Real Analysis Homework 3 (Midterm)

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Throughout, we will consider X to be a nonempty set, $\mathcal{M}$ to be a σ-algebra on X, and μ to be a measure on X.

Problem 1

(a) Let $(Z, \mathcal{N})$ be a measurable space, $f : X \to Z$ a $(\mathcal{M}, \mathcal{N})$-measurable function, and $A \subset X$. Prove that $f\mid_A : A \to Z$ is $(\mathcal{M}_A, \mathcal{N})$-measurable, where $\mathcal{M}_A = \{A \cap M : M \in \mathcal{M}\}$.

Proof. Let us consider some $Y \subset \mathcal{N}$. Since the preimage of a measurable set is measurable, we have that $f^{-1}(Y) \in \mathcal{M}$. Then if we restrict the preimage to A, that is $f^{-1}\mid_A (Y) = f^{-1}(Y) \cap A$. By definition, we have that $f^{-1}\mid_A (Y) \in \mathcal{M}_A$ and therefore we have that $f\mid_A$ is indeed $\mathcal{M}_A$-measurable. $\square$

(b) Prove that if $f : X \to \mathbb{R}$ is such that $\{x \in X : f(x) \geq r\} \in \mathcal{M}$ for all $r \in \mathbb{Q}$ then f is $\mathcal{M}$-measurable.

Proof. We are given that $\{x \in X : f(x) \geq r, r \in \mathbb{Q}\} \in \mathcal{M}$, and to show our result we aim to show that this holds for not just all rational r, but all real r. So, let us consider some $a \in \mathbb{R}$, and take the sequence $\{r_n\}_{n=1}^{\infty} \subset \mathbb{Q}$, which is an enumeration of the rationals $r_n > a$. We aim to show that

$$\{x \in X : f(x) \geq a \in \mathbb{R}\} = \bigcup_{n=1}^{\infty} \{x \in X : f(x) \geq r_n\}.$$

The inclusion ($\supset$) is trivial, due to how we defined our r_n's to be greater than a. For the other inclusion ($\subset$), we know that there exists some subsequence $\{r_{n_i}\}$ of $\{r_n\}$ which converges to a since $\mathbb{Q}$ is dense in $\mathbb{R}$. So, we have equality. Since we know that each $\{x \in X : f(x) \geq r_n\}$ is measurable, certainly a countable union of them is also measurable. So, we have shown that $\{x \in X : f(x) \geq a \in \mathbb{R}\} \in \mathcal{M}$. $\square$

(c) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is monotone then f is Borel measurable.

Proof. Take $a \in \mathbb{R}$, and without loss of generality, take f to be monotonically increasing. Then the intervals $\{x : f(x) > a\}$ are either of the form (x, ∞) or of the form $[x, \infty)$. Both of these are open sets in $\mathcal{B}_{\mathbb{R}}$, so indeed f is Borel measurable. $\square$

Problem 2

(a) Let $f : X \to \overline{\mathbb{R}}$ and $g : X \to \overline{\mathbb{R}}$ be $\mathcal{M}$-measurable. Show that $\{x \in X : f(x) = g(x)\}$ and $\{x \in X : f(x) < g(x)\}$ are measurable sets.

Proof. Let us begin with the set $\{x \in X : f(x) = g(x)\}$. Certainly, we have

$$\begin{aligned}\{x \in X : f(x) = g(x)\} = \{x \in X : f(x) = g(x) = \infty\} &\cup \{x \in X : f(x) = g(x) = -\infty\} \\ &\cup \{x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}) : f(x) - g(x) = 0\}\end{aligned}$$

Notice that the first two subsets are indeed contained in $\mathcal{M}$, since both $\{\pm\infty\} \in \mathcal{B}_{\overline{\mathbb{R}}}$. So, it suffices to show that $\{x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}) : f(x) - g(x) = 0\} \in \mathcal{M}$. To do so, first define $W := f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R})$. Then we can rewrite $\{x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}) : f(x) - g(x) = 0\}$ as $(g_W - f_W)^{-1}(\{0\})$.

