Abstract Algebra II Homework 3

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Section 10.4

Throughout, let R denote a ring with identity.

1 Let $f : R \to S$ be a ring homomorphism from the ring R to the ring S with $f(1_R) = 1_S$. Verify the details that $sr = sf(r)$ defines a right R-action on S under which S is an (S, R)-bimodule.

Proof. First we aim to show that the right action gives S an R-module structure. Indeed, we see that

$$\begin{aligned} (s+s')r &= (s+s')f(r) = sf(r) + s'f(r) = sr + s'r, \\ s(r+r') &= sf(r+r') = s(f(r)+f(r')) = sf(r) + sf(r') = sr + sr', \\ (sr)r' &= sf(r)r' = sf(r)f(r') = sf(rr') = s(rr'), \end{aligned}$$

and therefore S does have the structure of a right R-module. Now we aim to show that the left action of S is compatible with the right action. Indeed,

$$(s's)r = (s's)f(r) = s'(sf(r)) = s'(sr).$$

We see that indeed S has an (S, R)-bimodule structure as desired. $\square$

2 Show that the element $2 \otimes 1$ is 0 in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ but is nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.

Proof. Notice that in $\mathbb{Z}$, we have a notion of dividing by 2, while in $\mathbb{Z}/2\mathbb{Z}$ we do not. In particular, we have that in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$,

$$2 \otimes 1 = (1 \cdot 2) \otimes 1 = 1 \otimes (2 \cdot 1) = 1 \otimes 2 = 1 \otimes 0 = 0.$$

In order to prove that $2 \otimes 1$ is nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, we claim that $\phi : 2\mathbb{Z} \otimes_{\mathbb{Z}} /2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ defined by $2k \otimes x \mapsto kx$, with $k \in \mathbb{Z}$ and $x \in \mathbb{Z}/2\mathbb{Z}$, is a group homomorphism. Letting $a, b, k, k' \in \mathbb{Z}$ and $x, x' \in \mathbb{Z}/2\mathbb{Z}$, we see that indeed

$$\begin{aligned} \phi(a(2k) + b(2k'), x) &= \phi(2(ak + bk'), x) \\ &= (ak + bk')x \\ &= akx + bk'x \\ &= a\phi(2k, x) + b\phi(2k', x), \\ \phi(2k, ax + bx') &= k(ax + bx') \\ &= akx + bkx' \\ &= a\phi(2k, x) + b\phi(2k, x'). \end{aligned}$$

So, we see that ϕ is linear in each coordinate and therefore is indeed a homomorphism. Notice now that $\phi(2 \otimes 1) = 1 \neq 0$ and therefore $2 \otimes 1 \notin \ker \phi$. Therefore, $2 \otimes 1$ is nonzero as desired. $\square$

3 Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are both left $\mathbb{R}$-modules but are not isomorphic as $\mathbb{R}$-modules.

Proof. Note first that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is a $\mathbb{C}$-module which contains the $\mathbb{R}$-module, $\mathbb{C}$, as a submodule. Similarly, note that $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ is a $\mathbb{C}$-module which contains the $\mathbb{C}$-module, $\mathbb{C}$. Therefore since $\mathbb{R} \subset \mathbb{C}$, we have that both of these tensor products are $\mathbb{R}$-modules. Next, we see the following:

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \cong (\mathbb{R} \oplus \mathbb{R})_{\mathbb{R}}(\mathbb{R} \oplus \mathbb{R}) \cong \bigoplus_{i=1,\ldots,4} (\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}) \cong \bigoplus_{i=1,\ldots,4} \mathbb{R} \cong \mathbb{R}^4,$$
$$\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C} \cong \mathbb{R}^2.$$

Certainly $\mathbb{R}^4 \not\cong \mathbb{R}^2$, and therefore the two tensor products are not isomorphic. $\square$

4 Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic left $\mathbb{Q}$-modules. [Show they are both one-dimensional vector spaces over $\mathbb{Q}$]

