Real Analysis Homework 1

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Problem 1

A nonempty family $\mathcal{R} \subset \mathcal{P}(X)$ is called a ring is it is closed under finite unions and differences. A ring that is closed under countable unions is called a σ-ring. Prove the following statements:

a) Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.

Proof. First, let us note that the proof is essentially the same for both rings and σ-rings. In the case of rings, take $m \in \mathbb{Z}_+$, and in the case of σ-rings, take $m = \infty$. If we take some $E_1, E_2 \in \mathcal{R}$, then we know by definition of (σ-)ring that $E_1 \setminus E_2 \in \mathcal{R}$. So, since ($\sigma$-)rings are closed under finite (countable) unions, we know that $\bigcup_{n=2}^m E_1 \setminus E_n \in \mathcal{R}$. Then certainly, taking another difference will maintain containment in $\mathcal{R}$: $E_1 \setminus (\bigcup_{n=2}^m E_1 \setminus E_n) \in \mathcal{R}$. Now by DeMorgan's Laws, we have that

$$\bigcup_{n=2}^{m} E_1 \setminus E_n = E_1 \setminus \left(\bigcap_{n=2}^{m} E_n \right).$$

So, inserting this into the preceding containment, we see

$$E_1 \setminus \left(E_1 \setminus \left(\bigcap_{n=2}^{m} E_n \right) \right) = E_1 \cap \bigcap_{n=2}^{m} E_n = \bigcap_{n=1}^{m} E_n \in \mathcal{R}.$$

$\square$

b) If $\mathcal{R}$ is a ring (resp. σ-ring), then $\mathcal{R}$ is an algebra (resp. σ-algebra) iff $X \in \mathcal{R}$.

Proof. Let $\mathcal{R}$ be a (σ-)ring.
($\Longrightarrow$) Let us assume that $\mathcal{R}$ is an (σ-)algebra. Thus, $\mathcal{R}$ is closed under finite (countable) unions, finite (countable) differences, and complements. So, consider some set $E \in \mathcal{R}$. Then we can take the complement of this set, $E^c \in \mathcal{R}$. Certainly, $E \cup E^c \in \mathcal{R}$.
($\Longleftarrow$) Let us assume that $X \in \mathcal{R}$. We know that $\mathcal{R}$ is closed under finite (countable) unions and differences. So, consider some $E \in \mathcal{R}$. Then we can take $X \setminus E \in \mathcal{R}$. Then by the definition of set subtraction, we see

$$X \setminus E = X \cap E^c = E^c \in \mathcal{R}.$$

As such, we see that $\mathcal{R}$ is closed under complement, and therefore $\mathcal{R}$ must be a (σ-)algebra. $\square$

c) If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.

Proof. Let us denote the set in the problem statement by S. Note $X^c = \emptyset = A \setminus A \in \mathcal{R}$ for any $A \in \mathcal{R}$ implies that $\emptyset, X \in S$. We want to show that S is closed under complements and countable union. So, for $A \in S$, for A^c to be in S, we need $A^c \vee A \in S$, which certainly holds as $A \in S$.

Now take some sequence $\{A_i\}_{i=1}^\infty$ in S. We aim to show that $\bigcup A_i$ is also in S. To do so, we consider three cases:

1. If $A_i \in \mathcal{R}$ for all i, then since a countable union is closed, $\bigcup_i A_i$ is in $\mathcal{R}$.
2. If $A_i^c \in \mathcal{R}$ for all i, then $\bigcup A_i^c = (\bigcap A_i)^c$ must be in the set S as defined earlier. Indeed, $\mathcal{R}$ is closed under countable intersections via part a) of the problem. So, since $\bigcap A_i \in \mathcal{R}$, we have that the complement is as well.
3. If $\{A_i\}$ has some elements in case 1 and some in case 2, then we can take subsequences $\{A_{n_i}\}$ consisting of sets such that $A_{n_i} \in \mathcal{R}$, and $\{A_{k_i}\}$ consists of sets such that $A_{k_i}^c \in \mathcal{R}$. We see that by the previous two cases that $\bigcup A_{n_i}$ and $\bigcup A_{k_i}$ are both in $\mathcal{R}$, and therefore a countable union of these two sets, $\bigcup A_i$, is also in $\mathcal{R}$.

We have shown all cases, and therefore S is closed under countable union. $\square$

d) If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

Proof. Let us denote the set in the problem statement by S. Trivially, we have $\emptyset, X \in S$. Now let us take $E \in S$ and $F \in \mathcal{R}$. Then we have that $E \cap F \in \mathcal{R}$, and since $\mathcal{R}$ is closed under differences, we have $E^c \cap F = F \setminus (E \cap F) \in \mathcal{R}$, and therefore, $E^c \in S$. Now if $\{E_i\}$ is a countable set in S, then for any $F \in \mathcal{R}$, we have that $(\bigcup E_i) \cap F = \bigcup (E_i \cap F)$. Since $E_i \cap F \in \mathcal{R}$, it follows that $\bigcup (E_i \cap F) \in \mathcal{R}$. Therefore $\bigcup E_i \in S$, and therefore $S = \{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is indeed a σ-algebra. $\square$

Problem 2

Let $\mathcal{M}$ be an infinite σ-algebra.

a) Prove that $\mathcal{M}$ contains an infinite sequence of disjoint non-empty sets.

