Real Analysis Homework 4

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Throughout, if (a,b) is an interval in $\mathbb{R}$, either bounded or unbounded, we will denote $\int_{(a,b)} f dm$ (the Lebesgue integral) by $\int_a^b f(x)dx$.

Problem 1

Consider the function $f(x) = x^{-1/2}\chi_{(0,1)}(x)$. Let $\{r_n\}$ be an enumeration of the rational numbers and define $g(x) = \sum_{n=1}^\infty 2^{-n} f(x - r_n)$ for $x \in \mathbb{R}$. Prove that $g \in L^1(\mathbb{R}, m)$ and therefore g is finite a.e. Show that g is unbounded on every interval, and it remains after any modification on a set of Lebesgue measure zero.

Proof. First, we aim to show that $\int_{\mathbb{R}} |g| dx < \infty$. Since f and therefore g are non-negative functions, we have that by Tonelli's Theorem, we can interchange the integral and the sum as follows:

$$\int_{\mathbb{R}} g(x)dx = \sum_{n=1}^\infty 2^{-n} \underbrace{\int_{\mathbb{R}} (x-r_n)^{-1/2}\chi_{(0,1)}(x-r_n)dx}_{\bigstar}$$

Let us now compute the integral $\bigstar$.

$$\begin{aligned}
\bigstar &= \int_{\mathbb{R}} (x-r_n)^{-1/2}\chi_{(0,1)}(x-r_n)dx \\
&= \int_{\mathbb{R}} (x-r_n)^{-1/2}\chi_{(r_n,r_n+1)}(x)dx && \text{rewrite characteristic function} \\
&= \lim_{k\to\infty} \int_{\mathbb{R}} (x-r_n)^{-1/2}\chi_{r_n+\frac{1}{k},r_n+1)}(x)dx && \text{via Monotone Convergence Theorem} \\
&= \lim_{k\to\infty} \int_{r_n+\frac{1}{k}}^{r_n+1} (x-r_n)^{-1/2}dx && \text{Riemann = Lebesgue integral here} \\
&= \lim_{k\to\infty} 2(x-r_n)^{1/2}\Big|_{r_n+\frac{1}{k}}^{r_k+1} && \text{take antiderivative} \\
&= \lim_{k\to\infty} 2 - 2\frac{1}{k^{1/2}} = 2 && \text{evaluate.}
\end{aligned}$$

Therefore, we can rewrite the integral of g as

$$\int_{\mathbb{R}} g(x)dx = \sum_{n=1}^\infty 2^{-n+1} < \infty.$$

Clearly the integral is finite, so $g \in L^1(\mathbb{R}, m)$. $\diamond$

Second, let us consider the interval $I = (a,b)$. We know that $\mathbb{Q}$ is dense in $\mathbb{R}$, so take an element $r_{n_0} \in \{r_n\} \subset \mathbb{Q}$ such that $r_{n_0} \in (a,b)$, and let $x \in (a,b) \cap (r_{n_0}, r_{n_0}+1)$. Since $2^{-n_0}(x-r_{n_0})^{-1/2}$ is non-negative, we have that $g(x) \geq 2^{-n_0}(x-r_{n_0})^{-1/2} > M$ for a sufficiently large $M \gg 0$ for which x is sufficiently close to r_{n_0}. So, given any $M > 0$, there exists some interval around x inside of (a, b) such that $g(x) > M$. This establishes that g is unbounded on every interval.

Take now some open interval (a, b), and consider some set E such that $m(E) = 0$. By the density of the rational numbers, we can pick some number $r_k \in (a, b)$. With this, there exists some sequence $\{x_k\}$ such that $x_k \notin E$ for any k, otherwise there would exist some $\epsilon > 0$ such that $(r_k - \epsilon, r_k + \epsilon) \subset E$, which contradicts the fact that E is a null set. $\square$

Problem 2

Consider the functions $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$. Show that

(a) $\sum_{n=1}^{\infty} \int_0^{\infty} |f_n(x)| dx = \infty$

(b) $\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x) dx = 0$

(c) $\sum_{n=1}^{\infty} f_n \in L^1([0, \infty), m)$ and $\int_0^{\infty} \sum_{n=1}^{\infty} dx = \log(b/a)$.

The example above is such that $\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x) dx \neq \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x) dx$. Why does this example not contradict any of the related propositions stated in class regarding interchanging integral and summation signs?

