Intro to Morse Homology and Gromov-Witten Theory on Orbifolds

Carson Connard

Contents

1 Morse-Smale functions, Morse homology on manifolds

1.1 Morse-Smale function definition

Definition. If each critical point $p \in \mathrm{crit}(f)$ is non-degenerate (i.e., $\mathrm{Hess}(f)|_p \neq 0$), then we say that f is *Morse*.

Further, consider a negative pseudo-gradient vector field determined by g. One can define the *stable and unstable manifolds* of a critical point $p \in \mathrm{crit}(f)$ to be

$$W^s(p) = \{x \in M : \lim_{s\to\infty} \phi^s(x) = p\}, \qquad W^u(p) = \{x \in M : \lim_{s\to-\infty} \phi^s(x) = p\},$$

with ϕ a flow of the pseudo-gradient field. If we have that $W^u(p) \pitchfork W^s(q)\ \forall p, q \in \mathrm{crit}(f)$, then we say that the pairing (f, g) is *Morse-Smale.*

The image to have in mind is that we start at the top critical point on a manifold, and follow gradient trajectories **downward** to the other critical points, down to the bottom. (*Draw dented sphere.*)

1.2 Morse Homology

Note that by definition, $W^s(p) \pitchfork W^u(p)$. We also have that $W^u(p) \cap W^s(q) = \emptyset$ if $p \neq q$ and $f(p) \leq f(q)$, which also gives transversality. Note that the Smale condition implies that $\mathrm{codim}\,(W^u(p) \cap W^s(q)) = \mathrm{codim}\, W^u(p) + \mathrm{codim}\, W^s(q)$. From the note in the definition of (un)stable manifolds, this then implies that $\dim(W^u(p) \cap W^s(q)) = \mathrm{ind}(p) - \mathrm{ind}(q)$.

We denote this intersection, which is a submanifold of V, as

$$\mathcal{M}(p, q) = W^u(p) \cap W^s(q) = \{x \in V : \lim_{s\to\infty} \phi^s(x) = p \wedge \lim_{s\to-\infty} \phi^s(x) = q\};$$

the set of all points on the trajectories which connect p and q.

Consider the group $\mathbb{R}$ of translations through time, acting on $\mathcal{M}(p, q)$ as $s \cdot x = \phi^s(x)$. This action is free if $p \neq q$. We define the manifold $\mathcal{L}(p, q) = \mathcal{M}(p, q)/\mathbb{R}$. Then $\dim \mathcal{L}(p, q) = \mathrm{ind}(p) - \mathrm{ind}(q) - 1$.

1.2.1 Definition over $\mathbb{Z}/2\mathbb{Z}$, Morse homology on dented sphere

Definition. Consider the vector space $C_k(M, f; \mathbb{Z}/2) = \{\sum_{c\in\mathrm{crit}_k(f)} a_c c : a_c \in \mathbb{Z}/2\}$, where $\mathrm{crit}_k(f)$ are the critical points of index k of f. Also, if $p \in \mathrm{crit}_{k+1}(f)$, consider the boundary operator given as

$$\partial_X(p) = \sum_{q\in\mathrm{crit}_k(f)} n_X(p, q)q, \quad \text{with } n_X(p, q) \in \mathbb{Z}/2.$$

Notice that $C_k(f)$ is a free $\mathbb{Z}/2$-module of formal sums of the critical points of index k, but we can take any commutative ring instead of $\mathbb{Z}/2$. Indeed, we define the "full" Morse homology over $\mathbb{Z}$ later.

We want to define $n_X(p, q)$ to be the number of trajectories of X from critical points p to q, mod 2. Later, we will use $N_X(p, q) \in \mathbb{Z}$, which are the signed cardinalities of $\mathcal{L}(p, q)$ as coefficients, but for now we can just deal with the mod 2 case. In other words, since we are in $\mathbb{Z}/2\mathbb{Z}$, we don't need to worry about orientation, but we will later.

For $\mathbb{Z}/2\mathbb{Z}$, we will sketch the proof that $\partial^2 = 0$, and we also compute the $\mathbb{Z}/2\mathbb{Z}$ Morse homology for an example: the dented sphere.

Theorem 1. $(C_\bullet(M, f; \mathbb{Z}/2\mathbb{Z}), \partial^\bullet)$ is a complex.

Proof. (Sketch) We need to show that the boundary maps are well defined, i.e., $\mathcal{L}(p_{k+1}, q_k)$ is finite for each p_{k+1}, q_k, and also $\partial^2 = 0$ (the subscript is just to keep track of index). In particular, since

$$\partial_X \circ \partial_X(p_{k+1}) = \sum_{r_{k-1} \in \mathrm{crit}_{k-1}(f)} \left(\sum_{q_k \in \mathrm{crit}_k(f)} n_X(p_{k+1}, q_k) n_X(q_k, r_{k-1}) \right) r,$$

we aim to show that

$$\sum_{q_k \in \mathrm{crit}_k(f)} n_X(p_{k+1}, q_k) n_X(q_k, r_{k-1}) = \left| \coprod_{q_k \in \mathrm{crit}_k(f)} \mathcal{L}(p_{k+1}, q_k) \times \mathcal{L}(q_k, r_{k-1}) \right| = 0.$$

