Problem Set: Non Parametric Statistics: Group 18

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Question 1

Question 2

Compare the MISE and AMISE criteria in three densities in `nor1mix` of your choice.

1. Code (2.33) and the AMISE expression for the normal kernel, and compare the two error curves.
2. Compare them for $n = 100, 200, 500$, adding a vertical line to represent the h_{MISE} and h_{AMISE} bandwidths. Describe in detail the results and the major takeaways.

For this exercise we require some mathematical ideas that we will developed briefly.

We start with the KDE estimator $\hat{f}(x;h) = \sum_{i=1}^{n} K_h(x - X_i)$. The expectation and variance for this estimator are given bi the following expressions:

(1) $E[\hat{f}(x;h)] = (K_h * f)(x)$.

(2) $\text{Var}[\hat{f}(x;h)] = \frac{1}{n}((K_h^2 * f)(x) - (K_h * f)^2(x))$

Then, we develop some assymptotic expression for (1) and (2):

(3) $E[\hat{f}(x;h)] - f(x) = Bias[\hat{f}(x;h)] = \frac{1}{2}\mu_2(K)f''(x)h^2 + o(h^2)$

(4) $\text{Var}[\hat{f}(x;h)] = \frac{R(K)}{nh}f(x) + o((nh)^{-1})$

Then, from whe equations (3) and (4) we obtain the following expression for the MSE:

(5) $\text{MSE}[\hat{f}(x;h)] = \frac{\mu_2^2(K)}{4}(f''(x))^2h^4 + \frac{R(K)}{nh}f(x) + o(h^4 + (nh)^{-1})$

In equations (3), (4) and (5) we have:

(6) $\mu_2(K) := \int z^2K(z)dz$

(7) $R(K) := \int (K(x))^2dx$

Then, we use $\text{MISE}[\hat{f}(\cdot;h)]$ as global error criteria for measuring the performance of $\hat{f}$ in relation to the target density f.

(8) $\text{MISE}[\hat{f}(\cdot;h)] = \int MSE[\hat{f}(x;h)]$

Then, we obtain the following asymptotic expansion for the MISE:

(9) $\text{MISE}[\hat{f}(\cdot;h)] = \frac{1}{4}\mu_2^2(K)R(f'')h^4 + \frac{R(K)}{nh} + o(h^4 + (nh)^{-1})$

We define the dominating part of equation (9) as $AMISE[\hat{f}(\cdot;h)]$ defined as:

(10) $\text{AMISE}[\hat{f}(\cdot;h)] = \frac{1}{4}\mu_2^2(K)R(f'')h^4 + \frac{R(K)}{nh}$

with the expression $R(f'')$ given by:

(11) $R(f'') = \int (f''(x))^2 dx$

Finally, the bandwidth the minimizes the AMISE is:

(12) $h_{AMISE} = \left[\frac{R(K)}{\mu_2^2(K) R(f'') n} \right]^{1/5}$

Now, we consider our particular case of study. In this case, we *reduce* our analysis considering:

a) A normal kernel $K_h(\cdot)$ with distribution $\mathcal{N}(0,1)$

b) The density function f is based on the family of normal r-mixtures:

(13) $f(x; \mu, \sigma, \boldsymbol{w}) = \sum_{j=1}^{r} w_j \phi_{\sigma_j}(x - \mu_j)$

where $w_j \geq 0$, $j = 1, ..., r$ and $\sum_{j=1}^{r} w_j = 1$.

