Abstract Algebra II Homework 4

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Section 10.5

Throughout, let R denote a ring with identity.

1 Suppose that

$$\begin{array}{ccccc} A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ A' & \xrightarrow[\psi']{} & B' & \xrightarrow[\varphi']{} & C' \end{array}$$

is a commutative diagram of groups and that the rows are exact. Prove the following:

(a) If φ, α are surjective, and β is injective, then γ is injective.

Proof. Consider $c \in \ker \gamma$. Since φ is surjective, we have that there exists some $b \in B$ such that $\varphi(b) = c$. Since the diagram commutes, we have that $\varphi'(\beta(b)) = \gamma(\varphi(b)) = \gamma(c) = 0$. Since $c \in \ker \gamma$, we have that $\beta(b) \in \ker \varphi' = \mathrm{im}\psi'$. So, there exists some $a' \in A'$ such that $\psi'(a') = \beta(b)$. Now since α is assumed to be surjective, we know that there exists some $a \in A$ such that $\alpha(a) = a'$. The commutativity of the diagram gives that $\beta(\psi(a)) = \psi'(\alpha(a)) = \psi'(a') = \beta(b)$. Now since β is injective, we have that $\psi(a) = b$. So, $c = \varphi(b) = \varphi(\psi(a)) = 0$ since $A \to B \to C$ is exact. Therefore γ is indeed injective. $\square$

(b) If ψ', α, γ are injective, then β is injective.

Proof. Consider $b \in \ker \beta$. Since the diagram commutes, we have that $0 = \varphi'(0) = \varphi'(\beta(b)) = \gamma(\varphi(b))$. Since γ is injective, we know that $\varphi(b) = 0$ and therefore $b \in \mathrm{im}\psi$ since $\ker \varphi = \mathrm{im}\psi$ by exactness. So, let $b = \psi(a)$ for some $a \in A$. Then we have that $\psi'(\alpha(a)) = \beta(\psi(a)) = 0$, therefore $\alpha(a) \in \ker \psi'$. Now since the diagram is exact, we have that $\ker \psi' = 0$ and therefore $\alpha(a) = 0$, however since α is taken to be injective, we must have that $a = 0$. So, $b = \psi(a) = 0$, and we have that β is injective. $\square$

(c) If φ, α, γ are surjective, then β is surjective.

Proof. Consider $b' \in B'$. Since γ is surjective, we have that there exists some $c \in C$ such that $\gamma(c) = \varphi'(b')$. Since φ is surjective, there exists $b \in B$ such that $\varphi(b) = c$. Since the diagram is commutative, we have that $\varphi'(b') = \gamma(\varphi(b)) = \varphi'(\beta(b))$, and therefore $b' - \beta(b) \in \ker \varphi' = \mathrm{im}\psi'$. Pick now some $a' \in A'$ such that $\psi'(a') = b' - \beta(b)$. Since α is surjective, there exists some $a \in A$ such that $\alpha(a) = a'$. Therefore, we have that $\beta(\psi(a)) = \psi'(\alpha(a)) = b' - \beta(b)$. Thus, $b' = \beta(\psi(a) + b)$, and so we have that β is surjective. $\square$

(d) If β is injective, and α, φ are surjective, then γ is injective.

Proof. Identical problem as part (a). $\square$

(e) If β is surjective, γ, ψ' are injective, then α is surjective.

Proof. Take $a' \in A'$. As β is surjective, we know that there exists $b \in B$ such that $\beta(b) = \psi'(a')$. Since the diagram is commutative, we have that $0 = \varphi'(\psi'(a')) = \varphi'(\beta(b)) = \gamma(\varphi(b))$. Since γ is injective, we have that $\varphi(b) = 0$. So, $b \in \mathrm{im}\psi$, and so we can let $b = \psi(a)$ for some $a \in A$. So, $\psi'(a') = \beta(b) = \beta(\psi(a)) = \psi'(\alpha(a))$, and since ψ' is injective, we have that $a' = \alpha(a)$ and so $a' = \alpha(a)$ is surjective. □

2 Suppose that

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' \end{array}$$

is a commutative diagram and that the rows are exact. Prove the following:

(a) If α is surjective and β, δ are injective, then γ is injective.

Proof. Take $c \in \ker \gamma$. Then we have that $h'(\gamma(c)) = \delta(h(c)) = 0$. Since δ is injective, we have that $h(c) = 0$, so $c \in \ker h = \mathrm{im} g$ since the rows are exact. Hence, we can write $c = g(b)$ for some $b \in B$. Then we have that $g'(\beta(b)) = \gamma(g(b)) = 0$, and so $\beta(b) \in \ker g' = \mathrm{im} f'$, again by exactness. So, we can write $\beta(b) = f'(a')$ for some $a' \in A'$. Since α is taken to be surjective, we know that $a' = \alpha(a)$ for some $a \in A$, and so $\beta(f(a)) = f'(\alpha(a)) = f'(a') = \beta(b)$. Since β is injective, this gives that $f(a) = b$. Finally by exactness of the top row, we have that $c = g(b) = g(f(a)) = 0$ and so γ is injective. □

(b) If δ is injective and α, γ are surjective, then β is surjective.