The strategy here is to appeal to some facts we know about 1 dimensional manifolds, namely that (compact) manifolds of dimension 1 with boundary have a boundary consisting of an even number of points. So, if we can show that if $\coprod_{q_k} \mathcal{L}(p_{k+1}, q_k) \times \mathcal{L}(q_k, r_{k-1})$ is a boundary of a 1 dimensional manifold, then it has an even number of points, i.e. the sum is 0 mod 2. We skip this compactification argument (which studies the behavior of broken trajectories through Morse charts) which completes the proof (cf. [AD13] §3.2.b). $\square$

Example: Consider the sphere with a "dent" on the top, so there is a minimum (a), a saddle (b), and two local maxima (c and d). Computing homology over $\mathbb{Z}/2$ gives the following:

1. First, there is one 0-cell (i.e. critical point of index 0) which is a, so $C_0 = \mathbb{Z}/2$, generated by a. Next, there is one 1-cell (i.e. critical point of index 1) which is b, so $C_1 = \mathbb{Z}/2$, generated by b. Last, there are two 2-cells (i.e. critical points of index 2) which are c and d, so $C_2 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, generated by c and d.

2. We now want to consider the differential maps, which send elements of $C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2$. Notice though that each differential acts as a coefficient giving the number of trajectories between critical points. So, $\partial_2 d = b$ and $\partial_2 c = b$ (local max to saddle), and $\partial_1 b = 2a = 0$ mod 2 (saddle down to minimum, two possible directions).

3. Last, we aim to compute the actual homology groups of the dented sphere. Recall that $H_k = \ker \partial_k / \mathrm{im} \partial_{k+1}$. So, we consider first $H_0 = \ker \partial_0 / \mathrm{im} \partial_1$. Notice that trivially, if $\partial_0 : C_0 \to C_{-1}$, we have that $\ker \partial_0 = C_0 = \mathbb{Z}/2$ since $C_{-1} = 0$. Also, notice that $\partial_1 = 0$, and therefore $\mathrm{im} \partial_1 = 0 = \mathrm{id}_{\mathbb{Z}/2}$. Certainly then, $H_0 = (\mathbb{Z}/2)/\mathrm{id}_{\mathbb{Z}/2} = \mathbb{Z}/2$.

   Next, we compute $H_1 = \ker \partial_1 / \mathrm{im} \partial_2$. Notice that ∂_1 is the zero map, and therefore $\ker \partial_1 = \mathbb{Z}/2$. Also, $\mathrm{im} \partial_2 = \mathbb{Z}/2$ since $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2$ is not just the zero map. Therefore $H_1 = (\mathbb{Z}/2)/(\mathbb{Z}/2) = 0$.

   Lastly, we compute $H_2 = \ker \partial_2 / \mathrm{im} \partial_3$. Notice that ∂_2 is not injective. If we denote $c = (1, 0) \in \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $d = (0, 1) \in \mathbb{Z}/2 \oplus \mathbb{Z}/2$, we know that the elements which map to 0 must be $(0, 0)$ and $(1, 1)$, which we can identify as $\mathbb{Z}/2$. Finally $\mathrm{im} \partial_3 = 0$ since it is a homomorphism, and $0 \mapsto 0$. Therefore $H_2 = (\mathbb{Z}/2)/0 = \mathbb{Z}/2$.

1.2.2 Definition over $\mathbb{Z}$

Homology over $\mathbb{Z}$ is similar. We present the construction but don't do anything with it, as computation is easier in $\mathbb{Z}/2\mathbb{Z}$.

Definition. Let $C_k(M, f; \mathbb{Z}) = \{\sum_{p\in \mathrm{crit}_k(f)} a_p p : a_p \in \mathbb{Z}\}$, in other words it is the free $\mathbb{Z}$-module generated by the critical points of f, i.e. $C_k(f, \mathbb{Z}) = \mathbb{Z}\mathrm{crit}_k(f)$.

In order to define the differential map $\partial_X : C_k(f) \to C_{k-1}(f)$, notice that we are going from a critical point of order k to a critical point of order $k-1$. We know that when $\mathrm{ind}(a) - \mathrm{ind}(b) = 1$, that $\mathcal{L}(a, b)$ is an oriented compact manifold of dimension 0. Therefore, it is a finite collection of points, each of which is endowed with a sign which comes from the orientation of the manifold. Denote by $N_X(p, q) \in \mathbb{Z}$ the sum of these signs. Then we can define the differential map

$$\begin{aligned}\partial_X : C_k(f) &\to C_{k-1}(f) \\ p &\mapsto \sum_{q\in \mathrm{crit}_{k-1}(f)} N_X(p, q)q.\end{aligned}$$

Note that $n_X(p, q) = N_X(p, q) \bmod 2$.

2 Equivariant Morse functions

2.1 Density of Morse functions on G-manifolds

In the case of "regular" manifolds, it turns out that the set of Morse functions and the set of Morse-Smale functions is dense in $C^\infty(M, \mathbb{R})$. However, in the case of manifolds acted upon by a group (which we denote V), we do not have the same result: density holds for Morse functions, whereas it does not for Morse-Smale functions.

We have the following result of Wasserman [Was69], with a proof from Fukaya et al. from 2015 [FOOO15], which considers the case of a single orbifold chart; this can be extended to an entire global quotient orbifold.