Solution. Let us begin by noting that f_n is continuous, and therefore on any bounded interval $[\alpha, \beta]$ the Riemann integral coincides with the Lebesgue integral, which we can calculate as:

$$\int_{[\alpha,\beta]} f_n dm = \int_{\alpha}^{\beta} f_n(x) dx = \frac{1}{n} \left[e^{-nbx} - e^{-nax} \right]_{\alpha}^{\beta}.$$

Also, as will come in use at various points later, the point x for which $f_n(x) = 0$ is given by:

$$\begin{aligned} ae^{-nax} &= be^{-nbx} \\ \log(a) - nax &= \log(b) - nbx \\ \log(a) - \log(b) &= (na - nb)x \\ \frac{\log(b) - \log(a)}{nb - na} &= x =: \ell_n. \end{aligned}$$

(a) *Proof.* By monotone convergence theorem, we have that $\int_{[0,\infty)} |f_n| dm = \lim_{k\to\infty} \int_{[0,k]} |f_n| dm$. Also, since $|f_n|$ is continuous, we have that the Riemann integral on any bounded interval exists and coincides with the Lebesgue integral. By our definition of ℓ_n (that is, the point at which the graph crosses the x-axis), we can write the following:

$$|f_n(x)| = \begin{cases} -f_n(x) & x \leq \ell_n, \\ f_n(x) & x \geq \ell_n \end{cases}$$

which ensures that the value of $|f_n(x)|$ is always greater than or equal to 0. So, for $k > \ell_n$, we have

$$\begin{aligned} \int_{[0,k]} |f_n| dm &= -\int_0^{\ell_n} f_n(x) dx + \int_{\ell_n}^{k} f_n(x) dx \\ &= -\frac{1}{n} \left[e^{-nbx} - e^{-nax} \right]_0^{\ell_n} + \frac{1}{n} \left[e^{-nbx} - e^{-nax} \right]_{\ell_n}^{k} \\ &= -\frac{2}{n} \left[e^{-nb\ell_n} - e^{-na\ell_n} \right] + \frac{1}{n} \left[e^{-nbk} - e^{-nak} \right]. \end{aligned}$$

If we take $k \to \infty$, then we have that the second term goes to 0, that is

$$\begin{aligned} \int_{[0,\infty)} |f_n| dm = -\frac{2}{n} \left[e^{-nb\ell_n} - e^{-na\ell_n} \right] &= -\frac{2}{n} \left[e^{-nb(\log(b/a)/(n(b-a))} - e^{-na(\log(b/a)/(n(b-a))} \right] \\ &= \frac{1}{n} \cdot 2 \cdot \left[e^{-b(\log(b/a)/(b-a)} + e^{-a(\log(b/a)/(b-a)} \right]. \end{aligned}$$

We see that the integral is simply a constant times $1/n$, whose sum diverges. Therefore we have that indeed $\sum_{n=1}^{\infty}\int_0^{\infty}|f_n(x)|dx = \infty$. $\square$

(b) *Proof.* We shall deal with the positive and negative parts of this function separately. First, the positive part, f_n^+. By the monotone convergence theorem, we have that

$$\int_{[0,\infty)} f_n^+ dm = \lim_{k\to\infty}\int_{[0,k]} f_n^+ dm.$$

Since the graph only crosses the x-axis at one point, ℓ_n, we have that $f_n^+ = f_n\chi_{[\ell_n,\infty)}$. Moreover, we have that for any $k > \ell_n$, we have a bounded interval $[\ell_n, k]$. As such, the Lebesgue integral coincides with the Riemann integral on each of these intervals:

$$\int_{[0,k]} f_n^+ dm = \int_{[\ell_n,k]} f_n dm = \int_{\ell_n}^{k} f_n(x)dx = \frac{1}{n}\left[e^{-nax} - e^{-nbx}\right]_{\ell_n}^{k}.$$

All together, we have

$$\int_{[0,\infty]} f_n^+ dm = \lim_{k\to\infty}\frac{1}{n}\left[e^{-nbx} - e^{-nax}\right]_{\ell_n}^{k} = \frac{1}{n}\left[e^{-na\ell_n} - e^{-nb\ell_n}\right].$$

A similar argument holds for f_n^-. Again, since the graph only crosses the x-axis at one point, ℓ_n, we have that $f_n^- = -f_n\chi_{(-\infty,\ell_n]}$, which is a non-negative function. As such, considering the lower bound given in the problem statement, 0, we have that $[0,\ell_n]$ is a bounded interval. So, on said interval, the Riemann and Lebesgue integrals coincide. That is,

$$\int_{[0,\infty)} f_n^- dm = -\int_{[0,\ell_n]} f_n dm = -\int_0^{\ell_n} f_n(x)dx = -\frac{1}{n}\left[e^{-nbx} - e^{-nax}\right]_0^{\ell_n} = -\frac{1}{n}\left[e^{-na\ell_n} - e^{-nb\ell_n}\right].$$