Proof. Let us pick some infinite, nonempty $A \in \mathcal{M}$ ($A \neq \mathcal{M}$). Now we can restrict $\mathcal{M}$ to A and A^c, that is:

$$\mathcal{M}|_A = \{F \cap A : F \in \mathcal{M}\}$$
$$\mathcal{M}|_{A^c} = \{F \cap A^c : F \in \mathcal{M}\}$$

Now since $\mathcal{M}$ is an infinite σ-algebra, we have that at least one of $\mathcal{M}|_A$ and $\mathcal{M}|_{A^c}$ is infinite. Without loss of generality, pick $\mathcal{M}|_{A^c}$ to be the infinite one, and take $\mathcal{M}|_A$ to be the "nonempty" one. So, we have that $\mathcal{M}|_{A^c}$ is an infinite σ-algebra over A^c, and we can continue from here. First, however, let us assign the set which was taken to restrict $\mathcal{M}$ to be nonempty, $A = X_1$. Now, let us repeat this process with $\mathcal{M}|_{A^c}$ over A^c; that is, pick some $B \in A^c$ and restrict $\mathcal{M}|_{A^c}$ to B and B^c, assign the set which yields the nonzero restriction of $\mathcal{M}$, (without loss of generality) $B = X_2$, and proceed in similar fashion with the resulting infinite σ-algebra admitted by the other restriction. We can continue this process infinitely since any two restrictions of an infinite σ-algebra admits at least one infinite σ-algebra. So, the set $\{X_i\}_{i=1}^{\infty}$ is an infinite sequence of nonempty sets. They are certainly disjoint as any two sets A, A^c are always disjoint. $\square$

b) Show that the cardinality of $\mathcal{M}$ is greater than or equal to the cardinality of the continuum.

Proof. Let $\{E_i\}_{i=1}^{\infty}$ be an infinite sequence of non-empty (without loss of generality) disjoint sets in $\mathcal{M}$. Let us define the power set $\mathcal{P}(\{E_i\} := \mathcal{E}$, and claim that $\mathcal{E} \cong \mathcal{P}(\mathbb{N})$. Let us now consider the map $\psi : \mathcal{E} \to \mathcal{P}(\mathbb{N})$, which we define by sending $\emptyset \mapsto \emptyset$, and $\bigcup_{j \in J} E_j \mapsto J$. We see ψ is certainly well defined since each $\{E_i\}$ is disjoint. We see ψ is surjective since for any non-empty $J \subset \mathbb{N}$, we have that $\bigcup_{j \in J} E_j$ is an element of $\mathcal{E}$. To achieve injection, see that $A \neq B \in \mathcal{E}$ implies that there exists some $E_a \in A \setminus B$ for $1 \leq a \leq \infty$. Then a coincides with a number in $\mathbb{N}$, and we must have that $a \in \psi(A) \setminus \psi(B)$, and therefore $\psi(A) \neq \psi(B)$. Therefore we see ψ is a bijection between $\mathcal{P}(\{E_i\})$ and $\mathcal{P}(\mathbb{N})$, and we have proven our claim. $\square$

Problem 3

Let $\{(X_\alpha, \mathcal{M}_\alpha)\}_{\alpha \in A}$ be a family of measurable spaces. Consider the product σ-algebra $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ in $X = \prod_{\alpha \in A} X_\alpha$.

a) Prove that if A is countable then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ coincides with the σ-algebra generated by $\left\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\right\}$.

Proof. Suppose that $\pi_\alpha^{-1}(E_\alpha) \in \bigotimes_{\alpha \in A} \mathcal{M}_\alpha$. Then observe that π_α^{-1} is the preimage of the projection map of α, which is any element in X such that the α-th component is contained in E_α. Therefore, such an element must be in $\prod_{\alpha \in A} A_\alpha$, where $A_\alpha = E_\alpha$ and $A_\beta = X_\beta$ for $\alpha \neq \beta$. Now since $\pi_\alpha^{-1}(E_\alpha)$ is an arbitrary basis element for the product σ-algebra, we must have $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ contained in $\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\}$.

Conversely, if $\prod_{\alpha \in A} E_\alpha$ is an element in the set in the problem statement, then we see that it coincides with the intersection $\bigcap \pi_\alpha^{-1}(E_\alpha)$. Indeed, letting A denote the product and B the intersection, we have that $x \in A$ if and only if each $x_\alpha \in E_\alpha$. Therefore $x \in \pi_\alpha^{-1} E_\alpha$ for all α and $x \in B$. $\square$

b) Suppose that $\mathcal{M}_\alpha$ is generated by $\mathcal{E}_\alpha$, $\alpha \in A$. Prove that $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by $\mathcal{G}_1 = \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha, \alpha \in A\}$.

