Abstract Algebra I Homework 4

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Section 5.1

1 Show that the center of a direct product is the direct product of the centers, that is

$$Z(G_1 \times \cdots \times G_n) = Z(G_1) \times \cdots \times Z(G_n).$$

Deduce that a direct product of groups is abelian if and only if each of the factors is abelian.

Proof. We shall prove via induction over n. First, we show our base case $n = 2$:
($\subset$) Take $(g, h) \in Z(G \times H)$. Then by definition of the center of a group, we know that for arbitrary $(x, y) \in G \times H$, we have $(gx, hy) = (g, h)(x, y) = (x, y)(g, h) = (xg, yh)$, that is $gx = xg$ and $hy = yh$. Therefore we see $g \in Z(G)$ and $h \in Z(H)$, so we can write $(g, h) \in Z(G) \times Z(H)$.
($\supset$) Take $(g, h) \in Z(G) \times Z(H)$. Then by definition of the center of a group, we know that for arbitrary $x \in G$ and $y \in H$, we have $(g, h)(x, y) = (gx, hy) = (xg, yh) = (x, y)(g, h)$, that is $(g, h) \in Z(G \times H)$. Both containments have been shown, and therefore we have equality as desired.

Proceeding with induction, assume that the statement holds for the direct product of $n - 1$ groups where $n \geq 2$. Appealing to the results we showed in our base case and considering the fact that the direct product of groups is by definition also a group, we have:

$$Z\left(\bigoplus_{i=1}^{n} G_i\right) = Z\left(\bigoplus_{i=1}^{n-1} G_i \times G_n\right) = Z\left(\bigoplus_{i=1}^{n-1} G_i\right) \times Z(G_n) = \bigoplus_{i=1}^{n-1} Z(G_i) \times Z(G_n) = \bigoplus_{i=1}^{n} Z(G_i).$$

So, we have our desired result. $\diamond$

Let us now define $G := \bigoplus_{i=1}^{n} G_i$ for $n < \infty$. ($\Longrightarrow$) Take G to be abelian. Then we have that

$$\bigoplus_{i=1}^{n} Z(G_i) = Z\left(\bigoplus_{i=1}^{n} G_i\right) = Z(G) = G = \bigoplus_{i=1}^{n} G_i.$$

Taking the i-th projection gives that $\pi_i(G) = G_i = Z(G_i)$. As such, each G_i is abelian. ($\Longleftarrow$) Take each G_i to be abelian. Then we have that

$$Z(G) = Z\left(\bigoplus_{i=1}^{n} G_i\right) = \bigoplus_{i=1}^{n} Z(G_i) = \bigoplus_{i=1}^{n} G_i = G.$$

Certainly then we see G is abelian. Both implications have been shown, so indeed a direct product of groups is abelian if and only if each of its factors is abelian. $\square$

14 Let $G = A_1 \times A_2 \times ... \times A_n$ and for each i let B_i be a normal subgroup of A_i. Prove that $B_1 \times \cdots \times B_n \trianglelefteq G$ and that

$$(A_1 \times A_2 \times \cdots \times A_n)/(B_1 \times B_2 \times \cdots \times B_n) \cong (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n).$$

Proof. We shall prove via induction over n. First, we show our base case $n = 2$, that is, we want to show $(A_1 \times A_2)/(B_1 \times B_2) \cong (A_1/B_1) \times (A_2/B_2)$.
Notice that the natural projection maps $\pi_1 : A_1 \to A_1/B_1$ and $\pi_2 : A_2 \to A_2/B_2$ are surjective homomorphisms by definition, therefore $\pi_1 \times \pi_2 : A_1 \times A_2 \to (A_1/B_1) \times (A_2/B_2)$ is a surjective homomorphism as well. Now notice that $\ker \pi_i = B_i$.

