Real Analysis Homework 2

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Problem 1

Let F be an increasing, right continuous function defined on $\mathbb{R}$ and consider the measure μ_F induced by F on $\mathcal{B}_\mathbb{R}$. Show that (a) $\mu_F(\{a\}) = F(a) - F(a^-)$; (b) $\mu_F([a,b]) = F(b) - F(a^-)$; (c) $\mu_F([a,b)) = F(b^-) - F(a^-)$; (d) $\mu_F((a,b)) = F(b^-) - F(a)$.

Solution. We will solve the first problem, and use the result in other problems.

(a) First, let us consider the fact that

$$\{a\} = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, a\right].$$

Moreover, if we consider each $J_n := (a - \frac{1}{n}, a]$, we see that $J_1 \supset J_2 \supset \cdots$. Let us invoke the property of continuity from above of $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu_F)$, then we have the following:

$$\begin{aligned} \mu_F(\{a\}) = \mu\left(\bigcap_{n=1}^{\infty}\left(a - \frac{1}{n}, a\right]\right) = \lim_{n\to\infty} \mu\left(\left(a - \frac{1}{n}, a\right]\right) &= \lim_{n\to\infty}(F(a) - F(a - 1/n)) \\ &= F(a) - \lim_{n\to\infty} F(a - 1/n). \end{aligned}$$

Since F is increasing, we have that $\lim_{n\to\infty} F(a - 1/n) = \lim_{x\to a^-} F(x) = F(a^-)$, and since F is only taken to be right continuous, we know that we cannot have that $\lim_{n\to\infty} F(a-1/n) = \lim_{x\to a} F(x) = F(a)$. So, $\mu(\{a\}) = F(a) - F(a^-)$.

(b) Observe that $[a,b] = (a,b] \cup \{a\}$. Then we have that $\mu_F([a,b]) = \mu_F((a,b]) + \mu_F(\{a\})$. Recall that in class we showed that $\mu_F((a,b]) = F(b) - F(a)$, and from part (a), we have

$$\mu_F([a,b]) = \mu_F((a,b]) + \mu_F(\{a\}) = (F(b) - F(a)) + (F(a) - F(a^-)) = F(b) - F(a^-).$$

(c) Observe that $[a,b] = [a,b) \cup \{b\}$. Then we have that $\mu_F([a,b]) = \mu_F([a,b)) + \mu_F(\{b\})$, or equivalently, $\mu_F([a,b)) = \mu_F([a,b]) - \mu_F(\{b\})$. From parts (a) and (b), we have

$$\mu_F([a,b)) = \mu_F([a,b]) - \mu_F(\{b\}) = (F(b) - F(a^-)) - (F(b) - F(b^-)) = F(b^-) - F(a^-).$$

(d) Observe that $(a,b) = (a,b] \setminus \{b\}$. Then we have that $\mu_F((a,b)) = \mu_F((a,b]) - \mu_F(\{b\})$. So from what we showed in class and from part (a), we have

$$\mu_F((a,b)) = \mu_F((a,b]) - \mu_F(\{b\}) = (F(b) - F(a)) - (F(b) - F(b^-)) = F(b^-) - F(a).$$

Problem 2

Let F be an increasing, right continuous function defined on $\mathbb{R}$. Let μ be the completion of the measure μ_F induced by F on $\mathcal{B}_\mathbb{R}$ and denote by $\mathcal{M}_\mu$ the domain of μ. So we have,

$$\mu(E) = \inf\left\{\sum_{j=1}^{\infty}(F(b_j) - F(a_j)) : E \subset \bigcup_{j=1}^{\infty}(a_j, b_j]\right\} \text{ for } E \in \mathcal{M}_\mu.$$

Prove the following statements. Let $E \in \mathcal{M}_\mu$.

(a) $\mu(E) = \inf\left\{\sum_{j=1}^{\infty}\mu((a_j,b_j)) : E \subset \bigcup_{j=1}^{\infty}(a_j,b_j)\right\}$.