Recall however that we are off by a sign due to how we defined f_n^-. So, we see that indeed the integrals of f_n^+ and f_n^- are equal, and as such,

$$\int_{[0,\infty)} f_n dm = \int_{[0,\infty)} f_n^+ dm - \int_{[0,\infty)} f_n^- dm = 0.$$

Certainly then $\sum_n 0 = 0$, and we have achieved our desired result. $\square$

(c) *Proof.* First note that since f_n is continuous for all $n \in \mathbb{N}$, f_n is Lebesgue measurable for all $n \in \mathbb{N}$. Moreover, we can write $\lim_{N\to\infty}\sum_{n=1}^{N} f_n = \sum_{n=1}^{\infty}$, and as such, $\sum_{n=1}^{\infty} f_n(x)$ is also Lebesgue measurable. Now via the geometric series test, we see that

$$\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty}(ae^{-nax} - be^{-nbx})dx = \frac{ae^{-ax}}{1-e^{-ax}} - \frac{be^{-bx}}{1-e^{-bx}} = \frac{a}{e^{ax}-1} - \frac{b}{e^{bx}-1}.$$

So, we have the following Lebesgue integral and estimate:

$$\int_0^{\infty}\left|\sum_{n=1}^{\infty} f_n(x)\right| dx = \int_0^{\infty}\left|\frac{a}{e^{ax}-1} - \frac{b}{e^{bx}-1}\right| dx < \int_0^{\infty}(b-a)dx = \infty.$$

So we see that indeed, $\sum_{n=1}^{\infty} f_n \in L^1([0,\infty), m)$ as desired.

Next, take $0 < s < t$, then since the series $\sum_{n=1}^{\infty} f_n$ is bounded by $b-a$ in the interval $[s,t]$ and is Riemann integrable over $[s,t]$, we can equate the Lebesgue and Riemann integral in this case. Since the sum of continuous functions is also continuous, we have that $\sum_{n=1}^{\infty} f_n$ is continuous, and is therefore Lebesgue measurable.

Now consider a function $\sum_{n=1}^{\infty} f_n \chi_{[s,t]}$. Then as $s \to 0$ and $t \to \infty$, this function approaches $\sum_{n=1}^{\infty} f_n$. Now we see

$$\left|\sum_{n=1}^{\infty} f_n \chi_{[s,t]}\right| \leq \left|\sum_{n=1}^{\infty} f_n\right| \in L^1([0,\infty), m).$$

So, we can apply the dominated convergence theorem to get:

$$\begin{aligned}
\int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)dx &= \int \lim_{\substack{s\to 0^+ \\ t\to\infty}} \sum_{n=1}^{\infty} f_n(x)\chi_{[s,t]}(x)dx \\
&= \lim_{\substack{s\to 0^+ \\ t\to\infty}} \int_s^t \sum_{n=1}^{\infty} f_n(x)dx \\
&= \lim_{\substack{s\to 0^+ \\ t\to\infty}} \left(\log|1-e^{-ax}| - \log|1-e^{-bx}|\right)\Big|_s^t \\
&= \lim_{\substack{s\to 0^+ \\ t\to\infty}} \log\left|\frac{1-e^{-ax}}{1-e^{-bx}}\right| \\
&= \lim_{t\to\infty} \log\left|\frac{1-e^{-at}}{1-e^{-bt}}\right| - \lim_{s\to 0^+} \log\left|\frac{1-e^{-as}}{1-e^{-bs}}\right| \\
&= \lim_{s\to 0^+} \log\left|\frac{1-e^{-as}}{1-e^{-bs}}\right|
\end{aligned}$$

By l'Hôpital's rule, we compute the limit to be

$$\lim_{s\to 0^+} \log\left|\frac{1-e^{-as}}{1-e^{-bs}}\right| = \lim_{s\to 0^+} \log\left|\frac{be^{-bs}}{ae^{-as}}\right| = \log(b/a).$$

So indeed, $\int_0^{\infty} \sum_{n=1}^{\infty} dx = \log(b/a)$ as desired. □

Finally, the example does not contradict Theorem 2.15 because it attains negative values, and therefore the theorem cannot be applied. Theorem 2.25 doesn't apply either because $\sum \int |f_n(x)|dx = \infty$, whereas the theorem requires finiteness.