In order to continue, we claim now that $\ker(\pi_1 \times \pi_2) = (\ker \pi_1) \times (\ker \pi_2)$.
Proof. ($\subset$) Take $(a_1, a_2) \in \ker(\pi_1 \times \pi_2)$. Then $(1, 1) = (\pi_1 \times \pi_2)(a_1, a_2) = (\pi_1(a_1), \pi_2(a_2))$, that is $\pi_1(a_1) = 1$ and $\pi_2(a_2) = 1$. Surely then $a_1 \in \ker \pi_1$ and $a_2 \in \ker \pi_2$, so $(a_1, a_2) \in (\ker \pi_1) \times (\ker \pi_2)$.
($\supset$) Take $(a_1, a_2) \in (\ker \pi_1) \times (\ker \pi_2)$. Then $(\pi_1 \times \pi_2)(a_1, a_2) = (1, 1)$ and therefore $(a_1, a_2) \in \ker(\pi_1 \times \pi_2)$. $\diamond$

Continuing, we see that in fact $\ker(\pi_1 \times \pi_2) = B_1 \times B_2$. Finally, an application of the First Isomorphism Theorem gives the result.

We extend this base case via induction, which gives our full result. So assume that the result holds for $n-1$ pairs of groups. Appealing to our base case and the fact that a direct product of groups is by definition a group, we have:

$$\begin{aligned}
\left(\bigoplus_{i=1}^{n} A_i\right) / \left(\bigoplus_{i=1}^{n} B_i\right) = \left(\bigoplus_{i=1}^{n-1} A_i \times A_n\right) / \left(\bigoplus_{i=1}^{n-1} B_i \times B_n\right) &\cong \left(\left(\bigoplus_{i=1}^{n-1} A_i\right) / \left(\bigoplus_{i=1}^{n-1} B_i\right)\right) \times (A_n/B_n) \\
&= \left(\left(\bigoplus_{i=1}^{n-2} A_i\right) / \left(\bigoplus_{i=1}^{n-2} B_i\right)\right) \times (A_{n-1}/B_{n-1}) \times (A_n/B_n) \\
&= \cdots \\
&= (A_1/B_1) \times \cdots \times (A_n/B_n).
\end{aligned}$$

We have shown our desired result. $\square$

Section 5.2

2 In each of parts (a)-(e) give the lists of invariant factors for all abelian groups of the specified order:

(a) Order 270

Solution. Notice $270 = 2 \cdot 3^3 \cdot 5$. Therefore the invariant factors (in the style of problem 5.2.4) are $\{270\}$, $\{90, 3\}$, and $\{30, 3, 3\}$.

(b) Order 9801

Solution. Notice $9801 = 3^4 \cdot 11^2$. Therefore the invariant factors (in the style of problem 5.2.4) are $\{33, 33, 3, 3\}$, $\{363, 3, 3, 3\}$, $\{99, 33, 3\}$, $\{1089, 3, 3\}$, $\{99, 99\}$, $\{1089, 9\}$, $\{297, 33\}$, $\{3267, 3\}$, $\{891, 11\}$, and $\{9801\}$.

(c) Order 320

Solution. Notice $320 = 2^6 \cdot 5$. Therefore the invariant factors (in the style of problem 5.2.4) are $\{10, 2, 2, 2, 2, 2\}$, $\{20, 2, 2, 2, 2\}$, $\{20, 4, 2, 2\}$, $\{20, 4, 4\}$, $\{40, 2, 2, 2\}$, $\{40, 4, 2\}$, $\{40, 8\}$, $\{80, 2, 2\}$, $\{80, 4\}$, $\{160, 2\}$, and $\{320\}$.

(d) Order 105

Solution. Notice $105 = 3 \cdot 5 \cdot 7$. The only invariant factor is $\{105\}$.

(e) Order 44100

Solution. Notice $44100 = 2^2 \cdot 3^3 \cdot 5^2 \cdot 7^2$. Therefore the invariant factors (in the style of problem 5.2.4) are $\{210, 210\}$, $\{1470, 30\}$, $\{1050, 42\}$, $\{7350, 6\}$, $\{630, 70\}$, $\{4410, 10\}$, $\{3150, 14\}$, $\{22050, 2\}$, $\{420, 105\}$, $\{2940, 15\}$, $\{2100, 21\}$, $\{14700, 3\}$, $\{1260, 35\}$, $\{8820, 5\}$, $\{6300, 7\}$, $\{44100\}$.