Proof. (*From Folland Lemma 1.17*) Let us define $\nu(E) := \inf\left\{\sum_{j=1}^{\infty}\mu((a_j,b_j)) : E \subset \bigcup_{j=1}^{\infty}(a_j,b_j)\right\}$. We aim to show that $\nu(E) = \mu(E)$. Now suppose that $E \subset \bigcup_{j=1}^{\infty}(a_j,b_j)$, in which each (a_j,b_j) is a countable (without loss of generality) disjoint union of half intervals of the form $(c_j^k, c_j^{k+1}]$, which we can denote I_j^k for $k \in \mathbb{N}$ and where $\{c_j\}$ is any sequence in which $c_j^1 = a_j$ and $c_j^k \to b_j$ as $k \to \infty$. Note that these I_j^k's are indeed measurable, as each interval is of the form $\bigcap_n (c_j^k, c_j^{k+1} + 1/n)$, and each of these sets are open in $\mathbb{R}$. Clearly by this construction, $E \subset \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty} I_j^k$, and so via countable additivity, we have

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mu(I_j^k) &= \sum_{j=1}^{\infty}\mu((a_j,b_j)) && \text{by construction of } I_j^k \\
&\geq \mu(E) && \text{def. of infimum.}
\end{aligned}$$

So, we have shown that $\nu(E) \geq \mu(E)$.

Conversely, take some arbitrary $\epsilon > 0$. Then there exists some $\{(a_j,b_j]\}_{j=1}^{\infty}$ such that $E \subset \bigcup_{j=1}^{\infty}(a_j,b_j]$ and $\mu(E) \leq \sum_{j=1}^{\infty}\mu((a_j,b_j]) \leq \mu(E) + \epsilon$. Also for each j there exists some $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < \epsilon/(2^j)$, which follows from the right continuity of F. Then we have $E \subset \bigcup_{j=1}^{\infty}(a_j, b_j + \delta_j)$ since we already have $E \subset \bigcup_{j=1}^{\infty}(a_j,b_j]$, and

$$\begin{aligned}
\sum_{j=1}^{\infty}\mu((a_j,b_j+\delta_j)) &\leq \sum_{j=1}^{\infty}\mu((a_j,b_j]) + \epsilon && \left(\text{since } (a_j,b_j+\delta_j) = (a_j,b_j] \cup [b_j, b_j+\delta_j) \text{ and } \sum_j \frac{\epsilon}{2^j} = \epsilon\right) \\
&\leq \mu(E) + 2\epsilon, && \left(\text{since } \sum_{j=1}^{\infty}\mu((a_j,b_j]) \leq \mu(E) + \epsilon\right)
\end{aligned}$$

and therefore $\nu(E) \leq \mu(E)$ since we can force $\epsilon \to 0$.

Therefore, $\mu(E) = \nu(E)$, and we have shown our desired result. $\square$

(b) $\mu(E) = \inf\{\mu(U) : E \subset U \text{ and } U \text{ is open}\}$.

Proof. By part (a), we have that for any $\epsilon > 0$, there exists a set of intervals $\{(a_j,b_j\}$ such that $E \subset \bigcup_{j=1}^{\infty}(a_j,b_j)$ and $\sum_{j=1}^{\infty}(a_j,b_j) \leq \mu(E) + \epsilon$. If we then take $U := \bigcup_{j=1}^{\infty}(a_j,b_j)$, then U must be open since it is a countable union of open sets, and we have that $E \subset U$ and $\mu(U) \leq \sum_{j=1}^{\infty}(a_j,b_j) \leq \mu(E)+\epsilon$. Since $E \subset U$ we have that by monotonicity, $\mu(E) \leq \mu(U)$, This implies that $\mu(E) \leq \mu(U) \leq \mu(E) + \epsilon$. Forcing ϵ to 0, we have that $\mu(E) = \inf\{\mu(U) : E \subset U, U \text{ open}\}$. $\square$

(c) $\mu(E) = \sup\{\mu(K) : K \subset E \text{ and } K \text{ is compact}\}$.

