

GRAVITATIONAL INSTANTONS WITH S^1 SYMMETRY

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ABSTRACT. Uniqueness results for asymptotically locally flat and asymptotically flat S^1 -symmetric gravitational instantons are proved using a divergence identity of the type used in uniqueness proofs for static black holes, combined with results derived from the G -signature theorem. Our results include a proof of the S^1 -symmetric version of the Euclidean Black Hole Uniqueness conjecture, a uniqueness result for the Taub-bolt family of instantons, as well as a proof that an ALF S^1 -symmetric instanton with the topology of the Chen–Teo family of instantons is Hermitian.

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1. INTRODUCTION

A complete Ricci-flat four-manifold with at least quadratic curvature decay is called a gravitational instanton. Apart from their intrinsic interest as geometric objects, the study of gravitational instantons is motivated by ideas from Yang–Mills theory and Quantum Gravity [29]¹. In this paper we contribute to the classification of instantons by proving a set of uniqueness results for asymptotically locally flat (ALF) S^1 -symmetric instantons, making assumptions only on their topology. One of our results is the S^1 -symmetric version of the Riemannian signature version of the Black Hole Uniqueness conjecture of General Relativity, known as the Euclidean Black Hole Uniqueness Conjecture, which states that an instanton on $S^4 \setminus S^1$ is in the Kerr family of gravitational instantons, the Wick-rotated version of the Kerr family of black hole solutions of General Relativity. We shall focus on ALF instantons that admit an effective S^1 -action by isometries, generated by a Killing

¹Although the notion of gravitational instanton is sometimes introduced with additional assumptions, for example that the space is hyperkähler, we make no such a priori assumptions in this paper.

field with uniformly bounded norm, and use the term S^1 -instantons for these, cf. Definition 2.10 below.

Some early results related to the present work were presented by one of the authors in [54, 43]. In those papers, additional assumptions on the structure of the fixed point set were needed, as well as some non-trivial technical assumptions that are removed here. The facts about the structure of the fixed point set that were used in the just mentioned papers are applications of the index theorems to instantons with S^1 symmetry, as presented by Gibbons and Hawking in [25], see also [28]. In this paper, we instead follow the approach of Jang. See [34] and references therein. This method, which makes use of the full power of the G -signature formula, gives more precise information and enables us to eliminate assumptions on the fixed point set of the S^1 -action.

Recall that an ALF instanton has cubic volume growth and boundary at infinity L of topology $S^1 \times S^2$, S^3 , or a quotient thereof, see Definition 2.1. The case when L is a circle bundle over S^2 or \mathbb{RP}^2 is called ALF- A_k or ALF- D_k , respectively. In the case of ALF- A_k instantons, $k = -e - 1$, where e is the Euler characteristic of the circle bundle over S^2 . The case ALF- A_{-1} when the boundary at infinity is a trivial circle bundle over S^2 is called asymptotically flat (AF).

The most well-known AF instanton is the Euclidean Kerr Instanton, a 2-parameter family of instantons on $S^4 \setminus S^1 \cong S^2 \times \mathbb{R}^2$, with the Wick-rotated Kerr metric [24, 26, 31].

The Euclidean Kerr Instanton is toric, that is it admits an effective T^2 -action by isometries, and is algebraically special, of Petrov type D^+D^- , but does not have special holonomy. In addition to the Kerr Instanton, there is the remarkable Chen–Teo Instanton [15], a 2-parameter family of toric AF instantons on $\mathbb{CP}^2 \setminus S^1$. Two of the authors have recently proved that the Chen–Teo Instanton is one-sided algebraically special [1], and hence also Hermitian [19], i.e. it admits an integrable complex structure compatible with the metric. In view of this fact, all known examples of instantons are Hermitian. Both the Kerr and Chen–Teo instantons are S^1 -symmetric for suitable values of the parameters satisfying a rationality condition².

The Chen–Teo instanton is a counterexample to one of the 1970’s era uniqueness conjectures for gravitational instantons, the Euclidean Black Hole Uniqueness Conjecture [23, Conjecture 2], which stated that a non-flat AF instanton is in the Kerr family³. This shows that the Euclidean Black Hole Uniqueness Conjecture can hold only with additional assumptions, and moreover indicates that the classification problem for gravitational instantons merits renewed attention.

Examples of ALF- A_k instantons include the hyperkähler Taub–NUT instanton [29, 37] with topology $S^4 \setminus \{\text{pt.}\}$ and the Petrov type D^+D^- Taub–bolt instanton [47] with topology $\mathbb{CP}^2 \setminus \{\text{pt.}\}$. These are both $SU(2)$ -symmetric, and in particular toric and S^1 -symmetric. The remaining ALF- A_k examples are the Gibbons–Hawking multi-Taub–NUT solutions and the ALF- D_k instantons with the S^1 -symmetric Atiyah–Hitchin metrics, and the Cherkis–Hitchin–Kapustin–Ivanov–Lindström–Roček metrics, see [16, 11], which are all hyperkähler. See also [9]. The collinear multi-Taub–NUT instantons are toric and S^1 -symmetric.

For gravitational instantons with special geometry, some classification results are known. There is a complete classification of hyperkähler instantons with total curvature in L^2 [12, 11, 13, 55]. Further, toric ALF Hermitian instantons were

²For general parameter values, both Kerr and Chen–Teo admit a Killing field with bounded norm, which however generates closed orbits only for a restricted set of parameters.

³Note that the Kerr family includes the spherically symmetric Schwarzschild Instanton, and that a flat ALF instanton is, up to a rescaling, isometric to $\mathbb{R}^3 \times S^1$.

recently classified by Biquard and Gauduchon [8], and shown to belong to one of the Kerr, Chen–Teo, (multi-)Taub–NUT, or Taub–bolt families. The general classification problem remains open. See also [36] for uniqueness results on toric instantons without assumption on special geometry.

The following theorem states the main results of this paper.

Theorem 1.1. *Let (\mathcal{M}, g_{ab}) be an S^1 -instanton of ALF- A_k type. Then the following holds.*

- (1) *If $\mathcal{M} \cong S^4 \setminus S^1$, then (\mathcal{M}, g_{ab}) is in the Kerr family of instantons.*
- (2) *If $\mathcal{M} \cong \mathbb{CP}^2 \setminus \{\text{pt.}\}$, then (\mathcal{M}, g_{ab}) is in the Taub–bolt family of instantons.*
- (3) *If $\mathcal{M} \cong \mathbb{CP}^2 \setminus S^1$, then (\mathcal{M}, g_{ab}) is Hermitian.*

Point 3 of Theorem 1.1 is not a rigidity result in the same sense as the first two points. However, it can be viewed as a step towards rigidity for the Chen–Teo family in view of the following conjecture.

Conjecture 1 ([1, Conjecture 1]). *Let (\mathcal{M}, g_{ab}) be a non-Kähler ALF Hermitian instanton. Then (\mathcal{M}, g_{ab}) is one of Kerr, Chen–Teo, Taub–bolt, or Taub–NUT with the orientation opposite to the hyperkähler orientation.*

Remark 1.2. *An ALF non-Kähler Hermitian instanton admits a bounded Killing field, and is locally conformal to an extremal Kähler space. By analogy with the classification of compact Hermitian-Einstein spaces [38], it is plausible that a non-Kähler Hermitian instanton is actually toric, which in view of the classification result of [8] would imply the conjecture.*

We further make the following progressively stronger conjectures, closely related to Conjecture 1.

Conjecture 2. *The conclusion of Conjecture 1 holds for a non-Kähler ALF gravitational instanton which admits a non-vanishing Killing field with bounded norm.*

Remark 1.3. *In view of the results of Yau [58], the existence of one Killing field with bounded norm for a non-flat ALF instanton implies that one has either S^1 or toric symmetry. See also [25, §2]. The results in the present paper imply that an ALF instanton with the topology of Kerr, Taub–bolt or Chen–Teo must be Hermitian. Therefore the only part of the conjecture that is open is the case of toric instantons without special geometry.*

Finally, we mention the following conjecture, analogous to those made in [23], and stronger than Conjectures 1 and 2.

Conjecture 3. *An ALF gravitational instanton is Hermitian.*

Remark 1.4. *The problem raised by Conjecture 3 can be expected to be difficult, analogous to the question whether all compact Ricci-flat spaces have special holonomy, see [7, p. 19]. However, the ALF assumption provides structure not present in the compact case.*

The Chen–Teo Instanton was found using the Belinski–Zakharov soliton method, dressing a flat seed solution with four solitons and coalescing one pair of those. The computational methods employed by Chen and Teo did not enable them to explore solutions with more than four solitons. Thus, a priori there could exist additional instantons which can be constructed along the same lines, but containing more solitons. This is an interesting direction of research.

In addition to the families of instantons mentioned above, some additional potential examples have been mentioned in the literature. These include the Euclidean signature of the Kerr–NUT family of metrics, for example the so-called

Kerr–Taub-bolt instanton, see [27, §III]. An argument showing that the Euclidean Kerr–Taub-bolt instanton must have line singularities was given in [14, §4.8], see also [18, Appendix A].

We shall now indicate some of the background for the technique used in this paper. Recall that the Schwarzschild and Kerr instantons were constructed by Wick rotating known Lorentzian black hole metrics. The problem of classifying gravitational instantons has important parallels to the Black Hole Uniqueness Problem, and indeed, the divergence identity used in this paper arose out of attempts to prove the Black Hole Uniqueness Conjecture by generalizing Israel’s method [33, 44, 51] for the uniqueness problem from the static Lorentzian case to the stationary Lorentzian case.

In the case of a hypersurface orthogonal Killing vector field ξ this method applies to spaces of both Riemannian and Lorentzian signature (with non-compact isometry group \mathbb{R} in the Lorentzian case). The key ingredients in the argument are divergence identities on the 3-dimensional orbit space, among which we select

$${}^{(3)}\nabla_i \left(\frac{(1+\lambda)^2}{\sqrt{\lambda}} {}^{(3)}\nabla^i \frac{\sqrt{W}}{1-\lambda^2} \right) = Q, \quad (1.1)$$

which is closely related for example to [44, Eqs. (2.12)-(2.13)]. Here ${}^3\nabla_i$ is the Levi-Civita covariant derivative on the orbit space, λ is the squared norm of the Killing field, $\lambda = |g_{ab}\xi^a\xi^b|$, W is given by $W = ({}^{(3)}\nabla_i \sqrt{\lambda})({}^{(3)}\nabla^i \sqrt{\lambda})$, and Q is a non-negative expression in curvature such that $Q = 0$ if and only if the orbit space is conformally flat. The integral of the left hand side of (1.1) can, after dealing with potential singularities resulting from possible zeros of W , be evaluated in terms of contributions from infinity and from the surface gravities computed from the fixed point data of the isometry. The latter, in turn, can be related to the topology of \mathcal{M} using index theorems. In case the left hand side vanishes, it is possible to conclude that the space(-time) under consideration is Schwarzschild. This perspective is further developed in Appendix C.

The above scheme can, with considerable effort, be generalized to the generic case when ξ fails to be hypersurface-orthogonal. Let \mathcal{F}^\pm_{ab} be the (anti)-self dual parts of $(d\xi)_{ab}$, let $\sigma^\pm_a = 2\mathcal{F}^\pm_{ab}\xi^b$, and define the Ernst potentials \mathcal{E}^\pm by $\nabla_a \mathcal{E}^\pm = \sigma^\pm_a$. In the Riemannian case, this yields a pair of real scalar fields \mathcal{E}^\pm , while in the Lorentzian case, we have the familiar complex Ernst potential and its complex conjugate.

Restricting ourselves to the Riemannian case, we define real currents Ψ^\pm_a in terms of \mathcal{E}^\pm , λ , and the norms of \mathcal{F}^\pm_{ab} , see Section 5, such that

$$\nabla_a \Psi^\pm a = Q^\pm. \quad (1.2)$$

Here Q^\pm are non-negative expressions defined in terms of λ , the Weyl tensor, and \mathcal{F}^\pm_{ab} , and such that $Q^\pm = 0$ if and only if (\mathcal{M}, g_{ab}) is algebraically special. This pair of currents was first introduced by one of the authors [54]. As in the case when ξ is hypersurface orthogonal, when (1.2) reduces to (1.1), the integral of the left hand side of (1.2) can be evaluated in terms of the fixed point data for the S^1 action and the length of the S^1 -orbits at infinity.

The G -signature theorem for an ALF- A_k instanton yields an identity which equates the signature of \mathcal{M} with a rational expression, with parameters determined by the fixed point data for the S^1 -action, including the orientation and weights of the nuts, the self-intersection numbers of the bolts, as well as the Euler characteristic of the circle bundle at infinity, see (3.14). The fact that this rational expression is actually constant implies strong restrictions on the fixed point data. This observation was used by Jang, see [34], following earlier work by Li and Liu [40], and Li [39] to derive strong conditions on the fixed point set in terms of the

topology, in the case of S^1 -actions with only isolated fixed points. See also [20] for earlier related work.

Here we generalize these ideas to give restrictions on the fixed point set for ALF- A_k spaces with an S^1 -action, without assuming that there are only isolated fixed points. Making use of the results on the structure of the fixed point set derived from the G -signature theorem in the manner mentioned above, it is possible to show that for suitable topologies, equality holds in (1.2), which then implies algebraic speciality.

In the particular cases of instantons with topology $S^4 \setminus S^1$ one has equality for both signs in (1.2), which implies that the space is Petrov type D^+D^- . This, in turn, by [8] implies that the space is in the Euclidean Kerr family. The same situation holds in the case of $\mathbb{CP}^2 \setminus \{\text{pt.}\}$, yielding a uniqueness result for the Taub-bolt instanton. The just discussed results are points 1, 2 of Theorem 1.1. Finally, for an instanton of topology $\mathbb{CP}^2 \setminus S^1$, one finds that equality holds only for one sign, yielding the statement that the instanton is Hermitian, point 3 of Theorem 1.1.

We end this introduction by pointing out that in the Lorentzian case, the (anti)-self dual parts \mathcal{F}^\pm of $d\xi$ are complex, which means that the Ernst potential, as well as the corresponding current Ψ_a , are complex. Therefore, although there is a complex counterpart to (1.2) in the Lorentzian case, see [53], it is not clear whether it is possible to proceed towards a proof of the Black Hole Uniqueness Conjecture along the lines sketched above. We remark, however, that a related wave-type equation was employed in a uniqueness proof of Kerr for small angular momenta via completely different methods, see [2, 32].

2. ALF INSTANTONS

In this section, we introduce the notions of ALF four-manifold and ALF instanton and prove some results on the asymptotics of ALF spaces that will be needed in the proof of the main results. We shall use the following conventions throughout the paper. We use abstract index notation following [48]. Unless otherwise stated, (\mathcal{M}, g_{ab}) is assumed to be a smooth, complete, orientable Riemannian four-manifold, with Levi-Civita covariant derivative ∇_a . The Riemann and Ricci tensors, and the scalar curvature are given by $R_{abc}{}^d \nu_d = (\nabla_a \nabla_b - \nabla_b \nabla_a) \nu_d$, $R_{ab} = R^c{}_{acb}$, $R = R^a{}_a$, respectively. We will sometimes use index-free notation, and for example write $g = g_{ab} dx^a dx^b$ for the metric tensor. The norm of a tensor $\varpi_{ab\dots d}$ is defined by $|\varpi|^2 = \varpi_{ab\dots d} \varpi^{ab\dots d}$.

Definition 2.1 ([8, Def. 1.1]). *We say that a complete Riemannian four-manifold (\mathcal{M}, g) is asymptotically locally flat (ALF) if the following holds.*

- (1) \mathcal{M} has an end diffeomorphic to $\mathring{\mathcal{M}} = (A, \infty) \times L$ where L is $S^1 \times S^2$, S^3 , or a finite quotient thereof.
- (2) There is a triple (η, T, γ) defined on L , where η is a 1-form, T a vector field, with

$$i_T \eta = 1, \quad i_T d\eta = 0 \tag{2.1}$$

and γ defines a T -invariant metric on the distribution $\ker \eta$.

- (3) The transverse metric γ has constant curvature +1.
- (4) Let

$$\dot{g} = dr^2 + r^2 \gamma + \eta^2 \tag{2.2}$$

be a metric on $(A, \infty) \times L$, with Levi-Civita covariant derivative $\dot{\nabla}$. Here γ is extended to a tensor on L with $\ker \gamma$ generated by T . We assume that

the metric g restricted to the end is of the form

$$g = \dot{g} + h \quad (2.3)$$

where h is a symmetric 2-tensor such that for all non-negative integers k , there are constants C_k with

$$|\overset{\circ}{\nabla}{}^k h|_{\dot{g}} \leq C_k r^{-1-k}. \quad (2.4)$$

An ALF manifold with $L = S^1 \times S^2$ is called asymptotically flat (AF). If (\mathcal{M}, g_{ab}) is ALF with model $\mathring{\mathcal{M}} = (A, \infty) \times L$, we shall refer to L as the boundary at infinity of (\mathcal{M}, g_{ab}) .

Definition 2.2 (ALF instanton). Let (\mathcal{M}, g_{ab}) be a complete four-dimensional Ricci-flat Riemannian space. Then, (\mathcal{M}, g_{ab}) is an ALF gravitational instanton if it is ALF in the sense of definition 2.1.

Remark 2.3. (1) In view of definition 2.1,

$$\mathcal{L}_T \eta = d(i_T \eta) + i_T d\eta = 0, \quad \mathcal{L}_T \gamma = 0, \quad (2.5)$$

and hence T is a Killing field for \dot{g} , $\mathcal{L}_T \dot{g} = 0$.

- (2) Definition 2.1 is compatible with the case when the orbits of T do not close. This holds for generic AF models.

The following remark highlights some features of the geometry of AF instantons.

Remark 2.4 (AF Instanton, see [1, Definition 2.1]). (1) Let $\kappa, \Omega \in \mathbb{R}$, $\kappa \neq 0$, $|\Omega/\kappa| < 1$. Consider \mathbb{R}^4 with coordinates $(\tau, r, \theta, \phi) \in \mathbb{R} \times \mathbb{R}_+ \times [0, \pi] \times [0, 2\pi]$ so that the flat metric takes the form

$$d\tau^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.6)$$

Let $(\mathring{\mathcal{M}}, \dot{g}_{ab})$ be the flat space defined as \mathbb{R}^4 / \sim where the equivalence relation \sim is given by the identification

$$(\tau, r, \theta, \phi) \sim (\tau + 2\pi/\kappa, r, \theta, \phi + 2\pi\Omega/\kappa). \quad (2.7)$$

Introducing new Killing coordinates $\tilde{\tau}, \tilde{\phi}$ by

$$\tau = \frac{1}{\kappa} \tilde{\tau}, \quad \phi = \frac{\Omega}{\kappa} \tilde{\tau} + \tilde{\phi}, \quad (2.8)$$

we have that the identification

$$(\tilde{\tau}, \tilde{\phi}) \sim (\tilde{\tau} + 2\pi, \tilde{\phi} + 2\pi) \quad (2.9)$$

corresponds to (2.7). A non-flat ALF instanton with model $(\mathring{\mathcal{M}}, \dot{g}_{ab})$ is said to be AF, with parameters κ, Ω .

- (2) If

$$\Omega/\kappa = q/p, \quad \text{with } q, p \text{ mutually prime integers.} \quad (2.10)$$

then the action of $T = \partial_\tau$ has closed orbits, and $(\mathring{\mathcal{M}}, \dot{g}_{ab})$ is a model of an AF S^1 -instanton. The generic orbits of ∂_τ have period $2\pi p/\kappa$ where p, κ are as in (2.10). At the poles $\theta = 0, \pi$, where ∂_ϕ vanishes, we have $\partial_\tau = \kappa \partial_{\tilde{\tau}}$ with the exceptional period $2\pi/\kappa$.

Lemma 2.5 ([8, Lemma 1.7]). Let (\mathcal{M}, g_{ab}) be an ALF instanton. Then $H_{\text{dR}}^1(\mathcal{M}) = 0$.

Definition 2.6. Let (\mathcal{M}, g_{ab}) be an ALF four-manifold and let r be as in Definition 2.1. For a tensor field t on (\mathcal{M}, g_{ab}) , we say that $t = O(r^\alpha)$ if there is a constant C such that $|t| \leq Cr^\alpha$ for $r \geq A$, and write

$$t = O^*(r^\alpha) \quad (2.11)$$

if $|\nabla^k t| = O(r^{\alpha-k})$, for all non-negative integers k .

Remark 2.7. Let g be an ALF metric as in Definition 2.4. As explained [8, Proof of Prop. 1.6], η has the local form $\eta = dt + \alpha$ where $dt(T) = 1$, and α is a 1-form on the quotient of L by the action of T , satisfying $\alpha = O^*(r^{-1})$. On sufficiently small domains in L of the form $U \times (-t_0, t_0)$ with $U \subset S^2$, we have

$$r^2\gamma + \eta^2 = r^2\sigma + dt^2 + O^*(r^{-1}), \quad (2.12)$$

where σ is the standard metric on S^2 , and hence

$$g = dr^2 + r^2\sigma + dt^2 + O^*(r^{-1}) = \delta + dt^2 + O^*(r^{-1}) \quad (2.13)$$

on

$$C \times (-t_0, t_0) \subset \mathbb{R}^3 \times (-t_0, t_0). \quad (2.14)$$

Here C is the cone

$$C = (A, \infty) \times U \subset \mathbb{R}^+ \times S^2 \subset \mathbb{R}^3 \quad (2.15)$$

and δ denotes the flat metric on \mathbb{R}^3 . It follows that the curvature of g satisfies $\text{Riem} = O^*(r^{-3})$.

Definition 2.8.

- (1) An ALF instanton with L an S^1 bundle over S^2 with Euler number e is referred to as ALF- A_k , with $k = -e - 1$. We refer to the particular case with $L = S^2 \times S^1$ as AF, or ALF- A_{-1} .
- (2) An ALF instanton with L a S^1 bundle over \mathbb{RP}^2 is said to be of type ALF- D_k .

Remark 2.9. An ALF instanton of topology $S^4 \setminus S^1$ or $\mathbb{CP}^2 \setminus S^1$ is ALF- A_{-1} , or equivalently AF. These cases correspond to the Kerr and Chen–Teo families of instantons, respectively. Similarly, an ALF instanton with topology for example $\mathbb{CP}^2 \setminus \{\text{pt.}\}$, corresponding to the Taub–bolt instanton, is ALF- A_0 .