Theorem 2 ([Was69], [FOOO15])**.** Suppose V is a manifold on which a finite group G acts effectively (i.e., elements of V are fixed only by 1_G), with a G-invariant Riemannian metric equipped. Denote by $C^k_G(V)$ the set of all G-invariant C^k functions on V. Then the set of all G-invariant, smooth Morse functions on V is a countable intersection of open, dense subsets in $C^\infty_G(V)$.

Proof. Let $K \subset V$ be compact. It suffices to show that the set of all functions in $C^2_G(V)$ which are Morse on K is open and dense. Openness is clear due to the natural topology which we equip C^k-spaces with, the compact-open topology. The rest of what follows is to show density.

We define the following sets in order to stratify our space by the number of group elements which fix points in each stratification. For $p \in X$, we define the isotropy groups $G_p = \{g \in G : gp = p\}$, and we define

$$\tilde{X}(n) = \{p \in X : \#G_p = n\}, \qquad X(n) = \{p \in X : \#G_p \geq n\}.$$

Note that $\tilde{X}(n)/G$ is a smooth manifold.

First, we claim that if $p \in \tilde{X}(n)$, then $p \in \mathrm{crit}(f)$ if and only if $p \in \mathrm{crit}(f|_{\tilde{X}(n)})$. Indeed, this is a consequence of the behavior of the directional derivative, in which $X[f]$ is zero if $X \in T_pX$ is orthogonal to $\tilde{X}(n)$. The claim is resolved by the G-invariance of f. (Note picture I took of blackboard 11/1) ◇

Let us now define the (open) sets

$$\begin{aligned}A(n) &= \{f \in C^\infty_G(V) : \text{all } p \in \mathrm{crit}(f_{X(n)\cap K}) \text{ are Morse}\}, \\ B(n) &= A(n+1) \cap \{f \in C^\infty_G(V) : f|_{\tilde{X}(n)\cap K} \text{ is Morse}\}.\end{aligned}$$

It will suffice to show that $A(1)$ is dense in order to prove the theorem. So, we first claim now that if $A(n+1)$ is dense, then so is $B(n)$.

Proof. Take $W \subset K \cap \tilde{X}(n)$ to be a relatively compact open subset (i.e., $\overline{W}$ is compact). We define a C^1 map which acts like the evaluation of a derivative as (take differential of f at x)

$$F : W \times C^2_G(\overline{W}) \to T^*\tilde{X}(n), \quad (x, f) \mapsto df_p : T_pW \to \mathbb{R}.$$

We have that $F \pitchfork \tilde{X}(n) \subset T^*\tilde{X}(n)$, and we identify $\tilde{X}(n)$ with the zero section of the cotangent bundle $T^*\tilde{X}(n)$. We define the following set of points and functions which give a derivation in this zero section:

$$\mathcal{W} = \{(x, f) \in W \times C^2_G(\overline{W}) : F(x, f) \in \tilde{X}(n) \subset T^*\tilde{X}(n)\}$$

to be a sub-Banach manifold of the Banach manifold $W \times C^2_G(\overline{W})$. It turns out the restriction of the projection $\pi : \mathcal{W} \to C^2_G(\overline{W})$ is a Fredholm map (i.e., the kernel and cokernel are finite dimensional), allowing us to apply the Sard-Smale theorem. This gives that the set of regular values of π is dense in $C^2_G(\overline{M})$.

As a subclaim, we say that if f is a regular value of π then $f|_{\tilde{X}(n)}$ is Morse on W. Indeed, let us take $x \in W$ to be a critical point of f. Then certainly $(x, f) \in \mathcal{W}$. Consider now the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & T_x\tilde{X}(n) \arrow[r, "H"] \arrow[d] & T^*_x\tilde{X}(n) \arrow[d] & \\
0 \arrow[r] & T_{(x,f)}\mathcal{W} \arrow[r] \arrow[d, "\eta"] & T_x\tilde{X}(n) \oplus T_fC^2_G(\overline{W}) \arrow[r, "\overline{D_{(x,f)}F}"] \arrow[d] & \frac{T_{(x,o)}T^*\tilde{X}(n)}{T_xT^*\tilde{X}(n)} = T^*_n\tilde{X}(n) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_fC^2_G(\overline{W}) \arrow[r] & T_fC^2_G(\overline{W}) \arrow[r] \arrow[d] & 0 & \\
 & & 0 & &
\end{tikzcd}$$

For brevity of notation within the diagram, we are denoting by $\overline{D_{(x,f)}F}$ the composition of the induced map $D_{(x,f)}F : T_x\tilde{X}(n) \oplus T_fC^2_G(\overline{W}) \to T_{(x,o)}T^*\tilde{X}(n)$ and the projection π. Since f is regular, we have that $\eta : T_{(x,f)}\mathcal{W} \to T_fC^2_G(\overline{W})$ is surjective. Diagram chasing yields that $\zeta : T_x\tilde{X}(n) \to T^*_x\tilde{X}(n)$ is surjective as well. This map is equivalently the Hessian at x of $f|_{\tilde{X}(n)}$, resolving the subclaim. $\diamond$

So, we observe that if $f \in A(n+1)$, then $\mathrm{crit}(f)$ is compact in $\tilde{X}(n) \cap K$, since it does not have accumulation points on $X(n+1) \cap K$. So, $B(n)$ is dense, which follows from the subclaim and the Sard-Smale theorem. $\square$

Lastly, in order to show that $A(1)$ is dense, we show that if $B(n)$ is dense, then so is $A(n)$.