Proof. If $\pi_\alpha^{-1}(E_\alpha)$ is an element of $\mathcal{G}_1$, then $E_\alpha \in \mathcal{E}_\alpha \subset \mathcal{M}_\alpha$. Therefore $\pi_\alpha^{-1}(E_\alpha) \in \bigotimes_{\alpha\in A}\mathcal{M}_\alpha$. Let us consider the set $\mathcal{F}_\alpha := \{E \subset X_\alpha : \pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{G}_1)\}$. This collection is nonempty since it trivially contains X. But if we take $F \in \mathcal{F}$, then we must have $\pi_\alpha^{-1}(F) \in \mathcal{G}_1$. Because $\mathcal{M}(\mathcal{G}_1)$ is a σ-algebra, we have that $\pi_\alpha^{-1}(F)^c \in \mathcal{M}(\mathcal{G}_1)$. We claim now that $\pi_\alpha^{-1}(F)^c = \pi_\alpha^{-1}(F^c)$. Indeed, this follows from the fact that $x \in \pi_\alpha^{-1}(F)^c$ if and only if the α-component of x is in $X \setminus F$. However, this holds if and only if $x \in \pi_\alpha^{-1}(F^c)$, and therefore $F^c \in \mathcal{F}_\alpha$.

Similar reasoning shows that if we consider the union of a family of sets $\{F_i\} \subset \mathcal{M}(\mathcal{G}_1)$, then we have that $\pi_\alpha^{-1}(\bigcup F_i) = \bigcup \pi_\alpha^{-1}(F_i)$. Since the latter is contained in $\mathcal{M}(\mathcal{G}_1)$, we have that the former must also be, and therefore $\mathcal{F}_\alpha$ is a σ-algebra. Moreover, if we take $E \in \mathcal{E}_\alpha$, then $\pi_\alpha^{-1}(E) \in \mathcal{G}_1$ and therefore $E \in \mathcal{F}_\alpha$. Therefore, $\mathcal{F}_\alpha \supset \mathcal{E}_\alpha$. Since $\mathcal{F}_\alpha$ is a σ-algebra, it must contain $\mathcal{M}_\alpha$.

Now if $\pi_\alpha^{-1}(E_\alpha)$ is an arbitrary generator contained in $\bigotimes_{\alpha\in A}\mathcal{M}_\alpha$, then $E_\alpha \in \mathcal{F}_\alpha$, and therefore $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{M}(\mathcal{G}_1)$. Therefore, $\bigotimes_{\alpha\in A}\mathcal{M}_\alpha$ is generated by $\mathcal{G}_1$. $\square$

c) If A is countable and $X_\alpha \in \mathcal{E}_\alpha\ \forall \alpha \in A$, prove that $\bigotimes_{\alpha\in A}\mathcal{M}_\alpha$ is generated by $\mathcal{G}_2 = \left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\right\}$.

Proof. We want to show that $\mathcal{M}(\mathcal{G}_1) = \mathcal{M}(\mathcal{G}_2)$. Given that some set A is countable, Folland Theorem 1.3 states that $\bigotimes_{\alpha\in A}\mathcal{M}_\alpha$ is generated by $\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\right\}$. Given that $\mathcal{M}_\alpha$ is generated by elementary families of sets $\mathcal{E}_\alpha$, we have that

$$\mathcal{G}_2 \subset \left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\right\} \subset \bigotimes_{\alpha\in A}\mathcal{M}_\alpha = \mathcal{M}(\mathcal{G}_1).$$

Recall now that $\mathcal{M}(\mathcal{G}_2)$ is by definition the intersection of all σ-algebras which contain $\mathcal{G}_2$. So, if $\mathcal{G}_2 \subset \mathcal{M}(\mathcal{G}_1)$, then $\mathcal{M}(\mathcal{G}_2) \subset \mathcal{M}(\mathcal{G}_1)$.

Now notice that $\pi_\alpha^{-1}(E_\alpha)$ generates the strips described in class, while $\prod_{\alpha\in A} E_\alpha$ generates the strips and the intersection of the E_α's, where $E_\alpha \in \mathcal{E}_\alpha$. Therefore, $\mathcal{G}_1 \subset \mathcal{G}_2$ implies that $\mathcal{M}(\mathcal{G}_1) \subset \mathcal{M}(\mathcal{G}_2)$.

Both inclusions have been shown, therefore $\mathcal{M}(\mathcal{G}_1) = \mathcal{M}(\mathcal{G}_2)$. $\square$

d) Prove that $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_1^n \mathcal{B}_{\mathbb{R}}$.

Proof. ($\subset$) First, we want to show that $\mathcal{B}_{\mathbb{R}^n} \subset \bigotimes_1^n \mathcal{B}_{\mathbb{R}}$. Let us recall the generators for $\bigotimes_1^n \mathcal{B}_{\mathbb{R}}$, which are of the form $\mathcal{M}\left(\{\pi_j^{-1}(U) : U \text{ open in } \mathbb{R}\}\right)$. Therefore, the coordinate maps $\pi_j^{-1}(U)$ are open, and therefore, $\mathcal{M}\left(\{\pi_j^{-1}(U) : U \text{ open in } \mathbb{R}\}\right) \subset \mathcal{B}_{\mathbb{R}}$.
($\supset$) Second, we want to show that $\mathcal{B}_{\mathbb{R}^n} \supset \bigotimes_1^n \mathcal{B}_{\mathbb{R}}$. Let us recall that $U = \bigcup_{i=1}^\infty (a_i, b_i)$ is open in $\mathbb{R}$. Now $U = \bigcup_{k=1}^\infty I_k$, where $I_k = \prod_{i=1}^\infty (a_i^k, b_i^k)$, is open in $\mathbb{R}^n$. So, if U is open in $\mathbb{R}^n$, then U is a countable union of sets of the form $\prod_{i=1}^n (a_i, b_i) \in \bigotimes \mathcal{B}_{\mathbb{R}}$. Then we have that $U \in \bigotimes \mathcal{B}_{\mathbb{R}}$. Then we have shown that $\mathcal{B}_{\mathbb{R}^n} \supset \bigotimes_1^n \mathcal{B}_{\mathbb{R}}$ holds. $\square$