Definition 2.10 (ALF S^1 -instanton). Let (\mathcal{M}, g_{ab}) be an ALF instanton. Then (\mathcal{M}, g_{ab}) is an S^1 -instanton if it admits an effective S^1 action by isometries, generated by a Killing field with uniformly bounded norm.

Remark 2.11. The Euclidean Kerr instanton is AF with parameters κ, Ω corresponding to surface gravity and rotation speed, respectively. The non-rotating Euclidean Schwarzschild instanton is the limit of Kerr with $\Omega = 0$ and is S^1 -symmetric. On the other hand, the Euclidean Kerr instanton is an S^1 -instanton only if $\kappa/\Omega = q/p$ for mutually prime integers q, p . See [31].

Remark 2.12. Recall that L in Definition 2.1 has topology $S^2 \times S^1$, S^3 or a quotient thereof. We can refer to these as lens spaces, see [21, §3].

Assuming that the orbits of T are closed, the manifold L in Definition 2.1 is a Seifert fibration, that is a circle fibration over a base orbifold O . Recall that by Proposition B.2, if (\mathcal{M}, g_{ab}) is an S^1 -instanton, then T has closed orbits.

The orbifold Euler characteristic is denoted by $\chi[O]$, see [52, S 2] for background. If O has \tilde{O} as underlying surface, and has m cone points with angles $2\pi/q_i$, $i = 1, \dots, m$, then $\chi[O] = \chi[\tilde{O}] - \sum(1 - 1/q_i)$, see [52, p. 427].

If O is the base of the Seifert fibration of a lens space, then either O is orientable in which case it has S^2 as underlying surface, with 2 cone points, $S^2(n, n)$, while if

it is non-orientable it has \mathbb{RP}^2 as underlying surface, with one cone point, $\mathbb{RP}^2(n)$. The orbifold Euler characteristics are

$$\chi[S^2(n, n)] = 2 - 2(1 - 1/n) = 2/n, \quad (2.16)$$

$$\chi[\mathbb{RP}^2(n)] = 1 - (1 - 1/n) = 1/n. \quad (2.17)$$

For a manifold of type ALF- A_k we have $O = S^2$ and $\chi[O] = 2$, while for a manifold of type ALF- D_k we have $O = \mathbb{RP}^2$ and $\chi[O] = 1$. For a Seifert fibration of a lens space, we have $O = \mathbb{RP}^2(n)$ for $n \in \mathbb{N}$, with orbifold Euler characteristic $\chi[O] = 1 - (1 - 1/n) = 1/n$, see [52, S 2]. For Seifert fibrations of $S^2 \times S^1$, see [52, §4]. The Gauss–Bonnet theorem for orbifolds [52, S 2] states that

$$\int_O K d\mu = 2\pi\chi[O], \quad (2.18)$$

where K is the Gauss curvature. In the present case, we have $K = 1$, which gives the following expressions for the area $A[O]$ of O ,

$$A[O] = 2\pi\chi[O]. \quad (2.19)$$

3. THE FIXED POINT SET

Let (\mathcal{M}, g_{ab}) be an orientable ALF four-manifold with an effective isometric S^1 -action generated by the Killing field ξ^a , and let F be the fixed point set of the S^1 action. The components of F consist of isolated points called nuts, $P_1, \dots, P_{n_{\text{nuts}}}$ and two-surfaces called bolts $B_1, \dots, B_{n_{\text{bolts}}}$. The fixed points set is orientable. Unless \mathcal{M} is flat, the fixed point set F is non-empty [57]. Hence, in view of the fact that the Euler characteristic of \mathcal{M} is even and that bolts are oriented and have even Euler characteristic, in case \mathcal{M} is not flat, F contains at least a bolt or two nuts.

Let P be a nut. Working in a suitable normal coordinate system, the skew-adjoint tensor $\nabla_a \xi^b$, evaluated at P can be put in the canonical form

$$\nabla_a \xi^b \Big|_P = \begin{pmatrix} 0 & -\kappa^1 & 0 & 0 \\ \kappa^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\kappa^2 \\ 0 & 0 & \kappa^2 & 0 \end{pmatrix} \quad (3.1)$$

and thus has non-zero skew eigenvalues, the surface gravities κ^1, κ^2 . Since ξ^a generates an S^1 -action, κ^1/κ^2 must be rational, and hence there are positive, mutually prime integers w_1, w_2 called weights of P , and $\epsilon(P) = \pm 1$ called the orientation of P such that

$$\frac{\kappa^1(P)}{\kappa^2(P)} = \epsilon(P) \frac{w^1(P)}{w^2(P)}. \quad (3.2)$$

A nut with $w^1 = w^2 = 1$ is called self dual. Near P , the S^1 -action has exceptional orbits with periods $2\pi/|\kappa^i(P)|$, $i = 1, 2$, while the generic orbit has period

$$\frac{2\pi w^1(P)}{|\kappa^1(P)|} = \frac{2\pi w^2(P)}{|\kappa^2(P)|} \quad (3.3)$$

The fixed point data of P is the triple $(\epsilon(P), w^1(P), w^2(P))$. When it is convenient we shall write simply $\epsilon = \pm$ instead of $\epsilon = \pm 1$.

A bolt B is an orientable, totally geodesic two-surface in (\mathcal{M}, g_{ab}) , and thus has even Euler characteristic $\chi(B)$. In addition, the Euler characteristic of the normal bundle of B is equal to the self-intersection number $B \cdot B$. For $p \in B$, working in a

suitable coordinate system, $\nabla_a \xi^b$ can be put in the canonical form

$$\nabla_a \xi^b \Big|_{p \in B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\kappa \\ 0 & 0 & \kappa & 0 \end{pmatrix} \quad (3.4)$$

and thus has skew eigenvalues $0, \kappa$ for some surface gravity $\kappa \neq 0$. The orbits of the S^1 -action near B all have period $2\pi/|\kappa|$.

Lemma 3.1 (Generic period). *There is $G \in \mathbb{R}$ such that for any component Z of F we have*

$$G = \frac{w^1(Z)}{|\kappa^1(Z)|} = \frac{w^2(Z)}{|\kappa^2(Z)|}, \quad \text{if } Z \text{ is a nut} \quad (3.5)$$

$$G = \frac{1}{|\kappa(Z)|}, \quad \text{if } Z \text{ is a bolt} \quad (3.6)$$

In particular, the generic period for the S^1 -action is $2\pi G$.

Proof. The exponential map based at any nut is surjective. The same statement holds for the normal exponential map at a bolt. It follows from this that all periods are mutually divisible. This proves the statement. \square

3.1. Topological invariants. Let (\mathcal{M}, g_{ab}) be an ALF S^1 instanton, then the Euler characteristic of \mathcal{M} can be computed in terms of the fixed point set as

$$\chi[\mathcal{M}] = n_{\text{nuts}} + \sum_{i=1}^{n_{\text{bolts}}} \chi[B_i], \quad (3.7)$$

see [30].

We also need to compute the signature of \mathcal{M} in terms of the fixed point set. Since this computation involves a contribution from infinity we will make the simplifying assumption that (\mathcal{M}, g_{ab}) is an ALF– $A_k S^1$ instanton. Then the manifold L in Definition 2.1 is the total space of an S^1 -bundle over S^2 with Euler number $e = -k - 1$. Let N be the oriented total space of the corresponding disk bundle over S^2 , so that $\partial N = L$. Denote N with the opposite orientation by \overline{N} , then the manifolds \mathcal{M} and \overline{N} can be glued along L to form an oriented manifold $\mathcal{M} \cup \overline{N}$.

The S^1 action on L extends to an S^1 action on N which rotates the fibres and has as fixed point set the single bolt B_N which is the zero section of the disk bundle and has self-intersection number $B_N \cdot B_N = e$. With the opposite orientation on N we have $B_{\overline{N}} \cdot B_{\overline{N}} = -e$. The second homology group of N is generated by B_N and the intersection form is thus given by the 1×1 matrix $[B_N \cdot B_N] = [e]$. We conclude that $\text{sign}[N] = \text{sgn}(e)$, where $\text{sgn}(e) = 0$ if $e = 0$ and $\text{sgn}(e) = e/|e|$ otherwise.

From Novikov additivity of the signature and the G -signature theorem [4, Theorem 6.12] and [4, Proposition 6.18], we get

$$\begin{aligned} \text{sign}[\mathcal{M}] - \text{sign}[N] &= \text{sign}[\mathcal{M}] + \text{sign}[\overline{N}] \\ &= \text{sign}[\mathcal{M} \cup \overline{N}] \\ &= \text{sign}[g, \mathcal{M} \cup \overline{N}] \\ &= - \sum_{i=1}^{n_{\text{nuts}}} \cot \frac{\kappa^1(P_i)t}{2} \cot \frac{\kappa^2(P_i)t}{2} \\ &\quad + \sum_{i=1}^{n_{\text{bolts}}} \csc^2 \frac{\kappa(B_i)t}{2} B_i \cdot B_i \\ &\quad + \csc^2 \frac{\kappa(B_N)t}{2} B_{\overline{N}} \cdot B_{\overline{N}}, \end{aligned} \quad (3.8)$$

where the notation we use is related to the notation in [4, Proposition 6.18] by $\alpha_i = \kappa^1(P_i)t$, $\beta_i = \kappa^2(P_i)t$ for nuts and $\theta_i = \kappa(B_i)t$ for bolts. Here t must be chosen so that none of the angles is a multiple of π so that the action on the normal bundle does not contain a term $N^g(-1)$ in the notation of [4], so that the proof of [4, Proposition 6.18] applies. We thus get

$$\begin{aligned} \text{sign}[\mathcal{M}] = & - \sum_{i=1}^{n_{\text{nuts}}} \cot \frac{\kappa^1(P_i)t}{2} \cot \frac{\kappa^2(P_i)t}{2} \\ & + \sum_{i=1}^{n_{\text{bolts}}} \csc^2 \frac{\kappa(B_i)t}{2} B_i \cdot B_i \\ & - \csc^2 \frac{\kappa(B_N)t}{2} e + \text{sgn}(e). \end{aligned} \quad (3.9)$$

Set

$$g = e^{it/G}, \quad (3.10)$$

where G is the generic period of Lemma 3.1. Then

$$\cot \frac{\kappa^j(P_i)t}{2} = i \text{sgn}(\kappa^j(P_i)) \frac{g^{w^j(P_i)} + 1}{g^{w^j(P_i)} - 1}, \quad (3.11)$$

so

$$\cot \frac{\kappa^1(P_i)t}{2} \cot \frac{\kappa^2(P_i)t}{2} = -\epsilon(P_i) \frac{g^{w^1(P_i)} + 1}{g^{w^1(P_i)} - 1} \cdot \frac{g^{w^2(P_i)} + 1}{g^{w^2(P_i)} - 1}. \quad (3.12)$$

Further,

$$\csc^2 \frac{\kappa(B_i)t}{2} = -\frac{4g}{(g-1)^2}. \quad (3.13)$$

Note that Lemma 3.1 applies also to the bolt B_N . Together we have

$$\begin{aligned} \text{sign}[\mathcal{M}] = & \sum_{i=1}^{n_{\text{nuts}}} \epsilon(P_i) \frac{g^{w^1(P_i)} + 1}{g^{w^1(P_i)} - 1} \cdot \frac{g^{w^2(P_i)} + 1}{g^{w^2(P_i)} - 1} \\ & + \frac{4g}{(g-1)^2} \left(e - \sum_{i=1}^{n_{\text{bolts}}} B_i \cdot B_i \right) + \text{sgn}(e). \end{aligned} \quad (3.14)$$

Importantly, (3.14) holds as an algebraic identity with g an independent variable, see [30, §5.8]. As remarked in the introduction, this fact plays an important role in [34], see Section 3.2 below. In particular we can set $g = 0$ and obtain

$$\text{sign}[\mathcal{M}] = \sum_{i=1}^{n_{\text{nuts}}} \epsilon(P_i) + \text{sgn}(e). \quad (3.15)$$

Remark 3.2. *This argument to compute the signature also works for a general ALF manifold \mathcal{M} . Then the manifold N bounding the Seifert fibered manifold L can be constructed using equivariant plumbing as in [46, Chapter 2]. The signature and the fixed point data of N can then be computed from the corresponding plumbing graph. Since we do not need this for the results of the present paper we leave the details for future work.*

3.2. Structure of the fixed point set. In [34], Jang has provided a complete analysis of the fixed point set for S^1 actions on compact, oriented four-manifolds. The proofs however generalize to the situation under consideration here, namely an

ALF S^1 -instanton. In particular, near a bolt, the S^1 action is free, with the generic period $2\pi G$, see Lemma 3.1. Further, the identity

$$\frac{4g}{(1-g)^2} = \frac{(1+g)^2}{(1-g)^2} - 1 \quad (3.16)$$

allows us to rewrite equation (3.14) as the G -signature formula for a compact, oriented four-manifold $\tilde{\mathcal{M}}$ with an S^1 -action that has nut struture corresponding to the nuts of \mathcal{M} with

$$e - \sum_{i=1}^{n_{\text{bolts}}} B_i \cdot B_i \quad (3.17)$$

additional (fictitious) nuts. The algebraic facts regarding the G -signature formula used in [34] therefore apply directly in our situation.

Taking these facts into account, one may verify that the proofs of the following lemmas generalize to the case of ALF S^1 -instantons.

Lemma 3.3 (Weight balance [34, Lemma 2.6]). *Let (\mathcal{M}, g_{ab}) be an ALF S^1 instanton, and assume the fixed point set F of \mathcal{M} contains nuts. Let w be a positive integer that occurs as the weight of a nut in F . Then the number of times w occurs as a weight among all nuts of \mathcal{M} , counted with multiplicity, is even.*

We shall make use of the following notation. Let \mathcal{M} be a space with an S^1 -action. For $w > 1$, let $\mathcal{M}^{\mathbb{Z}_w}$ be the fixed point set of the discrete subgroup $\mathbb{Z}_w \subset S^1$. By [34, Lemma 2.2], the proof of which applies in the current situation, we have that if a component $Z \subset \mathcal{M}^{\mathbb{Z}_w}$ contains a fixed point P for S^1/\mathbb{Z}_w , then $Z = S^2$, and Z contains exactly one more fixed point $Q \subset \mathbb{Z}_w$ which is also a nut. Further, one of the weights of Q is w .

Lemma 3.4 (Companion nuts [34, Lemma 3.5]). *Let (\mathcal{M}, g_{ab}) be an ALF S^1 -instanton with fixed point set F and let $P \in F$ be a nut with weights a, w , with $w > 1$. Then there is a unique nut $Q \in S^2 \subset \mathcal{M}^{\mathbb{Z}_w}$. Let b denote the other weight of Q . Then*

- (1) *If $\epsilon(Q) = \epsilon(P)$, then $a \equiv -b \pmod{w}$*
- (2) *If $\epsilon(Q) = -\epsilon(P)$, then $a \equiv b \pmod{w}$.*

Lemma 3.5 (Nuts with highest weights [34, Lemma 3.6]). *Let (\mathcal{M}, g_{ab}) be an ALF S^1 -instanton with fixed point set F and let $P \in F$ be a nut with weights a, w . Assume that w is the biggest of all weights, and that $w > 1$. Then there is another nut Q with weights b, w and the following holds.*

- (1) *If $\epsilon(Q) = \epsilon(P)$, then $w = a + b$.*
- (2) *If $\epsilon(Q) = -\epsilon(P)$, then $a = b$.*

Remark 3.6 (Examples of fixed point structures). *Let (M, g_{ab}) be a four-dimensional, oriented AF S^1 space, with only isolated fixed points.*

- (1) *In order for the right hand side of (3.14) to be constant, it must hold that $n_{\text{nuts}} \geq 2$. See [39, Remark 3.2].*
- (2) *If $n_{\text{nuts}} = 2$, then (3.14) implies $\text{sign}[\mathcal{M}] = 0$, and the nuts have weights*

$$\{\pm, a, b\}, \quad \{\mp, a, b\}, \quad (3.18)$$

see [39, Remark 3.4]. The Euclidean Kerr family provides an example with $n = 2$ and $R_{ab} = 0$. Euclidean Kerr has Petrov type D^+D^- .

- (3) *For $n = 3$, the weights of the nuts are*

$$\{\pm, a, b\}, \quad \{\mp, a, a+b\}, \quad \{\mp, b, a+b\} \quad (3.19)$$

for some mutually prime, positive integers a, b , and $\text{sign}[\mathcal{M}] = \pm 1$, agreeing with the orientation of nut 1. See [39, Theorem 1.8], [34, Theorem 7.1].

The Chen–Teo example [15] has $n = 3$ and $R_{ab} = 0$, and is one-sided type D.

- (4) See [34, Theorem 7.2] for the case of $n = 4$.

4. GEOMETRY WITH A KILLING FIELD

In this section, we derive some useful facts for Ricci-flat four-manifolds with a non-vanishing Killing field. The presentation here follows [25] but adds more detail. Let (\mathcal{M}, g_{ab}) be a Ricci-flat four-manifold and let ξ^a be a Killing vector,

$$\nabla_a \xi_b + \nabla_b \xi_a = 0. \quad (4.1)$$

The scaling of ξ^a is arbitrary throughout this section; it will be fixed via asymptotic conditions in Section 5. It is convenient to decompose certain fields into self dual (SD) and anti-self dual (ASD) parts, which we denote by superscripts + and –, respectively.

Definition 4.1. Define the following covariant expressions derived from ξ^a ,

$$\lambda = \xi_a \xi^a, \quad (4.2a)$$

$$F_{ab} = \nabla_a \xi_b, \quad (4.2b)$$

$$\mathcal{F}_{ab}^\pm = F_{ab} \pm \frac{1}{2} \epsilon_{ab}^{cd} F_{cd}, \quad (4.2c)$$

$$\mathcal{F}^{\pm 2} = \mathcal{F}_{ab}^\pm \mathcal{F}^{\pm ab}, \quad (4.2d)$$

$$\sigma_a^\pm = 2 \mathcal{F}_{ab}^\pm \xi^b, \quad (4.2e)$$

$$\mu = \frac{1}{2} (\nabla_a \xi_b) (\nabla^a \xi^b), \quad \nu = \frac{1}{4} \epsilon_{abcd} (\nabla^a \xi^b) (\nabla^c \xi^d), \quad (4.2f)$$

and define the SD/ASD Weyl tensor and its square via

$$\mathcal{W}_{abcd}^\pm = W_{abcd} \pm \frac{1}{2} \epsilon_{cd}^{ef} W_{abef}, \quad (4.3)$$

$$\mathcal{W}^{\pm 2} = \mathcal{W}_{abcd}^\pm \mathcal{W}^{\pm abcd} \quad (4.4)$$

Remark 4.2. The quantities μ, ν given in (4.2f) are related to $\mathcal{F}^{\pm 2}$ via

$$\mathcal{F}^{\pm 2} = 4(\mu \pm \nu). \quad (4.5)$$

The next proposition states a number of identities for the fields that have just been introduced. The Lorentzian signature version is well-known, see [42] and references therein. The proof is deferred to Appendix A.

Proposition 4.3. The fields defined above have the following algebraic and differential properties

$$\mathcal{F}_{ab}^\pm \sigma_a^\pm b = -\frac{1}{2} \mathcal{F}^{\pm 2} \xi_a, \quad (4.6a)$$

$$\sigma_a^\pm \sigma_a^\pm a = \mathcal{F}^{\pm 2} \lambda, \quad (4.6b)$$

$$\nabla_c \mathcal{F}_{ab}^\pm = -\mathcal{W}_{abcd}^\pm \xi^d, \quad (4.6c)$$

$$\nabla_c \mathcal{F}^{\pm 2} = -2 \mathcal{W}_{cabd}^\pm \xi^a \mathcal{F}^{\pm bd}, \quad (4.6d)$$

$$\nabla_{[a} \sigma_{b]}^\pm = 0, \quad (4.6e)$$

$$\nabla_a \sigma_a^\pm = \mathcal{F}^{\pm 2}. \quad (4.6f)$$

$$\nabla^a \nabla_a \mathcal{F}^{\pm 2} = -\mathcal{W}_{abcd}^\pm \mathcal{F}^{\pm ab} \mathcal{F}^{\pm cd} + \frac{1}{2} \lambda \mathcal{W}^{\pm 2} \quad (4.6g)$$

Due to (4.6e) and Lemma 2.5 we can introduce the Ernst potentials \mathcal{E}^\pm via

$$\nabla_a \mathcal{E}^\pm = \sigma_a^\pm, \quad (4.7)$$

which, by (4.6f), (4.6b), solve the Ernst equations

$$\lambda \nabla^a \nabla_a \mathcal{E}^\pm = \nabla_a \mathcal{E}^\pm \nabla^a \mathcal{E}^\pm. \quad (4.8)$$

With the twist 1-form

$$\omega_a = \epsilon_{abcd} F^{cd} \xi^b, \quad (4.9)$$

we have the decomposition

$$\sigma^\pm_a = \nabla_a \lambda \pm \omega_a. \quad (4.10)$$

In view of (4.10) and (4.6e), we have that ω_a is closed,

$$\nabla_{[a} \omega_{b]} = 0. \quad (4.11)$$

and hence by Lemma 2.5, there is a twist potential ω such that

$$\nabla_a \omega = \omega_a. \quad (4.12)$$

The Ernst potentials \mathcal{E}^\pm take the form

$$\mathcal{E}^\pm = \lambda \pm \omega, \quad (4.13)$$

and we also find

$$\lambda F_{ab} = -\frac{1}{2} \epsilon_{abcd} \xi^c \omega^d - \xi_{[a} \nabla_{b]} \lambda. \quad (4.14)$$

4.1. Charges.

Following [25, §5], let

$$J_T^a = \frac{\sigma^{-a} - \sigma^{+a}}{2\lambda^2} = -\frac{\omega^a}{\lambda^2}, \quad (4.15a)$$

$$J_D^a = \frac{\mathcal{E}^+ \sigma^{-a} + \mathcal{E}^- \sigma^{+a}}{2\lambda^2} = \frac{-\omega \omega^a + \lambda \nabla^a \lambda}{\lambda^2}, \quad (4.15b)$$

$$J_E^a = \frac{\mathcal{E}^{+2} \sigma^{-a} - \mathcal{E}^{-2} \sigma^{+a}}{2\lambda^2} = \frac{-(\lambda^2 + \omega^2) \omega^a + 2\lambda \omega \nabla^a \lambda}{\lambda^2}. \quad (4.15c)$$

With the assumptions and definitions above, the currents J_T^a, J_D^a, J_E^a , called Translation, Dilation and Ehlers currents, respectively, are conserved, that is $\nabla_a J_T^a = \nabla_a J_D^a = \nabla_a J_E^a = 0$. Hence, integrating the currents over closed 3-surfaces enclosing but not intersecting nuts and bolts defines quasilocally conserved charges.