Proof. Notice that both of the tensor products are modules over $\mathbb{Q}$. We note first that $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \cong \mathbb{Q}$, and therefore is of dimension 1. For the second tensor product, let us take the simple tensor $q \otimes_{\mathbb{Z}} p \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, where $p = n/m$ and $q = a/b$, with $a, b, n, m \in \mathbb{Z}$. We see the following:

$$q \otimes p = \frac{a}{b} \otimes \frac{n}{m} = an\left(\frac{1}{b} \otimes \frac{1}{m}\right) = an\left(\frac{m}{bm} \otimes \frac{1}{m}\right) = an\left(\frac{1}{bm} \otimes \frac{m}{m}\right) = \frac{an}{bm}bm\left(\frac{1}{bm} \otimes 1\right) = \frac{an}{bm} \otimes 1.$$

So, any element of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ is obtained by multiplying some rational number and the simple tensor $1 \otimes 1$, and therefore this module is also of dimension 1.

Now since modules of finite dimension are isomorphic if and only if they have the same dimension, we have that indeed the two tensor products are isomorphic. $\square$

6 If R is any integral domain with quotient field Q, prove that $(Q/R) \otimes_R (Q/R) = 0$.

Proof. We shall show that all simple tensors are 0. Consider some arbitrary $(a/b)\otimes(c/d) \in (Q/R)\otimes_R(Q/R)$, with $a, b, c, d \in R$. We see that

$$\frac{a}{b} \otimes \frac{c}{d} = \frac{a}{b} \otimes b\frac{c}{db} = \frac{ab}{b} \otimes \frac{c}{db} = a \otimes \frac{c}{db} = 0 \otimes \frac{c}{db} = 0.$$

Indeed all simple tensors are 0 and therefore $(Q/R) \otimes_R (Q/R) = 0$. $\square$

7 If R is any integral domain with quotient field Q, and N is a left R-module, prove that every element of the tensor product $Q \otimes_R N$ can be written as a simple tensor of the form $(1/d) \otimes n$ for some nonzero $d \in R$ and $n \in N$.

Proof. Note that simple tensors are of this form since we have $(a/d) \otimes n = (1/d) \otimes an$. Each element in $Q \otimes_R N$ is a finite sum of simple tensors, which we write as

$$\sum_{i=1}^{k} \frac{1}{d_i} \otimes n_i.$$

If we now let $d := \prod_i d_i$ and $k_i := \prod_{j \neq i} d_i$, we see

$$\sum_{i=0}^{k} \frac{1}{d_i} \otimes n_i = \sum_{i=1}^{n} \frac{k_i}{d} \otimes n_i = \sum_{i=1}^{n} \frac{1}{d} \otimes a_i n_i = \frac{1}{d} \otimes \sum_{i=1}^{n} a_i n_i.$$

Since $a_i n_i \in N$, we have shown the result. $\square$

8 Suppose R is an integral domain with quotient field Q, and let N be any R-module. Let $U = R^{\times}$ be the set of nonzero elements in R and define $U^{-1}N$ to be the set of equivalence classes of ordered pairs of elements (u, n) with $u \in U$ and $n \in N$ under the equivalence relation $(u, n) \sim (u', n)$ if and only if $u'n = un'$ in N.

(a) Prove that $U^{-1}N$ is an abelian group under the addition defined by $\overline{(u_1, n_1)} + \overline{(u_2, n_2)} = \overline{(u_1u_2, u_2n_1 + u_1n_2)}$. Prove that $r\overline{(u, n)} = \overline{(u, rn)}$ defines an action of R on $U^{-1}N$ making it into an R-module. [This is an example of *localization* considered in general in Chapter 15 §4, cf. also Chapter 7 §5.]