Problem 3

Compute the following limits. Justify your computations.

(a) $\lim_{n\to\infty} \int_0^{\infty} n(1+n^2x^2)^{-1}dx$

Proof. Let $f_n(x) = n(1+n^2x^2)^{-1}$ for $n \in \mathbb{N}$. Since f_n is continuous for all $n \in \mathbb{N}$, it is a Lebesgue integrable function for all $n \in \mathbb{N}$. Taking $a \leq x < \infty$, we have the following estimate:

$$|f_n(x)| = \frac{n}{1+n^2x^2} < \frac{n}{n^2x^2} = \frac{1}{nx^2} \leq \frac{1}{x^2} \leq \frac{1}{a^2}.$$

So, we see that indeed $f_n(x)$ is bounded in $[a,\infty)$, and therefore the Lebesgue integral is equivalent to the Riemann integral in this case. Moreover, we see that the integral of $|f_n(x)|$ is finite, in particular

$$|f_n(x)| < \frac{1}{a^2} \implies \int_a^{\infty} |f_n(x)|dx < \int_a^{\infty} \frac{1}{a^2}dx = \infty.$$

Therefore, we see that $\{f_n(x)\} \subset L^1([0,\infty), m)$.

Consider now a function $f_n(x)\chi_{[a,t]}$ where $a < t < \infty$. As $t \to \infty$, we see that $f_n(x)\chi_{[a,t]} \to f_n(x)$. Therefore, we have that $|f_n(x)\chi_{[a,t]}| \le |f_n(x)| \in L^1$, so we can apply the dominated convergence theorem. That is,

$$\int_a^\infty f_n(x)dx = \lim_{t\to\infty} \int_a^\infty f_n(x)\chi_{[a,t]}(x)dx = \lim_{t\to\infty}\int_a^t f_n(x)dx$$
$$\implies \int_a^\infty \frac{n}{1+n^2x^2}dx = \lim_{t\to\infty}\int_a^t \frac{n}{1+n^2x^2}dx = \lim_{t\to\infty} \arctan(nx)\Big|_a^t = \frac{\pi}{2} - \arctan(na).$$

We see the following cases, which gives the desired limit:

$$\lim_{n\to\infty}\int_a^\infty f_n(x)dx = \begin{cases} 0 & a > 0, \\ \pi/2 & a = 0, \\ \pi & a < 0. \end{cases}$$

□

(b) $\lim_{n\to\infty}\int_0^\infty (1+x/n)^{-n}\sin(x/n)dx$.

Proof. Let $f_n(x) := (1+x/n)^{-n}\sin(x/n)$. Notice that due to the periodicity of $\sin(x/n)$, we can write the following bound to use later for the application of Dominated Convergence Theorem: $|f_n(x)| \le (1+(x/n))^{-n} =: g_n(x)$. Also, for $n \ge 2$, WolframAlpha says that $\int_0^\infty g_n(x) = n/(n-1)$. Notice that $g_n(x)$ is equivalent to a 'term' in the limit definition of e^{-x}, so define $g(x) := \lim_{n\to\infty} g_n(x) = e^{-x}$. Surely this function is integrable on $(0,\infty)$, so by the Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}\int_0^\infty f_n(x)dx = \int_0^\infty \lim_{n\to\infty}(1+x/n)^{-n}\sin(x/n)dx = \int_0^\infty 0dx = 0$$

□

Problem 4

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to \mathbb{C}$, $g : X \to \mathbb{C}$, and $f_n : X \to \mathbb{C}$, for $n \in \mathbb{N}$, be measurable functions. Suppose $g \in L^1(X,\mu)$, $|f_n| \le g$ a.e., and $f_n \to f$ in measure. Prove that:

(a) $\int_X f d\mu = \lim_{n\to\infty}\int_X f_n d\mu$

(b) $f_n \to f$ in $L^1(X,\mu)$

Proof. We shall take a unified approach to this problem. Notice the following line of implications:

$$f_n \to f \text{ in } L^1 \implies \int |f_n - f| \to 0 \implies \left|\int f_n - \int f\right| = \left|\int (f_n - f)\right|$$
$$\le \int |f_n - f| \to 0 \implies \int f = \lim \int f_n.$$

As such, it suffices to show that $f_n \to f$ in L^1, and we will automatically have part a.