Next, we will compute the charges corresponding to J_T^a .

Definition 4.4. Let K be a nut or a bolt in \mathcal{M} . Let U be an open set containing K with smooth boundary S , so that neither U nor S intersects any other nut or bolt. Let n^a be the inward pointing unit normal to S . The charge of K is

$$N(K) = \frac{1}{8\pi} \int_S J_a^T n^a d\mu = -\frac{1}{8\pi} \int_S \lambda^{-2} \omega_a n^a d\mu. \quad (4.16)$$

We now discuss the geometry of bolts in more detail. For a bolt B we denote the volume form of its tangent bundle by ϵ_{ab}^\perp and the volume form of its normal bundle by ϵ_{ab}^\parallel . They are both two-forms on $T\mathcal{M}$ defined at points in B , and they are related by

$$\epsilon_{abcd} \epsilon^{\perp cd} = 2\epsilon_{ab}^\parallel, \quad \epsilon_{abcd} \epsilon^{\parallel cd} = 2\epsilon_{ab}^\perp. \quad (4.17)$$

On B we have

$$F_{ab} = \nabla_a \xi_b = \kappa \epsilon_{ab}^\perp \quad (4.18)$$

where κ is constant, and

$$\mathcal{F}_{ab}^\pm = \kappa (\epsilon_{ab}^\perp \pm \epsilon_{ab}^\parallel). \quad (4.19)$$

Define the curvature scalars

$$\mathcal{R}^{\parallel\parallel} = \frac{1}{2} R_{abcd} \epsilon^{\parallel ab} \epsilon^{\parallel cd}, \quad (4.20)$$

$$\mathcal{R}^{\parallel\perp} = \frac{1}{2} R_{abcd} \epsilon^{\parallel ab} \epsilon^{\perp cd}, \quad (4.21)$$

$$\mathcal{R}^{\perp\perp} = \frac{1}{2} R_{abcd} \epsilon^{\perp ab} \epsilon^{\perp cd}. \quad (4.22)$$

Since B is totally geodesic we have

$$\mathcal{R}^{\parallel\parallel} = R^B = 2\mathcal{K}^B, \quad (4.23)$$

and since \mathcal{M} is Ricci-flat,

$$\mathcal{R}^{\perp\perp} = R^B = 2\mathcal{K}^B. \quad (4.24)$$

Proposition 4.5. *For a nut P with surface gravities κ^1, κ^2 we have*

$$N(P) = \frac{\pi}{2\kappa^1 \kappa^2}. \quad (4.25)$$

For a bolt B with surface gravity κ we have

$$\begin{aligned} N(B) &= -\frac{1}{8\kappa^2} \int_B \mathcal{R}^{\parallel\perp} d\mu \\ &= -\frac{\pi}{2\kappa^2} B \cdot B. \end{aligned} \quad (4.26)$$

Proof. We will compute the integral (4.16) when $S = S(\rho)$ is a connected component of the level set $\{\lambda = \rho\}$. Then $n^a = -\frac{\nabla^a \lambda}{|\nabla^a \lambda|}$. Since $\lambda^{-2}\omega_a$ is divergence free we have

$$\begin{aligned} 0 &= \nabla^a (\lambda^{-2}\omega_a) \\ &= -2\lambda^{-3}(\nabla^a \lambda)\omega_a + \lambda^{-2}\nabla^a \omega_a \\ &= 2\lambda^{-3}|\nabla \lambda| n^a \omega_a + \lambda^{-2}\nabla^a \omega_a, \end{aligned} \quad (4.27)$$

so

$$\lambda^{-2} n^a \omega_a = -\frac{1}{2\lambda|\nabla \lambda|} \nabla^a \omega_a = -\frac{1}{|\nabla \lambda|^2} \nabla^a \omega_a. \quad (4.28)$$

Here,

$$\begin{aligned} \nabla^a \omega_a &= \nabla^a (\epsilon_{abcd} \xi^b (\nabla^c \xi^d)) \\ &= \epsilon_{abcd} (\nabla^a \xi^b) (\nabla^c \xi^d) + \epsilon_{abcd} \xi^b (\nabla^a \nabla^c \xi^d) \\ &= 4\nu + \epsilon_{abcd} \xi^b R^{dca}_f \xi^f \\ &= 4\nu \end{aligned} \quad (4.29)$$

where the second term vanishes by the Bianchi identity.

Let $\alpha \in \mathbb{R}$, and let A, B be two expressions which satisfy $A = O(\lambda^\alpha)$, $B = O(\lambda^\alpha)$ near a component of the fixed point set. We write $A \simeq B$ if $A - B = O(\lambda^{\alpha+\delta})$ for some $\delta > 0$.

Near a nut P we have

$$\nu \simeq 2\kappa^1 \kappa^2 \quad (4.30)$$

and in normal coordinates,

$$\lambda \simeq (\kappa^1)^2 ((y^1)^2 + (y^2)^2) + (\kappa^2)^2 ((y^3)^2 + (y^4)^2). \quad (4.31)$$

Let $D(\delta)$ be the connected set containing P where $\lambda \leq \delta$. For δ small, the coarea formula applied to the sets $S(\rho^{1/2})$ where $\lambda^2 = \rho$ tells us that

$$\begin{aligned} \delta \cdot N(P) &= \int_0^\delta N(P) d\rho \\ &= -\frac{1}{8\pi} \int_0^\delta \int_{S(\rho^{1/2})} \lambda^{-2} \omega_a n^a d\mu d\rho \\ &= \frac{1}{8\pi} \int_0^\delta \int_{S(\rho^{1/2})} \frac{4\nu}{|\nabla \lambda^2|} d\mu d\rho \\ &\simeq \frac{8\kappa^1 \kappa^2}{8\pi} \int_0^\delta \int_{S(\rho^{1/2})} \frac{1}{|\nabla \lambda^2|} d\mu d\rho \\ &= \frac{\kappa^1 \kappa^2}{\pi} \int_{D(\delta^{1/2})} 1 d\mu \\ &= \frac{\kappa^1 \kappa^2}{\pi} \text{Vol}(D(\delta^{1/2})). \end{aligned} \quad (4.32)$$

With a linear change of coordinates $(x^1, x^2, x^3, x^4) = (\kappa^1 y^1, \kappa^1 y^2, \kappa^2 y^3, \kappa^2 y^4)$ with determinant $(\kappa^1 \kappa^2)^2$ we have that $\lambda \leq \delta^{1/2}$ corresponds to $|x| \leq \delta^{1/4}$ and we find

$$\text{Vol}(D(\delta^{1/2})) = \frac{1}{(\kappa^1 \kappa^2)^2} \cdot \frac{\pi^2 (\delta^{1/4})^4}{2} = \frac{\delta \pi^2}{2(\kappa^1 \kappa^2)^2}. \quad (4.33)$$

Together we have

$$N(P) = \frac{\pi}{2\kappa^1 \kappa^2}. \quad (4.34)$$

Next, working near a bolt B , introduce local coordinates x^a on \mathcal{B} and coordinates y^a in the normal direction, with $|y(p)| \cong d(p, \mathcal{B})$. Using the identity

$$\epsilon^{\perp h} [f y^g] y^h = \frac{1}{2} |y|^2 \epsilon^{\perp f g}, \quad (4.35)$$

we compute the Taylor expansion of $\nabla^a \xi^b$ to second order in the normal directions to B ,

$$\begin{aligned} \nabla^a \xi^b &\simeq (\nabla^a \xi^b)|_B + (\nabla_g \nabla^a \xi^b)|_B y^g + \frac{1}{2} (\nabla_h \nabla_g \nabla^a \xi^b)|_B y^g y^h \\ &= \kappa \epsilon^{\perp ab} + (R^{ba}_{gf} \xi^f)|_B y^g + \frac{1}{2} (\nabla_h (R^{ba}_{gf} \xi^f))|_B y^g y^h \\ &= \kappa \epsilon^{\perp ab} + \frac{1}{2} R^{ba}_{gf} (\nabla_h \xi^f)|_B y^g y^h \\ &= \kappa \epsilon^{\perp ab} + \frac{1}{2} \kappa R^{ba}_{gf} \epsilon^{\perp h} y^g y^h \\ &= \kappa \left(\epsilon^{\perp ab} + \frac{1}{4} |y|^2 R^{ba}_{gf} \epsilon^{\perp f g} \right). \end{aligned} \quad (4.36)$$

We find that

$$\begin{aligned} 4\nu &\simeq \kappa^2 \epsilon_{abcd} \left(\epsilon^{\perp ab} + \frac{1}{4} |y|^2 R^{ba}_{gf} \epsilon^{\perp f g} \right) \left(\epsilon^{\perp cd} + \frac{1}{4} |y|^2 R^{dc}_{hi} \epsilon^{\perp i h} \right) \\ &\simeq \kappa^2 \epsilon_{abcd} \epsilon^{\perp ab} \epsilon^{\perp cd} + \frac{1}{2} \kappa^2 |y|^2 \epsilon_{abcd} \epsilon^{\perp ab} R^{dc}_{hi} \epsilon^{\perp i h} \\ &= 0 + \kappa^2 |y|^2 \epsilon^{\parallel}_{cd} R^{dc}_{hi} \epsilon^{\perp ih} \\ &= -2\kappa^2 |y|^2 \mathcal{R}^{\parallel \perp}. \end{aligned} \quad (4.37)$$

Near the bolt B we further have

$$\lambda \simeq \kappa^2 |y|^2, \quad \nabla^a \lambda \simeq 2\kappa^2 y^a, \quad |\nabla^a \lambda| \simeq 2\kappa^2 |y|, \quad (4.38)$$

so

$$\begin{aligned} \lambda^{-2} n^a \omega_a &= -\frac{1}{2\lambda |\nabla^a \lambda|} 4\nu \\ &\simeq \frac{1}{2\kappa^2 |y|^2 \cdot 2\kappa^2 |y|} 2\kappa^2 |y|^2 \mathcal{R}^{\parallel\perp} \\ &= \frac{1}{2\kappa^2 |y|} \mathcal{R}^{\parallel\perp}. \end{aligned} \quad (4.39)$$

On the surface S around B we have $\epsilon = \lambda \simeq \kappa^2 |y|^2$, so the surface S consists of circles of circumference $\simeq 2\pi |y| \simeq 2\pi |\kappa|^{-1} \epsilon^{1/2}$ normal to B . We thus find that

$$\begin{aligned} N(B) &= -\frac{1}{8\pi} \int_S \lambda^{-2} \omega_a n^a d\mu \\ &\simeq -\frac{1}{8\pi} \int_S \frac{1}{2\kappa^2 |y|} \mathcal{R}^{\parallel\perp} d\mu \\ &\simeq -\frac{1}{8\pi} \int_B 2\pi |\kappa|^{-1} \epsilon^{1/2} \frac{1}{2|\kappa| \epsilon^{1/2}} \mathcal{R}^{\parallel\perp} d\mu \\ &= -\frac{1}{8\kappa^2} \int_B \mathcal{R}^{\parallel\perp} d\mu. \end{aligned} \quad (4.40)$$

Next, $\mathcal{R}^{\parallel\perp} d\mu = 4\pi e(NB)$, where $e(NB)$ is the Euler class of the normal bundle NB of B in \mathcal{M} . From [10, Theorem 11.17] we have

$$\int_B \mathcal{R}^{\parallel\perp} d\mu = 4\pi \int_B e(NB) = 4\pi B \cdot B, \quad (4.41)$$

which gives the result. \square

5. DIVERGENCE IDENTITIES

In this section we derive a family of divergence identities, see (5.22) below, depending on a parameter β taking real values. An alternative derivation on the orbit space, following [53, 54], will be given in Appendix C. Equation (5.22) with $\beta = 1$ is the key ingredient of the uniqueness arguments presented in Section 7.

After a suitable normalization of ξ , we may without loss of generality assume that the supremum of λ is 1.

Lemma 5.1. *Assume that the supremum of λ is 1. The following holds.*

(1)

$$\lambda = 1 + O^*(r^{-1}) \quad (5.1a)$$

The twist potential ω can be normalized so that $\omega \rightarrow 0$ at ∞ and in this case,

$$\omega = O^*(r^{-1}) \quad (5.1b)$$

(2) There are numbers $b^\pm \leq 0$ and $\delta > 0$ so that

$$\mathcal{E}^\pm = 1 + b^\pm r^{-1} + O^*(r^{-1-\delta}) \quad (5.2)$$

If \mathcal{E}^\pm is non-constant, then $b^\pm < 0$.

(3) $\mathcal{E}^\pm \equiv 1$ or $-1 < \mathcal{E}^\pm < 1$

Proof. Since g_{ab} is Ricci-flat we have $\Delta\xi = 0$, and hence

$$\Delta|\xi|^2 = 2\langle\nabla\xi, \nabla\xi\rangle. \quad (5.3)$$

By the maximum principle, we have $|\xi| < 1$. By Proposition B.2 it follows that $|\xi| \rightarrow 1$ and hence also $\lambda \rightarrow 1$ at infinity. We also have by Proposition B.2 that

$$\nabla_a\xi_b = O^*(r^{-2}) \quad (5.4)$$

By Lemma B.1 we may normalize ω to have $\omega \rightarrow 0$ at ∞ . Equations (5.1) now follow from the definitions of λ and ω , Equation (5.4), and Lemma B.1. From the definition of \mathcal{E}^\pm and (5.4), we have that $\mathcal{E}^\pm = 1 + O^*(r^{-1})$, and hence, by (4.7), (4.6f), and (5.1a), we have

$$\Delta\mathcal{E}^\pm = \mathcal{F}^{\pm 2}, \quad \text{where } \mathcal{F}^{\pm 2} = O^*(r^{-4}). \quad (5.5)$$

By Proposition B.4, it follows that \mathcal{E}^\pm has an expansion of the form (5.2). For R large, define the set $U_R = \{r \leq R\}$ with boundary $\partial U_R = S_R = \{r = R\}$. From Gauss's theorem we get

$$\begin{aligned} \int_{\mathcal{M}} \mathcal{F}^{\pm 2} d\mu &= \int_{\mathcal{M}} \Delta\mathcal{E}^\pm d\mu \\ &= \lim_{R \rightarrow \infty} \int_{U_R} \Delta\mathcal{E}^\pm d\mu \\ &= \lim_{R \rightarrow \infty} \int_{S_R} n^a \nabla_a \mathcal{E}^\pm d\mu \\ &= \lim_{R \rightarrow \infty} \int_{S_R} (-b^\pm r^{-2} + O^*(r^{-2-\delta})) d\mu \\ &= -Cb^\pm, \end{aligned} \quad (5.6)$$

where n^a is the outward pointing normal to S_r . Further, $C > 0$ is a constant related to the volume of L in Definition 2.1, see similar computations in the proof of Lemma 6.12. It follows that $b^\pm < 0$ unless $\mathcal{F}^{\pm 2} \equiv 0$ in which case $\mathcal{E}^\pm \equiv 1$. This proves point (2). If $b^\pm < 0$, the maximum principle implies that $\mathcal{E}^\pm < 1$. We thus get $1 > \mathcal{E}^\pm = -\mathcal{E}^\mp + 2\lambda \geq -\mathcal{E}^\mp > -1$. This proves point (3). \square

With the normalization used in Lemma 5.1, we have

$$\lambda \rightarrow 1 \quad \text{at } \infty, \quad (5.7a)$$

$$\omega \rightarrow 0 \quad \text{at } \infty, \quad (5.7b)$$

$$\mathcal{E}^\pm \rightarrow 1 \quad \text{at } \infty. \quad (5.7c)$$

These condition will be imposed throughout the rest of the paper.

We also note the following equivalent conditions for (\mathcal{M}, g_{ab}) to be half-flat, i.e. $\mathcal{W}^+{}_{abcd} \equiv 0$ or $\mathcal{W}^-{}_{abcd} \equiv 0$.

Lemma 5.2. *For any sign + or -, the following are equivalent.*

- (1) (\mathcal{M}, g_{ab}) is half-flat, that is $\mathcal{W}^\pm{}_{abcd} \equiv 0$,
- (2) $\mathcal{F}^\pm \equiv 0$,
- (3) $\mathcal{E}^\pm \equiv 1$.

Proof. If $\mathcal{W}^\pm{}_{abcd} \equiv 0$ it follows from (4.6d) that $\nabla_c \mathcal{F}^{\pm 2} \equiv 0$, that is $\mathcal{F}^{\pm 2}$ is constant. By Proposition B.2 we have that $\mathcal{F}^{\pm 2} \rightarrow 0$ at infinity, and hence $\mathcal{F}^{\pm 2} \equiv 0$. On the other hand, if $\mathcal{F}^{\pm 2} \equiv 0$ we have $\mathcal{F}^\pm{}_{ab} \equiv 0$ and from (4.6g) it follows that $\mathcal{W}^\pm{}_{abcd} \equiv 0$. This shows (1) \Leftrightarrow (2). From (4.6b) it follows that $\mathcal{F}^\pm \equiv 0$ if and only if $\sigma^\pm{}_a \equiv 0$, which by (4.7) is equivalent to $\nabla_a \mathcal{E}^\pm \equiv 0$. Since $\mathcal{E}^\pm \rightarrow 1$ at ∞ by (5.7c), this means that $\mathcal{E}^\pm \equiv 1$. Hence (2) \Leftrightarrow (3). \square

Definition 5.3. On domains where $\mathcal{E}^\pm \neq 1$ we introduce the objects

$$\mathcal{S}^\pm_{abcd} = \mathcal{W}^\pm_{abcd} - \frac{6(\mathcal{F}^\pm_{ab}\mathcal{F}^\pm_{cd} - \frac{1}{3}\mathcal{F}^{\pm 2}\mathcal{J}^\pm_{abcd})}{1 - \mathcal{E}^\pm}, \quad (5.8a)$$

$$\mathcal{S}^{\pm 2} = \mathcal{S}^\pm_{abcd}\mathcal{S}^{+abcd}, \quad (5.8b)$$

$$\mathbf{s}^{\pm 2} = \frac{\mathcal{F}^{\pm 2}}{(1 - \mathcal{E}^\pm)^4}, \quad (5.8c)$$

respectively. Here $\mathcal{J}^\pm_{abcd} = \frac{1}{4}(g_{ac}g_{bd} - g_{ad}g_{bc} \pm \epsilon_{abcd})$ is the metric on SD and ASD 2-forms.

5.1. Characterizations of special geometry. The following result holds independently for the SD and ASD sides. Recall the Petrov classification of Weyl tensors on Riemannian 4-manifolds [35], see also [49, 1].

Lemma 5.4. \mathcal{W}^\pm_{abcd} is of Petrov type D^\pm if one of the following equivalent conditions hold.

- (1) $\mathcal{S}^\pm_{abcd} = 0$ and $\mathcal{W}^\pm_{abcd} \neq 0$.
- (2) $\mathbf{s}^{\pm 2} = \text{const.} \neq 0$.

Proof. We first show the equivalence of the conditions. From (5.13) and $\mathcal{F}^{\pm 2} \neq 0$ we have (1) \Rightarrow (2). From (5.15) we have (2) \Rightarrow $\mathcal{S}^{\pm 2} = 0$ which implies (1).

Suppose condition (1) holds. Then we have

$$\mathcal{W}^\pm_{abcd} = \frac{6(\mathcal{F}^\pm_{ab}\mathcal{F}^\pm_{cd} - \frac{1}{3}\mathcal{F}^{\pm 2}\mathcal{J}^\pm_{abcd})}{1 - \mathcal{E}^\pm} \neq 0, \quad (5.9)$$

which implies that \mathcal{W}^\pm_{abcd} has degenerate eigenvalues and hence is of Petrov type D^\pm . \square

Proposition 5.5. For $\beta \in \mathbb{R}$, the scalar \mathbf{s}^\pm satisfies the modified Laplace equation

$$(\nabla^a - \Gamma^{\pm a})\nabla_a|\mathbf{s}^\pm|^\beta = V^\pm(\beta)|\mathbf{s}^\pm|^\beta, \quad (5.10)$$

with

$$\Gamma^\pm_a = \frac{(1 + \mathcal{E}^\mp)}{(1 - \mathcal{E}^\pm)\lambda}\sigma^\pm_a, \quad (5.11)$$

$$V^\pm(\beta) = \frac{\beta\lambda(\mathcal{F}^{\pm 2}\mathcal{S}^{\pm 2} + (\beta - 2)(\mathcal{F}^\pm\mathcal{S}^\pm)^2)}{4(\mathcal{F}^{\pm 2})^2}, \quad (5.12)$$

and where $(\mathcal{F}^\pm\mathcal{S}^\pm)^2 = \mathcal{F}^{\pm ab}\mathcal{S}^{\pm ef}_{ab}\mathcal{F}^{\pm cd}\mathcal{S}^{\pm efcd}$.

