

ON SOLUTIONS OF CERTAIN NON-LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

GARIMA PANT AND SANJAY KUMAR PANT

ABSTRACT. In this paper, we study about solutions of certain kind of non-linear differential difference equations

$$f^n(z) + wf^{n-1}(z)f'(z) + f^{(k)}(z+c) = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}$$

and

$$f^n(z) + wf^{n-1}(z)f'(z) + q(z)e^{Q(z)}f(z+c) = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z},$$

where $n \geq 2$, $k \geq 0$ are integers, w, p_1, p_2, α_1 & α_2 are non-zero constants satisfying $\alpha_1 \neq \alpha_2$, $0 \not\equiv q$ is a polynomial and Q is a non-constant polynomial.

1. Introduction

It is assumed that readers are familiar with the standard notations of Nevanlinna theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$, and $n(r, f)$ are called characteristic function of f , proximity function of f , counting function of f and un-integrated counting function of f respectively, here f denotes a meromorphic function. Recall that for a meromorphic function f , Nevanlinna's first main theorem expressed as

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1),$$

for all $a \in \mathbb{C}$. Here $O(1)$ denotes bounded error term depends on a , [5]. Also the quantities which are of growth $o(T(r, f))$ as $r \rightarrow \infty$, outside a set of finite linear measure, are denoted by $S(r, f)$. It means we say that a meromorphic function $h(z)$ is said to be a small function of $f(z)$ if $T(r, h) = S(r, f)$ and converse is also true. Note that the finite sum of $S(r, f)$ quantities is again forms $S(r, f)$.

Next the terms order of growth $\rho(f)$, hyper-order of growth $\rho_2(f)$ and exponent of convergence $\lambda(f)$ of a meromorphic function f are defined by

$$\begin{aligned} \rho(f) &= \limsup_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r}, \\ \rho_2(f) &= \limsup_{r \rightarrow \infty} \frac{\log^+ \log^+ T(r, f)}{\log r} \end{aligned}$$

and

$$\lambda(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ n(r, 1/f)}{\log r},$$

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respectively. In this paper, we use the above notations, definitions and facts in a frequent manner. For the standard results of Nevanlinna theory we refer [5, 12, 14]. A differential-difference polynomial $Q(z, f)$ in f is defined as a finite sum of products of f , derivatives of f and their shifts, with all the coefficients must be a small function of f , here f is a meromorphic function.

Many researchers have been studied about the solvability and existence of solutions of certain kind of non-linear differential-difference equations, one can see [1, 2, 6–10]. In this sequence, Li and Huang studied a certain differential-difference equation

$$f^n(z) + wf^{n-1}(z)f'(z) + f^{(k)}(z+c) = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}, \quad (1)$$

where n is a natural number, $k \geq 0$ is an integer, w, p_1, p_2, α_1 & α_2 are non-zero constant satisfying $\alpha_1 \neq \alpha_2$, and they provided the following result:

Theorem 1. [4] Suppose that $n \geq 5$, $\alpha_1/\alpha_2 \neq n$ and $\alpha_2/\alpha_1 \neq n$, then equation (1) has no transcendental entire solutions.

Motivated by the above result, we prove the following result:

Theorem 2. Suppose that $n \geq 5$ and f is a transcendental entire solution of finite order of the differential-difference equation (1), then it must be of $f(z) = Ce^{az}$ form, where a and C are non-zero constants satisfying $a = \alpha_i$, $na = \alpha_j$; $i \neq j$ and $C = (1+aw)^{-1/n}p_i^{1/n}$ for some $i = 1, 2$.

In the next result, we add a condition $N(r, 1/f) = S(r, f)$ in the hypothesis of the above theorem and we prove the same conclusion of the above theorem when $n = 4$ or 3.

Theorem 3. Let $n = 4$ or 3 and f be a finite order transcendental entire solution of the differential-difference equation (1) with $N(r, 1/f) = S(r, f)$. Then f must be of $f(z) = Ce^{az}$ form, where a and C are non-zero constants satisfying $a = \alpha_i$, $na = \alpha_j$; $i \neq j$ and $C = (1+aw)^{-1/n}p_i^{1/n}$ for some $i = 1, 2$.

