back ξ ; ξε ... Obs(x; e, ... e1.x).

$\Delta\Gamma \vdash e : \forall t.\tau' \quad \Delta \vdash \tau \text{ Type}$

App{τ}(e) → [τ/t]τ'

System F

mig

→ e1'

x.e2) → let (e1'; x.e2)

e1 val]

(e1; x.e2) → [e1/x]e

Lam

Δ, t ty

Δ, t typ

ΔΓ, x:τ ⊢ e':

(e1'; e2)

e2 → e2'

(e1; e2) → (e1; e2')

Karan Rathore
110661622

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash join(e_1; e_2) : \tau_1 \otimes \tau_2} \qquad \frac{\Gamma \vdash e_0 : \tau_1 \otimes \tau_2 \quad \Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash e : \tau}{\Gamma \vdash split(e_0; x_1 . x_2 . e) : \tau}$$

$$\frac{e_1 \text{ val} \quad e_2 \text{ val}}{split(join(e_1; e_2); x_1 . x_2 . e) \longmapsto [e_1/x_1][e_2/x_2]e}$$

eager Dynamic ~~of~~ join & split

(Right to left)

$$\frac{e_2 \to e_2'}{join(e_1; e_2) \to join(e_1; e_2')}$$

$$\frac{e_2 \text{ val} \quad e_1 \to e_1'}{join(e_1; e_2) \to join(e_1'; e_2)}$$

(Right to left)

$$\frac{e_0 \to e_0'}{split(e_0; x_1 . x_2 . e) \longrightarrow split(e_0'; x_1 . x_2 . e)}$$

~~Lazy rule~~ Lazy rule

$$\frac{e_1 \to e_1'}{join(e_1; e_2) \to join(e_1'; e_2)}$$

(Right to Left)

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Destructor is Never a Value

Lazy

$\overline{split(join(e_1;e_2); x_1.x_2.e)} \mapsto [e_1/x_1][e_2/x_2]e$

Constructor

Progress for join $\dfrac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash join(e_1;e_2) : \tau_1 \otimes \tau_2}$ (1)

1

Case: If $\cdot \vdash e : \tau$ then either $\boxed{e \text{ val}}$ or $\exists e'$ $\boxed{e \mapsto e'}$

$\Gamma = \cdot$, $e = join(e_1;e_2)$, $\tau = \tau_1 \otimes \tau_2$, $\Gamma \vdash e_1 : \tau_1$, $\rightarrow H_1$

$\Gamma \vdash e_2 : \tau_2 \rightarrow H_2$

By IH on H_1

e_1 is val or $\exists e_1'$ $e_1 \vdash e_1'$

Case 1 $e_1 \mapsto e_1'$

take e_1' in $join(e_1';e_2)$ $\{e_1' = e_1\}$

then $e_1 \rightarrow e_1'$ By rule (1)

(9) $\dfrac{e_1 \rightarrow e_1'}{join(e_1;e_2) \rightarrow join(e_1';e_2)}$

Case 2 e_1 val

By IH on H_2

either e_2 val or $\exists e_2'$, $e_2 \rightarrow e_2'$

take e_2' as $join(e_1;e_2')$

then $e_2 \mapsto e_2'$ By rule (1)

By rule (11)

(11) $\dfrac{e_1 \text{ val} \quad e_2 \rightarrow e_2'}{join(e_1;e_2) \rightarrow join(e_1;e_2')}$

e_1 val proved

Similary, e_2 val, $\dfrac{e_1 \text{ val} \quad e_2 \text{ val}}{join(e_1;e_2) \text{ is value}}$

$\boxed{e \text{ is value}}$

$nat = t.unit + t$
$list = t.unit + nat \times t$
$tree = t.unit + nat \times t \times t$

$fold\{t.\tau\}(e) : ind(t.\tau)$

$nat : ind(t.unit + t)$

$\underset{\text{nat}}{\text{for unit}}$

if we want $z = fold\{t.unit + t\}(inl\{unit; \cancel{nat}\}(triv))$

$$\{t.unit + t\} \qquad \text{Unit} + nat \xrightarrow{fold} nat$$

$s(n) : fold\{t.unit + t\}(inr\{unit; \cancel{nat}\ nat\}(n))$

$list : ind(t.unit + nat \times t)$

For empty list / nil $= fold\{t.unit + nat \times t\}(inl\{unit; nat \times t\}(triv))$

For ~~list~~ cons $(\overset{nat}{x}; \overset{list}{l})$

$$\Rightarrow fold\{t.unit + nat \times t\}(inr\{unit; nat \times list\}(Pair(x; l)))$$

$\cancel{nat \Rightarrow} t.unit \times t$

Example $\quad triv \times s(z)$

$fold(t.unit \times t)(pair(triv; s(z)))$

Case → Destructor
Inl | Inr → Constructor

Where No variable is mentioned or void is needed
Abort {P} (void)

Tautologies

(1) $T * P \to P$

$lam\{T * P\}(x.e). snd(x)$ — x: $T*P$

(5) $F * P \to F$ ✓

$Void * P \to Void$
$lam\{Void * P\}(x.e)\ fst(x)$

(?) $P \to T * P$
$lam\{P\}(x.e). pair(e_1; x)$
e_1 : unit

(6) $F \to F * P$
$lam\{F\}(x.e)\ pair(x; unit)$

(11) $T * P = T$
$lam\{T * P\}(x.e). fst(x)$

(?) $T \to T + P$
$lam\{T\}(x.e). inl\{T + P\}(x)$

we don't need F in goal

(7) $F \vee P \to P \;=\; F + P \to P$ or $Void + P \not\to P + Void$
$lam\{F + P\}(x.e). case(x; \underbrace{a.abort\{P\} void}_{Void}; \underbrace{b.b}_{P})$

(8) $P \to F + P$

$lam\{P\}(x.e). inl\{\underbrace{abort\{F\} void}\}(void). inr\{F + P\}(x)$ ✓

(9) $(T \to P) \to P$ — $T \to P \Rightarrow P$ (result)

$lam\{T \to P\}(x.e). ap(x, \langle\rangle)$ — null

$. lam\{T\}(y.e). ap(x; y)$ ✗

$\text{abort}\{\}(\underset{-}{e})$ $\quad V \to + \quad \wedge \quad \times$

(10) $P \to (F \to P)$

$\text{lam}\{P\}(x.e).\ \text{lam}\{F\}(y.e_1).x$ | $\text{lam}\{P\}(x.$

(11) $P \to T \to T$ $\qquad$ $\text{Void} \to P = \text{Unit}$

$\text{lam}\{P \to T\}(x.e).\ \langle\rangle$ || $\text{lam}\{P \to T\}(x.\langle\rangle)$

(12) $T \to (F \to P)$ or $\overset{T}{\cancel{\text{Void}}} \to (\overset{\text{Void}}{F} \to P)$

$\text{lam}\{T\}(x.e).\ \text{lam}\{\text{Void}\}(y.e_1).\ \text{abort}\{P\}(y)$

(13) $P \times P \to P$

$\text{lam}\{P \times P\}(x.\ \text{fst}(P))$

(14) $P \to P \times P$ $\qquad$ | $T \to P \to P$

$\text{lam}\{P\}(x.\ \text{pair}(x;x))$ | $\text{lam}\{T\}(x.\ \text{lam}\{P\}(y.y))$

(15) $P \times O = O \times P$ or $P \times O \to O \times P$

$\text{lam}\{P \times O\}(x.\ \text{pair}(\text{snd}(x);\ \text{fst}(x)))$

(16) $P + O \to O + P$

$\text{lam}\{P + O\}(x.\ \text{case}(x;\ \overset{P}{a}.e_1;\ \overset{O}{b}.e_2))$

$$\left.\begin{aligned} e_1 &= \text{inr}\{P,O\}(a) = P \\ e_2 &= \text{inl}\{P,O\}(b) = O \end{aligned}\right\rangle O + P$$

back{ ; }{ obsr(x; t; ...)

lam{ }(x.e)

$T \wedge P = P$ | $T \times P \to P$ lam{T×P}(x.snd(x))

$T + P \to T$ lam{T+P}(x.case(y; a.inl{T; P}(a); b.abort{void}(b)))✗

$T \to T + P$

lam{T}(a.inl{T; P}(x))

(5) $F \wedge P \to F$: lam{F×P}(x.fst(x))

(6) $F \to F \times P$ → lam{F}(x.pair(x; abort{P}(void)))

void → void × P

(7) $F + P \to P$ lam{F+P}(x.case(y; a.abort{F}(void); b.b))

$P \to F + P$ lam{P}(x.inl{F, P}(abort{F}(void)).inr{F, P}(x))

a void → void + P

P

(8) $T \to P \to P$: lam{T→P}(x.lam{T}(y.y))

lam{T→P}(x.ap(x; <>));

Null or void

reverse

$P = T \to P$

$P \to (T \to P)$ → lam{P ~~→T~~}(x.lam{T}(y.x))

(9) $P \to T = T$

(a) $(P \to T) \to T$: lam{P→T}(x.<>)

T

P→T

(b) $T \to (P \to T)$: lam{T}(x.lam{P}(y.x))

$(F \to P) \to T$: lam{F→P}(x. <>)

(7) $F \to P = T$; $T \to (F \to P)$: lam{T}(x. lam{F}(y. abort{P}(y)))

(8) $P * P \to P$: lam{P*P}(x. fst(x))

(9) $P \to P \times P$: lam{P}(x. pair(x; x))

(10) $P + P \to P$: lam{P+P}(x. case(x; a.a; b. ~~abort{P}(void)~~ b))

(11) $(P \to P) \to T$: lam{P→P}(x. <>)

(12) $T \to (P \to P)$: lam{T}(x. lam{P}(y.y))

(13) $P \times (Q \times R) \to (P \times Q) \times R$

lam{P×(Q×R)}(x. pair(e₁; e₂))

e_1 : pair(fst(x); snd(fst(x)))

e_2 : snd(snd(x))

(14) $P \vee (Q \vee R) \to (P \vee Q) \vee R$

lam{P+(Q+R)}(x. case(y; a.e₁; b.e₂))

e_1 : ~~inl(a)~~ inl(inl(a))

e_2 : case(z; a. inl{Q;R}(a); b. inr{Q;R}(b))

inr{Q;R}(b)

$\underbrace{(P+Q)}_{Left} + \underbrace{R}_{Right}$

$A*(B+C) = AB + AC$

(15) $P*(Q+R) \to (P*Q)+(P*R)$

$lam\{P*(Q+R)\}(x.\ inl\{R;Q\}(pair(e_1;e_2)).\ inr\{P;Q\}(pair(e_3;e_4))$

$e_1:\ fst(x)$
$e_2:\ Case(\cancel{x}\ snd(x);\ a.a;\ b.\langle\rangle)$

$lam\{P*(Q+R)\}(x.Case(snd(x);\ \overset{Q}{a}.e_1;\ \overset{R}{b}.e_2))$

$e_1:\ inl(pair(fst(x);\ a))$
$e_2:\ inr(pair(fst(x);\ b))$

(16) $P+(Q\times R) \to (P+Q)*(P+R)$

$lam\{P+(Q*R)\}(x.\ pair(e_1;e_2))$

$e_1:\ Case(x;\ \overset{P}{a}.\ inl\{P;Q\}(a);\ \overset{Q\times R}{b}.inr(fst(b)))$
$e_2:\ Case(x;\ a.inl\{P;Q\}(a);\ b.inr(snd(b)))$

(16) $(P+Q)*(P+R) \to P+(Q\times R)$

$lam\{(P+Q)\times(P+R)\}(x.Case(x;\ \overset{P+Q}{a}.e_1;\ \overset{R+R}{b}.e_2))$

$e_1 = Case(a;\ \overset{P}{a_1}.inl\{P\}(a_1);\ a_2.)$
$e_2 = Case(b;\ b_1.$

$lam\{(P+Q)*(P+R)\}(x.Case(fst(x);\ \overset{P}{a_1}.\ \overset{\{P;Q\times R\}}{inl}(a);\ \overset{Q}{b_1}.e_2))$

$e_1 = Case(snd(x);\ \overset{P}{a_2}.\underset{\{P;Q\times R\}}{inl}(a);\ \overset{R}{b_2}.inr(pair(b_1;b_2)))$

$(P \to Q) * (P \to R) \to \underbrace{P \to (Q * R)}$

lam{(P→Q)*(P→R)}(x. lam{P}(y. pair(e₁; e₂)));

e_1: ap(fst(x); y)
e_2: ap(snd(x); y)

$P \to (Q * R) \to (P \to Q) * (P \to R)$

lam{P→(Q*R)}(x. pair(e₁; e₂));

e_1: ~~lam{P}(y. ap(x; y))~~ lam{P}(y. fst(ap(x; y)))

e_2: lam{P}(y. snd(ap(x; y)))

$\underbrace{(P \vee Q)}$ $\to R \to (P \to R) * (Q \to R)$

lam{(P∨Q)→R}(x. pair(e₁; e₂));

e_1: lam{P}(a. ap(x; inl{P;Q}(a)))
e_2: lam{Q}(b. ap(x; inr{P;Q}(b)))

$(A \times B) \to (B \times A)$

lam{A×B}(x. split(x; a. b. join(b, a)))

back { ; }{ }{ } obsr(x; t; y; e)

$(P \to R) * (Q \to R) \to (P \vee Q) \to R$

lam{(P→R)*(Q→R)}(x. lam{P∨Q}(y. e))

e : Case(y; $\overset{P}{a}$. ap(fst(x); a); b. ap(snd(x); b))

$(P \to (Q \to R)) \to ((P * Q) \to R)$

lam{(P→(Q→R))}(x. lam{P*Q}(y. ap(D ; snd(y))))

error ap(x; fst(y)) — (Q→R)

$((P * Q) \to R) \to (P \to (Q \to R))$

lam{((P*Q)→R)}(x. lam{P}(y. lam{Q}(z. ap(x; pair(y; z)))))

$((P \to Q) * (P \to R)) \to (P \to (Q + R))$

lam{((P→Q) + (P→R))}(x. lam{P}(y. Case(x; $\overset{P \to Q}{a}$. inl{Q;R}(ap(a; $\overset{P}{y}$))
b. inr{Q;R}(ap($\underset{P \to R}{b}$; y))))))

$((P \to R) + (Q \to R)) \to ((P \times Q) \to R)$

lam{((P→R) + (Q→R))}(x. lam{P×Q}(y. Case(x; $\overset{P \to R}{a}$. e_1; $\overset{Q \to R}{b}$. e_2)))

e_1 : ap(a; fst(y))
e_2 : ap(b; snd(y))

Ques 8

(i) $(B \to void) \to (A \to (A \times B)) \to (A \to C)$

(ii) $((A + C) * (B + D)) \to ((D \times C) + (A + B))$

(iii) $(A \to B \to C) \to (B \to A \to C)$

lam{

$lam\{B \to void\}(x.\,(A \to (A \times B))\}(y.\,lam\{A\}(abort\{C\}(void))))$

$void : ap(x; snd(ap(y; \overset{a}{z})))$

$\underbrace{\qquad}_{(A \times B)}$

(ii) $lam\{(A+C) * (B+D)\}(x; case(snd(x); \overset{B}{a} \cdot inr\{(D \times C); (A+B)\}(inr\{A;B\}(a))$

$\overset{D}{b} \cdot case(fst(x); \overset{A}{u} \cdot inr\{(D \times C)\}(A+B)(inl\{A;B\}(u)); \overset{C}{v} \cdot$
$inl\{(D \times C); (A+B)\}(pair(b; c)))))$

(iii) $lam\{A \to (B \to C)\}(x.\,lam\{B \to A\}(\overset{A}{y}.\ \underbrace{ap(x; y)}_{B \to C})$

$lam\{A \to B \to C\}(x.\,lam\{B\}(y.\,lam\{A\}.(a.\,ap(\underbrace{ap(x; a)}_{B \to C}; y)))$

back { ; s² open(x, ...

Difficult

Q-2: $((A+C) \times (B+D)) \to ((D \times C) + (A+B))$

lam{(A+C)×(B+D)}(x·~~Case(fst(x);~~ Case(a; u·inr{(D×C);(A+B)}(

inl{A;B}(u); v

lam{(A+C)×(B+D)}x· Case(fst(x); a·inr{(D×C);(A+B)}(inl{A;B}(a))
; b·Case(snd(x); u·inr{(D×C);(A+B)}(inr{A;B}(u)); v·
inl{(D×C);(A+B)}(pair(v;b))))))

App & Lam

App{t}(a) // to pack ∀t

Lam(t·e) to open ∀t

(5) ∀t.∀u·s(t,u) ⟶ ∀u.∀t s(t,u)

lam{∀t.∀u·s(t,u)}(x·Lam(u·e).~~[illegible]~~)

~~e = App{t,u}(x)~~ e = Lam(t·e₂)

$e_2 = $ App{u}(App{t}(x))

(1) $\forall u . \forall t . S(t,u) \longrightarrow \forall t . \forall u . S(t,u)$

$lam\{\forall u . \forall t . S(t,u)\}(x . Lam(t . Lam(u . e)))$

$e = App\{t\}(App\{u\}(x)) \Rightarrow (S(t,u))$ Goal

(2) $\forall t . P(t) \times O(t) \rightarrow (\forall t . P(t)) \times (\forall t . O(t))$

$lam\{\forall t . P(t) \times O(t)\}(x . pair(e_1 ; e_2))$

$P(t) \times O(t)$

$e_1 = Lam(t . fst(App\{t\}(x)))$

$e_2 = Lam(t . snd(App\{t\}(x)))$

(3) $(\forall t . P(t)) \times (\forall t . O(t)) \longrightarrow \forall t . P(t) \times O(t)$

$lam\{(\forall t . P(t)) \times (\forall t . O(t))\}(x . Lam(t . pair(e_1 ; e_2)))$

$e_1 = \cancel{fst(x)}$ $App\{t\}(fst(x))$

e_2 $App\{t\}(snd(x))$

(4) $\cancel{\forall A}$ $\forall t . A \rightarrow P(t) \rightarrow A \rightarrow \forall t . P(t)$

$lam\{\forall t . A \rightarrow P(t)\}(lam\{A\}(y . Lam(t . e)))$

$A \rightarrow P(t)$ A

$e = ap(x, App\{t\}(y))$ ✗ e_2 $ap(App\{t\}(x) ; y)$

(5) $A \rightarrow \forall t . P(t) \longrightarrow \forall t . A \rightarrow P(t)$

$lam\{A \rightarrow \forall t . P(t)\}(x . \cancel{Lam\{t\}}\ lam\{\forall t . A\}(y . e))$

$e = App\{t\}(ap(\underbrace{App\{t\}(y)}_{A}, x)) : P(t)$

back ... Obm ...

(8) $(A \to \forall t \cdot P(t)) \longrightarrow (\forall t \cdot A \to P(t))$

$\text{lam}\{A \to \forall t \cdot P(t)\}(x \cdot \text{Lam}\{t\} \cdot \text{lam}\{A\}(y \cdot e)))$

$e = \underbrace{\text{App}\{t\}(\text{ap}(x, y))}_{P(t)}$ where $\text{ap}(x, y) : \forall t \cdot P(t)$

(9) $(A \to \forall t \cdot B(t)) \times (A \to \forall t \cdot C(t)) \longrightarrow \forall u \cdot A \to (B(u) \times C(u))$

$\text{lam}\{(A \to \forall t \cdot B(t)) \times (A \to \forall t \cdot C(t))\}(x \cdot \text{Lam}(u \cdot \text{lam}\{A\}(y \cdot \text{pair}(e_1; e_2))))$

~~$e_1 = \text{ap}(x, \text{fst}(x))$~~

$e_1 = \text{App}\{u\}(\text{ap}(\underbrace{\text{fst}(x), y}_{\forall t \cdot B(t)})) = B(t)$

$e_2 = \text{App}\{u\}(\text{ap}(\text{snd}(x), y))$

* Very Difficult

(10) $\forall t \cdot [(A \to t) \to (B \to t) \to t) \longrightarrow (A + B)$

$\text{lam}\{\forall t\ (A \to t) \to (B \to t) \to t\}(x \cdot \cancel{\text{App}\{t\}}\ e)$

$\text{ap}(\underbrace{\text{ap}(\text{App}\{A+B\}(x), e_1)}_{(B \to t) \to t}, e_2)$ where $e_1 : (A \to t)$, $e_2 : (B \to t)$

$e_1 = \text{lam}\{A \to t\}(x \cdot$ ~~$\text{lam}\{B \to t\}$~~ $(y \cdot \text{inl}\{A, B\}(x$

previous ... what we are open ... Next value

$\mathrm{open}(e_1; t, x \cdot e_2)$ $\quad$ $\mathrm{pack}\{t.\tau\}\{\rho\}(e)$

$e_1\ (A \to t) = \mathrm{lam}\{A\}(x \cdot \mathrm{inl}\{A;B\}(x))$

$e_2\ (B \to t) = \mathrm{lam}\{B\}(y \cdot \mathrm{inr}\{A;B\}(y))$

$A \to \mathrm{coi}(t \cdot A \times t)$

gen $\quad$ $\{T\}$ $\quad$ x

$\mathrm{lam}\{A\}(x \cdot \mathrm{fold}\{t \cdot A \times t\}(\mathrm{triv}; y \cdot \mathrm{pair}(x; y)))$

rest y $\quad$ $\exists t \cdot \exists u \cdot S(t,u) \to \exists u \cdot \exists t \cdot S(t,u)$

$\mathrm{open}(y \cdot t \cdot x \cdot \mathrm{open}(x \cdot u \cdot y \cdot \mathrm{pack}\{u, \exists t \cdot S(t,u)\}$

$x = \exists u \cdot S(t,u)$

1. $\exists t \cdot \exists u \cdot S(t,u) \to \exists u \cdot \exists t \cdot S(t,u)$

$\exists u \cdot S(t,u)$ $\quad$ $S(t,u)$ $\quad$ Rest Value

$\mathrm{lam}\{\ \}(x \cdot \mathrm{open}(x; t, a \cdot \mathrm{open}(a; u, b \cdot \mathrm{pack}\{u, \exists t \cdot S(t,u)\}\{u\}$
$(\mathrm{pack}\{t, S(t,u)\}\{t\}(b)))))$

2. $\exists t \cdot P(t) + O(t) \to (\exists t \cdot P(t)) + (\exists t \cdot O(t))$

$P(t)+O(t)$ $\quad$ $P(t)$ $\quad$ $O(t)$

$\mathrm{lam}\{\exists t \cdot P(t) + O(t)\}(x \cdot \mathrm{open}(x; t, a \cdot \mathrm{Case}(a; m \cdot e_1; n \cdot e_2)))$

$e_1 = \mathrm{inl}\{(\exists t \cdot P(t)); (\exists t \cdot O(t))\}(\mathrm{pack}(t, P(t)\}\{t\}(m))$

$e_2 = \mathrm{inr}\{(\exists t \cdot P(t)); (\exists t \cdot O(t))\}(\mathrm{pack}(t, O(t)\}\{t\}(n))$

back{ ; }t open(x, ... e1 . x)

(3) $(\exists t\cdot P(t)) + (\exists t\cdot O(t)) \to \exists t\cdot P(t) + O(t)$

$\text{lam}\{(\exists t\cdot P(t)) + (\exists\cdot O(t))\}(x\cdot \text{Case}(x;\ \overset{\exists t\cdot P(t)}{a}\cdot e_1;\ \overset{\exists t\cdot O(t)}{b}\cdot e_2))$

$e_1 = \text{open}(a; t, m$

~~lam{(∃·P(t))+(∃·O(t))}(x· open(x; t, δ.~~

$\text{lam}\{(\exists t\cdot P(t)) + (\exists t\cdot O(t))\}(x\cdot \text{Case}(x;\ \overset{\exists t\cdot P(t)}{a}\cdot e_1;\ \overset{\exists t\cdot O(t)}{b}\cdot e_2))$

$e_1 = \text{open}(a; t, \overset{P(t)}{m}\cdot \text{Pack}\{t, P(t)\}\{t\}(\text{inl}\{P(t); O(t)\}(m)))$

$e_2 = \text{open}(b; t, \overset{O(t)}{m}\cdot \text{Pack}\{t, O(t)\}\{t\}(\text{inr}\{P(t); O(t)\}(m)))$

(4) $(\exists t\cdot P(t) \wedge O(t)) \to ((\exists t\cdot P(t)) \times (\exists t\cdot O(t)))$

$\text{lam}\{\exists t\cdot P(t)\times O(t)\}(x\cdot \text{pair}(e_1; e_2))$

$e_1 = \text{open}(x; t, \overset{P(t)\times O(t)}{a}\cdot \text{pack}\{t, P(t)\times O(t)\}\{t\}(\text{fst}(a)));$

$e_2 : \text{open}(x; t,\ b\cdot \text{pack}\{t, P(t)\times O(t)\}\{t\}(\text{snd}(b)));$

(5) $((\forall t\cdot P(t) \to O(t)) \to \forall t\, P(t) \to (\forall t\cdot O(t))$

$\text{lam}\{((\forall t\cdot P(t)) \to O(t))\}(x\cdot \text{lam}\{\forall t\, P(t)\}(y\cdot \text{Lam}(t\cdot e)))$

$e = \text{ap}(x; y);$

$\exists t \cdot \tau \{ \tau_2 \} \quad \{ \}$ $\quad t, \; u \;)$

④ $((\forall t \cdot P(t)) \to O(t)) \to ((\exists t \cdot P(t)) \to (\exists t \cdot O(t)))$

$\text{lam}\{(\forall t \cdot P(t) \to O(t)\}(x \cdot \text{lam}\{\exists t \cdot P(t)\}(y \cdot \text{open}(y; t, \overset{P(t)}{a} \cdot e)))$

$e = \text{pack}\{t, O(t)\}\{t\}(\text{ap}(x; \underbrace{\text{App}\{\forall t \cdot P(t)\}(a)}_{P(t)}))$

$\text{App}\{t\}(P(t))$

⑤ $(\exists t \cdot P(t) \to O(t)) \to (\forall t \cdot P(t)) \to (\exists t \cdot O(t))$

$\text{lam}\{\exists t \cdot P(t) \to O(t))\}(x \cdot \cancel{\text{lam}\{\ldots\}}\ \text{lam}\{\forall t \cdot P(t)\}(y \cdot e))$

$e = \text{open}(x; t, \overset{(P(t) \to O(t))}{a} \cdot \text{pack}\{t, O(t)\}\{t\}(\text{ap}(a; \underbrace{\text{Ap}\{t\}(y)}_{P(t)})))$

$e = \text{open}$

$x \cdot e$

⑥ $(\exists t \cdot \forall u \cdot S(t,u)) \to (\forall u\, \exists t \cdot S(t,u))$

$\text{lam}\{(\exists t \cdot \forall u \cdot S(t,u))\}(x \cdot \cancel{\text{Lam}(u} \cdot \text{open}(x; \cancel{t}, \overset{\forall u \cdot S(t,u)}{a} \cdot \text{pack}\{t, S(t,u)\}\{t\}(a)))$

⑦ $\forall t \cdot A \to P(t) \to A \to \forall t \cdot P(t)$

$\text{lam}\{\forall t \cdot A \to P(t)\}(x \cdot \text{lam}\{A\}(y \cdot \text{Lam}\{t \cdot \text{ap}(x; e)))))$

$e = \text{Ap}\{t\}(y)$

⑧ $A \to \forall t \cdot P(t) \longrightarrow \forall t \cdot A \to P(t)$

$\text{lam}\{A \to \forall t \cdot P(t)\}(x \cdot \text{lam}\{\forall t \cdot A\}(y \cdot \cancel{\text{ap}(x}\ e))$

$\cancel{e = \text{App}\{t\}(\text{ap}(x; \text{App}\{t\}(y)))}$

$\underbrace{\qquad}_{A} \; \vdash \forall t \cdot P(t)$

back {; ∃t ... open(x; t, ... x).

