Abstract Algebra II Homework 6

Carson Connard

Section 11.3

1 Let V be a finite dimensional vector space. Prove that the map $\varphi \mapsto \varphi^*$ in Theorem 20 gives a ring isomorphism of $\mathrm{End}(V)$ with $\mathrm{End}(V^*)$.

Proof. First, recall that $\dim V = \dim V^*$ implies that $\dim \mathrm{End}(V) = \dim \mathrm{End}(V^*)$. So, it suffices to show that the map in the problem statement, which we can call ψ, is a linear transformation and an injective ring homomorphism. In order to show linearity, we see $\psi(\varphi_1 + \alpha\varphi_2) = \psi(\varphi_1) + \alpha\psi(\varphi_2)$. Indeed,

$$\begin{aligned} \psi(\varphi_1 + \alpha\varphi_2) &= (\varphi_1 + \alpha\varphi_2)^*(f) \quad \forall f \in V^* \\ \implies (\varphi_1 + \alpha\varphi_2)^*(f)(v) &= \varphi_1^*(f)(v) + (\alpha\varphi_2^*)(f)(v) \quad \forall v \in V \\ \implies f \circ (\varphi_1 + \alpha\varphi_2)(v) &= f \circ \varphi_1^*(v) + f \circ \alpha\varphi_2^*(v), \end{aligned}$$

which gives the result, as well as the addition preservation of a ring homomorphism. We also want to show that our second operation, composition, is preserved. Indeed,

$$\begin{aligned} \psi(\varphi_1 \circ \varphi_2)(f) &= f \circ (\varphi_1 \circ \varphi_2) \\ &= (f \circ \varphi_1) \circ \varphi_2 \\ &= \psi(\varphi_2)(f \circ \varphi_1) \\ &= \psi(\varphi_2)(\psi(\varphi_1)(f)) \\ &= (\psi(\varphi_2) \circ \psi(\varphi_1))(f). \end{aligned}$$

Therefore, we have that composition is preserved, and ψ is a ring homomorphism. It suffices to show now that ψ is injective, as injective linear maps are automatically isomorphisms between vector spaces of the same dimension, which is the case as detailed earlier. Indeed, $\psi(\varphi) = 0$ implies that $f \circ \varphi = 0$ for all $f \in \mathrm{End}(V^*)$, implying that $\varphi = 0$. Thus $\ker \psi = 0$ and so ψ is injective. $\square$

2 Let V be the collection of polynomials with coefficients in $\mathbb{Q}$ in the variable x of degree at most 5 with $B = \{1, x, x^2, ..., x^5\}$ as basis. Prove that the following are elements of the dual space of V and express them as linear combinations of the dual basis:

(a) $E : V \to \mathbb{Q}$ defined by $E(p(x)) = p(3)$ (i.e. evaluation at $x = 3$)

(b) $\varphi : V \to \mathbb{Q}$ defined by $\varphi(p(x)) = \int_0^1 p(t)\,dt$

(c) $\varphi : V \to \mathbb{Q}$ defined by $\varphi(p(x)) = \int_0^1 t^2 p(t)\,dt$

(d) $\varphi : V \to \mathbb{Q}$ defined by $\varphi(p(x)) = p'(5)$ where p' denotes the usual derivative of p

Proof. Consider the following basis for the dual space:

$$v_i^*(v_j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$$

where v_i is the ith element of the basis B.

(a) Let $\alpha p_1 + \beta p_2 \in V$. Notice $E(\alpha p_1 + \beta p_2) = (\alpha p_1, \beta p_2)(3) = \alpha p_1(3) + \beta p_2(3) = \alpha E(p_1) + \beta E(p_2)$. Since E is linear, we have that $E \in V^*$. Note that $E(x^i) = 3^i$, and so in the dual basis, we have $E = \sum_{i=0}^{5} 3^i v_i^*$.

(b) Let $\alpha p_1 + \beta p_2 \in V$. Notice

$$\varphi(\alpha p_1 + \beta p_2) = \int_0^1 \alpha p_1 + \beta p_2 \, dt = \alpha \int_0^1 p_1 \, dt + \beta \int_0^1 p_2 \, dt = \alpha\varphi(p_1) + \beta\varphi(p_2).$$

Since φ is linear, $\varphi \in V^*$. Note that $\varphi(x^i) = \int_0^1 x^i \, dx = 1/(i+1)$, so in the dual basis, $\varphi = \sum_{i=0}^{5} v_i^*/(i+1)$.