Proof. Take elements $\overline{(u_1, n_1)},\ \overline{(u_2, n_2)},\ \overline{(u_3, n_3)} \in U^{-1}N$. We show associativity as follows:

$$\begin{aligned}
\overline{(u_1, n_1)} + \left(\overline{(u_2, n_2)} + \overline{(u_3, n_3)}\right) &= \overline{(u_1, n_1)} + \overline{(u_2u_3, u_3n_2 + u_2n_3)} \\
&= \overline{(u_1(u_2u_3), (u_2u_u)n_1 + u_1(u_3n_2 + u_2n_3))} \\
&= \overline{(u_1u_2u_3, u_3(u_2n_1 + u_1n_2) + u_1u_2u_3)} \\
&= \overline{(u_1u_2, u_2n_1 + u_1n_2)} + \overline{(u_3, n_3)} \\
&= \left(\overline{(u_1, n_1)} + \overline{(u_2, n_2)}\right) + \overline{(u_3, n_3)}.
\end{aligned}$$

Take now $\overline{(u_1, n_1)},\ \overline{(u_2, n_2)} \in U^{-1}N$. We show commutativity as follows:

$$\overline{(u_1, n_1)} + \overline{(u_2, n_2)} = \overline{(u_1u_2, u_2n_1 + u_1n_2)} = \overline{(u_2u_1, u_1n_2 + u_2n_1)} = \overline{(u_2, n_2)} + \overline{(u_1, n_2)}.$$

Consider now the element $\overline{(1, 0)} \in U^{-1}N$. Notice $1 \in U$ since $1 \in R^\times$ and $0 \in N \in R$-mod. So for $\overline{(u, n)} \in U^{-1}N$ we have that $\overline{(u, n)} + \overline{(1, 0)} = \overline{(u \cdot 1, 1 \cdot n + u \cdot 0)} = \overline{(u, n)}$. Also consider both $\overline{(u, n)},\ \overline{(u, -n)} \in U^{-1}N$. We see that $\overline{(u, n)} + \overline{(u, -n)} = \overline{(u^2, un + u(-n))} = \overline{(u^2, u(n - n))} = \overline{(u^2, u(0))} = (1, 0)$ since $(u^2, 0) \sim (1, 0)$. So, we have that $U^{-1}N$ has inverses and is an abelian group under addition as desired.

Now, we want to show that $r\overline{(u, n)} = \overline{(u, rn)}$ defines an action on $U^{-1}N$. Taking $r, s \in R$ and $\overline{(u, n)} \in U^{-1}N$, we see that $(r+s)\overline{(u, n)} = \overline{(u, (r + s)n)} = \overline{(u, rn + sn)}$. Notice now that $(u, rn+sn) \sim (u^2, urn + usn)$ since $u(urn + usn) = u^2(rn + sn)$. So,

$$\begin{aligned}
\overline{(u, rn + sn)} &= (u^2, urn + usn) = \overline{(u, rn)} + \overline{(u, sn)} = r\overline{(u, n)} + s\overline{(u, n)}; \text{ also,} \\
rs\overline{(u, n)} &= \overline{(u, (rs)n} = \overline{(u, r(sn)} = r\overline{(u, sn)} = r(s(\overline{(u, n)})).
\end{aligned}$$

Take now $\overline{(w, m)} \in U^{-1}N$. We see that

$$\begin{aligned}
r(\overline{(u, n)} + \overline{(w, m)}) &= r(\overline{(uw, wn + um)} \\
&= \overline{(uw, r(wn + um))} \\
&= \overline{(uw, rwn + rum)} \\
&= \overline{(uw, w(rn) + u(rm))} \\
&= \overline{(u, rn)} + \overline{(w, rm)} \\
&= r\overline{(u, n)} + r\overline{(w, m)}.
\end{aligned}$$

Finally, take $1 \in R$. We have that $1\overline{(u, n)} = \overline{(u, 1n)} = \overline{(u, n)}$. So, we have that indeed this is an action on $U^{-1}N$, and so it is an R-module as desired. ☐

(b) Show that the map from $Q \times N$ to $U^{-1}N$ defined by sending $(a/b, n)$ to $\overline{(b, an)}$, where $a \in R$, $b \in U$, $n \in N$, is an R-balanced map, so induces a homomorphism from $Q \otimes_R N$ to $U^{-1}N$. Show that the map g from $U^{-1}N$ to $Q \otimes_R N$ defined by $g(\overline{(u, n)}) = (1/u) \otimes n$ is well defined and is an inverse homomorphism to f. Conclude that $Q \otimes_R N \cong U^{-1}N$ as R-modules.