Problem 4

Let X be an uncountable set and consider $\mathcal{M} = \{E \subset X : E \text{ or } E^c \text{ is countable}\}$.

a) Show that $\mathcal{M}$ is a σ-algebra in X.

Proof. In order to show that $\mathcal{M}$ is a σ-algebra, one must show that $\mathcal{M}$ is closed under countable unions and that $\mathcal{M}$ is closed under complement.

Let us define some family of sets $\{E_j\}_{j=1}^\infty \subset \mathcal{M}$. We will consider three cases in order to show closure under countable union:

(a) If all E_j's are countable, then we have that since a countable union of countable sets is countable, that $\bigcup_{j=1}^\infty E_j$ is countable. Therefore, $\bigcup_{j=1}^\infty E_j \in \mathcal{M}$.

(b) If all E_j^c's are countable, then so is $\bigcap_{j=1}^\infty E_j^c$, since $\bigcap_{j=1}^\infty E_j^c \subset E_1^c$, where without loss of generality, E_1^c is the "smallest" member of the E_j^c's. Then by DeMorgan's Laws, we have

$$\bigcap_{j=1}^\infty E_j^c = \left(\bigcup_{j=1}^\infty E_j\right)^c \quad \text{is countable.}$$

Then by definition of $\mathcal{M}$, we have that $\bigcup_{j=1}^\infty E_j \in \mathcal{M}$.

(c) If some sets are countable and some are uncountable, let us define two families of sets, $\{E_j\}_{j=1}^{\infty}, \{F_j\}_{j=1}^{\infty} \subset \mathcal{M}$, where all E_j's are countable, and all F_j^c's are countable. Then we want to show that $\bigcup_{j=1}^{\infty}(E_j \cup F_j) \in \mathcal{M}$. So, given that X is uncountable, we know that each E_j^c is uncountable. Since $E_j^c \cap F_j^c \subset F_j^c$ by definition of intersection, we have that $E_j^c \cap F_j^c$ must be countable. Furthermore, $\bigcap_{j=1}^{\infty} E_j^c \cap F_j^c$ is countable. Then by DeMorgan's Laws, we have that

$$\bigcap_{j=1}^{\infty} E_j^c \cap F_j^c = \left(\bigcup_{j=1}^{\infty}(E_j \cup F_j)\right)^c \quad \text{is countable.}$$

Then by definition of $\mathcal{M}$, we have that $\bigcup_{j=1}^{\infty}(E_j \cup F_j) \in \mathcal{M}$.

So, closure under countable union has been shown.

We now aim to show closure under complement. Consider some $E \in \mathcal{M}$. Then certainly $E^c \in \mathcal{M}$, since $(E^c)^c = E \in \mathcal{M}$. $\square$

b) For $E \in \mathcal{M}$, define $\mu(E) = 0$ if E is countable, and $\mu(E) = 1$ if E^c is countable. Prove that μ is a measure.

Proof. In order to show that μ is a measure, we need to show that (1) $\mu(\emptyset) = 0$, and (2) for some pairwise disjoint sequence $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$, we have that

$$\sum_{j=1}^{\infty} \mu(E_j) = \mu\left(\bigcup_{j=1}^{\infty} E_j\right).$$

We have (1) trivially. That is, since $\emptyset$ is countable, then by definition of μ we have that $\mu(\emptyset) = 0$. This satisfies (1).

For (2), we have two cases which will be shown separately:

(a) Take all E_j's to be countable. Then since a countable union of countable sets is countable, $\bigcup_{j=1}^{\infty} E_j$ is countable. So, $\mu(\bigcup_{j=1}^{\infty} E_j) = 0$ by definition of μ. Also, since $\mu(E_j) = 0$ for all j, we have that certainly $\sum_{j=1}^{\infty} E_j = \sum_{j=1}^{\infty}(0) = 0$. So, the left and right hand sides match, and in this case, μ is a measure.