Proof. Let us suppose that E is bounded. If E is closed, then by definition we have that E is compact, and therefore equality holds since K is then compact as well. If E is open, then let us take some open U such that $\overline{E} \setminus E \subset U$ such that $\mu(U) \leq \mu(\overline{E} \setminus E) + \epsilon$. Now define $K = \overline{E} \setminus U$. Then certainly K is compact since $\overline{E}$ is bounded and via DeMorgan's Laws we have that $\overline{E} \setminus U = \overline{E} \cap U^c$ is closed, and also $K \subset E$, and

$$\mu(K) = \mu(E) - \mu(E \cap U) = \mu(E) - [\mu(U) - \mu(U \setminus E)] \geq \mu(E) - \mu(U) + \mu(\overline{E} \setminus E) \geq \mu(E) - \epsilon.$$

If E is unbounded, then take $E_j = E \cap (j, j+1]$. By the preceding argument, we have that for any $\epsilon > 0$ there exists some compact $K_j \subset E_j$ such that $\mu(K_j) \geq \mu(E_j) - \epsilon/(2^j)$. Denote $H_n = \bigcup_{j=-n}^{n} K_j$. Then H_n is also compact, $H_n \subset E$, and $\mu(H_n) \geq \mu(\bigcup_{j=-n}^{n} E_j) - \epsilon$. Since $\mu(E) = \lim_{n\to\infty}\mu(\bigcup_{j=-n}^{n} E_j)$, we have equality. $\square$

Problem 3

Let F be an increasing, right continuous function defined on $\mathbb{R}$. Let μ be the completion of the measure μ_F induced by F on $\mathcal{B}_{\mathbb{R}}$ and denote by $\mathcal{M}_\mu$ the domain of μ. If $E \subset X$ the following are equivalent:

(a) $E \in \mathcal{M}_\mu$

(b) $E = V \setminus N_1$, where V is a G_δ set and $\mu(N_1) = 0$

(c) $E = H \cup N_2$, where H is an F_σ set and $\mu(N_2) = 0$

Proof. We will show $a \iff b$ and $a \iff c$.
($a \implies b$) Consider some bounded set $E \in \mathcal{M}_\mu$. Then by parts (b) and (c) of Problem 2, we have that for any $j \in \mathbb{N}$, there exists some sequence of open sets $\{U_j\}$ such that $E \subset U_j \subset U_{j-1}\ \forall j$ and a compact set K_j such that

$$\mu(U_j) - \frac{1}{2^j} \leq \mu(E) \leq \mu(K_j) + \frac{1}{2^j}.$$

Now let $V = \bigcap_{j=1}^\infty U_j$ and $H = \bigcup_{j=1}^\infty K_j$. Clearly V is a G_δ set by definition. Also we have that $E \subset V$ and $\mu(V) = \mu(E)$ due to how we defined our sets U_j. Let us take $N_1 = V \setminus E$, then $N_1 \in \mathcal{M}_\mu$ since $E, V \in \mathcal{M}_\mu$. Also since $E \subset V$, we have that $E = V \setminus N_1$. So if $\mu(E) < \infty$, we have that $\mu(N_1) = \mu(V) - \mu(E) = 0$.
Let us now consider some unbounded set E. Since μ is a σ-finite measure, we have that there exists a family of sets $\{E_k\}_{k\in\mathbb{N}} \subset \mathcal{M}_\mu$ such that $\mu(E_k) < \infty$ for all k, and that $E = \bigcup_{k=1}^\infty E_k$. Now by Problem 2a, we have that for each $k, n \in \mathbb{N}$ there exists some open set $U_{k,n}$ depending on k and n such that $E_k \subset U_{k,n}$, and $\mu(E_k) \leq \mu(U_{k,n}) \leq \mu(E_k) + 1/(n2^k)$. Let us define $U_n = \bigcup_{k=1}^\infty U_{k,n}$. Then certainly U_n is open since each $U_{k,n}$ is open, and $E \subset U_{k,n} \subset U_n$. Then we have the following:

$$\mu(E) \leq \mu(U_n) \leq \sum_{k=1}^\infty \mu(U_{k,n}) \leq \sum_{k=1}^\infty \mu(E_k) + \sum_{k=1}^\infty \frac{1}{n2^k} = \mu(E) + \frac{1}{n}.$$

