Real Analysis Homework 6

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Unless otherwise stated, we consider below L^p spaces on a fixed measure space $(X, \mathcal{M}, \mu)$.

Problem 1

If $0 < p_1 < p < p_2 \leq \infty$, prove that any element in $L^p(\mu)$ is the sum of an element of $L^{p_1}(\mu)$ and an element of $L^{p_2}(\mu)$. This is $L^p(\mu) \subset L^{p_1}(\mu) + L^{p_2}(\mu)$. **Hint**. Consider $h = f\chi_E$ and $g = f\chi_{E^c}$, where $E = \{x : |f(x)| > 1\}$.

Proof. Take $f \in L^p(\mu)$. Then we have that $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable and $\|f\|_p < \infty$. Define the set $E := \{x \in X : |f(x)| > 1\}$ and the functions $g := f\chi_E$ and $h := f\chi_{E^c}$. Certainly, this gives the following definitions:

$$g(x) = f\chi_E(x) = \begin{cases} f(x) & x \in E, \\ 0 & x \in E^c, \end{cases} \quad \text{and} \quad h(x) = f\chi_{E^c}(x) = \begin{cases} 0 & x \in E^c, \\ f(x) & x \in E. \end{cases}$$

Clearly then for all x, we have that $f(x) = g(x) + h(x)$. Since g, h are the products of two $\mathcal{M}$-measurable functions, they too must also be $\mathcal{M}$-measurable.

We now shall estimate the L^{p_1} norm of g, with $0 < p_1 < p \leq \infty$:

$$\|g\|_{p_1}^{p_1} = \int_X |f(x)\chi_E(x)|^{p_1} d\mu(x) = \int_E |f(x)|^{p_1} d\mu(x) \leq \int_E |f(x)|^{p_1} d\mu(x) < \int_E |f(x)|^p d\mu(x) = \|f\|_p^p.$$

Now since $f \in L^p(\mu)$, we have that $\|f\|_p^p < \infty$ and therefore by our estimate we have that $\|g\|_{p_1}^{p_1} < \infty$ as well, and as such, $g \in L^{p_1}(\mu)$. Similarly, we estimate the L^{p_2} norm of h:

$$\|h\|_{p_2}^{p_2} = \int_X |f(x)\chi_{E^c}(x)|^{p_2} = \int_X |f(x)|^{p_2}\chi_{E^c}^{p_2}(x) d\mu(x) = \int_{E^c} |f(x)|^{p_2} d\mu(x) \leq \int_{E^c} |f(x)|^p d\mu(x).$$

Indeed, since $\|h\|_{p_2}^{p_2} < \infty$, we have that $h \in L^{p_2}(\mu)$. $\square$

Problem 2

Logarithmic convexity of the norm in L^p spaces. If $0 < p_1 < p < p_2 \leq \infty$, then $L^{p_1}(\mu) \cap L^{p_2}(\mu) \subset L^p(\mu)$ and $\|f\|_p \leq \|f\|_{p_1}^{\theta} \|f\|_{p_2}^{1-\theta}$, where $\theta \in (0,1)$ is such that $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. **Hint**. Note that $\int_X |f|^p d\mu = \int_X |f|^{\theta p} |f|^{(1-\theta)p} d\mu$ and use Hölder's inequality appropriately.

Proof. Let $f \in L^{p_1}(\mu) \cap L^{p_2}(\mu)$. Then we have that $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable and $\|f\|_{p_1}$ and $\|f\|_{p_2}$ are finite. We have the following implication:

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \Longrightarrow 1 = \frac{\theta p}{p_1} + \frac{(1-\theta)p}{p_2} = \frac{1}{p_1/(\theta p)} + \frac{1}{p_2/((1-\theta)p)}.$$