In the same paper [4], Li and Huang studied one more type of certain differential-difference equation

$$f^n(z) + wf^{n-1}(z)f'(z) + q(z)e^{Q(z)}f(z+c) = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}, \quad (2)$$

where n is a natural number, $k \geq 0$ is an integer, $w, c, p_1, p_2, \alpha_1, \alpha_2$ are non-zero constants satisfying $\alpha_1 \neq \alpha_2$, $q \not\equiv 0$ is a polynomial and Q is a non-constant polynomial. They provided the following result:

Theorem 4. [4] Suppose that $n \geq 4$ and f is a transcendental entire solution with finite order of equation (2) with $\lambda(f) < \rho(f)$. Then the following conclusions hold:

- (i) Each solution f satisfies $\rho(f) = \deg Q = 1$.
- (ii) If $n \geq 1$ and f is a solution which belongs to $\Gamma_0 = \{e^{\alpha(z)} : \alpha(z) \text{ is a non-constant polynomial}\}$, then

$$f(z) = e^{(\alpha_2 z/n)+\beta}, \quad Q(z) = (\alpha_1 - \frac{\alpha_2}{n})z + b$$

or

$$f(z) = e^{(\alpha_1 z/n)+\beta}, \quad Q(z) = (\alpha_2 - \frac{\alpha_1}{n})z + b,$$

where β and b are constants.

In the next result, we prove that the same conclusions hold when $n = 3$ or 2 under the same hypothesis as given in the above theorem.

Theorem 5. Suppose $n = 3$ or 2 and f is a finite order transcendental entire solution of equation (2) with $\lambda(f) < \rho(f)$. Then the following conclusions hold:

- (i) Each solution f satisfies $\rho(f) = \deg Q = 1$.
- (ii) If $n \geq 1$ and f is a solution which belongs to $\Gamma_0 = \{e^{\alpha(z)} : \alpha(z) \text{ is a non-constant polynomial}\}$, then

$$f(z) = e^{(\alpha_2 z/n) + \beta}, \quad Q(z) = (\alpha_1 - \frac{\alpha_2}{n})z + b$$

or

$$f(z) = e^{(\alpha_1 z/n) + \beta}, \quad Q(z) = (\alpha_2 - \frac{\alpha_1}{n})z + b,$$

where β and b are constants.

Prior to [4], Chen et.al [1] studied the equation (2) when $\alpha_2 = -\alpha_1$ and they didn't take $\lambda(f) < \rho(f)$ condition in their hypothesis and proved the same conclusions.

2. Preliminary Results

The following lemma gives proximity function of logarithmic derivative of a meromorphic function f .

Lemma 1. [5] Suppose f is a transcendental meromorphic function and k is a natural number. Then

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f).$$

If f is of finite order growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

Next lemma estimates the characteristic function of a shift of a meromorphic function f .

Lemma 2. [3] Suppose f is a meromorphic function of finite order ρ and c is a non-zero complex number. Then for every $\epsilon > 0$,

$$T(r, f(z+c)) = T(r, f) + O(r^{\rho-1+\epsilon}) + O(\log r).$$

The following lemma gives the difference analogue of the lemma on the logarithmic derivative of a meromorphic function f having finite order.

Lemma 3. [3] Suppose f is a meromorphic function with $\rho(f) < \infty$ and $c_1, c_2 \in \mathbb{C}$ such that $c_1 \neq c_2$, then for each $\epsilon > 0$, we obtain

$$m\left(r, \frac{f(z+c_1)}{f(z+c_2)}\right) = O(r^{\rho-1+\epsilon}).$$

Next lemma plays an important part in the study of complex differential-difference equations and it can be seen in [13].

Lemma 4. [13] Suppose that g is a transcendental entire solution of finite order of a differential-difference equation of the form

$$f^n P(z, g) = Q(z, g),$$

where $P(z, g)$ and $Q(z, g)$ are polynomials in $g(z)$, its derivatives and its shifts with small meromorphic coefficients. If the total degree of $Q(z, g)$ is less than or equal to n , then

$$m(r, P(z, g)) = S(r, g),$$

for all r outside of a possible exceptional set of finite logarithmic measure.

The following lemma plays a vital role in the study of uniqueness of meromorphic functions.

