1. Differentiation

Definition. Let $f : [a, b] \to \mathbb{R}$ be a function. For any $x \in [a, b]$, we define

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad a < t < b, t \neq x$$

Define the **derivative of** f **at** x to be

$$f'(x) = \lim_{t \to x} \phi(t) \in \mathbb{R}$$

provided that the limit exists.

Remark.
We think of f' as a function $x \mapsto f'(x)$ where $\operatorname{dom} f' \subset \operatorname{dom} f$. In general, f needs to be defined in a neighborhood of x in order for $f'(x)$ to be defined. △

Lemma.
Let $f : [a, b] \to \mathbb{R}$. If f is differentiable at $x \in (a, b)$, then it is continuous at x.

Proof.

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x}(t - x) = \phi(t)(t - x)$$

Now taking limits,

$$\lim_{t \to x}(\phi(t)(t - x)) = f'(x) \cdot 0 = 0$$

Thus $\lim_{t \to x} f(t) = f(x)$. □

Remark.
Continuity is a necessary, but not sufficient, condition for differentiability.
To see this, consider $f(x) = |x|$:

$$\begin{cases} \phi(t) \to 1 & \text{as } t \to 0^+ \\ \phi(t) \to -1 & \text{as } t \to 0^- \end{cases} \implies f'(0) \text{ DNE}$$

△

Proposition.
Let $f, g : [a, b] \to \mathbb{R}$ be differentiable at x. Then

(1) $f + g$, fg, f/g $(g \neq 0)$ are all differentiable at x.

(2) These derivatives can be written in terms of f and g:

$$(f + g)' = f' + g' \qquad (fg)' = f'g + fg' \qquad (f/g)' = (gf' - fg')/(g'^2)$$

[**N.B.** Respectively we call these formulas Linearity, Product Rule, and Quotient Rule]

Proof.
We compute the formulas directly. Note that Quotient Rule is just an application of Product Rule via $f/g = f(1/g)$, so it suffices to prove the first two formulas.

1. Let $h = f + g$. Then

$$\frac{h(t)-h(x)}{t-x} = \frac{f(t)+g(t)-f(x)-g(x)}{t-x} = \frac{f(t)-f(x)}{t-x} + \frac{g(t)-g(x)}{t-x} \overset{\lim t\to x}{\Longrightarrow} h'(x) = f'(x) + g'(x)$$

2. Let $h = fg$. Note that

$$\begin{aligned} h(t) - h(x) = f(t)g(t) - f(x)g(x) &= f(t)g(t) + [-f(t)g(x) + f(t)g(x)] - f(x)g(x) \\ &= f(t)[g(t) - g(x)] + [f(t) - f(x)]g(x) \end{aligned}$$

   Then

$$\frac{h(t)-h(x)}{t-x} = f(t)\left(\frac{g(t)-g(x)}{t-x}\right) + g(x)\left(\frac{f(t)-f(x)}{t-x}\right)$$

$$\overset{t\to x}{\Longrightarrow} h'(x) = \lim_{t\to x} f(t)g'(x) + \lim_{t\to x} g(t)f'(x) = f(x)g'(x) + f'(x)g(x)$$

   where the final equality holds by continuity of f and g by the previous Lemma.

$\square$

Example.

(1) $f(x) =$ constant:

   Then $\phi(t) = 0$ for all t, hence $f'(x) = 0$ for all x.

(2) $f(x) = x$:

   Then $\phi(t) = 1$ for all t, hence $f'(x) = 1$ for all x.

(3) $f(x) = x^n$:

   We claim $f'(x) = nx^{n-1}$. Use induction on n. Our base case is $n = 1$, and $f'(x) = 1 = 1x^0$. Now by product rule

$$\begin{aligned} f(x) = x^n = xx^{n-1} \implies f'(x) = \tfrac{d}{dx}(x)x^{n-1} + x\tfrac{d}{dx}(x^{n-1}) &\overset{IH}{=} 1x^{n-1} + x((n-1)x^{n-2}) \\ &= x^{n-1} + (n-1)x^{n-1} \\ &= nx^{n-1} \end{aligned}$$

(4) If $f(x) = cg(x)$ for some constant c, then by product rule

$$f'(x) = \tfrac{d}{dx}(c)g(x) + c\tfrac{d}{dx}(g(x)) = 0g(x) + cg'(x) = cg'(x)$$

(5) By linearity, and the above examples, any polynomial is differentiable.