Therefore, we see that $p_1/\theta p$ and $p_2/(1-\theta)p$ are conjugate exponents. Indeed,

$$\int_X |f|^p d\mu = \int_X |f|^{\theta p} |f|^{(1-\theta)p} d\mu.$$

Using Hölder's inequality, we estimate $\|f\|_p^p$ as follows:

$$\begin{aligned}\|f\|_p^p = \int_X |f^{\theta p} f^{(1-\theta)p}| d\mu = \|f^{\theta p} f^{(1-\theta)p}\|_1 &\leq \|f^{\theta p}\|_{p_1/\theta p} \|f^{(1-\theta)p}\|_{p_2/(1-\theta)p} \\ &= \left(\int_X |f^{\theta p}|^{p_1/\theta p} d\mu \right)^{\theta p/p_1} \left(\int_X |f^{(1-\theta)p}|^{p_2/(1-\theta)p} d\mu \right)^{(1-\theta)p/p_2} \\ &= \left(\int_X |f|^{p_1} d\mu \right)^{\theta p/p_1} \left(\int_X |f|^{p_2} d\mu \right)^{(1-\theta)p/p_2} \\ &= \|f\|_{p_1}^{\theta p} \|f\|_{p_2}^{(1-\theta)p}.\end{aligned}$$

We see that indeed, $\|f\|_p \leq \|f\|_{p_1}^{\theta} \|f\|_{p_2}^{1-\theta}$ as desired. Moreover, since $\|f\|_{p_1}, \|f\|_{p_2} < \infty$, we have that $\|f\|_p < \infty$. Therefore $f \in L^p(\mu)$ as desired, so the containment $L^{p_1}(\mu) \cap L^{p_2}(\mu) \subset L^p(\mu)$ is shown. □

Problem 3

Prove the following relations among L^p spaces:

(a) If $\mu(X) < \infty$ and $0 < p < q \leq \infty$ then $L^q(\mu) \subset L^p(\mu)$ and $\|f\|_p \leq \|f\|_q \mu(X)^{\frac{1}{p} - \frac{1}{q}}$.

Proof. We can rewrite the relation given in the problem statement as $q - p > 0$, and we can consider the following:

$$1 = \frac{p}{q} + \frac{q-p}{q} = \frac{1}{q/p} + \frac{1}{q/(q-p)}.$$

So, we see that q/p and $q/(q-p)$ are conjugate exponents, and therefore by Hölder's inequality we have the following line of implications:

$$\begin{aligned}\|f^p\|_1 \leq \|f^p\|_{q/p} \|1\|_{q/(q-p)} &\implies \int_X |f^p| d\mu \leq \left(\int_X |f^p|^{q/p} d\mu \right)^{p/q} \left(\int_X d\mu \right)^{(q-p)/q} \\ &\implies \int_X |f|^p d\mu \leq \left(\int_X |f|^q d\mu \right)^{p/q} \left(\int_X d\mu \right)^{(q-p)/q} \\ &\implies \|f\|_p^p \leq \|f\|_q^p \mu(X)^{(q-p)/q} \\ &\implies \|f\|_p \leq \|f\|_q \mu(X)^{(q-p)/qp} = \|f\|_q \mu(X)^{1/p - 1/q}.\end{aligned}$$

The inequality has been shown, and since $\mu(X) < \infty$, we have that $L^q \subset L^p$.

For the case in which $q = \infty$, we aim to show that $\|f\|_p \leq \|f\|_\infty \mu(X)^{\frac{1}{p} - \frac{1}{q}}$. Certainly $|f(x)| \leq \|f\|_\infty$, and so

$$\int_X |f(x)|^p d\mu \leq \int_X \|f\|_\infty^p d\mu \implies \|f\|_p \leq \|f\|_\infty \mu(X)^{1/p}.$$

□

(b) Suppose that $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$, and μ is the counting measure. Recall that in this case $L^p(\mu)$ is denoted by ℓ^p. Prove that if $0 < p < q \leq \infty$ then $\ell^p \subset \ell^q$ and $\|f\|_q \leq \|f\|_p$ for any sequence $f = \{f(n)\}_{n \in \mathbb{N}}$. **Hint**. Consider first the case $q = \infty$. For $q < \infty$, use Problem 2.