Proof. The gradient of (5.8c), using (4.6d) and (4.7), is equal to

$$\nabla_a\mathbf{s}^{\pm 2} = -\frac{2\mathcal{F}^{\pm cd}\mathcal{S}^\pm_{abcd}\xi^b}{(1 - \mathcal{E}^\pm)^4}. \quad (5.13)$$

By the Maxwell-type equations (4.6c) and the Bianchi identity applied to (5.8a), the contracted second derivative is given by

$$\nabla_a\nabla^a\mathbf{s}^{\pm 2} = \frac{\mathcal{S}^{\pm 2}\lambda}{2(1 - \mathcal{E}^\pm)^4} - \frac{(1 + \mathcal{E}^\mp)}{(1 - \mathcal{E}^\pm)^5}\mathcal{F}^{\pm ab}\mathcal{F}^{\pm cd}\mathcal{S}^\pm_{abcd}. \quad (5.14)$$

We notice that the last term can be absorbed by a lower order term involving (5.11), such that

$$(\nabla^a - \Gamma^{\pm a})\nabla_a\mathbf{s}^{\pm 2} = \frac{\mathcal{S}^{\pm 2}\lambda}{2\mathcal{F}^{\pm 2}}\mathbf{s}^{\pm 2}. \quad (5.15)$$

Now (5.10) follows by expanding in β as follows,

$$\begin{aligned} (\nabla^a - \Gamma^{\pm a})\nabla_a |\mathbf{s}^\pm|^\beta &= \frac{\beta}{2}(\mathbf{s}^\pm)^{\beta/2-1} \left((\nabla^a - \Gamma^{\pm a})\nabla_a \mathbf{s}^\pm + \left(\frac{\beta}{2} - 1\right)(\nabla \mathbf{s}^\pm)^2 / \mathbf{s}^\pm \right) \\ &= \frac{\beta \mathcal{S}^\pm{}^2 \lambda}{4 \mathcal{F}^\pm{}^2} |\mathbf{s}^\pm|^\beta + \frac{\beta(\beta-2)}{4} |\mathbf{s}^\pm|^{\beta-4} \frac{4 \mathcal{F}^\pm{}^{cd} \mathcal{S}^\pm{}_{abcd} \xi^b \mathcal{F}^\pm{}^{ef} \mathcal{S}^\pm{}_{hef} \xi^h}{(1-\mathcal{E}^\pm)^8} \\ &= \frac{\beta \mathcal{S}^\pm{}^2 \lambda}{4 \mathcal{F}^\pm{}^2} |\mathbf{s}^\pm|^\beta + \frac{\beta(\beta-2)}{4} |\mathbf{s}^\pm|^{\beta-4} \frac{\mathcal{F}^\pm{}^{cd} \mathcal{S}^\pm{}_{abcd} \mathcal{F}^\pm{}^{ef} \mathcal{S}^\pm{}_{ef}{}^{ab} \lambda}{(\mathcal{F}^\pm{}^2)^2}. \end{aligned} \quad (5.16)$$

Here we used (5.15) and (5.13) for the second equality. The third equality uses (5.8c) and $(\mathcal{S}^\pm{}_{ibcd} \mathcal{F}^\pm{}^{cd})(\mathcal{S}^\pm{}^i{}_{hef} \mathcal{F}^\pm{}^{ef}) = \frac{1}{4}(\mathcal{S}^\pm{}_{ijcd} \mathcal{F}^\pm{}^{cd})(\mathcal{S}^\pm{}^{ij}{}_{ef} \mathcal{F}^\pm{}^{ef}) g_{bh}$ which is analogous to (A.3) for two SD/ASD two-forms with one index contraction. \square

Remark 5.6. *The quantities defined in Definition 5.3 are derived from the Weyl tensor in a manner analogous to the complex tensors used in characterizations of the Lorentzian Kerr geometry by Mars [41]. Identities analogous to (5.14) as well as wave equations for $\mathcal{S}^\pm{}_{abcd}$ were applied to the black hole uniqueness problem in [2, 32].*

Proposition 5.7. *For $\beta \geq 1/2$, the potentials $V^\pm(\beta)$ given in (5.12) satisfy*

$$V^\pm(\beta) \geq 0. \quad (5.17)$$

Proof. For $\beta \geq 2$, (5.12) is a sum of squares and hence non-negative. For $\beta \in \mathbb{R}$ we introduce the quantities

$$\mathcal{P}^\pm{}_{abc} = \gamma_{a[b} \mathcal{W}^\pm{}_{c]efk} \xi^e \mathcal{F}^\pm{}^{fk} + 4 \xi^e \xi^f \mathcal{W}^\pm{}_{eaf[b} \sigma^\pm{}_{c]}, \quad (5.18)$$

where $\gamma_{ab} = \lambda g_{ab} - \xi_a \xi_b$. The tensors $\mathcal{P}^\pm{}_{abc}$ introduced in (5.18) are the Riemannian counterparts of the (complex) ‘‘spacetime Simon tensor’’, see [41, Definition 1]. Their squares decompose into

$$2 \mathcal{P}^\pm{}^2 \lambda^{-3} = \mathcal{F}^\pm{}^2 \mathcal{S}^\pm{}^2 - \frac{3}{2} (\mathcal{F}^\pm{} \mathcal{S}^\pm{})^2, \quad (5.19)$$

which shows that (5.12), is a sum of squares for $\beta \geq 1/2$. \square

Lemma 5.8 (Divergence Identity). *Let $J_T{}^a, J_D{}^a, J_E{}^a$ be as in (4.15), and let*

$$J^\pm{}^a = \pm J_T{}^a - 2 J_D{}^a \pm J_E{}^a = \frac{\pm(1 \mp \mathcal{E}^\pm)^2 \sigma^{-a} \mp (1 \pm \mathcal{E}^\pm)^2 \sigma^{+a}}{2\lambda^2}. \quad (5.20)$$

For $\beta \in \mathbb{R}$, the vector fields

$$\Psi^\pm(\beta)a = \frac{1}{2} J^\pm{}^a |\mathbf{s}^\pm|^\beta + \frac{(1 + \mathcal{E}^\pm)(1 - \mathcal{E}^\pm)}{2\lambda} \nabla^a |\mathbf{s}^\pm|^\beta \quad (5.21)$$

satisfy

$$\nabla_a \Psi^\pm(\beta)a = \frac{(1 + \mathcal{E}^\pm)(1 - \mathcal{E}^\pm)}{2\lambda} V^\pm(\beta) |\mathbf{s}^\pm|^\beta. \quad (5.22)$$

Proof. To obtain (5.22), recall that the current $J^\pm{}_a$ is conserved (cf. the remark after (4.15)) and that the gradient of \mathbf{s}^\pm is given by (5.13). The divergence of the second term on the right hand side of (5.21) follows from

$$(\nabla_a + \Gamma^\pm{}_a) \left(\frac{(1 + \mathcal{E}^\pm)(1 - \mathcal{E}^\pm)}{\lambda} \right) = -J^\pm{}_a \quad (5.23)$$

and (5.10). \square

Remark 5.9. *An alternative proof of (5.22) is given by Lemma C.5. See also point (1) of Remark C.6.*

Let $\Psi_a^{\pm(\beta)}$ be as in (5.22). We now restrict to the case $\beta = 1$, see Remark 6.1. Set

$$\Psi_a^{\pm} = \Psi_a^{\pm(1)}, \quad V^{\pm} = V^{\pm}(1), \quad (5.24)$$

so that

$$\Psi_a^{\pm} = \frac{1}{2} J_a^{\pm} s^{\pm} + \frac{(1 + \mathcal{E}^{\mp})(1 - \mathcal{E}^{\pm})}{2\lambda} \nabla_a s^{\pm}. \quad (5.25)$$

Lemma 5.10. *Let (\mathcal{M}, g_{ab}) be an ALF S^1 -instanton. Then Ψ_a^{\pm} satisfies*

$$\nabla^a \Psi_a^{\pm} = \frac{(1 + \mathcal{E}^{\mp})(1 - \mathcal{E}^{\pm})}{2\lambda} V^{\pm} |s^{\pm}|, \quad (5.26)$$

where s^{\pm} is given by (5.8c), and

$$V^{\pm} = \frac{\lambda (\mathcal{F}^{\pm 2} s^{\pm 2} - (\mathcal{F}^{\pm} s^{\pm})^2)}{4(\mathcal{F}^{\pm 2})^2}, \quad (5.27)$$

In particular,

$$\nabla^a \Psi_a^{\pm} \geq 0 \quad (5.28)$$

with equality only if \mathcal{W}_{abcd}^{\pm} is algebraically special.

Proof. Equation (5.26) is the case $\beta = 1$ of (5.22). By Lemma 5.1, we have that $-1 < \mathcal{E}^{\pm} < 1$. Further, by Lemma 5.4 and Proposition 5.7 with $\beta = 1$, $V^{\pm} \geq 0$, with equality only if \mathcal{W}_{abcd}^{\pm} is algebraically special. \square

6. INTEGRATING THE DIVERGENCE IDENTITY

In this section we shall investigate the consequences of the divergence identity (5.26) presented in Lemma 5.10. By Lemma 5.1, $-1 < \mathcal{E}^{\pm} < 1$ and hence Ψ_a^{\pm} is smooth for $\lambda > 0$ and $\mathcal{F}^{\pm} > 0$, while it is singular on the set

$$\mathcal{Z}^{\pm} = \{\mathcal{F}^{\pm} = 0\} = \{s^{\pm} = 0\} \quad (6.1)$$

and on the fixed point set $F = \{\lambda = 0\}$ of the S^1 -action. Integrating (5.26) over a bounded domain $\Omega \subset \mathcal{M} \setminus (\mathcal{Z}^{\pm} \cup F)$ yields

$$\int_{\Omega} \nabla^a \Psi_a^{\pm} d\mu = \int_{\partial\Omega} n^a \Psi_a^{\pm} d\mu \quad (6.2)$$

where n^a is the outward pointing normal to $\partial\Omega$. We then consider a family Ω_{ϵ} exhausting $\mathcal{M}(\mathcal{Z}^{\pm} \cup F)$ and evaluate the limiting boundary contributions in terms of fixed point data. Using the G -signature formula, this is then used to give conditions under which (\mathcal{M}, g_{ab}) must be algebraically special, which in turn yields uniqueness results.

Remark 6.1. *In this paper we shall consider applications of the identity (5.22) for $\beta = 1$. It is only in this case that the boundary terms at the fixed point set are local, in the sense that they do not depend on the pointwise values of globally defined quantities such as \mathcal{E}^{\pm} , see Lemma 6.8.*

Lemma 6.2. *Near a nut P with surface gravities κ^1, κ^2 , we have*

$$\mathcal{F}^{\pm} = \mathcal{F}^{\pm}|_P + O(\lambda) = 2|\kappa^1 \pm \kappa^2| + O(\lambda). \quad (6.3)$$

Near a bolt B with surface gravity κ we have

$$\mathcal{F}^{\pm} = \mathcal{F}^{\pm}|_B + O(\lambda) = 2|\kappa| + O(\lambda). \quad (6.4)$$

Proof. From (4.6d) we have $\nabla_c \mathcal{F}^{\pm 2} = 0$ at points where $\xi = 0$. So $\mathcal{F}^{\pm} = \mathcal{F}^{\pm}|_P + O(\lambda)$ resp. $\mathcal{F}^{\pm} = \mathcal{F}^{\pm}|_B + O(\lambda)$. Further, $\mathcal{F}^{\pm}|_B$ is constant along a bolt. A computation using (3.1) and (3.4) gives us $\mathcal{F}^{\pm}|_P = 2|\kappa^1 \pm \kappa^2|$ and $\mathcal{F}^{\pm}|_B = 2|\kappa|$. \square

Proposition 6.3. *Let \mathcal{Z}^\pm be the singular set given in (6.1) and let $F = \{\lambda = 0\}$ be the fixed point set of the S^1 action. Assume that (\mathcal{M}, g_{ab}) is not half-flat. Then*

- (1) \mathcal{Z}^\pm is compact,
- (2) \mathcal{Z}^\pm and F are disjoint.

Proof. Since \mathcal{Z}^\pm is the zero set of a continuous function, it is closed. By (4.6b) and Lemma 5.1 we have

$$\mathcal{F}^{\pm 2} = \lambda^{-1} \sigma^\pm_a \sigma^{\pm a} = \lambda^{-1} (\nabla_a \mathcal{E}^\pm) (\nabla^a \mathcal{E}^\pm) = b^{\pm 2} r^{-4} + O^*(r^{-4-\delta}) \quad (6.5)$$

and thus $r^4 \mathcal{F}^{\pm 2} \rightarrow b^{\pm 2} \neq 0$ at infinity. Hence \mathcal{Z}^\pm is bounded and therefore compact.

Suppose $\lambda = \mathcal{F}^\pm = 0$ at a point p . From (4.6d) it follows that the derivative of $\mathcal{F}^{\pm 2}$ vanishes at p . Taking further derivatives of (4.6d) and using (4.2b), (4.2c) and (4.6c) we see that every term in every derivative of $\mathcal{F}^{\pm 2}$ contains either a factor ξ_a or a factor $\mathcal{F}^{\pm bd}$. It follows that all derivatives of $\mathcal{F}^{\pm 2}$ vanish at p . Since (\mathcal{M}, g_{ab}) is Ricci-flat we may work in coordinates where all geometric quantities are analytic. It follows that $\mathcal{F}^{\pm 2} = 0$ everywhere, which by Proposition 5.2 contradicts the assumption that (\mathcal{M}, g_{ab}) is not half-flat. \square

Proposition 6.4. *The set \mathcal{Z}^\pm is countably 2-rectifiable, and therefore has Hausdorff dimension at most 2.*

Remark 6.5. *The fact that \mathcal{Z}^\pm is 2-rectifiable means that it can be written as a countable union of sets of the form $\Phi(\Xi)$, where $\Xi \subset \mathbb{R}^2$ is bounded and $\Phi : \Xi \rightarrow \mathcal{M}$ is a Lipschitz map. In particular, the Hausdorff dimension of \mathcal{Z}^\pm is at most 2.*

Proof. Taking the total anti-symmetrization respectively the trace of (4.6c) and using the first Bianchi identity respectively the fact that \mathcal{W}^\pm_{abcd} is trace-free, we find that the 2-form \mathcal{F}^\pm_{ab} is closed and coclosed. From [5, Main Theorem] we thus know that \mathcal{Z}^\pm , which is the zero set of \mathcal{F}^\pm_{ab} , is countably 2-rectifiable. \square

Fix small $\delta, \epsilon > 0$. Since the Hausdorff dimension of \mathcal{Z}^\pm is at most 2 there is a finite cover of \mathcal{Z}^\pm by balls $B_{p_i}(r_i)$ of radius r_i such that $\sum_i r_i^{2+\delta} < \epsilon$. Define

$$U_{\mathcal{Z}^\pm, \epsilon} = \cup_i B_{p_i}(r_i). \quad (6.6)$$

For $\epsilon > 0$ define

$$U_{\infty, \epsilon} = \{r > \epsilon^{-1}\}, \quad U_{F, \epsilon} = \{\lambda < \epsilon\}. \quad (6.7)$$

If ϵ is sufficiently small, the sets $U_{\mathcal{Z}^\pm, \epsilon}$, $U_{F, \epsilon}$, $U_{\infty, \epsilon}$ are all disjoint. Further, define

$$\Omega^\pm_\epsilon = \mathcal{M} \setminus (U_{\infty, \epsilon} \cup U_{F, \epsilon} \cup U_{\mathcal{Z}^\pm, \epsilon}). \quad (6.8)$$

The sets $\mathcal{M} \setminus U_{\infty, \epsilon}$ give an exhaustion of \mathcal{M} as $\epsilon \rightarrow 0$ with the domains $U_{\infty, \epsilon}$ receding to infinity along the end. The domains $U_{F, \epsilon}$ surround the fixed point set F , and for $\epsilon \rightarrow 0$ converge to the nuts and bolts. Finally, the domains $U_{\mathcal{Z}^\pm, \epsilon}$ surround the set \mathcal{Z}^\pm . It follows from the just mentioned facts that the family of domains Ω^\pm_ϵ forms an exhaustion of $\mathcal{M} \setminus (F \cup \mathcal{Z}^\pm)$. We have

$$\partial \Omega^\pm_\epsilon = \partial U_{\infty, \epsilon} \cup \partial U_{F, \epsilon} \cup \partial U_{\mathcal{Z}^\pm, \epsilon}. \quad (6.9)$$

By construction, Ψ^\pm_a is smooth in Ω^\pm_ϵ . Therefore, in order to evaluate the right-hand side of (6.2), we may consider fluxes of Ψ^\pm_a through each of these boundary components in (6.9), and take the limit $\epsilon \rightarrow 0$. The boundaries $\partial U_{\infty, \epsilon}$ and $\partial U_{F, \epsilon}$ are smooth hypersurfaces for small ϵ , while the boundary $\partial U_{\mathcal{Z}^\pm, \epsilon}$ is smooth almost everywhere. The flux through $\partial U_{\infty, \epsilon}$ gives the contribution from infinity, while that through $\partial U_{F, \epsilon}$ gives the contribution from the nuts and bolts. Finally, the flux through $\partial U_{\mathcal{Z}^\pm, \epsilon}$ yields the contribution from the set where $\mathcal{F}^\pm = 0$.

Definition 6.6 (Length at infinity). *For $p \in M$, let $\ell(p)$ denote the length of the S^1 orbit through p . The length at infinity of (\mathcal{M}, g_{ab}) is defined as*

$$\ell_\infty = \liminf_{p \rightarrow \infty} \ell(p). \quad (6.10)$$

The remainder of this section is devoted to the proof of the following result.

Proposition 6.7 (Boundary terms). *Let (\mathcal{M}, g_{ab}) be an ALF S^1 -instanton that is not half-flat. Let O be the base of the boundary at infinity of (\mathcal{M}, g_{ab}) and let $\chi[O]$ be its orbifold Euler characteristic, see Remark 2.12. Then*

$$-2\pi\ell_\infty\chi[O] + 4\pi^2 \left(\sum_{i=1}^{n_{\text{bolts}}} \frac{\chi[B_i]}{|\kappa(B_i)|} \pm \sum_{i=1}^{n_{\text{nuts}}} \epsilon(P_i) \frac{|\kappa^1(P_i) \pm \kappa^2(P_i)|}{|\kappa^1(P_i)\kappa^2(P_i)|} \right) \geq 0 \quad (6.11)$$

with equality if and only if \mathcal{M} is algebraically special.

Lemma 6.8. *Let (\mathcal{M}, g_{ab}) be an ALF S^1 -instanton that is not half-flat. Let Ψ^\pm_a be given by (5.25). Then there is a locally bounded one-form Ξ_a such that*

$$\Psi^\pm_a = \pm \frac{\mathcal{F}^\pm}{2} J_a^T + \frac{\nabla_a \mathcal{F}^\pm}{2\lambda} + \Xi_a. \quad (6.12)$$

Proof. We note

$$\frac{1 + \mathcal{E}^\mp}{1 - \mathcal{E}^\pm} = 1 + \frac{\mathcal{E}^\pm + \mathcal{E}^\mp}{1 - \mathcal{E}^\pm} = 1 + \frac{2\lambda}{1 - \mathcal{E}^\pm}, \quad (6.13)$$

and with J^\pm_a given by (5.20) we have

$$\begin{aligned} J^\pm_a \cdot \frac{1}{(1 - \mathcal{E}^\pm)^2} &= \frac{\pm(1 \mp \mathcal{E}^\pm)^2 \sigma^-_a \mp (1 \pm \mathcal{E}^-)^2 \sigma^+_a}{2\lambda^2} \cdot \frac{1}{(1 - \mathcal{E}^\pm)^2} \\ &= \frac{1}{2\lambda^2} \left(\sigma^\mp_a - \frac{(1 + \mathcal{E}^\mp)^2}{(1 - \mathcal{E}^\pm)^2} \sigma^\pm_a \right) \\ &= \frac{1}{2\lambda^2} \left(\mp 2\omega_a - \frac{4\lambda}{1 - \mathcal{E}^\pm} \left(1 + \frac{\lambda}{1 - \mathcal{E}^\pm} \right) \sigma^\pm_a \right). \end{aligned} \quad (6.14)$$

Using (5.8c), (5.25), and (5.20) we then have

$$\begin{aligned} \Psi^\pm_a &= \frac{1}{2} J^\pm_a \mathbf{s}^\pm + \frac{(1 + \mathcal{E}^\mp)(1 - \mathcal{E}^\pm)}{2\lambda} \nabla_a \mathbf{s}^\pm \\ &= \frac{\mathcal{F}^\pm}{4\lambda^2} \left(\mp 2\omega_a - \frac{4\lambda}{1 - \mathcal{E}^\pm} \left(1 + \frac{\lambda}{1 - \mathcal{E}^\pm} \right) \sigma^\pm_a \right) \\ &\quad + \frac{(1 + \mathcal{E}^\mp)(1 - \mathcal{E}^\pm)}{2\lambda} \cdot \frac{2\mathcal{F}^\pm}{(1 - \mathcal{E}^\pm)^3} \nabla_a \mathcal{E}^\pm \\ &\quad + \frac{(1 + \mathcal{E}^\mp)(1 - \mathcal{E}^\pm)}{2\lambda} \cdot \frac{\nabla_a \mathcal{F}^\pm}{(1 - \mathcal{E}^\pm)^2} \\ &= \mp \frac{\mathcal{F}^\pm}{2} \cdot \frac{\omega_a}{\lambda^2} \\ &\quad + \frac{\mathcal{F}^\pm}{\lambda(1 - \mathcal{E}^\pm)} \left(- \left(1 + \frac{\lambda}{1 - \mathcal{E}^\pm} \right) + \frac{1 + \mathcal{E}^\mp}{1 - \mathcal{E}^\pm} \right) \sigma^\pm_a \\ &\quad + \frac{1 + \mathcal{E}^\mp}{1 - \mathcal{E}^\pm} \cdot \frac{\nabla_a \mathcal{F}^\pm}{2\lambda} \\ &= \pm \frac{\mathcal{F}^\pm}{2} J_a^T + \frac{\mathcal{F}^\pm}{(1 - \mathcal{E}^\pm)^2} \sigma^\pm_a + \left(1 + \frac{2\lambda}{1 - \mathcal{E}^\pm} \right) \frac{\nabla_a \mathcal{F}^\pm}{2\lambda}. \end{aligned} \quad (6.15)$$

The result follows. \square

6.1. Contribution from the fixed point set.

Lemma 6.9 (Boundary term at the nuts). *Let P be a nut with surface gravities κ^1, κ^2 . Let $U_{P,\epsilon}$ be the connected neighbourhood of P where $\lambda < \epsilon$ and let $S_{P,\epsilon} = \partial U_{P,\epsilon}$ be its boundary with inward pointing unit normal n^a . Then*

$$\lim_{\epsilon \rightarrow 0} \int_{S_{P,\epsilon}} n^a \Psi^\pm_a d\mu = \pm 4\pi^2 \frac{|\kappa^1 \pm \kappa^2|}{\kappa^1 \kappa^2}. \quad (6.16)$$