(c) Let $\alpha p_1 + \beta p_2 \in V$. Notice

$$\varphi(\alpha p_1 + \beta p_2) = \int_0^1 t^2(\alpha p_1 + \beta p_2) \, dt = \alpha \int_0^1 t^2 p_1 \, dt + \beta \int_0^1 t^2 p_2 \, dt = \alpha\varphi(p_1) + \beta\varphi(p_2).$$

Since φ is linear, we have that $\varphi \in V^*$. Note that $\varphi(x^i) = \int_0^1 x^2(x^i) \, dx = 1/(i+3)$, so in the dual basis, $\varphi = \sum_{i=0}^{5} v_i^*/(i+3)$.

(d) Let $\alpha p_1 + \beta p_2 \in V$. Notice $\varphi(\alpha p_1 + \beta p_2) = (\alpha p_1 + \beta p_2)'(5) = \alpha p_1'(5) + \beta p_2'(5) = \alpha\varphi(p_1) + \beta\varphi(p_2)$. Since φ is linear, $\varphi \in V^*$. Note that $\varphi(x^i) = (x^i)'(5) = i \cdot 5^{i-1}$, so in the dual basis, $\varphi = \sum_{i=0}^{5} i \cdot 5^{i-1} v_i^*$.

$\square$

3 Let S be any subset of V^* for some finite dimensional space V. Define $\mathrm{Ann}(S) = \{v \in V : f(v) = 0 \, \forall f \in S\}$.

(a) Prove that $\mathrm{Ann}(S)$ is a subspace of V.

(b) Let W_1, W_2 be subspaces of V^*. Prove that $\mathrm{Ann}(W_1 + W_2) = \mathrm{Ann}(W_1) \cap \mathrm{Ann}(W_2)$ and $\mathrm{Ann}(W_1 \cap W_2) = \mathrm{Ann}(W_1) + \mathrm{Ann}(W_2)$.

(c) Let W_1 and W_2 be subspaces of V^*. Prove that $W_1 = W_2$ if and only if $\mathrm{Ann}(W_1) = \mathrm{Ann}(W_2)$.

(d) Prove that the annihilator of S is the same as the annihilator of the subspace of V^* spanned by S.

(e) Assume that V is finite dimensional with basis $\{v_1, ..., v_n\}$. Prove that if $S = \{v_1^*, ..., v_k^*\}$ for some $k \leq n$, then $\mathrm{Ann}(S)$ is the subspace spanned by $\{v_{k+1}, ..., v_n\}$.

(f) Assume V is finite dimensional. Prove that if W^* is any subspace of V^*, then $\dim \mathrm{Ann}(W^*) = \dim V - \dim W^*$.

Proof. We can use certain parts to imply others, so we do this out of order. First, we see (a) because if $v_1, v_2 \in \mathrm{Ann}(S)$, then $f(v_1 + \alpha v_2) = f(v_1) + \alpha f(v_2) = 0$ for all $f \in S$. Since $0 \in \mathrm{Ann}(S)$ trivially, we are done.

Part (d) is obvious, as one just considers linear combinations of elements of S. Certainly this preserves the annihilator.

Next, we show (f). Consider S to be a finite set of linearly independent linear functionals $\{v_1^*, ..., v_k^*\}$. Define $T : V \to F^k$ by $T(v) = (v_1^*(v), ..., v_k^*(v))$. We see that we can write $\mathrm{Ann}(S) = \ker T$, as covered in class. If A is the reduced $k \times n$ matrix of T from basis $B_V = \{e_1, ..., e_n\}$ to $B_{F^k} = \{e_1, ..., e_k\}$, we see that the linear functionals are lineraly independent rows. Therefore, the kernel is $n - k = \dim V - \dim \mathrm{span}(S)$ by rank-nullity. This along with (d) gives the result.

When $S \subset V$, define $\mathrm{Ann}(S)^* = \{v^* \in V^* : v^*(S) = 0 \, \forall s \in S\}$ to be the dual to the annihilator defined in the problem statement. So, as used above, $\dim \mathrm{Ann}(S)^* = \dim V^* - \dim \mathrm{span}(S)$. When S is a subset of either V or V^*, we know that $\mathrm{Ann}(\mathrm{Ann}(S)^*) = S$ since $\supset$ is trivial and $\subset$ is given by restricting dimension.