(4) $\exists t. \exists u. S(t,u) \longrightarrow \exists u. \exists t. S(t,u)$

$\text{lam}\{\exists t. \exists u. S(t,u)\}(x. \text{open}(x; t, \overset{\exists u. S(t,u)}{a}. \text{open}(a; t, \overset{S(t,u)}{b}. e)))$

$e = \text{pack}\{u; \overset{\exists t.}{} S(t,u)\}\{t\}\ \text{pack}\{t; \overset{\exists t.}{} S(t,u)\}\{t\}(b)$

(5) $\exists t. P(t) + O(t) \longrightarrow (\exists t. P(t)) + (\exists t. O(t))$

$\text{lam}\{\exists t. P(t) + O(t)\}(x. \text{open}(x; t, \overset{P(t)+O(t)}{a}. \text{case}(a; \overset{P(t)}{a}. e_1; b. e_2)))$

$e_1 = \text{inl}\{\exists t. P(t); (\exists t. O(t))\}(\text{pack}\{t, P(t)\}\{t\}(\overset{P(t)}{a}))$

$e_2 = \text{inr}\{\exists t. O(t)\}(\text{pack}\{t, O(t)\}\{t\}(\overset{O(t)}{b}))$

Questions

(6) $\cancel{\forall t. P(t)}.\ \forall t. A \to P(t) \longrightarrow A \to \forall t. P(t)$ ✓

(7) $(A \to \forall t. B(t)) \times (A \to \forall t. C(t)) \longrightarrow \forall u. A \to (B(u) \times C(u))$

(8) $\exists t. A \times P(t) \longrightarrow A \times \exists t. P(t)$

(9) $A \times \exists t. P(t) \to \exists t. A \times P(t)$

(10) $((\exists t. P(t)) \to A) \rightleftarrows \forall t. P(t) \to A$

Solutions $\quad \forall t \cdot B(t) \quad [C]^{2}$

(6) $(A \to \forall t \cdot B(t)) \times (A \to \forall t \cdot C(t)) \longrightarrow \forall u \cdot A \to (B(u) \times C(u))$

$\text{lam}\{(A \to \forall t \cdot B(t)) \times (A \to \forall t \cdot C(t))\}(x \cdot \overset{\text{Lam}(u \cdot}{}\text{lam}\{A\}(y \cdot \text{pair}(e_1; e_2)))$;

$e_1 \quad \text{App}\{u\}(\text{ap}(\underbrace{\text{fst}(x); \overbrace{y}^{A}}_{\forall t \cdot B(t)}))$

$e_2 \quad \text{App}\{u\}(\text{ap}(\text{snd}(x); y))$

(7) $\exists t \cdot A \times P(t) \longrightarrow A \times \exists t \cdot P(t)$

$\text{lam}\{\exists t \cdot A \times P(t)\}(x \cdot \text{pair}(e_1; e_2))$;

$e_1 = \text{open}(x; t, a \cdot \text{fst}(a))$

$e_2 = \text{open}(x; t, a \cdot \text{pack}\{P(t); t\}\{t\}(\text{snd}(a)))$

(8) $A \times \exists t \cdot P(t) \to \exists t \cdot A \times P(t)$

$\text{lam}\{A \times \exists t \cdot P(t)\}(x \cdot \text{open}(\text{snd}(x); t, \overset{P(t)}{a} \cdot \text{pack}\{A \times P(t); t\}\{t\}(\text{pair}(\text{fst}(x); a))))$

(10) $(\exists t \cdot P(t)) \to A) \longrightarrow \forall t \cdot P(t) \to A \qquad P(t) \text{ — } A \text{ (Goal)}$

$\text{lam}\{(\exists t \cdot P(t)) \to A\}(x \cdot \text{Lam}\{P(t)\}(y \cdot e))$

$e = \text{ap}(x; \text{pack}\{P(t); t\}\{t\}(y))$

(11) $\forall t \cdot P(t) \to A \longrightarrow (\exists t \cdot P(t)) \to A$

$\text{lam}\{\forall t \cdot P(t) \to A\}(x \cdot \text{lam}\{\exists t \cdot P(t)\}(y \cdot \text{open}(y; t, \overset{P(t)}{a} \cdot \text{ap}(x; \text{App}\{t\}(a)))))$

rec

(7) ind (t. unit×t) → ind (t. unit × t) — inductive function

lam {ind (t. unit×t)} (x. fold {t. unit × t} (pair (triv; x)))

rec system T — τ (what to produce), prod rec

nat → nat

$\text{rec}\{e_0; x, y . e\}(e)$ — nat

Example 1 Factorial

$\text{fact } 0 = 1 (s(z))$
$\text{fact } s(n) = \underbrace{s(n)}_{x} \times \underbrace{\text{fact}(n)}_{y}$ — nat → nat

lam{nat}(n. rec{s(z); x . y : (s(x) × y)}(n))

(2) plus n z = n
plus n s(m) = s (plus n m)

$2+0 = 2$, $2+1 = 3$ (s(plus (2+0))) = 3

n + m = nat

nat → nat → nat

plus = lam{nat}(n. lam{nat}(m. rec{n; x, y . s(y)}(m))

(3) Time n z = z
Times n s(m) = plus (times n m) n

$2 \times 0 = \boxed{0}$
$2 \times 1 = \boxed{2}$

plus (2×0) 2 = 2

nat → nat → nat

lam{nat}(n. lam{m}(m. rec{z; x . y . plus (y, n)}(m)) m - n ≥

nat → nat → nat

nat → nat

(3) prod z = z

prod s(n) = n

lam{nat}(n. rec{z; x, y. x}(n)) nat

(4) double z = z

double s(n) = s(s(double n)) nat → nat

lam{nat}(n. rec{z; x. y. s(s(y))}(n))

(4) exp n z = s(z)

exp n s(m) = Times (exp n m) n

nat → nat → nat

lam{nat}(n. lam{nat}(m. rec{s(z); x. y. Times y n}(m on n)))

(5) leq z m = s(z) = 1

leq s(n) z = z = 0

leq s(n) s(m) = leq n m

nat → nat → nat

lam{nat}(n. lam{nat}(m. rec{s(z); x. y. rec{z; p, q. y(p)}(m)}.n

ap(y; m)

$gt\ z\ m = 0$

$gt\ s(n)\ z = 1$

$gt\ s(n)\ s(m) = gt\ n\ m$

(6) Strict greater $nat \to (nat \to nat)$

$lam\{nat\}(n.\ rec\{n;\ lam\{nat\}(m.\ z);\ y,\ rec\{m;\ x$

Preservation: If $e:\tau$ & $e \mapsto e'$ then $e':\tau$

Let rule - Case: 1

$$\frac{e_1 \to e_1'}{let(e_1; x.e_2) \mapsto let(e_1'; x.e_2)}$$

$e = let(e_1; x.e_2)$

$e' = let(e_1'; x.e_2)$

Above the line $e_1 \mapsto e_1'$ — H_1

By Inversion on $e:\tau$

Above the line

$\Gamma \vdash e_1 : \tau_1$ — H_2

$\Gamma \vdash e_2 : \tau_2$ — H_3

$$\frac{\Gamma \vdash e_1' : \tau_1}{\Gamma \vdash let(e_1'; x.e_2) : \tau_2}$$

By IH on H_1 & H_2

$\Gamma \vdash e_1' : \tau_1$

By rule of let $=$ $let(e_1'; x.e_2) : \tau_2$

Case (2)

$$\frac{[e_1\ val]}{let(e_1; x.e_2) \mapsto [e_1/x]e_2}$$

$e = let(e_1; x.e_2)$, $e' = [e_1/x]e_2$

Above the line e_1 val — H_1

By Inversion on $e:\tau$

Above the line

$\Gamma \vdash e_1 : \tau_1$ — H_2

$\Gamma, x:\tau_1 \vdash e_2 : \tau_2$ — H_3

$\boxed{\text{Substitution lemma: if } \Gamma, x:\tau \vdash e':\tau' \;\&\; \Gamma \vdash e:\tau \text{ then } [e/x]e' : \tau'}$

We got $[e_1/x]e_2 : \tau_2$

Progress: If $e:\tau$ then either e val or there exist e' where $e \mapsto e'$

Case

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x:\tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash let(e_1; x.e_2) : \tau_2}$$

$e = let(e_1; x.e_2)$

$\tau = \tau_2$

Above the line

$\Gamma \vdash e_1 : \tau_1$ — H_1

$\Gamma, x:\tau_1 \vdash e_2 : \tau_2$ — H_2

By IH on H_1 either e_1 is value or there exist e_1', $e_1 \mapsto e_1'$

Case $e_1 \mapsto e_1'$

By rule $e_1' = let(e_1'; x.e_2)$

$$\frac{e_1 \mapsto e_1'}{let(e_1; x.e_2) \mapsto let(e_1'; x.e_2)}$$

then $e_1 \mapsto e_1'$

Case e_1 val.

Apply IH on H_2

Take $e_2' = [e_1/x]e_2$

So e_1 val

e_1 val

~~Subcase 1 e_2 val~~

Case 2 $e_2 \mapsto e_2'$

$\cancel{let(e_1; x.e_2)}$ ~~take~~ e_2' as

$e_2 \mapsto e_2'$ $let(e_1; x.e_2')$

$e_2 \mapsto e_2'$ $\cancel{let(e_1; x.e_2)}$

$$\frac{e_1 \text{ val}}{let(e_1; x.e_2) \mapsto \underbrace{[e_1/x]e_2}_{e_2'}}$$ ✓

back ... obs ...

Preservation of let rule
if $e : \tau$ & $e \mapsto e'$ then $e' : \tau$

Case
$$\frac{e_1 \mapsto e_2}{let(e_1; x.e_2) \mapsto let(e_1'; x.e_2)}$$

$e = let(e_1; x.e_2)$
$e' = let(e_1; x.e_2)$

Above the line $e_1 \mapsto e_2$ — H1

By Inversion on e
$\Gamma \vdash e_1 : \tau_1$ — H2
$\Gamma, x : \tau_1 \vdash e_2 : \tau_2$ — H3

By IH on H1 & H2
$\Gamma \vdash e_1' : \tau_1$

$$\frac{e_1 \mapsto e_2}{let(e_1; x.e_2) \mapsto let(e_1'; x.e_2)}$$

By rule
also, $let(e_1'; x.e_2) : \tau_2$

$\boxed{e' : \tau_2}$

Case
$$\frac{e_1 \; val}{let(e_1; x.e_2) \mapsto [e_1/x]e_2}$$

$e = let(e_1; x.e_2)$
$e' = [e_1/x]e_2$

Above the line e_1 val — H1

By Inversion on $e : \tau$
$\Gamma \vdash e_1 : \tau_1$ — H2
$\Gamma, x : \tau_1 \vdash e_2 : \tau_2$ — H3

Substitution lemma on H_2 & H_3

$\Gamma \vdash e_1 : \tau_1$ & $\Gamma, x:\tau_1 \vdash e_2'$ then $[x/e_1]e_2 : \tau_2$

$[x/e_1]e_2 : \tau_2$

$\boxed{e' : \tau_2}$

Progress If $e : \tau$ then there either e val or there exist e'
$e \mapsto e'$

$$\frac{\Gamma \vdash e_1 : \tau_1 \; , \; \Gamma, x:\tau_1 \vdash e_2 : \boxed{\tau_2}}{let(e_1 ; x.e_2) : \boxed{\tau_2}}$$

$e = let(e_1 ; x.e_2)$

$\tau = \tau_2$

Above the line

$\Gamma \vdash e_1 : \tau_1$ — H_1

$\Gamma, x : \tau_1 \vdash e_2 : \tau_2$ — H_2

By IH on H_1

either e_1 val or $\exists e'$ where $e \mapsto e'$.

Case $e \mapsto e'$:

By let rule $$\frac{e_1 \mapsto e_2}{let(e_1 ; x.e_2) \mapsto let(e_1' ; x.e_2)}$$

$e_1' = let(e_1' ; x.e_2)$

then $e_1 \mapsto e_2$ i.e $\boxed{e_1 \mapsto e_1'}$

Case: e_1 val

Apply IH on H_2

$e_2' = [e_1/x]e_2$ $$\frac{e_1 \text{ val}}{let(e_1 ; x.e_2) \mapsto [e_1/x]e_2}$$
$[e_2']$

e_1 val

proved

app, arr

Typing Rule

~~appII~~ lam/arr = $\dfrac{\Gamma, x:\tau_1 \vdash e_1 . \tau_2}{\Gamma \vdash lam\{\tau\}(x.e_1) : arr(\tau_1; \tau_2)}$

Dynamic rule

Case 1: $\dfrac{e_2 \ val}{ap(lam\{\tau_1\}(x.e_1); e_2) \mapsto [e_2/x]e_1}$

$\boxed{\text{Preservation;}}$ If $e:\tau$ & $e \mapsto e'$ then $e':\tau$

$e = ap(lam\{\tau_1\}(x.e_1); e_2)$
$e' = [e_2/x]e_1$

Above the line e_2 val — H_1

By ~~...~~ Inversion on $e:\tau$

~~$\Gamma, x:\tau_1 \vdash e_1:\tau_2$~~

$\Gamma \vdash lam\{\tau\}(x.e_1) : arr(\tau_2; \tau)$ — H_2

$\Gamma \vdash e_2 : \tau_2$ — H_3

By Inversion of H_2

$\Gamma, x:\tau_2 \vdash e_1 : \tau$ — H_4

By substitution Lemma

$\Gamma, x:\tau \vdash e':\tau'$ & $e:\tau$ the $[e/x]e' = \tau'$

$[e_2/x]e_1 = \tau$

$\boxed{e':\tau}$

Progress: If $e : \tau$ then either e is val or $\exists e'$ where $e \mapsto e'$.

$$\frac{\Gamma, x : \tau_1 \vdash e_1 : \tau_2}{\Gamma \vdash \text{lam}\{\tau\}(x.e_1) : \text{arr}(\tau_1 ; \tau_2)}$$

$e = \text{lam}\{\tau\}(x.e_1)$

$\tau = \text{arr}(\tau_1 ; \tau_2)$

Above the line

$\Gamma, x : \tau_1 \vdash e_1 : \tau_2$ — H1

By IH on H1

either e_1 val or $e_1 \mapsto e_1'$

Case 1: $e_1 \mapsto e_1'$ $\cancel{}$

By rule $\text{lam}\{\tau\}(x.e_1) \mapsto \text{lam}\{\tau\}(x.e_1')$

$e_1' = \text{lam}\{\tau\}(x.e_1')$ $\cancel{}$ There is no stepping rule for lam

then $e_1 \mapsto e_1'$

Case 2: e val

By rule $\frac{}{\text{lam}\{\tau\}(x.e)\ \text{val}}$

e is val

back { ; }{ }{ } open(x; t; y; c)

Preservation $\exists e'$

If $e : \tau$ & $e \mapsto e'$ then e' is also τ

Case: $\dfrac{e_1 \mapsto e_1'}{ap(e_1; e_2) \mapsto ap(e_1'; e_2)}$ Laze Case

$e = ap(e_1; e_2)$, $e' = ap(e_1'; e_2)$

$\dfrac{\Gamma \vdash e_1 : \tau_2 \to \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash ap(e_1; e_2)}$ arr(τ_2; τ)

above the line $e_1 \mapsto e_1'$ — H_1

By inversion on $e : \tau$

(nat → nat)
$((A \Rightarrow B), A) = B$

$\Gamma \vdash e_1 : arr(\tau_2; \tau)$ — H_2

$\Gamma \vdash e_2 : \tau_2$ — H_3

By ~~inversion~~ o.

By IH on H_2 & H_1

$e_1' : arr(\tau_2; \tau)$

$\boxed{ap(e_1'; e_2) : \tau}$ proved

$\dfrac{\Gamma \vdash e_1' : \quad e_2 : \tau_2}{\Gamma \vdash ap(e_1'; e_2) : \tau}$

Case $\dfrac{e_1 \text{ val} \quad e_2 \mapsto e_2'}{ap(e_1; e_2) \mapsto ap(e_1; e_2')}$

nat → nat
arr{nat} C

$e = ap(e_1; e_2)$
$e' = ap(e_1; e_2')$

Above the line

e_1 val — H_1

$e_2 \mapsto e_2'$ — H_2

By inversion on $e : \tau$

$e_1 : arr(\tau_2; \tau)$ — H_3

$e_2 : \tau_2$ — H_4

By IH on H_2 & H_4 ($e_2 \mapsto e_2'$) & $e_2 : \tau_2$ $e_2' : \tau_2$

$e_2' : \tau_2$

By ap rule $ap(e_1; e_2') : \tau$

Case: $\dfrac{[e_2 \text{ val}]}{ap(lam\{\tau\}(x.e_1); e_2) \mapsto [e_2/x]e_1}$

$e = ap(lam\{\tau\}(x.e_1); e_2)$

$e' = [e_2/x]e_1$

We have to prove: $[e_2/x]e_1 : \tau$

Above the line e_2 val — H_1

By Inversion

$\Gamma \vdash e_1 : arr(\tau_2; \tau)$ — H_2

$\Gamma \vdash e_2 : \tau_2$ — H_3

By Inversion on H_2

$\Gamma, x : \tau_2 \vdash e_1 : \tau$ — H_4

By Substitution Lemma on H_3 & H_4,

$[e_2/x]e_1 : \tau$ ✓

$\boxed{\text{Progress}}$ If $e : \tau$ then either e val or $\exists e'$ where $e \mapsto e'$

$e = lam\{\tau_1\}(x.e)$

Case: $\dfrac{\Gamma, x:\tau_1 \vdash e : \tau_2}{\Gamma \vdash lam\{\tau_1\}(x.e_1) : arr(\tau_1, \tau_2)}$

Above the line $\Gamma, x : \tau_1 \vdash e : \tau_2$ — H_1

By IH on H_1 (e', $e \mapsto e'$ or e val)

Case $e \to e'$

la

Case: e val

$\dfrac{\quad}{lam\{\tau\}(x.e) \text{ val}}$

So e val

back { ; }{ } obs(x; t; y, ...

Progress $e : \tau$ then $\underbrace{e \text{ val or } e \mapsto e'}_{}$ $\exists e'$

$\underline{\Gamma, x:\tau_1 \vdash e_1 : \tau_2}$ $\qquad$ $lam\{\tau\}(x.e)$

Case: $\dfrac{\Gamma \vdash e_1 : arr(\tau_2; \tau_1) \quad \Gamma \vdash e_2 : \tau_2}{ap(e_1; e_2) : \tau_1}$

$e = ap(e_1; e_2)$

Above the line $\Gamma \vdash e_1 :: arr(\tau_2; \tau_1)$ — H1

$\Gamma \vdash e_2 : \tau_2$ — H2

By IH on H1

$\therefore$ either e_1 val or $\exists e'$ $e_1 \mapsto e_1'$

Case $e_1 \mapsto e_1'$ — take e_1' as $ap(e_1'; e_2)$ By rule $e_1 \mapsto e_1'$

case e_1 val → By IH on H_2

case e_2 val either e_2 val or $\exists e'$, $e_2 \mapsto e_2'$

By C.F.L (Canonical Form lemma) on H1

$e_1 = lam\{\tau\}(x.e_1)$

$$\dfrac{e_2 \text{ val}}{ap(lam\{\tau\}(x.e_1); e_2) \mapsto [e_2/x]e_1} \quad R_1$$

Inversion on H1

$\Gamma, x:\tau_1 \vdash e_1 : \tau_2$ → H3

By Substitution on H_2 & H_3

$[e_2/x]e_1$

let $e_2' = [e_2'/x]e_1$

using Rule R_1

$\boxed{e_2 \text{ val}}$

case $e_2' \mapsto e_2'$

Take e_2' as $ap(e_1; e_2')$

then $e_2 \mapsto e_2'$

& e_1 val by Rule $\dfrac{e_1 \text{ val} \quad e_2 \mapsto e_2'}{ap(e_1; e_2) \mapsto ap(e_1; e_2')}$

Case 2 e_1 val

proved by rule

System T

Preservation $e : \tau$ & $\exists e'$, $e \mapsto e'$ then $e' : \tau$

Case:
$$\frac{e \mapsto e'}{rec\{e_0; x, y.e_1\}(e) \mapsto rec\{e_0; x, y.e_1\}(e')}$$

$E = rec\{e_0; x, y.e_1\}(e)$
$E' : rec\{e_0; x, y.e_1\}(e')$

Above the line $e \mapsto e'$. — H_1

By inversion of $e : \tau$

$\Gamma \vdash e : nat$ — H_2

$\Gamma \vdash e_0 : \tau$ — H_2

$\Gamma, x : nat, y : \tau \vdash e_1 : \tau$ — H_3

By IH on H_2 & H_1

$\Gamma \vdash e' : nat$

By rec rule $\Gamma \vdash rec\{e_0; x, y.e_1\}(e) : \tau$ — H_4

By IH on H_1 & H_4

$E' : \tau$

Case:
$$\frac{}{rec\{e_0; x, y.e_1\}(z) \mapsto e_0}$$

$e = rec\{e_0; x, y.e_1\}(z)$ Goal $e' : \tau$

$e' = e_0$

By Inversion of e

$\Gamma \vdash z : nat$ — H_1

$\Gamma \vdash e_0 : \tau$ H_2 proved

$\Gamma, x : nat, y : \tau \vdash e_1 : \tau$ — H_3

back ε ; $\exists e$, obsn

$$\frac{\Gamma \vdash e : nat, \; \Gamma \vdash e_0 : \tau_1, \; \Gamma, x : nat, y : \tau_1 \vdash e_1 : \tau_1}{\Gamma \vdash rec\{e_0; x, y.e_1\}(e) : \tau_1}$$

Goal

$\boxed{\text{if } e : \tau \text{ then either } e \text{ val or } \exists e', e \mapsto e'}$

Progress: $e = rec\{e_0; x, y.e_1\}(e)$

$\tau = \tau_1$

Above the line

$\Gamma \vdash e : nat$ — H_1

$\Gamma \vdash e_0 : \tau_1$ — H_2

$\Gamma, x : nat, y : \tau_1 \vdash e_1 : \tau_1$ — H_3

By IH on H_1

e is either val or $\exists e'$, $e \mapsto e'$

Case $e \mapsto e'$

take $e' = rec\{e_0; x, y.e_1\}(e')$

By rule $\dfrac{e \mapsto e'}{rec\{e_0; x, y.e_1\}(e) \mapsto rec\{e_0; x, y.e_1\}(e')}$

$e \mapsto e'$

Case e val

By Canonical Form Lemma on H_1, $e : nat$

$e = z$ or $s(e')$

Subcase $e = z$

Take e' as $\cancel{z}$ e_0

$$\frac{}{rec\{e_0; x, y.e_1\}(z) \mapsto e_0}$$

e val, or z val

Subcase $e = s(e')$

take e' as $[e'/x][rec\{e_0; x, y.e_1\}(e')/y]\, e_1$

By rule $\dfrac{s(e) \text{ val}}{rec\{e_0; x, y.e_1\}(s(e))}$

$s(e)$ val

Preservation If $e : \tau$ & $e \mapsto e'$ then $e' : \tau$

Case: $$\frac{e \mapsto e_1}{rec\{e_0; x, y.e_1\}(e) \mapsto rec\{e_0; x, y.e\}(e')}$$

$e = rec\{e_0; x, y.e_1\}(e)$ — H_k $\boxed{\text{Goal } e' : \tau}$

$e' = rec\{e_0; x, y.e\}(e')$

Above the line $e \mapsto e_1$ — H_1

By Inversion on e

$\Gamma \vdash e : nat$ — H_2

$\Gamma \vdash e_0 : \tau$ — H_3

$\Gamma, x : nat, y : \tau \vdash e_1 : \tau$ — H_4

By IH on H_1 & H_2

$e' : nat$

By rec rule $rec\{e_0; x, y.e_1\}(e') : \tau$

Case: $$\frac{s(e)\ val}{rec\{e_0; x, y.e_1\}(s(e)) \mapsto [e, rec\{e_0; x, y.e_1\}(e)/x, y]e_1}$$

$e = rec\{e_0; x, y.e_1\}(s(e))$

$e' = [e, rec\{e_0; x, y.e_1\}(e)/x, y]e_1$ $\boxed{\text{Prove } e' : \tau}$

Above the line $s(e)$ val — H_1

Inversion on ~~e_1~~ ~~e val~~ (H_2)

$\Gamma \vdash s(e) : nat$ — H_2 , Inversion on H_2 $\Gamma \vdash e : nat$ — H_3

$\Gamma \vdash e_1 : \tau$ — H_3

$\Gamma, x : nat, y : \tau \vdash e_1 : \tau$ — H_4

By Apply weakening on H_3 i.e adding ~~Γ~~ $y : \tau$

$\Gamma, y : \tau \vdash e : nat$ — H_5

By applying substituting Lemma H_4 & H_5

$y : \tau \vdash [e/x]e_1 : \tau$ — H_6

back

$\Gamma\, y:\tau \vdash [e/x]e_1 : \tau$ — H_6

By 2, 3, 4

$rec\{e_0; x, y.e_1\}(e) : \tau$ H_7

By S.L on H_6 & H_7

$[e/x][rec\{e_0; x, y.e_1\}(e)/y]e_1 : \tau$

By ~~rec~~ Rule

$\boxed{e' : \tau}$

Case $\dfrac{e \mapsto e'}{s(e) \mapsto s(e')}$

$e = s(e)$

$e' = s(e')$

Above the line $e \mapsto e'$ — H_1

By Inversion on e

$\Gamma \vdash e : nat$ — H_2

By IH on H_2

$\Gamma \vdash e' : nat$ H_3

By rule $s(e') : nat$

$\dfrac{\Gamma \vdash e : nat}{\Gamma \vdash s(e) : nat}$

Progress If $e:\tau$ then either e val or $\exists e', e \mapsto e'$

Case $\dfrac{}{s(e) \; val}$ $\quad \dfrac{\Gamma \vdash e : nat}{\Gamma \vdash s(e) : nat}$ $= e$ $\quad = e'$

$\boxed{e' : \tau}$ Goal

Above the line
$\Gamma \vdash e : nat$ — H_1

By IH on H_1
either e is val or $\exists e', e \mapsto e'$

Case $e \mapsto e'$
take e' as $s(e')$
By rule $\dfrac{e \mapsto e'}{s(e) \mapsto s(e')}$

Case e val
$\underbrace{\text{if } \Gamma \vdash e : nat \text{ the By rule } \Gamma \vdash s(e) : nat}$

Case $\dfrac{\Gamma \vdash e_0 : \tau \quad \Gamma \vdash e : nat, \quad \Gamma, x : nat, y : \tau \vdash e_1 : \tau}{\Gamma \vdash rec\{e_0; x, y.e_1\}(e) : \tau}$
$e = \Gamma \vdash rec\{e_0; x, y.e_1\}(e)$
$e' = \tau$
Above the line → $\Gamma \vdash e : nat$ — H_1
$\Gamma \vdash e_0 : \tau$ — H_2
$\Gamma, x : nat, y : \tau \vdash e_1 : \tau$ — H_3
By ~~HG~~ IH on H_1
either e val or $\exists e', e \mapsto e'$
Case $e \mapsto e'$
take e' as $rec\{e_0; x, y.e_1\}(e)$
By rule $e \mapsto e'$
Case e val
By CFL on on H_1 $\quad e \begin{cases} z \\ s(e') \end{cases}$
Case I z
take $e' = e^0$
Value By rule $\dfrac{}{rec\{e_0; x, y.e_1\}(z) \mapsto e_0}$

back { ; }{ ... open(x; t, ...

