Abstract Algebra II Homework 5

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Section 11.1

6 Let V be a vector space of finite dimension. If φ is any linear transformation from V to V, prove there is an integer m such that the intersection of the image of φ^m and the kernel of φ^m is $\{0\}$.

Proof. Note that for all i, we have the containment $\ker\varphi^i \subseteq \ker\varphi^{i+1}$ and $\text{im}\varphi^{i+1} \subseteq \text{im}\varphi^i$. Since $\dim V < \infty$, we know that $\dim\ker\varphi, \dim\text{im}\varphi < \infty$ as well. Therefore, there must exist a and b such that $\ker\varphi^a = \ker\varphi^{a+1}$ and $\text{im}\varphi^{b+1} = \text{im}\varphi^b$. Without loss of generality, take $a \leq b$. We aim to show that $\ker\varphi^b \cap \text{im}\varphi^b = 0$. Any element $v \in \ker\varphi^b \cap \text{im}\varphi^b$ must be such that (1) $v = \varphi^b$ for some $w \in V$, and (2) $\varphi^b(v) = 0$. Substituting, we see $0 = \varphi^b(v) = \varphi^b(\varphi^b(w)) = \varphi^{2b}(w)$. So, $w \in \ker\varphi^{2b}$. Since $2b > b$, we know that $w \in \ker\varphi^b$ too. Thus, $v = \varphi^b(w) = 0$. □

8 Let V be a vector space over F and let φ be a linear transformation of the vector space V to itself. A nonzero element $v \in V$ satisfying $\varphi(v) = \lambda v$ for some $\lambda \in F$ is called an *eigenvector* of φ with *eigenvalue* λ. Prove that for any fixed $\lambda \in F$, the collection of eigenvectors of φ with eigenvalue λ together with 0 forms a subspace of V.

Proof. Take v_1, v_2 to be eigenvectors of φ with eigenvalue λ and take $c_1, c_2 \in F$. We see:

$$\varphi(c_1v_1 + c_2v_2) = c_1\varphi(v_1) + c_2\varphi(v_2) = c_1(\lambda v_1) + c_2(\lambda v_2) = \lambda(c_1v_1 + c_2v_2).$$

So, the space is closed under multiplication by a scalar and under addition by a vector. By construction, 0 is also in the space. □

9 Let V be a vector space over F and let φ be a linear transformation of the vector space V to itself. Suppose for $i = 1, 2, \ldots, k$ that $v_i \in V$ is an eigenvector for φ with eigenvalue $\lambda_i \in F$ (cf. the previous exercise) and that all the eigenvalues λ_i are distinct. Prove that $v_1, v_2, \ldots, v_k$ are linearly independent. Conclude that any linear transformation on an n-dimensional vector space has at most n distinct eigenvalues.

Proof. We induct over k. For the base case, $k = 1$, we have that v_1 is a linearly independent set trivially since $v_1 \neq 0$. Assume now that $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent set, and consider $\{v_1, v_2, \ldots, v_k\} \cup v_{k+1}$. Consider now the linear combination

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k + c_{k+1}v_{k+1} = 0.$$

We aim to show that the only solution to this equation is when $c_i = 0$, $i = 1, \ldots, k, k+1$. Indeed, we see that

$$\begin{cases} \lambda_1(c_1v_1 + c_2v_2 + \cdots + c_kv_k + c_{k+1}v_{k+1}) = \lambda_1(0) \\ \varphi(c_1v_1 + c_2v_2 + \cdots + c_kv_k + c_{k+1}v_{k+1}) = \varphi(0) \end{cases}$$
$$\implies \begin{cases} \lambda_1c_1v_1 + \lambda_1c_2v_2 + \cdots + \lambda_1c_kv_k + \lambda_1c_{k+1}v_{k+1} = 0 \\ \lambda_1c_1v_1 + \lambda_2c_2v_2 + \cdots + \lambda_kc_kv_k + \lambda_{k+1}c_{k+1}v_{k+1} = 0 \end{cases}$$

Subtracting equation 2 from equation 1, we have that

$$(\lambda_1 - \lambda_2)c_2v_2 + (\lambda_1 - \lambda_{3c3}v_3 + \cdots + (\lambda_1 - \lambda_k)c_kv_k + (\lambda_1 - \lambda_{k+1})c_{k+1}v_{k+1} = 0.$$

Notice that since the eigenvalues were taken to be distinct, we have that each $\lambda_1 - \lambda_i \neq 0$. Further, the induction hypothesis gives that the only solution to the previous linear combination is trivial. So, $c_i = 0$ for $i = 2, \cdots, k+1$. Plugging back in to either of the equations above, this gives that $c_1 = 0$ as well. Therefore indeed, the v_i are linearly independent.