Proof. Take $f \in B(n)$. Notice that the set of critical points of $f|_{\tilde{X}(n)}$ on $\tilde{X} \cap K$ is finite, since $f|_{\tilde{X}(n)}$ is a Morse function on $\tilde{X} \cap K$ and doesn't have accumulation points on $X(n+1) \cap K$. Say there are m critical points; let $\{p_1, ..., p_m\} = \mathrm{crit}(f|_{\tilde{X}(n)})$ on $\tilde{X}(n) \cap K$. The Hessian of f at these points are non-degenerate on $T_{p_i}\tilde{X}(n)$, but could be degenerate in the normal direction to $\tilde{X}(n)$. Let us choose functions $\{\chi_i\}$ and sets $\{V_i\}$, for $i = 1, ..., m$, such that the following conditions hold:

(i) V_i is a neighborhood of p_i.

(ii) $\mathrm{supp}(\chi_i) \subset V_i$.

(iii) $\chi_i \equiv 1$ in a neighborhood of p_i.

(iv) The $\overline{V}_i$ for $i = 1, ..., m$ are disjoint.

(v) $\overline{V}_i \cap X(n+1) = \emptyset$.

(vi) $gp_i = p_j$ implies $gV_i = V_j$ and $\chi_j \circ g = \chi_i$.

Utilizing the G-invariant Riemannian metric which V is equipped with, we write $f_n(x) = d(x, \tilde{X}(n))^2$. We choose V_i to be sufficiently small such that $\chi_i f_n$ is smooth, satisfying (v). Now a perturbation

$$f_\epsilon = f + \epsilon \sum_{i=1}^{m} \chi_i f_n$$

is a Morse function for sufficiently small $\epsilon > 0$. G-invariance is then given by (vi), and so $f_\epsilon \in A(n)$. Further, $f_\epsilon \to f$ as $\epsilon \to 0$. □

Since the density of $A(n+1)$ implies the density of $B(n)$, which implies the density of $A(n)$, we have that $A(1)$ must be dense. Therefore, we have that the set of smooth, G-invariant Morse functions is indeed dense. □

2.2 Nondensity of Morse-Smale functions on G-manifolds

2.2.1 Example

As mentioned before, Morse-Smale functions are not always dense in the case of orbifolds. We study the following example of such an orbifold:

Set $M = \mathbb{R} \times \mathbb{R}^2$ and $G = \mathbb{Z}/n\mathbb{Z}$. Define the action $[k] \cdot (x, w) = (x, \exp(2\pi i k/n) w)$ with $k \in G$, $x \in \mathbb{R}$, and $w \in \mathbb{R}^2 \cong \mathbb{C}$ and the function $F(x, y, z) = f(x) + g(x)(y^2 + z^2)$ such that:

1. $f(x)$ has two non-degenerate critical points with indices $\lambda_f(p) = 1$ and $\lambda_f(q) = 0$, connected by a unique negative gradient flow line $\ell : p \to q$. Think positive cubic with local min/max.

2. $g(x)$ is such that $g(p) > 0$ and $g(q) < 0$. An easy choice is $g(x) = -x$.

Via the definition of $F(x, y, z)$,

$$\begin{aligned}
\nabla F &= (f'(x) + g'(x)(y^2+z^2), 2yg(x), 2zg(x)),\\
\mathrm{crit}(F) &= \{(x,y,z) : \nabla F = 0\} = \{(p,0,0), (q,0,0)\},\\
\mathrm{Hess}(F) &= \begin{pmatrix} f''(x) + g''(x)(y^2+z^2) & 2yg'(x) & 2zg'(x) \\ 2yg'(x) & 2g(x) & 0 \\ 2zg'(x) & 0 & 2g(x) \end{pmatrix}\\
&\overset{\mathrm{crit}(F)}{=} \begin{pmatrix} f''(x) & 0 & 0 \\ 0 & -2x & 0 \\ 0 & 0 & -2x \end{pmatrix}.
\end{aligned}$$

Thus, $\lambda_F(p,0,0) = \lambda_F(q,0,0) = 2$. Since $(p,0,0), (q,0,0)$ are nondegenerate, F is Morse. But, the trajectory $(\ell, 0, 0)$ does not flow "high to low," so F is not Morse-Smale.

Further, an equivariant perturbation of F is also not Morse-Smale, as the indices of the critical points would not change. Therefore, Morse-Smale functions are not dense on $\mathcal{X} = [(\mathbb{R} \times \mathbb{R}^2)/(\mathbb{Z}/n\mathbb{Z})]$.

2.2.2 Construction of Morse-Smale-Witten complex for orbifolds

As discussed, we do not necessarily have a dense set of Morse-Smale functions on G-spaces, particularly orbifolds. In [CH14], this is remedied by constructing a standalone sort of Morse homology for orbifolds. In particular, such a complex on an orbifold $\mathcal{X} = [M/G]$ can be understood as the G-invariant part of the Morse complex of M.