Case $e = s(e')$

take $e' = [e, rec\{e_0; x, y.e_1\}(e')/x, y]e'$

By rule $s(e')$ ~~is val~~ is Value

the $\boxed{e' \text{ val}}$

preservation pair$(e_1; e_2)$

$e : \tau$ & $e \mapsto e'$ then $e' : \tau$

Case: $\dfrac{e_1 \mapsto e_1'}{pair(e_1; e_2) \mapsto pair(e_1'; e_2)}$

$e = pair(e_1; e_2)$

$e' = pair(e_1'; e_2)$

$\boxed{pair(e_1'; e_2) : \tau}$ Goal

Above the line $e_1 \mapsto e_1'$ — H1

By Inversion on e

$\dfrac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash pair(e_1; e_2) : \tau_1 \times \tau_2}$

$\Gamma \vdash e_1 : \tau_1$ — H2

$\Gamma \vdash e_2 : \tau_2$ — H3

By Rule

~~$\Gamma \vdash pair(e_1; e_2) : \tau_1 \times \tau_2$ — H4~~

By IH on H_2 & ~~H4~~ H1

~~$\tau_1 \times \tau_2$~~ ✓

$\Gamma \vdash e_1' : \tau_1$

By pair rule

$pair(e_1'; e_2) : \tau_1 \times \tau_2$

Preservation

Case: $\dfrac{e_1\ \mathrm{val} \quad e_2 \mapsto e_2'}{\mathrm{pair}(e_1;e_2) \mapsto \mathrm{pair}(e_1;e_2')}$

Goal
$e' : \tau$

$e = \mathrm{pair}(e_1;e_2)$
$e' = \mathrm{pair}(e_1;e_2')$

Above the line
e_1 val — H1
$e_2 \mapsto e_2'$ — H2

~~By IH on~~ e By Inversion on e
$\Gamma \vdash e_1 : \tau_1$ — H3
$\Gamma \vdash e_2 : \tau_2$ — H4

By IH on H2 & H4 ~~By~~
$e_2' : \tau_2$ — H5

Using H3 & H5 By rule
$\mathrm{pair}(e_1;e_2) : \tau_1 \times \tau_2$

~~wa~~ By H3 & H5 $\dfrac{\Gamma \vdash e_1 : \tau_1,\ \Gamma \vdash e_2 : \tau_2}{\mathrm{pair}(e_1;e_2) : \tau_1 \times \tau_2}$

$\underbrace{\mathrm{pair}(e_1;e_2') : \tau_1 \times \tau_2 \ \text{proved} \quad e' : \tau}$

Case: $\dfrac{e \to e'}{\mathrm{fst}(e) \to \mathrm{fst}(e')}$ $e = \mathrm{fst}(e)$, $e' = \mathrm{fst}(e')$

Goal
$\boxed{\mathrm{fst}(e') : \tau}$

Above the line
$e \mapsto e'$ — H1

Inversion on e $\dfrac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \mathrm{fst}(e) : \tau_1}$

$\Gamma \vdash e : \tau_1 \times \tau_2$ — H2

By IH on H1 & H2
$\Gamma \vdash e' : \tau_1 \times \tau_2$

By prod Rule $\dfrac{\Gamma \vdash e' : \mathrm{prod}(\tau_1;\tau_2)}{\Gamma \vdash \mathrm{fst}(e') : \tau_1}$

$\boxed{e' : \tau_1}$

back $\{ ; \}t$ obs$(x; t, 0, \ldots)$

Case: $\dfrac{[e_1 \ val] \quad [e_2 \ val]}{fst(e_1; e_2) \mapsto e_1}$

$e = fst(e_1; e_2)$

$e' = e_1$ $\quad \boxed{e' : \tau}$ Goal

Above the line

e_1 val — H_1

e_2 Val H_3

By Inversion on e $\quad \Gamma \vdash e : pair(\tau_1; \tau_2)$

$\Gamma \vdash e : pair(\tau_1; \tau_2)$ — H_3

~~since~~ By Inversion on H3 $\quad$ By rule $\dfrac{\Gamma \vdash e : prod(\tau_1; \tau_2)}{\Gamma \vdash fst(e) : \tau_1}$

$\boxed{\Gamma \vdash e_1 : \tau_1}$ — H_5 proved

$\Gamma \vdash e_2 : \tau_2$ — H_6

Progress: $e : \tau$ then e val or there exist e', $e \mapsto e'$

Case: $\dfrac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash pair(e_1; e_2) : \tau_1 \times \tau_2}$

$e = \Gamma \vdash pair(e_1; e_2)$

$e' = \tau_1 \times \tau_2$

Above the line

$\Gamma \vdash e_1 : \tau_1$ — H_1

$\Gamma \vdash e_2 : \tau_2$ — H_2

By IH on e_1 we get either $pair(e_1; e_2)$ val

or $e_1 \mapsto e_1'$

Case $e_1 \mapsto e_1'$

take e_1' as $pair(e_1'; e_2)$

By rule $\dfrac{e_1 \mapsto e_1'}{pair(e_1; e_2) \mapsto pair(e_1'; e_2)}$ Done ✓

Case ~~pair(e_1;e_2)~~ e_1 val

Take IH on H2

Case $e_2 \mapsto e_2'$

take e_2' as $pair(e_1; e_2')$

then $e_2 \mapsto e_2'$

$\boxed{e_1 \text{ val}}$

Case : e_2 val

~~that~~. we have e_1 val & e_2 val

then pair$(e_1; e_2)$ val

Preservation $e : \tau$ & $e \mapsto e'$ then $e' : \tau$

Case

$$\frac{s(e) \text{ val}}{\text{rec}\{e_0; x, y.e_1\}(s(e)) \to [e, \text{rec}\{e_0; x, y.e_1\}/x, y]e_1}$$

$e : \text{rec}\{e_0; x, y.e_1\}(s(e))$

$e' = [e, \text{rec}\{e_0; x, y.e_1\}/x, y]e_1$

By Inversion on e — $\Gamma \vdash e_0 : \tau$ — (i)

$\Gamma \vdash s(e) : \text{nat}$ — (ii)

$\Gamma, x : \text{nat}, y : \tau \vdash e_1 : \tau$ — H_3

By Inversion on (2)

$\Gamma \vdash e : \text{nat}$ — H_4

By Apply weaking on H_4 $\Gamma \vdash y : \tau \vdash e : \text{nat}$ — H_5

By substitution Lemma on H_3 & H_5

$\Gamma, y : \tau \vdash [e/x]e_1 : \tau$ — H_6

By (i), H_4, H_3

By rec rule → $\text{rec}\{e_0; x, y.e_1\}(e) : \tau$ — H_7

By Substition lemma on H_6 & H_7

$\Gamma, \text{rec}\{e_0; x, y.e_1\}(e) / y : \tau \vdash [e/x]e : \tau$

Progress: $\dfrac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash fst(e) : \tau}$

$e : \tau$ then either $e \mapsto e'$ or e val

$e = \Gamma \vdash fst(e)$

Case Above the line

$\Gamma \vdash e : \tau_1 \times \tau_2$ — H_1

By IH on H_1 either e val or $e \mapsto e'$

case $e \mapsto e'$

~~By CFL on H_1~~

~~$e = pair(e_1; e_2)$~~ — H_2

~~take e' as $pair(e'_1; e_2)$~~

~~then $e \mapsto e'$~~ ~~By rule $pair(e_1; e_2) \mapsto pair(e'_1; e_2)$~~

Case e val

~~By rule~~ $\dfrac{\cancel{e_1 \text{ val}} \quad \cancel{v_2 \text{ val}}}{\cancel{fst(e_1; e_2) \mapsto e_1}}$

Case $e \mapsto e'$

Take $e' = fst(e')$

then $e \mapsto e'$ — H_2

Case e val

By CFL Lemma if $e = pa(\tau_1 \times \tau_2)$ & e val

then $e = pair(e_1; e_2)$

e_1 val & e_2 val

if e_1 & e_2 are val $\qquad \dfrac{e_1 \text{ val} \quad e_2 \text{ val}}{pair(e_1; e_2) \text{ val}}$

$e = fst(e_1; e_2)$ — H_3

By IH on H_3

case e val

take e' as e_1 $\qquad \dfrac{e_1 \text{ val} \quad e_2 \text{ val}}{fst(e_1; e_2) \mapsto e_1}$

then $e \mapsto e'$

Case: $\dfrac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash snd(e) : \tau_2}$

$e = snd(e)$

$\tau = \tau_2$

Above the line

$\Gamma \vdash e : \tau_1 \times \tau_2$ — H1

By IH on H1 either eval or $\exists e'$, $e \mapsto e'$

Case: $e \mapsto e'$

Take e' as $snd(e')$

then $e \mapsto e'$

Case: e val

By CFL if $e : prod(\tau_1; \tau_2)$ then e val $e = pair(e_1; e_2)$

By rule e_1 val & e_2 val

Subclass $snd(e_1; e_2) \mapsto e_2$

take e' as e_2

then $e \mapsto e'$

Case: $\dfrac{\Gamma \vdash e_1 : \tau_1 , \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash pair(e_1; e_2) : \tau_1 \times \tau_2}$

$e = \Gamma \vdash pair(e_1; e_2)$

$e' = \tau_1 \times \tau_2$

Above the line

$\Gamma \vdash e_1 : \tau_1$ — H1

$\Gamma \vdash e_2 : \tau_2$ — H2

By IH on H1

either e_1 val or $e_1 \mapsto e_1'$

Case $e_1 \mapsto e_1'$

take e_1' as $pair(e_1'; e_2)$

By rule

then $e_1 \mapsto e_1'$

$$\frac{e_1 \mapsto e_1'}{pair(e_1; e_2) \mapsto pair(e_1'; e_2)}$$

Case e_1 val

IH on H2

either e val or $e_2 \mapsto e_2'$

case $e_2 \mapsto e_2'$

CFL if $e : \tau_1 \times \tau_2$ & e val then $e = \text{pair}(e_1; e_2)$
e_1 & e_2 is also valid

take $\text{pair}(e_1; e_2')$ as e'
By rule $\text{pair}(e_1;e_2) \mapsto \text{pair}(e_1; e_2')$
$e_2 \mapsto e_2'$
$\boxed{e_1 \text{ val}}$

Subcase e_2 val
we have e_1 & e_2 val
then $\text{pair}(e_1; e_2)$ val
e' val

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \text{pair}(e_1; e_2) : \tau_1 \times \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash fst(e) : \tau_1}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash snd(e) : \tau_2}$$

$$\frac{e_1 \text{ val} \quad e_2 \text{ val}}{\text{pair}(e_1; e_2) \text{ val}}$$

$$\frac{e_1 \text{ val} \quad e_2 \text{ val}}{fst(e_1; e_2) \mapsto e_1} \quad \Bigg| \quad \frac{e_1 \text{ val} \quad e_2 \text{ val}}{snd(e_1; e_2) \mapsto e_2}$$

$$\frac{e \mapsto e'}{fst(e) \mapsto fst(e')} \quad \Bigg| \quad \frac{e \mapsto e'}{snd(e) \mapsto snd(e')}$$

$$\frac{e_1 \mapsto e_1'}{\text{pair}(e_1; e_2) \mapsto \text{pair}(e_1'; e_2)}$$

$$\frac{e_1 \text{ val} \quad e_2 \mapsto e_2'}{\text{pair}(e_1; e_2) \mapsto \text{pair}(e_1; e_2')}$$

Preservation $\dfrac{e_1 \text{ Val} \quad e_2 \text{ Val}}{\text{Snd}(e_1; e_2) \mapsto e_2}$

$e : \tau$ & $\exists e'$, $e \mapsto e'$ then $e' : \tau$

$e = \text{Snd}(e_1; e_2)$

$e' = e_2$ we need $e_2 : \tau$ or $e_2 : \tau_2$

Above the line

e_1 Val — H_1

e_2 Val → H_2

By Inversion on e

$\Gamma \vdash e : \tau_1 \times \tau_2$ — H_3

By Inversion on H_3

$\Gamma \vdash e_1 : \tau_1$ — H_4

$\boxed{\Gamma \vdash e_2 : \tau_2}$ — H_5

Similarly for $\dfrac{e_1 \text{Val} \quad e_2 \text{Val}}{\text{fst}(e_1; e_2) \mapsto e_1}$

Case: $\dfrac{e \mapsto e'}{\text{Snd}(e) \mapsto \text{Snd}(e')}$

$e = \text{Snd}(e)$

$e' = \text{Snd}(e')$ To be prove $e' : \tau$ or $\text{Snd}(e') = \tau_2$

Above the line $e \mapsto e'$ — H_1

By Inversion on e

$\Gamma \vdash e : \tau_1 \times \tau_2$ — H_2

By IH on H_2 & H_1

$\Gamma \vdash e' : \tau_1 \times \tau_2$

By rule $\dfrac{\Gamma \vdash e' : \tau_1 \times \tau_2}{\text{Snd}(e') : \tau_2}$

~~take e' as $\text{Snd}(e')$~~

for $\dfrac{e \mapsto e'}{\text{fst}(e) \mapsto \text{fst}(e')}$

use rule $\dfrac{\Gamma \vdash e' : \tau_1 \times \tau_2}{\text{fst}(e') : \tau_1}$

back ... ; ... case(x; t; y; ...

Case $\dfrac{e_1\ val \quad e_2 \mapsto e_2'}{pair(e_1; e_2) \mapsto pair(e_1; e_2')}$

$e = pair(e_1; e_2)$; Goal
$e' = pair(e_1; e_2')$ $\boxed{e' : \tau}$

By Inversion e

$\Gamma \vdash e_1 : \tau_1$ — H_1

$\Gamma \vdash e_2 : \tau_2$ — H_2

Above the line $e_2 \mapsto e_2'$ H_3

By IH on H_2 & H_3

$e_2' : \tau_2$ — H_4

By rule : $\dfrac{\Gamma \vdash e_1 : \tau_1 ; \Gamma \vdash e_2' : \tau_1}{pair(e_1; e_2') : \tau_1 \times \tau_2}$

$pair(e_1; e_2') : \tau_1 \times \tau_2$

Progress $\dfrac{\Gamma \vdash e : \tau_1 \times \tau_2}{fst(e) : \tau_1}$

If $e : \tau$ then either $e\ val$ or $\exists e', e \mapsto e'$

$e = fst(e)$

$e' = \tau_1$

Above the line

$\Gamma \vdash e : \tau_1 \times \tau_2$ — H_1

By IH on e

Also

Case $e \mapsto e'$

take e' as $fst(e')$

By rule $\dfrac{e \mapsto e'}{fst(e) \mapsto fst(e')}$

Case $e\ val$

By CFL on e, $\Gamma \vdash e : \tau_1 \times \tau_2$ & $e\ val$ then

$e = pair(e_1; e_2)$

By pair rule

$e_1\ val$ & $e_2\ val$

Similarly take, e' as ~~fst(e)~~ e_1

$[f, e/t, x]e_2$

$$\frac{e_1 \text{ val} \quad e_2 \text{ val}}{fst(e_1; e_2) \mapsto e_1}$$ By rule

fst(e) val. Progress

Progress $e : \tau$ then either e val or $\exists e', e \mapsto e'$

Case: rec $$\frac{\Gamma \vdash e_0 : \tau, \ \Gamma \vdash e : nat, \ \Gamma, x : nat, y : \tau \vdash e_1 : \tau}{rec\{e_0; x, y . e_1\}(e) : \tau}$$

$e : rec\{e_0; x, y . e_1\}(e)$

$\cancel{e} : \tau$

Above the line

$\Gamma \vdash e_0 : \tau$ — H_1

$\Gamma \vdash e : nat$ — H_2

$\Gamma, x : nat, y : \tau \vdash e_1 : \tau$ — H_3

By IH on e

Case $e \mapsto e'$

take e' as $rec\{e_0; x, y . e_1\}(e')$

By rec rule $$\frac{e \mapsto e'}{rec\{e_0; x, y . e_1\}(e) \longmapsto rec\{e_0; x, y . e_1\}(e')}$$

$\boxed{e \mapsto e'}$

Case e val

By CFL on $e : nat$ & e val

then $e = Z$ or $S(e')$

Case: $e = Z$

$$\frac{\text{take } e' \text{ as } e_0}{rec\{e_0; x, y . e_1\}(z) \to e_0}$$

Case $e = S(e)$

take $e' = [e, rec\{e_0; x, y . e_0\}(e) / x, y]e_1$

By rule

$S(e)$ val

Sum Types

$$\frac{\Gamma \vdash e : \text{void}}{\text{Abort}\{\tau\}(e) : \tau}$$

$$\frac{e \mapsto e'}{\text{Abort}\{\tau\}(e) \mapsto \text{Abort}\{\tau\}(e')}$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \text{inl}\{\tau_1 ; \tau_2\}(e) : \tau_1 + \tau_2}$$

$$\frac{e \mapsto e'}{\text{inl}\{\tau_1 ; \tau_2\}(e) \mapsto \text{inl}\{\tau_1 ; \tau_2\}(e')}$$

$$\frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \text{inr}\{\tau_1 ; \tau_2\}(e) : \tau_1 + \tau_2}$$

same

$$\text{inr}\{\tau_1 ; \tau_2\}(e) \mapsto \text{inr}\{\tau_1 ; \tau_2\}(e')$$

$$\frac{\Gamma \vdash e : \text{sum}(\tau_1 ; \tau_2) \quad \Gamma, x : \tau_1 \vdash e_1 : \tau \quad \Gamma, y : \tau_2 \vdash e_2 : \tau}{\text{case}(e ; x.e_1 , y.e_2) : \tau}$$

$$\frac{e_0 \vdash e_0'}{\text{Case}(e_0 ; x.e_1 ; y.e_2) \longmapsto \text{Case}(e_0' ; x.e_1 , y.e_2)}$$

$$\frac{[e \text{ val}]}{\text{Case}(\text{inl}\{\tau_1 ; \tau_2\}(e) ; x.e_1 , y.e_2) \mapsto [e/x]e_1}$$

or

$$\frac{[e \text{ val}]}{\text{Case}(\text{inl}\{\tau_1 ; \tau_2\}(e) ; x.e_1 , y.e_2) \mapsto [e/x]e_2}$$

$$\frac{e \text{ val}}{\text{inl}\{\tau_1 ; \tau_2\}(e)} , \quad \frac{e \text{ val}}{\text{inr}\{\tau_1 ; \tau_2\}(e)}$$

Preservation

$e : \tau$ & $e \mapsto e'$ then $e' : \tau$

~~Theorem~~ $e \mapsto e'$

Case ~~for~~ $\mathrm{Abort}\{\tau\}(e)$ ~~e~~ $\mapsto \mathrm{Abort}\{\tau\}(e')$

$e = \mathrm{Abort}\{\tau\}(e)$

$e' = \mathrm{Abort}\{\tau\}(e')$

Goal $\mathrm{Abort}\{\tau\}(e') : \tau$

~~Above the line~~ By Inversion on e

$\Gamma \vdash e : \mathrm{Void}$ — H1

Above the line $e \mapsto e'$ — H2

By IH on H1 & H2

$\Gamma \vdash e' : \mathrm{Void}$

By $\mathrm{Abort}\{\tau\}(e')$ rule

$\mathrm{Abort}\{\tau\}(e') : \tau$

Case:
$$\frac{e \mapsto e'}{\mathrm{inl}\{\tau_1;\tau_2\}(e) \mapsto \mathrm{inl}\{\tau_1;\tau_2\}(e')}$$

$e = \mathrm{inl}\{\tau_1;\tau_2\}(e)$

$e' = \mathrm{inl}\{\tau_1;\tau_2\}(e')$

Above the line, $e \mapsto e'$ — H1

Inversion on e

$\Gamma \vdash e_1 : \tau_1$ — H2

By IH on H1 & H2

$\Gamma \vdash e' : \tau_1$

By rule

$\mathrm{inl}\{\tau_1,\tau_2\}(e) : \underline{\mathrm{Sum}}(\tau_1;\tau_2)$

$\mathrm{inl}\{\tau_1,\tau_2\}(e') : \mathrm{Sum}(\tau_1;\tau_2)$ ✓

back & ; ... open(...