Assume for the sake of contradiction that $\int |f_n - f| \not\to 0$. In particular, there must exist some subsequence $\{f_{n_k}\} \subset \{f_n\}$ such that $\int |f_{n_k} - f_n| \ge \epsilon$ for some fixed arbitrary $\epsilon > 0$. As given in the problem statement, we know $f_n \to f$ in measure. Now when we have a dominated function, which in this case is g. Then we have that $f_n \to f$ in measure implies that $f_{n_k} \to f$ in L^1, which implies that $f_{n_k} \to f$ in measure. Then we have that there exists some subsequence $\{f_{n_{k_j}}\} \subset \{f_{n_k}\}$ such that $f_{n_{k_j}} \to f$ μ-almost everywhere.

Now, we have that $|f_{n_{k_j}} - f| \le |f_{n_{k_j}}| + |f| \le 2g \in L^1$, which means that $\int |f_{n_{k_j}} - f| \to 0$, or more specifically, we have that

$$\lim_j \int |f_{n_{k_j}} - f| = \int |f_{n_k} - f| < \epsilon. \quad \Rightarrow\Leftarrow$$

□

Problem 5

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f_n : X \to \mathbb{C}$ for $n \in \mathbb{N}$ and $f : X \to \mathbb{C}$ be measurable functions. Prove that if $f_n \to f$ almost uniformly, then $f_n \to f$ μ-a.e. and $f_n \to f$ in measure.

Proof. Assume that $f_n \to f$ almost uniformly. Let us take a sequence of sets $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ such that $\mu(E_n) < 1/n$ for all n and such that $f_n \to f$ uniformly on E_n^c. Let us denote $E = \bigcap_n E_n$. By monotonicity, we have that $\mu(E) = 0$ since $\mu(E) \leq \mu(E_n)$ for any n, which we can force to be very large. Now if we take $x \in E^c$, then there exists some n such that $x \in E_n^c$ by definition of intersection. This is equivalent to $x \notin E_n$ for some n. So, this means that $\mu(\{x \in X : f_n(x) \not\to f(x)\}) = 0$. Then by definition we have that f must be convergent everywhere except on a set of measure 0, that is, $f_n \to f$ almost everywhere. $\square$

Problem 6

Let μ be the counting measure on $\mathbb{N}$, $f_n : \mathbb{N} \to \mathbb{C}$ for $n \in \mathbb{N}$, and $f : \mathbb{N} \to \mathbb{C}$. Prove that $f_n \to f$ in measure if and only if $f_n \to f$ uniformly on $\mathbb{N}$.

Proof. ($\Longrightarrow$) Let us fix some $\epsilon > 0$ and consider the set $\{x \in \mathbb{N} : |f_n(x) - f(x)| \geq \epsilon\}$. Take now n large enough such that $\mu(\{x \in \mathbb{N} : |f_n(x) - f(x)| \geq \epsilon\})$ is strictly less than, say, 1. But, since μ is the counting measure, we have that for some set A, if $\mu(A) < 1$ then $A = \emptyset$. So, for n sufficiently large, we have $|f_n(x) - f(x)| < \epsilon$ for all x, and therefore we have uniform convergence.
($\Longleftarrow$) Let us fix some $\epsilon > 0$ and some sufficiently large n_0 such that $\sup(|f_n(x) - f(x)|) < \epsilon$ for $n \geq n_0$. Certainly then we have $\mu(\{x \in \mathbb{N} : |f_n(x) - f(x)| \geq \epsilon,\ n \geq n_0\} = 0$. Since ϵ was chosen arbitrarily, we have convergence in measure. $\square$

Real Analysis Homework 5

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Problem 1

We showed in class that the hypothesis of σ-finiteness of $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ in the Fubini-Tonelli Theorem can not be dropped. The following counterexample shows that the hypothesis $f \in L^1(X \times Y, \mu \times \nu)$ is also needed: Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, and $\mu = \nu =$counting measure. Define the function

$$f(m,n) = \begin{cases} 1 & m = n, \\ -1 & m = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\int_{X\times Y} |f| d\mu \times \nu = \infty$ and that $\int_Y (\int_X f d\mu) d\nu \neq \int_X (\int_Y f d\nu) d\mu$.

Proof. In Homework 3, Problem 11, we saw that integrals with respect to the counting measure can be expressed as sums. In particular, we see

$$\int_{X\times Y} |f(m,n)| d\mu \times \nu = \sum_{m=n} |1| + \sum_{m=n+1} |-1| \geq \sum_{m=n} |1| = \infty.$$

Clearly then our estimate gives $\int_{X\times Y} |f| d\mu \times \nu = \infty$ as desired.