We will first construct a Morse-Smale-Witten complex for G-invariant Morse-Smale functions. Our approach is to define a "type" of critical point, either orientable or non-orientable, and consider a G-action on the subcomplex generated by the orientable critical points. This will be done for global quotient orbifolds, and can be extended to "full" orbifolds; we will omit this for time purposes though, since its construction is notationally dense, and I have other things to talk about. Throughout, assume that $f : M \to \mathbb{R}$ is Morse-Smale, and consider the following definitions.

Definition. A smooth function $\bar{f} : X \to \mathbb{R}$ is *Morse* if every $\bar{x}$ in the orbifold has a uniformizing chart $(\tilde{U}_{\bar{x}}, G_{\bar{x}}, \pi_{\bar{x}})$ where $\tilde{U} \subset \mathbb{R}^n$ is open and connected, $G \curvearrowright \tilde{U}$, and $\pi : \tilde{U} \to U \subset X$, such that $\bar{f} \circ \pi_{\bar{x}} : \tilde{U} \to \mathbb{R}$ is Morse (in the usual sense) on $\tilde{U}_{\bar{x}}$.

Definition. We say that there exists $\bar{p} \in \operatorname{crit}(\bar{f})$ if there exists $p \in \operatorname{crit}(f)$ such that $\pi(p) = \bar{p}$, with π the natural projection to the quotient space. If the G_p-action on the unstable manifold $W^-(p)$ at $p \in \pi^{-1}(\bar{p})$ is orientation preserving, then p is orientable; we say that $p \in \operatorname{crit}^+(f)$, and otherwise, $p \in \operatorname{crit}^-(f)$.

We can now decompose $CM_\bullet(M, f)$ by G-action preservation and orientability.

Definition. Denote by $CM_\bullet(M, f)$ the complex of vector spaces freely generated by critical points. Let $\operatorname{crit}_i^\pm(f) = \operatorname{crit}_i(f) \cap \operatorname{crit}^\pm(f)$. This allows us to decompose

$$CM_i(M, f) = CM_i^+(M, f) \oplus CM_i^-(M, f).$$

By definition, we have that the G-action preserves $CM_i^+(M, f)$, and so we define the complex

$$CM_i^+(X, \bar{f}) := CM_i^+(M, f)^G.$$

Further, for $\bar{p} \in \operatorname{crit}(\bar{f})$, we write the formal sum

$$[\bar{p}] := \sum_{p \in \pi^{-1}(\bar{p})} p.$$

Thus, we have that $CM^+(X, \bar{f})$ is freely generated by the $[\bar{p}]$'s with $\bar{p} \in \operatorname{crit}^+(\bar{f})$.

Now, we define the boundary map $\partial_i : CM_i^+(X, \bar{f}) \to CM_{i-1}^+(X, \bar{f})$. For each $\bar{p} \in \operatorname{crit}^+(\bar{f})$, take a G-invariant orientation on $\{W^-(p) : p \in \pi^{-1}(\bar{p})\}$. If $\bar{p} \in \operatorname{crit}^-(\bar{f})$, take an arbitrary orientation.

Recall the definition of the Morse complex discussed at the beginning of the lecture, that is,

$$\partial p = \sum_{q \in \operatorname{crit}_{k-1}(f)} N_X(p, q) q.$$

Definition. For a differential on $\bar{f} : \mathbf{X} \to \mathbb{R}$ on $[\bar{p}]$, we write

$$\partial[\bar{p}] = \sum_{p \in \pi^{-1}(\bar{p})} \partial p,$$

where ∂p is the differential for the function $f : M \to \mathbb{R}$.

In order to show that this boundary map defines a complex, note the following two lemmata, for which we will omit the proofs:

Lemma 1. If $\bar{p} \in \text{crit}^+(\bar{f})$, then $\partial[\bar{p}] \in CM^+(X, \bar{f})$. That is, $\partial[\bar{p}]$ has nonzero coefficients only at orientable critical points.

Proof. (Sketch) We aim to show that the differential applied to a non-orientable critical point is zero. We can induce an orientation on the moduli space $\mathcal{M}(p, q)$ by fixing an orientation on the manifold M and all of the unstable manifolds on M.

We then can fix a regular value $s \in (f(p), f(q))$ as in the first fundamental Morse theorem, and consider the set $S^-(p) = f^{-1}(s) \cap W^-(p)$ as a boundary of $D^-(p) = f^{-1}([s, \infty)) \cap W^-(p)$; we do the same with the stable manifold of q, giving some $\partial D^+(q) = S^+(q)$.

If the orbifold version of the moduli space is given as $\mathcal{M}(\bar{p}, q) = \bigcup_{p \in \pi^{-1}(\bar{p})} \mathcal{M}(p, q)$, then we can decompose into the sets $\mathcal{M}(\bar{p}, q)^+ \sqcup \mathcal{M}(\bar{p}, q)^-$ with respect to their sign. We then aim to show that $|\mathcal{M}(\bar{p}, q)^+| = |\mathcal{M}(\bar{p}, q)^-|$, which is done by considering how G-actions affect the orientations of the sets $S^-(p)$, $S^+(q)$, $D^-(p)$, and $D^+(q)$. □

Lemma 2. The expression $\partial(\sum_{p \in \pi^{-1}(\bar{p})} p) = \partial[\bar{p}]$ is G-invariant if $\bar{p}$ is orientable.