Notice that $f_W : W \to \mathbb{R}$ and $g_W : W \to \mathbb{R}$, which by Problem 1 gives that f_W, g_W are $\mathcal{M}_W$-measurable functions, where $\mathcal{M}_W = \{W \cap M : M \in \mathcal{M}\}$. Since the difference of two measurable functions is measurable, we have that $(g_W - f_W)^{-1}(\{0\}) \in \mathcal{M}_W \subset \mathcal{M}$ as desired. $\diamond$

Let us consider the set $\{x \in X : f(x) < g(x)\}$. Certainly, we have

$$\{x \in X : f(x) < g(x)\} = \{x \in X : f(x) < \infty,\ g(x) = \infty\} \cup \{x \in X : f(x) = -\infty,\ g(x) > -\infty\} \cup \{x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}) : f(x) - g(x) < 0\}$$

Notice that the first two subsets can be rewritten as $f^{-1}([-\infty, \infty))$ and $g^{-1}((-\infty, \infty])$ respectively. Since $[-\infty, \infty), (-\infty, \infty] \in \mathcal{B}_{\overline{\mathbb{R}}}$, we have that their preimages are in $\mathcal{M}$. Therefore the first two subsets are measurable. So, it suffices to show that $\{x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}) : f(x) - g(x) < 0\} \in \mathcal{M}$. Let us again define $W := f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R})$, so that we can write

$$\{x \in W : f(x) - g(x) < 0\} = \{x \in W : -\infty < f(x) - g(x) < 0\} = (f_W - g_W)^{-1}((-\infty, 0)).$$

Again by Problem 1, we see that f_W and g_W are both $\mathcal{M}_W$-measurable. Since the difference of two measurable functions is measurable, we have that $f_W - g_W$ is $\mathcal{M}_W$-measurable, and since $(-\infty, 0) \in \mathcal{B}_{\overline{\mathbb{R}}}$, we have that indeed $(f_W - g_W)^{-1}((-\infty, 0)) \in \mathcal{M}_W \subset \mathcal{M}$ as desired. □

(b) Let $f_n : X \to \overline{\mathbb{R}}$ be $\mathcal{M}$-measurable for all $n \in \mathbb{N}$. Show that the sets $\{x \in X : \lim_{n\to\infty} f_n(x)$ exists as a value in $\mathbb{R}\}$ and $\{x \in X : \lim_{n\to\infty} f_n(x)$ exists as a value in $\overline{\mathbb{R}}\}$ are measurable.

Proof. For the first set, let us define $f(x) := \limsup_n f_n(x)$ and $g(x) := \liminf_n f_n(x)$. Certainly, if there exists a limit in $\overline{\mathbb{R}}$, we must have $f(x) = g(x)$ where $f(x), g(x) \in \overline{\mathbb{R}}$. Via Part (a), we know that the set $\{x \in X : f(x) = g(x)\}$ is $\mathcal{M}$-measurable. Therefore the set $\{x \in X : \exists\ \lim_n f_n(x) \in \overline{\mathbb{R}}\}$ is measurable. ⋄

For the second set, recall from Part (a) that

$$\{x \in X : f(x) = g(x)\} = \{x \in X : f(x) = g(x) = \infty\} \cup \{x \in X : f(x) = g(x) = -\infty\} \cup \{x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}) : f(x) - g(x) = 0\},$$

with each of the subsets being measurable. Notice that the last subset is just the previous set in this problem, and implies that $\limsup_n f_n(x) = \liminf_n f_n(x)$, with both being contained in $\mathbb{R}$ due to our restriction. Due to the inequality, we have that there does exist some $\lim_n f_n(x) \in \mathbb{R}$, and therefore the set $\{x \in X : \lim_{n\to\infty} f_n(x)$ exists as a value in $\overline{\mathbb{R}}\}$ is measurable. □

Problem 3

Let $f : X \to \overline{\mathbb{R}}$ and $Z = f^{-1}(\mathbb{R})$. Prove that f is $\mathcal{M}$-measurable if and only if $f^{-1}(\{-\infty\}) \in \mathcal{M}$, $f^{-1}(\{\infty\}) \in \mathcal{M}$, and $f\,|_Z : Z \to \mathbb{R}$ is $\mathcal{M}_Z$-measurable, Here $\mathcal{M}_Z = \{Z \cap M : M \in \mathcal{M}\}$.