(b) For this case, we claim that there is only one $E_n \in \{E_j\}$ such that E_n^c is countable, which will indeed yield our result; so, for the sake of contradiction, assume that there exist two E_j's, say E_n and E_m $(n \neq m)$ such that E_n^c and E_m^c are both countable. Then we have:

$$\begin{aligned}
(E_n^c) \cup (E_m^c) &= (E_n \cap E_m)^c && \text{by DeMorgan's Laws} \\
&= \emptyset^c && \text{since the } E_j\text{'s are disjoint} \\
&= X \quad \Rightarrow\Leftarrow
\end{aligned}$$

However, we see that a contradiction arises, as a countable union of countable sets cannot be uncountable (which X is). As such, we must have that there is exactly only one uncountable set, without loss of generality say E_n, in $\bigcup_{j=1}^{\infty} E_j$. Therefore, since E_n^c is indeed countable and thus $\mu(E_n) = 1$, we have:

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = 1 = \mu(E_n) = \sum_{j=1}^{\infty} \mu(E_j).$$

So, we have shown that μ is a measure. $\square$

Problem 5

Given a nonempty set X, consider the measurable space $(X, \mathcal{P}(X))$ and a function $f : X \to [0, \infty]$. Define $\mu(\emptyset) = 0$ and $\mu(E) = \sum_{x \in E} f(x)$ for $E \subset X$, $E \neq \emptyset$. Recall that $\sum_{x \in E} f(x) = \sup\{\sum_{x \in I} f(x) : I \subset E \text{ and } I \text{ is finite}\}$.

a) Prove that μ is a measure.

Proof. In order to show that μ is a measure, we want to show that $\mu(\emptyset) = 0$ and that it is countable additive. Trivially, we have that $\mu(\emptyset) = 0$.

To show that μ is countable additive, that is

$$\mu(E) = \sum_{x \in E} f(x) = \sup\{\sum_{x \in I} f(x), I \subset E, I \text{ finite}\}$$

Let's say that we have a family $\{E_i\}$ of (without loss of generality) disjoint subsets of X. Let $I \subset \bigcup_{i=1}^{\infty} E_i$, with I finite. Certainly then, $I = \bigcup_{i=1}^{\infty} I \cap E_i$. Note that each $I \cap E_i$, which we can call I_i, is a finite subset of E_i. First, we want to show that $\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$. So take

$$\sum_{x \in I} f(x) = \sum_{i=1}^{\infty} \sum_{x \in I_i} f(x).$$

But certainly, since $I_i \subset E_i$, we have that $\sum_{x \in I_i} f(x) \leq \mu(E_i)$, and as such, we can bound the above sum as follows:

$$\sum_{x \in I} f(x) = \sum_{i=1}^{\infty} \sum_{x \in I_i} f(x) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

As a result, we now have that

$$\sum_{x \in I} f(x) = \sup \left\{ \sum_{x \in I} f(x) : I \subset E, I \text{ finite} \right\} = \mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

Now since I is arbitrary, $\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$.

Now we want to show $\mu(\bigcup_{i=1}^{\infty} E_i) \geq \sum_{i=1}^{\infty} \mu(E_i)$. Let us first assume that $\mu(E_i) < \infty$ for all i. If not, say $\mu(E_{i_0}) = \infty$, then it follows by monotonicity that $\mu(\bigcup E_i) = \infty$. Now let $\epsilon > 0$, and choose some finite $I_i \subset E_i$ be such that

$$\mu(E_i) < \sum_{x \in I_i} f(x) + \frac{\epsilon}{2^i}.$$

Now adding from $i = 1$ to $i = n$, we can estimate the following partial sum:

$$\begin{aligned} \sum_{i=1}^{n} \mu(E_i) &\leq \sum_{i=1}^{n} \sum_{x \in I_i} f(x) + \sum_{i=1}^{n} \frac{\epsilon}{2^i} \\ &< \sum_{x \in \bigcup_{i=1}^{n} I_i} f(x) + \epsilon \\ &\leq \mu\left(\bigcup_{i=1}^{\infty} E_i\right) + \epsilon. \end{aligned}$$

Now if we take $n \to \infty$, we have

$$\sum_{i=1}^{\infty} \mu(E_i) \leq \mu\left(\bigcup_{i=1}^{\infty} E_i\right) + \epsilon,$$

and since ϵ is arbitrary, it can approach 0, and our desired inequality is achieved.

We showed that the sum of the measures is less than or equal to and greater than or equal to the measure of the union, therefore we have shown that $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ and μ is countably additive. $\square$

b) Show that μ is σ-finite if and only if $0 \leq f(x) < \infty$ for all $x \in X$ and the set $\{x \in X : f(x) > 0\}$ is countable.

Proof. ($\Longleftarrow$) Let us take $A = \{x : f(x) = 0\}$ be countable. Also, redefine $\{x \in X : f(x) > 0\} := \{x_i\}$. Then certainly by definition, $\mu(A) = 0$. Then $X = A \cup \bigcup_{i=1}^{\infty} x_i$. Now note that the measure of a singleton is the following:

$$\mu(\{x_i\}) = \sup \left\{ \sum_{x \in I} f(x) : I \subset \{x_i\} \right\} = f(x_i) < \infty.$$

Therefore, we have by definition that μ is σ-finite.

($\Longrightarrow$) We want to show that μ is σ-finite implies that $0 \leq f(x) < \infty$, and that $\{x : f(x) > 0\}$ is countable. So, having that μ is σ-finite implies that $X = \bigcup_{i=1}^{\infty} E_i$ such that $\mu(E_i) < \infty$. Then $x \in X$ implies that there exists some i such that $x \in E_i$ implies that $\mu(\{x\}) \leq \mu(E_i) < \infty$. But note that $\mu(\{x\}) = f(x)$, so we have shown that $f(x)$ is indeed finite.