Proof. Recall that $\|f\|_{\ell^p} := (\sum_{n \in \mathbb{N}} |f(n)|^p)^{1/p}$. If $q = \infty$, then $\|f\|_{\ell^p} = \sup |f(n)|$, and we have the following:

$$|f(n)|^p \leq \sum_{n \in \mathbb{N}} |f(n)|^p \implies |f(n)| \leq \left(\sum_{n \in \mathbb{N}} |f(n)|^p \right)^{1/p} \implies \sup |f(n)| \leq \left(\sum_{n \in \mathbb{N}} |f(n)|^p \right)^{1/p} \implies \|f\|_{\ell^q} \leq \|f\|_{\ell^p}.$$

For the finite case, that is $q < \infty$, notice that by Problem 2, we have that if $1/p = \theta/p + (1-\theta)/\infty = \theta/p$, then $\theta = p/q$. As such, we have the following estimate:

$$\|f\|_{\ell^q} \leq \|f\|_{\ell^p}^{\theta}\|f\|_{\ell^\infty}^{1-\theta} = \|f\|_{\ell^p}^{p/q}\|f\|_{\ell^\infty}^{1-p/q} \leq \|f\|_{\ell^p}^{p/q}\|f\|_{\ell^p}^{1-p/q} = \|f\|_{\ell^p}.$$

Indeed, for all q, the desired result holds. Moreover, since $f \in L^p$, that is f is finite, and $\|f\|_q \leq \|f\|_p$, we know that $\ell^p \subset \ell^q$. $\square$

Problem 4

Chebyshev's Inequality. If $0 < p < \infty$ and $f \in L^p(\mu)$ then for all $\lambda > 0$,

$$\mu(\{x : |f(x)| > \lambda\}) \leq \left(\frac{\|f\|_p}{\lambda}\right)^p.$$

Proof. First, notice that $\|f\|_p = (\int f^p d\mu)^{1/p}$, and so $\|f\|_p^p = \int |f|^p d\mu$. So, we can rewrite the right hand side as $(1/\lambda^p)\int |f|^p d\mu$. Also, since it is not given, let us assume that $\{x : |f(x)| > \lambda\} \subset X$. We have the following estimates:

$$\begin{aligned}
\int_X |f|^p \, d\mu &\geq \int_{\{x:|f(x)|>\lambda\}} |f|^p \, d\mu && \text{monotonicity of integral} \\
&> \int_{\{x:|f(x)|>\lambda\}} \lambda^p \, d\mu && \text{since } |f(x)| > \lambda \Longrightarrow |f(x)|^p > \lambda^p \text{ for } p > 0 \\
&= \lambda^p \int_{\{x:|f(x)|>\lambda\}} d\mu && \text{linearity of integral} \\
&= \lambda^p \cdot m\{x : |f(x)| > \lambda\} && \text{def. of Lebesgue measure.}
\end{aligned}$$

Dividing by λ^p on both sides yields the desired result. $\square$

Problem 5

Let $1 \leq p < \infty$.

(a) If $f_n, f \in L^p(\mu)$ and $f_n \to f$ in $L^p(\mu)$, then $f_n \to f$ in measure. **Hint**. Use Chebyshev's inequality.

Proof. Let $f_n \to f$ in $L^p(\mu)$. By definition, we have that $\lim_{n\to\infty}\left(\int_X |f_n - f|^p d\mu\right)^{1/p} = 0$. Now recall that $f_n \to f$ in measure if $\forall \epsilon > 0$, we have that $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) = 0$ as $n \to \infty$. By Chebyshev's inequality, we have that

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \leq \left(\frac{\|f_n - f\|_p}{\epsilon}\right)^p = \frac{1}{\epsilon^p}\int_X |f_n - f|^p d\mu.$$

Then as $n \to \infty$, we see that $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) = 0$, and therefore $f_n \to f$ in measure. $\square$

(b) If $f_n, f \in L^p(\mu)$ and $f_n \to f$ in $L^p(\mu)$ then there exists a subsequence of $\{f_n\}$ which converges to f almost everywhere.

Proof. From part (a), we know that $f_n \to f$ in measure. The following diagram (where dotted lines denote implication for a subsequence) which we used in class gives the result.

$$\begin{tikzcd}
& \text{u.c.} \arrow[dd, Rightarrow] \arrow[dr, Rightarrow] & \\
L^1 \arrow[rr, Rightarrow, dashed] \arrow[dr, Rightarrow] & & \text{a.e.} \\
& \mu \arrow[ur, Rightarrow, dashed] &
\end{tikzcd}$$

$\square$

(c) If $f_n \to f$ in measure and $|f_n| \leq |g|$ a.e. for all n, and $g \in L^p(\mu)$, then $f_n, f \in L^p(\mu)$ and $f_n \to f$ in $L^p(\mu)$.