Proof. Let us define a map $\phi : Q \times N \to U^{-1}N$ by $(a/b, n) \mapsto \overline{(b, an)}$. Letting $a/b, c/d \in Q$ and $m, n \in N$, we have that

$$\phi\left(\left(\frac{a}{b} + \frac{c}{d}, n\right)\right) = \phi\left(\left(\frac{ad + bc}{bd}, n\right)\right) = \overline{(bd, (ad + bc)n)}$$

$$
\begin{aligned}
&= \overline{(bd, adn_b cn)} = \overline{(bd, dan + bcn)} \\
&= \overline{(b, an)} + \overline{(d, cn)} = \phi\left(\left(\frac{a}{b}, n\right)\right) + \phi\left(\left(\frac{c}{d}, n\right)\right), \text{ and} \\
\phi\left(\left(\frac{a}{b}, n+m\right)\right) &= \overline{(b, a(n+m))} = \overline{(b, an + am)} \\
&\stackrel{\sim}{=} \overline{(b^2, ban + bam)} = \overline{(b, an)} + \overline{b, am)} \\
&= \phi\left(\left(\frac{a}{b}, n\right)\right) + \phi\left(\left(\frac{a}{b}, m\right)\right).
\end{aligned}
$$

Indeed, we have that ϕ is biadditive. Take now $r \in R$, and we see that

$$
\phi\left(\left(\frac{a}{b}r, n\right)\right) = \phi\left(\left(\frac{ar}{b}, n\right)\right) = \overline{(b, arn)} = \overline{(b, a(rn))} = \phi\left(\left(\frac{a}{b}, rn\right)\right).
$$

Indeed, we have that ϕ is R-balanced. Let us consider now a map $g : U^{-1}N \to Q \otimes_R N$ defined as $\overline{(u, n)} \mapsto (1/u) \otimes n$. We aim to show that g is well defined. So, take $\overline{(u, n)} = \overline{(w, m)}$. We have that

$$
\begin{aligned}
g(\overline{(u, n)}) &= \frac{1}{u} \otimes n = \left(\frac{1}{u}\frac{w}{w}\right) \otimes n = \frac{1}{uw} \otimes wn \\
&\stackrel{\sim}{=} \frac{1}{uw} \otimes um = \frac{u}{uw} \otimes m = \frac{1}{w} \otimes m \\
&= g(\overline{(w, m)}).
\end{aligned}
$$

So, g is indeed well defined. Now we show that g is an R-module homomorphism. Take elements $\overline{(u, n)}, \overline{(w, m)} \in U^{-1}N$, and take $r \in R$. We see that:

$$
\begin{aligned}
g(r\overline{(u, n)} + \overline{(w, m)} &= g(\overline{(u, rn)} + \overline{(w, m)}) = g(\overline{(uw, wrn + um)}) \\
&= \frac{1}{uw} \otimes (wrn + um) = \frac{1}{uw} \otimes (wrn) + \frac{1}{uw} \otimes (um) \\
&= \frac{1}{uw} \otimes r(wn) + \frac{1}{uw} \otimes (um) = \frac{rw}{uw} \otimes n + \frac{u}{uw} \otimes m \\
&= r\left(\frac{1}{u} \otimes n\right) + \frac{1}{w} \otimes m = rg(\overline{(u, n)}) + g(\overline{(w, m)}).
\end{aligned}
$$