Finally to show (b), we start with the $\supset$ direction. Let $v \in \mathrm{Ann}(W_1) \cap \mathrm{Ann}(W_2)$. Then we have that $(w_1 + w_2)(v) = w_1(v) + w_2(v) = 0$ as desired. For $\subset$, let $v \in \mathrm{Ann}(W_1 + W_2)$. Then we see $w_1(v) = (w_1 + 0)(v) = 0$ and similar for w_2. Then we have

$$\begin{aligned} \mathrm{Ann}(W_1 \cap W_2) = \mathrm{Ann}(W_1) + \mathrm{Ann}(W_2) &\iff \mathrm{Ann}(\mathrm{Ann}(W_1 \cap W_2)) = \mathrm{Ann}(\mathrm{Ann}(W_1) + \mathrm{Ann}(W_2)) \\ &\iff W_1 \cap W_2 = \mathrm{Ann}(\mathrm{Ann}(W_1)) \cap \mathrm{Ann}(\mathrm{Ann}(W_2)). \end{aligned}$$

□

4 If V is infinite dimensional with basis $\mathcal{A}$, prove that $\mathcal{A}^* = \{v^* : v \in \mathcal{A}\}$ does *not* span V^*.

Proof. Let $\varphi : V \to F$ be defined as $\varphi(v_i) = 1$ for all $v_i \in \mathcal{A}$. Then $\varphi = \sum_{i \in \mathcal{A}} v_i^*$ can only have finitely many nonzero terms. That is, for some finite $\mathcal{A}' \subset \mathcal{A}$, we have $\varphi = \sum_{i \in \mathcal{A}'} v_i^*$. This then implies that $\varphi(x) = 0$ when $x \notin \mathcal{A}$, and so $\varphi \notin \mathrm{span}\mathcal{A}^*$. □

Section 11.4

1 Formulate and prove the cofactor expansion formula along the jth column of a square matrix A.

Solution. Take F to be a field and A to be a square matrix with values in F. Let $A = [\alpha_{ij}]$. If

$$A = \begin{bmatrix} A_{11} & X_1 & A_{12} \\ Y_1 & \alpha_{ij} & Y_2 \\ A_{21} & X_2 & A_{22} \end{bmatrix},$$

then if A_{11} is of dimension $(i-1) \times (j-1)$, then the (i, j) minor of A is

$$A_{ij} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Recall that the cofactor expansion formular for $\det A$ along the ith row is given as $\det A = \sum_{j=1}^{n} (-1)^{i+j} \alpha_{ij} \det A_{ij}$. We claim that the same is for along the jth column, but indexed by i. We can prove this by appealing to the transpose:

$$\begin{aligned} \det(A) &= \det(A^T) \\ &= \sum_{j=1}^{n} (-1)^{i+j} \alpha_{ji} \det((A^T)_{ji}) \\ &= \sum_{j=1}^{n} (-1)^{i+j} \alpha_{ji} \det((A_{ji})^T) \\ &= \sum_{j=1}^{n} (-1)^{i+j} \alpha_{ji} \det(A_{ji}) \\ &= \sum_{i=1}^{n} (-1)^{i+j} \alpha_{ij} \det(A_{ij}). \end{aligned}$$

2 Let F be a field and let $A_1, A_2, ..., A_n$ be (column) vectors in F^n. Form the matrix A whose ith column is A_i. Prove that these vectors form a basis of F^n if and only if $\det A \neq 0$.

Proof. Suppose that the vectors form a basis. Then the A_i are all linearly independent, and so the linear transformation represented by A is injective. Since A is a square matrix, we have that A must be invertible, and therefore we have that $\det A \neq 0$.

Now suppose that $\det A \neq 0$. Then we have that $Ax \neq 0$ unless $x = 0$. If A is comprised of column vectors $A_1, ..., A_n$, then we can express vector multiplication Ax as $x_1 A_1 + x_2 A_2 + \cdots + x_n A_n$. Since $Ax \neq 0$, we know that this linear combination is also nonzero. Therefore the A_i are linearly independent. □

3 Let R be any commutative ring with identity, let V be an R-module and let $x_1, x_2, ..., x_n \in V$. Assume that for some $A \in M_{n\times n}(R)$,

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0.$$

Prove that $(\det A)x_i = 0$ for all $i \in \{1, 2, ..., n\}$.