Proof. From the previous lemma, $\partial[\bar{p}]$ contains only orientable critical points. If we consider two orientable points q and $q' = g \cdot q$, it suffices to show that the coefficients of q, q' are equal. This is clear since the action of g and g^{-1} determine sign preserving isomorphisms between $\mathcal{M}(\bar{p}, q)$ and $\mathcal{M}(\bar{p}, q')$, as we chose orientations of the unstable manifolds of f to be G-invariant in the proof of the previous lemma. □

These two lemmata show that $CM^+_\bullet(\mathbf{X}, f)$ is a subcomplex of $CM_\bullet(M, f)$. Note that ∂_0 follows from the definition of the usual $\partial(p)$ inherited from the manifold case. Notably, we have the following theorem of [CH14], whose proof is omitted since it is four pages long and deals with the full orbifold case, and moreover, Lino told me to skip it since it uses some pretty heavy machinery (spectral sequences, a bunch of other things in Bott & Tu) which I am (currently) unfamiliar with:

Theorem 3. $HM_\bullet(\mathbf{X}, \bar{f}) \cong H_\bullet(M/G) = H_\bullet(X)$.

Example. Consider the dented sphere with Morse function h given by the height function, with two maxima p, q, one minimum s, and a saddle point r. Suppose we quotient by the action of $\mathbb{Z}/2\mathbb{Z}$, which preserves r, s and interchanges p, q. Note that topologically, $S^2/(\mathbb{Z}/2\mathbb{Z}) \cong S^2$. We can write the chain complex obtained (naïvely) by the G-action on critical points, without considering orientability, as

$$0 \to \langle p + q \rangle \to \langle r \rangle \to \langle s \rangle \to 0,$$

where $(p + q)$ is taken as a single generator. However, notice that the differential does not square to be 0, and so homology is not well defined.

Instead, write

$$0 \to \langle p + q \rangle \to 0 \to \langle s \rangle \to 0.$$

This is because of the non-orientability of r; that is, the rotation reverses the orientation of $W^-(r)$, and we must omit $\langle r \rangle$ and not use it as a generator. This obtains the correct Morse homology, which is indeed isomorphic to the homology of the quotient space (topologically the same as S^2):

$$HM_\bullet(S^2/(\mathbb{Z}/2\mathbb{Z})) = \begin{cases} \mathbb{Z}, & i = 0, 2, \\ 0, & \text{else} \end{cases} = H_\bullet(S^2/(\mathbb{Z}/2\mathbb{Z})) = H_\bullet(S^2).$$

3 Gromov-Witten theory on symplectic manifolds

3.1 Chen-Ruan/Stringy Cohomology and its Product

The results of this section are from the exposition [Kim07], based in part on the work of [CR04].

Consider an almost complex manifold X with the action of a finite group G, denoted $\rho(\gamma) : X \to X$ for all $\gamma \in G$, which preserves the structure. Then the product space $G \times X$ inherits the action of G where γ takes $(m, x) \mapsto (\gamma m \gamma^{-1}, \rho(\gamma)x)$.

Definition. The inertia manifold IX of X is the submanifold of $G \times X$ consisting of pairs (m, x) such that $\rho(m)x = x$. It decomposes as $IX = \coprod_{m \in G} X^m$, with $X^m = IX \cap (\{m\} \times X)$ for all $m \in G$ (submanifold of fixed points of m in X). Moreover, IX has a G-invariant involution $\sigma : IX \to IX$ sending $X^m \to X^{m^{-1}}$ via $\sigma(m, x) = (m^{-1}, x)$.

Definition. The stringy cohomology $\mathcal{H}(X, G)$ of the G-manifold X is, as a G-module but not as a ring, equal to $H^\bullet(IX)$, that is

$$\mathcal{H}(X, G) := \bigoplus_{m \in G} \mathcal{H}_m(X) := \bigoplus_{m \in G} H^\bullet(X^m).$$

It is a self invariant, G-graded G-module. Further, it has a G-invariant, rationally valued metric $\eta : \mathcal{H}(X, G) \otimes \mathcal{H}(X, G) \to \mathbb{Q}$ defined by $\eta(v_1, v_2) = \int_{[IX]} v_1 \cup \sigma^* v_2$, which vanishes on G-homogeneous elements v_1 and v_2, unless they have inverse G-gradings.

Now, we want to figure out how to define the multiplication in stringy cohomology. We will consider the element S in the rational K-theory of IX, which is "half" of the normal bundle of IX in $G \times X$ (I'll discuss this later).