Case: $\dfrac{e \mapsto e'}{\text{case}(e; x.e_1; y.e_2) \mapsto \text{case}(e'; x.e_1; y.e_2)}$

$e = \text{case}(e; x.e_1, y.e_2)$
$e' = \text{case}(e'; x.e_1; y.e_2)$

Above the line
$e \mapsto e'$ — H_1

By Inversion on e
$\Gamma \vdash e : \text{sum}(\tau_1; \tau_2)$ — H_2
$\Gamma, x : \tau_1 \vdash e_1 : \tau$ — H_3
$\Gamma, y : \tau_2 \vdash e_2 : \tau$ — H_4

By IH on H_1 & H_2

$\Gamma \vdash e' : \text{sum}(\tau_1; \tau_2)$ — H_5

By case rule, H_5, H_3, H_4

$\text{case}(e'; x.e_1; y.e_2) : \tau$ proved

Case l
$$\frac{[e \text{ val}]}{\text{Case}(\text{inl}\{\tau_1;\tau_2\}(e); x.e_1, y.e_2) \mapsto [e/x]e_1}$$

$e = \text{Case}(\text{inl}\{\tau_1;\tau_2\}(e); x.e_1, y.e_2)$
$e' = [e/x]e_1$ To proof $[e/x]e_1 : \tau$

Above the line
e val
By Inversion on e
$\Gamma \vdash \text{inl}\{\tau_1;\tau_2\}(e) : \tau_1 + \tau_2$ — H_1
$\Gamma, x:\tau_1 \vdash e_1 : \tau$ — H_2
$\Gamma, y:\tau_2 \vdash e_2 : \tau$ — H_3

By Inversion on H_1
$\Gamma \vdash e_1 : \tau_1$ — H_4
By Substitution lemma on H_2 & H_4
$\Gamma \vdash [e/x]e_1 : \tau$ ✓

Progress
If $e:\tau$ then
$\exists e, e \mapsto e'$
e val

Case $\frac{\Gamma \vdash e : \text{Void}}{\text{Abort}\{\tau\}(e) : \tau}$
$e = \text{Abort}\{\tau\}(e)$
$e' = \tau$
Above the line → $\Gamma \vdash e : \text{Void}$ — H_1
By IH on H_1
Case $e \mapsto e'$
take e' as $\text{Abort}\{\tau\}(e')$
By rule $\text{Abort}\{\tau\}(e) \mapsto \text{Abort}\{\tau\}(e')$ then $e \mapsto e'$
Case: e val
If e : Void & e val then e is false by CFL

back $\{;\}\{$ obs $(x; t, \ldots$

Case: $\dfrac{\Gamma \vdash e : \tau_1}{inl\{\tau_1;\tau_2\}(e) : \tau_1 + \tau_2}$

$e = inl\{\tau_1;\tau_2\}(e)$

$\tau = \tau_1 + \tau_2$

Above the line $\Gamma \vdash e : \tau_1$ — H_1

By IH on H_1 => either e val or $\exists e', e \mapsto e'$

Case $e \mapsto e'$

take e' as $inl\{\tau_1;\tau_2\}(e')$

By inl rule

$e \mapsto e'$

Case e val

take e' as $[e/x]e_1$

By Rule $\dfrac{\cancel{e \text{ val}}}{case(inl\{\tau_1;\tau_2\}(e); x.e_1; y.e_2) \longmapsto [e/x]e_1}$ $\dfrac{e \text{ val}}{inl\{\tau_1;\tau_2\}(e}$

e val

Case: $\dfrac{\Gamma \vdash e : \tau_1 + \tau_2,\ \Gamma, x : \tau_1 \vdash e_1 : \tau,\ \Gamma, y : \tau_2 \vdash e_2 : \tau}{case(e; x.e_1, y.e_2) : \tau}$

$e = case(e; x.e_1, y.e_2)$

$\tau = \tau$

Above the line

$\Gamma \vdash e : \tau_1 + \tau_2$ — H_1

$\Gamma, x : \tau_1 \vdash e_1 : \tau$ — H_2

$\Gamma, y : \tau_2 \vdash e_2 : \tau$ — H_3

By IH on H_1

either e val or $e \mapsto e'$

Case $e \mapsto e'$

take e' as $case(e'; x.e_1, y.e_2)$

~~By~~ rule $e \mapsto e'$

Case: e val

if $e : \mathrm{Sum}(\tau_1; \tau_2)$ then there exist (e')

Subclass: $e = \mathrm{inl}\{\tau_1;\tau_2\}(e')$ — (14)

$e = \mathrm{inr}\{\tau_1;\tau_2\}(e')$ (15)

take e' as $[e'/x]e_1$

then $e \mapsto e'$ and e' val

$$\frac{[e' \text{ val}]}{\mathrm{Case}(\mathrm{inl}\{\tau_1;\tau_2\}(e'); x.e_1 \ldots) \mapsto [e'/x]e_1}$$

Subclass $e : \mathrm{inr}\{\tau_1;\tau_2\}(e')$

take e' as $[e'/y]e_2$

then By rule

$e \mapsto e'$ & e' val

$$\frac{[e' \text{ val}]}{\mathrm{Case}(\mathrm{inr}\{\tau_1;\tau_2\}(e'); y.e_2) \mapsto [e'/y]e_2}$$

$(A \to \forall t . P(t)) \to \forall t . (A \to P(t))$

$\mathrm{lam}\{(A \to \forall t . P(t))\}(x . \mathrm{lam}\{t\}(a : e))$ — $A \to P(t)$

$e = \mathrm{lam}\{A\}(y . \mathrm{App}\{t\}(\mathrm{ap}(x; y)))$

$\mathrm{Lam}\{t\}(x.e)$

$\mathrm{App}\{t\}(e)$

(4) $((P \to O) + (P \to R)) \to (P \to (O + R))$

$\mathrm{lam}\{(P \to O) + (P \to R)\}(x . \mathrm{lam}\{P\}(y . \mathrm{case}(x; \overset{P\to O}{a} . e_1, \overset{P\to R}{b} . e_2)))$

$e_1 = \mathrm{inl}\{O;R\}(\mathrm{ap}(a;y))$

$e_2 = \mathrm{inr}\{O;R\}(\mathrm{ap}(b;y))$

(11) $(A \to \forall t . B(t)) \times (A \to \forall t . C(t)) \to \forall u . A \to (B(u) \times C(u))$

$\mathrm{lam}\{\;\}(x . \mathrm{Lam}\{u\}(x . \mathrm{lam}\{A\}(y . \mathrm{pair}(e_1; e_2))))$

$e_1 =$ ~~fst~~ $\mathrm{App}\{u\}(\mathrm{ap}(\mathrm{fst}(x); y))$ | $e_2 = \mathrm{App}\{u\}(\mathrm{ap}(\mathrm{snd}(x); y))$

$\Gamma, x:\tau_1 \vdash e:\tau$ $\quad \Gamma \vdash e_1:\tau_1$ $\quad [e_1/x]e$ $\quad$ App{t}(e)

Lam(t.e)

(5) $(\forall t.P(t)) \times (\forall t.Q(t)) \to \forall t.\underline{P(t) \times Q(t)}$

lam{$(\forall t.P(t)) \times (\forall t.Q(t))$}(x.Lam(t. pair(App{t}(fst(x))); App{t}(snd(x)))))

$\boxed{\text{Substitution Lemma}}$

Typing substitution Lemma $\quad \Gamma, x:\tau_1 \vdash e:\tau$ & $\Gamma \vdash e_1:\tau$

then $[e_1/x]e$

I If Δ, t type $\vdash \tau'$ type & $\Delta \vdash \tau$ type

then $\Delta \vdash [\tau/t]\tau'$ type

II If Δ, t type $\Gamma \vdash e':\tau'$ & $\Delta \vdash \tau$ type then

$\Delta\ [\tau/t]\Gamma \vdash [\tau/t]e' : [\tau/t]\tau'$

III If $\Delta, x:\tau \vdash e':\tau'$ & $\Delta\,\Gamma \vdash e:\tau$, then

$\Delta\,\Gamma \vdash [e/x]e' : \tau'$

$$\frac{e \mapsto e'}{\text{App}\{\tau\}(e) \mapsto \text{App}\{\tau\}(e')}$$

$$\frac{\Delta, t\ \text{type} \vdash \tau\ \text{type}}{\Delta \vdash \forall(t.\tau)\ \text{type}}$$

$$\frac{}{\text{App}\{\tau\}(\text{Lam}(t.e)) \mapsto [\tau/t]e}$$

$$\frac{\Delta \vdash \tau_1\ \text{type} \quad \Delta \vdash \tau_2\ \text{type}}{\Delta \vdash \text{arr}(\tau_1;\tau_2)\ \text{type}}$$

lam i

$$\frac{\Delta, t\ \text{type}\ \Gamma \vdash e : \tau}{\Delta\ \Gamma \vdash \text{Lam}(t.e) : \forall(t.\tau)}$$

$$\frac{\Delta\ \Gamma \vdash e : \forall(t.\tau') \quad \Delta \vdash \tau\ \text{type}}{\Delta\ \Gamma \vdash \text{App}\{\tau\}(e) : [\tau/t]\tau'}$$

Preservation If $e : \tau$ & $e \mapsto e'$ then $e' : \tau$

Case: $\dfrac{e \mapsto e'}{\text{App}\{\tau\}(e) \mapsto \text{App}\{\tau\}(e')}$ $[\tau/t]\tau'$

$\boxed{e' : \tau}$ prove $= [\tau/t]\tau'$

$e = \text{App}\{\tau\}(e)$

$e' = \text{App}\{\tau\}(e')$

Above the line $e \mapsto e'$ — H1

By Inversion on $e : \tau$

$\Delta\ \Gamma \vdash e : \forall(t.\tau')$ — H2

$\Delta \vdash \tau$ type — H3

By IH on H1 & H2

$\Delta\ \Gamma \vdash e' : \forall(t.\tau')$ — H4

By rule of APP on H4 & H3

$\text{APP}\{\tau\}(e') : [\tau/t]\tau'$

Case: ~~$App\{\tau\}(Lam(t.e)) \mapsto [\tau/t]e$~~

$e = App\{\tau\}(Lam(t.e))$
$e' = [\tau/t]e$

By Inversion on $e : \tau$

$\Delta \Gamma \vdash Lam(t.e) : \forall(t.\tau')$ — H1

$\Delta \vdash \tau$ type — H2

$\dfrac{\Delta, t \text{ type } \Gamma \vdash e : \tau}{Lam(t.e) : \forall(t.\tau')}$

$\dfrac{\Delta \Gamma \vdash e : \forall(t.\tau') \quad \Delta \vdash \tau}{\Delta \Gamma \vdash App\{\tau\}(e) : [\tau/t]\tau'}$

By Inversion on H1 $e : all(t.\tau')$

$\Delta, t \text{ type } \Gamma \vdash e : \tau'$ — H3

By substitution lemma on H2 & H3

If $\Delta, t \text{ type } \Gamma \vdash e : \tau'$ & $\Delta \vdash \tau$ type

then $\Delta, [\tau/t] \Gamma \vdash [\tau/t]e : [\tau/t]\tau'$

So $[\tau/t]e : [\tau/t]\tau'$

$[\tau/t]\tau' : \tau'$

so $[\tau/t]e : \tau'$

Case: ~~App{t~~

$App\{\tau\}(Lam(t.e)) \mapsto [\tau/t]e$

$e = App\{\tau\}(Lam(t.e))$
$e' = [\tau/t]e$

By Inversion on $e : \tau$

$\Delta \Gamma \vdash Lam(t.e) : \forall(t.\tau')$ — H1

~~Δ⊢t~~ $\Delta \vdash \tau$ type — H2

By Inversion H1

$\Delta, t \text{ type } \Gamma \vdash e : \tau'$ — H3

$= \Delta, [\tau/t] \Gamma \vdash [\tau/t]e : [\tau/t]\tau'$ — H4

$\Delta \vdash \tau$ type $\quad \underline{\Delta\,\Gamma \vdash e : \tau}$

$$\frac{}{App\{\tau\}(Lam(t.e)) \mapsto [\tau/t]e} \qquad \frac{e \mapsto e'}{App\{\tau\}(e) \mapsto App\{\tau\}(e')}$$

Progress $e : \tau$ then either e val or $\exists e,\ e \mapsto e'$

Case $\dfrac{\Delta, t\ type\ \Gamma \vdash e : \tau}{\Delta\,\Gamma \vdash Lam(t.e) : \forall(t.\tau)}$

$e = \Delta\,\Gamma \vdash Lam(t.e)$

$\tau = \forall(t.\tau)$

Above the line

$\Delta, t\ type\ \Gamma \vdash e : \tau$ — H_1

Case e val

By rule $Lam(t.e)$ val

Case $e \mapsto e'$

Not possible

Case: $\dfrac{\Delta\,\Gamma \vdash e : \forall(t.\tau') \quad \Delta \vdash \tau\ type}{\Delta\,\Gamma \vdash App\{\tau\}(e) : [\tau/t]\tau'}$

$e = \Delta\,\Gamma \vdash App\{\tau\}(e)$

$\tau = [\tau/t]\tau'$

Above the line

$\Delta\,\Gamma \vdash e : \forall(t.\tau')$ — H_1

$\Delta \vdash \tau$ type

By I.H on H_1

Case $e \mapsto e'$

take e' as $App\{\tau\}(e')$

By rule $e \mapsto e'$

Case e val by $e : \forall(t.\tau')$ & e val

By CFL lemma

$e = Lam(t.e)$ ✓

take $e' = [\tau/t]e$

then $e \mapsto e'$

back ... ...

(I)

$$\frac{e \mapsto e'}{\Delta\,\Gamma \vdash \mathrm{App}\{\tau\}(e) \mapsto \mathrm{App}\{\tau\}(e')}$$

$$\frac{\Delta, t\ \mathrm{type}\ \Gamma \vdash e : \forall(t.\tau') \quad \Delta \vdash \tau\ \mathrm{type}}{\mathrm{App}\{\tau\}(\mathrm{Lam}(t.e)) : \cancel{\forall(t.e)}\ [\tau/t]e}$$

(II)

$$\frac{}{\mathrm{App}\{\tau\}(\mathrm{Lam}(t.e)) \mapsto [\tau/t]e}$$

(3)

$$\frac{\Delta \vdash \tau\ \mathrm{type} \quad \Delta;\Gamma \vdash e : \forall(t.\tau')}{\Delta\,\Gamma \vdash \mathrm{App}\{\tau\}(e) : [\tau/t]e}$$

(4)

$$\frac{\Delta, t\ \mathrm{type}\ \Gamma \vdash e : \tau}{\Delta\,\Gamma \vdash \mathrm{Lam}(t.e) : \forall(t.\tau)}$$

(5)

$$\frac{\Delta, \Gamma \vdash e : \forall(t.\tau') \quad \Delta \vdash \tau\ \mathrm{type}}{\Delta\,\Gamma \vdash \mathrm{App}\{\tau\}(e) \mapsto [\tau/t]\tau'}$$

(6)

$$\frac{\dfrac{\Delta, \mathrm{type}\ \Gamma \vdash e : \tau'}{\Delta, \Gamma \vdash e : \forall(t.\tau') \quad \Delta \vdash \tau\ \mathrm{type}}}{\Delta\,\Gamma \vdash \mathrm{App}\{\tau\}(e) \mapsto [\tau/t]\tau'}$$

(7)

$$\frac{\Delta, t\ \mathrm{type} \vdash \tau\ \mathrm{type}}{\Delta \vdash \forall(t.\tau)\ \mathrm{type}}$$

(8)

$$\frac{\Delta \vdash \tau_1\ \mathrm{type} \quad \Delta \vdash \tau_2\ \mathrm{type}}{\Delta \mapsto \mathrm{arr}(\tau_1;\tau_2)\ \mathrm{type}}$$

(9)

$$\frac{}{\Delta, t\ \mathrm{type} \vdash \tau\ \mathrm{type}}$$ — Type judgement

(10)

$$\frac{\Delta\,\Gamma \vdash e_1 : \mathrm{arr}(\tau_2;\tau) \quad \Delta\,\Gamma \vdash e_2 : \tau_2}{\Delta\,\Gamma \vdash \mathrm{ap}(e_1;e_2) : \tau}$$

(11)

$$\frac{\Delta \vdash \tau_1\ \mathrm{type},\ \Delta\,\Gamma \vdash x : \tau_1 \vdash e : \tau_2}{\Delta\,\Gamma \vdash \lambda\{\tau_1\}(x.e) : \mathrm{arr}(\tau_1;\tau_2)}$$

Preservation:

Case: $\dfrac{e \mapsto e'}{App\{\tau\}(e) \mapsto App\{\tau\}(e')}$ $\rightarrow$ $e = App\{\tau\}(e)$

τ': $App\{\tau\}(e') \rightarrow \tau$ (To prove)

If $\underbrace{e:\tau}$ ~~&~~ $e \mapsto e'$ then $e':\tau$

$e \mapsto e'$ Given (Above this ~~Rno~~) – H_1

By Inversion on $e:\tau$

$$\dfrac{\dfrac{\Delta, t\ type\ \Gamma \vdash e:\tau'}{\Delta\Gamma \vdash e:\forall(t.\tau') \quad \Delta \vdash \tau\ type}}{\Delta\Gamma \vdash \underbrace{App\{\tau\}(e)}:[\tau/t]\tau'}$$

$\Delta\Gamma \vdash e: \forall(t.\tau')$ (H_2)

$\Delta \vdash \tau\ type$ – (H_3)

~~Inversion on H_2~~

~~Δ, t type $\Gamma \vdash e:\tau'$~~ ~~(H_2)~~

~~By IH on H_1 & H_3~~

~~Δ, t type $\Gamma \vdash e':\tau'$~~

~~By Rule~~

$$\dfrac{\Delta, t\ type\ \Gamma \vdash e:\tau}{\Delta\Gamma \vdash \forall(t.\tau)\ type}$$

By IH on H_1 & H_2

$\Delta\Gamma \vdash e': \forall(t.\tau')$ – (H_4)

✓

By using rule (H_3) & H_4

$$\dfrac{\Delta\Gamma \vdash e':\forall(t.\tau') \quad \Delta \vdash \tau\ type}{\Delta\Gamma \vdash App\{\tau\}(e'):[\tau/t]\tau'}$$

Case $\_\_\_\_\_\_\_\_\_\_\_\_$

$App\{\tau\}(Lam(t.e)) \mapsto [\tau/t]e$

$$\frac{\Delta, t\ type\ \Gamma \vdash e : \tau'}{\frac{\Delta, t\ type\ \Gamma \vdash e : \forall(t.\tau') \quad \Delta \vdash \tau\ type}{\Delta\,\Gamma \vdash App\{\tau\}(e) : [\tau/t]\tau'}}$$

Case $e = App\{\tau\}(Lam(t.e))$

$e' = [\tau/t]e$

Inversion on $e : \tau$

$\therefore \Delta, \Gamma \vdash e : \forall(t.\tau')$ — H_1

$\Delta \vdash \tau\ type$ — H_2

By Inversion on H_1

$\Delta\ t\ type\ \Gamma \vdash e : \tau'$ — H_3

By SL on H_2 & H_3

$\Delta\ [\tau/t]\,\Gamma \vdash [\tau/t]e : [\tau/t]\tau'$ — H_4

By weakening on H_4

$\Delta\,\Gamma \vdash [\tau/t]e : [\tau/t]\tau'$

$\boxed{e' : \tau}$ By App rule

Progress: $\dfrac{\Delta, t\ type\ \Gamma \vdash e : \tau}{\Delta\,\Gamma \vdash Lam(t.e) : \forall(t.\tau)}$

$e : \Delta\,\Gamma \vdash Lam(t.e)$

$\tau : \forall(t.\tau)$

Above the

$\Delta, t\ type\ \Gamma \vdash e : \tau$

By IH on $e : \tau$

Case $e \mapsto e'$ $\times$	Case e val Lam rule

Case $\dfrac{\Delta\Gamma \vdash e : \forall(t.\tau') \quad \Delta \vdash \tau \text{ type}}{\Delta\Gamma \vdash App\{\tau\}(e) : [\tau/t]\tau'}$

$e = \Delta\Gamma \vdash App\{\tau\}(e)$ To prove $e \begin{cases} \text{val} \\ \exists e', e \mapsto e' \end{cases}$

$\tau = [\tau/t]\tau'$

By IH on $e : \tau$

Case $e \mapsto e'$

take e' as $App\{\tau\}(e')$

By rule $App\{\tau\}(e) \mapsto App\{\tau\}(e')$

$e \mapsto e'$

Case eval

By Above the line

$\Delta\Gamma \vdash e : \forall(t.\tau')$ — H_1

$\Delta \vdash \tau$ type — H_2

By C.F.L on H_1 & eval the $e = Lam(t.e)$

take $e' = [\tau/t]e$

$$\frac{\quad e \text{ val}}{App\{\tau\}(\underbrace{e}_{Lam(t.e)}) \mapsto [\tau/t]e}$$

proved

Regularity $\Delta\Gamma \vdash e:\tau$ then τ type, Γ valid

Case: $\dfrac{\Delta \vdash \tau_1 \text{ type} \quad \Delta, x \vdash x:\tau_1 \vdash e:\tau_2}{\Delta\Gamma \vdash \text{lam}\{\tau_1\}(x.e) : \text{arr}(\tau_1;\tau_2)}$

$e = \Delta\Gamma \vdash \text{lam}\{\tau_1\}(x.e)$
$\tau = \text{arr}(\tau_1;\tau_2)$

Above the line

$\Delta \vdash \tau_1$ type — H_1

$\Delta, \underbrace{\vdash x:\tau_1}\vdash e:\tau_2$ — H_2

By Strengthing on H_2

$\Delta\Gamma, e:\tau_2$ — H_3

By IH on H_3

$\Delta\Gamma \vdash e:\tau_2$ type — H_4

Using H_1 & H_4

$\text{arr}(\tau_1;\tau_2)$ type

Case $\dfrac{\Delta\Gamma \vdash e_1 : \text{arr}(\tau_2;\tau') \quad \Delta\Gamma \vdash e_2:\tau_2}{\Delta\Gamma \vdash \text{app}(e_1;e_2) : \tau'}$

$e = \Delta\Gamma \vdash \text{app}(e_1;e_2)$
$\tau = \tau'$

Above the

$\Delta\Gamma \vdash e_1 : \text{arr}(\tau_2:\tau')$ — H_1

$\Delta\Gamma \vdash e_2:\tau_2$ — H_2

By Inversion on H_1

$\Delta \vdash \tau_2$ Type — H_3

$\Delta\Gamma, \underbrace{\vdash x:\tau_2}\vdash e:\tau'$

Remove by Strengthing lemma

$\Delta\Gamma \vdash e:\tau'$ — H_4

By IH on H_4

$\Delta\Gamma \vdash e \boxed{: \tau' \text{ type}}$

Case: $\dfrac{\Delta, t\ \text{type}\ \Gamma \vdash e : \tau'}{\Delta \Gamma \vdash \text{Lam}(t.e) : \forall(t.\tau)}$

e

$\tau = \forall(t.\tau)$

Above the line

$\Delta, t\ \text{type}\ \Gamma \vdash e : \tau$ — H_1

By IH on H_1

$\Delta, \underbrace{t\ \text{type}}\ \Gamma \vdash e : \underbrace{\tau\ \text{type}}$ $\qquad \dfrac{\Delta, t\ \text{type}\ \Gamma \vdash e : \tau\ \text{type}}{\Delta \vdash \forall(t.\tau)\ \text{type}}$

$\forall(t.\tau)$ type

Case $\dfrac{\Delta\Gamma \vdash e : \forall(t.\tau') \quad \Delta \vdash \tau\ \text{type}}{\Delta\Gamma \vdash \text{App}\{\tau\}(e) : [\tau/t]\tau'}$

$e =$

$\tau = [\tau/t]\tau'$

Above the line $\Delta\Gamma \vdash e : \forall(t.\tau')$ — H_1

$\Delta \vdash \tau\ \text{type}$ — H_2

By Inversion on H_1

$\Delta, t\ \text{type}\ \Gamma \vdash e : \tau'$ H_3

By IH on H_3 $\quad \Delta, t\ \text{type}\ \Gamma \vdash e : \tau'\ \text{type}$ H_4

By Substituting Lemma on H_4 & H_2

$\Delta, [\tau/t]\ \text{type}\ \Gamma \vdash [\tau/t]e : \underbrace{[\tau/t]\tau'\ \text{type}}$
type

back{ ; }{ }{ } open(x; t; y; e)

Dynamic Step

$$\frac{e \mapsto e'}{App\{\tau\}(e) \mapsto App\{\tau\}(e')}$$

$$\frac{}{App\{\tau\}(Lam(t.e)) \mapsto [\tau/t]e} \text{ val}$$

$$\frac{\Delta\ t\ type\ \Gamma \vdash e : \tau}{\Delta\Gamma \vdash Lam\{t.e\} : \forall(t.\tau)}$$

$$\frac{\Delta\Gamma \vdash e : \forall(t.\tau') \quad \Delta \vdash t\ type}{\Delta\Gamma \vdash App\{\tau\}(e) : [\tau/t]\tau'}$$

$\Delta\Gamma \vdash e : \tau$ Typing

$\Delta \vdash \tau$ type Type formation

$$\frac{\Delta, t\ type \vdash \tau\ type}{\Delta \vdash \forall(t.\tau)\ type}$$

$$\frac{\Delta \vdash \tau_1\ type \quad \Delta \vdash \tau_2\ type}{\Delta \vdash arr(\tau_1; \tau_2)\ type}$$

Substitution Lemma

(i) Type Judgement → If $\Delta, t\ type\ \Gamma \vdash \tau'$ type & $\Delta \vdash \tau$ type
then $\Delta, [J/t] \vdash [\tau/t]\tau'$ type.

(ii) Mixed Judgement → $\Delta, t\ type\ \Gamma \vdash e : \tau'$ & $\Delta \vdash \tau$ type
then $\Delta, [\tau/t]\Gamma \vdash [\tau/t]e : [\tau/t]\tau'$ [exp]

(iii) Typing sub, $\Delta\Gamma, x : \tau \vdash e' : \tau'$ & $\Delta\Gamma \vdash e : \tau$
then $\Delta\Gamma \vdash [e/x]e' : \tau'$

Preservation $e : \tau$ & $e \mapsto e'$ then $e' : \tau$

Case $e \mapsto e'$

$App\{\tau\}(e) \mapsto App\{\tau\}(e')$

$e = App\{\tau\}(e)$
$e' = App\{\tau\}(e')$

Above the line

$e \mapsto e'$ — (H1)

By Inversion on $e : \tau$

$\Delta \Gamma \vdash e : \forall(t.\tau')$ — (H2)

$\Delta \vdash \tau$ type — (H3)

By IH on H1 & H2

$\Delta \Gamma \vdash e' : \forall(t.\tau')$ (H4)

~~By App Rule~~

$$R_1 \rightarrow \frac{\Delta \Gamma \vdash e : \forall(t.\tau') \quad \Delta \vdash \tau \text{ type}}{\Delta \Gamma \vdash App\{\tau\}(e) : [\tau/t]\tau'}$$

take e' as $[\tau/t]\tau'$

By App rule R_1

$\Delta \Gamma \vdash App\{\tau\}(e') : [\tau/t]\tau'$

$App\{\tau\}(Lam(t.e)) \mapsto [\tau/t]e$

$e = App\{\tau\}(Lam(t.e))$
$e' = [\tau/t]e$

By Inversion on $e : \tau$

$\Delta \Gamma \vdash Lam(t.e) : \forall(t.\tau')$ — (H1)

$\Delta \vdash \tau$ type — (H2)

By Inversion on H1

$\Delta, t \text{ type}\, \Gamma \vdash e : \tau'$ — H3

By Substitution Lemma on H2 & H3

$\Delta, [\tau/t]\Gamma \vdash [\tau/t]e : [\tau/t]\tau'$

By Strengthing Lemma

$[\tau/t]e : [\tau/t]\tau'$

$e' : [\tau/t]\tau'$

By App rule

$App\{\tau\}(e) : [\tau/t]\tau'$

~~take e' as τ~~

$(\forall t. P(t) \rightarrow O(t)) \rightarrow ((\forall t. P(t)) \rightarrow \forall t. O(t))$

$lam\{\forall t. P(t) \rightarrow O(t)\}(x. lam\{\forall t. P(t)\}(y. Lam(t. \overset{O(t)}{e})))$

$e = ap(App\{t\}(x); App\{t\}(y))$

back ... obsn(x; ...