3 In each of parts (a)-(e) give the lists of elementary divisors for all abelian groups of the specified order and the match each list with the corresponding list of invariant factors found in the preceding exercise.

(a) Order 270

Solution. The elementary divisors, in the same order as in problem 2, are: $\{2, 27, 5\}$, $\{2, 3, 5, 9\}$, and $\{2, 3, 3, 3, 5\}$.

(b) Order 9801

Solution. The elementary divisors, in the same order as in problem 2, are: $\{3, 3, 3, 3, 11, 11\}$, $\{3, 3, 3, 3, 121\}$, $\{3, 3, 9, 11, 11\}$, $\{3, 3, 9, 121\}$, $\{9, 9, 11, 11\}$, $\{9, 9, 121\}$, $\{3, 27, 11, 11\}$, $\{3, 27, 121\}$, $\{81, 11, 11\}$, and $\{81, 121\}$.

(c) Order 320

Solution. The elementary divisors, in the same order as in problem 2, are: $\{2, 2, 2, 2, 2, 2, 5\}$, $\{2, 2, 2, 2, 4, 5\}$, $\{2, 2, 4, 4, 5\}$, $\{4, 4, 4, 5\}$, $\{2, 2, 2, 8, 5\}$, $\{2, 4, 8, 5\}$, $\{8, 8, 5\}$, $\{2, 2, 16, 5\}$, $\{4, 16, 5\}$, $\{2, 32, 5\}$, and $\{64, 5\}$.

(d) Order 105

Solution. The only elementary divisors are $\{3, 5, 7\}$.

(e) Order 44100

Solution. The elementary divisors, in the same order as in problem 2, are: $\{2,2,3,3,5,5,7,7\}$, $\{2,2,3,3,5,5,49\}$, $\{2,2,3,3,25,7,7\}$, $\{2,2,3,3,25,49\}$, $\{2,2,9,5,5,7,7\}$, $\{2,2,9,5,5,49\}$, $\{2,2,9,25,49\}$, $\{2,2,9,25,49\}$, $\{4,3,3,5,5,7,7\}$, $\{4,3,3,5,5,49\}$, $\{4,3,3,25,7,7\}$, $\{4,3,3,25,49\}$, $\{4,9,5,5,7,7\}$, $\{4,9,5,5,49\}$, $\{4,9,25,7,7\}$, and $\{4,9,25,49\}$.

4 In each of parts (a)-(d) determine which pairs of abelian groups listed are isomorphic (here $\{a_1, a_2, ..., a_k\}$ denotes the abelian group $Z_{a_1} \times Z_{a_2} \times \cdots \times Z_{a_k}$).

(a) $\{4,9\}$, $\{6,6\}$, $\{8,3\}$, $\{9,4\}$, $\{6,4\}$, $\{64\}$

Solution. First, notice that $\{64\}$ is the only one with 64 elements, so it is not isomorphic to any of the others. Second, we see that $\{6,4\}$ and $\{8,3\}$ are the only groups with 24 elements, but $\{8,3\}$ contains an element of order 8 while $\{6,4\}$ does not, so $\{64\}$, $\{6,4\}$, and $\{8,3\}$ are all in isomorphism classes of their own. Certainly $\{9,4\} \cong \{4,9\}$. Finally, $\{6,6\}$ is in a class of its own as $\{4,9\} \cong \{9,4\}$ contains an element of order 4 while $\{6,6\}$ doesn't.

(b) $\{2^2, 2\cdot 3^2\}$, $\{2^2\cdot 3, 2\cdot 3\}$, $\{2^3, 3^2\}$, $\{2^2\cdot 3^2, 2\}$

Solution. First notice that $\{2^2, 2\cdot 3^2\} \cong \{2^2\cdot 3^2, 2\}$ since both have elementary divisors $\{4,2,9\}$. The remaining two groups have elementary divisors distinct from the others, so the remaining two groups are in isomorphism classes of their own.