Proof. By Lemma 6.2, Proposition 4.5, and Lemma 6.8 we have

$$\begin{aligned} \int_{S_{P,\epsilon}} n^a \Psi^\pm_a d\mu &= \int_{S_{P,\epsilon}} n^a \left(\pm \frac{\mathcal{F}^\pm}{2} J_a^T + \frac{\nabla_a \mathcal{F}^\pm}{2\lambda} + \Xi_a \right) d\mu \\ &= \int_{S_{P,\epsilon}} n^a \left(\pm \frac{2|\kappa^1 \pm \kappa^2| + O(\lambda)}{2} J_a^T + O(\lambda^{-1}) \right) d\mu \\ &= \pm |\kappa^1 \pm \kappa^2| \cdot 8\pi N(P) + O(\epsilon) + \int_{S_{P,\epsilon}} O(\lambda^{-1}) d\mu \\ &= \pm 4\pi^2 \frac{|\kappa^1 \pm \kappa^2|}{\kappa^1 \kappa^2} + O(\epsilon^{1/2}), \end{aligned} \quad (6.17)$$

and the result follows. \square

Lemma 6.10 (Boundary term at the bolts). *Let B be a bolt with surface gravity κ . Let $U_{B,\epsilon}$ be the connected neighbourhood of B where $\lambda < \epsilon$ and let $S_{B,\epsilon} = \partial U_{B,\epsilon}$ be its boundary with inward pointing unit normal n^a . Then*

$$\lim_{\epsilon \rightarrow 0} \int_{S_{B,\epsilon}} n^a \Psi^\pm_a d\mu = \frac{4\pi^2}{|\kappa|} \chi[B]. \quad (6.18)$$

Proof. By Lemma 6.2, Proposition 4.5, and Lemma 6.8 we have

$$\begin{aligned} \int_{S_{B,\epsilon}} n^a \Psi^\pm_a d\mu &= \int_{S_{B,\epsilon}} n^a \left(\pm \frac{\mathcal{F}^\pm}{2} J_a^T + \frac{\nabla_a \mathcal{F}^\pm}{2\lambda} + \Xi_a \right) d\mu \\ &= \pm \frac{2|\kappa| + O(\epsilon)}{2} 8\pi N(B) + O(\epsilon^{1/2}) \\ &\quad + \frac{1 + O(\epsilon)}{2\epsilon} \int_{S_{B,\epsilon}} \frac{1}{2\mathcal{F}^\pm} n^a \nabla_a \mathcal{F}^{\pm 2} d\mu \\ &= \mp (|\kappa| + O(\epsilon)) \cdot \frac{8\pi}{8\kappa^2} \int_B \mathcal{R}^{\parallel \perp} d\mu + O(\epsilon^{1/2}) \\ &\quad + \frac{1 + O(\epsilon)}{4\epsilon(2|\kappa| + O(\epsilon))} \int_{S_{B,\epsilon}} n^a \nabla_a \mathcal{F}^{\pm 2} d\mu. \end{aligned} \quad (6.19)$$

We use (4.19) and (4.17) to compute on B ,

$$\begin{aligned} \mathcal{W}_{abcd}^\pm \mathcal{F}^{\pm cd} &= (W_{abcd} \pm \frac{1}{2} \epsilon_{cd}^{ef} W_{abef}) \kappa (\epsilon^{\perp cd} \pm \epsilon^{\parallel cd}) \\ &= \kappa (R_{abcd} \pm \frac{1}{2} R_{abef} \epsilon_{cd}^{ef}) (\epsilon^{\perp cd} \pm \epsilon^{\parallel cd}) \\ &= \kappa (R_{abcd} \epsilon^{\perp cd} \pm \frac{1}{2} R_{abef} \epsilon_{cd}^{ef} \epsilon^{\perp cd}) \\ &\quad \pm \kappa (R_{abcd} \epsilon^{\parallel cd} \pm \frac{1}{2} R_{abef} \epsilon_{cd}^{ef} \epsilon^{\parallel cd}) \\ &= \kappa (R_{abcd} \epsilon^{\perp cd} \pm R_{abef} \epsilon^{\parallel ef}) \\ &\quad \pm \kappa (R_{abcd} \epsilon^{\parallel cd} \pm R_{abef} \epsilon^{\perp ef}) \\ &= 2\kappa (R_{abcd} \epsilon^{\perp cd} \pm R_{abcd} \epsilon^{\parallel cd}), \end{aligned} \quad (6.20)$$

and with (4.23) and (4.24) we get

$$\begin{aligned}
\mathcal{W}^\pm_{abcd}\mathcal{F}^{\pm ab}\mathcal{F}^{\pm cd} &= 2\kappa^2(R_{abcd}\epsilon^{\perp cd} \pm R_{abcd}\epsilon^{\parallel cd})(\epsilon^{\perp ab} \pm \epsilon^{\parallel ab}) \\
&= 2\kappa^2(R_{abcd}\epsilon^{\perp ab}\epsilon^{\perp cd} \pm R_{abcd}\epsilon^{\perp ab}\epsilon^{\parallel cd}) \\
&\quad \pm 2\kappa^2(R_{abcd}\epsilon^{\parallel ab}\epsilon^{\perp cd} \pm R_{abcd}\epsilon^{\parallel ab}\epsilon^{\parallel cd}) \\
&= 2\kappa^2(2\mathcal{R}^{\perp\perp} \pm 2\mathcal{R}^{\perp\parallel}) \pm 2\kappa^2(2\mathcal{R}^{\perp\parallel} \pm 2\mathcal{R}^{\parallel\parallel}) \\
&= 4\kappa^2(\mathcal{R}^{\perp\perp} + \mathcal{R}^{\parallel\parallel}) \pm 8\kappa^2\mathcal{R}^{\perp\parallel} \\
&= 8\kappa^2(R^B \pm \mathcal{R}^{\perp\parallel}). \tag{6.21}
\end{aligned}$$

Taking into account the fact that n^a is the inward pointing unit normal to $S_{B,\epsilon}$ with respect to $U_{B,\epsilon}$, we have using (4.6g),

$$\begin{aligned}
\int_{S_{B,\epsilon}} n^a \nabla_a \mathcal{F}^{\pm 2} d\mu &= - \int_{U_{B,\epsilon}} \nabla^a \nabla_a \mathcal{F}^{\pm 2} d\mu \\
&= - \int_{U_{B,\epsilon}} \left(-\mathcal{W}^\pm_{abcd}\mathcal{F}^{\pm ab}\mathcal{F}^{\pm cd} + \frac{1}{2}\lambda\mathcal{W}^{\pm 2} \right) d\mu \\
&= \int_{U_{B,\epsilon}} \mathcal{W}^\pm_{abcd}\mathcal{F}^{\pm ab}\mathcal{F}^{\pm cd} d\mu - \frac{\epsilon}{2} \int_{U_{B,\epsilon}} \mathcal{W}^{\pm 2} d\mu \\
&= \frac{\pi\epsilon}{\kappa^2} \int_B 8\kappa^2(R^B \pm \mathcal{R}^{\perp\parallel}) d\mu + \frac{\epsilon}{2} O(\epsilon^{1/2}). \tag{6.22}
\end{aligned}$$

In the last step here, the integral over the tube $U_{B,\epsilon}$ is approximated with the integral over B multiplied with the area $\frac{\pi\epsilon}{\kappa^2}$ of the normal disk. Together we find

$$\begin{aligned}
\int_{S_{B,\epsilon}} n^a \Psi^\pm_a d\mu &= \mp\pi \frac{|\kappa| + O(\epsilon)}{\kappa^2} \int_B \mathcal{R}^{\parallel\perp} d\mu + O(\epsilon^{1/2}) \\
&\quad + \frac{1 + O(\epsilon)}{4\epsilon(2|\kappa| + O(\epsilon))} \left(\frac{\pi\epsilon}{\kappa^2} \int_B 8\kappa^2(R^B \pm \mathcal{R}^{\perp\parallel}) d\mu + \frac{\epsilon}{2} O(\epsilon^{1/2}) \right) \\
&= \mp\pi \frac{|\kappa| + O(\epsilon)}{\kappa^2} \int_B \mathcal{R}^{\parallel\perp} d\mu + O(\epsilon^{1/2}) \\
&\quad + \pi \frac{1 + O(\epsilon)}{|\kappa| + O(\epsilon)} \left(\int_B (R^B \pm \mathcal{R}^{\perp\parallel}) d\mu + \frac{\epsilon}{2} O(\epsilon^{1/2}) \right). \tag{6.23}
\end{aligned}$$

Finally, the Gauss–Bonnet theorem tells us that

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} \int_{S_{B,\epsilon}} n^a \Psi^\pm_a d\mu &= \mp\frac{\pi}{|\kappa|} \int_B \mathcal{R}^{\parallel\perp} d\mu + \frac{\pi}{|\kappa|} \int_B (R^B \pm \mathcal{R}^{\parallel\perp}) d\mu \\
&= \frac{\pi}{|\kappa|} \int_B R^B d\mu \\
&= \frac{2\pi}{|\kappa|} \int_B \mathcal{K}^B d\mu \\
&= \frac{4\pi^2}{|\kappa|} \chi[B]. \tag{6.24}
\end{aligned}$$

□

6.2. Contribution from infinity.

Lemma 6.11 (Length at infinity). *Let (M, g) be an ALF- A_k S^1 -instanton, with S^1 action generated by ξ , with $\lambda \rightarrow 1$ at ∞ . Let Π be the minimum period of the linearized S^1 action at the fixed point set,*

$$\Pi = \min \left\{ \frac{2\pi}{|\kappa(P)|}, \frac{2\pi}{|\kappa(B)|} \right\}, \tag{6.25}$$

where the minimum is taken over all nuts and bolts in the fixed point set F , and all surface gravities. The length at infinity ℓ_∞ , see Definition 6.6, satisfies

$$\ell_\infty \geq \Pi. \quad (6.26)$$

Proof. Assume for a contradiction there is an $\epsilon > 0$ such that for any r_0 sufficiently large there is $p \in M$, $r(p) > r_0$ with $\ell(p) < (1 - \epsilon)\Pi$. We may without loss of generality assume $\lambda(p) > 1 - \epsilon$ for $r(p) > r_0$, by choosing r_0 large enough.

Since $\lambda \rightarrow 1$ at ∞ , F is compact. Hence, there is a minimizing geodesic γ connecting p to F . Let $\gamma(0) \in F$ be the starting point of γ . The period of the S^1 action on γ is determined by the period of the linearized action at x on the initial velocity vector $\dot{\gamma}(0)$. Let the period of the S^1 action on γ be Π_γ . In view of (6.25), $\Pi_\gamma \geq \Pi$. We have that

$$(1 - \epsilon)\Pi > \ell(p) = \lambda^{1/2}(p)\Pi_\gamma > (1 - \epsilon/2)\Pi \quad (6.27)$$

which is a contradiction. This proves the lemma. \square

Lemma 6.12 (The boundary term at infinity). *Let $S_{\infty,\epsilon} = \partial U_{\infty,\epsilon} = \{r = \epsilon^{-1}\}$ with outward outward-pointing unit normal n^a . Then*

$$\lim_{\epsilon \rightarrow 0} \int_{S_{\infty,\epsilon}} n^a \Psi^\pm_a d\mu \leq -2\pi \ell_\infty \chi[O]. \quad (6.28)$$

Proof. By (5.2) we have

$$\begin{aligned} \mathcal{E}^\pm &= 1 + b^\pm r^{-1} + O^*(r^{-1-\delta}), \\ \sigma^\pm_a &= \nabla_a \mathcal{E}^\pm = -b^\pm r^{-2} \nabla_a r + O^*(r^{-2-\delta}), \end{aligned} \quad (6.29)$$

as $r \rightarrow \infty$. By (6.5) we further have

$$\begin{aligned} \mathcal{F}^\pm &= |b^\pm| r^{-2} + O^*(r^{-2-\delta}) = -b^\pm r^{-2} + O^*(r^{-2-\delta}), \\ \nabla_a \mathcal{F}^\pm &= 2b^\pm r^{-3} \nabla_a r + O^*(r^{-3-\delta}). \end{aligned} \quad (6.30)$$

We find that

$$\begin{aligned} \Psi^\pm_a &= \frac{1}{2} J^\pm_a \frac{\mathcal{F}^\pm}{(1 - \mathcal{E}^\pm)^2} + \frac{(1 + \mathcal{E}^\mp)(1 - \mathcal{E}^\pm)}{2\lambda} \nabla_a \left(\frac{\mathcal{F}^\pm}{(1 - \mathcal{E}^\pm)^2} \right) \\ &= \frac{1}{4\lambda^2} \left(\sigma^\mp_a - \frac{(1 + \mathcal{E}^\mp)^2}{(1 - \mathcal{E}^\pm)^2} \sigma^\pm_a \right) \mathcal{F}^\pm \\ &\quad + \frac{1 + \mathcal{E}^\mp}{2\lambda(1 - \mathcal{E}^\pm)} \nabla_a \mathcal{F}^\pm + \frac{1 + \mathcal{E}^\mp}{\lambda(1 - \mathcal{E}^\pm)^2} \mathcal{F}^\pm \nabla_a \mathcal{E}^\pm \\ &= -r^{-2} \nabla_a r + 2r^{-2} \nabla_a r - 2r^{-2} \nabla_a r + O^*(r^{-2-\delta}) \\ &= -r^{-2} \nabla_a r + O^*(r^{-2-\delta}). \end{aligned} \quad (6.31)$$

We have $n^a = \frac{\nabla^a r}{|\nabla r|}$, where $|\nabla r| \rightarrow 1$ as follows from (2.2) and (2.3). From (6.31) we find that

$$\begin{aligned} \int_{S_{\infty,\epsilon}} n^a \Psi^\pm_a d\mu &= \int_{S_{\infty,\epsilon}} (-\epsilon^2 + O(\epsilon^{2+\delta})) d\mu \\ &= -(1 + O(\epsilon^\delta)) \epsilon^2 \text{Vol}(S_{\infty,\epsilon}, g) \\ &= -(1 + O(\epsilon^\delta)) \epsilon^2 \text{Vol}(L, \sigma_{\epsilon^{-1}}) \end{aligned} \quad (6.32)$$

The manifold L is a circle fibration over the orbifold O . From Definition 2.1 we have that the transverse metric γ gives a metric on O with constant curvature +1. With respect to this fibration, the metric $\sigma_r = r^2 \gamma + \eta^2$, and its volume form is

$d\mu^{\sigma_r} = r^2 d\mu^\gamma \wedge \eta$. By Fubini's theorem and the orbifold Gauss–Bonnet theorem [52, S 2], the volume of L is

$$\begin{aligned} \text{Vol}(L, \sigma_{\epsilon^{-1}}) &= \int_O \left(\epsilon^{-2} \int_{\text{fiber}} \eta \right) d\mu^\gamma \\ &\geq \epsilon^{-2} \ell_\infty \int_O 1 d\mu^\gamma \\ &= \epsilon^{-2} \ell_\infty \int_O \mathcal{K}^O d\mu^\gamma \\ &= 2\pi \epsilon^{-2} \ell_\infty \chi[O]. \end{aligned} \quad (6.33)$$

We conclude that

$$\lim_{\epsilon \rightarrow 0} \int_{S_{\infty, \epsilon}} n^a \Psi_a^\pm d\mu \leq -2\pi \ell_\infty \chi[O]. \quad (6.34)$$

□

6.3. The singular term.

Lemma 6.13 (The singular term).

$$\lim_{\epsilon \rightarrow 0} \int_{\partial U_{z^\pm, \epsilon}} n^a \Psi_a^\pm d\mu = 0. \quad (6.35)$$

Proof. Proposition 6.3 tells us that λ is bounded away from zero near \mathcal{Z}^\pm . From (5.25) and (5.8c) we therefore see that $|n^a \Psi_a^\pm| \leq C$ independently of $\epsilon > 0$ near \mathcal{Z}^\pm . Thus

$$\left| \int_{\partial U_{z^\pm, \epsilon}} n^a \Psi_a^\pm d\mu \right| \leq C \text{Vol}(\partial U_{z^\pm, \epsilon}). \quad (6.36)$$

By (6.6) we have for sufficiently small $\delta > 0$

$$\text{Vol}(\partial U_{z^\pm, \epsilon}) \leq D \sum_i r_i^3 \leq D \sum_i r_i^{2+\delta} < D\epsilon, \quad (6.37)$$

and the claim follows. □

Remark 6.14. *The singular set \mathcal{Z}^\pm was also an issue in the black hole uniqueness proofs carried out on the orbit space in the hypersurface-orthogonal case, as sketched in Section 1. While in his proof Israel [33] assumed the absence of such a set, the latter was admitted and subject of a technically subtle discussion in Müller zum Hagen et. al. [44]. On the other hand, we are not aware of suitable generalizations of Robinson's identities [51], which avoid the singular set, to the non-hypersurface orthogonal case.*

6.4. Proof of Proposition 6.7. Proposition 6.7 follows from Lemmas 5.10, 6.9, 6.10, 6.12, 6.13.

7. PROOF OF THE MAIN THEOREM

In this section, we will analyze the terms in the divergence identity derived in Proposition 6.7 and prove our main theorem. The first step is the following corollary to Proposition 6.7.

Definition 7.1. *For a nut P with fixed point data $\{\epsilon, a, b\}$ with $a \leq b$, let*

$$Z^\pm(P) = \pm \epsilon \frac{1}{a} + \frac{1}{b}. \quad (7.1)$$

Corollary 7.2. *Let (\mathcal{M}, g_{ab}) be an ALF- $A_k S^1$ -instanton which is not half-flat. Let $w \geq 1$ be the maximal weight of all nuts in \mathcal{M} . If \mathcal{M} has no nuts, we set $w = 1$. It holds that*

$$-\frac{2}{w} + \chi[\mathcal{M}] - n_{\text{nuts}} + \sum_{i=1}^{n_{\text{nuts}}} Z^\pm(P_i) \geq 0 \quad (7.2)$$

with equality if and only if \mathcal{W}_{abcd}^\pm is algebraically special.

Proof. Since (\mathcal{M}, g_{ab}) is ALF- A_k we have $\chi[O] = 2$. Let G be as in Lemma 3.1. Divide (6.11) by $4\pi^2 G$. The corollary then follows from (3.7) and Lemma 6.11. \square

Lemma 7.3. *Let P be a nut. If $\epsilon(P) = 1$, then $Z^+(P) \leq 2$ and $Z^-(P) \leq 0$. If $\epsilon(P) = -1$, then $Z^-(P) \leq 2$ and $Z^+(P) \leq 0$. For any of the inequalities, equality holds if and only if the weights of P are $\{1, 1\}$.*

Proof. Let P have weights a, b , where $a \leq b$. If $\epsilon(P) = 1$, then

$$Z^+(P) = \frac{1}{a} + \frac{1}{b}, \quad (7.3)$$

and since a and b are positive integers, we have $Z^+(P) \leq 2$ with equality if and only if $a = b = 1$. Since $a \leq b$ we see that

$$Z^-(P) = -\frac{1}{a} + \frac{1}{b} \leq 0 \quad (7.4)$$

where equality occurs if and only if $a = b$, which happens only when $a = b = 1$ since a and b are coprime. The reasoning in the case $\epsilon(P) = -1$ is similar. \square

We are now ready to apply inequality (7.2) to known spaces with ALF- $A_k S^1$ -instanton metrics. We first consider the case with Kerr topology.

Lemma 7.4. *Let (\mathcal{M}, g_{ab}) be an AF S^1 -instanton on $S^4 \setminus S^1$. Then equality holds in (7.2), for both signs.*

Proof. In the present case we have $e = 0$, $\chi[\mathcal{M}] = 2$ and $\text{sign}[\mathcal{M}] = 0$. Equation (3.15) with $e = 0$ tells us that the number of positively oriented nuts is equal to the number of negatively oriented nuts. We denote this number by k . Corollary 7.2 yields

$$-\frac{2}{w} + 2 - 2k + \sum_{i=1}^{n_{\text{nuts}}} Z^\pm(P_i) \geq 0. \quad (7.5)$$

In case there are no nuts, $w = 1$ and $k = 0$, and we have equality in (7.5) and hence in (7.2).

We now assume $k > 0$. In the case $w = 1$, all weights are 1, so we get $Z^\pm(P_i) = \pm\epsilon(P_i) + 1$ for all i , and thus the left hand side of (7.5) is equal to

$$-2 + 2 - 2k + \sum_{i=1}^{n_{\text{nuts}}} (\pm\epsilon(P_i) + 1) = \pm \sum_{i=1}^{n_{\text{nuts}}} \epsilon(P_i) = 0. \quad (7.6)$$

This means that equality holds in (7.5) and (7.2), for both signs.

Now consider the case $w > 1$, and assume without loss of generality that P_1 has nut data $\{\epsilon(P_1), a, w\}$ for some positive integer a . By Lemma 3.5, there exists another nut, with nut data either $\{-\epsilon(P_1), a, w\}$ or $\{\epsilon(P_1), b, w\}$, where $a + b = w$; without loss of generality, assume that this nut is P_2 . In the case where P_2 has nut data $\{-\epsilon(P_1), a, w\}$, we have

$$Z^\pm(P_1) + Z^\pm(P_2) = \pm \frac{\epsilon(P_1)}{a} + \frac{1}{w} \mp \frac{\epsilon(P_1)}{a} + \frac{1}{w} = \frac{2}{w}, \quad (7.7)$$

and then (7.5) is equivalent to

$$2 - 2k + \sum_{i=3}^{n_{\text{nuts}}} Z^\pm(P_i) \geq 0. \quad (7.8)$$

On the other hand, since the set $\{P_3, \dots, P_{n_{\text{nuts}}}\}$ contains $k - 1$ nuts of each orientation, Lemma 7.3 implies that

$$2 - 2k + \sum_{i=3}^{n_{\text{nuts}}} Z^\pm(P_i) \leq 0 \quad (7.9)$$

which means that equality must hold in view of Lemma 7.3. Consider now the case where P_2 has nut data $\{\epsilon(P_1), b, w\}$, with $a + b = w$. Then

$$Z^\pm(P_1) + Z^\pm(P_2) = \frac{2}{w} \pm \epsilon(P_1) \left(\frac{1}{a} + \frac{1}{b} \right), \quad (7.10)$$

so that (7.5) is equivalent to

$$\pm \epsilon(P_1) \left(\frac{1}{a} + \frac{1}{b} \right) + 2 - 2k + \sum_{i=3}^{n_{\text{nuts}}} Z^\pm(P_i) \geq 0. \quad (7.11)$$

From (3.15) we find that the set $\{P_3, \dots, P_{n_{\text{nuts}}}\}$ must contain $k - 2$ ($k \geq 2$) nuts with the same orientation as P_1 , and k nuts with the opposite orientation. Assuming that $\epsilon(P_3) = \epsilon(P_4) = -\epsilon(P_1)$, Lemma 7.3 tells us that

$$\sum_{i=5}^{n_{\text{nuts}}} Z^\pm(P_i) \leq 2k - 4. \quad (7.12)$$

In case $\epsilon(P_1) = 1$, thus, the + version of (7.11) implies that

$$0 \leq \frac{1}{a} + \frac{1}{b} + Z^+(P_3) + Z^+(P_4) - 2 \leq \frac{1}{a} + \frac{1}{b} - 2 \leq 0, \quad (7.13)$$

where the middle inequality uses again Lemma 7.3. Thus, $a = b = 1$ and $Z^+(P_3) = Z^+(P_4) = 0$, which, once again by Lemma 7.3, means that P_3 and P_4 have weights $\{1, 1\}$. In the case $\epsilon(P_1) = -1$, similarly, the - version of (7.11) implies that

$$0 \leq \frac{1}{a} + \frac{1}{b} + Z^-(P_3) + Z^-(P_4) - 2 \leq \frac{1}{a} + \frac{1}{b} - 2 \leq 0. \quad (7.14)$$

In either case, thus, $a = b = 1$, and P_3 and P_4 have weights $\{1, 1\}$. Hence the left hand side of (7.5) is equal to

$$\begin{aligned} & \pm 2\epsilon(P_1) + 2(\mp\epsilon(P_1) + 1) + 2 - 2k + \sum_{i=5}^{n_{\text{nuts}}} Z^\pm(P_i) \\ & \leq 2 + 2 - 2k + (2k - 4) = 0, \end{aligned} \quad (7.15)$$

which finishes the proof. \square

We next consider the case of Taub-bolt topology, which turns out to be rather similar to the Kerr case.