Proof. ($\Longrightarrow$) First, consider the sets $\{x \in X : f(x) > a\}$ and $\{x \in X : f(x) < -b\}$, for $a, b \in \mathbb{N}$. These sets are measurable, since f is taken to be $\mathcal{M}$-measurable. Then we can take the following:

$$\bigcap_{a\in\mathbb{N}} \{x \in X : f(x) > a\} = \{x \in X : f(x) = \infty\} = f^{-1}(\{\infty\})$$
$$\bigcap_{b\in\mathbb{N}} \{x \in X : f(x) < -b\} = \{x \in X : f(x) = -\infty\} = f^{-1}(\{-\infty\})$$

Since a countable intersection of $\mathcal{M}$-measurable sets is $\mathcal{M}$-measurable, we have that $f^{-1}(\{\infty\}), f^{-1}(\{-\infty\}) \in \mathcal{M}$. The final claim is immediate, as if f is measurable, a restriction of f is measurable: in particular, if $Z \subset X$, then $f\,|_Z$ is $\mathcal{M}_Z$-measurable, where $\mathcal{M}_Z = \{Z \cap M : M \in \mathcal{M}\}$.
($\Longleftarrow$) Take a set $B \in \mathcal{B}_{\overline{\mathbb{R}}}$. Then

$$\begin{aligned} f^{-1}(B) &= f^{-1}(B \cap \mathbb{R}) \cup f^{-1}(B \cap \{-\infty\}) \cup f^{-1}(B \cap \{\infty\}) \\ &= \underbrace{(f^{-1}(B) \cap f^{-1}(\mathbb{R}))}_{=f^{-1}(B)\cap Z = f|_Z^{-1}(B)} \cup \underbrace{(f^{-1}(\{-\infty\}) \vee \emptyset)}_{\in\mathcal{M}} \cup \underbrace{(f^{-1}(\{\infty\}) \vee \emptyset)}_{\in\mathcal{M}} \end{aligned}$$

We see that the two terms on the right are certainly measurable via assumption. So, we aim to show that $f|_Z^{-1}(B) \in \mathcal{M}$. Note that by assumption, we have that $f|_Z$ is $\mathcal{M}_Z$-measurable. Then

$$\begin{aligned} B \cap \mathbb{R} \in \mathcal{B}_{\overline{\mathbb{R}}} &\Longrightarrow f|_Z^{-1}(B \cap \mathbb{R}) \in \mathcal{M}_Z \\ &\Longrightarrow f|_Z^{-1}(B) \cap f|_Z^{-1}(\mathbb{R}) \in \mathcal{M}_Z \\ &\Longrightarrow f|_Z^{-1}(B) \cap Z \in \mathcal{M}_Z \\ &\Longrightarrow f|_Z^{-1}(B) \in \mathcal{M}. \end{aligned}$$

It is clear that each term in our expansion of $f^{-1}(B)$ is $\mathcal{M}$-measurable, therefore $f^{-1}(B) \in \mathcal{M}$. □

Problem 4 Not confident

Let $f : X \to \overline{\mathbb{R}}$ and $g : X \to \overline{\mathbb{R}}$ be measurable functions.

(a) Prove that fg is measurable (note $0 \cdot \pm\infty = 0$)

Proof. Since measurable functions are the limits of sequences of simple functions, let us take sequences $\{f_n\}$ and $\{g_n\}$ such that $f = \lim_n f_n$ and $g = \lim_n g_n$ pointwise. Then for each $x \in X$, we have that $f(x)g(x) = \lim_n f_n(x)g_n(x)$. Since the product of simple functions is simple, we have that fg is the limit of a sequence of simple functions. So, since the limit of a sequence of simple functions is measurable, we have that fg is measurable. $\square$

ALTERNATE:

Proof. Observe that $fg(x) = f(x)g(x)$ is defined to make the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "fg"] \arrow[d, hook, "D"'] & \overline{\mathbb{R}} \\ X \times X \arrow[r, "f \times g"'] & \overline{\mathbb{R}} \times \overline{\mathbb{R}} \arrow[u, "\Pi"'] \end{tikzcd}$$

Where D is the map $D(x) = (x, x)$, and Π is the multiplication map $\Pi(x, y) = xy$. Observe that D and $f \times g$ are measurable, so to show that fg is measurable, it suffices to show that Π is measurable. From problem 3, it suffices to show that $\Pi^{-1}(\{-\infty\}), \Pi^{-1}(\{\infty\}) \in \mathcal{M}$ and $\Pi|_Z : Z \to \mathbb{R}$ is $\mathcal{M}_Z$-measurable, where $Z = \Pi^{-1}(\mathbb{R})$ and $\mathcal{M}_Z = \{Z \cap M : M \in \mathcal{M}\}$.