Now since $\mu(E_k) < \infty$, we see that

$$\mu(U_n \setminus E) \leq \sum_{k=1}^\infty \mu(U_{k,n} \setminus E_k) = \sum_{k=1}^\infty (\mu(U_{k,n}) - \mu(E_k)) \leq \sum_{k=1}^\infty \frac{1}{2^k} = \frac{1}{n}.$$

So taking now $V := \bigcap_{k=1}^\infty U_n$ (which is clearly a G_δ set), we see that $E \subset V$ and also

$$\mu(E) \geq \mu(V) \geq \mu(U_n) \geq \mu(E) + \frac{1}{n},$$

which yields $\mu(V) = \mu(E)$. Moreover, we have that $V \setminus E \in \mathcal{M}_\mu$ since both V and E are sets in $\mathcal{M}_\mu$, and with that, we see

$$\mu(V \setminus E) \leq \mu(U_n \setminus E) \leq \frac{1}{n},$$

which gives that $\mu(V \setminus E) = 0$ when $n \to \infty$. Setting $N_1 := V \setminus E$, we see that $E = V \setminus N_1$.

($a \implies c$) Take some $E \in \mathcal{M}_\mu$. Then since σ-algebras are closed under complement, we have that $E^c \in \mathcal{M}_\mu$. By the previous section of the proof ($a \implies b$), we have that for some G_δ set V and some set N_1 such that $\mu(N_1) = 0$, we have that $E^c = V \setminus N_1 = V \cap N_1^c$ by DeMorgan's Laws. This implies that $E = (V \cap N_1^c)^c = V^c \cup N_1$. Since a complement of a G_δ set is an F_σ set, we have that V^c must be an F_σ set. If we then take $H = V^c$ and redefine $N_2 := N_1$, then we see that $E = H \cup N_2$ as desired.

($b \implies a$) Suppose $E = V \setminus N_1$ where V is a G_δ set and $\mu(N_1) = 0$. We know that $V \in \mathcal{M}_\mu$ as it is a G_δ set, and since μ is by definition a complete measure, we have that $N_1 \in \mathcal{M}_\mu$. So then we have $E = V \setminus N_1 \in \mathcal{M}_\mu$ since both both V and N_1 are in $\mathcal{M}_\mu$.

($c \implies a$) Take some $E \in \mathcal{M}_\mu$ with $E = H \cup N_2$ where H is an F_σ set and $\mu(N_2) = 0$. We know that $H \in \mathcal{M}_\mu$ since it is an F_σ set, and since μ is by definition a complete measure, we have that $N_2 \in \mathcal{M}_\mu$ as well. Therefore we have that $E = H \cup N_2 \in \mathcal{M}_\mu$.

We have shown $b \iff a \iff c$, and we are done. $\square$

Problem 4

The following shows that Lebesgue measure in $\mathbb{R}$ is translation invariant and has a simple behavior under dilations: If $E \subset \mathbb{R}$, and $s, r \in \mathbb{R}$ define the sets $E + s = \{x + s : x \in E\}$, and $rE = \{rx : x \in E\}$. Prove that if $E \in \mathcal{L}$, then $E + s \in \mathcal{L}$ for all $s, r \in \mathbb{R}$. Moreover, $m(E + s) = m(E)$ and $m(rE) = |r|m(E)$.

Proof. Let us recall that $\mathcal{L}$ is the μ_F completion of $\mathbb{R}$, and the Lebesgue measure $m = \overline{\mu_F}$ where $F(x) = x$. In particular, that is $\mathcal{L} = \{E \cap F : E \in \mathcal{B}_\mathbb{R}, F \subset N, N \in \mathcal{B}_\mathbb{R}, m(N) = 0\}$. So, given that $E \in \mathcal{L}$, take $E = A \cup F$ where $A \in \mathcal{B}_\mathbb{R}$ and $F \subset N$ for some $N \in \mathcal{B}_\mathbb{R}$ such that $m(N) = 0$.

Now without loss of generality, we can take A and F to be disjoint via the constructions we have used in class and in previous homework. So, we have the following:

$$\begin{aligned} E+s &= (A\cup F)+s = (A+s)\cup(F+s), \\ rE &= r(A\cup F) = (rA)\cup(rF). \end{aligned}$$

Since Borel σ-algebras are invariant under translation and dilation, we have that $E+s$ and rE are both contained in $\mathcal{L}$.