Proof. Assume for the sake of contradiction that $(\int |f_n - f|^p)^{1/p} \not\to 0$. In particular, there must exist some subsequence $\{f_{n_k}\} \subset \{f_n\}$ such that $(\int |f_{n_k} - f_n|^p)^{1/p} \geq \epsilon$ for some fixed arbitrary epsilon. As given, we know that $f_n \to f$ in measure. When we have a dominating function, in this case g, we have that $f_n \to f$ in measure implies that $f_{n_k} \to f$ in L^1, which implies that $f_{n_k} \to f$ in measure. Then we have that there exists some subsequence $\{f_{n_{k_j}}\} \subset \{f_{n_k}\}$ such that $f_{n_{k_j}} \to f$ μ-a.e. Now we have that $|f_{n_{k_j}} - f| \leq |f_{n_{k_j}}| + |f| \leq 2g \in L^p$ which implies that $\left(\int |f_{n_{k_j}} - f|^p\right)^{1/p} \to 0$, or more specifically,

$$\lim_j \left(\int |f_{n_{k_j}} - f|^p\right)^{1/p} = \left(\int |f_{n_k} - f|^p\right)^{1/p} < \epsilon. \quad \Rightarrow\Leftarrow$$

Since $g \in L^p$ dominates f_n a.e., we have that $f_n \in L^p$. Then since $f_n \to f$ in measure, we have that there exists some subsequence $\{f_{n_k}\}$ which converges pointwise to f a.e., and therefore $f \in L^p$ as well. $\square$

Problem 6

Prove the following generalization of Hölder's inequality. If $\sum_{i=1}^k 1/p_i = 1/p$, $0 < p_i$, $p \leq \infty$, and $f_i \in L^{p_i}(\mu)$ for $i = 1, ..., k$, then $\|f_1 f_2 \cdots f_k\|_p \leq \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}$.

Proof. For the sake of induction, take the base case $k = 2$. When $k = 2$, we have that $f_1 \in L^{p_1}(\mu)$, $f_2 \in L^{p_2}(\mu)$, and $1/p_1 + 1/p_2 = 1$. We have the following line of implications:

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \Longrightarrow \frac{p}{p_1} + \frac{p}{p_2} = 1 \Longrightarrow \frac{1}{p_1/p} + \frac{1}{p_2/p} = 1.$$

Indeed, p_1/p and p_2/p are conjugate exponents. Then by Hölder's inequality, we have the following implications:

$$\begin{aligned}
\|f_1^p f_2^p\| \leq \|f_1^p\|_{p_1/p} \|f_2^p\|_{p_2/p} &\Longrightarrow \|f_1^p f_2^p\|_1^{1/p} \leq \|f_1^p\|_{p_1/p}^{1/p} \|f_1^p\|_{p_2/p}^{1/p} \\
&\Longrightarrow \left(\int_X |f_1 f_2|^p d\mu\right)^{1/p} \leq \left(\int_X |f_1^p|^{p_1/p} d\mu\right)^{1/p_1} \left(\int_X |f_1^p|^{p_2/p} d\mu\right)^{1/p_2} \\
&\Longrightarrow \|f_1 f_2\|_p \leq \left(\int_X |f_1|^{p_1} d\mu\right)^{1/p_1} \left(\int_X |f_1^p|^{p_2} d\mu\right)^{1/p_2} = \|f_1\|_{p_1} \|f_2\|_{p_2}.
\end{aligned}$$

The base case holds. Assume now that $\|f_1 f_2 \cdots f_k\|_p \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_k\|_{p_k}$, where $\sum_{i=1}^k 1/p_i = 1/p$ with $0 < p, p_i < \infty$, and each $f_i \in L^{p_i}(\mu)$.

