Research Statement

Carson Connard

1 Morse-Smale Functions on Orbifolds

The Witten approach to Morse theory considers the stable and unstable (alternatively, descending and ascending) manifolds $W^s(p)$ and $W^u(p)$ of critical points on a Riemannian manifold M. In particular, one can study a more rigid notion of Morse functions, called *Morse-Smale functions*, in which for any two critical points p, q, one has that $W^u(p) \pitchfork W^s(q)$ – this is called the *Smale transversality condition.*

Such an approach allows one to understand the geometry of the manifold further than in the seminal work of Morse, in which the manifold can be handle-decomposed; rather, studying Morse-Smale functions allows for the construction of a chain complex whose homology $HM_\bullet(M, f)$ turns out to be isomorphic to the singular homology of M.

Certainly, a natural extension of this powerful theory would be to more general objects, namely orbifolds. Indeed, in [CH14], an analogous construction to Morse homology is formed for orbifolds by considering just the "orientable" critical points (i.e., those $p \in \mathrm{crit}(f)$ whose isotropy groups do not reverse the orientation of $W^u(p)$). The result that it is isomorphic to singular homology still holds. However, when one studies the properties of Morse and Morse-Smale functions on orbifolds, certain issues arise.

In the case of manifolds, it turns out that the set of Morse functions *and* the set of Morse-Smale functions (although this latter set is more dependent on the Riemannian metric than the actual information of the function f) is dense in $C^\infty(M, \mathbb{R})$. However, in the case of manifolds acted upon by a group (G-manifolds, which we denote V), we do not have the same result: density holds for Morse functions, whereas it does not for Morse-Smale functions. Indeed, we have the following result, which can be considered as the case for a single orbifold chart; this can be extended to an entire orbifold ([FOOO15]):

Theorem 1 ([Was69])**.** For any finite dimensional G-manifold V, the set of G-invariant Morse functions is dense in $C^\infty_G(V, \mathbb{R})$ (i.e., smooth G-invariant functions $V \to \mathbb{R}$).

As for the lack of an analogous result for Morse-Smale functions, I constructed the following example of an orbifold and a function on such an orbifold for which the function (and any perturbation of it, contradicting a density claim) does not satisfy the Smale transversality condition.

Example. Set $M = \mathbb{R} \times \mathbb{R}^2$ and $G = \mathbb{Z}/n\mathbb{Z}$. Define the action $[k] \cdot (x, w) = (x, \exp(2\pi ik/n)w)$ with $k \in G$, $x \in \mathbb{R}$, and $w \in \mathbb{R}^2 \cong \mathbb{C}$ and the function $F(x, y, z) = f(x) + g(x)(y^2 + z^2)$ such that:

i. $f(x)$ has two non-degenerate critical points with indices $\lambda_f(p) = 1$ and $\lambda_f(q) = 0$, connected by a unique negative gradient flow line $\ell : p \to q$. For example, take a positive cubic with a local minimum and maximum.

ii. $g(x)$ is such that $g(p) > 0$ and $g(q) < 0$. An easy choice is $g(x) = -x$.

Via the definition of $F(x, y, z)$,

$$\begin{aligned}
\nabla F &= (f'(x) - (y^2 + z^2), -2xy, -2xz),\\
\mathrm{crit}(F) &= \{(x, y, z) : \nabla F = 0\} = \{(p, 0, 0), (q, 0, 0)\},\\
\mathrm{Hess}(F) &= \begin{pmatrix} f''(x) & -2y & -2z \\ -2y & -2x & 0 \\ -2z & 0 & -2x \end{pmatrix} \overset{(x,0,0)}{=} \begin{pmatrix} f''(x) & 0 & 0 \\ 0 & -2x & 0 \\ 0 & 0 & -2x \end{pmatrix}.
\end{aligned}$$

Thus, $\lambda_F(p, 0, 0) = \lambda_F(q, 0, 0) = 2$. Since $(p, 0, 0), (q, 0, 0)$ are nondegenerate, F is Morse. But, the trajectory $(\ell, 0, 0)$ does not flow "high to low," so F is not Morse-Smale.