So, indeed, g is an R-module homomorphism. Now we want to show that $g = f^{-1}$. By Theorem 10 in the book, we have that f is a unique homomorphism such that $\phi = f \circ i$, with $i : Q \times N \to Q \otimes_R N$ being R-balanced. Taking $\overline{(u, n)} \in U^{-1}N$, we have

$$
(f \circ g)(\overline{(u, n)}) = f\left(\frac{1}{u} \otimes n\right) = f(i\left(\frac{1}{u} \otimes n\right)) = \phi\left(\left(\frac{1}{u} \otimes n\right)\right) = \overline{(u, n)}.
$$

So, $f \circ g = \text{id}$. Similarly, take now $\sum_i^n a_i/b_i \otimes n_i \in Q \otimes_R N$. We see that

$$
\begin{aligned}
(g \circ f)\left(\sum_i^n \frac{a_i}{b_i} \otimes n_i\right) &= g\left(f\left(\sum_i^n \frac{a_i}{b_i} \otimes n_i\right)\right) = g\left(\phi\left(\sum_i^n \frac{a_i}{b_i} \otimes n_i\right)\right) = g\left(\sum_i^n \overline{(b_i, a_i n_i)}\right) \\
&= \sum_i^n g(\overline{(b_i, a_i n_i)}) = \sum_i^n \left(\frac{1}{b_i} \otimes a_i n_i\right) = \sum_i^n \left(\frac{a_i}{b_i} \otimes n_i\right).
\end{aligned}
$$

So, $g \circ f = \text{id}$ as well, and therefore we have that f and g are inverses. So, $Q \otimes_R N \cong U^{-1}N$ as desired. □

(c) Conclude from (b) that $(1/d) \otimes n$ is 0 in $Q \otimes_R N$ if and only if $rn = 0$ for some nonzero $r \in R$.

Proof. ($\Longrightarrow$) First, assume that $(1/d) \otimes n = 0$. Then there exists some $\overline{(u, n)} \in U^{-1}N$ where $g(\overline{(u, n)}) = (1/d) \otimes n$, and so $g(\overline{(u, n)}) = 0$, $f(g(\overline{(u, n)})) = f(0)$, and $\overline{(u, n)} = f(0)$. Now since

f is a homomorphism, we have that $f(0) = 0 \in U^{-1}N$, and we know from part (a) that $\overline{(1,0)}$ is the additive identity of $U^{-1}N$. Therefore $f(0) = \overline{(1,0)}$, giving that $\overline{(1,0)} = \overline{(u,n)}$. So, we have that $1 \cdot n = 0$ by the given equivalence relation. Thus, we have that $1 = r \in R$ is such that $rn = 0$.

($\Longleftarrow$) Assume now that there exists some $r \in R$ such that $rn = 0$. Then we have that

$$\frac{1}{d} \otimes n = \frac{1}{d}\frac{r}{r} \otimes n = \frac{1}{dr} \otimes rn = \frac{1}{dr} \otimes 0 = \frac{1}{dr} \otimes 0 \cdot 0 = \frac{1}{dr} 0 \otimes 0 = 0 \otimes 0.$$

So, $(1/d) \otimes n = 0 \in Q \otimes_R N$ if and only if $rn = 0$ for some nonzero $r \in R$. $\square$

(d) If A is an abelian group, show that $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$ if and only if A is a torsion abelian group (i.e. every element of A has finite order).

Proof. ($\Longleftarrow$) First, assume A is a torsion abelian group, and take some simple tensor $r \otimes a \in \mathbb{Q} \otimes_{\mathbb{Z}} A$. Then there exists some $n \in \mathbb{Z} \setminus \{0\}$ such that $an = 0$. So we see that

$$r \otimes a = \left(\frac{r}{n} n\right) \otimes a = \frac{r}{n} \otimes na = \frac{r}{n} \otimes 0 = 0.$$

Therefore, any simple tensor is 0, and therefore the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$.