Lemma 5. [14] Let $f_1, f_2, \dots, f_n (n \geq 2)$ be meromorphic functions and h_1, h_2, \dots, h_n be entire functions satisfying

- (1) $\sum_{i=1}^n f_i e^{h_i} \equiv 0$.
- (2) For $1 \leq j < k \leq n$, $h_j - h_k$ are not constants .
- (3) For $1 \leq i \leq n, 1 \leq m < k \leq n$,
 $T(r, f_i) = o(T(r, e^{(h_m - h_k)}))$ as $r \rightarrow \infty$, outside a set of finite logarithmic measure.

Then $f_i \equiv 0 (i = 1, 2, \dots, n)$.

Next lemma estimates the characteristic function of an exponential polynomial f . This lemma can be seen in [11].

Lemma 6. Suppose f is an entire function given by

$$f(z) = A_0(z) + A_1(z)e^{w_1 z^s} + A_2(z)e^{w_2 z^s} + \dots + A_m(z)e^{w_m z^s},$$

where $A_i(z); 0 \leq i \leq m$ denote either exponential polynomial of degree $< s$ or polynomial in z , $w_i; 1 \leq i \leq m$ denote the constants and s denotes a natural number. Then

$$T(r, f) = C(Co(W_0)) \frac{r^s}{2\pi} + o(r^s),$$

Here $C(Co(W_0))$ is the perimeter of the convex hull of the set $W_0 = \{0, \bar{w}_1, \bar{w}_2, \dots, \bar{w}_m\}$.

Lemma 7. If f is a transcendental meromorphic function of finite order $\rho(f)$ and satisfying $\lambda(f) < \rho(f)$, then $N(r, 1/f) = S(r, f)$.

Proof. Given that f is a transcendental meromorphic function of finite order $\rho(f)$ and satisfying $\lambda(f) < \rho(f)$. We prove this lemma by contradiction.

Suppose f is a finite order transcendental meromorphic function such that $N(r, 1/f) \neq S(r, f)$. This means $N(r, 1/f) \neq o(T(r, f))$, outside a set of finite measure, and this gives

$$\lim_{r \rightarrow \infty} \frac{N(r, 1/f)}{T(r, f)} \not\rightarrow 0.$$

Let

$$\lim_{r \rightarrow \infty} \frac{N(r, 1/f)}{T(r, f)} \rightarrow a, \quad (3)$$

where $a > 0$. First we use the definition of \limsup in equation (3), for every $\epsilon > 0$, there exist r_0 such that

$$\frac{N(r, 1/f)}{T(r, f)} \leq a + \epsilon,$$

for $r \geq r_0$. This gives

$$\lambda(f) \leq \rho(f). \quad (4)$$

Similarly, we use the definition of \liminf in equation(3) and we get

$$\lambda(f) \geq \rho(f). \quad (5)$$

From equation (4) and (5), we get a contradiction to the fact $\lambda(f) < \rho(f)$. \square

3. Proof of Theorems

Proof of Theorem 2. Let f be a finite order transcendental entire solution of equation (1).

Set $M = f^n(z) + wf^{n-1}(z)f'(z)$ and $N = f^{(k)}(z + c)$, then equation (1) can be rewritten as

$$M + N = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}. \quad (6)$$

Differentiating equation (6), we get

$$M' + N' = \alpha_1 p_1 e^{\alpha_1 z} + \alpha_2 p_2 e^{\alpha_2 z}. \quad (7)$$

From equation (6) and (7), we eliminate $e^{\alpha_2 z}$ and we get

$$\alpha_2 M + \alpha_2 N - M' - N' = (\alpha_2 - \alpha_1) p_1 e^{\alpha_1 z}. \quad (8)$$

After differentiating above equation, we get

$$\alpha_2 M' + \alpha_2 N' - M'' - N'' = (\alpha_2 - \alpha_1) \alpha_1 p_1 e^{\alpha_1 z}. \quad (9)$$