Further, an equivariant perturbation of F is also not Morse-Smale, as the indices of the critical points would not change. This gives then that Morse-Smale functions are not dense in $C^\infty_G(\mathcal{X}, \mathbb{R})$ on the global quotient orbifold presented as $\mathcal{X} = [(\mathbb{R} \times \mathbb{R}^2)/(\mathbb{Z}/n\mathbb{Z})]$. $\diamond$

Certainly, however, there do exist orbifolds in which the Smale transversality condition globally applies (consider the trivial group action!). In this case, one could use any function that they please (following from Theorem 1) to construct the Morse-Smale-Witten complex from [CH14]. An interesting question and potential research direction is as follows:

Question 1. Can one classify orbifolds which guarantee the density of Morse-Smale functions in $C^\infty_G(\mathcal{X}, \mathbb{R})$?

In the case of manifolds, it is known that a "bumpy" Riemannian metric gives that a Morse function automatically satisfies the Smale transversality condition. An important question to ask in pursuit of the above question is: "does there exist an analogous result for orbifolds, or does the involvement of the G-action cause this to not be the case?"

2 Exotic Cohomology Product for Lagrangian Suborbifolds

Notation: X is an almost-complex manifold, equipped with the action of a finite group G denoted $X \overset{\rho(\gamma)}{\to} X$ for all $\gamma \in G$; this action preserves the almost-complex structure on the manifold. Thus, the product space $G \times X$ inherits the action $\rho(\gamma) : (m, x) \mapsto (\gamma m \gamma^{-1}, \rho(\gamma)x)$. Also, denote by $\mathcal{X}$ an orbifold, and (X, ω) a symplectic manifold. For the latter, we will simply write X if the context is clear.

Quantum cohomology is a powerful invariant on symplectic manifolds which extends the standard cohomology of the manifold. Of particular interest in this theory is the *quantum product*, which provides a "quantum" version of the cup product, which describes how submanifolds intersect each other. Namely, the quantum product considers intersections up to pseudo-holomorphic curves; naturally then, one can utilize Gromov-Witten invariants to define this product. In the seminal work of Chen and Ruan ([CR04]), these ideas are extended to orbifolds in order to construct the Chen-Ruan (and stringy) cohomology and its associative product structure.

The first goal in this section is to provide a construction of the *stringy cohomology* of the manifold X. Second, we will define its product structure. Third, we describe the Chen-Ruan cohomology of an orbifold and its product structure, a construction highly analogous to that of the case of stringy cohomology on manifolds. These first three goals will use techniques independent of Gromov-Witten theory, due to [JKK06], and will be referenced later as a possible approach towards a problem (note, the survey paper [Kim07] describes many of these results). Fourth, we provide an alternate definition of the cohomology product structures via an identification with Gromov-Witten invariants and the Poincaré pairing which the cohomology is equipped with. This will leave room to provide a conjecture for future research.

2.1 Stringy Cohomology

We first define inertia manifolds and their role in constructing stringy cohomology.

Definition. The inertia orbifold IX of X is the submanifold of $G \times X$ consisting of pairs (m, x) such that $\rho(m)x = x$. It decomposes as $IX = \bigsqcup_{m \in G} X^m$, with $X^m = IX \cap (\{m\} \times X)$ for all $m \in G$. Equivalently, this is the submanifold of fixed points of m in X. Moreover, IX has a G-invariant involution $\sigma : IX \to IX$ sending $X^m \to X^{m^{-1}}$ via $\sigma(m, x) = (m^{-1}, x)$.

This immediately yields the definition of the stringy cohomology of X:

Definition. The stringy cohomology $\mathcal{H}(X, G)$ of the G-manifold X is, as a G-module but not as a ring, equal to $H^\bullet(IX)$, that is

$$\mathcal{H}(X, G) := \bigoplus_{m \in G} \mathcal{H}_m(X) := \bigoplus_{m \in G} H^\bullet(X^m).$$

Note that the stringy cohomology is a self invariant, G-graded G-module. Further, it has a G-invariant, rationally valued Poincaré pairing $\langle \cdot, \cdot \rangle : \mathcal{H}(X, G) \otimes \mathcal{H}(X, G) \to \mathbb{Q}$ defined by $\langle v_1, v_2 \rangle = \int_{[IX]} v_1 \cup \sigma^* v_2$, which vanishes on G-homogeneous elements v_1 and v_2, unless they have inverse G-gradings.