Progress $e:\tau$ then $\exists e', e \mapsto e'$ or e val

Case $\dfrac{\Delta, t\ \text{type}\ \Gamma \vdash e : \tau}{\Delta\ \Gamma \vdash \text{Lam}(t.e) : \forall(t.\tau)}$

$e = \Delta\Gamma \vdash \text{Lam}(t.e)$
$\tau = \forall(t.\tau)$

Above the line
$\Delta, t\ \text{type}\ \Gamma \vdash e : \tau$ — (H1)

By IH on (H1)

case $e \mapsto e'$
Not possible

case e val
$\dfrac{\text{By Lam rule}}{\text{Lam}(t.e)\ \text{val}}$

Case: $\dfrac{\Delta\Gamma \vdash e : \forall(t.\tau') \quad \Delta \vdash \tau\ \text{type}}{\Delta\Gamma \vdash \text{App}\{\tau\}(e) : [\tau/t]\tau'}$

$e : \Delta\Gamma \vdash \text{App}\{\tau\}(e)$
$\tau = [\tau/t]\tau'$
Above the line
$\Delta\Gamma \vdash e : \forall(t.\tau')$ — (H1)
$\Delta \vdash \tau\ \text{type}$ — (H2)

By IH on H1
case $e \mapsto e'$
take e' as $\text{App}\{\tau\}(e')$
By rule $e \mapsto e'$

Case e val:
By CFL e val, $e : \forall(t.\tau')$
the $e = \text{Lam}(t.e')$

take e' as $[\tau/t]e'$
then By $\text{App}\{\tau\}(\text{Lam}(t.e')) \mapsto [\tau/t]e'$
$e \mapsto e'$
e is val

~~Regularity~~

Case: $\dfrac{\Delta, \Gamma \vdash x : \tau_1 ; e : \tau_2 \quad \Delta \vdash \tau_1\ \text{type}}{\Delta\Gamma \vdash \text{lam}\{\tau\}(x.e) : \text{arr}(\tau_1;\tau_2)}$

$e = \Delta\Gamma \vdash \text{lam}\{\tau\}(x.e)$
$\tau = \text{arr}(\tau_1;\tau_2)$
Above the line
$\Delta\Gamma, x : \tau_1 \vdash e : \tau_2$ (H1)
$\Delta \vdash \tau_1$ type (H2)

$\dfrac{\tau_1\ \text{type} \quad \tau_2\ \text{type}}{\Delta \vdash \text{arr}(\tau_1 \tau_2)\ \text{type}}$

By Strengthing on H1
$\Delta\Gamma \vdash e : \tau_2$ (H3)

By IH on H3
$\Delta\Gamma \vdash e : \tau_2\ \text{type}$

~~$\Delta \vdash \tau_1$ type $\Delta \vdash \tau_2$ type~~
$\Delta \vdash \text{arr}(\tau_1; \tau_2)$ type

Case $\dfrac{\Delta\Gamma \vdash e_1 : arr(\tau_2;\tau) \quad \Delta\Gamma \vdash e_2 : \tau_2}{\Delta\Gamma \vdash app(e_1;e_2) : \tau}$

$e = \Delta\Gamma \vdash app(e_1;e_2)$
$\tau = \tau$

Above the line
$\Delta\Gamma \vdash e_1 : arr(\tau_2 : \tau)$ - H_1
$\Delta\Gamma \vdash e_2 : \tau_2$ - H_2

By Inversion on H_1
$\Delta\Gamma \vdash \tau_2$ type - H_3
$\boxed{\Delta \vdash \tau \text{ type}}$ - H_4
- proved

Case $\dfrac{\Delta\Gamma \vdash e : \forall(t.\tau') \quad \Delta \vdash \tau \text{ type}}{\Delta\Gamma \vdash App\{\tau\}(e) : [\tau/t]\tau'}$

$e = \Delta\Gamma \vdash App\{\tau\}(e)$
$\tau = [\tau/t]\tau'$

Above the line
$\Delta\Gamma \vdash e : \forall(t.\tau')$ - H_1
$\Delta \vdash \tau$ type - H_2

By Inversion on H_1

$\Delta\Gamma, t \text{ type} \vdash e : \tau'$ - H_3

~~By substitution lemma~~
By IH on $H3$
$\Delta\Gamma,$ ~~$\{[\tau/t]\}$~~
$\Delta, t \text{ type } \Gamma \vdash \tau' \text{ type}$ - H_4
$e : \underbrace{[\tau/t]\tau'}$

By substitution lemma
$\Delta, [\tau/t] \text{ type } \Gamma \vdash \underbrace{[\tau/t]\tau' \text{ type}}$
proved

Case $\dfrac{\Delta, t \text{ type } \Gamma \vdash e : \tau}{\Delta\Gamma \vdash Lam(t.e) : \forall(t.\tau)}$

$e = \Delta\Gamma \vdash Lam(t.e)$
$\tau = \forall(t.\tau)$

To prove
$\boxed{\forall(t.\tau) \text{ type}}$

~~By inversion on~~. Above the line
$\Delta, t \text{ type } \Gamma \vdash e : \tau$ - H_1

By IH on H_1
$\Delta, t \text{ type}$ ~~Γ~~ $\vdash \tau$ type

By rule $\dfrac{\Delta, t \text{ type} \vdash \tau \text{ type}}{\Delta \vdash \forall(t.\tau) \text{ type}}$ Done

$$\frac{\Delta, t\ \text{type} \vdash \tau\ \text{type}}{\Delta \vdash \exists(t.\tau)\ \text{type}}$$

$$\frac{\Delta \vdash \rho\ \text{type} \quad \Delta, t\ \text{type} \vdash \tau\ \text{type} \quad \Delta\,\Gamma \vdash e : [\rho/t]\tau}{\Delta\,\Gamma \vdash \text{pack}\{t.\tau\}\{\rho\}(e) : \exists(t.\tau)}$$

$$\frac{\Delta\,\Gamma \vdash e_1 : \exists(t.\tau) \quad \Delta, t\ \text{type}\ \Gamma, x : \tau \vdash e_2 : \tau_2 \quad \Delta \vdash \tau_2\ \text{type}}{\Delta\,\Gamma \vdash \text{open}\{t.\tau\}\{\tau_2\}(e_1; t, x.e_2) : \tau_2}$$

Dynamics

$$\frac{e\ \text{val}}{\text{pack}\{t.\tau\}\{\rho\}(e)\ \text{val}}$$

$$\frac{e \mapsto e'}{\text{pack}\{t.\tau\}\{\rho\}(e) \longmapsto \text{pack}\{t.\tau\}\{\rho\}(e')}$$

$$\frac{e_1 \mapsto e_1'}{\text{open}\{t.\tau\}\{\tau_2\}(e_1; t, x.e_2) \mapsto \text{open}\{t.\tau\}\{\tau_2\}(e_1'; t, x.e_2)}$$

$$\frac{e\ \text{val}}{\text{open}\{t.\tau\}\{\tau_2\}(\text{pack}\{t.\tau\}\{\rho\}(e); t, x.e_2) \mapsto [\rho, e/t, x]e_2}$$

Open-Pack

$\exists P(t)\ \Pi \times B$

Preservation $e : \tau$ & $\exists e',\ e \mapsto e'$ then $e' : \tau$

Case:

$$\frac{e \mapsto e'}{\text{pack}\{t.\tau\}\{\rho\}(e) \mapsto \text{pack}\{t.\tau\}\{\rho\}(e')}$$

$e = \text{pack}\{t.\tau\}\{\rho\}(e)$

$e' = \text{pack}\{t.\tau\}\{\rho\}(e')$

Above the line

$e \mapsto e'$ — (H1)

By Inversion on $e : \tau$

$\Delta \vdash \rho\ type$ — (H2)

$\Delta, t\ type \vdash \tau\ type$ — (H3)

$\Delta\,\Gamma \vdash e : [\rho/t]\tau$ — (H4)

By IH on H1 & H4

$\Delta\,\Gamma \vdash e' : [\rho/t]\tau$ — H5

By using Pack Typing rule on H2, H3 & H5

$\text{pack}\{t.\tau\}\{\rho\}(e') : \exists(t.\tau)$

$\Delta\,\Gamma \vdash e' : [\rho/t]\tau$

$$\frac{\Delta \vdash \rho\ typ \quad \Delta, t\ typ \vdash \tau\ typ}{\text{pack}\{t.\tau\}\{\rho\}(e') : \exists(t.\tau)}$$

$e' = [\rho/t]\tau$

$$\frac{\Delta, t\ typ \vdash \tau\ typ}{\Delta \vdash \exists(t.\tau)}$$

Case

$$\frac{e_1 \mapsto e_1'}{\text{open}\{t.\tau\}\{\tau_2\}(e_1; t, x.e_2) \mapsto \text{open}\{t.\tau\}\{\tau_2\}(e_1'; t, x.e_2)}$$

$e = \text{open}\{t.\tau\}\{\tau_2\}(e_1; t, x.e_2)$

$e' = \text{open}\{t.\tau\}\{\tau_2\}(e_1'; t, x.e_2)$

Above the line

$e_1 \mapsto e_1'$

By Inversion on $e : \tau$

$\Delta\,\Gamma \vdash e_1 : \exists(t.\tau)$ — (H1)

$\Delta, t\ type\ \Gamma, x : \tau \vdash e_2 : \tau_2$ — (H2)

$\Delta \vdash \tau_2\ type$ — H3

We need to prove $e' : \tau_2$

$$\frac{\Delta\,\Gamma \vdash \exists(t.\tau) \quad \Delta, t\ typ\ \Gamma, x : \tau \vdash e_2 : \tau_2}{\Delta\,\Gamma \vdash \text{open}\{t.\tau\}\{\tau_2\}(e_1'; t, x.e_2)}$$

pack { ; } { } { } open(x; t;

By IH on H_1

$e_1' : \exists(t.\tau)$

$\therefore$ By rule of open $\text{open}\{t.\tau\}\{\rho\}(e_1'; t, x.e_2) : \tau_2$

Case

$$\frac{e\ \text{val}}{\text{open}\{t.\tau\}\{\rho\}(\text{pack}\{t.\tau\}\{\rho\}(e); t, x.e_2) \mapsto [\rho, e/t, x]e_2}$$

$e = \text{open}\{t.\tau\}\{\rho\}(\text{pack}\{t.\tau\}\{\rho\}(e); t, x.e_2)$

$e' = [\rho, e/t, x]e_2$; $\boxed{e_2 : \tau_2}$

Above the line

e val $\to H_1$

By Inversion on $e:\tau$

$\Delta\,\Gamma \vdash \text{pack}\{t.\tau\}\{\rho\}(e) : \exists(t.\tau)$ — H_2

$\Delta, t\ \text{type}\ \Gamma, x:\tau \vdash e_2 : \tau_2$ — H_3

$\Delta \vdash \tau\ \text{type} \to H_4$

By Inversion on H_2

$\Delta \vdash \rho\ \text{type}$ — H_5

$\Delta, t\ \text{type} \vdash \tau\ \text{type}$ — H_6

$\Delta, \Gamma \vdash e : [\rho/t]\tau$ — H_7

$e^2\ \rho/t$

$[e/x]e_2$

By substituting Lemma (2) H_7 & H_3

$x : [\rho/t]\tau \vdash [\rho/t]e_2 : [\rho/t]\tau_2$

$[\rho, e/t, x]e_2 = \tau_2$

$e : \tau_2$

$e \quad 2 \quad s(e). \; [e, x^{e_1}/[x, y] e_2$

Preservation $\quad [\ell, \_, x, -] e_2$

Case $\quad open\{t.\tau\}\{\ell\}(e_1; t, x.e_2) \mapsto op_n$

typing rule

$$\frac{\Delta\Gamma \vdash e_1 : \exists(t.\tau) \quad \Delta, t\ type\ \Gamma, x:\tau \vdash e_2 : \tau_2}{open\{t.\tau\}\{\tau_2\}(e_1; t, x.e_2) : \tau_2}$$

$\rightarrow$ Destructor

$$\frac{\Delta, t\ type \vdash \tau\ type}{\Delta \vdash \exists(t.\tau)\ type}$$

$$\frac{\Delta, t\ type \vdash \tau\ type}{\Delta \vdash \forall(t.\tau)\ type}$$

$$\frac{\Delta \vdash \rho\ type \quad \Delta, t\ type : \tau\ type \quad \Delta\Gamma \vdash e : [\rho/t]\tau}{\Delta\Gamma \vdash pack\{t.\tau\}\{\rho\}(e) : \exists(t.\tau)}$$

Constructor

Progress

Case : $\dfrac{\Delta\Gamma \vdash e_1 : \exists(t.\tau) \quad \Delta, t\ type \vdash \tau\ type \quad \Delta, \Gamma \vdash x:\tau \vdash e_2 : \tau_2}{\Delta\Gamma \vdash open\{t.\tau\}\{\tau_2\}(e_1; t, x.e_2) : \tau_2}$

$e : \Delta\Gamma \vdash open\{t.\tau\}\{\tau_2\}(e_1; t, x.e_2)$

$\tau : \tau_2$

Above the line

$\Delta\Gamma \vdash e_1 : \exists(t.\tau) \quad - H_1$

$\Delta, t\ type \vdash \tau\ type \quad - H_2$

$\Delta, \Gamma \vdash x:\tau \vdash e_2 : \tau \quad - H_3$

By IH on e_1 (H_1)

Case $e_1 \mapsto e'$

take e' as $open\{t.\tau\}\{\tau_2\}(e_1'; t, x.e_2)$

then $e_1 \mapsto e_1'$

Case e val

$e_1 : \exists(t.\tau)$ & e_1 val then By CFL

$e_1 = pack\{t.\tau\}\{\rho\}(e)$

take e' as $[\rho, e/t, x]e_2 \qquad e_1 \mapsto e_1'$

e val

Case: $\dfrac{\Delta \vdash \rho \text{ type} \quad \Delta, t \text{ type} \vdash \tau \text{ type} \quad \Delta, \Gamma \vdash e : [\rho/t]\tau}{\text{pack}\{t.\tau\}\{\rho\}(e) : \exists(t.\tau)}$

e

$\tau = \exists(t.\tau)$

Above the line

$\Delta \vdash \rho$ type — H_1

Δ, t type $\vdash \tau$ type — H_2

$\Delta, \Gamma \vdash e : [\rho/t]\tau$ — H_3

$\therefore$ By IH on H_3

Case: $e \mapsto e'$

take e' as pack$\{t.\tau\}\{\rho\}(e')$

By rule

Case e val

pack rule

Regularity

$\Delta\Gamma \vdash e : \tau$ then Γ is valid or τ type

Case: $\dfrac{\cancel{\Delta \vdash \rho \text{ type}} \quad \Delta\Gamma \vdash e_1 : \exists(t.\tau) \quad \cancel{\Delta, t \text{ type} \vdash \tau \text{ type}} \quad \Delta, \Gamma \vdash x : \tau \vdash e_2 : \tau_2}{\Delta\Gamma \vdash \text{open}\{t.\tau\}\{\tau_2\}(e_1; t, x.e_2) : \tau_2}$

$e =$

$\tau = \tau_2$

Above the line

$\Delta\Gamma \vdash e_1 : \exists(t.\tau)$ — H_1

$\cancel{\Delta \vdash \text{type} \vdash \tau \text{ type}}$ — H_2

$\Delta, \Gamma \vdash x : \tau \vdash e_2 : \tau_2$ — H_3

Δ, t type $\Gamma, x : \tau \vdash e_2 : \tau_2$

$\boxed{\Delta \vdash \tau_2 \text{ type}}$ ✓

By IH on H_3

$\cancel{\Delta\Gamma e_2 : \tau_2 \text{ type}}$

$[s, e/t, x]$

$$\text{Case } \frac{\Delta \vdash s \text{ type} \quad \Delta, t \text{ type} \vdash \tau \text{ type} \quad \Delta\Gamma \vdash e : [s/t]\tau}{\text{pack}\{t.\tau\}\{s\}(e) : \exists(t.\tau)}$$

e

$\tau = \exists(t.\tau)$

Above the line

$\Delta \vdash s$ type — H_1

Δ, t type $\vdash \tau$ type — H_2

$\Delta\Gamma \vdash e : [s/t]\tau$

By rule $\dfrac{\Delta, t \text{ type} \vdash \tau \text{ type}}{\Delta \vdash \exists(t.\tau) \text{ type}}$

$$\frac{t \text{ type} \vdash \tau \text{ type}}{\forall(t.\tau) \text{ type}}$$

Typing rules

Progress

$$\text{Case rec } \frac{\Gamma \vdash e_0 : \tau \quad \Gamma \vdash x : nat, \quad \Gamma \vdash e : nat, \quad \Gamma, y : \tau \vdash e ; \tau}{\Gamma \vdash \text{rec}\{e_0; x; y.e_1\}(e) : \tau}$$

e

$\tau = \tau$

$\Gamma, x : nat, y : \tau \vdash e_1 : \tau$

Above the line

$\Gamma \vdash e_0 : \tau$ — H_1

$\Gamma \vdash e : nat$ — H_2

$\Gamma, x : nat, y : \tau \vdash e_1 : \tau$ — H_3

By CFL on H_2 $\Gamma \vdash e : nat$ & e val $\quad e = z$

or $e = s(z)$

By IH on H_2

$e \mapsto e'$

take e' as $\text{rec}\{e_0; x, y.e_1\}(e')$

$e \mapsto e'$

e val

Case, $e = z$ take e' as e_0

By rule

Case $e = s(z)$ take e' as $[e, \text{rec}\{e_0; x, y.e_1\}e/x, y]e_1$

pack {; }{ open(x; t; y; ...

(2) $(\exists t.P(t)) \wedge O(t) \to (\exists t.P(t)) \wedge (\exists t.O(t))$

$\text{lam}\{(\exists t.P(t)) \wedge O(t)\}(x.\ \overset{\text{open}(x;t,y.}{}\ \text{pair}(e_1; e_2)));$

$e_1 = \cancel{\text{fst}}\ \ \cancel{\text{fst}(x)}\ \text{pack}\{\exists t.P(t)\}\{t\}(\text{fst}(y));$

$e_2 = \text{pack}\{\exists t.P(t)\}\{t\}(\text{snd}(y));$

(3) $(\forall t.P(t) \to O(t)) \to ((\forall t.P(t)) \to (\forall t.O(t)))$

$\text{lam}\{\forall t.P(t) \to O(t)\}(x.\ \text{Lam}(t.\ \text{lam}\{P(t)\}(y.\ e)))$

$e = \text{ap}(x; \text{App}\{t\}(y));$

(4) $(\forall.P(t) \to O(t)) \to ((\exists t.P(t)) \to (\exists t.O(t))$

$\text{lam}\{(\forall.P(t) \to O(t))\}(x.\ \text{lam}\{\exists t.P(t)\}(y.\ \text{open}(y; t, a.\ e)$

$e = \text{pack}\{\exists t.O(t)\}\{t\}(\text{ap}(x; \text{App}\{t\}(a)))$

(5) $(\exists t.\forall u.S(t,u)) \to (\forall u.\exists t.S(t,u))$

$\text{lam}\{\exists t.\forall u.S(t,u)\}(x.\ \text{Lam}(u.\ \text{open}(x; t, y.\ \text{pack}\{\exists t, S(t,u)\}\{t\}(y$

$$\frac{\Gamma \vdash e_0 : \text{sum}(\tau_1;\tau_2) \quad \Gamma, x:\tau \vdash e_1 : \tau_1 \quad \Gamma, y:\tau \vdash e_2 : \tau_2}{\Gamma \vdash \text{case}(e_0; x.e_1; y.e_2) : \tau} \qquad \frac{\Gamma \vdash e : \text{nat}}{\Gamma \vdash S(e) : \text{nat}}$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \text{inl}\{\tau_1;\tau_2\}(e) : \tau_1 + \tau_2} \quad \Bigg| \quad \frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \text{inr}\{\tau_1;\tau_2\} : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e : \text{void}}{\Gamma \vdash \text{Abort}\{\tau\}(e) : \tau} \quad \Bigg| \quad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \text{pair}(e_1; e_2) : \tau_1 \times \tau_2} \quad \Bigg| \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \text{fst}(e) : \tau_1}$$

$$\frac{\Delta \vdash \tau_1 \text{ type} \quad \Delta \vdash \tau_2 \text{ type}}{\Delta \vdash arr(\tau_1;\tau_2) \text{ type}}$$

$(\tau_2) - nat \to nat \; (5)$

$$\frac{\Delta \vdash t \text{ type} \vdash \tau \text{ type}}{\Delta \vdash \forall(t.\tau) \text{ type} \quad or \quad \Delta \vdash \exists(t.\tau) \text{ type}}$$

$$\frac{\Delta \vdash \tau_1 \text{ type} \quad \Delta, \Gamma \vdash x : \tau_1 \vdash e : \tau_2}{\Delta\Gamma \vdash lam\{\tau_1\}(x.e) : \to(\tau_1, \tau_2)}$$

$$\frac{\Delta\Gamma \vdash e_1 : arr(\tau_2;\tau_2) \quad \Delta\Gamma \vdash e_2 : \tau_2}{\Delta\Gamma \vdash app(e_1;e_2) : \tau_2}$$

$$\frac{\Delta\Gamma \vdash e_0 : \exists(t.\tau') \quad \Delta, t \text{ type } \Gamma \vdash x : \tau \vdash e_1 : \tau_2 \quad , \Delta \vdash \tau_2 \text{ type}}{open\{t.\tau\}\{\tau_2\}(e_0; t; x.e_1) : \tau_2}$$ Destructor

$$\frac{\Delta\Gamma \vdash e_0 : \exists(t.\tau) \quad \Delta, t \text{ type } \Gamma \vdash x : \tau \vdash e_1 : \tau_2 \quad \Delta \vdash \tau_2 \text{ type}}{open\{t.\tau\}\{\tau_2\}(e_0; t, x.e_1) :}$$

$$\frac{\Delta \vdash \rho \text{ type} \quad \Delta, t \text{ type} \vdash \tau \text{ type} \quad \Delta\Gamma \vdash e : [\rho/t]\tau}{pack\{t.\tau\}\{\rho\}(e) : \exists(t.\tau)}$$

$$\frac{}{App\{\tau\}(Lam(t.e)) \longmapsto [\tau/t]e}$$

$$\frac{e \mapsto e'}{App\{\tau\}(e) \longmapsto App\{\tau\}(e')}$$

$$\frac{e_1 \mapsto e_1'}{app(e_1;e_2) \mapsto app(e_1';e_2)}$$

$$\frac{e_1 \text{ val} \quad e_2 \mapsto e_2'}{app(e_1;e_2) \mapsto app(e_1;e_2')}$$

$$\frac{e_2 \text{ val}}{ap(lam\{\tau\}(x.e); e_2) \mapsto [e_2/x]e}$$

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{lam\{\tau\}(x.e)}$$

list $\to$ nat

if e_2 = list then nat

CFL Lemma

(i) If $e : \text{nat}$ the e val then $e = Z$ or $e = S(e')$

(ii) If $e : \to(\tau_1;\tau_2)$ & e val the $e = \text{lam}\{\tau\}(x.e')$ for some e'

(iii) If $e : (\tau_1 + \tau_2)$ & e val then $e = \text{inl}\{\tau_1;\tau_2\}(e)$ $\vee$ $\text{inr}\{\tau_1;\tau_2\}(e')$

(iv) If $e : \exists(t.\tau)$ & e val then $e = \text{pack}\{t.\tau\}\{t\}(e)$

(v) If $e : \forall(t.\tau)$ & e val then $e = \text{Lam}(t.e)$

Substitution Lemma $\quad [e/x]e_1$

If $\Gamma \vdash e : \tau$ & $\Gamma, x:\tau \vdash e_1 : \tau_1$ then $\Gamma \vdash [e/x]e_1 : \tau_1$

If $\Delta, t \text{ type} \vdash \tau' \text{ type}$ & $\Delta \vdash \tau \text{ type}$ then $\Delta \vdash [\tau/t]\tau' \text{ type}$

If $\Delta, t \text{ type}, \Gamma \vdash e' : \tau'$ & $\Delta \vdash \tau \text{ type}$ then $\Delta, [\tau/t]\Gamma \vdash [\tau/t]e' : [\tau/t]\tau'$

$\Delta\,\Gamma, x:\tau \vdash e' : \tau'$ & $\Delta\,\Gamma \vdash e : \tau$ then $\Delta\,\Gamma \vdash [e/x]e' : \tau'$

Substi

$\Gamma \vdash e : \tau$ & $\Gamma, x : \tau \vdash e' : \tau$ then $[e/x]e'$

$\Delta, t \text{ type} \vdash \tau' \text{ type}$ & τ type then $\Delta \vdash [\tau/t]\tau'$ type

$\Delta, t \text{ type}\ \Gamma \vdash e' : \tau'$ & τ type then $\Delta, [\tau/t]\Gamma \vdash [\tau/t]e' : [\tau/t]\tau'$

$\Delta\,\Gamma, x:\tau \vdash e' : \tau'$ & $\Delta\,\Gamma \vdash e : \tau$ then $\Delta\,\Gamma, [e/x]e' : \tau'$

(i) Blanky
(ii) Donughts
(iii) Shirt > Flower
(iv) Flower

Preservation $e:\tau$ & $e \mapsto e'$, then $e':\tau$

Case: $e_2 \mapsto e_2'$

$rec\{t.\tau\}(x.e_1; e_2) \mapsto rec\{t.\tau\}(x.e_1; e_2')$

$e = rec\{t.\tau\}(x.e_1; e_2)$

$e' = rec\{t.\tau\}(x.e_1; e_2')$

$e_2 \mapsto e_2'$ (Given) (H1)

By Inversion on $e:\tau$

$\Gamma, x:[\tau'/t]\tau \vdash e_1 : \tau'$ — H2

$\Gamma \vdash e_2 : ind(t.\tau)$ — H3

By IH on H1 & H3

$\Gamma \vdash e_2' : ind(t.\tau)$ — H4

By rec rule use H2 & H4

$rec\{t.\tau\}(x.e_1; e_2') : \tau'$ proved

Case: $e \mapsto e'$

$unfold\{t.\tau\}(e) \mapsto unfold\{t.\tau\}e'$

$e : unfold\{t.\tau\}(e)$

$e' : unfold\{t.\tau\}(e')$

$e \mapsto e'$ (Given) (H1)

By Inversion on $e:\tau$

$\Gamma \vdash e : coi(t.\tau)$ — H2

By IH on H1 & H2

$\Gamma \vdash e' : coi(t.\tau)$ (H3)

Using unfold rule

$$\frac{\Gamma \vdash e : coi(t.\tau)}{\Gamma \vdash unfold\{t.\tau\}(e) : [}$$

$\Gamma \vdash unfold\{t.\tau\}(e') : [coi(t.\tau)/t]\tau$

proved

back{ ; }{ }{ } $case(x; t; y; e)$

$\Gamma \vdash e_1 : \tau_1$ & $\Gamma, x : \tau_1 \vdash e' : \tau$

$[e/x]e' : \tau$ $x = (a \to b)$ $ap(x; a) = b$

(i) $(P \to (Q \to R)) \to (P \times Q \to R)$

$lam\{(P \to (Q \to R))\}(x \cdot lam\{P \times Q\}(y \cdot ap(ap(x; fst(y)); snd(y))))$

(ii) $[(P + Q) \to R] \to (P \to R) \times (Q \to R)$

$lam\{P+Q\}(x \cdot lam\{R\}(y \cdot pair(e_1; e_2)))$

Preservation: $e : \tau$ & $\exists e', e \mapsto e'$ then $e' : \tau$

$e = split(join(e_1; e_2); x_1 \cdot x_2 \cdot e) : \tau$ $\dfrac{e_1 \, val \quad e_2 \, val}{split(join(e_1; e_2); x_1 \cdot x_2 \cdot e) \mapsto [e_1/x_1][e_2/x_2]e}$

$e' = [e_1/x][e_2/x]e : \tau$ [To be prove]

Above the line
e_1 val — H_1
e_2 val — H_2

By Inversion on $e : \tau$

$\cdot \Gamma \vdash e_0 : \tau_1 \times \tau_2 \; = \; \Gamma \vdash join(e_1; e_2) : \tau_1 \times \tau_2$