From equation (8) and (9), we eliminate $e^{\alpha_2 z}$ and we get

$$\alpha_1 \alpha_2 M - (\alpha_1 + \alpha_2) M' + M'' = -(\alpha_1 \alpha_2 N - (\alpha_1 + \alpha_2) N' + N''), \quad (10)$$

where

$$\begin{aligned} M &= f^n(z) + wf^{n-1}(z)f'(z) \\ M' &= nf^{n-1}f' + w[(n-1)f^{n-2}(f')^2 + f^{n-1}f''] \\ M'' &= n(n-1)f^{n-2}(f')^2 + nf^{n-1}f'' + (n-1)(n-2)wf^{n-3}(f')^3 + \\ &\quad [2(n-1) + w(n-1)]f^{n-2}f'f'' + wf^{n-1}f'''. \end{aligned}$$

After substituting the values of M , M' and M'' in the equation (10), we get

$$f^{n-3}\phi = -(\alpha_1 \alpha_2 N - (\alpha_1 + \alpha_2) N' + N''), \quad (11)$$

where

$$\begin{aligned} \phi &= \alpha_1 \alpha_2 f^3 + (w\alpha_1 \alpha_2 - n(\alpha_1 + \alpha_2))f^2 f' + (n - w(\alpha_1 + \alpha_2))f^2 f'' + wf^2 f''' + \\ &\quad (n-1)(n-w(\alpha_1 + \alpha_2))f(f')^2 + w(n-1)(n-2)(f')^3 + 3w(n-1)ff'f''. \end{aligned} \quad (12)$$

Given that $n - 3 \geq 2$ and as we set $N = f^{(k)}(z + c)$, then applying Lemma 4 to equation (11), we get

$$m(r, \phi) = S(r, f) \quad \text{and} \quad m(r, f\phi) = S(r, f).$$

Now there are two cases:

1: If $\phi \not\equiv 0$, then

$$\begin{aligned} T(r, f) &= m(r, f) = m\left(r, \frac{f\phi}{\phi}\right) \\ &\leq m(r, f\phi) + m\left(r, \frac{1}{\phi}\right) \\ &\leq S(r, f). \end{aligned}$$

This is not possible.

2: If $\phi \equiv 0$, then from equation (12), we have

$$\begin{aligned} \alpha_1\alpha_2f^3 &\equiv -[(w\alpha_1\alpha_2 - n(\alpha_1 + \alpha_2))f^2f' + (n - w(\alpha_1 + \alpha_2))f^2f'' + wf^2f''' + \\ &\quad (n-1)(n-w(\alpha_1 + \alpha_2))f(f')^2 + w(n-1)(n-2)(f')^3 + 3w(n-1)ff'f'']. \end{aligned} \quad (13)$$

Suppose f has infinitely many zeros, then from the above equation, it is very clear that zeros of f have multiplicity greater or equal to 2. Let z_0 be a zero of f with multiplicity $m \geq 2$, then left side of (13) has zero at z_0 with multiplicity $3m$, while right side of the same has zero at z_0 with multiplicity at most $3m - 3$, which is not possible. Hence f has finitely many zeros, now applying Hadamard factorisation theorem, f must be of the form

$$f(z) = \beta(z)e^{P(z)}, \quad (14)$$

where $P(z)$ is a non-constant polynomial and $\beta(z)$ is an entire function satisfying $\rho(\beta) < \deg(P)$.

Now substituting the value of f into equation (6), we have

$$[\beta^n(z) + w\beta^{n-1}(z)(\beta'(z) + \beta(z)P'(z))]e^{nP(z)} + [\gamma'(z) + P'(z+c)\gamma(z)]e^{P(z+c)} = p_1e^{\alpha_1z} + p_2e^{\alpha_2z}, \quad (15)$$

where $\gamma(z)$ is the coefficient of $f^{(k-1)}(z+c)$ which is in the terms of $\beta(z+c)$, $P(z+c)$ and their derivatives, and $\gamma'(z)$ is the derivative of $\gamma(z)$.