2.2 Product Structure on Stringy Cohomology

Now, we can explicitly define the multiplication in stringy cohomology. Consider the G-equivariant vector bundle $\mathcal{N} \to IX$ whose restriction to X^m, which we denote $\mathcal{N}_m$, is the normal bundle of $X^m \hookrightarrow X$ for all $m \in G$. Take $r = |m|$ for $m \in G$. Then $\mathcal{N}_m \to X^m$ is an $\langle m \rangle$-equivariant complex vector bundle, where $\langle m \rangle$ acts preserving the fibers of $\mathcal{N}_m$. We can therefore decompose $\mathcal{N}_m$ into eigenbundles

$$\mathcal{N}_m := \bigoplus_{k=0}^{r-1} \mathcal{N}_{m,k},$$

where m acts upon the fibers of $\mathcal{N}_{m,k}$ with eigenvalue ζ_r^k, where $\zeta_r = \exp(2\pi i/r)$. The key to defining the product on stringy cohomology is to consider the element $\mathcal{S}$ in the K-theory of the inertia manifold which can be considered as "half" of $\mathcal{N}$:

Definition. Let $K(IX)$ denote the rational K-theory of complex vector bundles over IX. Consider $\mathcal{S}$ in $K(IX)$ whose restriction in $K(X^m)$ is

$$\mathcal{S}_m := S|_{X^m} := \bigoplus_{k=0}^{r-1} \frac{k}{r} \mathcal{N}_{m,k},$$

with $m \in G$ and $|m| = r$.

We can consider $\mathcal{S}$ to be half of the normal bundle in the K-theoretic sense. In particular, this means that $\mathcal{S} \oplus \sigma^* \mathcal{S} = \mathcal{N}$. This comes from the fact that $\sigma^* \mathcal{N}_{m,k} = \mathcal{N}_{m^{-1}, r-k}$ for all $k = 1, ..., r-1$. So, for all $m \in G$, we have the equality

$$\sigma^* \mathcal{S}_{m^{-1}} = \bigoplus_{\ell=0}^{r-1} \frac{\ell}{r} \sigma^* \mathcal{N}_{m^{-1},\ell} = \bigoplus_{\ell=0}^{r-1} \frac{\ell}{r} \mathcal{N}_{m,r-1} = \bigoplus_{\ell=0}^{r-1} \frac{r-k}{r} \mathcal{N}_{m,k} = \mathcal{N}_m \ominus \mathcal{S}_m.$$

In order to then define the product structure on the stringy cohomology $\mathcal{H}(X, G)$, we need a notion of a *double inertia manifold*, and an obstruction class on its K-theory (its definition requires the use of $\mathcal{S}$). Consider the set $G^{[3]} := \{(m_1, m_2, m_3) \in G^3 : m_1 m_2 m_3 = 1\}$, where $G \curvearrowright G^{[3]}$ via diagonal conjugation. Hence, we have that $G^{[3]} \times X$ inherits a diagonal G action.

Definition. The *double inertia manifold* of X, denoted IIX, is the G-submanifold of $G^{[3]} \times X$,

$$IIX := \bigsqcup_{m \in G^{[3]}} X^m,$$

where $\mathbf{m} = (m_1, m_2, m_3)$ and $X^{\mathbf{m}} = \{(\mathbf{m}, x) : \rho(m_1)x = \rho(m_2)x = \rho(m_3)x = x\}$. That is, $X^{\mathbf{m}}$ is the submanifold of points in X simultaneously fixed by m_i.

Also, we define G-equivariant evaluation maps $e_i : IIX \to IX$ by $(\mathbf{m}, x) \mapsto (m_i, x)$. A *twisted evaluation map* is the composition of an e_i with the involution on IX, that is $\check{e}_i := \sigma \circ e_i$.

Let $\mathbf{N} \to IIX$ denote the G-equivariant complex vector bundle whose restriction $\mathbf{N}_{\mathbf{m}}$ to $X^{\mathbf{m}}$ is the normal bundle of the inclusion $X^{\mathbf{m}} \hookrightarrow X$. (This mirrors $\mathcal{N} \to IX$, but it is the "double inertia" version).