($\Longrightarrow$) Now, let us suppose for the sake of contradiction that $an \neq 0$ for all $n \in \mathbb{Z}$. Then $a\mathbb{Z}$ is a free $\mathbb{Z}$-module of rank 1, and therefore $\mathbb{Q} \otimes_{\mathbb{Z}} a\mathbb{Z}$ is a free $\mathbb{Q}$ module of dimension 1. Recall now that some simple tensor $a \otimes b \in A \otimes B$ is zero if and only if $(a, b) \mapsto 0$ via any bilinear map sourced at $A \otimes B$. Since $1 \otimes a \neq 0$ in $\mathbb{Q} \otimes \mathbb{Z} a\mathbb{Z}$, there exists some $\mathbb{Z}$-bilinear map $\mathbb{Q} \otimes_{\mathbb{Z}} a\mathbb{Z} \to M$, for M some $\mathbb{Z}$-module, which does not annihilate $(1, a)$. Extending, we have that there exists a map $\mathbb{Q} \otimes_{\mathbb{Z}} A \to M$, and therefore $1 \otimes a \neq 0$. $\square$

9 Prove that $U^{-1}N$ is an abelian group under the addition defined by $\overline{(u_1, n_1)} + \overline{(u_2, n_2)} = \overline{(u_1u_2, u_2n_1 + u_1n_2)}$. prove that $r\overline{(u, n)} = \overline{(u, rn)}$ defines an action of R on $U^{-1}N$ making it into an R-module. [This is an example of *localization* considered in general in Ch. 15 §4, cf. also Ch. 7 §5]

Proof. Let us consider the R-module homomorphism $\phi : N \to Q \otimes_R N$ defined by $n \mapsto 1 \otimes n$. Taking some element $n \in \ker \phi$, we have that

$$\begin{aligned} \phi(n) &= 1 \otimes n \\ \frac{r}{r}(1 \otimes n) &= 0 = \frac{r}{r} \otimes n = \frac{1}{r} \otimes rn. \end{aligned}$$

By problem 8(c), we have that $(1/r) \otimes rn = 0$ if and only if $rn = 0$. Therefore there must exist some $r \in R$ such that $rn = 0$ so that $\ker \phi \subset \mathrm{Tor} N$. So, take some $n \in \mathrm{Tor} N$. There then exists some $r \in R$ such that $rn = 0$. We have the following:

$$\phi(n) = 1 \otimes n = \frac{r}{r}(1 \otimes n) = \frac{r}{r} \otimes n = \frac{1}{r} \otimes rn = 0.$$

So indeed, $n \in \ker \phi$. Therefore $\ker \phi = \mathrm{Tor} N$ as desired. $\square$

14 Let I be an arbitrary nonempty index set and for each $i \in I$ let N_i be a left R-module. Let M be a right R-module. Prove the group isomorphism: $M \otimes (\bigoplus_{i \in I} N_i) \cong \bigoplus_{i \in I}(M \otimes N_i)$, where the direct sum of an arbitrary collection of modules is defined in Exercise 10.3.20. [Use the same argument as for the direct sum of two modules, taking care to note where the direct *sum* hypothesis is needed – cf. the next exercise.]

Proof. **Lemma**: M is the direct sum of modules M_i if and only if there exist injections $\iota_i \in \hom_{\mathbf{Mod}}(M_i, M)$ and projections $\pi_i \in \hom_{\mathbf{Mod}}(M, M_i)$ such that $\pi_i \circ \iota_j$ is the identity on M_i if $i = j$ and the 0 map otherwise, and such that $\sum \iota_i \circ \pi_i = \mathrm{id}_M$.

Proof. ($\Longrightarrow$) Suppose that $M = \bigoplus M_i$. There exist projection and injection maps $\pi_i : M \to M_i$ and $\iota_i : M_i \to M$ respectively which satisfy the required conditions via universal properties of coproducts.