Proof. We have that there is a system

$$\begin{cases} \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n = 0 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n = 0 \\ \vdots \\ \alpha_{n1}x_1 + \alpha_{n2}x_2 + \cdots + \alpha_{nn}x_n = 0. \end{cases}$$

We can construct the cofactor formula for the determinant along the first column. Indeed, we have

$$\begin{aligned} (\det A)x_1 + \sum_{j=2}^{n} \left[\alpha_{1j} \det A_{11} + \sum_{k=2}^{n} (-1)^{k+1} \alpha_{kj} \det A_{k1} \right] x_j = 0 \\ \implies (\det A)x_1 + \sum_{j=2}^{n} \sum_{k=1}^{n} (-1)^{k+1} \alpha_{kj} \det A_{k1} \cdot x_j = 0 \end{aligned}$$

For each j, take B_j to be A with the first column replaced with the jth column. Therefore we have

$$0 = \det B_j = \sum_{k=1}^{n} (-1)^{k+1} \beta_{k1} \det B_{k1} = \sum_{k=1}^{n} (-1)^{k+1} \alpha_{kj} \det A_{k1}.$$

So, the cofactor expansion from above collapses to $\det(A)x_1 = 0$. So, interchanging arbitrary x_i with x_1 negates the determinant, we have that $-(\det A)x_i = 0 = (\det A)x_i$ as desired. $\square$

4 Let A be an $n \times n$ matrix.

(a) Prove that the elementary row operations have the following effect on determinants:

(i) Interchanging two rows changes the sign of the determinant

(ii) Adding a multiple of one row to another does not alter the determinant

(iii) Multiplying any row by a nonzero element $u \in F$ multiplies the determinant by u

Proof. (a) We will induct over n. Indeed, for $n = 2$, the result holds, as $\det A = ad - bc$, but if we swap the rows, we have that $\det \tilde{A} = cb - da = -(ad - bc)$. Assume that this works up to $(n-1) \times (n-1)$ dimensional matrices. We show that this works for an $n \times n$ matrix A now. Suppose that the rows p and $p+1$ are swapped, forming a matrix B, where $1 \leq p < p+1 \leq n$. Indeed, notice that $a_{ij} = b_{ij}$ for all $i \neq p, p+1$. Since B_{ij} is obtained from A_{ij}, we have that the induction hypothesis gives $\det B_{ij} = -\det A_{ij}$. We can now compute $\det B$ by cofactor expansion as follows:

$$\begin{aligned} \det B &= b_{11}(-1)^{1+1} \det B_{11} + \cdots + b_{i1}(-1)^{i+1} \det B_{i1} + \cdots + b_{p1}(-1)^{p+1} \det B_{p1} \\ &\quad + b_{(p+1)1}(-1)^{(p+1)+1} \det B_{(p+1)1} + \cdots + b_{n1}(-1)^{n+1} \det B_{n1} \\ &= a_{11}(-1)^{1+1}(-1) \det A_{11} + \cdots + a_{i1}(-1)^{i+1}(-1) \det A_{i1} + \cdots + a_{(p+1)1}(-1)^{(p+1)+1} \det A_{(p+1)1} \\ &\quad + a_{p1}(-1)^{(p+1)+1} \det A_{p1} + \cdots + a_{n1}(-1)^{n+1}(-1) \det A_{n1} \\ &= a_{11}(-1)^{1+1}(-1) \det A_{11} + \cdots + a_{i1}(-1)^{i+1}(-1) \det A_{i1} + \cdots + a_{(p+1)1}(-1)^{p+1}(-1) \det A_{(p+1)1} \end{aligned}$$

$$
\begin{aligned}
&\quad + a_{p1}(-1)^{p+1}(-1)\det A_{p1} + a_{n1}(-1)^{n+1}(-1)\det A_{n1} \\
&= (-1\big(a_{11}(-1)^{1+1}\det A_{11} + \ldots + a_{i1}(-1)^{i+1}\det A_{i1} + \cdots \\
&\quad + a_{(p+1)1}(-1)^{(p+1)+1}\det A_{(p+1)1} + a_{p1}(-1)^{p+1}\det A_{p1} + \cdots + a_{n1}(-1)^{n+1}\det A_{n1}\big) \\
&= -\det A
\end{aligned}
$$

If we were to instead swap two non adjacent rows (i.e. not p and $p+1$), then we would equivalently have an odd number of adjacent row changes and so the result still holds.