We now want to show that $\{x : f(x) > 0\}$ is countable. So for the sake of contradiction, take $\{x : f(x) > 0\}$ to be uncountable. Then

$$\{x : f(x) > 0\} = \bigcup_{k=1}^{\infty} \left\{x : f(x) > \frac{1}{k}\right\}.$$

Note that by our assumption, the left hand side is uncountable. Then we see that the right hand side is a countable union of sets, in which at least one of the sets $\{x : f(x) > 1/k\}$ must be uncountable; in other words, there exists some k such that $\{x : f(x) > 1/k\}$ is uncountable. Now define $A_k = \{x : f(x) > 1/k\}$. Then we have that $A_k = \bigcup_{i=1}^{\infty} E_i \cap A_k$, which implies that there exists some i such that $E_i \cap A_k$ is uncountable by assumption. Now note that since $E_i \cap A_k \subset E_i$, we have $\mu(E_i \cap A_k) < \infty$ by monotonicity of μ. Let us now compute $\mu(E_i \cap A_k)$. By definition, we have

$$\mu(E_i \cap A_k) = \sup \left\{ \sum_{x \in I} f(x) : I \subset E_i \cap A_k, I \text{ finite} \right\}.$$

Let us now consider the set I. Certainly, $I \subset E_i \cap A_k$, with I finite. Then we can bound the following,

$$\sum_{x \in I} f(x) > \frac{1}{k}|I|,$$

however we see that we can force $|I|$ to be as high as we want, say ∞. This yields a contradiction, as we saw that the supremum described before should be finite, and we just found a sum of $f(x)$ which is infinite. So, we must have that $\{x : f(x) > 0\}$ is countable. □

Problem 6

Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{E_i\}_{i \in \mathbb{N}} \subset \mathcal{M}$. Prove that:

a) $\mu(\liminf E_i) \leq \liminf \mu(E_i)$

Proof. Define some family of sets $F_n := \bigcap_{k=n}^{\infty} E_k$, for $n \in \mathbb{N}$. Then certainly by definition of intersection, we have that $F_1 \subset F_2 \subset \cdots$. Then by the property of continuity from below of measure and the definition of F_n, we have

$$\mu\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_n \mu(F_n) = \lim_n \mu\left(\bigcap_{k=n}^{\infty} E_k\right).$$

Now since $\bigcap_{k=n}^{\infty} E_k \subset E_k$ for arbitrary $k \geq n$ by definition of intersection, we have by monotonicity of μ that $\mu\left(\bigcap_{k=n}^{\infty} E_k\right) \leq \mu(E_k)$. Now since k was chosen arbitrarily, let us take the infimum on the right hand side; this will still yield a valid inequality:

$$\mu\left(\bigcap_{k=n}^{\infty} E_k\right) \leq \inf \mu(E_k).$$

Therefore, taking the limit as $n \to \infty$ on both sides and recognizing the definition of limit infimum yields the following:

$$\lim_n \mu\left(\bigcap_{k=n}^{\infty} E_k\right) = \mu\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k\right) = \mu\left(\liminf_n E_n\right) \leq \liminf_n \mu(E_n).$$

Therefore, we have shown that indeed, $\mu(\liminf E_i) \leq \liminf \mu(E_i)$. □

b) $\mu(\limsup E_i) \geq \limsup \mu(E_i)$ if $\mu(\bigcup_{i=1}^{\infty} E_i) < \infty$.

Proof. Similarly to the previous part, let us define some family of sets $G_n := \bigcup_{k=n}^{\infty} E_k$, for $n \in \mathbb{N}$. Then certainly by definition of union, we have that $G_1 \supset G_2 \supset \cdots$. Then using the fact that μ is finite, we have that the measure of the "largest" G_n is finite, that is $\mu(G_1) < \infty$. Then by the property of continuity from above of μ and the definition of G_n, we have:

$$\mu\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right) = \mu\left(\bigcap_{n=1}^{\infty} G_n\right) = \lim_n \mu(G_n) = \lim_n \mu\left(\bigcup_{k=n}^{\infty} E_k\right).$$

Now since $\bigcup_{k=n} E_k \supset E_k$ for arbitrary $k \geq n$ by definition of union, then by monotonicity of μ, we have that $\mu(\bigcup_{k=n}^{\infty} E_k) \geq \mu(E_k)$ for arbitrary $k \geq n$. Since k was chosen arbitrarily, let us take the supremum on the right hand side; this will still yield a valid inequality:

$$\mu\left(\bigcup_{k=n}^{\infty} E_k\right) \geq \sup(\mu(E_k)).$$

Therefore, taking the limit as $n \to \infty$ on both sides and recognizing the definition of limit supremum yields the following:

$$\lim_n\left(\mu\left(\bigcup_{k=n}^{\infty} E_k\right)\right) = \mu\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right) = \mu\left(\limsup_n(E_n)\right) \geq \limsup_n(\mu(E_n)).$$

Therefore, we have shown that indeed, for finite μ, that $\mu(\limsup E_i) \geq \limsup \mu(E_i)$. □

Problem 7

Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $\{E_k\}_{k\in\mathbb{N}}$ is a sequence of measurable sets in X such that $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Prove that almost all $x \in X$ lie in at most finitely many of the sets E_k.