We aim to show that the statement holds up to $k+1$. Let $\sum_{i=1}^{k+1} 1/p_i = 1/p$ with $0 < p, p_i < \infty$, and each $f_i \in L^{p_i}(\mu)$. Then we have

$$\sum_{i=1}^{k+1} \frac{1}{p_i} = \frac{1}{p} \Longrightarrow \sum_{i=1}^{k} \frac{1}{p_i} = \frac{1}{p} - \frac{1}{p_{k+1}} = \frac{p_{k+1} - p}{p \cdot p_{k+1}} = \frac{1}{\frac{p \cdot p_{k+1}}{p_{k+1} - p}}.$$

Since $p_i > 0$ for all i between 1 and $k+1$, we have that $\sum_{i=1}^k 1/p_i > 0$, implying that $p_{k+1} - p > 0$. So by our assumption, we have $\|f_1 f_2 \cdots f_k\|_{\frac{p \cdot p_{k+1}}{p_{k+1} - p}} \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_k\|_{p_k}$. Notice the following implication:

$$\frac{1}{\frac{p \cdot p_{k+1}}{p_{k+1} - p}} + \frac{1}{p_{k+1}} = \frac{1}{p} \Longrightarrow \frac{1}{\frac{p \cdot p_{k+1}}{p_{k+1} - p}} + \frac{1}{\frac{p_{k+1}}{p_{k+1} - p}} + \frac{1}{\frac{p_{k+1}}{p}} = 1.$$

So, applying Hölder's inequality, we see the following:

$$\begin{aligned}
\|f_1^p f_2^p \cdots f_{k+1}^p\|_1 &\leq \|f_1^p f_2^p \cdots f_k^p\|_{\frac{p_{k+1}}{p_{k+1}-p}} \|f_{k+1}\|_{\frac{p_{k+1}}{p}} \\
\int_X |f_1^p f_2^p \cdots f_{k+1}^p| d\mu &\leq \left(\int_X |f_1^p f_2^p \cdots f_k^p|^{\frac{p_{k+1}}{p_{k+1}-p}} d\mu \right)^{\frac{p_{k+1}-p}{p_{k+1}}} \left(\int_X |f_{k+1}^p|^{\frac{p_{k+1}}{p}} d\mu \right)^{\frac{p}{p_{k+1}}} \\
\int_X |f_1 f_2 \cdots f_{k+1}|^p d\mu &\leq \left(\int_X |f_1 f_2 \cdots f_k|^{\frac{p \cdot p_{k+1}}{p_{k+1}-p}} d\mu \right)^{\frac{p_{k+1}-p}{p_{k+1}}} \left(\int_X |f_{k+1}|^{p_{k+1}} d\mu \right)^{\frac{p}{p_{k+1}}} \\
\|f_1 f_2 \cdots f_{k+1}\|_p^p &\leq \|f_1 f_2 \cdots f_k\|_{\frac{p \cdot p_{k+1}}{p_{k+1}-p}}^p \|f_{k+1}\|_{p_{k+1}}^p \\
\|f_1 f_2 \cdots f_{k+1}\|_p &\leq \|f_1 f_2 \cdots f_k\|_{\frac{p \cdot p_{k+1}}{p_{k+1}-p}} \|f_{k+1}\|_{p_{k+1}}
\end{aligned}$$

So we have the following estimates:

$$\|f_1 f_2 \cdots f_{k+1}\|_p \leq \|f_1 f_2 \cdots f_k\|_{\frac{p \cdot p_{k+1}}{p_{k+1}-p}} \|f_{k+1}\|_{p_{k+1}} \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_{k+1}\|_{p_{k+1}}.$$

So by induction, we have shown that the generalized Hölder's inequality holds for all k. $\square$

Problem 7

Let $f, g : \mathbb{R}^n \to \mathbb{C}$ be measurable functions. Define the convolution, $f * g$, of f and g (when it exists) as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(t) g(x-t) dt, \quad x \in \mathbb{R}^n.$$

Prove Young's inequality: Let $1 \leq p,\, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} \geq 1$, and r defined by $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g \in L^r(\mathbb{R}^n)$ and $\|f * g\|_r \leq \|f\|_p \|g\|_q$. **Hint**. Do first the cases $q = p = r = 1$, $q = 1$ and $r = p = \infty$, $q = 1$ and $p = r \in (1, \infty)$. For the general case note that for $p, q, r < \infty$ and $f, g \geq 0$,

$$f * g(x) = \int f(t)^{p/r} g(x-t)^{q/r} f(t)^{p(1/p-1/r)} g(x-t)^{q(1/q-1/r)} dt.$$

Apply Problem 6 with $k = 3$, $\frac{1}{p_1} = \frac{1}{p} - \frac{1}{r}$, $\frac{1}{p_2} = \frac{1}{q} - \frac{1}{r}$ and $p_3 = r$.