Lemma 7.5. *Let (\mathcal{M}, g_{ab}) be an ALF- A_0 S^1 -instanton on $\mathbb{C}P^2 \setminus \{\text{pt.}\}$. Then equality holds in (7.2), for both signs.*

Proof. From the assumption on the topology, we have $\chi[\mathcal{M}] = 2$ and $\text{sign}[\mathcal{M}] = 1$, and since \mathcal{M} is ALF- A_0 , the circle fibration at infinity has Euler number $e = 1$. By (3.15) we see that, as in the proof of the previous lemma, there is an equal number of positively and negatively oriented nuts. The rest of that proof goes through without any modification. \square

Finally, we consider the case of an S^1 -instanton with the same topology as the Chen–Teo instanton.

Lemma 7.6. *Let (\mathcal{M}, g_{ab}) be an AF S^1 -instanton on $\mathbb{C}P^2 \setminus S^1$. Then equality holds in (7.2), for $-$.*

Proof. Let Φ^\pm be the left hand side of (7.2), so $\Phi^\pm \geq 0$ by Corollary 7.2. We wish to show that $\Phi^- = 0$. By the assumption on the topology, we have $\chi[\mathcal{M}] = 3$ and $\text{sign}[\mathcal{M}] = 1$. From (3.15) we see that

$$\sum_{i=1}^{n_{\text{nuts}}} \epsilon(P_i) = 1, \quad (7.16)$$

and thus the numbers of positively oriented and negatively oriented nuts are $k+1$ and k , respectively, for some integer $k \geq 0$. If all nuts have weights $\{1, 1\}$, then $\Phi^- = 0$ and we are done, so assume that $w > 1$, where w is the largest weight, and assume without loss of generality that P_1 has this weight; then P_1 has nut data $\{\epsilon(P_1), b, w\}$ for some b . If there exists another nut with nut data $\{-\epsilon(P_1), b, w\}$, then the same argument as in the proof of (7.9) works to show that $\Phi^- \leq 0$, since there are $k-1$ remaining negatively oriented nuts in this case.

Thus, assume that there exists no nut with nut data $\{-\epsilon(P_1), b, w\}$. By Lemma 3.4 there exists another nut with nut data $\{\epsilon(P_1), a, w\}$, where $a+b=w$. Assume without loss of generality that this nut is P_2 , and that $b \geq a$. If $a=b=1$ and $\epsilon(P_1)=1$, then using Lemma 7.3 we see that

$$\Phi^- = -2 + 2 - 2k + \sum_{i=3}^{n_{\text{nuts}}} Z^-(P_i) \leq -2 + 2 - 2k + 2k = 0. \quad (7.17)$$

If $a=b=1$ and $\epsilon(P_1)=-1$, we similarly find

$$\Phi^- = 2 + 2 - 2k + \sum_{i=3}^{n_{\text{nuts}}} Z^-(P_i) \leq 2 + 2 - 2k + 2(k-2) = 0. \quad (7.18)$$

Henceforth, we can thus assume that $b > a$. By applying Lemma 3.4 to the nut P_1 , there is a nut with data either $\{\epsilon(P_1), b, c\}$, where $a+b \equiv -c \pmod{b}$, or data $\{-\epsilon(P_1), b, c\}$, where $a+b \equiv c \pmod{b}$. Since b is equal to neither a nor $a+b$, this nut cannot be P_2 , so we can assume without loss of generality that it is P_3 . If $\epsilon(P_3) = \epsilon(P_1)$, then since $a+c \equiv a+b+c \equiv 0 \pmod{b}$ and since $0 < a+c < b+(a+b) < 3b$, we have $c \in \{b-a, 2b-a\}$. On the other hand, if $\epsilon(P_3) = -\epsilon(P_1)$, then since $a-c \equiv a+b-c \equiv 0 \pmod{b}$ and since $a+b$ is by assumption the highest weight in the configuration, we have $c \in \{a+b, a\}$. However, the case $c=a+b$ in which P_3 would have nut data $\{-\epsilon(P_1), b, w\}$ has also been excluded by assumption above. This implies $c=a$.

Together with the possibilities of the choice of $\epsilon(P_1)$, we are left with six cases each of which will be considered separately. In the list below, the ordering of the nuts is P_1 , P_2 and P_3 , and the weights of each nut are displayed in increasing order.

Case: $\{+, b, a+b\}, \{+, a, a+b\}, \{+, b-a, b\}$. By Lemma 7.3 we have

$$\Phi^+ = \frac{1}{a} + \frac{2}{b} + \frac{1}{b-a} + 2 - 2k + \sum_{i=4}^{n_{\text{nuts}}} Z^+(P_i) \leq \frac{1}{a} + \frac{2}{b} + \frac{1}{b-a} - 2, \quad (7.19)$$

$$\Phi^- = -\frac{1}{a} - \frac{1}{b-a} + 2 - 2k + \sum_{i=4}^{n_{\text{nuts}}} Z^-(P_i) \leq -\frac{1}{a} - \frac{1}{b-a} + 2. \quad (7.20)$$

The right hand side of (7.19) is negative for $b > 3$ which is thus impossible. The remaining cases $(a, b) = (1, 2)$, $(a, b) = (1, 3)$ and $(a, b) = (2, 3)$ will be handled separately. When $(a, b) = (1, 2)$, (7.20) yields $\Phi^- = 0$ and we are done.

When $(a, b) = (1, 3)$, we have the nut configuration

$$\{+, 3, 4\}, \quad \{+, 1, 4\}, \quad \{+, 2, 3\}, \quad (7.21)$$

and since the weight 2 occurs only once in this configuration, by Lemma 3.3 there must exist another nut, say P_4 , with nut data $\{\epsilon(P_4), 2, c\}$ for some c . If $\epsilon(P_4) = 1$, then

$$\Phi^+ = \frac{8}{3} + \frac{1}{c} + 2 - 2k + \sum_{i=5}^{n_{\text{nuts}}} Z^+(P_i) \leq -\frac{4}{3} + \frac{1}{c} < 0 \quad (7.22)$$

which is a contradiction, while if $\epsilon(P_4) = -1$, then

$$\Phi^- = -1 + \frac{1}{c} + 2 - 2k + \sum_{i=5}^{n_{\text{nuts}}} Z^-(P_i) \leq -1 + \frac{1}{c} \leq 0 \quad (7.23)$$

and we conclude that $\Phi^- = 0$.

Finally, when $(a, b) = (2, 3)$, we have the nut configuration

$$\{+, 3, 5\}, \quad \{+, 2, 5\}, \quad \{+, 1, 3\}, \quad (7.24)$$

and as in the previous case, there is a nut P_4 with data $\{\epsilon(P_4), 2, c\}$. Reasoning as in the previous case, we find that $\Phi^+ < 0$ or $\Phi^- < 0$ holds, depending on whether $\epsilon(P_4) = 1$ or $\epsilon(P_4) = -1$.

Case: $\{+, b, a+b\}, \{+, a, a+b\}, \{+, b, 2b-a\}$. In this case,

$$\Phi^+ = \frac{1}{a} + \frac{2}{b} + \frac{1}{2b-a} + 2 - 2k + \sum_{i=4}^{n_{\text{nuts}}} Z^+(P_i) \leq \frac{1}{a} + \frac{2}{b} + \frac{1}{2b-a} - 2, \quad (7.25)$$

and the right hand side is negative for $b > 2$, which rules out this case.

For the case $(a, b) = (1, 2)$, we have the nut configuration

$$\{+, 2, 3\}, \quad \{+, 1, 3\}, \quad \{+, 2, 3\}, \quad (7.26)$$

and since the weight 3 occurs an odd number of times, Lemma 3.3 tells us that there exists another nut, say P_4 , with nut data $\{\epsilon(P_4), 3, c\}$, for some c . If $\epsilon(P_4) = 1$, then

$$\Phi^+ = \frac{8}{3} + \frac{1}{c} + 2 - 2k + \sum_{i=5}^{n_{\text{nuts}}} Z^+(P_i) \leq -\frac{4}{3} + \frac{1}{c} < 0, \quad (7.27)$$

while if $\epsilon(P_4) = -1$, then

$$\Phi^- = -\frac{4}{3} + \frac{1}{c} + 2 - 2k + \sum_{i=5}^{n_{\text{nuts}}} Z^-(P_i) \leq -\frac{4}{3} + \frac{1}{c} < 0, \quad (7.28)$$

which leads to a contradiction.

Case: $\{+, b, a+b\}, \{+, a, a+b\}, \{-, a, b\}$. In this case,

$$\Phi^- = \sum_{i=4}^{n_{\text{nuts}}} Z^-(P_i) + 2 - 2k \leq 0, \quad (7.29)$$

so $\Phi^- = 0$.

Case: $\{-, b, a+b\}, \{-, a, a+b\}, \{-, b-a, b\}$. In this case,

$$\Phi^- = \frac{1}{a} + \frac{2}{b} + \frac{1}{b-a} + 2 - 2k + \sum_{i=4}^{n_{\text{nuts}}} Z^-(P_i) \leq \frac{1}{a} + \frac{2}{b} + \frac{1}{b-a} - 4 < 0, \quad (7.30)$$

which is impossible.

Case: $\{-, b, a+b\}, \{-, a, a+b\}, \{-, b, 2b-a\}$. In this case,

$$\Phi^- = \frac{1}{a} + \frac{2}{b} + \frac{1}{2b-a} + 2 - 2k + \sum_{i=4}^{n_{\text{nuts}}} Z^-(P_i) \leq \frac{1}{a} + \frac{2}{b} + \frac{1}{2b-a} - 4 < 0, \quad (7.31)$$

which is a contradiction.

Case: $\{-, b, a+b\}, \{-, a, a+b\}, \{+, a, b\}$. In this case,

$$\Phi^- = \frac{2}{b} + 2 - 2k + \sum_{i=4}^{n_{\text{nuts}}} Z^-(P_i) \leq \frac{2}{b} - 2 < 0, \quad (7.32)$$

which again is a contradiction. \square

7.1. Proof of Theorem 1.1. First assume that (\mathcal{M}, g_{ab}) is an ALF S^1 -instanton on $S^4 \setminus S^1$. Then, by Lemma 7.4, we have that equality holds in (7.2) for both signs. By Corollary 7.2, it follows that (\mathcal{M}, g_{ab}) is Petrov type D and hence toric [3]. The classification of [8] now applies to show that (\mathcal{M}, g_{ab}) is in the Kerr family. Similarly, if (\mathcal{M}, g_{ab}) is an ALF S^1 -instanton on $\mathbb{CP}^2 \setminus \{\text{pt.}\}$, we have by Lemma 7.5 that it is Petrov type D and by a similar argument toric and hence by [8], (\mathcal{M}, g_{ab}) is in the Taub-bolt family. Finally, Lemma 7.6 gives point 3 of Theorem 1.1, in particular an AF S^1 -instanton on $\mathbb{CP}^2 \setminus S^1$ is one-sided type D and therefore Hermitian.

8. CONCLUDING REMARKS

In this paper we have applied the divergence identity derived in Section 5 to the case of ALF S^1 -instantons, restricting to those topologies where there are known Hermitian examples. However, it is worth emphasizing that the method can be applied more broadly. The G -signature can, together with the approach of Jang, be used to classify S^1 -actions on gravitational instantons. This allows one to apply the divergence identity to general S^1 -instantons. These applications will be considered in forthcoming work.

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APPENDIX A. PROOF OF PROPOSITION 4.3

First we recall the form of partly contracted volume forms,

$$\begin{aligned} \epsilon_{abcd} \epsilon_{efh}^d &= g_{ah}g_{bf}g_{ce} - g_{af}g_{bh}g_{ce} - g_{ah}g_{be}g_{cf} \\ &\quad + g_{ae}g_{bh}g_{cf} + g_{af}g_{be}g_{ch} - g_{ae}g_{bf}g_{ch}, \end{aligned} \quad (\text{A.1})$$

$$\epsilon_{abcd} \epsilon_{fh}^{cd} = -2g_{ah}g_{bf} + 2g_{af}g_{bh}. \quad (\text{A.2})$$

By (A.1), the SD/ASD field $\frac{1}{2}\epsilon_{ab}^{cd}\mathcal{F}_{cd}^\pm = \pm\mathcal{F}_{ab}^\pm$ satisfies

$$\begin{aligned} \mathcal{F}_{ab}^\pm \mathcal{F}_{c}^{\pm b} &= \frac{1}{4}\epsilon_{ab}^{ef}\epsilon_c^{b\,hi}\mathcal{F}_{ef}^\pm \mathcal{F}_{hi}^\pm \\ &= -\frac{1}{4}\mathcal{F}^{\pm 2}g_{ac}. \end{aligned} \quad (\text{A.3})$$

By (A.3),

$$\begin{aligned} \mathcal{F}_{ab}^\pm \sigma^{\pm b} &= 2\mathcal{F}_{ab}^\pm \mathcal{F}_{c}^{\pm b} \xi^c \\ &= -\frac{1}{2}\mathcal{F}^{\pm 2}\xi_a, \end{aligned} \quad (\text{A.4})$$

which is (4.6a) and similarly, (4.6b) follows via

$$\begin{aligned}\sigma^\pm_a \sigma^\pm_a &= 4\mathcal{F}^\pm_{ab} \mathcal{F}^\pm_{c}{}^b \xi^c \\ &= \mathcal{F}^{\pm 2} \lambda.\end{aligned}\tag{A.5}$$

By Ricci flatness, we have $\nabla_b F_{ac} = \nabla_b \nabla_a \xi_c = -W_{acbd} \xi^d$, see [50, Proposition 8.1.3], and hence (4.6c) follows. This also leads to (4.6d) via

$$\begin{aligned}\nabla_c \mathcal{F}^{\pm 2} &= 2\mathcal{F}^{\pm ab} \nabla_c \mathcal{F}^\pm_{ab} \\ &= -2\mathcal{F}^{\pm ab} \mathcal{W}^\pm_{abcd} \xi^d.\end{aligned}\tag{A.6}$$

To prove (4.6e), it is convenient to start with its dual,

$$\begin{aligned}\epsilon_{fh}^{ab} \nabla_a \sigma^\pm_b &= 2\epsilon_{fh}^{ab} \nabla_a (\mathcal{F}^\pm_{bc} \xi^c) \\ &= \underbrace{-2\epsilon_{fh}^{ab} \mathcal{W}^\pm_{bcd} \xi^c \xi^d}_{=0} + 2\epsilon_{fh}^{ab} \mathcal{F}^\pm_{bc} F_a{}^c \\ &= \underbrace{2\epsilon_{fh}^{ab} F_{bc} F_a{}^c}_{=0} \pm \epsilon_{fh}^{ab} \epsilon_{bc}^{ij} F_{ij} F_a{}^c \\ &= 0.\end{aligned}\tag{A.7}$$

The last step used (A.1) and dualizing the equation shows (4.6e). The divergence, (4.6f), follows from

$$\begin{aligned}\nabla_a \sigma^\pm_a &= 2\nabla_a (\mathcal{F}^\pm_{ab} \xi_b) \\ &= \underbrace{-2\mathcal{W}^\pm_{ad} \xi_b \xi^d}_{=0} + 2\mathcal{F}^\pm_{ab} F_{ab} \\ &= \mathcal{F}^\pm_{ab} (\mathcal{F}^\pm_{ab} + \mathcal{F}^\mp_{ab}) \\ &= \mathcal{F}^{\pm 2}.\end{aligned}\tag{A.8}$$

Finally, to prove (4.6g), apply a derivative to (4.6d),

$$\begin{aligned}\nabla^c \nabla_c \mathcal{F}^{\pm 2} &= -2\nabla^c (\mathcal{W}^\pm_{abcd} \xi^d \mathcal{F}^{\pm ab}) \\ &= -2 \left(\underbrace{(\nabla^c \mathcal{W}^\pm_{abcd}) \xi^d \mathcal{F}^{\pm ab}}_{=0} + \mathcal{W}^\pm_{abcd} \mathcal{F}^{\pm ab} F^{cd} - \underbrace{\mathcal{W}^\pm_{abcd} \mathcal{W}^\pm{}^{abc}{}_e \xi^d \xi^e}_{=\frac{1}{4} g_{de} \mathcal{W}^{\pm 2}} \right) \\ &= -\mathcal{W}^\pm_{abcd} \mathcal{F}^{\pm ab} \mathcal{F}^{\pm cd} - \underbrace{\mathcal{W}^\pm_{abcd} \mathcal{F}^{\pm ab} \mathcal{F}^{\mp cd}}_{=0} + \frac{1}{2} \lambda \mathcal{W}^{\pm 2}.\end{aligned}\tag{A.9}$$

In the second line, we used the Bianchi identity for the first term and a decomposition of contracted Weyl tensors, analogous to (A.3), for the last term. The second term in the last line vanishes because it is SD and ASD in the index pair cd .

APPENDIX B. ASYMPTOTICS AT INFINITY

In this appendix, we prove a couple of technical results on the asymptotic geometry of ALF S^1 -spaces.

B.1. Asymptotics of Killing vector fields. We will prove that a bounded Killing field with closed orbits for a general ALF metric g must be asymptotic to a multiple of the Killing field T of the background metric \dot{g} . For a similar result, see [6, Proposition 2.1]. We first formulate a lemma which is [17, Lemma, Appendix A], adapted to our setting.

Lemma B.1. *Let f be a function on $(A, \infty) \times L$ with $|df| = O(r^{-1-\alpha})$, $\alpha > 0$. Then there is a constant f_∞ so that $f = f_\infty + O(r^{-\alpha})$.*

Proposition B.2. *Let (\mathcal{M}, g_{ab}) be an ALF space. Assume ξ^a is a Killing field with uniformly bounded norm, all of whose orbits are closed. Then*

$$\xi^a = cT^a + O^*(r^{-1}) \quad (\text{B.1})$$

for some constant c and T has closed orbits in L . In particular,

$$\nabla\xi = O^*(r^{-2}). \quad (\text{B.2})$$

Proof. Since $\mathcal{L}_\xi g = 0$ and ξ is bounded we have

$$\nabla^2\xi = \text{Riem } \xi = O(r^{-3}), \quad (\text{B.3})$$

see [50, Proposition 8.1.3]. Using the Kato inequality we find that

$$|d|\nabla\xi|| \leq |\nabla^2\xi| = |\text{Riem } \xi| = O(r^{-3}). \quad (\text{B.4})$$

By Lemma B.1 there is a constant w_∞ so that

$$|\nabla\xi| = w_\infty + O(r^{-2}). \quad (\text{B.5})$$

and in particular $|\nabla\xi|$ is bounded.

We next work in local coordinates as discussed in Remark 2.7 on

$$C \times (-t_0, t_0) = (A, \infty) \times U \times (-t_0, t_0) \subset \mathbb{R}^3 \times (-t_0, t_0) \quad (\text{B.6})$$

for U an open subset of S^2 . We denote the standard coordinates on \mathbb{R}^3 by (x_1, x_2, x_3) and the coordinate on $(-t_0, t_0)$ by $x_0 = t$. In these coordinates we have $g = \delta + dt^2 + O(r^{-1})$. Then

$$\nabla\xi = d\xi + \Gamma\xi = W + O(r^{-2}) \quad (\text{B.7})$$

where $\Gamma = O(r^{-2})$ represents the Christoffel symbols for g , and W is a 4×4 -matrix of functions representing $d\xi$. From (B.3) we find that

$$dW = \nabla^2\xi - \Gamma\nabla\xi = \text{Riem } \xi - \Gamma\nabla\xi. \quad (\text{B.8})$$

Since $\nabla\xi$ is bounded we find that $|dW| = O(r^{-2})$ and from Lemma B.1 we conclude that $W = W^\infty + O(r^{-1})$ for a constant 4×4 -matrix W^∞ . We thus get

$$d\xi = W^\infty + O(r^{-1}). \quad (\text{B.9})$$

Since ξ is a Killing vector field for g we have

$$0 = g(\nabla_{\partial_i}\xi, \partial_j) + g(\partial_i, \nabla_{\partial_j}\xi) = W_{ij}^\infty + W_{ji}^\infty + O(r^{-1}) \quad (\text{B.10})$$

for $0 \leq i, j \leq 3$. We conclude that W^∞ is skew-symmetric.

Let the point $p_0 \in C$ be such that the points $p_0 + e_i$, $i = 1, 2, 3$ are also in C . Then since C is a cone the line segments $R(p_0 + se_i)$, $0 \leq s \leq 1$, are also contained in C for $R \geq 1$. Let c_i^R be the curve

$$c_i^R(s) = (R(p_0 + se_i), 0) \subset C \times (-t_0, t_0), \quad 0 \leq s \leq 1. \quad (\text{B.11})$$

Then

$$\begin{aligned} \xi(c_i^R(1)) - \xi(c_i^R(0)) &= \int_0^1 d\xi(Re_i) ds \\ &= R \int_0^1 d\xi(e_i) ds \\ &= R \int_0^1 (W^\infty + O(r^{-1})) (e_i) ds \\ &= R(W_{ij}^\infty e_j + O(R^{-1})) \\ &= RW_{ij}^\infty e_j + O(1). \end{aligned} \quad (\text{B.12})$$

Since ξ is bounded as $R \rightarrow \infty$ we find that $W_{ij}^\infty = 0$ for $i = 1, 2, 3$. Since W^∞ is skew-symmetric we conclude that $W^\infty = 0$.