Consider the G-equivariant vector bundle $\mathcal{N} \to IX$ whose restriction to X^m, which we denote $\mathcal{N}_m$, is the normal bundle of $X^m \hookrightarrow X$ for all $m \in G$. Take $r = |m|$ for $m \in G$. Then $\mathcal{N}_m \to X^m$ is an $\langle m \rangle$-equivariant complex vector bundle, where $\langle m \rangle$ acts preserving the fibers of $\mathcal{N}_m$. We can therefore decompose $\mathcal{N}_m$ into eigenbundles

$$\mathcal{N}_m := \bigoplus_{k=0}^{r-1} \mathcal{N}_{m,k},$$

where m acts upon the fibers of $\mathcal{N}_{m,k}$ with eigenvalue ζ_r^k and $\zeta_r = \exp(2\pi i/r)$. This is analogous to eigenvalue decomposition of a matrix or linear operator. This is "K-theorized" by considering the negative portion as well:

Definition. Let $K(IX)$ denote the rational K-theory of complex vector bundles over IX. Consider $\mathcal{S}$ in $K(IX)$ whose restriction in $K(X^m)$ is

$$\mathcal{S}_m := S|_{X^m} := \bigoplus_{k=0}^{r-1} \frac{k}{r} \mathcal{N}_{m,k},$$

with $m \in G$ and $|m| = r$. We define the *age* of m to be $a(m) = \mathrm{rk}(\mathcal{S}_m)$. It is a rationally valued locally constant function on X^m, and can be considered as the sum of ranks of each eigenbundle (with the rational weight):

$$a(m) = \sum_{k=0}^{r-1} \frac{k}{r} \mathrm{rk}(\mathcal{N}_{m,k}).$$

Now we can consider $\mathcal{S}$ to be half of the normal bundle in the K-theoretic sense, that is, $\mathcal{S} \oplus \sigma^*\mathcal{S} = \mathcal{N}$. This comes from the fact that $\sigma^*\mathcal{N}_{m,k} = \mathcal{N}_{m^{-1},r-k}$ for all $k = 1, ..., r-1$. So, for all $m \in G$,

$$\sigma^*\mathcal{S}_{m^{-1}} = \bigoplus_{\ell=0}^{r-1} \frac{\ell}{r}\sigma^*\mathcal{N}_{m^{-1},\ell} = \bigoplus_{\ell=0}^{r-1} \frac{\ell}{r}\mathcal{N}_{m,r-1} = \bigoplus_{\ell=0}^{r-1} \frac{r-k}{r}\mathcal{N}_{m,k} = \mathcal{N}_m \ominus \mathcal{S}_m.$$

Consider now the set $G^{[3]} := \{(m_1, m_2, m_3) \in G^3 : m_1m_2m_3 = 1\}$, where $G \curvearrowright G^{[3]}$ via diagonal conjugation. Hence, we have that $G^{[3]} \times X$ inherits a diagonal G action.

Definition. The *double inertia manifold* of X, denoted IIX, is the G-submanifold of $G^{[3]} \times X$,

$$IIX := \bigsqcup_{m \in G^{[3]}} X^m,$$

where $\mathbf{m} = (m_1, m_2, m_3)$ and $X^{\mathbf{m}} = \{(\mathbf{m}, x) : \rho(m_1)x = \rho(m_2)x = \rho(m_3)x = x\}$. $X^{\mathbf{m}}$ is the submanifold of points in X simultaneously fixed by m_i.

Also, we define G-equivariant evaluation maps $e_i : IIX \to IX$ by $(\mathbf{m}, x) \mapsto (m_i, x)$. A *twisted evaluation map* is $\check{e}_i := \sigma \circ e_i$.

Let $\mathbf{N} \to IIX$ denote the G-equivariant complex vector bundle whose restriction $\mathbf{N}_{\mathbf{m}}$ to $X^{\mathbf{m}}$ is the normal bundle of the inclusion $X^{\mathbf{m}} \hookrightarrow X$. (This mirrors $\mathcal{N} \to IX$, but it is the "double inertia" version).

Definition. The *obstruction class* $\mathcal{R}$ in $K(IIX)$ is defined by

$$\mathcal{R} := \ominus\mathbf{N} \oplus \bigoplus_{i=1}^{3} e_i^*\mathcal{S}.$$

In particular, for all $\mathbf{m} \in G^{[3]}$, in $K(X^{\mathbf{m}})$, we have

$$\mathcal{R}(\mathbf{m}) = \ominus\mathbf{N}_{\mathbf{m}} \oplus \bigoplus_{i=1}^{3} \mathcal{S}_{m_i}|_{X^{\mathbf{m}}}.$$

Definition. Take X to be an almost complex manifold with the action of a finite group G. The *multiplication on stringy cohomology* $\mathcal{H}(X, G) \otimes \mathcal{H}(X, G) \to \mathcal{H}(X, G)$ taking $v_1 \otimes v_2 \mapsto v_1 \star v_2$ is defined by

$$v_1 \star v_2 := \check{e}_{3*}(e_1^*v_1 \cup e_2^*v_2 \cup \epsilon(\mathcal{R})),$$

with $\epsilon(\mathcal{R})$ being the Euler class of $\mathcal{R}$.

3.2 Extracting cohomology products on symplectic manifolds

This part of the talk will shift towards certain applications of Gromov-Witten theory to extract exotic cohomology products on manifolds, and further, orbifolds.

Let (X, ω) be a closed symplectic manifold equipped with a Poincare pairing $\langle \cdot, \cdot \rangle$, and consider its cohomology $H^\bullet(X)$. Through the approaches of Gromov-Witten theory and the theory of pseudo-holomorphic curves, we can study a moduli space of holomorphic maps from $\mathbb{C}P^1$ (which we can geometrically think of as a sphere) to X, with three marked points.