Proof. We shall first consider three cases for when $q = 1$: (a) $p = r = 1$, (b) $p = r = \infty$, (c) $p = r \in (1, \infty)$.

(a) We estimate the following:

$$\begin{aligned}
\|f * g\| = \int_{\mathbb{R}^n} |f * g(x)| dx &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)||g(y)| dy \right) dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)||g(y)| dx dy && \text{Tonelli} \\
&= \int_{\mathbb{R}^n} |g(y)| \left(\int_{\mathbb{R}^n} |f(x-y)| dx \right) dy \\
&= \int_{\mathbb{R}^n} |g(y)| \left(\int_{\mathbb{R}^n} |f(x)| dx \right) dy && \text{translation invariance of } \textstyle\int \\
&= \|f\|_1 \|g\|_1.
\end{aligned}$$

(b) We estimate the following:

$$|f * g(x)| \leq \int_{\mathbb{R}^n} |f(x-y)||g(y)| dy \leq \|f\|_\infty \int_{\mathbb{R}^n} |g(y)| dy = \|f\|_\infty \|g\|_1.$$

So, taking the supremum norm on the left hand side, we see that $\|f * g\|_\infty \leq \|f\|_\infty \|g\|_1$.

(c) We aim to show that $\|f * g\|_r \leq \|f\|_r \|g\|_1$. We estimate the following:

$$\begin{aligned}
|f * g(x)| &= \left| \int_{\mathbb{R}^n} f(y)g(x-y)dy \right| \\
&= \left| \int_{\mathbb{R}^n} f(y)g(x-y)^{1/p} g(x-y)^{1/p'} dy \right| \\
&\leq \left(\int_{\mathbb{R}^n} |f(y)g(x-y)^{1/p}|^p \right)^{1/p} \left(\int_{\mathbb{R}^n} |g(x-y)^{1/p'}|^{p'} \right)^{1/p'} && \text{Hölder} \\
&= \left(\int_{\mathbb{R}^n} |f(y)|^p |g(x-y)| \right)^{1/p} \left(\int_{\mathbb{R}^n} |g(x-y)| \right)^{1/p'} \\
&= (|f|^p * g(x))^{1/p} \|g\|_1^{1/p'} && \text{translation invariance of } \int
\end{aligned}$$

With this, we can now estimate

$$\begin{aligned}
\|f * g\|_p^p &\leq \|(|f|^p * g)^{1/p}\|_p^p \|g\|_1^{p/p'} \\
&= \left(\int_{\mathbb{R}^n} \left((|f(x)|^p g(x-y))^{1/p} \right)^p dy \right) \left(\int_{\mathbb{R}^n} g(x-y) \right)^{p/p'} \\
&= \left(\int_{\mathbb{R}^n} |f(x)|^p g(x-y) dy \right) \left(\int_{\mathbb{R}^n} g(y) dy \right)^{p/p'} && \text{translation invariance of } \int \\
&= \||f|^p + g\|_1 \|g\|_1^{p/p'} \\
&\leq \||f|^p\|_1 \|g\|_1 \|g\|_1^{p/p'} && \text{by case (a)} \\
&= \|f\|_p^p \|g\|_1 \|g\|_1^{p/p'}.
\end{aligned}$$

Therefore, we have that $\|f * g\|_p \leq \|f\|_p \|g\|_1^{1/p} \|g\|_1^{1/p'}$. By Hölder's inequality, we have that this gives $\|f * g\|_p \leq \|f\|_p \|g\|_1$.