Inserting $W = W^\infty + O(r^{-1}) = O(r^{-1})$ into (B.7) we get that $\nabla \xi = O(r^{-1})$, which we in turn insert into (B.8) to see that $|dW| = O(r^{-3})$. From Lemma B.1 we conclude that

$$W = W^\infty + O(r^{-2}) = O(r^{-2}) \quad (\text{B.13})$$

and

$$\nabla \xi = O(r^{-2}). \quad (\text{B.14})$$

We next write $\xi = V$ in coordinates, where V is a vector of functions. From

$$\nabla \xi = dV + \Gamma V \quad (\text{B.15})$$

we see that $|dV| = O(r^{-2})$, so from Lemma B.1 we get $V = V^\infty + O(r^{-1})$ for a constant vector V^∞ .

In the local coordinate system we have found that

$$\begin{aligned} \xi &= \sum_{i=0}^3 V_i^\infty e_i + O(r^{-1}) \\ &= V_0^\infty \partial_t + \sum_{i=1}^3 V_i^\infty e_i + O(r^{-1}) \\ &= V_0^\infty T + \sum_{i=1}^3 V_i^\infty e_i + O(r^{-1}). \end{aligned} \quad (\text{B.16})$$

By continuity the coefficients $V_0^\infty = c$ must be independent of the choice of local coordinates. Patching together with a partition of unity we find

$$\xi = cT + \eta + O(r^{-1}) \quad (\text{B.17})$$

where $\eta = Z\partial_r + \zeta$ is a vector field on $\mathring{M} = (A, \infty) \times L$ with Z a constant and ζ a (r -independent) vector field on L , and $\dot{g}(T, \eta) = 0$.

By assumption, the orbits of ξ are all closed. Since ξ is bounded the orbits have uniformly bounded lengths. If now $\eta \neq 0$ it is easy to see that ξ can have arbitrarily long orbits by choosing a starting point with sufficiently large r . We conclude that $\eta = 0$ and $\xi = cT + O(r^{-1})$. Using the above estimates together with $\text{Riem} = O^*(r^{-3})$ and (B.3) we can now conclude that in fact (B.1) and (B.2) hold.

It remains to show that T has closed orbits on L . To see this, let $p \in L$ be a point whose T -orbit is not closed, and consider a sequence of points $p_i = (r_i, p) \in \mathring{\mathcal{M}}$, for $r_i \rightarrow \infty$. The ξ -orbits of p_i are closed with uniformly bounded lengths ℓ_i . Passing to a subsequence, we may assume that ℓ_i tends to a limit ℓ_∞ . Flowing p along T a time ℓ_∞ yields a point q with $d(p, q) > 0$. This contradicts (B.1). \square

B.2. Asymptotics of solutions to Poisson equations. Let (\mathcal{M}, g_{ab}) be an ALF S^1 instanton. In this section, we shall analyze the asymptotics at infinity for solutions of the Poisson equation of the form

$$\Delta u = O^*(r^{-4}), \quad \text{with } u = O^*(r^{-1}). \quad (\text{B.18})$$

First we prove a lemma for almost invariant functions on (\mathring{M}, \dot{g}) .

Lemma B.3. *Let $(\mathring{\mathcal{M}}, \dot{g}_{ab})$, T be as in Definition 2.1. Suppose $u \in C_{loc}^2(\mathcal{M})$, $u = O^*(r^{-1})$ satisfies $Tu = O^*(r^{-3})$ and*

$$\Delta^{\dot{g}} u = O^*(r^{-4}). \quad (\text{B.19})$$

Then there are constants $b, \delta \in \mathbb{R}$ so that

$$u = br^{-1} + O(r^{-1-\delta}). \quad (\text{B.20})$$

Proof. We have $\dot{g} = dr^2 + \sigma_r$ where $\sigma_r = r^2\gamma + \eta^2$. Let $\sigma_1 = \gamma + \eta^2$. Then

$$\Delta^{\dot{g}} u = r^{-2} \partial_r(r^2 \partial_r u) + \Delta^{\sigma_r} u. \quad (\text{B.21})$$

Since T is a Killing vector field for \dot{g} we get

$$r^2 \Delta^{\sigma_r} u = \Delta^{\sigma_1} u + (r^2 - 1)T^2 u. \quad (\text{B.22})$$

We thus have

$$r^{-2} \partial_r(r^2 \partial_r u) + r^{-2} \Delta^{\sigma_1} u + r^{-2}(r^2 - 1)T^2 u = O(r^{-4}) \quad (\text{B.23})$$

or

$$r^2 \partial_r^2 u + 2r \partial_r u + \Delta^{\sigma_1} u + (r^2 - 1)T^2 u = O(r^{-2}). \quad (\text{B.24})$$

By assumption $Tu = O^*(r^{-3})$, and hence $T^2 u = O^*(r^{-4})$ and $(r^2 - 1)T^2 u = O^*(r^{-2})$. This leads to an equation of the form

$$r^2 \partial_r^2 u + 2r \partial_r u + \Delta^{\sigma_1} u = f \quad (\text{B.25})$$

for some f satisfying

$$f = O^*(r^{-2}). \quad (\text{B.26})$$

Let $D = \sqrt{1 - 4\Delta^{\sigma_1}}$ as a first order pseudodifferential operator. Then the equation factorizes as

$$r^{\frac{1}{2}(1+D)} \partial_r \left(r^{1-D} \partial_r \left(r^{\frac{1}{2}(1+D)} u \right) \right) = f, \quad (\text{B.27})$$

or

$$\partial_r \left(r^{1-D} \partial_r \left(r^{\frac{1}{2}(1+D)} u \right) \right) = r^{-\frac{1}{2}(1+D)} f. \quad (\text{B.28})$$

Since $D \geq 1$ we have

$$\left\| r^{-\frac{1}{2}(D-1)} \right\|_{\text{Op}(H_s \rightarrow H_s)} \leq 1 \quad (\text{B.29})$$

for $r \geq 1$. Since $u = O^*(r^{-1})$ we get

$$\begin{aligned} \left\| r^{-\frac{1}{2}(D-1)} \partial_r u \right\|_{L^\infty} &\leq C \left\| r^{-\frac{1}{2}(D-1)} \partial_r u \right\|_{H_s} \\ &\leq C \|\partial_r u\|_{H_s} \\ &= O(r^{-2}) \end{aligned} \quad (\text{B.30})$$

and

$$\begin{aligned} \left\| r^{-\frac{1}{2}(D-1)} \left(\frac{1}{2}(1+D)u \right) \right\|_{L^\infty} &\leq C \left\| r^{-\frac{1}{2}(D-1)} \left(\frac{1}{2}(1+D)u \right) \right\|_{H_s} \\ &\leq C \left\| \left(\frac{1}{2}(1+D)u \right) \right\|_{H_s} \\ &\leq C \|u\|_{H_{s+1}} \\ &= O(r^{-1}), \end{aligned} \quad (\text{B.31})$$

so we find that

$$\begin{aligned} r^{1-D} \partial_r \left(r^{\frac{1}{2}(1+D)} u \right) &= r^{1-\frac{1}{2}(D-1)} \partial_r u + r^{-\frac{1}{2}(D-1)} \left(\frac{1}{2}(1+D)u \right) \\ &= rO(r^{-2}) + O(r^{-1}) \\ &= O(r^{-1}). \end{aligned} \quad (\text{B.32})$$

We may thus integrate from r to ∞ to conclude that

$$0 - r^{1-D} \partial_r \left(r^{\frac{1}{2}(1+D)} u \right) = \int_r^\infty s^{-\frac{1}{2}(1+D)} f(s) ds \quad (\text{B.33})$$

or

$$\partial_r \left(r^{\frac{1}{2}(1+D)} u \right) = -r^{D-1} \int_r^\infty s^{-\frac{1}{2}(1+D)} f(s) ds. \quad (\text{B.34})$$

Let

$$1 = \delta_0 < \delta_1 \leq \dots \leq \delta_k \leq 3 \quad (\text{B.35})$$

be the eigenvalues of D with value less than or equal to 3 and let ϕ_0, \dots, ϕ_k be the corresponding orthonormal eigenfunctions. Set

$$u_i(r) = \int_L u \phi_i d\mu^{\sigma_1}, \quad f_i(r) = \int_L f \phi_i d\mu^{\sigma_1}. \quad (\text{B.36})$$

Then

$$\partial_r \left(r^{\frac{1}{2}(1+\delta_i)} u_i(r) \right) = -r^{\delta_i-1} \int_r^\infty s^{-\frac{1}{2}(1+\delta_i)} f_i(s) ds. \quad (\text{B.37})$$

If $\delta_i < 3$ we integrate from r to q , $r < q$, and get

$$q^{\frac{1}{2}(1+\delta_i)} u_i(q) - r^{\frac{1}{2}(1+\delta_i)} u_i(r) = - \int_r^q t^{\delta_i-1} \int_t^\infty s^{-\frac{1}{2}(1+\delta_i)} f_i(s) ds dt \quad (\text{B.38})$$

so

$$\begin{aligned} \left| q^{\frac{1}{2}(1+\delta_i)} u_i(q) - r^{\frac{1}{2}(1+\delta_i)} u_i(r) \right| &\leq \int_r^q t^{\delta_i-1} \int_t^\infty s^{-\frac{1}{2}(1+\delta_i)} |f_i(s)| ds dt \\ &\leq C \int_r^q t^{\delta_i-1} \int_t^\infty s^{-\frac{1}{2}(1+\delta_i)} s^{-2} ds dt \\ &\leq C \int_r^q t^{\delta_i-1} t^{-\frac{1}{2}(3+\delta_i)} dt \\ &= C \int_r^q t^{-\frac{1}{2}(5-\delta_i)} dt \\ &\leq C \left(r^{-\frac{1}{2}(3-\delta_i)} - q^{-\frac{1}{2}(3-\delta_i)} \right). \end{aligned} \quad (\text{B.39})$$

and we conclude that the function $q \mapsto q^{\frac{1}{2}(1+\beta)} u_i(q)$ is bounded and the limit

$$\lim_{q \rightarrow \infty} q^{\frac{1}{2}(1+\delta_i)} u_i(q) = A_i \quad (\text{B.40})$$

exists. Taking the limit $q \rightarrow \infty$ of the above we get

$$\left| A_i - r^{\frac{1}{2}(1+\delta_i)} u_i(r) \right| \leq C r^{-\frac{1}{2}(3-\delta_i)} \quad (\text{B.41})$$

or

$$u_i(r) = r^{-\frac{1}{2}(1+\delta_i)} A_i + O(r^{-2}). \quad (\text{B.42})$$

If $\delta_i = 3$ we integrate from 1 to r to get

$$r^2 u_i(r) - u_i(1) = - \int_1^r t^2 \int_t^\infty s^{-2} f_i(s) ds dt \quad (\text{B.43})$$

so

$$\begin{aligned} \left| r^2 u_i(r) - u_i(1) \right| &\leq \int_1^r t^2 \int_t^\infty s^{-2} |f_i(s)| ds dt \\ &\leq C \int_1^r t^2 \int_t^\infty s^{-2} s^{-2} ds dt \\ &\leq C \int_1^r t^2 t^{-3} dt \\ &= C \ln r \end{aligned} \quad (\text{B.44})$$

and

$$u_i(r) = r^{-2}u_i(1) + O(r^{-2}\ln r) = O(r^{-2}\ln r). \quad (\text{B.45})$$

Let δ be the smallest eigenvalue of D which is > 3 and let P be the spectral projection for D corresponding to eigenvalues $\geq \delta$. Let $U = Pu$ and $F = Pf$. Then

$$\partial_r \left(r^{\frac{1}{2}(1+D)} U \right) = -r^{D-1} \int_r^\infty s^{-\frac{1}{2}(1+D)} F(s) ds \quad (\text{B.46})$$

which we integrate from 1 to r to get

$$r^{\frac{1}{2}(1+D)} U(r) - U(1) = - \int_1^r t^{D-1} \int_t^\infty s^{-\frac{1}{2}(1+D)} F(s) ds dt \quad (\text{B.47})$$

or

$$\begin{aligned} U(r) &= r^{-\frac{1}{2}(1+D)} U(1) - r^{-\frac{1}{2}(1+D)} \int_1^r t^{D-1} \int_t^\infty s^{-\frac{1}{2}(1+D)} F(s) ds dt \\ &= K(r) + I(r). \end{aligned} \quad (\text{B.48})$$

Since $D - \delta \geq 0$ on the image of P we have

$$\left\| r^{-\frac{1}{2}(D-\delta)} \right\|_{\text{Op}(H_s \cap \text{Im } P \rightarrow H_s \cap \text{Im } P)} \leq 1 \quad (\text{B.49})$$

for $r \geq 1$. Here Op is the operator norm. For $K(r)$ we have

$$K(r) = r^{-\frac{1}{2}(1+D)} U(1) = r^{-\frac{1}{2}(\delta+1)} r^{-\frac{1}{2}(D-\delta)} U(1) \quad (\text{B.50})$$

so

$$\begin{aligned} \|K(r)\|_{L^\infty} &\leq C r^{-\frac{1}{2}(\delta+1)} \left\| r^{-\frac{1}{2}(D-\delta)} U(1) \right\|_{H_s} \\ &\leq C r^{-\frac{1}{2}(\delta+1)} \|U(1)\|_{H_s} \\ &\leq C r^{-\frac{1}{2}(\delta+1)}. \end{aligned} \quad (\text{B.51})$$

For $I(r)$ we have

$$\begin{aligned} I(r) &= -r^{-\frac{1}{2}(1+D)} \int_1^r t^{D-1} \int_t^\infty s^{-\frac{1}{2}(1+D)} F(s) ds dt \\ &= - \int_1^r \int_t^\infty r^{-\frac{1}{2}(1+D)} t^{D-1} s^{-\frac{1}{2}(1+D)} F(s) ds dt \\ &= - \int_1^r \int_t^\infty r^{-\frac{1}{2}(D-\delta+\delta+1)} t^{D-\delta+\delta-1} s^{-\frac{1}{2}(D-\delta+\delta+1)} F(s) ds dt \\ &= - \int_1^r \int_t^\infty r^{-\frac{1}{2}(\delta+1)} t^{\delta-1} s^{-\frac{1}{2}(\delta+1)} F(s) r^{-\frac{1}{2}(D-\delta)} t^{D-\delta} s^{-\frac{1}{2}(D-\delta)} F(s) ds dt \\ &= -r^{-\frac{1}{2}(\delta+1)} \int_1^r t^{\delta-1} \int_t^\infty s^{-\frac{1}{2}(\delta+1)} \left(\frac{r}{t}\right)^{-\frac{1}{2}(D-\delta)} \left(\frac{s}{t}\right)^{-\frac{1}{2}(D-\delta)} F(s) ds dt. \end{aligned} \quad (\text{B.52})$$

Since $s \geq t$ and $t \leq r$ we have

$$\left\| \left(\frac{s}{t}\right)^{-\frac{1}{2}(D-\delta)} \right\|_{\text{Op}(H_s \cap \text{Im } P \rightarrow H_s \cap \text{Im } P)} \leq 1, \quad (\text{B.53})$$

$$\left\| \left(\frac{r}{t}\right)^{-\frac{1}{2}(D-\delta)} \right\|_{\text{Op}(H_s \cap \text{Im } P \rightarrow H_s \cap \text{Im } P)} \leq 1, \quad (\text{B.54})$$

and since $F(r) = O(r^{-2})$,

$$\begin{aligned}
\|I(r)\|_{L^\infty} &\leq r^{-\frac{1}{2}(\delta+1)} \int_1^r t^{\delta-1} \int_t^\infty s^{-\frac{1}{2}(\delta+1)} \left\| \left(\frac{r}{t}\right)^{-\frac{1}{2}(D-\delta)} \left(\frac{s}{t}\right)^{-\frac{1}{2}(D-\delta)} F(s) \right\|_{L^\infty} ds dt \\
&\leq Cr^{-\frac{1}{2}(\delta+1)} \int_1^r t^{\delta-1} \int_t^\infty s^{-\frac{1}{2}(\delta+1)} \left\| \left(\frac{r}{t}\right)^{-\frac{1}{2}(D-\delta)} \left(\frac{s}{t}\right)^{-\frac{1}{2}(D-\delta)} F(s) \right\|_{H_s} ds dt \\
&\leq Cr^{-\frac{1}{2}(\delta+1)} \int_1^r t^{\delta-1} \int_t^\infty s^{-\frac{1}{2}(\delta+1)} \|F(s)\|_{H_s} ds dt \\
&\leq Cr^{-\frac{1}{2}(\delta+1)} \int_1^r t^{\delta-1} \int_t^\infty s^{-\frac{1}{2}(\delta+1)} s^{-2} ds dt \\
&= Cr^{-\frac{1}{2}(\delta+1)} \int_1^r t^{\delta-1} \int_t^\infty s^{-\frac{1}{2}(\delta+5)} ds dt \\
&= Cr^{-\frac{1}{2}(\delta+1)} \int_1^r t^{\delta-1} t^{-\frac{1}{2}(\delta+3)} dt \\
&= Cr^{-\frac{1}{2}(\delta+1)} \int_1^r t^{\frac{1}{2}(\delta-5)} dt \\
&= Cr^{-\frac{1}{2}(\delta+1)} \left(r^{\frac{1}{2}(\delta-3)} - 1 \right) \\
&\leq Cr^{-2}.
\end{aligned} \tag{B.55}$$

Together we have

$$U(r) = O(r^{-2}). \tag{B.56}$$

Finally,

$$\begin{aligned}
u(r) &= u_0(r)\phi_0 + \sum_{0 < \delta_i < 3} u_i(r)\phi_i + \sum_{\delta_i=3} u_i(r)\phi_i + U(r) \\
&= r^{-1} A_0 \phi_0 + \sum_{0 < \delta_i < 3} \left(r^{-\frac{1}{2}(1+\delta_i)} A_i + O(r^{-2}) \right) \phi_i \\
&\quad + \sum_{\delta_i=3} O(r^{-2} \ln r) \phi_i + O(r^{-2}) \\
&= br^{-1} + O(r^{-1-\delta})
\end{aligned} \tag{B.57}$$

for some $\delta > 0$. \square

Next we consider the general case of S^1 -invariant functions on an S^1 -ALF manifold.

Proposition B.4. *Let (M, g) be an S^1 -ALF manifold where the circle action is generated by the bounded Killing field ξ . Suppose $u = O^*(r^{-1})$ is an S^1 -invariant solution to*

$$\Delta^g u = O^*(r^{-4}). \tag{B.58}$$

Then there are constants $\delta > 0$ and $b \in \mathbb{R}$ so that

$$u = br^{-1} + O(r^{-1-\delta}). \tag{B.59}$$

Proof. Since $u = O(r^{-1})$ and $\xi u = 0$ it follows from Proposition B.2 that $Tu = O(r^{-3})$. From $g = \dot{g} + O(r^{-1})$ and the standard formula for the Laplacian

$$\Delta^g u = \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g} g^{ij} \partial_j u \right) \tag{B.60}$$

we see that $\Delta^g u = O^*(r^{-4})$. The result thus follows from Lemma B.3. \square

APPENDIX C. THE QUOTIENT

C.1. Generalities. In this section we consider the orbit space $\mathcal{N} = \mathcal{M}/S^1$, on which the family of divergence identities was found originally [53, 54]. We recall its algebraic derivation from these papers; as to its relation to the lifted family (5.22) we make use of [41]. However, before doing this we give a mathematically sound introduction of the orbit space. The key is the following result

Theorem C.1 ([56]). *For a compact Lie group G acting smoothly on (\mathcal{M}, g_{ab}) the following holds.*

- (1) *There exists a unique maximal orbit type.*
- (2) *The union \mathcal{M}_0 of maximal orbits is open and dense in \mathcal{M} .*
- (3) *The G -action on \mathcal{M} restricts to \mathcal{M}_0 , and $\mathcal{M}_0 \rightarrow \mathcal{M}_0/G$ is a Riemannian submersion. It is also a fibre bundle with fibre G/H where H is a principal isotropy group.*
- (4) *The quotient $\mathcal{N}_0 = \mathcal{M}_0/G$ of the principal part is open, dense and connected in \mathcal{M}/G .*

In the present case we have $G = H = S^1$. Maximal orbits in the above sense all have generic period in the sense of Lemma 3.1. Orbits which are not maximal are called exceptional. While all orbits in a neighborhood of a bolt have generic period, there are exceptional orbits in the neighborhood of a non-self dual nut.

We now recall [22] that, for an S^1 -isometry with Killing field ξ^a , there is a local diffeomorphism between

- tensor fields and differential forms on \mathcal{N}_0 and
- tensor fields and forms on \mathcal{M}_0 which are orthogonal to ξ^a and commute with ξ^a .

With a slight abuse of notation, we shall say that latter objects are also “on \mathcal{N}_0 ” (that is we drop the term “push-forward”). For objects which are defined directly on \mathcal{N}_0 we use greek indices $(\alpha, \beta, \dots = 1, 2, 3)$.

In particular, we obtain a non-degenerate metric $\gamma_{\mu\nu}$ ($\mu, \nu = 1, 2, 3$) on \mathcal{N}_0 via

$$\gamma_{ab} = \lambda g_{ab} - \xi_a \xi_b. \quad (\text{C.1})$$

We denote by D_μ , $\mathcal{R}_{\mu\nu}$ and \mathcal{R} the covariant derivative, the Ricci tensor and Ricci scalar with respect to $\gamma_{\mu\nu}$.