($\Longleftarrow$) Suppose now that the specified maps exist. Then there exists some map $\pi : M \to \bigoplus M_i$ defined by $m \mapsto \sum \pi_i(m)$. Moreover, the injection maps ι_i can be combined to give a map $\iota : \bigoplus M_i \to M$. By construction, we have that these are inverse isomorphisms as desired. □

We now aim to show that $M \otimes (\bigoplus_{i\in I} N_i) \cong \bigoplus_{i\in I}(M \otimes N_i)$. To do so, we consider the homomorphisms

$$1 \otimes \iota_i : M \otimes N_i \to M \otimes N \quad \text{and} \quad 1 \otimes \pi_j : M \otimes N \to M \otimes N_j.$$

We see that these homomorphisms can be composed and summed as in the hypothesis of the lemma:

$$\begin{gathered}(1 \otimes \pi_j)(1 \otimes \iota_i) = 1 \otimes \pi_j\iota_i = (\pi_j\iota_i)(1 \otimes 1),\\ \sum_i (1 \otimes \iota_i)(1 \otimes \pi_i) = 1 \otimes \sum_i \iota_i\pi_i = 1 \otimes 1 = \mathrm{id}_{M\otimes N}.\end{gathered}$$

By the preceding lemma, we have that indeed $M \otimes (\bigoplus_{i\in I} N_i) \cong \bigoplus_{i\in I}(M \otimes N_i)$. □

15 Show that tensor products do not commute with direct products in general. [Consider the extension of scalars from $\mathbb{Z}$ to $\mathbb{Q}$ of the direct product of the modules $M_i = \mathbb{Z}/2^i\mathbb{Z}$ with $i = 1, 2, \cdots$]

Proof. **Lemma 1**: if G is an abelian group and contains no elements of infinite order, then $\mathbb{Q} \otimes_{\mathbb{Z}} G = 0$. Indeed, if $g \in G$ with order n, then for $q \in \mathbb{Q}$ we have that

$$q \otimes g = (nr/n) \otimes g = (r/n) \otimes (ng) = (r/n) \otimes 0 = 0.$$

Lemma 2: if G is an abelian group with an element of infinite order, then $\mathbb{Q} \otimes_{\mathbb{Z}} G \neq 0$. Indeed, if we take $g \in G$ to be such that $|g| = \infty$, then it suffices to show that $\mathbb{Q} \otimes \langle g\rangle \neq 0$. This is true since g having infinite order gives that $\langle g\rangle \cong \mathbb{Z}$,, and so $\mathbb{Q} \otimes \langle g\rangle \cong \mathbb{Q} \otimes \mathbb{Z} \cong \mathbb{Q} \neq 0$.

Let us consider the direct product of modules $M = \prod_{i=1}^{\infty} M_i$, with $M_i = \mathbb{Z}/2^i\mathbb{Z}$. We claim that $\mathbb{Q} \otimes_{\mathbb{Z}} \prod_{i=1}^{\infty} M_i \ncong \prod_{i=1}^{\infty}(\mathbb{Q} \otimes_{\mathbb{Z}} M_i)$. Notice that each M_i is a finite abelian group, and therefore $\mathbb{Q} \otimes_{\mathbb{Z}} M_i = 0$ by the first lemma. Therefore $\prod_{i=1}^{\infty}(\mathbb{Q} \otimes_{\mathbb{Z}} M_i) = 0$. Let now $m = (1 + 2^i\mathbb{Z}) \in M$, and suppose $|m| = n$, where n is finite. Then we have that $n + 2^i\mathbb{Z} = 0$ for all $i \in \mathbb{N}$. So, $2^i \mid n$ for all $i \in \mathbb{N}$, but this is a contradiction. Therefore $|m| = \infty$, and so the second lemma gives that $\mathbb{Q} \otimes_{\mathbb{Z}} M \neq 0$. Indeed we see that tensor products do not always commute with direct products, and we are done. □

16 Suppose R is commutative and let I and J be ideals of R, so R/I and R/J are naturally R-modules.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 \bmod I) \otimes (r \bmod J)$.