For the second part, take $E\in\mathcal{L}$ and $r,s\in\mathbb{R}$, and define $m_s = m$ and $m_r = |r|m$. Then we have that $m_s = m$ and $m_r = |r|m$ on finite disjoint unions of half intervals. So, let $\mathcal{A}$ be an algebra of finite disjoint unions of half intervals, then we see that $\mathcal{B}_{\mathbb{R}}$ is the σ-algebra generated by $\mathcal{A}$. Now by Folland Theorem 1.14, we have that indeed, $m_s = m$ and $m_r = |r|m$ are restrictions on $\mathcal{B}_{\mathbb{R}}$.

Now for all $F\subset N$, we have that $m(F+s) = m(rF) = m(F) = 0$. Thus, we have the following.

$$\begin{aligned} m(E+s) &= m((A\cup F)+s) = m((A+s)\cup(F+s)) \\ &= m(A+s)+m(F+s) = m(A+s) = m(A) = m(A)+m(F) = m(A\cup F) = m(E), \\ m(rE) &= m(r(A\cup F)) = m((rA)\cup(rF)) = m(rA)+m(rF) \\ &= m(rA) = |r|m(A) = |r|m(A\cup F) = |r|m(E). \end{aligned}$$

We see that indeed $m(E+s) = m(E)$ and $m(rE) = |r|m(E)$. □

Problem 5

Nonmeasurable sets in $\mathbb{R}$. Consider in $[0,1)$ the equivalence relation $x\sim y$ if and only if $x-y\in\mathbb{Q}$. Define N as the subset of $[0,1)$ that contains exactly one element of each equivalence class. Show that N is not Lebesgue measurable.

Proof. Let us consider said set $N\subset[0,1)$ which contains exactly one member of each equivalence class. Let us take now $R=\mathbb{Q}\cap[0,1)$, and take $N_r := \{x+r : x\in N\cap[0,1-r)\}\cup\{x+r-1 : x\in N\cap[1-r,1)\}$ for each $r\in R$. In other words, each N_r is obtained by shifting N to the right by r units, and then bringing the part of N which does not intersect with $[0,1)$ to the left of N, which is equivalent to shifting the section of N which does not intersect with $[0,1)$ to the left by one unit. Certainly by construction we have that $N_r\subset[0,1)$ for each $r\in R$, and every $x\in[0,1)$ belongs to precisely one N_r.

Now, if $y\in N$ and $y\sim x$, then we see that $x\in N_r$ where $r=x-y$ for $x\geq y$, or where $r=x-y+1$ for $x<y$. Note that for distinct r and r', we have that $N_r\cap N_{r'}=\emptyset$, as if there were some element in $N_r\cap N_{r'}$, then we have that there are two distinct elements of N which belong to the same equivalence class: this is a contradiction via the construction of N.

Consider now the Lebesgue measure m. Notice that by the countable additivity of measure and the property that $E\cong F$ implies that $m(E)=m(F)$, we see that $m(N) = m(N\cap[0,1-r)) + m(N\cap[1-r,1)) = m(N_r)$. Moreover, since R is countable (it is simply a subset of the rationals) and $[0,1)=\bigsqcup_{r\in R} N_r$, we have

$$m([0,1)) = \sum_{r\in R} m(N_r) = \sum_{n=1}^{\infty} m(N).$$

But recall that $m([0,1))$ is defined to be equal to $1-0=1$: if we have that $m(N)>0$, then $\sum_{n=1}^{\infty} m(N)=\infty$, and if $m(N)=0$ then $\sum_{n=1}^{\infty} m(N) = 0$. Certainly $0\neq 1\neq\infty$, and we have a contradiction. □

Problem 6

Let $(X,\mathcal{M})$ be a measurable space.

(a) Prove that if $f: X\to\mathbb{R}$ is such that $\{x\in X : f(x)\geq r\}$ for all $r\in\mathbb{Q}$, then f is $\mathcal{M}$-measurable.