In local coordinates $x^a = (t, x^\mu)$ adapted to the isometry this reduction corresponds to a decomposition of g_{ab} as

$$ds^2 = \lambda(dt + \sigma_\mu dx^\mu)^2 + \lambda^{-1}\gamma_{\mu\nu}dx^\mu dx^\nu \quad (\text{C.2})$$

where σ_μ is related to the pullbacks of ω_a and $\epsilon_{abcd}\xi^d$ to \mathcal{N}_0 by

$$\omega_\mu = -\lambda^2 \epsilon_{\mu\nu\tau} D^\nu \sigma^\tau. \quad (\text{C.3})$$

In the coordinates introduced above, 1-forms V_a on \mathcal{M}_0 take the form $(0, V_\mu)$ and the dual vector fields split as $V^a = (0, \lambda V^\mu)$. The volume forms are related via $\sqrt{\det g} = \lambda^{-1} \sqrt{\det \gamma}$. For the divergence operators ∇_a and D_μ acting on vectors V^a on \mathcal{N}_0 we obtain

$$\nabla_a V^a = \lambda D_\mu V^\mu. \quad (\text{C.4})$$

From (C.3) it follows that

$$D_\mu(\lambda^{-2}\omega^\mu) = 0. \quad (\text{C.5})$$

which corresponds to the earlier observation that the current (4.15a) is conserved.

Excluding the half-flat case we recall from (3) of Lemma 5.1 that $|\mathcal{E}^\pm| < 1$ on \mathcal{N}_0 , and we introduce w, Θ, A_μ via

$$w^\pm = (1 + \mathcal{E}^\pm)^{-1}(1 - \mathcal{E}^\pm) \quad (\text{C.6})$$

$$\Theta = 1 - w^+ w^- = 4\lambda(1 + \mathcal{E}^+)^{-1}(1 + \mathcal{E}^-)^{-1} \quad (\text{C.7})$$

$$A_\mu = \frac{1}{2}(w^+ D_\mu w^- - w^- D_\mu w^+) = \frac{(1 - \mathcal{E}^-)^2 \sigma^+_\mu - (1 - \mathcal{E}_+^2) \sigma^-_\mu}{(1 + \mathcal{E}^+)^2(1 + \mathcal{E}^-)^2} \quad (\text{C.8})$$

Rewriting [25, Eq. (5.18)] we recall that the field equations on \mathcal{N}_0 can be derived by varying the Lagrangian

$$L = \sqrt{\det \gamma} [\mathcal{R} - 2\Theta^{-2} \gamma^{\mu\nu} (D_\mu w^+) (D_\nu w^-)] \quad (\text{C.9})$$

with respect to w^\pm and $\gamma_{\mu\nu}$. This results in

$$D_\mu \left(\Theta^{-1} \widehat{D}^{\pm\mu} w^\pm \right) = 0 \quad (\text{C.10})$$

$$\mathcal{R}_{\mu\nu} = 2\Theta^{-2} (D_{(\mu} w^-) (D_{\nu)} w^+). \quad (\text{C.11})$$

where we have introduced

$$\widehat{D}_\mu^\pm = D_\mu \mp 2\Theta^{-1} A_\mu. \quad (\text{C.12})$$

The $\widehat{D}^{\pm\mu}$ is related to the operator \mathcal{D}_\pm^μ of [54] via $\mathcal{D}_\pm^\mu = \Theta^{-1} \widehat{D}^{\pm\mu}$.

As a consequence of (C.10) we also obtain

$$D_\mu (\Theta^{-2} A^\mu) = 0. \quad (\text{C.13})$$

Remark C.2.

- (1) As discussed in [25, Sect. 5], equations (C.10) and (C.11) are invariant under the (Geroch-) $SL(2, \mathbb{R})$ -group [22] which can be represented by the shift $\omega \rightarrow \omega + a$ with fixed λ , the rescalings $\mathcal{E}^\pm \rightarrow b\mathcal{E}^\pm$ and the Ehlers transform $w^\pm \rightarrow c^{\pm 1} w^\pm$ with constants a, b and c , and with fixed $\gamma_{\mu\nu}$. The Noether currents corresponding to these symmetries on \mathcal{M} are given by (4.15).
- (2) A special class of solutions is characterized by $d^+ w^+ = d^- w^-$ for some constants $d^\pm \in \mathbb{R}$. This includes the cases in which one of w^\pm (and hence d^\pm) vanishes identically, which have been called ‘‘Multi-NUT’’, see [25, (3.19)]. From (C.11), $\gamma_{\mu\nu}$ is flat for these solutions and using (C.7) and (C.8) we see that (C.10) reduces to an ordinary Laplace equation for the non-vanishing w^\pm . The generic solutions where $d^+ w^+ = d^- w^-$ but $w^+ \not\equiv 0$ and $w^- \not\equiv 0$ can be transformed by an Ehlers transformation to the case $w^+ = w^-$ which means $\omega \equiv 0$ and hence corresponds to a hypersurface-orthogonal Killing field ξ^a on \mathcal{M}_0 .

C.2. The divergence identity on the quotient. We now recall the derivation of the family of divergence identities on the quotient \mathcal{N}_0 . We first introduce the pairs of quantities

Definition C.3.

$$k^{\pm 4} = D^\mu w^\pm D_\mu w^\pm \quad (\text{C.14})$$

$$B^\pm_{\mu\nu} = 4\Theta^{-2} \mathcal{C} (D_\mu D_\nu w^\pm - (3(w^\pm)^{-1} + \Theta^{-1} w^\mp) D_\mu w^\pm D_\nu w^\pm) \quad (\text{C.15})$$

$$C^\pm_{\mu\nu\sigma} = 4\Theta^{-2} (D_\mu D_{[\nu} w^\pm D_{\sigma]} w^\pm - \gamma_{\mu[\nu} w^\pm_{\sigma]}) \quad (\text{C.16})$$

where

$$w^\pm_\sigma = \gamma^{\mu\nu} D_\mu D_{[\nu} w^\pm D_{\sigma]} w^\pm \quad (\text{C.17})$$

and \mathcal{C} denotes the trace-free part of a symmetric tensor.

Remark C.4. The quantities k^\pm and the push-forwards C_{abc} of $C_{\mu\nu\nu}$ to \mathcal{M}_0 are related to the ones defined in (4.2f), (5.8c) and (5.18) via

$$k^{\pm 4} = \frac{16(\mu \pm \nu)}{(1 + \mathcal{E}^\pm)^4} = 4w^{\pm 4}\mathbf{s}^{\pm 2} \quad (\text{C.18})$$

$$C^{\pm}_{abc} = -\frac{(1 + \mathcal{E}^\pm)^2}{8\lambda^2(1 + \mathcal{E}^\pm)^2}\mathcal{P}^{\pm}_{abc} \quad (\text{C.19})$$

Furthermore, $B^\pm_{\mu\nu}$ and $C^\pm_{\mu\nu\sigma}$ are related via

$$C^\pm_{\mu\nu\sigma} = B^\pm_{\mu[\nu}D_{\sigma]}w^\pm + \frac{1}{2}\gamma_{\mu[\nu}B^\pm_{\sigma]\tau}D^\tau w^\pm. \quad (\text{C.20})$$

We obtain the following identities [54]

Lemma C.5. On sets where $k^{\pm 4} \neq 0$ (C.10) and (C.11) imply, for each $\alpha \in \mathbb{R}$,

$$\begin{aligned} D_\mu \left(\Theta^{-1} \widehat{D}^{\pm\mu} \frac{k^{\pm\alpha+1}}{w^{\pm\alpha}} \right) &= \alpha(\alpha+1) \frac{k^{\pm\alpha-1}}{\Theta w^{\pm\alpha}} (D_\mu k^\pm - \frac{k^\pm}{w^\pm} D_\mu w^\pm)(D^\mu k^\pm - \frac{k^\pm}{w^\pm} D^\mu w^\pm) \\ &\quad + \frac{\alpha+1}{16} \frac{k^{\pm\alpha-7}}{w^{\pm\alpha}} \Theta^3 C^\pm_{\mu\nu\sigma} C^{\pm\mu\nu\sigma}. \end{aligned} \quad (\text{C.21})$$

Proof. We first sketch the straightforward calculation in the case $\alpha = 0$.

$$\begin{aligned} D_\mu \left(\Theta^{-1} \widehat{D}^{\pm\mu} k^\pm \right) &= 2\Theta^{-2} w^\mp D_\mu w^\pm D^\mu k^\pm + \Theta^{-1} \Delta k^\pm \\ &= 2\Theta^{-2} w^\mp D_\mu w^\pm D^\mu k^\pm - \frac{3}{2} \Theta^{-1} k^{\pm-4} D^\mu k^\pm D_\mu D_\nu w^\pm D^\nu w^\pm \\ &\quad + \frac{1}{2} \Theta^{-1} k^{\pm-3} D_\mu \Delta w^\pm D^\mu w^\pm + \frac{1}{2} \Theta^{-1} k^{\pm-3} \mathcal{R}_{\mu\nu} D^\mu w^\pm D^\nu w^\pm \\ &\quad + \frac{1}{2} \Theta^{-1} k^{\pm-3} D_\mu D_\nu w^\pm D^\mu D^\nu w^\pm \\ &= \frac{1}{2} k^{\pm-7} \Theta^{-1} (-4\Theta^{-1} k^{\pm 7} w^\mp D_\mu w^\pm D^\mu k^\pm - 6k^{\pm 6} D_\mu k^\pm D^\mu k^\pm \\ &\quad - 2\Theta^{-2} w^\mp k^{\pm 12} + k^{\pm 4} D_\mu D_\nu w^\pm D^\mu D^\nu w^\pm) \\ &= \frac{1}{16} \Theta^3 k^{\pm-7} C^\pm_{\mu\nu\sigma} C^{\pm\mu\nu\sigma}, \end{aligned} \quad (\text{C.22})$$

where $\Delta = D_\mu D^\mu$. The general case $\alpha \in \mathbb{R}$ follows from (C.22) by splitting the argument on the left hand side of (C.21) as $k^{\pm\alpha+1}/w^{\pm\alpha} = k^\pm (k^\pm/w^\pm)^\alpha$ and applying the chain rule. \square

Remark C.6.

- (1) The identity (C.21) is equivalent to (5.22) for $\alpha = 2\beta - 1$; hence remarks made in Section 5 on the latter family for various values of β carry over to the former.
- (2) For $\alpha = 3$ the right hand side of (C.21) simplifies to $\frac{1}{8}\Theta^3 B^\pm_{\mu\nu} B^{\pm\mu\nu}/w^{\pm 3}$.
- (3) When ξ^a is hypersurface-orthogonal (which implies $\omega \equiv 0$, and $w^+ = w^- = w$) the objects $B^\pm_{\mu\nu}$ coincide and we have

$$B^\pm_{\mu\nu} = B_{\mu\nu} = -4\Theta^{-1} w^{-1} \left(1 \pm \Theta^{1/2} \right) \mathcal{R}^\pm_{\mu\nu} \quad (\text{C.23})$$

where $\mathcal{R}^\pm_{\mu\nu}$ are the Ricci tensors with respect to the metrics $\gamma^\pm_{\mu\nu} = \frac{1}{4}(\Theta^{-1/2} \pm 1)^2 \gamma_{\mu\nu}$.

- (4) Still for $\omega \equiv 0$, each of $C^\pm_{\mu\nu\sigma}$ reduces, via the field equations (C.10), (C.11) to the Bach–Cotton tensor whose vanishing characterizes conformal flatness. The corresponding characterizations of the Lorentzian Schwarzschild metric and the restriction of (C.21) for certain values of α were employed

in uniqueness proofs [33, 44]. For example, for $\alpha = 0$ (C.21) reduces to a linear combination of [44, Eq. (2.12)-(2.13)]

- (5) In the general case $B^\pm_{\mu\nu}$, k^\pm and $C^\pm_{\mu\nu\sigma}$ have complex Lorentzian counterparts $B_{\mu\nu}$, k and $C_{\mu\nu\sigma}$ which have analogous properties. They have been employed in local characterizations of the Kerr metric among the AF ones [53] and of larger classes of metrics if the asymptotic assumption is dropped, see [45] and references therein.

C.3. The divergence identities on \mathcal{M} . We can now give an alternative proof of the divergence identity (5.22).

Proof. In suitable coordinates, both sides of (5.22) are analytic on U^\pm_ϵ and agree with the respective sides of (C.21) on the subset $\mathcal{M}_0 \cap U^\pm_\epsilon$ which is dense, open and connected by Theorem C.1. Hence the validity of (5.22) extends to U^\pm_ϵ as required. \square

Remark C.7. The family (C.21) suggests an attempt for direct integration on \mathcal{N}_0 , without going the detour via \mathcal{M} . In fact, this would directly yield relations (7.2) which are just what is used in the uniqueness arguments. There are problems with this strategy, however, which can be exemplified via the definition of the nut charge: Recall that on \mathcal{M} , N is defined as the flux of (4.15a) through a hypersurface which may contain or intersect both maximal as well as exceptional orbits on \mathcal{M} ; accordingly N it is not related in an obvious way to an analogous quantity defined from (C.5) via a surface integral on \mathcal{N}_0 . This applies in particular to the key relation (4.25) for the surface gravities of the nut. To see this in concrete terms, relation (7.2) would arise via integration of (C.5) on \mathcal{N} with a “quotient nut charge” given by $N\kappa^1/2\pi = 1/4\kappa^2$, where $|\kappa^1| \geq |\kappa^2|$, that is via division of N by the length of the longer exceptional orbit. In contrast, the “quotient nut charge” [25, Eq. (5.9)] is equal to N divided by the length of the maximal orbit and thus reads $N\kappa^1/2\pi w^1 = 1/(4\kappa^2 w^1) = 1/(4\kappa^1 w^2)$. In short, the integration procedure on \mathcal{N} is bound to fail without proper understanding of the integration process on the orbit space. We will not expose this in this work.

REFERENCES

- [1] S. Aksteiner and L. Andersson. Gravitational instantons and special geometry. *J. Differential Geom.*, 128(3):928–958, 2024.
- [2] S. Alexakis, A. D. Ionescu, and S. Klainerman. Uniqueness of smooth stationary black holes in vacuum: small perturbations of the Kerr spaces. *Comm. Math. Phys.*, 299(1):89–127, 2010.
- [3] V. Apostolov, D. M. J. Calderbank, and P. Gauduchon. Ambitoric geometry I: Einstein metrics and extremal ambikähler structures. *J. Reine Angew. Math.*, 721:109–147, 2016.
- [4] M. F. Atiyah and I. M. Singer. The index of elliptic operators. III. *Ann. of Math. (2)*, 87:546–604, 1968.
- [5] C. Bär. On nodal sets for Dirac and Laplace operators. *Comm. Math. Phys.*, 188(3):709–721, 1997.
- [6] R. Beig and P. T. Chruściel. Killing vectors in asymptotically flat space-times. I. Asymptotically translational Killing vectors and the rigid positive energy theorem. *J. Math. Phys.*, 37(4):1939–1961, 1996.
- [7] A. L. Besse. *Einstein manifolds*. Classics in Mathematics. Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition.
- [8] O. Biquard and P. Gauduchon. On toric hermitian ALF gravitational instantons. *Communications in Mathematical Physics*, 399(1):389–422, 2023.
- [9] O. Biquard and V. Minerbe. A Kummer construction for gravitational instantons. *Comm. Math. Phys.*, 308(3):773–794, 2011.
- [10] R. Bott and L. W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.
- [11] G. Chen and X. Chen. Gravitational instantons with faster than quadratic curvature decay (ii). *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2019(756):259–284, 2019.

- [12] G. Chen and X. Chen. Gravitational instantons with faster than quadratic curvature decay. I. *Acta Math.*, 227(2):263–307, 2021.
- [13] G. Chen and X. Chen. Gravitational instantons with faster than quadratic curvature decay (iii). *Mathematische Annalen*, 380(1):687–717, 2021.
- [14] Y. Chen and E. Teo. Rod-structure classification of gravitational instantons with $U(1) \times U(1)$ isometry. *Nuclear Physics B*, 838(1):207–237, Oct 2010.
- [15] Y. Chen and E. Teo. A new AF gravitational instanton. *Physics Letters B*, 703(3):359–362, Sept. 2011.
- [16] S. A. Cherkis and N. J. Hitchin. Gravitational instantons of type D_k . *Comm. Math. Phys.*, 260(2):299–317, 2005.
- [17] P. T. Chruściel. On the invariant mass conjecture in general relativity. *Comm. Math. Phys.*, 120(2):233–248, 1988.
- [18] G. Clément and D. Gal'tsov. On the smarr formulas for electrovac spacetimes with line singularities. *Physics Letters B*, 802:135270, 2020.
- [19] A. Derdziński. Self-dual Kähler manifolds and Einstein manifolds of dimension four. *Compositio Math.*, 49(3):405–433, 1983.
- [20] E. Dobrowolski and T. Januszkiewicz. Arithmetic of an elliptic curve and circle actions on four-manifolds. *Colloq. Math.*, 64(1):13–18, 1993.
- [21] H. Geiges and C. Lange. Seifert fibrations of lens spaces. *Abh. Math. Semin. Univ. Hambg.*, 88(1):1–22, 2018.
- [22] R. Geroch. A Method for Generating Solutions of Einstein's Equations. *Journal of Mathematical Physics*, 12(6):918–924, June 1971.
- [23] G. W. Gibbons. Gravitational instantons: a survey. In *Mathematical problems in theoretical physics (Proc. Internat. Conf. Math. Phys., Lausanne, 1979)*, volume 116 of *Lecture Notes in Phys.*, pages 282–287. Springer, Berlin, 1980.
- [24] G. W. Gibbons and S. W. Hawking. Action integrals and partition functions in quantum gravity. *Phys. Rev. D*, 15(10):2752–2756, May 1977.
- [25] G. W. Gibbons and S. W. Hawking. Classification of gravitational instanton symmetries. *Comm. Math. Phys.*, 66(3):291–310, 1979.
- [26] G. W. Gibbons and M. J. Perry. Black Holes and Thermal Green Functions. *Proceedings of the Royal Society of London Series A*, 358(1695):467–494, Jan. 1978.
- [27] G. W. Gibbons and M. J. Perry. New gravitational instantons and their interactions. *Phys. Rev. D*, 22:313–321, July 1980.
- [28] G. W. Gibbons, C. N. Pope, and H. Römer. Index theorem boundary terms for gravitational instantons. *Nuclear Physics B*, 157(3):377–386, Oct. 1979.
- [29] S. W. Hawking. Gravitational instantons. *Phys. Lett. A*, 60(2):81–83, 1977.
- [30] F. Hirzebruch, T. Berger, and R. Jung. *Manifolds and modular forms*. Aspects of Mathematics, E20. Friedr. Vieweg & Sohn, Braunschweig, 1992. With appendices by Nils-Peter Skoruppa and by Paul Baum.
- [31] C. J. Hunter. Action of instantons with a nut charge. *Phys. Rev. D*, 59(2):024009, Dec. 1998.
- [32] A. D. Ionescu and S. Klainerman. On the uniqueness of smooth, stationary black holes in vacuum. *Invent. Math.*, 175(1):35–102, 2009.
- [33] W. Israel. Event Horizons in Static Vacuum Space-Times. *Physical Review*, 164(5):1776–1779, Dec. 1967.
- [34] D. Jang. Circle actions on oriented manifolds with discrete fixed point sets and classification in dimension 4. *Journal of Geometry and Physics*, 133:181–194, Nov. 2018.
- [35] A. Karlhede. Classification of Euclidean metrics. *Classical Quantum Gravity*, 3(1):L1–L4, 1986.
- [36] H. K. Kunduri and J. Lucietti. Existence and uniqueness of asymptotically flat toric gravitational instantons. *Letters in Mathematical Physics*, 111(5):133, Oct. 2021.
- [37] C. LeBrun. Complete Ricci-flat Kähler metrics on \mathbf{C}^n need not be flat. In *Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989)*, volume 52 of *Proc. Sympos. Pure Math.*, pages 297–304. Amer. Math. Soc., Providence, RI, 1991.
- [38] C. LeBrun. On Einstein, Hermitian 4-manifolds. *J. Differential Geom.*, 90(2):277–302, 2012.
- [39] P. Li. Circle action with prescribed number of fixed points. *Acta Math. Sin. (Engl. Ser.)*, 31(6):1035–1042, 2015.
- [40] P. Li and K. Liu. Circle action and some vanishing results on manifolds. *Internat. J. Math.*, 22(11):1603–1610, 2011.
- [41] M. Mars. A spacetime characterization of the Kerr metric. *Classical Quantum Gravity*, 16(7):2507–2523, 1999.
- [42] M. Mars. Uniqueness properties of the Kerr metric. *Classical Quantum Gravity*, 17(16):3353–3373, 2000.

- [43] M. Mars and W. Simon. A proof of uniqueness of the Taub-bolt instanton. *Journal of Geometry and Physics*, 32:211–226, 1999.
- [44] H. Müller zum Hagen, D. C. Robinson, and H. J. Seifert. Black holes in static vacuum space-times. *Gen. Relativity Gravitation*, 4:53–78, 1973.
- [45] M. Nozawa. An alternative to the simon tensor. *Classical and Quantum Gravity*, 38(15):155001, 2021.
- [46] P. Orlik. *Seifert manifolds*. Lecture Notes in Mathematics, Vol. 291. Springer-Verlag, Berlin-New York, 1972.
- [47] D. N. Page. Taub-NUT instanton with an horizon. *Physics Letters B*, 78(2-3):249–251, Sept. 1978.
- [48] R. Penrose and W. Rindler. *Spinors and space-time. Vol. 1*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1987. Two-spinor calculus and relativistic fields.
- [49] R. Penrose and W. Rindler. *Spinors and space-time. Vol. 2*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, second edition, 1988. Spinor and twistor methods in space-time geometry.
- [50] P. Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, Cham, third edition, 2016.
- [51] D. C. Robinson. A simple proof of the generalization of Israel’s theorem. *General Relativity and Gravitation*, 8(8):695–698, Aug. 1977.
- [52] P. Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15(5):401–487, 1983.
- [53] W. Simon. Characterizations of the Kerr metric. *Gen. Relativity Gravitation*, 16(5):465–476, 1984.
- [54] W. Simon. LETTER TO THE EDITOR: Nuts have no hair. *Classical and Quantum Gravity*, 12:L125–L130, Dec. 1995.
- [55] S. Sun and R. Zhang. Collapsing geometry of hyperkähler 4-manifolds and applications. *Acta Math.*, 232(2):325–424, 2024.
- [56] T. tom Dieck. *Transformation groups*, volume 8 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1987.
- [57] B. Weber. Energy and asymptotics of Ricci-flat 4-manifolds with a Killing field. *Proc. Amer. Math. Soc.*, 147(7):3117–3130, 2019.
- [58] S. T. Yau. Remarks on the group of isometries of a Riemannian manifold. *Topology*, 16(3):239–247, 1977.

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