Indeed, we know that any subset of an n-dimensional vector space has at most n linearly independent vectors, and therefore the result must follow that there are at most n distinct eigenvalues. $\square$

14 Let $\mathcal{A}$ be a basis for the infinite dimensional space V. Prove that V is isomorphic to the direct sum of copies of the field F indexed by the set $\mathcal{A}$. Prove that the direct product of copies of F indexed by $\mathcal{A}$ is a vector space over F and it has strictly larger dimension than the dimension of V.

Proof. Recall that the direct sum in $\mathbf{Vect}_F$ is the set of tuples with only finitely many nonzero entries, whereas the direct product in $\mathbf{Vect}_F$ is the set of tuples without bound on the number of nonzero entries.

Note that for any $v \in V$, we can write $v = c_1a_1 + c_2a_2 + \cdots + c_na_n$ with $a_i \in \mathcal{A}$ and $c_i \in F$. Since a linear combination is always finite by definition, and so isomorphism to the direct sum is clear, since we can think of free modules (in particular, vector spaces) as being a direct sum of k copies of the base ring (in particular, field), with $k = |\mathcal{A}|$.

As for the direct product, this is a vector space since we can consider $\prod_{\mathcal{A}} F = \{\alpha : \mathcal{A} \xrightarrow{\alpha} F\}$. This is a set of functionals, and therefore is a vector space. Notice though that the cardinality of $\bigoplus_{\mathcal{A}} F$ is $\max(|\mathcal{A}|, |F|)$. But, the cardinality of $\prod_{\mathcal{A}} F$ is $|F|^{|\mathcal{A}|} > |\mathcal{A}|, |F|$. $\square$

Section 11.2

2 Let V be the vector space given by the collection of polynomials with coefficients in $\mathbb{Q}$ in the variable x of degree at most 5. Let $\varphi = d/dx$ be the linear transformation of V to itself given by the usual differentiation of a polynomial with respect to x. Determine the matrix of φ with respect to the two bases $B = \{1, x, x^2, \ldots, x^5\}$ and $B' = \{1, 1+x, 1+x+x^2, \ldots, 1+x+x^2+x^3+x^4+x^5\}$.

Solution. Notice first that $\varphi(1) = 0$, $\varphi(x) = 1$, $\varphi(x^2) = 2x$, $\varphi(x^3) = 3x^2$, $\varphi(x^4) = 4x^3$, and $\varphi(x^5) = 5x^4$. So, the matrix M_B^B is given by

$$M_B^B(\varphi) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Next, notice that

$$\begin{aligned}
\varphi(1) &= 0 \\
\varphi(x+1) &= 1 \\
\varphi(x^2+x+1) &= 2x+1 \\
&= 2(x+1) - 1 \\
\varphi(x^3+x^2+x+1) &= 3x^2+2x+1 \\
&= 3(x^2+x+1) - 1(x+1) - 1 \\
\varphi(x^4+x^3+x^2+x+1) &= 4x^3+3x^2+2x+1 \\
&= 4(x^3+x^2+x+1) - 1(x^2+x+1) - 1(x+1) - 1 \\
\varphi(x^5+x^4+x^3+x^2+x+1) &= 5x^4+4x^3+3x^2+2x+1 \\
&= 5(x^4+x^3+x^2+x+1) - 1(x^3+x^2+x+1) - 1(x^2+x+1) - 1(x+1) - 1.
\end{aligned}$$

So, the matrix $M_{B'}^{B'}$ is given by

$$\begin{bmatrix} 0 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3 Let V be the collection of polynomials with coefficients in F in the variable x of degree at most n. Determine the transition matrix from the basis $\{1, x, x^2, \dots, x^n\}$ for V to the elements $1, x - \lambda, \dots, (x - \lambda)^{n-1}, (x - \lambda)^n$, where $\lambda \in F$ is fixed. Conclude that these elements are a basis for V.