Proof. Consider an element of the tensor product, $\sum_{i=1}^{n}(a_i + I) \otimes (b_i + J)$, where the $a_i, b_i \in R$. We see the following:

$$\begin{aligned}\sum_{i=1}^{n}(a_i + I) \otimes (b_i + J) &= \sum_{i=1}^{n} a_i(1 + I) \otimes (b_i + J) = \sum_{i=1}^{n}(1 + I) \otimes a_i(b_i + J)\\ &= \sum_{i=1}^{n}(1 + I) \otimes (a_ib_i + J) = (1 + I) \otimes \left(\sum_{i=1}^{n} a_ib_i + J\right).\end{aligned}$$

Certainly $\sum_{i=1}^{n} a_ib_i \in R$, so we let that be the r as in the problem statement, and we are done. □

(b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r \bmod I) \otimes (r' \bmod J)$ to $rr' \bmod (I + J)$.

Proof. Let us consider the map $\phi : R/I \times R/J \to R/(I+J)$ defined as $(a+I, b+J) \mapsto ab + I + J$. This is well defined, as if $a + I = a' + I$ and $b + J = b' + J$, then $a - a' \in I$ and $b - b' \in J$. Thus, $ab - a'b' = (a - a')b + a'(b - b') \in I + J$, which gives that

$$\phi(a + I, b + J) = ab + I + J = a'b' + I + J = \phi(a' + I, b' + J)$$

as desired. Next, we show that ϕ is bilinear. Indeed, taking $a, b, c, d, r, s \in R$, we have

$$\begin{aligned}
\phi(r(a+I) + s(c+I), b+J) = \phi(ra + sc + I, b + J) &= (ra + sc)b \\
&= rab + scb \\
&= r\phi(a + I, b + J) + s\phi(c + I, b + J).
\end{aligned}$$

Indeed, this gives that there exists an R-module homomorphism $\psi : R/I \otimes_R R/J \to R/(I+J)$ which sends $(a + I, b + J) \mapsto ab + I + J$. We aim to show that this is a bijection. Indeed, ψ is surjective since for any $r \in R$, we have that $\psi((1 + I) \otimes (r + J)) = r + I + J \in R/(I + J)$. For injectivity, let us consider some element $x \in R/I \otimes_R R/J$ such that $x \in \ker \psi$. Part (a) tells us that we can write $x = (1 + J) \otimes (r + J)$ for some $r \in R$. So, $\psi((1 + I) \otimes (r + J)) = r + I + J = 0$, and so $r \in I + J$. So, we see

$$\begin{aligned}
x &= (1 + I) \otimes (r + J) = (1 + I) \otimes (a + b + J) \\
&= (1 + I) \otimes (a + J) = (1 + I) \otimes a(1 + J) \\
&= a(1 + I) \otimes (1 + J) = (a + I) \otimes (1 + J) \\
&= (0 + I) \otimes (1 + J) = 0 \cdot (0 + I) \otimes (1 + J) \\
&= 0_{R/I \otimes_R R/J}.
\end{aligned}$$

So, we have that ψ is bijective, and is therefore an isomorphism as desired. $\square$

25 Let R be a subring of the commutative ring S and let x be indeterminate over S. Prove that $S[x]$ and $S \otimes_R R[x]$ are isomorphic as S-algebras.

Proof. Consider the universal property stated in Theorem 10.4.8 which states that for any bilinear φ, there exists a unique R-module homomorphism Φ such that the following diagram commutes:

$$\begin{tikzcd}
S \times R[x] \arrow[r, "\iota"] \arrow[d, "f"'] \arrow[dr, "\forall\varphi"] & S \otimes_R R[x] \arrow[d, "\exists!\Phi"] \\
N & S[x] \arrow[l, "\phi"]
\end{tikzcd}$$

In particular, φ factors through Φ, that is $\varphi = \Phi \circ \iota$. We also added an arbitrary R-module, N, to the diagram, for which f is a bilinear map which factors through ϕ, that is $f = \phi \circ \varphi$.

In order to prove the claim, we want to show that indeed φ is a bilinear map, and we want to show that Φ is an R-module homomorphism. Incomplete. $\square$