Next, we shall compute both the left and right hand sides of the inequality $\int_Y (\int_X f d\mu) d\nu \neq \int_X (\int_Y f d\nu) d\mu$. First, we shall deal with the left hand side.

$$\begin{aligned}
\int_Y \left(\int_X f d\mu \right) d\nu &= \int_Y \left(\int_X f^n(m) d\mu(m) \right) d\nu(n) \\
&= \int_{\{1\}} \left(\int_X f^n(m) d\mu(m) \right) d\nu(n) + \int_{Y\setminus\{1\}} \left(\int_X f^n(m) d\mu(m) \right) d\nu(n) \\
&= \int_{\{1\}} 1 d\nu(n) + \int_{Y\setminus\{1\}} 0 d\nu(n) \\
&= 1.
\end{aligned}$$

Now for the right hand side.

$$\begin{aligned}
\int_X \left(\int_Y f d\nu \right) d\mu &= \int_X \left(\int_Y f_m(n) d\nu(n) \right) d\mu(m) \\
&= \int_{\mathbb{N}\setminus\{1\}} \left[\int_{\{m\}} f_m d\nu(n) + \int_{\{m-1\}} f_m(n) d\nu(n) + \int_{\mathbb{N}\setminus\{m,m+1\}} f_m(n) d\nu(n) \right] d\mu(m) + \\
&\quad + \int_{\{1\}} \left[\int_{\{1\}} f_m(n) d\nu(n) + \int_{\mathbb{N}\setminus\{1\}} f_m(n) d\nu(n) \right] d\mu(m) \\
&= \int_{\mathbb{N}\setminus\{1\}} (1 - 1 + 0) d\mu(m) + \int_{\{1\}} (1 + 0) d\mu(m) \\
&= \int_{\mathbb{N}\setminus\{1\}} 0 d\mu(m) + \int_{\{1\}} 1 d\mu(m) \\
&= 0 + 0 = 0.
\end{aligned}$$

Certainly $1 \neq 0$, so we have shown the desired result. ∎

Problem 2

"*The integral of a function is the area under its graph.*" Consider a σ-finite measure space $(X, \mathcal{M}, \mu)$ and let $f : X \to [0, \infty)$ belong to $L^1(X, \mu)$. Consider the set

$$G_f = \{(x, y) \in X \times \mathbb{R} : 0 \leq y \leq f(x)\}.$$

Show that $G_f \in \mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$ and that $(\mu \times m)(G_f) = \int_X f d\mu$. The same is true if one replaces $y \leq f(X)$ with $y < f(x)$ in the definition of G_f.

Proof. Let us first consider the function $F_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $F_2(z, y) = z - y$. We have that F_2 is $(\mathcal{B}_{\mathbb{R}^2}, \mathcal{B}_{\mathbb{R}})$-measurable since F_2 is continuous (that is, the preimage of an open set in $\mathcal{B}_{\mathbb{R}}$ is going to be open in $\mathcal{B}_{\mathbb{R}^2}$). Now consider $F_1 : X \times \mathbb{R} \to \mathbb{R}^2$ defined by $F_1(x, y) = (f(x), y)$. We have that F_1 is $(\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}^2})$-measurable since each component is $(\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}^2})$-measurable. We now compose $F_2 \circ F_1 = F$ given by $(x, y) \mapsto f(x) - y$. Indeed, F is $(\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}^2})$-measurable since both F_1 and F_2 are measurable (composition of measurable functions is measurable). Now we aim to show that $G_f \in \mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$. Notice that

$$\begin{aligned} G_f = \{(x, y) \in X \times \mathbb{R} : 0 \leq y \leq f(x)\} &= \{(x, y) \in X \times \mathbb{R} : F(x, 0) \geq 0,\ y \geq 0\} \\ &= F^{-1}([0, \infty)) \cap (X \times [0, \infty)). \end{aligned}$$

Notice that both $F^{-1}([0, \infty))$ and $X \times [0, \infty)$ are both contained in $\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$, and therefore their intersection must be as well. As such, we have $G_f \in \mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$ as desired.