Solution.

8 Let V be an n-dimensional vector space over F and let φ be a linear transformation of V to itself.

(a) Prove that if V has a basis consisting of eigenvectors for φ (cf. §11.1 Ex. 8), then the matrix representing φ with respect to this basis (for both domain and range) is diagonal with the eigenvalues as diagonal entries.

Proof. Let us take $B = (v_1, \cdots, v_b)$ to be a basis for V, and we shall assume that each v_i is an eigenvector. That is, $\varphi v_i = \lambda_i v_i$ for some $\lambda_i \in F$. To compute the jth column of M_B^B, we have

$$\varphi v_j = \lambda_j v_j = 0v_1 + 0v_2 + \cdots + \lambda_j v_j + 0v_{j+1} + \cdots + 0v_n.$$

Therefore, the jth column is all zeroes except for λ_j in the jth row (and therefore, on the diagonal). □

(b) If A is the $n \times n$ matrix representing φ with respect to a given basis for V (for both domain and range), prove that A is similar to a diagonal matrix if and only if V has a basis of eigenvectors for φ.

Proof. ($\Longrightarrow$) Almost trivially, if A is diagonalized by the matrix of eigenvectors $S = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$, then the columns of S form a basis of eigenvectors for A.

($\Longleftarrow$) Suppose that there exists a basis of eigenvectors for φ, say $B = (v_1, \dots, v_n)$. Then the matrix given by $S = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ is such that

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & = & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix},$$

where the λ_i are the eigenvalues associated with each v_i. □

9 If W is a subspace of the vector space V stable under the linear transformation φ (i.e. $\varphi(W) \subseteq W$), show that φ induces linear transformations $\varphi|_W$ on W and $\bar{\varphi}$ on the quotient vector space V/W. If $\varphi|_W$ and $\bar{\varphi}$ are nonsingular, prove that φ is nonsingular. Prove the converse holds if V has finite dimension and give a counterexample with V infinite dimensional.

Proof. Certainly $\varphi|_W$ is a linear transformation as we assume that W is φ-stable. Define now $\bar{\varphi}(\bar{v}) = \overline{\varphi(v)}$. We see that it is well defined, as if $\bar{v}_1 = \bar{v}_2$, then $v_1 - v_2 \in W$ and so $v_1 = v_2 + w$ ($w \in W$) and so $\overline{\varphi(v_1)} = \overline{\varphi(v_2 + w)} = \overline{\varphi(v_2)}$.

For the sake of contradiction, let us assume that both $\varphi|_W, \bar{\varphi}$ are nonsingular and assume $\varphi(v) = 0$. If $v \in W$ then $v = 0$ by the nonsingularity of $\varphi|_W$. Otherwise, if $v \notin W$, then $\bar{v} \neq 0$ and we have $\varphi(v) \notin W$ by the nonsingularity of $\bar{\varphi}$. This means that $\varphi(v) \neq 0$ which is a contradiction. Therefore φ is nonsingular.

Conversely, we certainly have that φ being nonsingular implies that $\varphi|_W$ is nonsingular. Now if $\dim V < \infty$, then we can show that $\bar{\varphi}$ is nonsingular. So, this means that $\dim W < \infty$ and therefore we can assume $\varphi(v) \in W$. Since $\varphi|_W$ is nonsingular and therefore surjective, we can find some $w \in W$ such that $\varphi(w) = \varphi(v)$, which implies that $v = w$, therefore $v \in W$ and $\bar{v} = 0$. □

Counterexample: Let us assume now that $V = \bigoplus_I \mathbb{R}$, where I is a countably infinite index set. If we take φ to be the right shift operator with W being the subspace which consists of vectors of the form $(0, v_2, v_3, ...)$, we see that φ is nonsingular but that $\bar{\varphi}$ is nonsingular since $\bar{\varphi}(\bar{e}_1) = 0$.

10 Let V be an n-dimensional vector space and let φ be a linear transformation of V to itself. Suppose W is a subspace of V of dimension m that is stable under φ.