The proof for $\Pi^{-1}(\{-\infty\})$ is similar to $\Pi^{-1}(\{\infty\})$. Observe that

$$\Pi^{-1}(\{\infty\}) = (\mathbb{R} \setminus \{0\}) \times \{\infty\} \cup \{\infty\} \times (\mathbb{R} \setminus \{0\}) \cup \{(-\infty, -\infty)\}.$$

Which is a finite union of measurable sets in $\mathcal{M}$. Thus $\Pi^{-1}(\{\infty\})$ is measurable. Similarly, $\Pi^{-1}(\{-\infty\})$ is also measurable. Now consider $\Pi|_Z : Z \to \mathbb{R}$. By Theorem 2.6 in Folland, the multiplication map is $\mathcal{M}_Z$-measurable. Therefore Π must be measurable and we have fg being measurable. $\square$

(b) Let $c \in \overline{\mathbb{R}}$ and

$$h(x) = \begin{cases} c & \text{if } f(x) = -g(x) = \pm\infty, \\ f(x) + g(x) & \text{otherwise.} \end{cases}$$

Prove that h is measurable.

Proof. We provide a similar argument we used for the previous part. h is defined to make the following diagram commute

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, hook, "D"'] & \overline{\mathbb{R}} \\ X \times X \arrow[r, "f \times g"'] & \overline{\mathbb{R}} \times \overline{\mathbb{R}} \arrow[u, "\Sigma"'] \end{tikzcd}$$

Here, we have Σ being the expected summation map with the exception that $\Sigma(-\infty, \infty) = \Sigma(\infty, -\infty) = c$. Then to see that h is measurable, it suffices to show that Σ is measurable. Using problem three again, we can prove this by showing that $\Sigma^{-1}(\{-\infty\}), \Sigma^{-1}(\{\infty\}) \in \mathcal{M}$ and $\Sigma|_Z : Z \to \mathbb{R}$ is $\mathcal{M}_Z$-measurable, where $Z = \Sigma^{-1}(\mathbb{R})$.

Again, the proof for $\Sigma^{-1}(\{-\infty\})$ is similar to $\Sigma^{-1}(\{\infty\})$. But $\Sigma^{-1}(\{\infty\}) = (\overline{\mathbb{R}} \setminus \{-\infty\}) \times \{\infty\} \cup \{\infty\} \times (\overline{\mathbb{R}}) \in \mathcal{B}_{\overline{\mathbb{R}}} \otimes \mathcal{B}_{\overline{\mathbb{R}}}$. For $\Sigma|_Z : Z \to \mathbb{R}$, we have Folland's Theorem 2.6 again telling us that this must be measurable. $\square$

Problem 5

Let $(X, \mathcal{M}^*, \overline{\mu})$ be the completion of $(X, \mathcal{M}, \mu)$ and set $\mathbb{K} = \mathbb{R}, \overline{\mathbb{R}}$, or $\mathbb{C}$. If $f : X \to \mathbb{K}$ is $\mathcal{M}^*$-measurable, prove that there exists an $\mathcal{M}$-measurable function $g : X \to \mathbb{K}$ such that $g = f$ μ-a.e.

Proof. We will first show the result for characteristic functions, and then for simple functions, and then in the general case. For the characteristic function case, consider $f := \chi_A$, where $A \in \mathcal{M}^*$ can be written as $A = M \cup N$, where $M \in \mathcal{M}$ and $N \in \mathcal{N} \subset \mathcal{M}$ such that $\mu(N) = 0$. Certainly then the function $g := \chi_M$ is a $\mathcal{M}$-measurable function is such that $f = g$ μ-a.e.

For simple functions, consider $f(x) = \sum_{i=1}^{n} a_i \chi_{A_i}$ for $A_i \subset X$ for all $i = 1, ..., n$. Since by assumption we have f is $\mathcal{M}^*$-measurable, we have that $A_i \in \mathcal{M}^*$ for all $i = 1, ...n$. Similarly to in the characteristic function case, we can take $A_i = M_i \cup N_i$, where N_i is a null set for all i.

For the general case, consider our f, which is $\mathcal{M}^*$-measurable. Then there exists some sequence $\{s_n\}$ of $\mathcal{M}^*$-measurable simple functions such that $s_n \to f$ $\overline{\mu}$-a.e. Via our simple function case, for each n, let $\tilde{s_n}$ be a $\mathcal{M}$-measurable simple function where $s_n = \tilde{s_n}$ except on some set $E_n \in \mathcal{M}^*$ with $\mu(E_n) = 0$ (that is, $\tilde{s_n}(x) = s_n(x)$ for $x \in E_n^c$). Now choose $N \in \mathcal{M}$ such that $\mu(N) = 0$ and such that $\bigcup_{n \in \mathbb{N}} E_n = N$, and take $g = \lim \tilde{s_n} \chi_{N^c} = f \chi_{N^c}$. This g is such that $f = g$ μ-a.e. $\square$

Problem 6

Folland Prop. 2.13. Prove the following statements. Let $f : X \to [0, \infty)$ and $g : X \to [0, \infty)$ be $\mathcal{M}$-measurable simple functions.