Proof. Let us begin by noting that the claim in this problem statement is equivalent to saying that $\mu(\limsup_k E_k) = 0$. This follows from the definition of limit supremum on sets, which is $\limsup(E_i) = \{x : x \in E_i \text{ for infinitely many } i\}$. Surely, we want to show that the measure of this is 0, which aligns with the problem statement.

As used in the previous problem, we know that by definition,

$$\limsup_k E_k = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k.$$

Recall from the previous problem that we can construct a monotone sequence $F_n = \bigcup_{k=n}^{\infty} E_k$ such that $F_1 \supset F_2 \supset \cdots$. This means that we can use continuity from above as a property of μ. So, taking the measure, we have:

$$\begin{aligned}
\mu(\limsup_{k\to\infty} E_k) &= \mu\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right) && \text{by def. of } \limsup_k E_k \\
&= \lim_{n\to\infty} \mu\left(\bigcup_{k=n} E_k\right) && \mu \text{ is continuous from above} \\
&\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k) && \text{by subadditivity of } \mu
\end{aligned}$$

Notice however that the last term is essentially the 'tail' of the series $\sum_{k=1}^{\infty} \mu(E_k)$, which is finite (and therefore convergent) by the problem statement. So, since the tail of a convergent sequence goes to 0 as $n \to \infty$, we have that $\lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k) = 0$. This forces $\mu(\limsup_{k\to\infty} E_k) = 0$, and we have shown our desired result. □

Problem 8

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose that the measure μ is semifinite (i.e. for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \subset E$ such that $F \in \mathcal{M}$ and $0 < \mu(F) < \infty$). Prove that if $E \in \mathcal{M}$ and $\mu(E) = \infty$, for any $C > 0$ there exists $F \subset E$ such that $F \in \mathcal{M}$ and $C < \mu(F) < \infty$.

Proof. Let us denote $S := \sup\{\mu(F) : F \subset E, 0 < \mu(F) < \infty\}$. We know that S exists since E is semifinite. Let's suppose that $S < \infty$. Then there exists some $B \subset E$ such that $\mu(B) = S$. Indeed, for each $n \in \mathbb{N}$, there exists some $B_n \subset E$ such that $S - 1/n < \mu(B_n) \leq S$. Then $\bigcup_{i=1}^{\infty} B_n \subset E$ has measure S, since for every $\epsilon > 0$, we have an $n \in \mathbb{N}$ where $\frac{1}{n} < \epsilon$ and hence $S - \epsilon < \mu(B_n) \leq \mu(B) \leq S$.

Now let $B \subset E$ denote the set in which $\mu(B) = S$. Then $E \setminus B$ must have infinite measure since $\mu(E \setminus B) + \mu(B) = \mu(E) = \infty$. Since μ is semifinite, there exists some $HE \setminus B$ such that $0 < \mu(H) < \infty$. Because H and B are disjoint, $\mu(H \cup B) = \mu(H) + \mu(B) > \mu(B) = S$. Therefore the supremum cannot be finite. Therefore, $S = \infty$, and thus for each C, there exists some $F \subset E$ such that $F \in \mathcal{M}$ and $C < \mu(F) < \infty$. □

Problem 9

Given a measure space $(X, \mathcal{M}, \mu)$ define μ_0 on $\mathcal{M}$ by $\mu_0(E) = \sup\{\mu(F) : F \subset E, F \in \mathcal{M}, \mu(F) < \infty\}$. Prove the following:

a) μ_0 is a semifinite measure (it is called the semifinite part of μ).

Proof. We will prove that μ_0 is a measure, as semifiniteness follows from its definition. We begin by noting that $E \in \mathcal{M}$ implies $\emptyset \in E$, and as such, $\mu(\emptyset) = 0$. Then we have that $0 \in \{\mu(F) : F \subset E, \mu(F) < \infty\}$ and $\mu_0(E) \geq 0$. Now if $F \subset \emptyset$, we know that $F = \emptyset$, and therefore $\mu_0(\emptyset) = \{\mu(F) : F \subset \emptyset, (F) < \infty\}$=0.

So, to show μ_0 is a measure, we aim to show that $\mu_0(\bigcup_i E_i) = \sum_i \mu_0(E_i)$. Let us take the family $\{E_i\}_{i=1}^{\infty}$ of disjoint sets such that $E = \bigcup_{i=1}^{\infty} E_i$. Let us note that if we take some $F \in \mathcal{M}$, we have $\{F \cap E_i\}_{i=1}^{\infty}$ is also a family of disjoint sets; moreover, $F = F \cap E = \bigcap_{i=1}^{\infty}(F \cap E_i)$. So then we have

$$\mu(F) = \sum_{i=1}^{\infty} \mu(F \cap E_i).$$

Since we have that $F \cap E_i \subset E_i$ and $\mu(F)$ is finite, we know that $\mu(F \cap E_i)$ is finite and therefore $F \cap E_i \in \{G : G \subset E_i, \mu(G) < \infty\}$. This implies that $\mu(F \cap E_i) \leq \sup\{\mu(G) : G \subset E_i, \mu(G) < \infty\}$ for all i, and also

$$\mu(F) = \sum_{i=1}^{\infty} \mu(F \cap E_i) \leq \sum_{i=1}^{\infty} \mu_0(E_i).$$

As such, we have that $\mu_0(E) = \sup\{\mu(F) : F \subset E, \mu(F) < \infty\} \leq \sum_{i=1}^{\infty} \mu_0(E_i)$. This will be referenced later.