(b) Suppose the matrix B is obtained by adding u times row p to row q. We form A' by replacing the qth row of A with u times the pth row. We notice then that without the qth row, the matrices A, A', B are identical. The qth row of B, however, is the sum of the qth rows of A and A'. So, we have that $\det B = \det A + \det A'$ by properties of the determinant. But since A' contains a scalar multiple of another row, we know that $\det A' = 0$. Therefore $\det B = \det A$.

(c) We induct on n again. We see that for $n = 2$, if we multiply the top row by u, we have $\det A = (ua)d - (ub)c = u(ad - bc) = u\det A$. Suppose this works up to dimension $(n-1) \times (n-1)$. We aim to show it works for $n \times n$ dimensional matrices. Indeed, suppose A is the base matrix and B is obtained by multiplying teh pth row by some scalar u. Then we can compute the determinant of B as follows:

$$
\begin{aligned}
\det B &= a_{11}(-1)^{1+1}\det B_{11} + \cdots + ua_{p1}(-1)^{p+1}\det B_{p1} + \cdots + a_{n1}(-1)^{n+1}\det B_{n1} \\
&= ua_{11}(-1)^{1+1}\det A_{11} + \cdots + ua_{p1}(-1)^{p+1}\det A_{p1} + \cdots + ua_{n1}(-1)^{n+1}\det A_{n1} \\
&= u\big(a_{11}(-1)^{1+1}\det A_{11} + \cdots + a_{p1}(-1)^{p+1}\det A_{p1} + \cdots + a_{n1}(-1)^{n+1}\det A_{n1}\big) \\
&= u\det A.
\end{aligned}
$$

□

(b) Prove that $\det A$ is nonzero if and only if A is row equivalent to the $n \times n$ identity matrix. Suppose A can be row reduced to the identity matrix using a total of s row interchanges as in (i) and by multiplying rows by the nonzero elements $u_1, u_2, ..., u_t$ as in (iii). Prove that $\det A = (-1)^s(u_1u_2\cdots u_y)^{-1}$.

Proof. Equivalently, we aim to show that if A is an $n \times n$ matrix in reduced row echelon form such that $\det A \neq 0$, then A is the identity matrix. Indeed, we know that the leading coefficient in the first row is in the first column, so $a_{11} = 1$ and all $a_{1k} = 0$ for $1 < k \leq n$. Suppose that $a_{kk} = 1$ for all $1 \leq k < n$, and that all entries above and below these diagonal values are 0. That is, we can express A as a block decomposition

$$
A = \begin{bmatrix} I_k & M \\ 0 & A' \end{bmatrix},
$$

where I_k is the $k \times k$ dimensional identity matrix, and A' is a reduced $(n-k) \times (n-k)$ dimensional matrix. So, again we must have that $a'_{11} = 1$, and so $a_{(k+1)(k+1)} = 1$ with all 0s above and below. Inducting over k then gives that $a_{kk} = 1$ for all k, and that all entries above and below are 0. Therefore we have that A must be equivalent to the identity. □

Section 11.5

4 Prove that $m \wedge n_1 \wedge n_2 \wedge \cdots \wedge n_k = (-1)^k(n_1 \wedge n_2 \wedge \cdots \wedge n_k \wedge m)$. In particular, $x \wedge (y \wedge z) = (y \wedge z) \wedge x$ for all $x, y, z \in M$.

Proof. We prove by induction. First, $k = 0$ is trivial, and $k = 1$ gives $m \wedge n_1 = (-1)^1(n_1 \wedge m) = m \wedge n_1$ as well. So, suppose that the result holds up to $k - 1$, and we want to show that it holds for k. Indeed,

$$
m \wedge n_1 \wedge n_2 \wedge \cdots \wedge n_{k-1} \wedge n_k = (m \wedge n_1 \wedge n_2 \wedge \cdots \wedge n_{k-1}) \wedge n_k
$$

$$= (-1)^{k-1}(n_1 \wedge n_2 \wedge \cdots \wedge n_{k-1} \wedge m) \wedge n_k$$
$$= (-1)^k (n_1 \wedge n_2 \wedge \cdots \wedge n_{k-1} \wedge n_k).$$

□

5 Prove that if M is a free R-module of rank n then $\bigwedge^i(M)$ is a free R-module of rank $\binom{n}{i}$ for $i = 0, 1, 2, \cdots$.