Now we shall solve the general case:

$$f * g(x) = \int f(y)g(x-y)dy = \int \underbrace{f(y)^{p/r} g(x-y)^{q/r}}_{p_3} \underbrace{f(y)^{p\left(\frac{1}{p}-\frac{1}{r}\right)}}_{p_1} \underbrace{g(x-y)^{q\left(\frac{1}{q}-\frac{1}{r}\right)}}_{p_2} dy$$

(I cannot remember what she said to solve here... something about reciprocals of p_1, p_2, p_3.) By Hölder's inequality, we have the estimate

$$\begin{aligned}
|f * g(x)| &\leq \left(\int_{\mathbb{R}^n} \left(f(y)^{p/r} q(x-y)^{q/r} \right)^r \right)^{1/r} \left(\int_{\mathbb{R}^n} \left(f(y)^{p\left(\frac{1}{p}-\frac{1}{r}\right)} \right)^{p_1} \right)^{1/p_1} \left(\int_{\mathbb{R}^n} \left(g(x-y)^{q\left(\frac{1}{q}-\frac{1}{r}\right)} \right)^{p_2} \right)^{1/p_2} \\
&= \left(\int_{\mathbb{R}^n} f(y)^p g(x-y)^q dy \right)^{1/r} \left(\int_{\mathbb{R}^n} f(y)^p dy \right)^{1/p - 1/r} \left(\int_{\mathbb{R}^n} g(x-y)^q dy \right)^{1/q - 1/r} \\
&= \left(\int_{\mathbb{R}^n} f(y)^p g(x-y)^q dy \right)^{1/r} \left(\int_{\mathbb{R}^n} f(y)^p dy \right)^{1/p_1} \left(\int_{\mathbb{R}^n} g(x-y)^q dy \right)^{1/p_2}.
\end{aligned}$$

Raising everything to the power r, we see

$$|f * g(x)|^r \leq \left(\int_{\mathbb{R}^n} f(y)^p g(x-y)^q dy \right) \left(\int_{\mathbb{R}^n} f(y)^p dy \right)^{r/p_1} \left(\int_{\mathbb{R}^n} g(x-y)^q dy \right)^{r/p_2}.$$

Integrating, we have

$$\int_{\mathbb{R}^n} |f * g(x)|^r dx \leq \|f\|^{pr/p_1} \|g\|_q^{qr/p_2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y)^p g(x-y)^q dy dx$$

$$
\begin{aligned}
&= \|f\|^{pr/p_1}\|g\|_q^{qr/p_2}\int_{\mathbb{R}^n} f(y)\left(\int_{\mathbb{R}^n} g(x-y)^q dx\right)dy \\
&= \|f\|_p^{pr/p_1}\|g\|_q^{qr/p_2}\|f\|_p^p\|g\|_q^q \\
&= \left(\int |f|^p\right)^{\frac{1}{p}\frac{pr}{p_1}}\left(\int |g|^q\right)^{\frac{1}{q}\frac{qr}{p_2}}\left(\int |f|^p\right)^{\frac{1}{p}p}\left(\int |g|^q\right)^{\frac{1}{q}q} \\
&= \left(\int |f|^p\right)^{\frac{r}{p_1}}\left(\int |g|^q\right)^{\frac{r}{p_2}}\left(\int |f|^p\right)\left(\int |g|^q\right) \\
&= \left(\int |f|^p\right)^{\frac{r}{p_1}+1}\left(\int |g|^q\right)^{\frac{r}{p_2}+1} \\
&= \left(\int |f|^p\right)^{\frac{r}{p}}\left(\int |g|^q\right)^{\frac{r}{p}} && \text{e.g. } \tfrac{1}{p_1}=\tfrac{1}{p}-\tfrac{1}{r}\Rightarrow\tfrac{r}{p_1}=\tfrac{r}{p}-1 \text{ etc.} \\
&= \|f\|_p^r\|g\|_q^r.
\end{aligned}
$$

Dividing by r gives $\|f*g\|_r \leq \|f\|^p\|g\|_q$ as desired. □

Problem 8

Continuity in $L^p(\mathbb{R}^n)$. If $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then $\lim_{|h|\to 0}\|f(\cdot+h)-f(\cdot)\|_p = 0$. **Hint**. Assume first that $f \in C_c(\mathbb{R}^n)$.

Proof. Let us first consider the case in which $f \in C_c(\mathbb{R}^n)$. We aim to show that such an f is uniformly continuous, which would give our result since we are removing dependence on x values. In particular, we want to show that

$$\lim_{|h|\to 0}\int |f(\cdot+h)-f(\cdot)| = \int \lim_{|h|\to 0}|f(\cdot+h)-f(\cdot)|.$$

This works by the dominated convergence theorem where the fact that $|f(\cdot+h)-f(\cdot)| \leq |f(\cdot+h)|+|f(\cdot)|$ by the triangle inequality, and since $f \in L^p$, we have the desired hypothesis. To show uniform continuity, recall the Heine-Cantor theorem, which states that for a function $f: X \to Y$ with X a compact metric space, Y a metric space, and f continuous, then f is uniformly continuous.