With this two expression, we can obtain a specific value for the AMISE in equation (10). With assumption a), we obating the following expressions for equations (6) and (7)

(6.1) $\mu_2(K) = 1$

(7.1) $R(K) = \frac{1}{2\sqrt{\pi}}$

Expression for equation (11) can be obtained from the following adaption of expression given in *Theorem 4.1* given by *Marron and Wand (1992)*

(11.1) $R(f'') = \int (f''(x))^2 dx =$

With this expression, we obtain the reduced form of the AMISE:

(10.1) $\text{AMISE}[\hat{f}(\cdot; h)] = \frac{1}{4} R(f'') h^4 + \frac{1}{2nh\sqrt{\pi}}$

With optimal bandwidth h_{AMISE}

(12.1) $h_{AMISE} = \left[\frac{(2\sqrt{\pi})^{-1}}{R(f'') n} \right]^{1/5}$

Finally, under this assumptions, we obtain a explicit and exact MISE expression of equation (8):

(14) $\text{MISE}_r[\hat{f}(\cdot; h)] = (2\sqrt{\pi} n h)^{-1} + \boldsymbol{w}' \{(1 - n^{-1})\Omega_2 - \Omega_1 + \Omega_0\} \boldsymbol{w}$

with $(\Omega_a)_{i,j} = \phi_{(ah^2 + \sigma_i^2 + \sigma_j^2)^{1/2}}(\mu_i - \mu_j)$ for $i, j = 1, ..., r$

Finally, we can proceed with a numeric approach over equation (14) and obtain:

(15) $\arg\min_{h>0} \text{MISE}[\hat{f}(\cdot; h)]$

We start with a easy example using the dataset `nor1mix::MW.nm1` that is distributed according to $\mathcal{N}(0,1)$

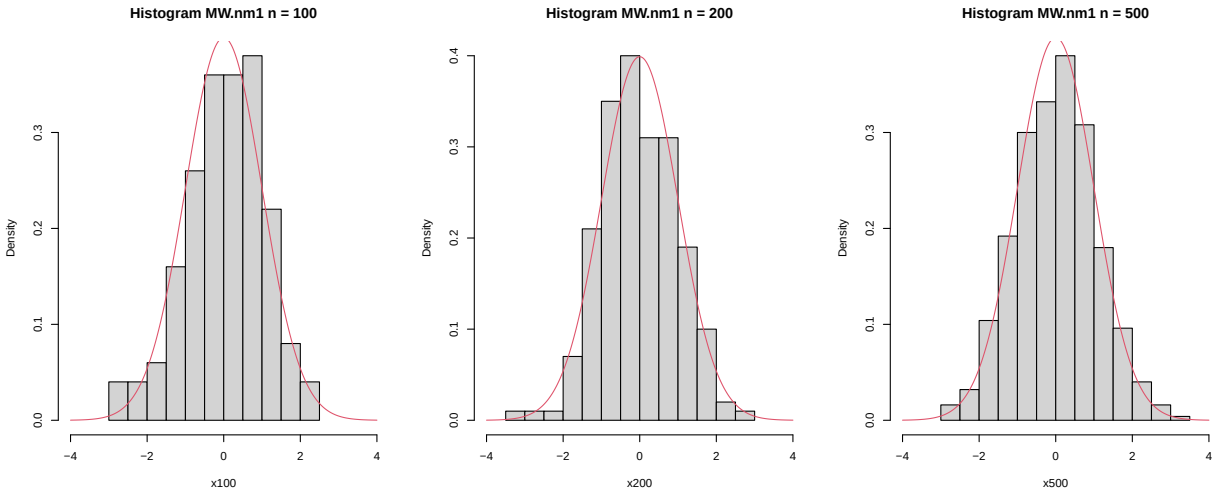

Figure 1: Histograms from sample obtained from n = 100, 200 and 500 for object MW.nm1