(c) $\{5^2\cdot 7^2, 3^2\cdot 5\cdot 7\}$, $\{3^2\cdot 5^2\cdot 7, 5\cdot 7^2\}$, $\{3\cdot 5^2, 7^2, 3\cdot 5\cdot 7\}$, $\{5^2\cdot 7, 3^2\cdot 5, 7^2\}$

Solution. First notice that the groups $\{5^2\cdot 7^2, 3^2\cdot 5\cdot 7\}$, $\{3^2\cdot 5^2\cdot 7, 5\cdot 7^2\}$, and $\{5^2\cdot 7, 3^2\cdot 5, 7^2\}$ are isomorphic since they have the same elementary divisors, which are $\{9,25,5,49,7\}$. So, these three are isomorphic to each other, while $\{3\cdot 5^2, 7^2, 3\cdot 5\cdot 7\}$ is in an isomorphism class of its own.

(d) $\{2^2\cdot 5\cdot 7, 2^3\cdot 5^3, 2\cdot 5^2\}$, $\{2^3\cdot 5^3\cdot 7, 2^3\cdot 5^3\}$, $\{2^2, 2\cdot 7, 2^3, 5^3, 5^3\}$, $\{2\cdot 5^3, 2^2\cdot 5^3, 2^3, 7\}$

Solution. First notice that $\{2^2\cdot 5\cdot 7, 2^3\cdot 5^3, 2\cdot 5^2\} \cong \{2^2, 2\cdot 7, 2^3, 5^3, 5^3\}$ since both have elementary divisors of $\{8,4,2,125,125,7\}$. The remaining two groups have elementary divisors distinct from all of the others, so the remaining two groups are in isomorphism classes of their own.

Section 5.4

11 Prove that if $G = HK$ where $H, K \operatorname{char} G$ with $H \cap K = 1$ then $\mathrm{Aut}(G) \cong \mathrm{Aut}(H) \times \mathrm{Aut}(K)$. Deduce that if G is an abelian group of finite order then $\mathrm{Aut}(G)$ is isomorphic to the direct product of the automorphism group of its Sylow subgroups.

Proof. First, we note that $H \operatorname{char} G$ means that $\phi \in \mathrm{Aut}(G)$ is such that $\phi(H) = H$, likewise for $K \operatorname{char} G$. We can rewrite this restriction as $\phi|_H(H) = H$, which means that $\phi|_H \in \mathrm{Aut}(H)$, likewise $\phi|_K \in \mathrm{Aut}(K)$. So by definition of automorphism, we have that both $\phi|_H, \phi|_K$ are bijective homomorphisms. We aim to show that the morphism $\psi : \mathrm{Aut}(G) \to \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ defined by $\phi \mapsto (\phi|_H, \phi|_K)$ is an isomorphism.

First, we show that ψ is a homomorphism:

$$\psi(\phi_1\phi_2) = ((\phi_1\phi_2)|_H, (\phi_1\phi_2)|_K) = (\phi_1|_H, \phi_1|_K)(\phi_2|_H, \phi_2|_K) = \psi(\phi_1)\psi(\phi_2).$$

Indeed, ψ is a homomorphism.

Next, we show that ψ is injective. Suppose we take $\phi \in \ker\psi$. Then $\psi(\phi) = (\phi|_H, \phi|_K) = (\mathrm{id}_H, \mathrm{id}_K)$. So, let us take an element $hk \in HK = G$, so then $\phi(hk) = \phi(h)\phi(k) = hk$. Therefore $\phi = \mathrm{id}_G$ and ψ is injective.