$\Gamma, x_1 : \tau_1$
$\Gamma, x_2 : \tau_2 \vdash \boxed{e : \tau}$

$\Gamma \vdash e_1 : \tau_1$ — H_3
$\Gamma \vdash e_2 : \tau_2$ — H_4

$\Gamma, e : \tau$

$\Gamma, x_1 : \tau_1, x : \tau_2 \vdash e : \tau$ — H_5

By substitution lemm H_3 & H_5

$\Gamma, [e_1/x_1], x_2 : \tau_2 \vdash e : \tau$ H_6

$\Gamma, [e_1/x_1][e_2/x_2] \vdash e : \tau$ By Substitution H_4 & H_6
$e' : \tau$

$$\frac{\Gamma \vdash e_0 : \tau_1 \otimes \tau_2 \quad \Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash e : \tau}{}$$

$\Gamma \vdash a : \text{int}$

$\Gamma \vdash b : \text{int} \vdash c : \text{float}$

$\Gamma, [a/b]c : \text{float}$

Progress $\Gamma \vdash \text{split}(e_0; x_1 . x_2 . e) : \underbrace{\tau}$

$e = \Gamma \vdash \text{split}(e_0; x_1 . x_2 . e)$

$\tau = \tau$

$e : \tau$ then either $\exists e', e \mapsto e'$ or e val

Above the line

$\Gamma \vdash e_0 : \tau_1 \otimes \tau_2$ - H_1

$\Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash \underbrace{e : \tau}$ - H_2

Strengthening Lemma

$\Gamma, x_2 : \tau_2 \vdash e : \tau$

Strengthening Lemma

$\Gamma \vdash e : \tau$

By IH on H_1

e_0 val

$\boxed{e_0 \text{ val}}$

$\Gamma \vdash e_0 : \underbrace{\tau_1 \otimes \tau_2}$

By CFL

If e_0 val & $e_0 : \tau_1 \otimes \tau_2$

$e_0 = \text{Join}(e_1; e_2)$

$$\frac{e_1 \text{ val} \quad e_2 \text{ val}}{\text{join}(e_1; e_2) \text{ val}}$$

$\left.\begin{matrix} e_1 \text{ val} \\ e_2 \text{ val} \end{matrix}\right] = \text{join}(e_1; e_2) \text{ val}$

~~take~~ $e' = \underbrace{\cancel{[e_1/x_1][e_2/x_2]e}}$

$e_0 = \text{join}(e_1; e_2)$ val

e_0 val

$\cancel{\dfrac{e_0 \mapsto e_0'}{\text{split}(e_0; x_1 . x_2 . e) \mapsto \text{split}(e_0'; x_1 . x_2 . e)}}$

$e_0 \mapsto \underbrace{e_0'}$

take e' as $\underbrace{\text{split}(e_0'; x_1 . x_2 . e')}$

then $e_0 \mapsto e_0'$

back $\{ ; \}$ ... open$(x; t; y, ...)$

Progress: $e : \tau$ then either e val or $\exists e', e \mapsto e'$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash join(e_1; e_2) : \tau_1 \otimes \tau_2}$$

$$\frac{e_1 \; val}{fst(e_1; e_2) \mapsto e_1}$$

$e = join(e_1; e_2)$

$\tau : \tau_1 \otimes \tau_2$

Above the line

$\Gamma \vdash e_1 : \tau_1 \quad - H_1$

$\Gamma \vdash e_2 : \tau_2 \quad - H_2$

By IH on H_1

case $e_1 \mapsto e_1'$

take e' as $join(e_1'; e_2)$

then By Rule

$e_1 \mapsto e_1'$

$$\frac{e_1 \mapsto e_1'}{join(e_1; e_2) \mapsto join(e_1'; e_2)}$$

case e_1 val

Take IH on H_2

case $e_2 \mapsto e_2'$

take e' as $join(e_1; e_2')$

then By rule

$e_2 \mapsto e_2'$

$\boxed{e_1 \; val}$ ✓

$$\frac{e_1 \; val \quad e_2 \mapsto e_2'}{join(e_1; e_2) \mapsto join(e_1; e_2')}$$

case e_2 val

we have e_1 val e_2 val

then $join(e_1; e_2)$ val

$$\frac{e_1 \; val \quad e_2 \; val}{join(e_1; e_2) \; val}$$ ✓

Case: $\dfrac{\Gamma \vdash e_0 : \tau_1 \otimes \tau_2 \quad \Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash \tau}{\Gamma \vdash split(e_0; x_1 . x_2 . e) : \tau}$

$e = split(e_0; x_1 . x_2 . e)$
$\tau = \tau$

Above the line
$\Gamma \vdash e_0 = \tau_1 \otimes \tau_2$ — H_1
$\Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash e : \tau$ — H_2

By IH on claim H_1

Case: $e_0 \mapsto e_0'$
take e_0' as $split(e_0'; x_1 . x_2 . e)$
then By rule $\dfrac{e_0 \mapsto e_0'}{split(e_0; x_1 . x_2 . e) \mapsto split(e_0'; x_1 . x_2 . e)}$
$e \mapsto e'$
we get, $e_0 \mapsto e_0'$ (Above the line)

Case e_0 Val
If e_0 val & $e_0 : \tau_1 \otimes \tau_2$ then By CFL $e_0 = join(e_1 \otimes e_2)$
take e' as $[e_1 / x_1][e_2 / x_2]e$
$e \mapsto e'$ $\dfrac{e_1 \text{ Val} \quad e_2 \text{ Val}}{split(join(e_1 \otimes e_2); x_1 . x_2 . e) \to [e_1/x_1][e_2/x_2]e}$
and e_1 val & e_2 val

By join rule $\dfrac{e_1 \text{ val} \quad e_2 \text{ Val}}{join(e_1; e_2) \text{ Val}}$

$join(e_1; e_2)$ Val

$$\frac{}{Z\ \text{even}} \qquad \frac{}{S(Z)\ \text{odd}} \qquad \frac{n\ \text{even}}{S(S(n))\ \text{even}} \qquad \frac{n\ \text{odd}}{S(S(n))\ \text{odd}}$$

(i) $\forall n$, even $n \rightarrow$ odd $S(n)$
By Induction on even n

Case Z : even Z = ~~odd S(Z)~~
By IH
odd $S(Z)$
By Rule odd $S(Z)$ is odd

Case even $S(S(n))$
Above the line
n even — H_1
By IH on H_1
$S(n)$ is odd — H_2

$\therefore$ odd $S(n)$ = odd $S(S(S(n)))$ = $\boxed{\text{even}}$

(ii) $n + (m + l) = (n + m) + l$

By Induction on n

Case Z : $n = Z$
$Z + (m + l) = (Z + m) + l$
$m + l$ ✓
$\therefore$ It could be written as
$(Z + m) + l$

Case e: $n = S(n)$
$S(n) + (m + l)$
$S(n + (m + l))$
By IH
$S((n + m) + l)$
$(S(n) + m) + l$

Q :

$Z + m = m$ — A_1

$S(n) + m = S(n+m)$ — A_2

$p(n) + m = p(n+m)$ — A_3

$S(p(n)) = n$ — R_1

$p(S(n)) = n$ — R_2

$p(S(n)) + m$

$Z + p(m) = p(Z+m)$

$p(m) = p(m)$

Rule

$s(n+m)$

To prove $n + p(m) = p(n+m)$

By Induction on n

Case 1 $n = Z$

$Z + p(m) = p(Z+m)$

$p(m) = p(m)$

Proved

$S(n) + m$

By plus rule

Case $n = S(n)$

$S(n) + p(m) = \cancel{p(S(n)+m)}$

By A_2 $= S(n + p(m))$

$\cancel{S(n+p(m))}$ $\cancel{\text{By IH}}$

By IH

$\cancel{p(S(n)+m)}$ $S(p(n+m)) = n+m$

By R_1

$= n+m$

$p(S(n+m))$ By R_1 & R_2

plus rule

$p(S(n)+m)$

proved #

R.H.S $p(n+m)$

$n = S(n)$

$p(S(n)+m)$

$p(S(n+m))$

$= n+m$

Q

$\forall n \forall m \forall l \quad n + (m+l) = (n+m)+l$

By Induction on n

Case 1: $n = Z$

$Z + (m+l) = (Z+m)+l$

$m+l$

Case $n = S(n)$

$S(n) + (m+l)$

$S(n + (m+l))$ By IH on this

$S((n+m)+l)$

$\Rightarrow (S(n)+m)+l$

$S(S(n)+m)$

Q $\Rightarrow \forall n \quad n+Z = n \quad , \forall n \forall m \quad n + S(m) = S(n+m)$

Using $n+Z = n$ — (1)

By Induction on n

Case $n = Z$

$Z + Z = Z$

Case $n = S(n)$

$S(n) + Z = S(n)$ By Induction (1)

By Induction on $n + S(m)$

Case $n = Z$

$Z + S(m)$

$S(m) = S(Z+m)$

Case $n = S(n)$

$n + S(m)$

$S(n) + S(m)$

$S(n + S(m))$ — (11)

$S(S(n+m))$ By IH on (11)

$S(S(n) + m)$ Done

RHS

$S(S(n)+m)$

① $\forall n \ \forall m - n + m = m + n$

By Induction on n

Case $n = Z$

$Z + m$

m By Rule ①

$m + Z$

Proved

Case $n = S(n)$

$S(n) + m$

$S(n + m)$ — By IH

$S(m + n)$

$m + S(n)$ Done

RHS

$m + n$

$m + S(n)$

$n = S(n)$

① $= n \times (m \times l) = (n \times m) \times l$

By Induction on n

Case $n = Z$

$Z \times (m \times l) = Z$

R.HS $= (Z \times m) \times l = Z$

Case $n = S(n)$

$S(n) : n + 1$

$S(n) \times (m \times l)$

~~$S(n \times (m \times l))$~~ — By IH, we get

~~$S((n \times m) \times l)$~~

~~$(S(n) \times m) \times l$~~

$(n + 1) \times (m \cdot l)$

$n \times (m \times l)$ nm+

We also need to use add prop

$\underbrace{n \times (m \times l)} + m \times l$ → By IH we get

$\Rightarrow (n \times m) \times l + m \times l$

$(n \times m + m) l$

$(S(n) \cdot m) l$

$(S(n) = n + 1$

$(n + 1) m$

$(nm + m)$

$l + (m' \cdot l + n \cdot l)$

$(l + m' \cdot l) + n \cdot l$

$S(m) \cdot l + n \cdot l$

$(m+n) \cdot l = ml + n \cdot l$

$(S(m) + n) \cdot l = S(m) + nl$

$n \cdot m = m \cdot n$

By Induction on n

Case $n = Z$

$Z \cdot m = Z$

$m \cdot Z = Z$

Case $n = S(n)$

$S(n) \cdot m =$

sol.

Hence $S(n) = n' + 1$

$(n' + 1) \cdot m$

$\underbrace{n'm} + m$ — By IH on This

$mn' + m$

$m \cdot S(n)$

Q: # length

- $|nill| = Z$ — R_1
- $|cons(n; l)| = |l| + 1$ or $S|l|$ — $R2$

Concat $l_1 \cdot l_2$

$= nill \cdot l_2 = l_2$ — $R3$

$Cons(n; l_1) \cdot l_2 = Cons(n; l_1 \cdot l_2)$ — $R4$

rev (l)

- rev (nill) = nill — $R5$
- rev (Cons (n; l) = rev(l) + Cons (n; nill) — $R6$

Ques: $|l_1 \cdot l_2| = |l_1| + |l_2|$
By Induction on l_1

Case l_1 : nill
$nill \cdot l_2 = l_2$ By Rule 3
$= |nill| + |l_2|$

Case 2 l_1 : Cons (n; l_1)

$|Cons(n; l_1) \cdot l_2| = |Cons(n; l_1 \cdot l_2|$ By R4
$= S(|l_1| + |l_2|)$, By ~~R2~~ IH
$= S(|l_1| + |l_2|)$ By R2

$|Cons(n; l)| = S(|l|)$
$= |l| + 1$

$= S|l_1| + |l_2|$
Cons (n; l_1) + $|l_2|$

Ques: $|rev(l)| = |l|$
By Induction on l
Case l = nill
$|rev(nill)| = |(nill)|$

Case l : Cons (n; l)
rev (Cons (n; l))

|Cons(n; l)|
$|l|$ + Cons (n; nill)
$|l|$ + S(nill)
$|l|$ + Z+1
S(l) + Z
Cons(n; l) = S

rev(|l|) + Cons (n; nill)
By IH
S(Z)

Cons (n;

(13) $rev(l_1 ; l_2) = rev(l_2) \cdot rev(l_1)$

By Induction on l_1

Case $l_1 = nill$

$= rev(nill \cdot l_1)$

$rev(l_2)$

Also, $rev(l_2) \cdot rev(nill)$

Case $l_1 = cons(n ; l_1)$

$rev(cons(n ; l_1) ; l_2)$

$= rev(cons(n ; l_1 ; l_2))$ By Concat $cons(n ; l_1) \cdot l_2 = cons(n ; l_1 \cdot l_2)$

$rev(l_1 \cdot l_2) + cons(n ; nill)$

By IH

$rev(l_2) \cdot rev(l_1) + cons(n ; nill)$

$= rev(l_2) \cdot rev(l_1) + (nill + 1)$ Rev(cons(n ; l)) $= rev(l) + cons(n ; nill)$

$(rev(l_2) \cdot rev(l_1) + 1) + nill$

$rev(l_2) \cdot rev(l_1) + 1$

$rev(l_2) \cdot rev(l_1 + 1)$ $|cons(n ; l)| = |l| + 1$

$rev(l_2) \cdot rev(cons(n ; l_1))$

rev (l) = l

(13) rev(rev(l)) = l

By Induction on l

Case 1 : l = nill

rev (rev (nill))

= rev(nill)

= nill

Case 2 : l = Cons (n; l)

rev (rev(Cons(n; l)))

rev (rev(l) + Cons (n; nill))

Cons (n; nill) : rev(rev(nill))

rev (rev (l) + rev (rev (nill)))

By IH

l + Cons (n; nill)

l + nill + 1

Cons (n; nill) = nill + 1

S(l) + nill

S(l) = Cons (n; l)

[1, 2] [ ]

[1, 2]

rev (rev (Cons (n; l)))

rev (rev l · Cons (n; nill))

rev (rev l) · rev (Cons (n; nill))

l · rev (Cons (n; nill))

l · Cons (n; nill)

= Cons (n; l)

By Concatenati

Inductive type.

Nat = Unit + nat
list = Unit + nat × list
Tree = unit + nat × Tree × Tree

nat = 0, 1, 2, ... ⇒ z $s(z)$ $s(s(z))$

list [ ] [0] [1, 0]

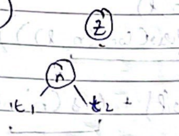
nill, cons(z; nill) cons(s(z); cons(z; nill))

[1] = cons[s(z); s(z))

[1, 1] = cons(cons(s(z); s(z)); cons(s(z); s(z))

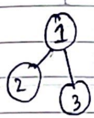
tree — empty = n(z; empty; empty)

— node(n; t_1; t_2)



tree =

= node(s(z); node(s(s(z)); empty; empty); empty)



= node(s(z); node(s(s(z)); empty; empty); node(s(s(s(z))); empty; empty))

Inductive type nat = ind(t. unit + t) inductive nat

list = ind(t. unit + nat × t) list

tree = ind(t. unit + nat × t × t)

Inductive rec rec{t.τ}(x.e_1; e_2) inductive type

rec{t. unit + t}(e; y.e_1) : nat

rec{t. unit + nat × t}(e; y.e_1) : list

rec{t. unit + nat × t × t}(e; y.e_1) : Tree

Ques 1 Double function

$d(z) = z$

$d(s(n)) = s(s(\underbrace{d(n)}_{y}))$

: nat → nat

$lam\{nat\}(n \cdot rec\{t \cdot unit + t\}(n; y \cdot case(y; u \cdot z; \overset{nat}{v} \cdot s(s(v))))$ unit + nat

Ques 2 len : list → nat

$lam\{list\}(l \cdot lrec(l; z; h \cdot y \cdot s(y)))$ head of list

$|l| = s(l)$
length

By Inductive rec

$len = lam\{list\}(l \cdot rec\{t \cdot unit + nat \times t\}(l \cdot y \cdot case(y; u \cdot z; v \cdot s(\underbrace{snd(v)}_{nat \times nat}))))$

t: unit + nat × t; unit·t; nat × nat

Ques 3 : Sum of Nodes

Sum(empty) = 0

Sum(node($n; t_1, t_2$)) = $n + Sum(t_1) + Sum(t_2)$

tree = ind(t · unit + nat × t × t)

tree → tree → tree

$len = s(nat)$

Sum = $lam\{tree\}(t_1 \cdot lam\{tree\}(t_2 \cdot rec\{t \cdot unit + nat \times (t \times t)\}(x \cdot e; t)$; tree

x : unit + nat × tree × tree

$e = case(x; a \cdot z; b \cdot plus(fst(b); plus(snd(fst(b)); snd(snd(b))))$;

$rec\{\}(x.e; e_1)$ — inductive type

plus z m = m
plus s(n) m = s(plus (n m))

(i) plus: (nat → nat) → nat

~~lam{nat}(n. lam{nat}(m. rec{unit + nat}(x;~~

lam{nat}(n. lam{nat}(m. rec{t. unit + t}(x. e; n)))

x = unit + nat

e = case(x; a. m; b. s(b)) = s(plus n m)

~~n = fold{t. unit + t}(~~

(ii) time: (nat → nat) → nat

time z m = z

time s(n) m = plus (time n m) m

(2×2) + 2 = 6

lam{nat}(n. lam{nat}(m. rec{t. unit + t}(x. e; n)));

x = unit + nat

e = case(x; a. z; b. plus(b; m))

z = fold{t. unit + t}(inl{unit; nat}(triv));

plus(b; m) = fold{t. unit + t}(inr{unit; nat}(m));

(iii) lens: list → nat

len nil → z

len cons(n; l) → |l| + 1 or s(|l|)

lam{list}(n. rec{t. unit + nat × t}(x. e; l))

x = unit + nat × list

e = case(x; a. z; b. (snd(b)))

z = fold{t. unit + nat × t}(inl{unit; nat × nat}(triv));

$b = fold\{t.unit + nat \times nat\}(inr\{unit; nat \times nat\}(pair(fst(b); snd(b)));$

(iv) Q: Sum of list

list → nat

sum nil : z

sum : cons(n; l) : plus(sum l);

$lam\{list\}(l.\ rec\{t.unit + nat \times t\}(x.e; l)$

$x = t.unit + nat \times$ ~~list~~ nat

$e = case(x;\ a.z;\ b.\ plus(b));$ fst(b); snd(b)

plus : nat → nat → nat

ap(ap(plus; fst(b)); snd(b))

$z = fold\{t.unit + nat \times nat\}(inl\{unit; nat \times nat\}(triv));$

$plus = fold\{t.unit + nat \times t\}(inr\{unit; nat \times nat\}(pair(fst(b); snd(b)));$

(v) Cat : list → list → list

cat nil l_2 l_2

cat con(n; l_1) l_2 : cons(n; cat(l_1; l_2))

$lam\{list\}(l_1.\ lam\{list\}(l_2.\ rec\{t.unit + nat \times t\}(x.e; l_1)$

$x = t.unit + nat \times list$

$e = case(x;\ a.\ nil;\ b.\ cons(n; rec\ snd(b)))$ nat × list, fst(b)

$nil = fold\{t.unit + nat \times t\}(inl\{unit; nat \times list\}(triv));$

$cons(n; l) = fold\{unit + nat \times t\}(inr\{unit; nat \times list\}(pair(fst(b); snd(b)));$

K

$[1, 2, 3] \to [3, 2, 1] = [1, 2\ 3\ 3\ 2\ 1]$

Rev nill $l_2 \Rightarrow l_2$

Rev Cons$(n; l_1)\ l_2$ = Rev l_2 Cons $(n; l_2)$

Reverse of list

Rev Cat: $(list \to list) \to list$

lam{list}(l, lam{list}(l_2. rec{t.unit + nat x t}(x.e; l)

x = unit + nat x (list → list)

e = case(x; a.nill; b. Cons(fst(b); ap(snd(b); Cons(fst(b); l_1)

f: list → list

Cons: fold{t.unit + nat.t}(inr{unit; nat x list}(pair(fst(b); l_1)))

Reverse of a list

list → list

lam{list}(l. rec{t.unit+nat x t}(x.e; l))

x = unit+ nat x list

e = case(x; a.nill; b. Cons(fst(b); snd(b)));

Rev (nil) → Nill

Rev (Cons(n; l)) → = Rev l. Con(n; nill)

Times Rev l Cons(n; nill)

s(nill)

nill = fold{t.unit + nat x t}(inl{unit; nat}(triv));

Cons(n; l) = fold{unit + nat x t}(inr{unit; nat}(snd(b)));

e = case(x; a.nill; b. Times(b; s(a)));

Rev l s(nill)

Cons(n; nill)

Cons(fst(b); a)

2

RevCat ~~list → (list → list)~~ | ~~(list → list) → list~~

RevCat nil r → r

RevCat cons(n; l) r = rev r + cons(n; r)

lam{list}(l · rec{t · unit + nat × (list → list)}(r · e; l));

x = t · unit + nat × (list → list)

e = case(x; a · r; b · ap(

$\underbrace{\hspace{20em}}_{(list \to (list \to list))}$

rev cat =

lam{list}(l_1 · rec{t · unit + nat × t}(l_1; r · e_1))

r = unit + nat × (list → list)

~~e_1 = case(r; a · r~~

e_1 = lam{list}(l_2 · case(x; v · ~~[illegible]~~ l_2; v · ap(snd(v); l_2 cons(fst(v); l_2)))

Rev nill l_2 → l_2

Rev cons(n; l_1) l_2 = Rev l_1 · cons(n; l_2)

list → list
l_2 result

(15) Height (LTR(t)) = Height (t)
By Induction on t

(i) t = empty
Height (LTR (empty))
Height (empty) = empty   By rule

Case t = node (n; t₁, t₂)
Height (LTR (node (n; t₁, t₂)))
Height (LTR node (n; LTR (t₁), LTR(t₂)))

= S (max (H(LTR(t₁)), H(LTR(t₂))))
By IH

S (max (H(t₁), H(t₂)))
↳ By Rule
H (node (n; t₁, t₂)) ✓

(16) Rev (Inorder (t)) = Inorder (LTR (t))
By Induction on t

Case t = empty
Rev(Inorder (empty)) = ~~Rev~~ (nil)
= Inorder (LTR (empty))

Case t = node (n; t₁, t₂)
Rev (Inorder (node (n; t₁, t₂)))
= rev (Inorder (t₁) · Cons(n; Inorder (t₂)))   Const
= rev (Cons (n; Inorder (t₂) · Inorder (t₁))) // By Const rule

rec{e_0; x.y.e}(n)

rec{t.τ}(x.e; e_2) ind

By rev

rev (Inorder (t_2)·Inorder(t_1))·Cons(n; nil)

= rev(in(t_2))·rev(in(t_1))·Cons(n; nil)

By IH

Inorder (LTR (t_2))·$\underbrace{\text{Inorder(LTR}(t_1))\cdot\text{Cons}(n;\text{nil})}_{\text{By Concat}}$

Inorder(LTR(t_2))·Cons(n; Inorder(t_1); nil)

By Inorde rule

Inorder (node (n; LTR (t_2); LTR(t_1))

By LTR Rule

Inorder (LTR (node (n; t_2, t_2)) =

$$\frac{\Gamma, x : [\tau'/t]\tau \vdash e_1 : \tau' \quad e_2 \text{ ind}}{\text{rec}\{t.\tau\}(x.e; e_1) : \tau'}$$

(t.τ)

lt z z = z.

lt z s(m) = s(z),

lt s(n) z = z

lt s(n) s(n_2) = lt n_1 n_2

n_1 m

n_2

lt n   lam{nat}(n_2.

lam{nat}(n. rec{lam{nat}(m. rec{z; x.y.s(z)}(m); p.q.rec{z; u.v.ap(q, v)}(n_2)}(n_1)

gt 0 0 = 0

gt s(m) 0 = s(z)

gt 0 s(n) = 0

gt s(n_2) s(n_1) = gt n_2 n_1

lam{nat}(n_2

lt n_2

lam{nat}(m. rec{lam{nat}(m. rec{z; x.y.s(z)}(m); p.q.rec{z; u.v.ap(q; v)}(n_2)}(n_1)

(12) $Size(LTR(t)) = Size(t)$

By Induction on t

Case t = empty

$Size(LTR(empty)) = Size(empty)$ By rule

$= empty$ ✓

Case $t = node(n; t_1; t_2)$

$Size(LTR(node(n; t_1, t_2)))$

$Size(node(n; LTR(t_1), LTR(t_2)))$

$S(Size(LTR(t_1)), Size(LTR(t_2)))$

By IH

$S(size(t_1), size(t_2)) = Size(node(n; t_1, t_2))$ ✓✓

(13) $rev(rev(l)) = l$

By Induction l

Case l = nill

$rev(rev(nil)) = nill$

Case $l = Cons(n; l)$

$rev(rev(Cons(n; l)))$

$rev(rev\, l \cdot Cons(n; nill))$

$rev(rev(l)) \cdot rev(Cons(n; nill))$

By IH

$l \cdot rev(Cons(n; nill))$

$l \cdot Cons(n; nill)$ By rule Cons

$Cons(n; l)$

(7) $Si(t) = |Pre(t)|$

By Induction on t

Case t = empty $\quad Size(empty) = empty$

also, By rule $\quad |pre(empty)|$

Case $t = node(n; t_1, t_2)$

$Size(node(n; t_1, t_2)) =$

$S(Size(t_1) + Size(t_2))$

By IH

$S(|Pre(t_1)| + |Pre(t_2)|)$

$S|Pre(t_1)| + |Pre(t_2)|$ By Plus rule

$|Cons(n; Pre(t_1))| + |Pre(t_2)|$ — (By Len Rule)

$|Cons(n; Pre(t_1))| + |Pre(t_2)|$

$= |Cons(n; Pre(t_1)) \cdot Pre(t_2)| \qquad |l_1 + l_2| = |l_1| \cdot |l_2|$

$|Pre(node(n; t_1, t_2))|$ ✓

(8) $Size(t) = |Post(t)|$

Case $t = node(n; t_1, t_2)$

$Size(node(n; t_1, t_2))$

$S(Size(t_1) + Size(t_2))$

$S(|Post(t_1)| + |Post(t_2)|)$ — By IH

$|Post(t_1)| + S|Post(t_2)|$

$|Post(t_1)| + |Cons(n; Post(t_2))|$

$|Post(t_1)| \cdot |Cons(n; Post(t_2))|$

$|Post(t_1)| \cdot |Post(t_2)| \cdot |Cons(n; nil)| \qquad |Post(t_2)| \cdot Cons(n; nil)|$