(a) If $c \geq 0$, then $\int_X cg d\mu = c \int_X g d\mu$.

Proof. Take $c \geq 0$. Then:

$$\int cg d\mu = \int c \sum_i a_i \chi_{E_i} d\mu = \int \sum_i c a_i \chi_{E_i} d\mu = \sum_i c a_i \mu(E_i) = c \left(\sum_i a_i \mu(E_i) \right) = c \int \sum_i a_i \chi_{E_i} d\mu = c \int g d\mu$$

$\square$

(b) $\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$.

Proof. Let us consider the standard representations of f, g, and $f + g$ into simple functions as follows:

$$f = \sum_{i=1}^{n} a_i \chi_{E_i}, \quad g = \sum_{j=1}^{m} b_j \chi_{F_j}, \quad \text{and} \quad f + g = \sum c_k \chi_{G_k}.$$

Then we have that $\{E_i\}$ is a finite family of measurable sets such that $E_i \neq E_{\tilde{i}}$, where $X = \bigcup_i E_i$. Similarly, we have that $\{F_j\}$ is a finite family of measurable sets such that $F_j \neq F_{\tilde{j}}$, where $X = \bigcup_j F_j$. Similarly again, we have that $\{G_k\}$ is a finite family of measurable sets such that $G_k \neq G_{\tilde{k}}$, where $X = \bigcup_k G_k$. Certainly then, $\{E_i \cap F_j \cap G_k\}_{i,j,k}$ is a finite family of disjoint measurable sets as well, such that:

$$\text{For all } i, \quad E_i = \bigcup_{j,k} E_i \cap F_j \cap G_k,$$
$$\text{For all } j, \quad F_j = \bigcup_{i,k} E_i \cap F_j \cap G_k,$$
$$\text{For all } k, \quad G_k = \bigcup_{i,j} E_i \cap F_j \cap G_k.$$

By additivity of μ, and since each $E_i \cap F_j \cap G_k$ is disjoint from another, we have

$$\mu(E_i) = \sum_{j,k} \mu(E_i \cap F_j \cap G_k),$$
$$\mu(F_j) = \sum_{i,k} \mu(E_i \cap F_j \cap G_k),$$
$$\mu(G_k) = \sum_{i,j} \mu(E_i \cap F_j \cap G_k).$$

Notice that $a_i + b_j = c_k$ for all i,j,k in which $E_i \cap F_j \cap G_k \neq \emptyset$. Indeed, let us assume that $E_i \cap F_j \cap G_k \neq \emptyset$. With the above measures, we can show the desired equality.

$$\begin{aligned}
\int_X (f+g)d\mu = \sum_k c_k \mu(G_k) &= \sum_{i,j,k} c_k \mu(E_i \cap F_j \cap G_k) = \sum_{i,j,k} (a_i + b_j)\mu(E_i \cap F_j \cap G_k) \\
&= \sum_{i,j,k} a_i \mu(E_i \cap F_j \cap G_k) + \sum_{i,j,k} b_j \mu(E_i \cap F_j \cap G_k) \\
&= \sum_i a_i \sum_{j,k} \mu(E_i \cap F_j \cap G_k) + \sum_k b_j \sum_{i,k} \mu(E_i \cap F_j \cap G_k) \\
&= \sum_i a_i \mu(E_i) + \sum_j b_j \mu(F_j) = \int_X f d\mu + \int_X g d\mu.
\end{aligned}$$

□

(c) If $f \leq g$, then $\int_X f d\mu \leq \int_x g d\mu$.