If $\deg(P) = l \geq 2$, then applying Lemma 6, the order of growth of the left side of the above equation would be l while right side of the same has 1 order of growth, which is not possible. Hence $\deg P = 1$, let $P(z) = az + b$, where a and b are constant with $a \neq 0$. After substitution the $P(z)$, equation (15) becomes

$$[\beta^n(z) + w\beta^{n-1}(z)(\beta'(z) + a\beta(z))]e^{n(az+b)} + [\gamma'(z) + a\gamma(z)]e^{a(z+c)+b} - p_1e^{\alpha_1z} - p_2e^{\alpha_2z} = 0, \quad (16)$$

Next we study the following cases:

- (1) If $na \neq \alpha_j$ and $a \neq \alpha_i; i \neq j$, then applying lemma 5, we get $p_1 \equiv 0$ and $p_2 \equiv 0$, which is a contradiction.
- (2) If $na = \alpha_j$ and $a \neq \alpha_i; i \neq j$, say $na = \alpha_1$ and $a \neq \alpha_2$, then again applying lemma 5, we get $p_2 \equiv 0$, which is a contradiction.

(3) If $na = \alpha_j$ and $a = \alpha_i; i \neq j$, say $na = \alpha_1$ and $a = \alpha_2$, then again applying lemma 5, we get

$$[\beta^n(z) + w\beta^{n-1}(z)(\beta'(z) + a\beta(z))]e^{nb} - p_1 \equiv 0, \quad (17)$$

$\implies \beta^{n-1}(z)[(1+wa)\beta(z) + w\beta'(z)]e^{nb} \equiv p_1$. This gives $\beta(z)$ must be a constant, say $\beta(z) = \beta_0$ and $a \neq -1/w$. Substituting this into equation (17), we get $\beta_0 = (1+aw)^{-1/n}e^{-b}p_1^{1/n}$. Thus from equation (14), $f(z) = (1+aw)^{-1/n}p_1^{1/n}e^{az} = Ce^{az}$ is the solution of equation (1). \square

Proof of Theorem 3. Let f be a finite order transcendental entire solution of equation (1) with $N(r, 1/f) = S(r, f)$. Set $M = f^n(z) + wf^{n-1}(z)f'(z)$ and $N = f^{(k)}(z+c)$, then equation (1) can be rewritten as

$$M + N = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}. \quad (18)$$

Proceeding to the similar lines as we done in the proof of Theorem 2, we get

$$f^{n-3}\phi = -(\alpha_1\alpha_2N - (\alpha_1 + \alpha_2)N' + N''), \quad (19)$$

in place of equation (11), where

$$\begin{aligned} \phi &= \alpha_1\alpha_2f^3 + (w\alpha_1\alpha_2 - n(\alpha_1 + \alpha_2))f^2f' + (n - w(\alpha_1 + \alpha_2))f^2f'' + wf^2f''' + \\ &\quad (n-1)(n - w(\alpha_1 + \alpha_2))f(f')^2 + w(n-1)(n-2)(f')^3 + 3w(n-1)ff'f''. \end{aligned} \quad (20)$$

(i) Let $n = 4$, we study the following two cases:

1: If $\phi \not\equiv 0$, then applying Lemma 4 to equation (19), we get

$$m(r, \phi) = S(r, f). \quad (21)$$

Applying Lemma 1 to equation (20), we get

$$m\left(r, \frac{\phi}{f^3}\right) = S(r, f). \quad (22)$$

Given that $N(r, 1/f) = S(r, f)$, so

$$N\left(r, \frac{\phi}{f^3}\right) = N\left(r, \frac{1}{f^3}\right) = 3N\left(r, \frac{1}{f}\right) = S(r, f). \quad (23)$$

Thus using first fundamental theorem and the above three equations, we get

$$\begin{aligned} T(r, f) &\leq 3T(r, f) = T(r, f^3) \\ &\leq T\left(r, \frac{f^3}{\phi}\right) + T(r, \phi) + O(1) \\ &= T\left(r, \frac{\phi}{f^3}\right) + m(r, \phi) + O(1) \\ &= S(r, f) \end{aligned}$$

This is not possible.

2: If $\phi \equiv 0$, then proceeding to the similar lines as we have done in the proof of Theorem 2, we get the required conclusion.