$|Post(t_1) \cdot Post(t_2) \cdot Cons(n; nil)|$

$|Post(n; t_1, t_2)|$

(16) $rev(inorder(t)) = inorder(LTR(t))$

Case $t = node(n; t_1, t_2)$

$rev(inorder(node(n; t_1, t_2)))$

$rev(inorder(t_1) \cdot Cons(n; Inorder(t_2)))$

$rev(inorder(t_1)) \cdot rev(Cons(n; Inorder(t_2)))$

$rev(inorder(t_1)) \cdot rev(Inorder(t_2)) \cdot Cons(n; nil)$

By IH

$inorder(LTR(t_1)) \cdot \underline{inorder(LTR(t_2)) \cdot Cons(n; nil)}$

$inorder(LTR(t_1)) \cdot Cons(n; LTR(t_2))$

$inorder(node(n; LTR(t_1), LTR(t_2)))$

$inorder(LTR(node(n; t_1, t_2)))$

$$\frac{\Gamma \vdash e : [ind(t.\tau)/t]\tau}{\Gamma \vdash fold\{t.\tau\}(e) : ind(t.\tau)}$$

$$\frac{\Gamma \vdash e_2 : \tau_2 \quad \Gamma, x : \tau_2 \vdash e_1 : [\tau_2/t]\tau}{\Gamma \vdash gen\{t.\tau\}(x.e_1; e_2) : coi(t.\tau)}$$

$$\frac{\Gamma, x : [\tau'/t]\tau \vdash e_1 : \tau' \quad \Gamma \vdash e_2 : ind(t.\tau)}{\Gamma \vdash rec\{t.\tau\}(x.e_1; e_2) : \tau'}$$

$$\frac{\Gamma \vdash e : coi(t.\tau)}{\Gamma \vdash unfold\{t.\tau\}(e) : [coi(t.\tau)/t]\tau}$$

$map\{t.\tau\}(x.e')(e)$

$\infty i \quad t \cdot A \times t) \to Coi\,(t \cdot A \times t)$

lam{ } $\{ (x \cdot unfold\{t \cdot A \times t\}(x;$ map{t.t}(x.e')(e) $t \cdot A \times t$

(i) $A \to Coi\,(t \cdot A \times t)$ ~~gen{t.t}(x.e_1; e_2)~~

unit or void — coind

lam{A}(x. gen{t.A×t}{T}(triv; y. pair(x;y)))

gen{t.T}{T'}(e₂; x.e)

(ii) ind (t·unit×t) → ind (t·unit×t)

inductive type

lam{ind(t·unit×t)}(x.fold{t·unit×t}(pair(triv;x)))

(iii) (A→B) → P(A) → P(B) assuming t.P(t) pos

lam{A→B}(x. lam{P(A)}(y. map{t.P(t)}{A;B}(a. ap(x;a))(y)))

~~lam{A→B}(y. lam{P(A)}(a. map{t.t}{A;B}(~~

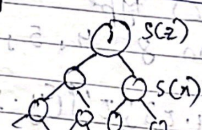
lam{A→B}(x. lam{P(A)}(a. map{t.P(t)}{A;B}(a.ap(x;a))(a)))

∞tree: coi (t·nat × (t×t))

stream = gen{t·nat × (t×t)}(e₂; (e₂ j× ξ e₁)

gen{t·nat×(t×t)}(s(z); x. pair(x; pair(s(x); s(x))))

e₁



nat × (t×t)

nat × (nat × nat)

$e : \tau$ & $\exists e'$, $e \mapsto e'$ then $e' : \tau$

Preservation

Case

$rec\{t.\tau\}(x.e_1; e_2) \mapsto [map\{t.\tau\}(y.rec\{t.\tau\}(x.e_1; y))(e_2)/x]e_1$

$e_2 = fold\{t.\tau\}(e_2)$

$e = rec\{t.\tau\}(x.e_1; e_2) : \tau'$

$e' = [map\{t.\tau\}(y.rec\{t.\tau\}(x.e_1; y))(e_2)/x]e_1$

$\boxed{e' = \tau'}$ To be prove

By inversion on, $e : \tau$

$\Gamma \vdash e_2 : fold\{t.\tau\}(e_2)$ — (1)

$\Gamma, x : [\tau'/t]\tau \vdash e_1 : \tau'$ — (2)

τ' type — (3)

By inversion on (1)

$\Gamma \vdash e_2 : [ind(t.\tau)/t]\tau$ — (4)

By variable rule, we have

$y : ind(t.\tau) \vdash y : ind(t.\tau)$ — (5)

By weakening on 2 & 3 we get

$y : ind(t.\tau), x : [\tau'/t]\tau \vdash e_1 : \tau'$ — (6)

$y : ind(t.\tau)$ type — (7)

By rec rule on (5), (6), (7), (4)

$y : ind(t.\tau) \vdash rec\{t.\tau\}\{\tau'\}(y; x.e_1) : \tau'$ — (8)

By Ind - ind rule on (5) we get

$ind(t.\tau)$ type — 9

By Map rule on 8, 4, 5, (9), (3)

$map\{t.\tau\}\{ind\{t.\tau\}; \tau'\}(y.rec\{t.\tau\}\{\tau'\}(y; x.e_1)(e_2) : [\tau'/t]\tau$ — (9)

By Substitution lemma on (2) & 9

$e' : \tau'$

Progress

$$\frac{\Gamma \vdash e_2 : \tau_2 \quad \Gamma, x : \tau_2 \vdash e_1 : [\tau_2/t]\tau}{\Gamma \vdash \text{gen}\{t.\tau\}(x.e_1; e_2) : \text{coi}(t.\tau)}$$

$e = \text{gen}\{t.\tau\}\{\tau'\}(e_2; x.e_1)$ $\Gamma \vdash \text{gen}\{t.\tau\}\{\tau'\}(e_2; x.e_1)$

$\tau = \text{coi}(t.\tau)$

Above the line

$\Gamma \vdash e_2 : \tau_2$ — (1)

$\Gamma, x : \tau_2 \vdash e_1 : [\tau_2/t]\tau$ — (11)

By IH on (1) (e_2) either e_2 val or $\exists e_2$, $e_2 \mapsto e_2'$

Case e_2 val

By rule $$\frac{e_2 \text{ val}}{\text{gen}\{t.\tau\}\{\tau'\}(e_2; x.e_1) \text{ val}}$$

$\text{gen}\{t.\tau\}\{\tau'\}(e_2; x.e_1)$ val

Case $e_2 \mapsto e_2'$

Take e_2' as $\text{gen}\{t.\tau\}\{\tau'\}(e_2'; x.e_1)$

By rule

$$\frac{e_2 \mapsto e_2'}{\text{gen}\{t.\tau\}\{\tau'\}(e_2; x.e_1) \mapsto \text{gen}\{t.\tau\}\{\tau'\}(e_2'; x.e_1)}$$

$l \to r$

eager lazy $\quad r \to l \quad$ eager

$$\frac{e_1 \mapsto e_1'}{ap(e_1; e_2) \mapsto ap(e_1'; e_2)}, \quad \left[\frac{e_1 \text{ val} \quad e_2 \mapsto e_2'}{ap(e_1; e_2) \mapsto ap(e_1; e_2')}\right]$$

$$\frac{[e_2 \text{ val}]}{ap(\text{lam}\{\tau_1\}(x.e_1); e_2) \mapsto [e_2/x]e_1}$$

$e \qquad e'$

$$\frac{}{\text{lam}\{\tau_1\}(x.e) \text{ val}}$$

$e_2 \mapsto e_2' \qquad e_2 \text{ val} \quad e_1 \mapsto e_1'$

e_1 val

$$\frac{e_1 \mapsto e'}{let(e_1;x.e_2) \mapsto let(e_1';x.e_2)}$$

$$\frac{\Gamma \vdash e_1 : \tau \quad \Gamma, x:\tau \vdash e_2 : \tau'}{\Gamma \vdash let(e_1; x.e_2) : \tau'}$$

$$\frac{e_1\ val \quad e_2 \mapsto e_2'}{let(e_1;x.e_2) \mapsto let(e_1;x.e_2')}$$

$$\frac{[e_1\ val]}{let(e_1;x.e_2) \mapsto [e_1/x]e_2}$$

Preservation $e:\tau$ & $\exists e, e \mapsto e'$, then $e':\tau$

$e = let(e_1; x.e_2)$

~~Suppose~~ $e' = [e_1/x]e_2$

Above the line e_1 val

~~Above the line~~ (By Inversion on $e:\tau$)

$\Gamma \vdash e_1 : \tau$ — (i)

$\Gamma, x:\tau \vdash e_2 : \tau'$ — (ii)

Using substitution theorem on (i) & (ii)

$\Gamma, \vdash [e_1/x]e_2 : \tau'$ — (iii)

By let rule

take e' as $[e_1/x]e_2$

$e \mapsto e'$

By rule $[e_1/x]e_2$ using (iii)

$[e_1/x]e_2 : \tau'$

(H) Progress

$e = let(e_1; x.e_2)$

$\tau = \tau'$

By Above the line

$\Gamma \vdash e_1 : \tau$ — (i)

$\Gamma, \vdash x:\tau \vdash e_2 : \tau'$ — (ii)

By IH on (i)

case $e_1 \mapsto e_1'$

take e' as $let(e_1'; x.e_2)$

By rule

$e_1 \mapsto e_1'$

Case e_1 val

take e_1' as $[e/x]e_2$

$let(e_1;x.e_2) \mapsto [e_1/x]e_2$

By rule

e_1 val

$$\frac{\Gamma, x:\tau_1 \vdash e : \tau_2}{lam\{\tau\}(x.e) : arr(\tau_1;\tau_2)}$$

$$\frac{e_2\ val}{ap(lam\{\tau\}(x.e_1);e_2) : [e_2/x]e_2}$$

$$\frac{\Gamma \vdash e_1 : arr(\tau_2, \tau) \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash ap(e_1;e_2) : \tau_2}$$

$$\frac{\Gamma \vdash e_1 : (\tau_2 : \tau) \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash ap(e_1; e_2) : \tau}$$

$$\frac{\Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash lam\{\tau\}(x.e) : arr(\tau_1, \tau_2)}$$

$$\frac{\Gamma \vdash e \text{ Nat}}{\Gamma \vdash S(e) : Nat}$$

Preservation: $e = ap(lam\{\tau\}(x.e_1); e_2)$

$e' = [e_2/x]e_1$

Above the line, e_2 Val — H_1

$$\frac{e_1 \mapsto e_1'}{ap(e_1; e_2) \mapsto ap(e_1'; e_2)}$$

$$\frac{e_1 \text{ Val} \quad e_2 \mapsto e_2'}{ap(e_1; e_2) \mapsto ap(e_1; e_2')}$$

By Inversion on $e : \tau$

$\Gamma \vdash lam\{\tau\}(x.e) : arr(\tau_2 : \tau)$ — H_2

$\Gamma \vdash e_2 : \tau_2$ — H_3

$$\frac{[e_2 \text{ Val}]}{ap(}$$

By Inversion on H_2

$\Gamma, x : \tau_2 \vdash e : \tau$ — H_4

By substitution on H_3 & H_4

$[e_2/x]e : \tau$

e'

$e' : \tau$ ✓

$ap(x)$

$(A \to (B + C)) \to D) \to ((A \to B) + (A \to C)) \to D$

$lam\{(A \to (B+C)) \to D)\}(x \cdot lam\{(A \to B) + (A \to C)\}(y \cdot Case(y; a : lam\{A\}(a \cdot e_1); b$ $A \to B$ $A \to C$

$lam\{B\}(c \cdot e_2))$

$e_1 =$ $inl\{B; C\}($

$= (t.\text{unit} + \text{nat}$
$= (t.\text{unit} + \text{nat} \times t$
$= (t.\text{unit} + \text{nat} \times t \times t$

$\text{lam}\{\tau\}(x.e$

$((A \to (B+C)) \to D) \to ((A \to B) + (A \to C)) \to D$

$\text{lam}\{A \to (B+C) \to D\}(x.\ \text{lam}\{(A \to B) + (A \to C)\}(y.\ \text{case}(y;\ a.\ (\text{lam}\{A\}(a'.e)$

$e_1 = \text{inl}\{B;C\}(\text{ap}(a;a'))$

$e_2 = \text{inr}\{B;C\}(\text{ap}(b;b')) = C$

$[2,3,4] \to \text{Snoc}(\text{Snoc}(\text{Snoc}(\text{nil};2)))$

Cat $\text{Snoc}(\text{nil};2)$ nil $= \text{Snoc}(\text{nil};2)$

Cat $\text{Snoc}(\text{nil};2)\ \text{Snoc}(\text{Snoc}(\text{nil};3);4) \leftarrow [3,4]$

$= \text{Snoc}(\ ;\ \text{Cont}(\ ;\ ))$

Concat(

list $\to$ list $\to$ list

$\text{lam}\{\text{list}\}(l_1.\ \text{lam}\{\text{list}\}(l_2.\ \text{rec}\{t.\text{unit} + \text{nat} \times t\}(x.e;\ l_1)$

$x = t.\text{unit} + \text{nat} \times \text{list}$

$e = \text{case}(x;\ a.\ \text{nil};\ b.\ \text{Snoc}(\text{fst}(b);\ \text{ap}(\ ;\ \text{snd}(b))$

Snoc(snd(b)

$\boxed{\text{Cat}}\ \text{Snoc}(\text{nil};2)\ \text{Snoc}(\underbrace{\text{Snoc}(\text{nil};3)}_{n};4)$

$= \text{Snoc}(n;\ \text{Concat}(\ ;\ )$

$\text{nil} = \text{fold}\{t.\text{unit} + \text{nat} \times t\}(\text{inl}\{\text{unit};\ \text{nat} \times \text{list}\}(\langle\rangle$

$\text{Snoc}(n;l) = \text{fold}\{t.\text{unit} + \text{nat} \times t\}(\text{inr}\{\text{unit};\ \text{nat} \times \text{list}\}(\text{pair}$

$plus = ap(ap(y; x); x);$

System T

$$\frac{\Gamma \vdash e : nat}{\Gamma \vdash S(e) : nat} \qquad \frac{\Gamma \vdash e_0 : \tau \quad \Gamma \vdash e : nat \quad \Gamma, x : nat, y : \tau \vdash e_1 : \tau}{\Gamma \vdash rec\{e_0; x.y.e_1\}(e) : \tau}$$

$$\frac{e \mapsto e'}{rec\{e_0; x.y.e_1\}(e) \mapsto rec\{e_0; x.y.e_1\}e'}$$

$$\frac{}{rec\{e_0; x.y.e_1\}(z) \mapsto e_0} \qquad \frac{[S(e)\ val]}{rec\{e_0; x.y.e_1\}(S(e)) \mapsto [e, rec\{e_0; x.y.e_1\}e/x, y]e_1}$$

$$\frac{e\ val}{S(e)\ val}$$

$map\{t.pos\}\{A; B\}(\overbrace{x.h}^{e'})(e)$

Construct B

Old Mapping $P(A)$

$$\frac{t.\tau\ poly \quad \Gamma, x : \rho \vdash e' : \rho' \quad \Gamma \vdash e : [\rho/t]\tau}{\Gamma \vdash map\{t.\tau\}(x.e')(e) : [\rho'/t]\tau}$$

$(A \to B) \to P(A) \to P(B)$

$\Delta \vdash App\{\tau\}(e) : [\tau/t]\tau'$ type

$App\{\tau\}(Lam(t.e)) \mapsto [\tau/t]e$

$$\frac{\Delta, t\ type\ \Gamma \vdash e : \tau}{\Gamma \vdash Lam(t.e) :}$$

$$\frac{e \mapsto e'}{App\{\tau\}(e) \mapsto App\{\tau\}(e')}$$

$$\frac{}{Lam\{t.e\}\ val}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash fst(e) : \tau_1} , \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash snd(e) : \tau_2}$$

$$\frac{e_1\ val \quad e_2\ val}{fst\langle e_1; e_2\rangle \mapsto e_1}$$

$$\frac{e \mapsto e'}{fst(e) \mapsto fst(e')}$$

$$\frac{e_1\ val \quad e_2\ val}{pair(e_1; e_2)\ val}$$

Progress $e : \tau$ then val or $e \mapsto e'$

$$\frac{\Gamma \vdash e : void}{\Gamma \vdash Abort\{\tau\}(e) : \tau'}$$

#

$e = fst(e)$

$\tau = \tau_1$

Above is line — (H_1)

$\Gamma \vdash e : \tau_1 \times \tau_2$

By IH on H_1

$e \mapsto e'$

Take e' as $fst(e')$

Done By rule

e val

By CFL $\Gamma \vdash e : pair(e_1; e_2)$

take e' as e_1

e_1 val e_2 val

pair( ) val

$$\frac{e : \tau_1}{\Gamma \vdash inl\{\tau_1;\tau_2\}(e) : \tau_1 + \tau_2} \qquad \frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash inr\{\tau_1;\tau_2\} : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma, x : \tau_1 \vdash e_1 : \tau \quad \Gamma, y : \tau_2 \vdash e_2 : \tau}{\Gamma \vdash case(e; x.e_1; y.e_2) : \tau} \qquad \frac{e \mapsto e'}{inl\{\tau_1;\tau_2\}(e) \mapsto inl\{\tau_1;\tau_2\}(e')}$$

$$\frac{e \mapsto e'}{inr\{\tau_1;\tau_2\}(e) \mapsto inr\{\tau_1;\tau_2\}(e')} \qquad \frac{e_0 \mapsto e_0'}{case(e_0; x.e_1; y.e_2) \mapsto case(e_0'; x.e_1; y.e_2)}$$

$$\frac{[e \text{ val}]}{case(inl\{\tau_1;\tau_2\}(e); x.e_1; y.e_2) \mapsto [e/x]e_1}$$

$$\frac{[e \text{ val}]}{case(inr\{\tau_1;\tau_2\}(e); x.e_1; y.e_2 \mapsto [e/y]e_2}$$

Preservation

$e : case(inl\{\tau_1;\tau_2\}(e); x.e_1; y.e_2)$

$e' = [e/x]e_1$

Allow the line

e val — H_1

By Inversion on $e : \tau$

$\Gamma \vdash e_0 : sum(\tau_1;\tau_2)$ — H_2

$\Gamma, x : \tau_1 \vdash e_1 : \tau$ — H_3

$\Gamma, y : \tau_2 \vdash e_2 : \tau$ — H_4

By Inversion on H_2

$\Gamma \vdash e : \tau_1$ — H_4

By Substitution H_4 & H_3

$\Gamma, [e/x]e_1 : \tau$ ✓

Progress

$e : case(e_0; x.e_1; y.e_2)$

$\tau = \tau$

Allow the line

By IH on H_2

e_0 val or $e_0 \mapsto e_0'$

Case 1: $e_0 \mapsto e_0'$

take e_0' as $case(e_0'; x.e_1; y.e_2)$

Case 2: e val

CFL on $e_0 : sum(\tau_1;\tau_2)$

Case 2.1 $e_0 = inl\{\tau_1;\tau_2\}(e')$

take $e_0 : [e'/x]e_1$

the e' val

Case 2.2 $e_0 = inr\{\tau_1, \tau_2\}(e')$

take $e_0 : [e'/y]e_2$

e' value

$$\frac{\Delta\,\Gamma \vdash e : \forall(t.\tau) \quad \Delta \vdash \tau\ \text{type}}{\Delta\,\Gamma \vdash \mathrm{App}\{\tau\}(e) : [\tau/t]\tau'}$$

$$\frac{\Delta, t\ \text{type}\ \Gamma \vdash e : \tau}{\mathrm{Lam}(t.e) : \forall(t.\tau)}$$

$$\frac{}{\mathrm{Lam}(t.e)\ \mathrm{val}} \qquad \mathrm{App}\{\tau\}(\mathrm{Lam}(t.e)) \mapsto [\tau/t]e$$

$$\frac{e \mapsto e'}{\mathrm{App}\{\tau\}(e) \mapsto \mathrm{App}\{\tau\}(e')}$$

Preservation

$$\frac{}{\mathrm{App}\{\tau\}(\mathrm{Lam}(t.e)) \mapsto [\tau/t]e}$$

$e = \mathrm{App}\{\tau\}(\mathrm{Lam}(t.e))$

$\tau = [\tau/t]e$

$e : \tau$ & $e \mapsto e'$ then $e' : \tau$

By Inversion on $e : \tau$

$\Delta\,\Gamma \vdash e : \forall(t.\tau')$ — H_1

$\Delta \vdash \tau$ type — H_2

Here $e = \mathrm{Lam}(t.e)$ — H_3

By Inversion on H_3

$\Delta, t\ \text{type}\ \Gamma \vdash e : \tau'$ — H_4

By Substitution Theorem on H_2 & H_4

$\Delta, [\tau/t]\,\Gamma \vdash [\tau/t]e : \boxed{[\tau/t]\tau'}$

Progress

$e = \Delta, \Gamma \vdash \mathrm{App}\{\tau\}(e) : [\tau/t]\tau'$

$\tau = [\tau/t]\tau'$

Above the line

By IH on H_1

Case $e \mapsto e'$

take e' as $\mathrm{App}\{\tau\}(e')$

Suppose e val

By CFL on $e : \forall(t.e)$

$e = \mathrm{Lam}(t.e)$

take e' as $[\tau/t]e$

By App rule

$\mathrm{App}\{\tau\}(\mathrm{Lam}(t.e)) \mapsto [\tau/t]e$

Value ✓

$$\frac{\Delta\,\Gamma \vdash e_1 : \exists(t.\tau) \quad \Delta, t\ \text{type}\ \Gamma, x:\tau \vdash e_2 : \tau_2 \quad \Delta \vdash \tau_2\ \text{type}}{\mathrm{open}\{t.\tau\}\{\tau_2\}(e_1;\, t, x.e_2) : \tau_2}$$

$$\frac{\Delta\,\Gamma \vdash e : [\rho/t]\tau \quad \Delta \vdash \rho\ \text{type} \quad \Delta, t\ \text{type} \vdash \tau\ \text{type}}{\mathrm{pack}\{t.\tau\}\{\rho\}(e) : \exists(t.\tau)}$$

$$\frac{\Delta\Gamma \vdash e_1 : \exists(t.\tau) \quad \Delta \vdash \tau_2 \text{ type} \quad \Delta, t \text{ type}\ \Gamma, x:\tau \vdash e_2 : \tau_2}{\Delta\Gamma \vdash \text{open}\{t.\tau\}\{\tau_2\}(e_1; t, x.e_2) : \tau_2}$$

$$\frac{\Delta, t \text{ type} \vdash \tau \text{ type} \quad \Delta \vdash \rho \text{ type} \quad \Delta\Gamma \vdash e : [\rho/t]\tau}{\Delta\Gamma \vdash \text{pack}\{t.\tau\}\{\rho\}(e) : \exists(t.\tau)}$$

$$\frac{\Delta\Gamma \vdash \qquad e \text{ val}}{\Delta\Gamma \vdash \text{open}\{t.\tau\}\{\tau_2\}(\text{pack}\{t.\tau\}\{\rho\}(e); t, x.e_2) ; \tau_2}$$

$$[\rho, e/t, x]e_2$$

plus $\qquad \hat{nat} \to \hat{nat} \to nat$

$\text{ap}(\text{ap}(\text{plus}; n); m)$

base case

$\text{gen}\{t.\tau\}\{T\}(e_2; x.e_1)$

* $(A \to B \to C) \to (B \to A \to C)$

$\text{lam}\{(A \to B \to C)\}(x.\ \text{lam}\{B\}(y.\ \text{lam}\{A\}(z.\ \text{ap}(\text{ap}(x; z); y))))$

* $(B \to \text{Void}) \to (A \to (A \times B)) \to (A \to C)$

$\text{lam}\{B \to \text{Void}\}(x.\ \text{lam}\{A \to (A\times B)\}(y.\ \text{lam}\{A\}(z.\ \text{Abort}\{C\}(\text{ap}(x; \text{snd}(\text{ap}(y; z)))$

* $((A+C) \times (B+D)) \to ((D \times C) + (A+B))$

$\text{lam}\{(A+C)\times(B+D)\}(x.\ \text{case}(\text{snd}(x);\ a^B.\ \text{inr}\{D\times C; A+B\}(\text{inr}\{A;B\}(a);$
$b^D.\ \text{case}(\text{fst}(x);\ p^A.\ \text{inr}\{D\times C; A+B\}(\text{inl}\{A;B\}(p);\ q^C.\ \text{inl}\{D\times C; A+B\}(\text{pair}$
$(b; q);$

$\underbrace{(A \to \forall(t.B(t))}\times(A\to\forall(t.C(t))) \to \forall(u.A\to(B(u)\times C(u)))$

lam{$A\to\forall(t.B(t))\times(A\to\forall(t.C(t)))$}(x. Lam(u. lam{A}(y. pair(e₁;e₂))))

$e_1 =$ ~~ap ap(x;y)~~ App{u}(ap(fst(x);y))

$e_2 =$ App{u}(ap(snd(x);y))

\# $\forall t.$ ~~$\forall$~~ $((A\to t)\to(B\to t)\to t)\to(A+B)$

lam{$\forall t.(A\to t)\to(B\to t)\to t$}(x.e) ~~App{A+B}~~(

e = ap(ap(App{A+B}(x); e_1); e_1)

e_1 = lam{A}(a. inl{A;B}(a))

e_2 = lam{B}(b. inr{A;B}(b))

\# ind(t. unit × t) → ind(t. unit × t)

lam{ind(t. unit × t)}(x. fold{t. unit × t}(pair(triv; x)))

\# A → coi(t. A × t)

lam{A}(x. gen{t. A × t}{T}(triv; \y. pair(x;y)))

(A → B) → P(A) → P(B)

lam{A → B}(x. lam{P(A)}(y. map{t. P(t)}{A;B}(a; ap(x;a))(y)))

mult(x; y)   mult = nat → nat → nat

plus = nat → nat → nat

ap(ap(mult; x); y)

plus(a; b)

ap(ap(plus; a); b)