Proof. Take $f = \sum_i a_i \chi_{E_i}$ and $g = \sum_j b_j \chi_{F_j}$ to be the standard representations of f and g into simple functions. Then we have that $\{E_i\}$ is a finite family of measurable sets such that $E_i \neq E_{\bar{i}}$, where $X = \bigcup_i E_i$. Similarly, we have that $\{F_j\}$ is a finite family of measurable sets such that $F_j \neq F_{\bar{j}}$, where $X = \bigcup_j F_j$. Certainly then we have that $\{E_i \cap F_j\}_{i,j}$ is a finite family of measurable sets as well, where:

$$\text{For all } i, \quad E_i = \bigcup_{j,k} E_i \cap F_j, \quad \text{and for all } j, \quad F_j = \bigcup_j E_i \cap F_j.$$

By additivity of μ, and since each $E_i \cap F_j$ is disjoint from another, we have

$$\mu(E_i) = \sum_j \mu(E_i \cap F_j) \quad \text{and} \quad \mu(F_j) = \sum_i \mu(E_i \cap F_j).$$

Moreover, for all i,j, we have that if $E_i \cap F_j \neq \emptyset$ then $a_i \leq b_k$. So, we can achieve our desired inequality:

$$\begin{aligned}
\int_X f d\mu = \sum_i a_i \mu(E_i) = \sum_{i,j} a_i \mu(E_i \cap F_j) &\leq \sum_{i,j} b_j \mu(E_i \cap F_j) \\
&= \sum_{i,j} b_j \mu(E_i \cap F_j) = \sum_j b_j \sum_i \mu(E_i \cap F_j) \\
&= \sum_j b_j \mu(F_j) = \int_X g d\mu.
\end{aligned}$$

□

(d) For $A \in \mathcal{M}$, let $\nu(A) = \int_A f d\mu$. Then ν is a measure on $\mathcal{M}$.

Proof. Notice that for all $A \in \mathcal{M}$, we have that $\int_A d\mu = \mu(A)$. So, the map : $A \to \int_A d\mu$ is simply equivalent to $A \to \mu(A)$, which is clearly a measure. □

Problem 7

Suppose $f : X \to [0,\infty]$ is $\mathcal{M}$-measurable and define $\nu : \mathcal{M} \to [0,\infty]$ by $\nu(E) = \int_E f d\mu$. Show that ν is a measure on $\mathcal{M}$ and that $\int_X g d\nu = \int_X g f d\mu$ for every $g : X \to [0,\infty]$ that is $\mathcal{M}$-measurable.

Proof. Let us first show that ν is a measure on $\mathcal{M}$. First, we see that indeed,

$$\nu(\emptyset) = \int_\emptyset f d\mu = \int_X f \chi_\emptyset d\mu = 0.$$

Next, take $\{E_i\}_i$ to be a collection of pairwise disjoint sets in $\mathcal{M}$, and take $E = \bigcup_i E_i$. Then we have

$$\nu\left(\bigcup_{i=1}^\infty E_i\right) = \nu(E) = \int_E f d\mu = \int_X f\chi_E d\mu = \int_X \sum_{i=1}^\infty f\chi_{E_i} d\mu = \sum_{i=1}^\infty \int_X f\chi_{E_j} d\mu = \sum_{i=1}^\infty \int_{E_j} f d\mu = \sum_{i=1}^\infty \nu(E_j)$$

We have shown the two conditions for ν to be a measure. $\diamond$

For the second part, take $g = \chi_E$ where $E \in \mathcal{M}$. Then we have

$$\int_X g d\nu = \int_X \chi_E d\nu = \int_E d\nu = \nu(E) = \int_E f d\mu = \int_X \chi_E f d\mu = \int_X g f d\mu$$

Clearly, we see that this holds for g being a characteristic function χ_E where $E \in \mathcal{M}$. Now consider g to be a non-negative $\mathcal{M}$-measurable simple function given by $g = \sum_{i=1}^n a_i \chi_{E_i}$, where $E_i \subset X$ for all $i \in \mathbb{N}$. Then we have:

$$\int_X g d\nu = \int_X \sum_{i=1}^n a_i \chi_{E_i} d\nu = \sum_{i=1}^n a_i \int_X \chi_{E_i} d\nu = \sum_{i=1}^n a_i \int_{E_j} d\nu = \sum_{i=1}^n a_i \nu(E_i)$$
$$\int_X g f d\mu = \int_X \sum_{i=1}^n a_i \chi_{E_i} f d\mu = \sum_{i=1}^n a_i \int_X \chi_{E_i} f d\mu = \sum_{i=1}^n a_i \int_{E_j} f d\mu = \sum_{i=1}^n a_i \nu(E_i)$$