   N.B. Moving forward, when showing different results we will not describe all details in our hypotheses, e.g. if we are mentioning f is differentiable, then you may assume that we actually have a function $f : [a, b] \to \mathbb{R}$ to begin with. The theorem statements in analysis are usually long as-is, so we want to emphasize the result much more than everything standard that we are assuming at the beginning.

Proposition. *(Chain Rule)*
Suppose f is continuous on $[a,b]$, f differentiable at x, $\operatorname{dom} g \supset \operatorname{im} f$, and g differentiable at $f(x)$. If $h = g \circ f : (a,b) \to \mathbb{R}$ is the composition, then h is differentiable at x and $h'(x) = g'(f(x)) \cdot f'(x)$.

Proof.
Note that for any t, there exists u such that

$$f'(x) + u = \frac{f(t) - f(x)}{t - x} \tag{$\star$}$$

(by just taking the difference of the known terms). We can think of this u as a function of t, and by definition of derivative we have $u(t) \to 0$ as $t \to x$. Now letting $y = f(x)$, we can similarly find

$$\frac{g(s) - g(y)}{s - y} = g'(y) + v(s) \quad \text{where } \lim_{s \to y} v(s) = 0 \tag{$\star\star$}$$

Now letting $s = f(t)$, note by continuity of f that $s \to y$ as $t \to x$. Then

$$\begin{aligned} h(t) - h(x) = g(f(t)) - g(f(x)) = g(s) - g(y) &\overset{(\star\star)}{=} (g'(y) + v(s))(s - y) \\ &= (g'(f(x)) + v(f(t)))(f(t) - f(x)) \\ &\overset{(\star)}{=} (g'(f(x)) + v(f(t)))(f'(x) + u(t))(t - x) \end{aligned}$$

$$\begin{aligned} &\implies \frac{h(t) - h(x)}{t - x} = (g'(f(x)) + v(f(t)))(f'(x) + u(t)) \xrightarrow{t \to x} (g'(f(x)) + \lim_{s \to y} v(s))(f'(x) + \lim_{t \to x} u(t)) \\ &\implies h'(x) = (g'(f(x)) + 0)(f'(x) + 0) = g'(f(x)) \cdot f'(x) \end{aligned}$$

$\square$

Example.

(1)

$$f(x) = \begin{cases} x \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

f is continuous away from zero as a composition of continuous functions, and

$$|f(x) - 0| = |f(x)| \leq |x| |\sin(\tfrac{1}{x})| \leq |x| 1 \xrightarrow{x \to 0} 0$$

Thus f is continuous everywhere.
Next, we determine where f is differentiable. Away from zero, we are good by the chain rule, but at zero we need to check by definition

$$\phi(t) = \frac{f(t) - f(0)}{t - 0} = \frac{t \sin(\frac{1}{t}) - 0}{t} = \sin(\tfrac{1}{t}) \overset{t \to 0}{\not\longrightarrow} 0$$

N.B. To more conclusively show the limit does not exist, we can construct sequences (a_n) and (b_n) that both tend to 0 such that $\sin(1/a_n) = 1$ and $\sin(1/b_n) = -1$ for all n. In particular, let

$$a_n = \frac{1}{\pi/2 + 2\pi n} \quad \text{and} \quad b_n = \frac{1}{3\pi/2 + 2\pi n}$$

Hence f' is not differentiable everywhere.

(2)

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

We claim f is continuous everywhere, as

$$|f(x) - 0| = |f(x)| \leq |x||\sin(\tfrac{1}{x})| \leq |x^2|1 \xrightarrow{x \to 0} 0$$

and we further claim that f' exists everywhere, as

$$\phi(t) = \frac{t^2 \sin(\frac{1}{t}) - 0}{t - 0} = t \sin(\tfrac{1}{t})$$

and

$$-1 \leq \sin(1/t) \leq 1 \implies -t \leq t \sin(1/t) \leq t \implies 0 = \lim_{t \to 0} -t \leq \lim_{t \to 0} \phi(t) \leq \lim_{t \to 0} t = 0 \implies f'(0) = 0$$

Thus we can write the derivative

$$f'(x) = \begin{cases} 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Note that this is not continuous at zero, by a similar argument to (1), so it is certainly not differentiable at zero.