Proof. Consider some arbitrary $a\in\mathbb{R}$. It suffices to show that $[a,\infty)\in\mathcal{B}_{\mathbb{R}}$ implies that $f^{-1}([1,\infty))\in\mathcal{M}$. We consider two cases: one for $a\in\mathbb{Q}$ and one for $a\in\mathbb{R}\setminus\mathbb{Q}$. If $a\in\mathbb{Q}$, then by our assumptions for f, we have that $f^{-1}([a,\infty)) = \{x\in X : f(x)\geq r\}\in\mathcal{M}$. If a is irrational, then we have that there exists some increasing sequence $\{a_i\}$ which converges to a. In this case, we have $f^{-1}([a_i,\infty))$ is measurable in $\mathcal{M}$ and as such $f^{-1}(\bigcap_{i=1}^{\infty}[a_i,\infty))$ is measurable in $\mathcal{M}$. So, we have the following:

$$f^{-1}([a,\infty)) = f^{-1}\left(\bigcap_{i=1}^{\infty}[a_i,\infty)\right) = \bigcap_{i=1}^{\infty} f^{-1}([a_i,\infty))\in\mathcal{M}.$$

So we have seen that $f^{-1}([a,\infty))\in\mathcal{M}$ for all $a\in\mathbb{R}$. So, f must be $\mathcal{M}$-measurable. □

(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

Proof. Consider $a \in \mathbb{R}$. It suffices to show that $f^{-1}([a, \infty)) \in \mathcal{B}_\mathbb{R}$. Since $\emptyset$ and $\mathbb{R}$ are included in $\mathcal{B}_\mathbb{R}$ by definition, let us assume for the sake of simplicity that $\emptyset \neq f^{-1}([a, \infty)) \neq \mathbb{R}$. Now, if there exists some $b \in \mathbb{R}$ such that for any $c < b$ we have that $f(c) < a$, then $f^{-1}([a, \infty)) = f^{-1}([f(b), \infty))$. Loosely speaking, we can say that b can be considered to be the smallest element such that $f(b)$ is greater than or equal to $f(a)$. We claim now that $f^{-1}([f(b), \infty)) = [b, \infty)$. If we have some $x \in f^{-1}([f(b), \infty))$, then certainly $f(x) \geq f(b)$. Furthermore, since f is a monotone function, we see $x \geq b$ and $x \in [b, \infty)$. On the other hand, if $x \in [b, \infty)$, we have that $x \geq b$ and since f is monotone, we have that $f(x) \geq f(b)$ and thus $x \in f^{-1}([f(b), \infty))$. Therefore, we have that

$$f^{-1}([a, \infty)) = f^{-1}([f(b), \infty)) = [b, \infty) \in \mathcal{B}_\mathbb{R}.$$

Now let us suppose that there does not exist such a $b \in \mathbb{R}$. Because $f^{-1})[a, \infty)) \neq \mathbb{R}$, we may assume that there exists a $c \in \mathbb{R}$ such that $f(c) < a$. Then there exists a bounded below, decreasing sequence $\{b_i\}$ such that $f(b_i) > a \; \forall i$. Indeed, $\{b_i\}$ is bounded below by c and it is decreasing since there does not exist a smallest element b such that $f(b) \geq a$. Now since f is monotone and $f^{-1}([a, \infty)) \neq \emptyset$, we may consider such a sequence which converges to some $b' := \inf\{f^{-1}([a, \infty))\}$.

We claim that $f^{-1}([a, \infty)) = (b', \infty)$. So, take $x \in f^{-1}([a, \infty))$. By assumption, we have that there is no smallest element b such that $f(b) \geq f(a)$, and as such, we must have some $y \in f^{-1}([a, \infty))$ such that $y < x$. Thus $b' < x$ and $x \in (b', \infty)$. On the other hand, take $x \in (b', \infty)$. Since $\{b_i\} \to b'$, we have some n such that $i \geq n$ implies that $x > b_i$. Because f is monotone, we have that $f(x) > f(b_i)$. Therefore $f(x) > a$ and $x \in f^{-1}([a, \infty))$. Now since $(b', \infty) \in \mathcal{B}_\mathbb{R}$, we have that $f^{-1}([a, \infty)) \in \mathcal{B}_\mathbb{R}$. So, we have that f is Borel measurable. $\square$