Finally, we aim to show that ψ is surjective. To do so, however, we will define the following, and show that it is an isomorphism. So, let $\phi \in \mathrm{Aut}(G)$ and take any $(f, g) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$. Now define $\phi(hk) = f(h)g(k)$. We show the following:

- Homomorphism: Note that since $H, K \trianglelefteq G$ and $H \cap K = 1$, elements commute.

$$\begin{aligned}\phi(h_1k_1h_2k_2) = \phi(h_1h_2k_1k_2) = f(h_1h_2)g(k_1k_2) &= f(h_1)f(h_2)g(k_1)g(k_2) \\ &= f(h_1)g(k_1)f(h_2)g(k_2) = \phi(h_1k_1)\phi(h_2k_2).\end{aligned}$$

- Injectivity: Let $hk \in \ker\phi$. Then we see that $f(h)g(k) = 1$ implies that $f(h) = g(k)^{-1}$. But, we have that $H \cap K = 1$, and as such, we must have that $f(h) = g(k) = 1$, so $h = k = 1$. Therefore we have $\ker\phi = 1$, and as a result ϕ is injective.

- Surjectivity: Since $f \in \mathrm{Aut}(H)$, we can write for any $h \in H$ that $h = f(h')$, likeiwse for any $k \in K$ we have $k = g(k')$. So, $\phi(h'k') = f(h')g(k') = hk$, so indeed ϕ is surjective.

Now that it has been shown that this ϕ defined as $\phi(hk) = f(h)g(k)$ is an isomorphism, we can write that $\phi|_H = f$ and $\phi|_K = g$, since

$$\phi|_H(h) = \phi(h \cdot 1) = f(h)g(1) = f(h),$$
$$\phi|_K(k) = \phi(1 \cdot k) = f(1)g(k) = g(k).$$

As such, $\psi(\phi) = (f, g)$, which establishes that ψ is surjective.

All criterion for ψ to be an isomorphism have been established, so indeed $\mathrm{Aut}(G) \cong \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ as desired. □

Section 5.5

6 Assume that K is a cyclic group, H is an arbitrary group, and $\phi_1, \phi_2 : K \to \mathrm{Aut}(H)$ are homomorphisms such that $\phi_1(K)$ and $\phi_2(K)$ are conjugate subgroups of $\mathrm{Aut}(H)$. If K is infinite, assume ϕ_1, ϕ_2 are injective. Prove by constructing an explicit isomorphism that $H \rtimes_{\phi_1} K \cong H \rtimes_{\phi_2} K$ (in particular, if the subgroups $\phi_1(K)$ and $\phi_2(K)$ are equal in $\mathrm{Aut}(H)$, then the resulting semidirect products are isomorphic). [Suppose $\sigma\phi_1(K)\sigma^{-1} = \phi_2(K)$ so that for some $a \in \mathbb{Z}$ we have $\sigma\phi_1(k)\sigma^{-1} = \phi_2(k)^a$ for all $k \in K$. Show that the map $\psi : H \rtimes_{\phi_1} K \to H \rtimes_{\phi_2} K$ defined by $\psi((h, k)) = (\sigma(h), k^a)$ is a homomorphism. Show ψ is bijective by constructing a 2-sided inverse.]

Proof. We aim to show that ψ is an isomorphism. We first show that ψ as defined in the problem statement is a homomorphism. Take $(h_1, k_1), (h_2, k_2) \in H \rtimes_{\phi_1} K$. Then we have the following:

$$\begin{aligned}
\psi((h_1, k_1), (h_2, k_2)) = \psi((h_1\phi_1(k_1)(h_2), k_1k_2)) &= (\sigma(h_1\phi_1(k_1)(h_2)), (k_1k_2)^a)) \\
&= (\sigma(h_1)\sigma(\phi_1(k_1)(h_2)), k_1^a k_2^a) \\
&= \sigma(h_1)(\sigma(\phi_1(k_1)(h_2)), k_1^a k_2^a) \\
&= (\sigma(h_1)(\phi_2(k_1)^a \circ \sigma)h_2, k_1^a k_2^a) \\
&= (\sigma(h_1)\phi_2(k_1^a)(\sigma(h_2)), k_1^a k_2^a) \\
&= (\sigma(h_1), k_1^a)(\sigma(h_2), k_2^a) \\
&= \psi((h_1, k_1))\psi((h_2, k_2)).
\end{aligned}$$

We see that indeed, ψ is a homomorphism. Now, we aim to show that ψ is bijective. First, we shall consider the infinite case.