If we consider our $f \in C_c(\mathbb{R}^n)$, then certainly $\operatorname{supp} f$ is compact. But, the total domain $\mathbb{R}^n$ may not be dense. So, let $\mathbb{R}^n = \operatorname{supp} f \cup X$, where X is disjoint from the support of f. Then consider a function $f' := f|_{\operatorname{supp} f}$. Certainly f' satisfies the hypothesis of Heine-Cantor, and therefore f' is uniformly continuous. Consider now $\hat{f} := f|_X$. By definition of support, we have that X is the set for which $f \equiv 0$. Certainly a constant function is uniformly continuous, regardless of compactness of the domain. So, we see that indeed f over the full domain $\mathbb{R}^n$ is uniformly continuous, and so we see that $|f(\cdot+h)-f(\cdot)| < \epsilon$, so certainly $\lim_{|h|\to 0}\|f(\cdot+h)-f(\cdot)\|_p = 0$.

Now for the general case, take some $f \in L^p$. Then since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, there exists a function $g \in C_c(\mathbb{R}^n)$ such that $\|f-g\|_p < \epsilon/3$ where $\epsilon > 0$. We have the following norm estimate:

$$
\begin{aligned}
\|f(\cdot+h)-f\|_p &= \|f(\cdot+h)-g(\cdot+h)+g(\cdot+h)-g+g-f\|_p \\
&\leq \|f(\cdot+h)-g(\cdot+h)\|_p + \|g(\cdot+h)-g\|_p + \|g-f\|_p \\
&= 2\|f-g\|_p + \|g(\cdot+h)-g\| && \text{by change of variables} \\
&< \frac{2\epsilon}{3} + \|g(\cdot+h)-g\|_p \\
&< \epsilon && \text{second term above is} < \frac{\epsilon}{3} \text{ if } 0 < |h| < \delta.
\end{aligned}
$$

Taking $\epsilon \to 0$ gives the desired result. □

Problem 9

Prove that $L^\infty(\mathbb{R}^n)$ is not separable (i.e. $L^\infty(\mathbb{R}^n)$ does not contain a countable dense subset). **Hint**. Show that there exists an uncountable family $\mathcal{G} \subset L^\infty(\mathbb{R}^n)$ such that if $f, g \in \mathcal{G}$ then $\|f - g\|_\infty = 1$.

Proof. First, consider a family of sets $\mathcal{G} := \{\chi_{(a,b)} : -\infty < a < b < \infty\}$. This is uncountable, and we see that

$$\|\chi_{(a,b)} - \chi_{(\tilde{a},\tilde{b})}\|_\infty = \begin{cases} 1 & (a,b) \neq (\tilde{a},\tilde{b}), \\ 0 & (a,b) = (\tilde{a},\tilde{b}). \end{cases}$$

Notice that the supremum norm implies that when the intervals are not equal, $\|\chi_{(a,b)} - \chi_{(\tilde{a},\tilde{b})}\|_\infty = 1$.

For the sake of contradiction, suppose that L^∞ is separable. Then let $\{f_n\}$ be the countable dense subset. Since $\{f_n\}$ is dense, there exists some f_{n_0} such that $\|\chi_{(a,b)} - f_{n_0}\|_\infty < \frac{1}{2}$. For said f_{n_0}, there exists intervals $(a,b) \neq (\tilde{a},\tilde{b})$ such that $\|\chi_{(a,b)} - f_{n_0}\|_\infty < 1/2$ and $\|\chi_{(\tilde{a},\tilde{b})} - f_{n_0}\|_\infty < 1/2$. Therefore, we have that $\|\chi_{(a,b)} - \chi_{(\tilde{a},\tilde{b})}\|_\infty < 1 \Rightarrow\Leftarrow$ □

Problem 10 (Optional)

Prove that $L^p(\mathbb{R}^n)$ is separable for $1 \leq p < \infty$.