Next, consider the measure space $(\mathbb{R}, L, m)$. We see that this space is σ-finite because for $k \in \mathbb{N}$,

$$\mathbb{R} = \left(\bigcup_{k=1}^{\infty} [k, k+1] \right) \cup \left(\bigcup_{k=1}^{\infty} [-k-1, -k] \right) \text{ where } m([k, k+1]) = m([-k-1, -k]) = 1,$$

and certainly $1 < \infty$. Now note that we can write the following:

$$\chi_{G_f}(x, y) = \chi_{\{y \in \mathbb{R} : 0 \leq y \leq f(x)\}}(x, y).$$

With σ-finiteness established, we can apply the Fubini-Tonelli Theorem as follows:

$$\begin{aligned} (\mu \times m)(G_f) &= \int_{X \times \mathbb{R}} \chi_{G_f} d\mu \times m \\ &= \int_X \left(\int_{\mathbb{R}} \chi_{G_f}(x, y) dm(x) \right) d\mu(y) \\ &= \int_X \left(\int_{\mathbb{R}} \chi_{\{y \in \mathbb{R} : 0 \leq y \leq f(x)\}}(x, y) dm(x) \right) d\mu(y) \\ &= \int_X f d\mu. \end{aligned}$$

The desired result has been shown. □

Problem 3

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces (not necessarily σ-finite).

(a) Show that if $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable and $g : Y \to \mathbb{C}$ is $\mathcal{N}$-measurable, then $h(x, y) = f(x)g(y)$ is $\mathcal{M} \otimes \mathcal{N}$-measurable.

Proof. Let us consider the function $F : X \times Y \to \mathbb{C}$ defined as $(x, y) \mapsto f(x)$. We aim to show that this function is $\mathcal{M} \otimes \mathcal{N}$-measurable. So, consider now $E \in \mathcal{B}_{\mathbb{C}}$. The preimage of E under F is as follows:

$$F^{-1}(E) = \{(x, y) \in X \times Y : f(x, y) \in E\} = \{(x, y) \in X \times Y : f(x) \in E\} = f^{-1}(E) \times Y.$$

Notice that $f^{-1}(E) \in \mathcal{M}$, and certainly $Y \in \mathcal{N}$. As such, we have that $f^{-1}(E) \times Y \in \mathcal{M} \otimes \mathcal{N}$. So, F is $\mathcal{M} \otimes \mathcal{N}$ measurable.

Similarly, consider the function $G : X \times Y \to \mathbb{C}$ defined as $(x, y) \mapsto g(y)$. We aim to show that this function is $\mathcal{M} \otimes \mathcal{N}$-measurable. So, consider now some $E \in \mathcal{B}_{\mathbb{C}}$. The preimage of E under G is as follows:

$$G^{-1}(E) = \{(x, y) \in X \times Y : g(x, y) \in E\} = \{(x, y) \in X \times Y : g(y) \in E\} = X \times g^{-1}(E).$$

Notice that $g^{-1}(E) \in \mathcal{N}$, and certainly $X \in \mathcal{M}$. As such, we have that $X \times g^{-1}(E) \in \mathcal{M} \otimes \mathcal{N}$. So, G is $\mathcal{M} \otimes \mathcal{N}$ measurable.

Finally, we have that $h(x, y) = f(x)g(y) = F(x, y)G(x, y)$. Since both F and G are $\mathcal{M} \otimes \mathcal{N}$-measurable, certainly their product is as well. So, $h(X, y)$ is $\mathcal{M} \otimes \mathcal{N}$-measurable. $\square$

(b) If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and $\int h d\mu \times \nu = (\int f d\mu)(\int g d\nu)$.

Proof. We start with the characteristic function case. So, take $f = \chi_A$ for $A \in \mathcal{M}$ and $g = \chi_B$ for $B \in \mathcal{N}$. Then $h(x, y) = f(x)g(y) = \chi_A(x)\chi_B(y) = \chi_{A\times B}(x, y)$. We have the following:

$$\begin{aligned}\int_{X\times Y} h d\mu \times \nu = \int_{X\times Y} \chi_{A\times B} d\mu \times \nu &= (\mu \times \nu)(A \times B)\\ &= \mu(A)\nu(B) = \left(\int_X \chi_A d\mu\right)\left(\int_Y \chi_B d\nu\right) = \left(\int_X f d\mu\right)\left(\int_Y g d\nu\right).\end{aligned}$$

We see that our result holds for the characteristic function case.

By linearity of the integral, we have that the result also holds for simple functions.