(a) Prove that there is a basis for V with respect to which the matrix for φ is of the form $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, where A is an $m \times m$ matrix, B is an $m \times (n-m)$ matrix, and C is an $(n-m) \times (n-m)$ matrix (such a matrix is called block upper triangular).

Proof. □

(b) Prove that if there is a subspace W' invariant under φ so that $V = W \oplus W'$ decomposes as a direct sum, then the bases for W and W' give a basis for V with respect to which the matrix for φ is block diagonal: $\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$, where A is an $m \times m$ matrix and C is an $(n-m) \times (n-m)$ matrix.

Proof. □

(c) Prove conversely that if there is a basis for V with respect to which φ is block diagonal as in (b), then there are φ-invariant subspaces W and W' of dimensions m and $n-m$ respectively, with $V = W \oplus W'$.

Proof. □

11 Let φ be a linear transformation from the finite dimensional vector space V to itself such that $\varphi^2 = \varphi$.

(a) Prove that $\mathrm{im}\varphi \cap \ker \varphi = 0$.

Proof. Suppose there is some $v \neq 0$ in $\mathrm{im}\varphi \cap \ker \varphi$. Then there exists some $w \in V$ such that $\varphi(w) = v$, and $\varphi(v) = 0$. If we apply φ to $\varphi(w) = v$, we see

$$\varphi(\varphi(w)) = \varphi(v) = 0 \implies \varphi^2(w) = \varphi(w) = v = 0. \quad \Rightarrow\Leftarrow$$

□

(b) Prove that $V = \mathrm{im}\varphi \oplus \ker \varphi$.

Proof. We write $v = \varphi(v) + (v - \varphi(v))$. Then

$$\varphi(v - \varphi(v)) = \varphi(v) - \varphi^2(v) = \varphi(v) - \varphi(v) = 0.$$

So $v - \varphi(v) \in \ker \varphi$, and certainly $\varphi(v) \in \mathrm{im}\varphi$. This is sufficient. □

(c) Prove that there is a basis of V such that the matrix of φ with respect to this basis is a diagonal matrix whose entries are all 0 or 1.

Proof. Take $\operatorname{im}\varphi$ to have basis B_1 and $\ker\varphi$ to have basis B_2. Part (b) implies that $B_1 \cup B_2$ is a basis for V. Also if $v \in B_1$ then there exists some w such that $\varphi(w) = v$, giving

$$\varphi(v) = \varphi^2(w) = \varphi(w) = v.$$

So, the matrix representation of $\varphi|_{B_1}$ is the identity. Certainly, the matrix of $\varphi|_{\ker\varphi}$ is 0. So, the matrix representation ought to be $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, where I is the identity matrix and 0 are zero matrices. Indeed, this is the desired result. $\square$

22 Suppose A, B are two row equivalent $m \times n$ matrices.

(a) Prove that the set $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$ of solutions to the homogeneous linear equations $AX = 0$ as in equation (4) above are the same as the set of solutions to the homogeneous linear equations $BX = 0$. [It suffices to prove this for two matrices differing by an elementary row operation.]

Proof. Suppose $AX = 0$. Then we have $PAX = P \cdot 0 = 0$, where P is an invertible matrix such that $PA = B$, since A, B are row equivalent. So $BX = 0$. Conversely if $BX = 0$ then $P^{-1}BX = P^{-1} \cdot 0 = 0$, so $AX = 0$. Thus $AX = 0 \iff BX = 0$. $\square$

(b) Prove that any linear dependence relation satisfied by the columns of A viewed as vectors in F^m is also satisfied by the columns of B.

Proof. Let us denote by A, B two column matrices such that $PA = B$. Then if we have $\sum \alpha_i A_i = 0$, then

$$P \sum_i \alpha_i A_i = 0 = \sum_i \alpha_i P A_i = \sum_i \alpha_i \beta_i.$$

Thus, the linear dependence of B is the same as the linear dependence of A. $\square$

(c) Conclude from (b) that the number of linearly independent columns of A is the same as the number of linearly independent columns of B.

Proof. The maximal number of linearly independent columns of A has to be the same number of linearly independent columns of B. This is because if P is such that $AP = B$, then it is surjective, and so the number of independent columns must be equivalent. $\square$