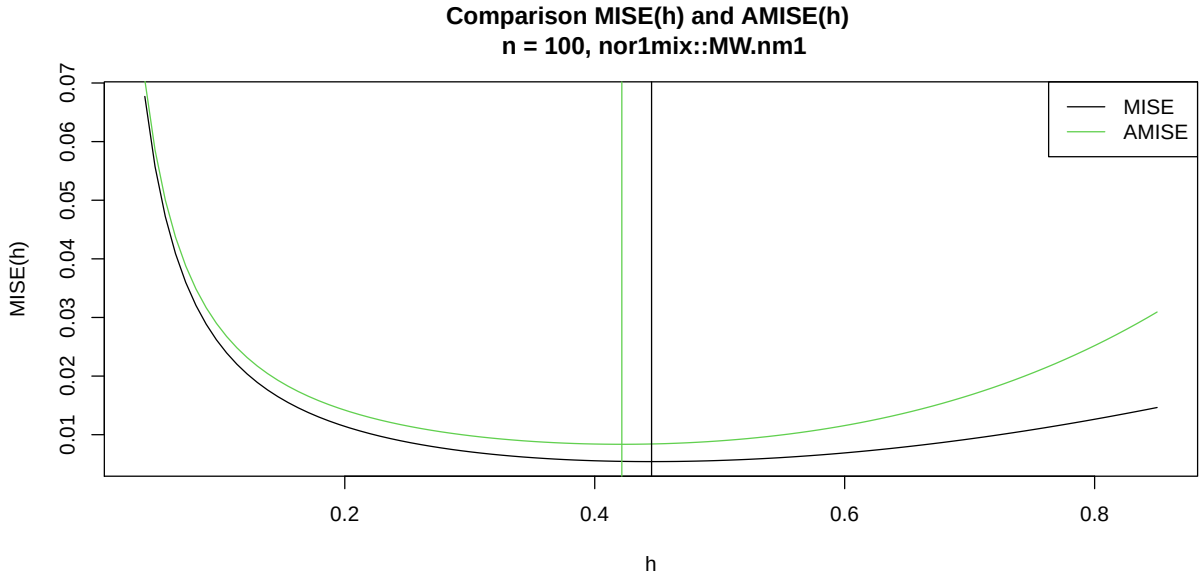

Figure 2: MISE and AMISE for range bandwith between 0.04 and 0.85 and n = 100

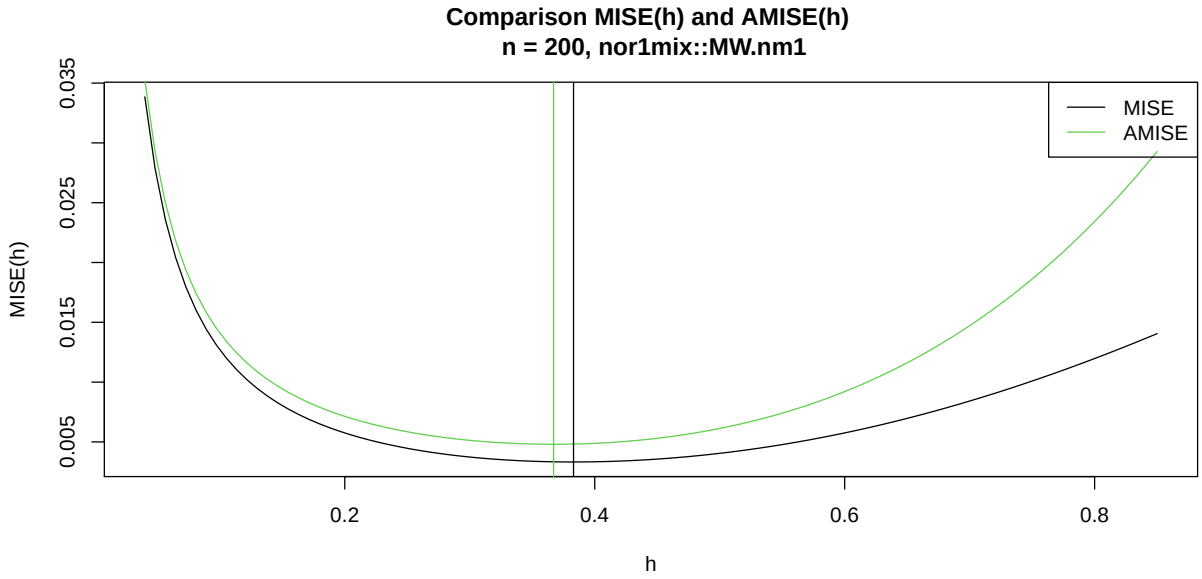

Figure 3: MISE and AMISE for range bandwith between 0.04 and 0.85 and n = 200

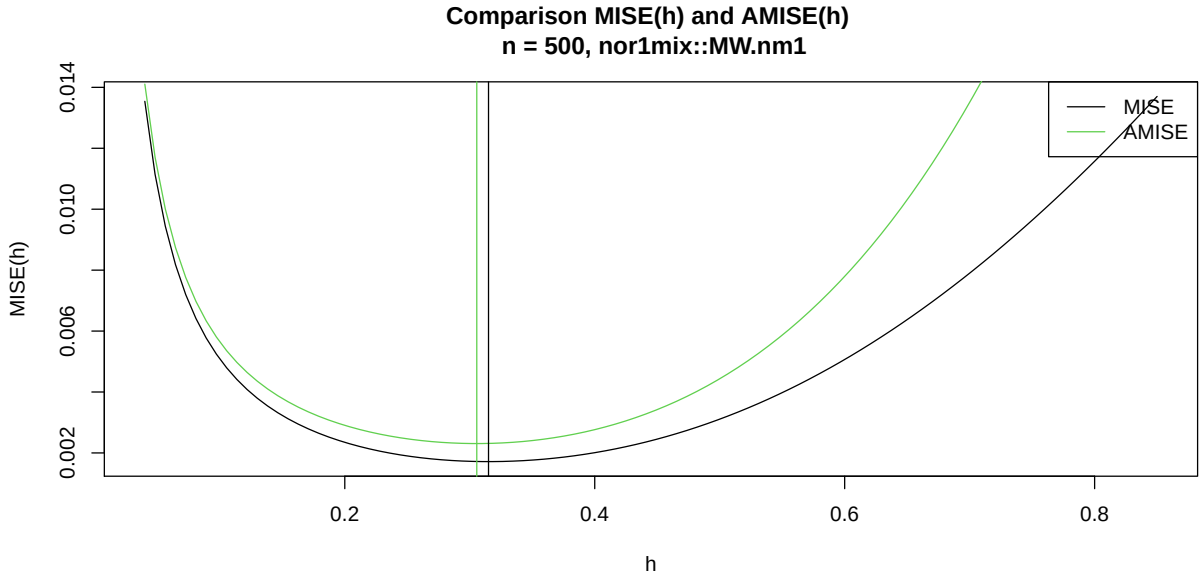

Figure 4: MISE and AMISE for range bandwith between 0.04 and 0.85 and n = 500

Question 3

Adapt the np_pred_CI function to include the argument type_boot, which can take either the value "naive" or "wild".

1) If `type_boot = "wild"`, then the function must perform the wild bootstrap algorithm described above, implemented from scratch following substeps i–iv.

2) Compare and validate the correct behavior of the confidence intervals, for the two specifications of type_boot, in the model considered in Exercise 5.8 (without performing the full simulation study).

For this question we describe briefly the necessary steps related with the implementation of the wild-bootstrap:

Fitted values: $\hat{Y}_i := \hat{m}(X_i; q, h)$ with $i = 1, ..., n$.

Prediction (Under a new grid): $\hat{m}(x, q, h)$. What about the uncertainty of $\hat{m}(x; q, h)$?

1. Asymptotic approach: $\hat{m}(x; q, h) \pm \hat{se}(\hat{m}(x; q, h))$
2. Bootstrap (we will focus on will bootstrap)
3. Compute $\hat{m}(x; q, h) = \sum_{i=1}^{n} W_i^q(x) Y_i$ from the original sample $(X_1, Y_1), ..., (X_n, Y_n)$
4. Enter the wild bootstrap. For $b = 1, ..., B$.

i. Simulate $V_1^{*b}, ..., V_n^{*b}$ to be iid copies of V such that $E[V] = 0$ and $Var[V] = 1$,

ii. Compute the *pertubed residuals* $e_i^{*b} = \hat{e}_i V_i^{*b}$ where $Y_i^{*b} := \hat{m}(X_i; q, h) + e_i^{*b}$ for $i = 1, ..., n$.