Definition. The *obstruction class* $\mathcal{R}$ in $K(IIX)$ is defined by

$$\mathcal{R} := \ominus \mathbf{N} \oplus \bigoplus_{i=1}^{3} e_i^* \mathcal{S}.$$

In particular, for all $\mathbf{m} \in G^{[3]}$, in $K(X^{\mathbf{m}})$, we have

$$\mathcal{R}(\mathbf{m}) = \ominus \mathbf{N}_{\mathbf{m}} \oplus \bigoplus_{i=1}^{3} \mathcal{S}_{m_i}|_{X^{\mathbf{m}}}.$$

As promised, we can now construct the *stringy product* on $\mathcal{H}(X,G)$ by utilizing these objects dependent on the double inertia manifold.

Definition. The *multiplication on stringy cohomology* $\star : \mathcal{H}(X,G)\otimes\mathcal{H}(X,G) \to \mathcal{H}(X,G)$, $v_1\otimes v_2 \mapsto v_1\star v_2$ is defined by

$$v_1 \star v_2 := \check{e}_{3*}(e_1^* v_1 \cup e_2^* v_2 \cup \epsilon(\mathcal{R})),$$

with $\epsilon(\mathcal{R})$ the Euler class of $\mathcal{R}$.

2.3 Chen-Ruan Cohomology

Considering the stringy cohomology and product, we can analogously give a definition for the Chen-Ruan cohomology for orbifolds.

Recall that an orbifold $\mathcal{X}$ is locally presented as an open set $U \subset \mathbb{R}^n$ which possesses an almost-complex structure, equipped with a finite group action which preserves this almost-complex structure. Further, the inertia orbifold $I\mathcal{X}$ is locally presented as IU, and the double inertia orbifold is locally presented as IIU.

Definition. The *Chen-Ruan cohomology* $H_{\text{orb}}(\mathcal{X})$ is constructed identically to the stringy cohomology, except by replacing the "manifoldy" objects with the "orbifoldy" objects listed above. However, $H_{\text{orb}}(\mathcal{X})$ is isomorphic to $H^\bullet(I\mathcal{X})$ as a vector space. As an interesting note, if $\mathcal{X} = [X/G]$ with X a manifold, $H_{\text{orb}}(\mathcal{X})$ is isomorphic to the G-invariants of the stringy cohomology $\mathcal{H}(X,G)$.

The Chen-Ruan product is also constructed similarly, via the evaluation maps $IIU \to IU$ and the obstruction class defined by the bundle over IIX.

2.4 Gromov-Witten Invariants Identification

Let (X,ω) be a closed symplectic manifold, and let $H^\bullet(X)$ be equipped with a Poincare pairing $\langle\cdot,\cdot\rangle$.

Gromov-Witten theory is a powerful tool in symplectic geometry which connects to mathematical physics, namely Type IIA string theory. The main point of study are the Gromov-Witten invariants, which "count" genus g pseudo-holomorphic curves with n marked points, mapping into X. That is, we consider *stable maps* $f : \mathbb{C}P^1 \to X$. It turns out that the set of these functions forms a moduli space, which allows us to define the Gromov-Witten invariants:

Definition. Consider the stable maps $\mathbb{C}P^1 \to X$ of genus g, where $\mathbb{C}P^1$ has n marked points. Then we can form a moduli space $\mathcal{M}_{g,n}(X)$ of such curves. Note that for our use, we only consider genus 0 curves with 3 marked points; that is, $\mathcal{M}_{0,3}(X)$. We define the *evaluation maps* on the moduli space, $\text{ev}_i : \mathcal{M}_{0,3}(X) \to X$, by evaluating a stable map at a marked point, $[f : (\mathbb{C}P^1, x_1, x_2, x_3) \to X] \mapsto f(x_i)$.

This moduli space and its evaluation maps are enough to define Gromov-Witten invariants. Consider now three cohomology classes α, β, γ on X. Then the *Gromov-Witten invariants* are given as

$$GW_{0,3}(\alpha,\beta,\gamma) = \int_{\mathcal{M}_{0,3}(X)} \text{ev}_1^*\alpha \wedge \text{ev}_2^*\beta \wedge \text{ev}_3^*\gamma.$$

Remarkably, however, Gromov-Witten invariants provide a definition of the quantum product, and for orbifolds, the Chen-Ruan product.