Sum : tree → nat

Sum empty = empty
Sum node(t₁; t₂) = Sum(t₁) + sum(t₂)

lam{tree}(t · rec{t·nat + (t×t)}(x · e; t))

χ = nat + (nat × nat)

e = case(x; a·n; b·ap(ap(plus; fst(b)); snd(b)))

$[e_2 \text{ val}]$

$ap(lam(x \cdot e_1); e_2) \mapsto [e_2/x]e_1$

$e = ap(lam(x \cdot e_1); e_2)$
$e' = [e_2/x]e_1$

Above the line
e_2 val − H_1

By Inversion on $e : \tau$
$\Gamma \vdash lam(x \cdot e_1) : arr(\tau_1; \tau_2)$ − H_2
$e_2 : \tau_1$ − H_3

By Inversion on H_2
$\Gamma, x : \tau_2 \vdash e_1 : \tau_2$ − H_4

By Substitution Lemma H_3 & H_4
$\Gamma, [e_2/x]e_1 : \tau_2$

$S(e)$ val

$rec\{e_0; x \cdot y \cdot e_1\}(S(e)) \mapsto$
$[x; rec\{e_0; x \cdot y \cdot e_1\}(e) \ldots]$

$e' = [e; rec\{e_0; x \cdot y \cdot e_1\}(e)/x, y]e_1$

By the above the line
$S(e)$ val

By Inversion
$\Gamma \vdash e_0 : \tau$ − H_1
$\Gamma, x : nat, y : \tau \vdash e_1 : \tau$ − H_2
$\Gamma \vdash e : nat$ − H_3
$S($

By Inversion on H_3
$\Gamma \vdash e : nat$ − H_4

By Weakening Lemma on H_4

$\Gamma, y : \tau \vdash e : nat$ − H_5

By substituting H_5 in H_2

$\Gamma, y : \tau, [e/x] \vdash [e/x]e_1 : \tau$ − H_6

By rec Rule
$\Gamma \vdash rec\{e_0; x \cdot y \cdot e_1\}(e) : \tau$
$\frac{e}{x}[rec\{e_0; x \cdot y \cdot e_1\}(e)/y]e_1 : \tau$

$$U \times V = \begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} \times \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} U_y V_z - U_z V_y \\ U_z V_x - U_x V_z \\ U_x V_y - U_y V_x \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ U = & (2, -2, 2) \\ V = & (2, 1, -5) \end{array} = \begin{bmatrix} -2\times -5 - 2 \\ 2\times 2 - 2\times -5 \\ 2\times(1) - (-2)\times 2 \end{bmatrix} = \begin{bmatrix} 10 - 2 \\ 4 + 10 \\ 2 + 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \\ 6 \end{bmatrix} \begin{array}{c} N_x \\ N_y \\ N_z \end{array}$$

$$l = \sqrt{8^2 + 14^2 + 6^2}$$

$$= \sqrt{64 + 196 + 36}$$

Let $S = \sin 45$

$C = \cos 45$

$$T_c \, S_\alpha \, R_\theta \, T_c^{-1}$$

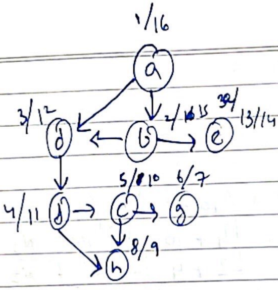
$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 1 & 1 & 1 \\ 1 & \alpha & 1 & 1 \\ 1 & 1 & \alpha & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & & 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha & 0 & 0 & -1 \\ 0 & \alpha & 0 & -1 \\ 0 & 0 & \alpha & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [\quad]$$

$$\begin{bmatrix} C\alpha & -S\alpha & 0 & -1 \\ S\alpha & C\alpha & 0 & -1 \\ 0 & 0 & \alpha & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\alpha & -S\alpha & 0 & (C\alpha - S\alpha - 1) \\ S\alpha & C\alpha & 0 & (S\alpha + C\alpha - 1) \\ 0 & 0 & \alpha & (\alpha - 1) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

//



	Discovery Time	Finish Time	
a	1 ✓	16	1
b	2 ✓	~~5~~ 15	2
c	9	12	5
d	5	14	3
e	3 ✓	4	8
f	~~4~~ 6	~~10~~ 13	4
g	10	11	6
h	~~8~~ 7	8	7

a b d e f c h g

$(P \to Q) \wedge (P \to R) \to P \to (Q \wedge R)$

$\mathrm{lam}\{(P\to Q)\times(P\to R)\}(x.\ \mathrm{lam}\{P\}(y.\ \mathrm{pair}(\overbrace{\mathrm{ap}(\mathrm{fst}(x); y)}^{ap}; \mathrm{ap}(\mathrm{snd}(x); y))))$

$(P \to (Q \to R)) \to ((P\times Q) \to R)$

$\mathrm{lam}\{P\to(Q\to R)\}(x.\ \mathrm{lam}\{P\times Q\}(y.\ \mathrm{ap}(\underbrace{\mathrm{ap}(x; \mathrm{fst}(y)}_{Q\to R}); \mathrm{snd}(y))))$

$((P\to Q) + (P\to R)) \to (P \to (Q + R))$

$\mathrm{lam}\{(P\to Q)+(P\to R)\}(x.\ \mathrm{lam}\{P\}(y.\ \mathrm{case}(x;\ \overset{P\to Q}{a}.\ \mathrm{inl}\{Q;R\}(\mathrm{ap}(a;y));\ b.\ \mathrm{inr}\{Q;R\}(\mathrm{ap}(b;y)))))$

$(P\to R) + (Q\to R) \to ((P\times Q) \to R)$

$\mathrm{lam}\{(P\to R)+(Q\to R)\}(x.\ \mathrm{lam}\{P\times Q\}(y.\ \mathrm{case}(x;\ \overset{P\to R}{a}.\ \mathrm{ap}(a; \mathrm{fst}(y));\ \overset{Q\to R}{b}.\ \mathrm{ap}(b; \mathrm{ap}(b; \mathrm{snd}(y))))))$

(i) $\forall t.\forall u.S(t.u) \to \forall u.\forall t\ S(t,u)$

$\mathrm{lam}\{\forall t.\forall u.S(t.u)\}(x.\ \cancel{\mathrm{Lam}(u.\ \mathrm{App}\{t\}}(\ e));$

$e = \mathrm{App}\{t;S(t,u)\}(\underbrace{\mathrm{App}\{t\}(x)}_{\forall u.S(t,u)});$

$\mathrm{lam}\{\forall t.\forall u.S(t.u)\}(x.\ \mathrm{Lam}(u.\ \mathrm{Lam}(t.\ \overset{S(t,u)}{e})))$

$e = \mathrm{App}\{u\}(\mathrm{App}\{t\}(x));$

(ii) $\exists t.\exists u.S(t,u) \to \exists u.\exists t.S(t,u)$

$\mathrm{lam}\{\exists t.\exists u.S(t,u)\}(x.\ \mathrm{open}\{\ x;\ t;\ \overset{\exists u.S(t,u)}{a}.\ \mathrm{open}(a;\ t;\ \overset{S(t,u)}{b}.\ \mathrm{pack}\{u; S(t.u)\}\{t\}$
$(\mathrm{pack}\{t; S(t,u)\}\{t\}(b)))$

(iii) $\exists t.\forall u\ S(t,u) \to (\forall u.\ \exists t.S(t,u))$

$\mathrm{lam}\{\exists t.\forall u.S(t,u)\}(x.\ \mathrm{Lam}(u.\ \mathrm{open}(x;\ t;\ \overset{\forall u.S(t,u)}{a}.\ \mathrm{pack}\{t, S(t,u)\}(\mathrm{App}\{u\}(a))))$

$e =$

$$\frac{\sigma \text{ type} \quad \Delta, t \text{ type} \; \Gamma \vdash \tau \text{ type} \quad \Delta \; \Gamma \vdash e : [\sigma/t]\tau}{\Gamma \vdash \text{pack}\{t.\tau\}\{\sigma\}(e) : \exists(t.\tau)}$$

$(\exists t \cdot P(t)) \to A \Rightarrow \forall t \cdot P(t) \to A$

lam{(∃t·P(t))→A}(x· Lam(t· lam{P(t)}(y· ap(x; e))))

e = pack{t, P(t)}{t}(y);

$A \times \exists t \cdot P(t) \to \exists t \cdot A \times P(t)$

lam{A × ∃t·P(t)}(x· open(snd(x); t, a· pair(e₁; a)))

e₁ = pack{t, A}{t}(fst(x))

Alternative

pack{t· A × P(t)}{t}(pair(fst(x); a))

* $(A+C) \times (B+D) \to (D \times C) + (A+B)$

lam{(A+C)×(B+D)}(x· case(snd(x); a· inr{(D×C)+(A+B)}(inr{A;B}(a));
b· case(fst(x); b'· inr{(D×C)+(A+B)}(inl{A;B}(b)); q· pair(b; q))))

$A \to \text{coi}(t \cdot A \times t)$

lam{A}(a· gen{t·A×t}{T}(z; x· pair(a; x)))

$((A \to (B+C)) \to D) \to ((A \to B) + (A \to C)) \to D$

lam{(A→(B+C))→D}(x· lam{(A→B)+(A→C)}(y· case(y; a· ap(x;
lam{A}(a'· inl{B;C}(ap(a; a'))); b· lam{C}(c· inr{B;C}(ap(b; c))))

$\underbrace{\qquad}_{A \to B} + \underbrace{\qquad}_{A \to C}$

$|l_1| + |l_2| = |l_1| \cdot |l_2|$

$rev(Cons(n; l)) = rev\ l \cdot Cons(n; nil)$

$times\ Z\ m = Z$

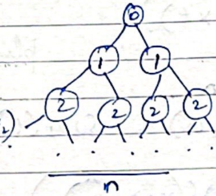
$2 \times 3 = 6$ $\quad Z + n = n$ $\quad |Cons(n; l)| = S(l) \approx |l| + 1.$ $\quad time\ n\ S(m) = S(plus\ (n\ m))$

$0 + 2 = 2$ $\quad n\ S(m) = S(plus\ n\ m)$

$2 + 3 = 5$

$node(n; t_1; t_2)$

$(t \cdot Unit + nat \times t_1 \times t_2)$



Co-Binary tree

$Coi\ (t \cdot nat \times (t \times t))$

$gen\ \{t \cdot \overset{nat}{\cancel{unit}} \times (t \times t)\}\ \{T\}\ (Z;\ x \cdot pair(\underbrace{fst(x);\ pair(fst(Snd(x)), Snd(Snd(x)))}_{Inductive}))$

Coinductive $(x;\ pair(S(x);\ S(x)))$

$Map\ \{t \cdot C\}\ \{\ \}\ \{\overset{x \cdot e'}{\cancel{x_0; x_1 \cdot e_2}}\}\ (e)$

pos $\quad$ Mapping Fuction $\quad f/t \to f'/t$

$$\frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash inr\{\tau_1; \tau_2\}(e) : \tau_1 + \tau_2}$$

$$\cancel{\frac{e_1\ Val \quad e_2}{\Gamma \vdash Case(e_0;\ a \cdot e_1;\ b \cdot e_2) \mapsto [e_0/a]e_1 \quad [e_0/b]e_2}}$$

$e_1\ Val$

$e_2\ Val$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash inl\{\tau_1; \tau_2\}(e) : \tau_1 + \tau_2}$$

$$\frac{e\ Val}{inl\{\tau_1; \tau_2\}(e)}$$

$$\frac{[e\ Val]}{\Gamma \vdash Case(inl\{\tau_1; \tau_2\}(e);\ x \cdot e_1;\ y \cdot e_2) \mapsto [e/x]e_1}$$

$$\frac{[e\ Val]}{\Gamma \vdash Case(inr\{\tau_1; \tau_2\}(e),\ x \cdot e_1;\ y \cdot e_2) \mapsto [e/y]e_2}$$

$x \; s(s(x))$ $o+1$ $= 2$ 1 — $\mathrm{rec}\{e_0; x.y.e_1\}(e)$ nat nat

#

tree = ind (t. nat + (t × t))

Sum = tree → nat   plus = nat → nat → nat

Sum (leaf (n)) = n

Sum node(n; t_1; t_2) = Sum (t_1) + Sum (t_2)

lam{tree}(t. rec{t. nat + (t×t)}(x.e; e_0)) plus (fst(b); snd(b))

x = nat + (nat × nat)

e_1 = ~~nat~~ (nat)

e = case (x; a.a; b. ap(ap(plus; fst(b)); snd(b));

---

f n 0 = n   f → nat → nat → nat

f n s(m) = m × (f n m) + 1   mult(m; y)

lam{nat}(n. lam{nat}(m. rec{n; x.y. s(ap(ap(mult; x); y))}(m)))

plus(a; b) →

plus : nat → nat → nat

ap(ap(plus; a); b)

lam{nat}(n. lam{nat}(m. rec{t. unit + nat}(x.e; n)

x = Unit + nat

e = case(x; a.n; b. s(ap(ap(mult; m); b)))

$n + p(m) = p(n+m)$

By Induction on n

Lemma.

$p(Z+m) = p(m)$

Case $n = Z$

$Z + p(m)$

$= p(m)$ By Rule A_1

proved

$p(Z+m) = p(m)$

Case $n = S(n)$

$S(n) + p(m)$

$S(n + p(m))$ — By A2

$S(p(n+m))$ By IH

$n+m$ By L1

By Lemma 2 — Lemma 1

RHS — $p(S(n)+m)$

$p(S(n+m))$ = By L2

$n+m$

Proved

4 2 2 2

5 3 2 2

$\downarrow \; \downarrow \quad n - m$

sub $Z \; m = Z$

sub $S(n) \; Z = S(n)$

sub $S(n) \; S(m) = (\text{sub}(n \; m))$

lam{nat}(n. lam{nat}(m. rec{Z; x.y. rec{S(n); p.q.

ap(q; p)}(m)}(n)

leq z m = S(z)
leq S(n) z = z
leq S(n) S(m) = leq n m

pred

lam{nat}(n. lam{nat}(m. rec{S(z); x.y. rec{z; p.q. ap(y; p)}(m)}(n)))

$fib(n-1)$ = ap(y; pred(x))

ap(pred; x)

fib z = z
fib S(z) = S(z)
fib S(n) = fib(n) + fib(n-1)

lam{nat}(n. rec{z; x.y. rec{S(z); p.q. plus(y; y(p))}(n)}

~~lam{nat}~~

y; y(p)

Pred(y) ap(y; p)

lam{nat}(n. rec{lam{nat}(m. rec{z; x.y. rec{lam{nat}(n_2.
rec{S(z); p.q. rec{z; v.v. ap(q; v)}}(n_2)

App{t}(p(t)) leq n_1

(7) $(\forall t . P(t) \to A) \to (\exists t . P(t)) \to A)$

lam{∀t . P(t)→A}(x. lam{∃t . P(t)}(y. open(y; t, p. e)))

e = ap(App{t}(x); p)

$P(t) \to A$

$p(t)$

or $\quad P \to 0 \quad \text{lam}\{P\}(\overset{n}{x}:P) \quad \underline{P \to 0}$ $\quad 2\ 0 = 2,\ 2\ 1 = 1$ prod $(T(sm))$

$$\left(\left(P \overset{N}{+} (P \to 0)\right) \to 0\right) \to 0$$

$$P \overset{y}{\to} 0$$

$$\text{lam}\{(P + (P \to 0)) \to 0\}(x \cdot ap(x;\ \text{inl}\{P;(P \to 0)\}(\text{lam}\{P\}(e))))$$

$$e = ap(x;\ \text{inl}\{P;\ P \to 0\}(y))$$

Height(LTR(t)) = Height(t)

By Induction on t

$t =$ empty

empty

Case $t = \text{node}(n; t_1; t_2)$

Height (LTR(node$(n; t_1; t_2)$))

Height (LTR Node$(n; LTR(t_1); LTR(t_2))$)

Height (Node$(n$; Height(t_1); Height$(t_2))$) By IH

$S(\text{Max}(H(t_1); H(t_2)))$

$H(\text{Node}(N; t_1; t_2))$

TSUB n z = pred(T n m)

TSUB $S(m)$ $S(m)$ =

$\text{lam}\{\text{nat}\}(n \cdot \text{lam}\{\text{nat}\}(m \cdot \text{rec}\{n; x \cdot y \cdot \text{pred}(y)\}(m)$

Sub n z = z

Sub $S(n)$ z = $S(n)$

Sub $S(n)$ $S(m)$ = Sub n m

Size empty = empty

Size node$(n; t_1; t_2)$ = node Size(t_1) + Size(t_2)

lt z z = z

lt z $S(m)$ = $S(z)$

lt $S(n)$ z = z

lt $S(n_1)$ $S(n_2)$ = lt n m

leq z m = $S(z)$

leq $S(n)$ z = z

leq $S(n)$ $S(m)$ = leq n m

$\text{lam}\{\text{nat}\}(n \cdot \text{lam}\{\text{nat}\}(m \cdot \text{rec}\{S(z); x \cdot y \cdot \text{rec}\{z; p \cdot q \cdot (y; p)\}(m)\}(n)$

$\text{lam}\{\text{nat}\}(n_1 \cdot \text{rec}\{\text{lam}\{\text{nat}\}(m \cdot \text{rec}\{z; x \cdot y \cdot S(z)\}(m)); p \cdot q \cdot$ $\text{lam}\{\text{nat}\}(n_2 \cdot \text{rec}\{z; u \cdot ap(q; u)\}(n_2))\}(n_1)$

rev (empty) = empty
rev (node (n; t_1; t_2)) = node (n; rev (t_1); rev (t_2))

Q → Height (rev (t)) = height (t)
By Induction on t

Case t = empty
height (empty)

Case t = node (n; t_1; t_2)
H (rev (node (n; t_1, t_2)))
Height (node (n; rev (t_1), rev (t_2)))
S (Max (Height (rev (t_1)), Height (rev (t_2))))
S (Max (H (t_1), H(t_2))) — By IH
node (n; t_1; t_2) — By rule

(m+n).l = m.l + n.l
By Induction on m

(m+1)l
ml + l + n.l

Case m = Z
(Z+n).l — Plus rule Z+n = n
~~n~~ n.l
= Z + n.l

Case m = S(m)
(S(m) + n).l = (m+1)l + n.l
S(m+n).l = m.l + l + n.l
m.l + (n+1)l
m.l.

inl(inl)
inl(inr)

$P+(Q+R) \to (P+Q)+R$ inr

LTR empty = empty
LTR node(n; t₁; t₂) = node(n; LTR(t₁); LTR(t₂))

$*\ [((P\to Q)\to Q)\to Q] \to (P\to Q)$

lam{((P→Q)→Q)→Q}(x. lam{P}(p. ap(x; lam{P→Q}(y. ap(y; p))))))

$*\ (P\to Q)\to((P\to Q)\to Q)\to Q$

lam{P→Q}(x. lam{(P→Q→Q}(y. ap(y; x)))

$*\ (P+(P\to R))\to R) \to R$

lam{(P+(P→R))→R}(x. ~~lam{P+(P→R)}~~(y. case(y; a. inl{P; (P→R)}(a); b. inr{P; P→R}(lam{P}(p. ap(b; p))))))

$*\ R \to ((P+(P\to R))\to R$

lam{R}(x. lam{P+(P→R)}(y.

lam{R}(x. ~~lam{P+(P→R)}(y. case(y; a. inl{P; (R→R}(a); b. inr{P; (R→R}~~ (ap(b; a))))

$(P+(P\to R))\to R)\to R$

lam{(P+(P→R))→R}(y. (inr{P; (P→R)}
ap(y; inr{P; P→R}(lam{P→R}(p. inl{P; P→R)

$\forall$ or

$$((P + (P \to R)) \to R) \to R$$

R?

$\mathrm{lam}\{(P+(P\to R))\to R\}(x . \mathrm{ap}(x; \mathrm{inr}\{P; P\to R\}(\mathrm{lam}\{P\}(p . \mathrm{ap}(x; \mathrm{inl}\{P; P\to R\}(p))))))$

take : Nat $\to$ (list $\to$ list)

take z l = nill

take S(n) nill = nill.

take S(n) Cons(n; l) = Cons(n; take n l)

take = lam{nat}(n . rec{t. unit + ~~nat x~~}(x . e ; n))

x = unit + (list $\to$ list)

nat

e = lam{list}(l_1 . Case(x; a . nill; b . e_2))

e_2 = rec{t. unit + nat x t}(y . e ; e_1)

list

y = unit + nat x list

take n l

e = Case(x; a . nill; b . Cons(fst(b); snd(b)))

plus $\cancel{n}$ m = m
plus $s(n)$ s(m) = s(plus n m)

2 0 = 1

Time n z = z
Times n s(m) plus (Time n m ; m)

pred z = Z
pred s(n) = n

double z = Z
double s(n) s(s(dn))

fact z = Z
fact s(n) = s(n) × fact n

TSub $\cancel{n}$ $\cancel{z}$ = n
TSub n s(m) = pred (TSub n m)

Sub z m = Z
Sub s(n) z = s(n)
Sub s(n) s(m) = sub n m

fib z z
fib s(z) = s(z)
fib s(n) = fib n + fib n-1

$$\frac{\Delta\ \Gamma \vdash e_0 : (\exists(t.\tau) \quad \Delta, t\ \text{type}\ \Gamma\ x:\tau \vdash e_1 : \tau_2 \quad \Delta \vdash \tau_2\ \text{type}}{\Gamma \vdash \text{open}\{t.\tau\}\{\tau_2\}(e_0;\ t;\ x.e_1) : \tau_2}$$

$$\frac{\Delta, t\ \text{type} \vdash \tau\ \text{type} \quad \rho\ \text{type} \quad \Gamma \vdash e : [\rho/t]\tau}{\Gamma \vdash \text{pack}\{t.\tau\}\{\rho\}(e) : (\exists(t.\tau)}$$

System T rec $\dfrac{\Gamma \vdash e : nat \quad \Gamma \vdash e_0 : \tau \quad \Gamma, x : nat, y : \tau \vdash e_1 : \tau}{\Gamma \vdash rec\{e; e_0; x.y.e_1\} : \tau}$

Progress

$e : \tau$ then $e \mapsto e'$ or e val

$e = rec\{e; e_0; x.y.e_1\}$

$\tau = \tau$

Above the line

$\Gamma \vdash e : nat$

$\Gamma \vdash e_0 : \tau$

$\Gamma, x : nat, y : \tau \vdash e_1 : \tau$

By IH on e

Case $e \mapsto e'$

take e' as $rec\{e'; e_0; x.y.e_1\}$

Done

Case e val

By C.F.L $e : nat$ and e val Hyp. $e = z$ / $e = s(e)$

Case $e = z$

take e' as e_0

$rec\{z; e_0; x.y.e_1\} \mapsto e_0$ e val

Case $e = s(e)$

take e' as $[e, rec\{e; e_0; x.y.e_1\}/x, y]$

$e \mapsto e'$

$s(e)$ val By rule

$open(pack\{t.\tau\}\{\rho\}(e); t, x.e_2) \mapsto [\rho, e/t, x]e_2$

$e = open(pack\{t.\tau\}\{\rho\}(e); t, x.e_2)$

$e' = [\rho, e/t, x]e_2$

Goal

$[\rho, e/t, x]e_2 : \tau_2$

By Inversion

$pack\{t.\tau\}\{\rho\}(e) : \exists(t.\tau)$ — H_1

$\Delta, t\ type\ \Gamma x : \tau \vdash e_2 : \tau_2$ — H_2

$\tau_2\ type$ — H_3

By substitution H5 & H2

$\Delta, [\rho/t]x : \tau \vdash [\rho/t]e_2 : \tau_2$ — H7

By Inversion on H_1

$\Delta, t\ type \vdash \tau\ type$ — H_4

$\Delta \vdash \rho\ type$ — H_5

$\Delta \Gamma \vdash e : [\rho/t]\tau$ — H_6

Again Substituting H6 & H7

$[\rho, [e/t], x]e_2 : \tau_2$

$a \to b$
$lam\{a\}(x.e)$

plus(a;b)

where plus nat → nat → nat

$\underbrace{\underbrace{\text{plus a}}_{E_1 \quad E_2}\ \text{b}}_{E_1 \quad E_2}$

ap(ap(plus ; a) ; b)

old ~~old~~

a: nat x nat

App{τ}(fst(a)) ⟶ fst(a) [τ]

old new

A → B τ

λa: A.e

New

(lam{A}(a.e))

old

~~By Strength on H2~~
~~$\Delta, \Gamma x : C \vdash e_2 : \tau$~~

By Substituting H5 & H2
~~$\Delta, [\beta/t] \Gamma x : C \vdash e_2 : C_2$ — H7~~
[β/t]

By Substituting H6 & H7
$\Delta \Gamma \vdash e : [\beta/t]$

$\Delta, \Gamma\ [e/x] \vdash e_2 :$ ~~$[\beta/t] C_2$~~

$S(e)$

$S(e)$ val

$rec\{e; e_0; x.y.e_1\} \longmapsto [e, rec\{e; e_0; x.y.e_1\}/x, y]$

$e = rec\{e_0; e_0; x.y.e_1\}$

$e' = rec\{e_0; e_0; x.y.e_1\}$

Above the line

$S(e)$ val — H_1

By Inversion on $e : \tau$

$\Gamma \vdash S(e) : nat$ — H_2

$\Gamma \vdash e_0 : \tau$ — H_3

$\Gamma, x : nat, y : \tau \vdash e_1 : \tau$ — H_4

By Inversion on H2

$\Gamma \vdash e : nat$ — H_5

By weakening on H5

$\Gamma, y : \tau \vdash e : nat$ — H6

By Substituting H6 & H5

~~$y : \tau \vdash [e/x] \vdash e_1 : \tau$~~

$y : \tau \vdash [e/x]e_1 : \tau$

By rule

$rec\{e; e_0; x.y.e_1\} : \tau$ — H_8

By Substituting H7 & H8

$[e, rec\{e; e_0; x.y.e_1\}/x, y] : \tau$

0!

plus Z m = m
plus n S(m) = S(plus n m)

times Z m = Z
times n S(m) plus(times n m) m

TSub Z m = Z
TSub n S(m) = Prod(TSub n m)

Sub Z m = Z
Sub S(n) Z = S(n)
Sub S(n) S(m) = Sub n m

Prod Z = Z
Prod S(n) = n

fact Z = ~~Z~~ S(Z)
fact S(n) = S(n) x fact n

fib Z = Z
fib S(Z) = S(Z)
fib S(n) = fib(n) + fib(n-1)