Injectivity of ϕ_1 and ϕ_2 for which $|K| = \infty$ is given in the problem statement. So, say there exists some b such that $\sigma^{-1}\phi_2(k)\sigma = \phi_1(k)^b$, similar to how $\sigma\phi_1(k)\sigma^{-1} = \phi_2(k)^a$, both for all k. Combining these equations yields that $\phi_2(k) = \phi_2(k^{ab})$. Since ϕ_2 is injective, we have that $k^{1-ab} = 1$. Since $|K| = \infty$ and k was taken arbitrarily, we have that $ab = 1$. So, either $a, b = -1$ or $a, b = 1$. Define now the morphism $\eta : H \rtimes_{\phi_2} K \to H \rtimes_{\phi_1} K$ by $\eta((h, k)) = (\sigma^{-1}(h), k^a)$. Then

$$(\eta \circ \psi)((h, k)) = \eta(\sigma(h), k^a) = ((\sigma^{-1}\sigma)(h), k^{a \cdot a}) = (h, k) \implies \eta \circ \psi = 1 = \psi \circ \eta.$$

So, we see that for the case in which K is infinite that ψ is bijective, and indeed $H \rtimes_{\phi_1} K \cong H \rtimes_{\phi_2} K$.

Now we consider the case in which K is finite (more precisely, $K \cong \mathbb{Z}_n$), and we again aim to show that ψ is a bijection. By Lagrange's theorem, $\mathrm{im}\phi_1$ is cyclic and is of order m, where $m \mid n$. Now if $K = \langle x \rangle$, then we know that $\phi_1(K) = \langle \phi_1(x) \rangle$. Moreover, since conjugation by $\sigma : \phi_1(K) \to \phi_2(K)$ is an isomorphism, we have that $\phi_2(K) = \langle \phi_2(x)^a \rangle$. So, $\gcd(a, m) = 1$. We claim now that there exists some $\bar{a}$ such that $\gcd(\bar{a}m) = 1$.

> *Proof.* Take $a, m, n \in \mathbb{F}$ with $m \mid n$ and $\gcd(a, m) = 1$. Take $d = \gcd(a, n)$, and take $n = mq$ for some q. Then $\gcd(d, m) = 1$, and therefore $d \mid q$. Now denote $a = a'd$ and $q = q'd$. Take t to be the product of all prime divisors of q' which do not divide d. Finally, take $\bar{a} = a + tm$. If p is a prime divisor of n, then we have the following cases:
>
> 1) If $p \mid m$ then $p \nmid a$ since $(a, m) = 1$. So $p \nmid \bar{a}$.
> 2) If $p \nmid m$ and $p \mid q'$ then we have that either of the following. **(a)** If $p \mid d$ then $p \nmid t$, so $p \nmid tm$; also we have that $p \mid a$, so $p \nmid a + tm\bar{a}$. **(b)** If $p \nmid d$ then $p \mid t$ and $p \nmid a$. So then $p \nmid \bar{a}$.
> 3) if $p \nmid m, q'$ then $p \mid d$ since $n = mq'd$. So $p \mid a$ and $p \nmid t$, so $p \nmid \bar{a}$.
>
> Since we do not have that $p \mid \bar{a}$ in an case, we must have $\gcd(\bar{a}, n) = 1$. □

So, by the lemma, we know that there exists some $\bar{a}$ such that $\gcd(\bar{a}, n) = 1$. Moreover, by the way we defined $\bar{a}$ in the proof of the lemma, there exists some b such that $\bar{a}b \equiv 1 \mod n$. We now define the function $\eta : H \rtimes_{\phi_2} K \to H \rtimes_{\phi_1} K$ by $\eta((h, k)) = (\sigma^{-1}(h), k^b)$, which similarly to the infinite case is also a two-sided inverse of ψ. So, we have for the finite case as well that indeed $H \rtimes_{\phi_1} K \cong H \rtimes_{\phi_2} K$. So, our result holds in general. $\square$