Consider now three cohomology classes on the target X, and integrate the pullbacks of the evaluation maps at these classes over the moduli space:

$$A(\alpha, \beta, \gamma) = \int_{\mathcal{M}} \mathrm{ev}_1^* \alpha \wedge \mathrm{ev}_2^* \beta \wedge \mathrm{ev}_3^* \gamma.$$

It turns out that this 3-tensor defines a product via the identification $A(\alpha, \beta, \gamma) = \langle \alpha \star \beta, \gamma \rangle$. This product ($\star$) is a deformation of the standard cup product on $H^\bullet(X)$; it is the quantum product on $H^\bullet(X)$.

As a side note, we can think of Gromov-Witten invariants as being the number of (in this case, degree 0) holomorphic spheres in X passing through generic cycles which are Poincare-dual to α, β, γ.

Pairing the cohomology with this product and the Poincare pairing, we retrieve a Frobenius algebra. This algebra is defined over formal power series $\mathbb{C}[\![q]\!]$; if we were to set $q = 0$, that is, we study constant or zero energy curves, we can retrieve the standard cohomology of X.

If we instead consider a Lagrangian submanifold and repeat the same process, we can again retrieve the standard cohomology after deforming the cup product and restricting to constant curves.

3.3 Extension to orbifolds

A natural extension of this phenomenon would be to symplectic orbifolds. If we consider a symplectic orbifold $\mathcal{X}$, we must consider the cohomology of its inertia orbifold, $I\mathcal{X}$. When we repeat the same constructions as before, we must note that the marked points on the spheres can be orbifold points (or singularities). Following with the cup product deformation and study the Frobenius algebra when restricted to constant curves, it turns out that the product which we extract is the Chen-Ruan (or stringy) product.

3.3.1 Dihedral twisted sectors; Conjecture 1

The appearance of this stringy product is a powerful result, as this is further structure which is not immediately retrievable. A desirable result would be to extend this to Lagrangian "suborbifolds," like we have for manifolds.

In [COW23], the notion of a *dihedral twisted sector* $I_{\mathcal{X}}L$ is introduced. This can be thought of, as the notation perhaps suggests, that this is like an inertia orbifold of L with respect to $\mathcal{X}$. Using the construction of the stringy product as motivation, we conjecture the following:

Conjecture 1. We can apply analogous ideas as in the construction of the stringy product to the cohomology $H^\bullet(I_{\mathcal{X}}L)$. Doing such a thing would require the use of some "double" dihedral twisted sector, $II_{\mathcal{X}}L$, whose construction is inspired by the construction of $I_{\mathcal{X}}L$ in [COW23].

In order to resolve this conjecture, I plan on applying the techniques found in [JKK06] in order to construct the Chow ring and K-theory for some newly constructed $II_{\mathcal{X}}L$. Further, to show the associativity of the conjectured product, [JKK06] introduces a technique to do such which is independent of Gromov-Witten theory (and is much more succinct). At least in the case for the stringy product, associativity follows from the proof of associativity for quantum cohomology, which utilizes an argument on the moduli space of genus zero curves with four marked points.

3.3.2 More general structure at play; Conjecture 2

The ability to do these constructions is known to hold in the case of symplectic manifolds and Lagrangian submanifolds. However, note the following theorem of Gromov (I think):

Theorem 4 (Gromov?)**.** Symplectic manifolds always have almost complex structures which are compatible with the symplectic structure. Further, there exists an infinite-dimensional space of compatible almost-complex structures; this space is contractible, that is, all of the almost-complex structures are homotopy equivalent.

This means that in the case of manifolds, up to homotopy, the crucial data which allows one to apply the ideas of Gromov-Witten theory to extract deformed cohomology products is dependent on the almost-complex structure of the manifold. Further, it is known that a Lagrangian submanifold is totally-real with respect to the almost-complex structure, and the same story holds in the Lagrangian case. These motivate the following conjecture:

Conjecture 2. If we consider a symplectic orbifold $\mathcal{X}$ and a Lagrangian suborbifold (dihedral twisted sector) $I_{\mathcal{X}}L$, then the data required to extract the stringy product or the product from Conjecture 1 is dependent on some almost-complex/totally-real structure instead of symplectic/Lagrangian structure.

4 Bibliography

References

[AD13] Michèle Audin and Mihai Damien. *Morse Theory and Floer Homology.* Springer London, 2013.

[CH14] Cheol-Hyun Cho and Hansol Hong. Orbifold Morse–Smale–Witten complexes. *International Journal of Mathematics*, 25(05):1450040, 2014.

[COW23] Bohui Chen, Kaoru Ono, and Bai-Ling Wang. Twisted sectors for lagrangian floer theory on symplectic orbifolds, Aug 2023.

[CR04] Weimin Chen and Yongbin Ruan. A new cohomology theory of orbifold. *Communications in Mathematical Physics*, 248(1):1–31, May 2004.

[FOOO15] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Kuranishi structure, pseudo-holomorphic curve, and virtual fundamental chain: Part 1, 2015.

[JKK06] Tyler J. Jarvis, Ralph Kaufmann, and Takashi Kimura. Stringy K-theory and the Chern character, Nov 2006.

[Kim07] Takashi Kimura. Orbifold cohomology reloaded, Nov 2007.

[Was69] Arthur G. Wasserman. Equivariant differential topology. *Topology*, 8(2):127–150, 1969.