(ii) Let $n = 3$, again we study the following two cases:

1: If $\phi \not\equiv 0$, then applying Lemma 1 and 4 to equation (19), we get

$$m\left(r, \frac{\phi}{f}\right) = S(r, f).$$

Applying Lemma 1 to equation (20), we get

$$m\left(r, \frac{\phi}{f^3}\right) = S(r, f).$$

Given that $N(r, 1/f) = S(r, f)$, so

$$N\left(r, \frac{1}{f^3}\right) = 3N\left(r, \frac{1}{f}\right) = S(r, f).$$

Thus using first fundamental theorem and the above three equations, we get

$$\begin{aligned} 3T(r, f) &= T(r, f^3) \\ &= m\left(r, \frac{1}{f^3}\right) + N\left(r, \frac{1}{f^3}\right) + O(1) \\ &\leq m\left(r, \frac{\phi}{f^3}\right) + m\left(r, \frac{1}{\phi}\right) + S(r, f) \\ &\leq T(r, \phi) + S(r, f) = m(r, \phi) + S(r, f) \\ &\leq m\left(r, \frac{\phi}{f}\right) + m(r, f) + S(r, f) \\ &= T(r, f) + S(r, f) \end{aligned}$$

This gives $2T(r, f) = S(r, f)$, which is not possible.

2: If $\phi \equiv 0$, then proceeding to the similar lines as we have done in the proof of Theorem 2, we get the required conclusion. \square

Proof of Theorem 5. Let f be a finite order transcendental entire solution of equation (2) satisfying $\lambda(f) < \rho(f)$.

Suppose $\rho(f) < 1$, then applying Lemma 1, 2, 3 and the first fundamental theorem of Nevanlinna to equation (2), we get

$$\begin{aligned} T(r, e^{Q(z)}) &= T\left(r, \frac{p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} - f^n - w f^{n-1} f'}{q f^{(k)}(z+c)}\right) \\ &\leq T\left(r, \frac{1}{q f^{(k)}(z+c)}\right) + T(r, p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}) + T\left(r, f^n \left(1 + \frac{f'}{f}\right)\right) + O(1) \\ &\leq T(r, q f^{(k)}(z+c)) + T(r, p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}) + n T(r, f) + O(\log r) \\ &\leq T(r, p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}) + S(r, p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}). \end{aligned}$$

This gives $\deg Q(z) \leq 1$, and we know that $\deg Q(z) \geq 1$, hence $\deg Q(z) = 1$ and say $Q(z) = az + b$; $a \neq 0$. Now equation (2) becomes

$$f^n(z) + w f^{n-1}(z) f'(z) + q(z) e^{az+b} f(z+c) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}.$$

On differentiating above equation, we get

$$nf^{n-1}f' + w[(n-1)f^{n-2}f'^2 + f^{n-1}f''] + \gamma(z)e^{az+b} = p_1\alpha_1e^{\alpha_1z} + p_2\alpha_2e^{\alpha_2z},$$

where $\gamma(z) = q'(z)f^{(k)}(z+c) + q(z)f^{(k+1)}(z+c) + aq(z)f^{(k)}(z+c)$.

Eliminating e^{α_1z} from above two equations, we obtain

$$\begin{aligned} \alpha_1f^n + (\alpha_1w - n)f^{n-1}f' - (n-1)wf^{n-2}f'^2 - wf^{n-1}f'' + (\alpha_1qf^{(k)}(z+c) - \gamma(z))e^{az+b} \\ = p_2(\alpha_2 - \alpha_1)e^{\alpha_2z}. \end{aligned} \quad (24)$$

Case 1: If $a \neq \alpha_2$, then applying Lemma 5, we get $p_2(\alpha_1 - \alpha_2) \equiv 0$, which is a contradiction.

Case 2: If $a = \alpha_2$, then equation (24) becomes

$$\begin{aligned} \alpha_1f^n + (\alpha_1w - n)f^{n-1}f' - (n-1)wf^{n-2}f'^2 - wf^{n-1}f'' + [\{\alpha_1qf^{(k)}(z+c) - \gamma(z)\}e^b - \\ p_2(\alpha_1 - \alpha_2)]e^{\alpha_2z} = 0. \end{aligned} \quad (25)$$

Now applying Lemma 5 to the above equation, we get

$$\alpha_1f^n + (\alpha_1w - n)f^{n-1}f' - (n-1)wf^{n-2}f'^2 - wf^{n-1}f'' \equiv 0$$

or

$$\alpha_1 + (\alpha_1w - n)\frac{f'}{f} - (n-1)w\left(\frac{f'}{f}\right)^2 - w\frac{f''}{f} \equiv 0.$$

Set $\frac{f'}{f} = s$, we get a Riccati differential equation

$$(\alpha_1w - n)s - ws' - nws^2 + \alpha_1 \equiv 0, \quad (26)$$

since $(\frac{f'}{f})' = \frac{f''}{f} - (\frac{f'}{f})^2$.