So, we have that this holds for g being a simple function. Now consider $g : X \to [0, 1]$ to be a $\mathcal{M}$-measurable function. Then we have a sequence $\{s_n\}$ of non-negative $\mathcal{M}$-measurable simple functions which is monotonically increasing and converges to g. Take $s_n = \sum_{i=1}^n b_i \chi_{E_i}$, where $E_i \subset X$ for all $i \in \mathbb{N}$. By the Monotone Convergence Theorem, we have that $\int_X g d\nu = \lim_n \int_X s_n d\nu$. So,

$$\begin{aligned}
\int_X g d\nu = \lim_n \int_X s_n d\nu = \lim_n \int_X \sum_{i=1}^n b_i \chi_{E_i} d\nu &= \lim_n \sum_{i=1}^n b_i \int_X \chi_{E_i} d\nu \\
\lim_n \sum_{i=1}^n b_i \int_{E_i} d\nu &= \lim_n \sum_{i=1}^n b_i \nu(E_i) \\
&= \lim_n \sum_{i=1}^n b_i \int_{E_i} f d\mu = \lim_n \sum_{i=1}^n b_i \int_X f \chi_{E_i} d\mu \\
&= \lim_n \int_X f \sum_{i=1}^n b_i \chi_{E_i} d\mu = \lim_n \int_X f s_n d\mu.
\end{aligned}$$

By the Dominated Convergence Theorem, we have that $\int_X f g d\mu = \lim_n \int_X f s_n d\mu$. So, the general case is shown. $\square$

Problem 8

Let $f : X \to [0, \infty]$ be an $\mathcal{M}$-measurable function such that $\int_X f d\mu < \infty$. Prove that the set $\{x \in X : f(x) = \infty\}$ has measure zero (i.e. f is finite μ-a.e.) and that the set $\{x \in X : f(x) > 0\}$ is σ-finite.

Proof. For the first part, let us define the sets $I := f^{-1}(\{\infty\})$ and $I_n := f^{-1}([n, \infty])$. Certainly we have that $I = \bigcap_n I_n$, and by definition of intersection, we have that $I \subset I_n$. Moreover, by monotonicity of measure, we have that $\mu(F) \le \mu(F_n)$ for each $n \in \mathbb{N}$. Next, let us note that for $x \in X$, we have $\chi_{I_n}(x) \le (1/n)f(x)$. In particular, if $x \notin I_n$, we have that $0 \le (1/n)f(x)$ since f is a positive function, and if $x \in I_n$, we have that $1 \le (1/n)f(x)$ since $f(x) \ge n$ by definition. So, we have the following for all n:

$$\mu(I) \le \mu(I_n) = \int \chi_{I_n}(x) d\mu \le \int \frac{1}{n} f(x) d\mu = \frac{1}{n} \int f(x) d\mu.$$

Therefore, we must have $\mu(I) = 0$, as we can force $n \to \infty$ while $\int f(x) d\mu < \infty$.

For the second part, let us define the sets $E := \{x \in X : f(x) > 0\}$ and $E_n := \{x \in X : f(x) \ge 1/n\}$. Certainly we have that $E = \bigcup_n E_n$. We want to show that $\mu(E_n) < \infty$ for all n. So, observe that $\chi_{E_n}(x) \le n f(x)$ for all $x \in X$. So, by the monotonicity of the integral, we have

$$\mu(E_n) = \int \chi_{E_n} d\mu \le \int n f d\mu = n \int f d\mu < \infty.$$

So, we have shown that $\mu(E_n) < \infty$, and as a result $\mu(E) < \infty$. $\square$

Problem 9

Let $f : X \to [0, \infty]$ be an $\mathcal{M}$-measurable function such that $\int_X f d\mu < \infty$. Prove that for every $\epsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f d\mu > (\int_X f d\mu) - \epsilon$.

Proof. Recall that $\{x \in X : f(x) > 0\} = \bigcup_n \{x \in X : f(x) > 1/n\}$, where $f \geq 0$ and $\int f < \infty$. Let us denote $A := \{x \in X : f(x) = 0\}$ and $A_n := \bigcup_n \{x \in X : f(x) > 1/n\}$. Consider the function $f_n := f\chi_{A_n} \nearrow f\chi_A$. By the monotone convergence theorem, we have $\lim_n \int_{A_n} f d\mu = \int_A f d\mu = \int_X f d\mu$. So, if $\epsilon > 0$, then there exists some n_0 such that $0 \leq \int_X f d\mu - \int_{A_{n_0}} f d\mu < \epsilon$, which rewriting as $\int_{A_{n_0}} f d\mu > (\int_X f d\mu) - \epsilon$ gives our result. □

Problem 10 Part (a) done

Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where μ is the counting measure.

(a) Show that if $f : \mathbb{N} \to [0, \infty)$ then $\int_\mathbb{N} f d\mu = \sum_{k \in \mathbb{N}} f(k)$.