iii. Obtain the bootstrap sample: $(X_1, Y_1^{*b}), ..., (X_n, Y_n^{*b})$ where $Y_i^{*b} := \hat{m}(X_i; q, h) + e_i^{*b}$ with $i = 1, ..., n$.

iv. Compute $\hat{m}^{*b}(x; q, h) = \sum_{i=1}^{n} W_i^q(x) Y_i^{*b}$ from $(X_1^*, Y_1^{*b}), ..., (X_n^8, Y_n^{*b})$

Our first simulation scenario is the following:

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## Multistart 1 of 1 |Multistart 1 of 1 |Multistart 1 of 1 |Multistart 1 of 1 /Multistart 1 of 1 |Multis
## Warning in par(mforw = c(1, 2)): "mforw" is not a graphical parameter
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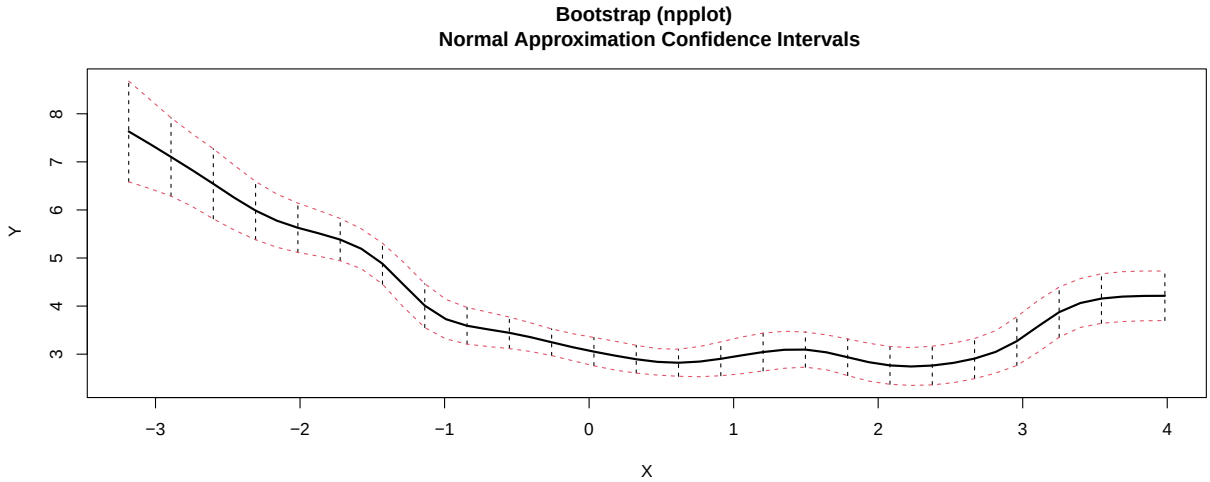