Proof. Let $E = \{e_i\}_{i=1,\dots,n}$ be a generating set for M. Recall that the kth tensor power is free with the free generating set $\{e_{i1} \otimes \cdots \otimes e_{ik} : i : k \to n\}$. We can construct a generating set for $\bigwedge^i(M)$ now. We know that this is generated by simple tensors of the form $m_1 \wedge \cdots \wedge m_i$, and let us assume that each m_i is generated from some $e_{i1} \wedge \cdots \wedge e_{ik}$. We show that these elements of the generating set for $\bigwedge^i(M)$ are distinct. Take some increasing function $\lambda : k \to n$ and define $\phi_\lambda : \mathcal{T}^k(M) \to R$. We see that $A^k(M)$ is contained in ϕ_λ, and so there exists an induced homomorphism $\bigwedge^i(M) \to R$. Also $\phi_\lambda = 0$ on all $e_{i1} \wedge \cdots \wedge e_{ik}$ except for the one which has indices given by λ. So, the elements are distinct. There are $\binom{n}{k}$ wants to choose the indicies i_j, and so there exists a generating set B of $\bigwedge^i(M)$ which is of order $\binom{n}{k}$. Since any nony non trivial linear combination in $\bigwedge^i(M)$ induces a nontrivial linear combination in the free basis on $\mathcal{T}^k(M)$, we have that the basis is free. This gives that $\bigwedge^i(M)$ is a free R-module of rank $\binom{n}{i}$. I have no clue... □

8 Let R be an integral domain and let F be its field of fractions.

(a) Considering F as an R-module, prove that $\bigwedge^2 F = 0$.

Proof. Let $(a/b) \otimes (c/d) \in \mathcal{T}^2(F)$ be a simple tensor. Then

$$\frac{a}{b} \otimes \frac{c}{d} = \frac{ad}{bd} \otimes \frac{cb}{bd} = abcd\left(\frac{1}{bd} \otimes \frac{1}{bd}\right) \in A^2(F).$$

Therefore, we have that $(a/b) \otimes (c/d) = 0$ in $\bigwedge^2(F)$. □

(b) Let I be any R-submodule of F (e.g. any ideal in R). Prove that $\bigwedge^i I$ is a torsion R-module for $i \geq 2$ (i.e. for every $x \in \bigwedge^i I$ there is some nonzero $r \in R$ with $rx = 0$).

Proof. Take the following to be nonzero:

$$\frac{a_1}{b_1} \wedge \frac{a_2}{b_2} \wedge \cdots \wedge \frac{a_k}{b_k} \in \bigwedge^k(I).$$

Then we have that $a_i, b_i \neq 0$. Take now $a_1 a_2 b_1 b_2$ to be zero in R, so we have

$$a_1 a_2 b_1 b_2 \left(\frac{a_1}{b_1} \wedge \frac{a_2}{b_2} \wedge \cdots \wedge \frac{a_k}{b_k}\right) = \frac{a_1 a_2}{1} \wedge \frac{a_1 a_2}{1} \wedge \cdots \wedge \frac{a_k}{b_k} = 0.$$

Therefore every element of $\bigwedge^k(I)$ is torsion and therefore $\bigwedge^k(I)$ is torsion as an R-module. □

(c) Give an example of an integral domain R and an R-module I in F with $\bigwedge^i I \neq 0$ for every $i \geq 0$ (cf. the example following Cor. 37).

Solution. Take $R = \mathbb{Z}[x_1, \cdots]$ and $I = (x_1, \cdots)$. It suffices to find some alternating n-linear map $\phi_n : I \times \cdots \times I \to \mathbb{Z}$ (n factors) such that $\phi_n(x_1, ..., x_n) = 1$ for all n. So, define ϕ_n as

$$\phi_n\left(\sum_i a_{1i}x_i, \cdots, \sum_i a_{ni}x_i\right) = \det(a'_{ij})_{ij},$$

where a'_{ij} is the constant term of a_{ij}, and $1 \leq i, j \leq n$. Notice that if two of these sums equal each other, then they share the constant term. So, $(a_{ij})_{ij}$ is uniquely determined and therefore ϕ_n is well defined. The map is alternating and multilinear on the components of $I \times \cdots \times I$, as det is alternating and multilinear on matrix rows. We see $(x_1, ..., x_n)$ represents the identity in this case, and therefore $\phi_n(x_1, ..., x_n) = 1$.

11 Prove that the image of Alt_k is the unique largest subspace of $\mathcal{T}^k(V)$ on which each permutation σ in the symmetric group S_k acts as multiplication by the scalar $\epsilon(\sigma)$. ???