \text{fix}\{\tau\}(x.e) : \tau}$$

$$\frac{e_0 \text{ val}}{\text{open}\{t.\tau\}\{\rho\}(\text{pack}\{t.\tau\}\{\rho\}(e_0); t.x.e_1) \mapsto [\rho, e/t, x]e_1}$$

$$\frac{e_0 \Downarrow v \quad \text{and} \quad \text{pack}\{t.\tau\}\{\rho\}(e_0; t.x.e_1) \Downarrow v}{}$$

$$\frac{e \mapsto e'}{\text{ifz}\{e_0; x.e_1\}(e) \mapsto \text{ifz}\{e_0; x.e_1\}(e')}$$

$$\frac{}{\text{pack}}$$

$$\frac{}{\text{ifz}\{e_0; x.e_1\}(z) \mapsto e_0}$$

$$\frac{}{p \subset \bar{\tau}}$$

$$\frac{s(e) \downarrow y}{\text{ifz}\{e_0; x.e_1\}(s(e)) \mapsto [e/x]e_1}$$

$$\frac{}{\Gamma, x : \tau \vdash x : \tau}$$

$$\frac{}{\text{fix}\{\tau\}(x.e) \mapsto [\text{fix}\{\tau\}(x.e)/x]e}$$

$$\frac{\Gamma \vdash e_0 : \text{nat} \quad \Gamma \vdash e_0 : \tau \quad \Gamma, x : \text{nat} \vdash e_1 : \tau}{\Gamma \vdash \text{ifz}\{e_0; x.e_1\}(e) : \tau}$$

$$\frac{\text{fix}\{\tau\}(x.e) \Downarrow v}{}$$

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \text{lam}\{\tau\}(x.e) : \text{pair}(\tau_1; \tau_2)}$$

if $\Gamma, x:\tau \vdash e' : \tau'$ & $\Gamma \vdash e : \tau$
then $[e/x]\, e' : \tau'$

→ $e = \underline{arr(\tau_1, \tau_2)}$ the $e =$
$lam\{\tau\}(x.e)$

if $\Gamma \vdash e' : \tau'$ then $\Gamma, x:\tau \vdash e' : \tau$
for any $x \in dom(\Gamma')$

$E \mapsto e'$ & $E' + L \mapsto E'$ in

$e : \tau$ & e val

$\tau = nat$ $e = s(s(s \cdots z))$

s.2

if $\Delta, t\ type \vdash \tau'\ type$ & $\Delta \vdash \tau\ type$
then $[\tau/t]\tau'\ type$

if $\Delta, t\ type \vdash e' : \tau'\ type$ & $\Delta \vdash \tau\ type$
$\Delta ; [\tau/t]\Gamma \vdash [\tau/t]e' : [\tau/t]\tau$

Recursive defn

$2\frac{3}{5}$

$n \underline{z}$

plus z n = n

plus $s(m)$ n = $s(\text{plus}(m\ n))$

$\text{plus} = \lambda n{:}nat\ \lambda m{:}nat.\ rec\{n; x, y.\ s(y)\}(m)$

$\text{plus} = \cancel{\lambda x{:}nat}\ fix\{nat \to nat \to nat\}\ (p.\ \lambda n{:}nat\ \lambda m{:}nat.$
$ifz\{m; n; x.\ s(p\ x\ n)\}\ )$

times n z = z

times n $s(m)$ = plus n (times n m)

$\text{times} = \lambda n{:}n\ \lambda m{:}nat\ rec\{z; x, y.$
$\text{plus}\ n\ y\}(m)$

$\text{times} = fix\{nat \to nat \to nat\}.\ (p.\ \lambda n{:}nat,$
$\lambda m{:}nat).\ ifz\{m; z; x.\ p\ n(\text{times}\ n\ x)\}$

plus 2 2 =

$ifz(2; 2; x.\ s(\text{plus}\ x\ 2))$

$= s(s(\text{plus}\ 0\ 2))$

$s(s(ifz(0; 2; x.\ \text{plus}\ x\ z))$

$= s(s(2))$

$= 4$

pred z = z

pred s(n) = n

pred = $\lambda n{:}nat.\ rec\{z; x, y.\ x\}$

pred = fix{nat → nat}. $\lambda n{:}nat.$
ifz (n; z; x. x)

→ Double z = z

Double s(n) = s(s(Double n))

Double = $\lambda n{:}nat.\ rec\{z; x, y.\ ss(y)\}(n)$

Double = fix{nat - nat} . (D . $\lambda n{:}nat.$
ifz (n; z; x . s(s(D x)))

2
4
8
3+3

→

$2^3 = 8$

expo n z = s(z)

expo n s(m) = times n (exp n m)

times 2 × 3

expo = $\lambda n{:}nat\ \lambda m{:}nat\ rec\{s(z); x, y.$
times (y) n

expo = fix{nat → nat → nat}.
(T, E; $\lambda n{:}nat\ \lambda m{:}nat.$ ifz{m; s(z); x
T (E n x) n})

→ iii

leq s(a) z = z

leq s(n) s(m) = leq n m

leq = λn: nat. rec{s(z); x. y. λm: nat.
rec{z; p, q: y(p)}m}n

leq = fix{nat → nat → nat}. (h. λn: nat.
ifz(n; s(z); x. ifz(m; z. zq. leq x y))

→ if z)

ifz z a b = a

ifz s(n) a b = b.

$\lambda a:\tau\ \lambda b:\tau$

λn: nat. rec{a; x, y. b}n

→ fact z = s(z)

fact s(n) = Times s(n) * fact(n)

fact: ~~λ nat~~ λn: nat. rec{s(z); x, y. Times s(x). y}(n

fact: fix{nat → nat}. (f. λn: nat. ifz(n; s(z).
x. T. s(x). (f. x)).