We can identify the Poincaré pairing on $H^\bullet(X)$ with Gromov-Witten invariants,

$$\langle \alpha \star \beta, \gamma\rangle = GW_{0,3}(\alpha,\beta,\gamma).$$

It turns out that the product $\star$ is exactly the quantum product on the quantum ring $H^\bullet(X)[\![q]\!]$. Now, if we were to restrict to just constant curves, that is, where $q = 0$, we have that this quantum product reduces to the standard cup product on $H^\bullet(X)$ (cf. [MS12]).

We have an analogous construction for a Lagrangian submanifold L, which considers a "disk" as the source for pseudo-holomorphic curves, instead of a projective sphere $\mathbb{C}P^1$. When doing so, we are able to again retrieve the standard cup product on $H^\bullet(L)[\![q]\!]$ when restricting to $q = 0$.

A natural extension of this phenomenon would be to symplectic orbifolds. If we consider a symplectic orbifold $\mathcal{X}$, we must instead consider the cohomology of its inertia orbifold, $I\mathcal{X}$. When we repeat the same constructions as before, we must note that the marked points on the spheres can be orbifold points (or singularities). Indeed, if α, β, γ are cohomology classes on $\mathcal{X}$, we identify

$$\langle \alpha \star \beta, \gamma \rangle = GW_{0,3}(\alpha, \beta, \gamma).$$

Amazingly, when we follow the same procedure as before, it turns out that when we restrict the product $\star$ to $H^\bullet(I\mathcal{X})[\![q]\!]$ when $q = 0$ we retrieve the Chen-Ruan product!

We would like to do a similar construction for some notion of a Lagrangian suborbifold $\mathcal{L}$. In order to do so, as inspired by the "full" orbifold case, we need some notion of a "Lagrangian inertia orbifold." Thankfully, [COW23] introduces the notion of the *dihedral twisted sector* $I_{\mathcal{X}}\mathcal{L}$, which acts as an inertia orbifold of $\mathcal{L}$ *with respect to* $\mathcal{X}$. The prior constructions inspire the following conjecture:

Question 2. If we follow a similar identification of Gromov-Witten invariants with the cohomology of $I_{\mathcal{X}}\mathcal{L}$, we should be able to extract some new, exotic product by restricting $H^\bullet(I_{\mathcal{X}}\mathcal{L})[\![q]\!]$ to $q = 0$. Further, the approach of [JKK06] to construct the product structure for orbifold cohomology omits the usage of Gromov-Witten invariants. As a second approach to exploring the cohomology of $I_{\mathcal{X}}\mathcal{L}$, we should be able to apply the same techniques as described between §2.1-2.3 (that is, by considering a double inertia variety and constructing an obstruction class). That is, there should exist some sort of a "double" dihedral twisted sector $II_{\mathcal{X}}\mathcal{L}$, which can be studied to retrieve results regarding $H^\bullet(I_{\mathcal{X}}\mathcal{L})$.

In order to resolve the latter part of this conjecture, I plan on applying the techniques found in [JKK06] in order to construct the Chow ring and K-theory for some newly constructed $II_{\mathcal{X}}\mathcal{L}$. Further, to show the associativity of the conjectured product, [JKK06] discusses a technique to do so on Chen-Ruan cohomology which is independent of Gromov-Witten theory, which could be adjusted to $H^\bullet(I_{\mathcal{X}}\mathcal{L})$.

The ability to perform these constructions is known to hold in the case of symplectic manifolds and Lagrangian submanifolds. However, note the following theorem, which generalizes the required structure in order to do so:

Theorem 2 (Gromov?)**.** Symplectic manifolds always have almost complex structures which are compatible with the symplectic structure.

Further, it turns out that this space of almost-complex structures is infinite-dimensional; moreover, this space is contractible, that is, all of the almost-complex structures are homotopy equivalent.

This means that in the case of manifolds, up to homotopy, the crucial data which allows one to apply the ideas of Gromov-Witten theory to extract deformed cohomology products is dependent on the almost-complex structure of the manifold. Further, it is known that a Lagrangian submanifold is totally-real with respect to the almost-complex structure, and the same story holds in the Lagrangian case. These motivate the following conjecture for the case of orbifolds:

Question 3. If we consider a symplectic orbifold $\mathcal{X}$ and a Lagrangian suborbifold with dihedral twisted sector $I_{\mathcal{X}}\mathcal{L}$, then the data required to extract the stringy product or the product from Question 2 is dependent on some almost-complex/totally-real structure instead of symplectic/Lagrangian structure.

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