By routine computation, we get $s_1 = \frac{-1}{w}$ and $s_2 = \frac{\alpha_1}{n}$ are the solutions of equation (26). Let $s \neq s_1$ and $s \neq s_2$, then

$$\frac{1}{(\frac{\alpha_1}{n} + \frac{1}{w})} \left(\frac{s'}{s + \frac{1}{w}} - \frac{s'}{s - \frac{\alpha_1}{n}} \right) = n.$$

On integrating the above equation, we get

$$\log \left(\frac{s + \frac{1}{w}}{s - \frac{\alpha_1}{n}} \right) = n \left(\frac{\alpha_1}{n} + \frac{1}{w} \right) + c_1,$$

where c_1 is an arbitrary constant.

This gives

$$s = \frac{\alpha_1}{n} + \frac{\alpha_1/n + 1/w}{e^{n(\alpha_1/n+1/w)z+c_1} - 1} = \frac{f'}{f}.$$

We observe that the zeros of $e^{n(\alpha_1/n+1/w)z+c_1} - 1$ are the zeros of f . Let z_0 be a zero of $e^{n(\alpha_1/n+1/w)z+c_1} - 1$ with multiplicity m , then

$$m = \text{Res} \left(\frac{f'}{f}, z_0 \right) = \text{Res} \left(\frac{\alpha_1}{n} + \frac{\alpha_1/n + 1/w}{e^{n(\alpha_1/n+1/w)z+c_1} - 1}, z_0 \right) = \frac{1}{n},$$

which is a contradiction.

If $s_1 = -1/w$, then $f'/f = -1/w$ and this gives $f = c_2 e^{-z/w}$, where c_2 is an arbitrary

non-zero constant. Thus $\rho(f) = 1$, which is a contradiction to $\rho(f) < 1$.

If $s_2 = \alpha_1/n$, then $f'/f = \alpha_1/n$ and this gives $f = c_3 e^{\alpha_1 z/n}$, where c_3 is an arbitrary non-zero constant. Thus $\rho(f) = 1$, which is a contradiction.

Hence $\rho(f) \geq 1$, so let $\rho(f) > 1$ and set $P(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}$ and $H(z) = q(z)f^{(k)}(z+c)$ for simplicity. Then equation (2) becomes

$$f^n + w f^{n-1} f' + H(z) e^{Q(z)} = P(z).$$

On differentiating above equation gives

$$n f^{n-1} f' + (n-1) w f^{n-2} f'^2 + w f^{n-1} f'' + (H'(z) + H(z) Q'(z)) e^{Q(z)} = P'(z).$$

Eliminating $e^{Q(z)}$ with the help of above two equation yields

$$f^{n-2} \phi = M(z) P(z) - P'(z) H(z), \quad (27)$$

where

$$M(z) = H'(z) + H(z) Q'(z)$$

and

$$\phi(z) = M(z) f^2 + w M(z) f f' - n H(z) f f' - w(n-1) H(z) f'^2 - w H(z) f f''. \quad (28)$$

(i) Let $n = 3$, then applying Lemma 2 to equation (27), we get

$$m(r, \phi) = S(r, f). \quad (29)$$

Also given that $\lambda(f) < \rho(f)$, applying Lemma 7 gives

$$N\left(r, \frac{1}{f}\right) = S(r, f),$$

hence

$$N\left(r, \frac{\phi}{f^3}\right) = N\left(r, \frac{1}{f^3}\right) = 3N\left(r, \frac{1}{f}\right) = S(r, f). \quad (30)$$

Applying Lemma 1 and 3 to equation (28), we get

$$m\left(r, \frac{\phi}{f^3}\right) = S(r, f). \quad (31)$$

If $\phi \not\equiv 0$, using equations (29), (30), (31) and first fundamental theorem of Nevanlinna, we get

$$\begin{aligned} 3T(r, f) &= T(r, f^3) = T\left(r, \frac{1}{f^3}\right) + O(1) \\ &\leq T\left(r, \frac{\phi}{f^3}\right) + T\left(r, \frac{1}{\phi}\right) + O(1) \\ &\leq T(r, \phi) + S(r, f) \\ &= S(r, f), \end{aligned}$$

which is a contradiction.