12 Classify the groups of order 20 (there are five isomorphism types).

Solution. We know the five isomorphism classes of order 20 are as follows:

$$\mathbb{Z}_{20} \qquad \mathbb{Z}_{10} \times \mathbb{Z}_2 \qquad D_{20} \qquad \mathbb{Z}_5 \rtimes \mathbb{Z}_4 \qquad F_{20}.$$

I need to keep playing with this and other examples, so I do not have this completed yet. It is still on my list of things to do though!!!

Section 6.1

7 Prove that subgroups and quotient groups of nilpotent groups are nilpotent (proof should work for infinite groups). Give an explicit example of a group G which possesses a normal subgroup H such that both H and G/H are nilpotent but G is not nilpotent.

Proof. First, take G to be nilpotent; that is for $n > 0$ we have $G^n = 1$, for where $G^0 = G$ and $G^{i+1} = [G^i, G]$. Take $H \leq G$. We claim that for all i, we have $H^i \leq G^i$. We use the fact that $H \leq G$ as the base case for induction, and assume that the claim holds up to $H^i \leq G^i$. Then we have the following:

$$\{xyx^{-1}y^{-1} : x \in H^i, y \in H\} \subset \{xyx^{-1}y^{-1} : x \in G^i, y \in G\},$$

which implies that $H^{i+1} \leq G^{i+1}$. Thus if $G^n = 1$ then $H^n = 1$, and so we have that the subgroup H is nilpotent.

Next, define $\phi : G \to H$ be some surjective homomorphism. We claim that $\phi(G^i) = H^i$. We can consider the base case for induction to be how the map is defined, and so we can assume for the sake of induction that $\phi(G^i) = H^i$. So appealing to the fact that ϕ is a homomorphism, we have

$$\{xyx^{-1}y^{-1} : x \in H^i, y \in H\} = \{xyx^{-1}y^{-1} : x \in \phi(G^i), y \in \phi(G)\} = \{\phi(hgh^{-1}g^{-1}) : h \in G^i, g \in G\} = \phi(G^{i+1}).$$

So, if $G^n = 1$ then $H^n = \phi(G^n) = 1$, and as such the image of ϕ is nilpotent. If we consider $K \trianglelefteq G$, then we can take $\phi : G \to G/K$, and therefore we have that G/K is nilpotent as desired. $\square$

Example. (Incomplete) I am unsure how to approach this, however an idea which might make sense is if we take $G = P \rtimes H$, where P is a p-subgroup (which is necessarily nilpotent) and H is some non-nilpotent group. I am curious to see what one such example might be!

17 Prove that $G^{(i)}$ is a characteristic subgroup of G for all i.

Proof. We will first show that this holds for the commutator subgroup, and then we will proceed via induction. So, first let us take $\phi \in \mathrm{Aut}(G)$ and $x, y \in G$. Then we see $\phi(x^{-1}y^{-1}xy) = \phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y) = [\phi(x), \phi(y)]$. Then we have that $\phi(G^{(1)}) \subset G^{(1)}$. Also, we have that $\phi^{-1}(G^{(1)}) \subset G^{(1)}$, which implies that $\phi(G^{(1)}) = G^{(1)}$. So, we see that $G^{(1)} \mathrm{char} G$.

Let us now assume for the sake of induction that this holds up to $G^{(i)} \mathrm{char} G$, and we aim to show that $G^{(i+1)} \mathrm{char} G$. So, consider $\phi \in \mathrm{Aut}(G)$. We have the following:

$$\phi(G^{(i+1)} = \phi([G, G^{(i)}]) = [\phi(G), \phi(G^{(i)}] = [G, G^{(i)}] = G^{(i+1)}.$$

We see that indeed $G^{(i+1)} \mathrm{char} G$ as desired. So, $G^{(i)} \mathrm{char} G$ for all i. $\square$