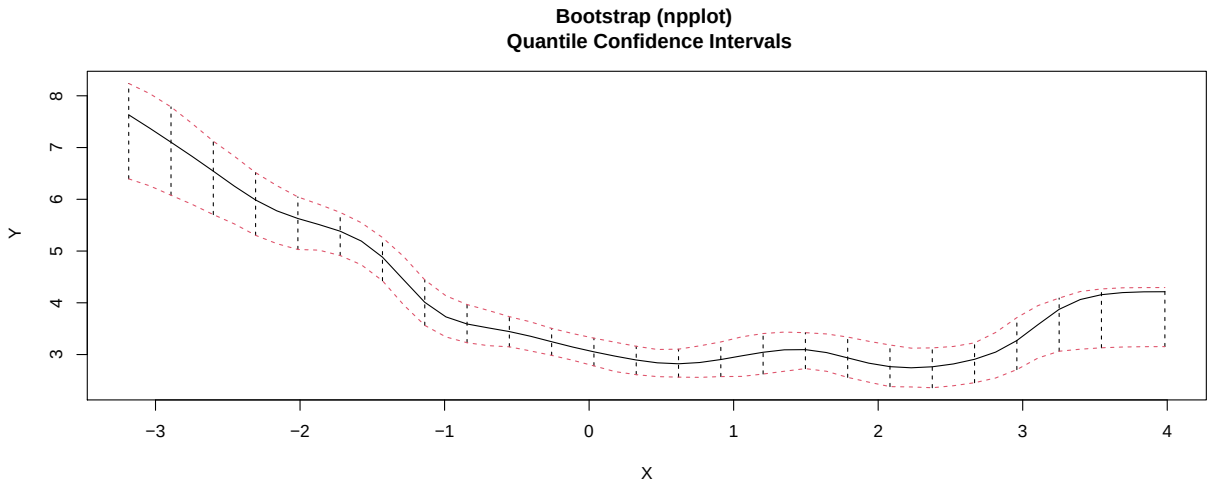

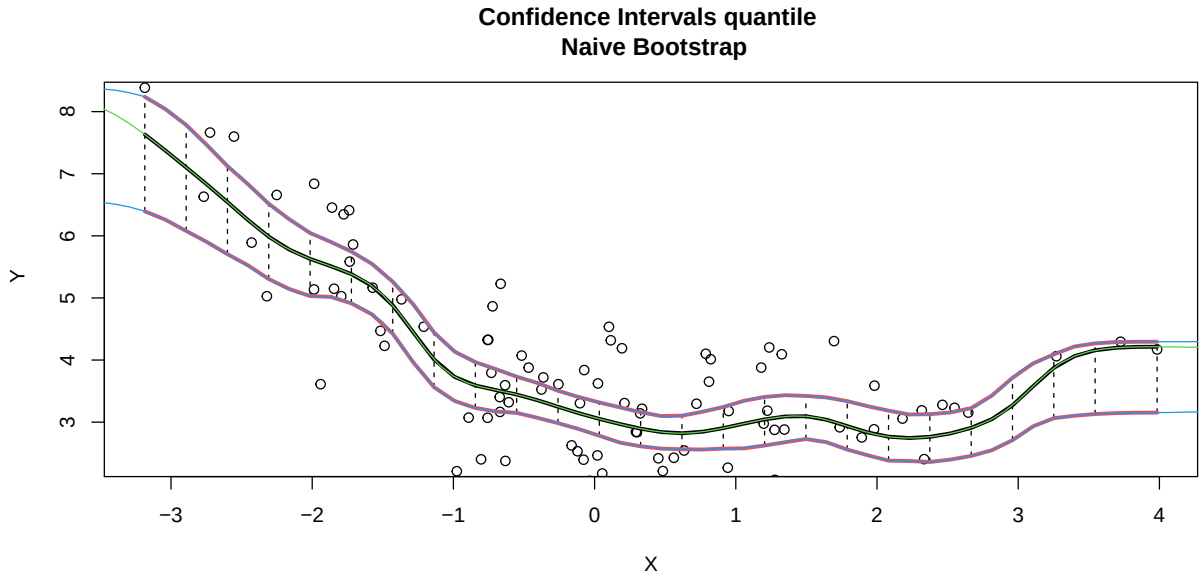

Figure 5: Function np_pred_CI: Naive Bootstrap

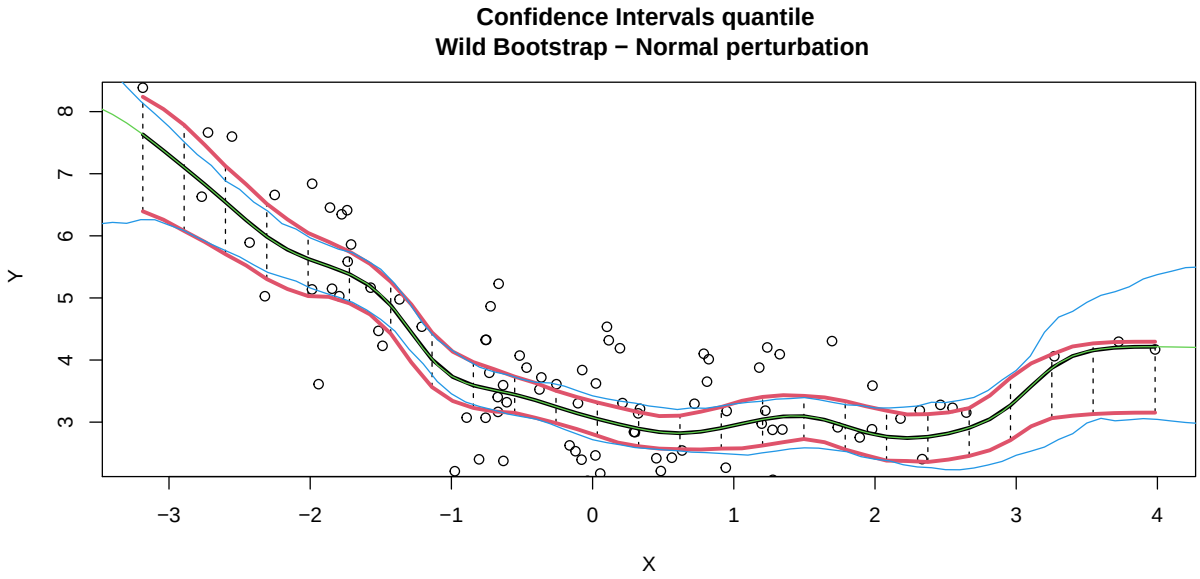