If $\phi \equiv 0$, from equation (27), we get $M(z) P(z) - P'(z) H(z) \equiv 0$. This gives

$$(H'(z) + H(z) Q'(z)) P(z) - P'(z) H(z) \equiv 0.$$

$$\begin{aligned} &\Rightarrow \frac{H'(z)}{H(z)} + Q'(z) - \frac{P'(z)}{P(z)} = 0. \\ &\Rightarrow \frac{q'(z)}{q(z)} + \frac{f^{(k+1)}(z+c)}{f^{(k)}(z+c)} + Q'(z) - \frac{P'(z)}{P(z)} = 0. \end{aligned}$$

On integrating above equation, we get

$$q(z)f^{(k)}(z+c)e^{Q(z)} = \frac{1}{c_4}P(z) = \frac{1}{c_4}(p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}). \quad (32)$$

Since f is a finite order transcendental entire solution satisfying $\lambda(f) < \rho(f)$, then using Hadamard factorisation theorem, f must be of

$$f(z) = g(z)e^{h(z)} \quad (33)$$

form, where $h(z)$ is a polynomial such that $\rho(f) = \deg(h) > 1$ and $g(z)$ is the canonical product of zeros of $f(z)$ with $\lambda(f) = \rho(g) < \rho(f)$.

Using equations (32) and (33) to the equation (2) gives

$$g^2(z)(g'(z) + g(z)h'(z) + g(z))e^{3h(z)} = (1 - 1/c_4)(p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}).$$

On applying Lemma 6 to the above equation, we get that the order of growth of the left side is greater than 1, while the order of growth of the right side is exactly 1. This is a contradiction, hence $\rho(f) = 1$.

(ii) Let $n = 2$, then applying Lemmas 1 and 3 to equations (27) and (28) give

$$m\left(r, \frac{\phi}{f}\right) = m\left(r, \frac{H'(z) + H(z)Q'(z)}{f}\right) = S(r, f) \quad (34)$$

and

$$m\left(r, \frac{\phi}{f^3}\right) = S(r, f). \quad (35)$$

Given that $\lambda(f) < \rho(f)$, applying Lemma 7 gives

$$N\left(r, \frac{1}{f}\right) = S(r, f). \quad (36)$$

If $\phi \not\equiv 0$, using the first fundamental theorem of Nevanlinna and equations (34), (35) & (36), we have

$$\begin{aligned} 3T(r, f) &= T(r, f^3) = m\left(r, \frac{1}{f^3}\right) + N\left(r, \frac{1}{f^3}\right) + O(1) \\ &\leq m\left(r, \frac{\phi}{f^3}\right) + m\left(r, \frac{1}{\phi}\right) + 3N\left(r, \frac{1}{f}\right) + O(1) \\ &\leq T(r, \phi) + S(r, f) \\ &= m(r, \phi) + S(r, f) \\ &\leq m\left(r, \frac{\phi}{f}\right) + m(r, f) + S(r, f) \\ &\leq T(r, f) + S(r, f). \end{aligned}$$

This gives $2T(r, f) = S(r, f)$, which is a contradiction.

If $\phi \equiv 0$, then proceeding similar manner as done in (i), we get the same contradiction. Thus $\rho(f) = 1$.

Next, to prove $\deg(Q) = 1$ and (ii) conclusion, we follow the same technique as done in [4, Proof of Theorem 7].

□

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GARIMA PANT; DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI, DELHI-110007, INDIA.
Email address: garimapant.m@gmail.com

SANJAY KUMAR PANT; DEPARTMENT OF MATHEMATICS, DEEN DAYAL UPADHYAYA COLLEGE, UNIVERSITY OF DELHI, NEW DELHI-110078, INDIA.
Email address: skpant@ddu.du.ac.in