Proof. Note that $f = \sum_{k=1}^\infty f(k)\chi_{\{k\}}$. So, consider $f : \mathbb{N} \to [0, \infty)$, then $f(j) = \sum_{k=1}^\infty f(k)\chi_{\{k\}}(j)$. Now we have the following:

$$\begin{aligned}
\int_\mathbb{N} f(j) d\mu(j) &= \int_\mathbb{N} \sum_{k=1}^\infty f(k)\chi_{\{k\}}(j) d\mu(j) && \text{rewriting } f(j) \\
&= \sum_{k=1}^\infty \int_\mathbb{N} f(k)\chi_{\{k\}}(j) d\mu(j) && \text{switch integral and summation} \\
&= \sum_{k=1}^\infty f(k) \int_\mathbb{N} \chi_{\{k\}} d\mu && \text{pull out constant} \\
&= \sum_{k=1}^\infty f(k)
\end{aligned}$$

We have shown $\int_\mathbb{N} f d\mu = \sum_{k \in \mathbb{N}} f(k)$ as desired. □

(b) Show that if $f : \mathbb{N} \to \mathbb{R}$ and $f \in L^1(\mathbb{N}, \mu)$ then $\int_\mathbb{N} f d\mu = \sum_{k \in \mathbb{N}} f(k)$.

Proof. Note that $f(n) = \sum_{k=1}^\infty f(k)\chi_{\{k\}}(n)$. Considering that $f \in L^1(\mathbb{N}, \mu)$ and that the integral of $|f|$ is finite, we have that we can interchange the integral and sum as follows:

$$\int_\mathbb{N} |f| d\mu = \int_\mathbb{N} \left| \sum_{k=1}^\infty f(k)\chi_{\{k\}}(n) \right| d\mu \leq \sum_{k=1}^\infty \int_\mathbb{N} |f(k)\chi_{\{k\}}(n)| d\mu < \infty.$$

Now given that μ is the counting measure, we have that

$$\int_\mathbb{N} f d\mu = \int_\mathbb{N} \sum_{k=1}^\infty f(k)\chi_{\{k\}}(n) d\mu = \sum_{k=1}^\infty f(k) \int_\mathbb{N} \chi_{\{k\}}(n) d\mu = \sum_{k=1}^\infty f(k).$$

We have shown our desired result. □

(c) Interpret Fatou's lemma, the MCT, and the DCT as statements about infinite series.

Problem 11 Proof given, add details

Folland Thm. 2.27. Let $-\infty < a < b < \infty$ and $f : X \times [a, b] \to \mathbb{C}$ such that $f(\cdot, t) \in L^1(X, \mu)$ for all $t \in [a, b]$. Set $F(t) = \int_X f(x, t) d\mu(x)$. Prove the following statements.

(a) Suppose there exists $g \in L^1(X, \mu)$ such that $|f(x, t)| \leq g(x)$ for all $x \in X$ and $t \in [a, b]$. If $f(x, \cdot)$ is continuous at t_0 for each $x \in X$ then F is continuous at t_0.

Proof. Let us begin by applying the Dominated Convergence Theorem to $f_n(x) := f(x, t_n)$, where $\{t_n\}$ is a sequence in $[a, b]$ which converges to t_0. □

(b) Suppose that $f(x,\cdot)$ is differentiable in $[a,b]$ for each $x \in X$ and that there exists $g \in L^1(X,\mu)$ such that $|\frac{\partial f}{\partial t}(x,t)| \leq g(x)$ for all $x \in X$ and $t \in [a,b]$. Then F is differentiable in $[a,b]$ and $F'(t) = \int_X \frac{\partial f}{\partial t} d\mu(x)$ for all $t \in [a,b]$.

Proof. Let us observe the following:

$$\frac{\partial f}{\partial t}(x,t_0) = \lim h_n(x) \text{ where } h_n(x) = \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0},$$

where again $\{t_n\} \subset [a,b]$ is some sequence which converges to t_0. It follows that $\partial f/\partial t$ is a measurable function, and applying the mean value theorem, we have

$$|h_n(x)| \leq \sup_{t \in [a,b]} \left| \frac{\partial f}{\partial t}(x,t) \right| \leq g(x).$$

We can invoke the Dominated Convergence Theorem again, which gives

$$F'(t_0) = \lim \frac{F(t_n) - F(t_0)}{t_n - t_0} = \lim \int h_n(x) d\mu(x) = \int \frac{\partial f}{\partial t}(x,t) d\mu(x).$$

$\square$