Figure 6: Function np_pred_CI: Wild Bootstrap and normal perturbation

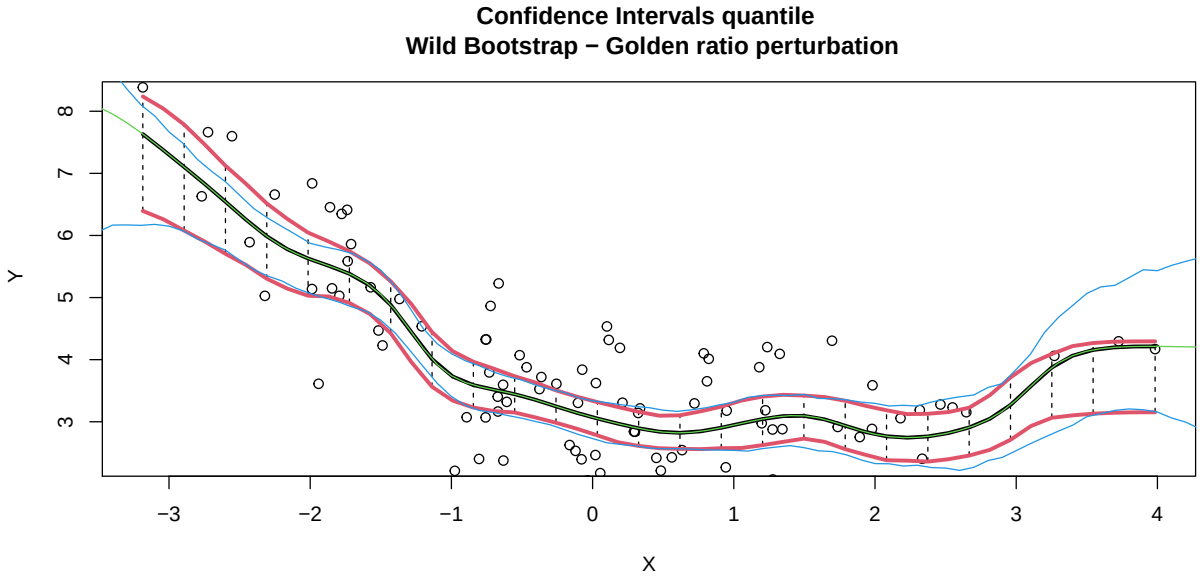

Figure 7: Function np_pred_CI: Wild Bootstrap and golden error perturbation