Now note that if there exists some j such that $\mu_0(E_j) = \infty$, then since $E_j \subset E$, we have $\{F : F \subset E_j, \mu(F) < \infty\} \subset \{F : F, \mu(F) < \infty\}$, and

$$\infty = \mu_0(E_j) = \sup\{\mu(F) : F \subset E_j, \mu(F) < \infty\} \leq \sup\{\mu(F) : F \subset E, \mu(F) < \infty\} = \mu_0(E) = \infty.$$

Therefore, $\mu_0(E) = \infty = \sum_{i=1}^{\infty} \mu_0(E_i)$. Now let us suppose that $\mu_0(E_i) < \infty$. Take $\epsilon > 0$. Then by definition of μ_0, there exists some family $\{F_i\}_{i=1}^{\infty}$ such that $F_i \subset E$ and $\mu_0(E_i) - \epsilon/2^i \leq \mu(F_i) \leq \mu_0(E_i) < \infty$. Now since $F_i \subset E_i$, we have that $\{F_i\}$ is a disjoint family of sets, and also $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i = E$. So, we have that

$$\begin{aligned}
\sum_{i=1}^{\infty} \mu_0(E_i) - \epsilon = \sum_{i=1}^{\infty} \left(\mu_0(E_i) - \frac{\epsilon}{2^i}\right) &\leq \sum_{i=1}^{\infty} \mu(F_i) \\
&= \mu\left(\bigcup_{i=1}^{\infty} F_i\right) \\
&= \sup\left\{\mu\left(\bigcup_{i=1}^{k} F_i\right) : 1 \leq k \leq \infty\right\} \\
&\leq \sup\{\mu(F) : F \subset E, \mu(F) < \infty\} \\
&= \mu_0(E)
\end{aligned}$$

Now since ϵ was chosen arbitrarily, we have that

$$\sum_{i=1}^{\infty} \mu_0(E_i) \leq \mu_0(E)$$

From what we showed previously, we have that $\sum_{i=1}^{\infty} \mu_0(E_i) = \mu_0(E)$, which gives that μ_0 is indeed a measure. It is also semifinite since there exists a set $B \subset E$ such that $\mu_0(B) < \infty$ by the way μ_0 is constructed. □

b) If μ is semifinite then $\mu = \mu_0$.

Proof. Suppose μ is semifinite. If $\mu(E) = \infty$, then we know from Problem 8 that for any $C \in \mathbb{R}_+$, there exists some set $F \subset E$ such that $C < \mu(F) < \infty$. So then we have $\mu_0(E) = \sup\{\mu(F) : F \subset E, \mu(F) < \infty\} = \infty = \mu(E)$. Thus, if μ is semifinite, then $\mu(E) = \mu_0(E)$. $\square$

Problem 10

a) Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that if μ is σ-finite then it is semifinite.

Proof. Let E_i be a cover of X such that each $\mu(E_i) < \infty$. Also let E be a set in $\mathcal{M}$ such that $\mu(E) = \infty$. Then $E_i \cap E \subset E_i$, and therefore $\mu(E_i \cap E) < \infty$ for each i. We want to show that at least one of these intersections has a positive image. Let us note that there is an E_i where $E_i \cap E$ is non-empty, as $\bigcup E_i = X$. Therefore $E \subset \bigcup E_i$, and there must exist some E_i where $E_i \cap E$ is non-empty. Let $\{E_j\}$ denote the subset of $\{E_i\}$ such that $E_j \cap E$ is non-empty for each j. Certainly, we have $\bigcup E_j \supset E$ and $(\bigcup E_j) \cap E = \bigcup(E_j \cap E) = E$. Therefore, we have that

$$\mu(E) = \mu\left(\bigcup(E_j \cap E)\right) \leq \sum \mu(E_j \cap E).$$

Now since $\mu(E)$ is infinite, then $\sum \mu(E_j \cap E)$ is infinite. This holds only if there exists some k where $0 < \mu(E_k \cap E) \leq \mu(E_k) < \infty$. So, μ is semifinite. $\square$

b) Give an example of a measure space $(X, \mathcal{M}, \mu)$ where μ is semifinite but not σ-finite.

Consider $(X, \mathcal{M}, \mu)$ to be $(\mathbb{R}, \mathcal{P}(\mathbb{R})$, counting measure $\mu)$, where we have $\mu(\emptyset) = 0$, $\mu(\{x_1, ..., x_n\}) = n$, and $\mu(E) = \infty$ for some infinite set $E \subset \mathbb{R}$. Then since $\mathbb{R}$ is uncountably infinite, and a countable union of countable sets is countable, we cannot have σ-finiteness. But, for any infinite set $E \subset \mathbb{R}$, we have an element $\in E$ so that we have $\{e\} \subset E$ and $\mu(\{e\}) = 1$.