Finally for the general case, consider sequences $\{f_n\}$ and $\{g_n\}$ of measurable simple functions such that $f_n \to f$ and $g_n \to g$ a.e. and $|f_n| \nearrow |f|$ and $|g_n| \nearrow |g|$ a.e. since the functions are complex-valued. By the previous case, we can write the following two lines in order to show that $(\star)$ $h \in L^1$, and $(\star\star)$ $\int h d\mu \times \nu = (\int f d\mu)(\int g d\nu)$:

$$\begin{aligned}(\star) \qquad & \int_{X\times Y} |f_n g_n| d\mu \times \nu = \left(\int_X |f_n| d\mu\right)\left(\int_Y |g_n| d\nu\right),\\ (\star\star) \qquad & \int_{X\times Y} f_n g_n d\mu \times \nu = \left(\int_X f_n d\mu\right)\left(\int_Y g_n d\nu\right).\end{aligned}$$

First for $(\star)$, an application of the monotone convergence theorem yields

$$\int_{X\times Y} |fg| d\mu \times \nu = \left(\underbrace{\int_X |f| d\mu}_{\in L^1}\right)\left(\underbrace{\int_Y |g| d\nu}_{\in L^1}\right) < \infty.$$

So, we see that indeed $h = fg \in L^1(\mu \times \nu)$.

Next for $(\star\star)$, we know that h is indeed in L^1. So, since $f_n g_n \to h$, we have that $|f_n g_n| \leq |fg| = |h| \in L^1(\mu \times \nu)$. So, an application of the dominated convergence theorem yields

$$\int_{X\times Y} f_n g_n d\mu \times \nu \overset{\text{DCT}}{\longrightarrow} \int_{X\times Y} fg d\mu \times \nu = \int_{X\times Y} h d\mu \times \nu.$$

Recall that we claimed that $|f_n| \nearrow |f|$ and $|g_n| \nearrow |g|$, and so certainly $|f_n| \leq |f| \in L^1$ and $|g_n| \leq |g| \in L^1$, so applying the dominated convergence theorem gives the following as well:

$$\int_X f_n d\mu \overset{\text{DCT}}{\longrightarrow} \int_X f d\mu \quad \text{and} \quad \int_Y g_n d\nu \overset{\text{DCT}}{\longrightarrow} \int_Y g d\nu.$$

So, we see that taking $n \to \infty$, we have that $\int h d\mu \times \nu = (\int f d\mu)(\int g d\nu)$ as desired. $\square$

Problem 4

The Fubini-Tonelli Theorem is valid when $(X, \mathcal{M}, \mu)$ is an arbitrary measure space and Y is a countable set, $\mathcal{N} = \mathcal{P}(Y)$, and ν is the counting measure on Y. See Theorems 2.15 and 2.25 in Folland's book, which were proved in class.

Solution. Recall Folland Theorem 2.15: If $\{f_n\}$ is a sequence of nonnegative $\mathcal{M}$-measurable functions and $f(x) = \sum_{n=1}^{\infty} f_n(x)$, then $\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$. Given that ν is the counting measure on Y, we have that $\int_{\mathbb{N}} f_n d\nu(n) = \sum_{n=1}^{\infty} f_n$. So, we have the following.

$$\begin{aligned} &\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu \\ \Longrightarrow &\int_X \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu \\ \Longrightarrow &\int_X \left(\int_{\mathbb{N}} f_n(x) d\nu(n) \right) d\mu(x) = \int_{\mathbb{N}} \left(\int_X f_n(x) d\mu(x) \right) d\nu(n). \end{aligned}$$

Indeed, we see that Folland 2.15 is a special case of the Fubini-Tonelli Theorem.

Now recall Folland Theorem 2.25: if $f_n \in L^1(\mu)$ such that $\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty$, then $\sum_{n=1}^{\infty} f_n = f \in L^1(\mu)$ almost everywhere and $\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$. Again given that ν is the counting measure on Y, we have that $\int_{\mathbb{N}} f_n d\nu(n) = \sum_{n=1}^{\infty} f_n$. So, we have the following:

$$\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty \Longrightarrow \int_{\mathbb{N}} \left(\int_X |f_n(x)| d\mu(x) \right) d\nu(n) < \infty.$$

We have the following implications now.

$$\begin{aligned} &\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu \\ \Longrightarrow &\int_X \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu \\ \Longrightarrow &\int_X \left(\int_{\mathbb{N}} f_n(x) d\nu(n) \right) d\mu(x) = \int_{\mathbb{N}} \left(\int_X f_n(x) d\mu(x) \right) d\nu(n). \end{aligned}$$

Indeed, we see that Folland 2.25 is a special case of the Fubini-Tonelli Theorem.