Proof. Take $b' \in B'$. Since γ is surjective, we have that there exists some $c \in C$ such that $\gamma(c) = g'(b')$. Since the rows are exact, we have $\delta(h(c)) = h'(\gamma(c)) = h'(g'(b)) = 0$, and therefore $h(c) \in \ker \delta$, which since δ is injective, we have $\ker \delta = 0$ so $h(c) = 0$. Therefore $c \in \ker h = \mathrm{im} g$, so we can write $c = g(b)$ for some $b \in B$. Since the diagram commutes, we have that $g'(b') = \gamma(g(b)) = g'(\beta(b))$, and therefore $b' - \beta(b) \in \ker g' = \mathrm{im} f'$. So, we can write $b' - \beta(b) = f'(a')$ for some $a' \in A'$. Now since α is surjective, we can write $a' = \alpha(a)$ for some $a \in A$. Then $\beta(f(a)) = f'(\alpha(a)) = f'(a') = b' - \beta(b)$, and so $b' = \beta(f(a) + b)$ and so β is indeed surjective. □

6 Prove that the following are equivalent for a ring R: (i) Every R-module is projective; (ii) Every R-module is injective.

Proof. Take $0 \to A \to B \to C \to 0$ to be a short exact sequence of arbitrary R-modules. If we assume that all R-modules are projective, then we have that C is projective and therefore splits by definition. Therefore by definition we have that A is an injective module. But, since A was taken arbitrarily, we have that all R-modules are injective.

Conversely, assume that all R-modules are injective. Then we have that A is injective, and so the exact sequence splits. By definition, we have then that C is projective. But, since C was taken arbitrarily, we have that all R-modules are projective. Therefore we have shown that (i) $\iff$ (ii). □

7 Let A be a nonzero finite abelian group.

(a) Prove that A is not a projective $\mathbb{Z}$-module.

Proof. Notice first that the only finite free $\mathbb{Z}$-module is 0, which we exclude due to the problem statement.

Recall first that a module is projective if it is free (or more generally, is a direct summand of a free module). Also recall that $\mathbb{Z}$ is a PID. We claim that a module M over a PID is projective if and only if it is free.

Certainly the ($\Longleftarrow$) direction is clear. So, suppose now that M is projective, and so it must be a direct summand of a free module. Now since we are over a PID, it is clear that if $N \subset M$ is a submodule, then it is free and of rank no greater than the rank of M. Incomplete. $\square$

(b) Prove that A is not an injective $\mathbb{Z}$-module. Incomplete.

9 Assume R is commutative with identity.

(a) Prove that the tensor product of two free R-modules is free.

Proof. Recall that a free module F is simply a direct sum of copies of R. That is, for example, $F \cong \bigoplus_i R$. Suppose we take two free modules F_1 and F_2, we aim to show that their tensor product is isomorphic to a number of direct sums of R. So, we can write

$$F_1 \otimes_R F_2 \cong \Big(\bigoplus_i R\Big) \otimes_R \Big(\bigoplus_j R\Big).$$

But, as we showed in the previous homework, for when R is commutative, we have that

$$\Big(\bigoplus_i R\Big) \otimes_R \Big(\bigoplus_j R\Big) \cong \bigoplus_{i,j} R \otimes_R R \cong \bigoplus_{i,j} R.$$

So, we have that indeed $F_1 \otimes_R F_2 \cong \bigoplus_{i,j} R$ as desired. $\square$

(b) Use (a) to prove that the tensor product of two projective R-modules is projective.

Proof. Let us take projective modules P_1, P_2. By definition, there exist modules P_1', P_2' such that both $P_i \oplus P_i'$ are free. Now since the tensor product distributes over direct sums, we write the following:

$$(P_1 \oplus P_1') \otimes_R (P_2 \oplus P_2') \cong (P_1 \otimes_R P_2) \oplus (P_1' \otimes_R P_2) \oplus (P_1 \otimes_R P_2') \oplus (P_1' \otimes_R P_2').$$

We know that a tensor product of free modules is free, and therefore since the left hand side is indeed a tensor product of free modules, we have that certainly the right hand side is too. Indeed, we have that $P_1 \otimes_R P_2$ is a direct summand of a free module, and so $P_1 \otimes_R P_2$ is projective as desired. $\square$

12 Let A be an R-module, let I be any nonempty index set, and for each $i \in I$ let B_i be an R-module. Prove the following isomorphisms of abelian groups; when R is commutative prove also that these are R-module isomorphisms.

(a) $\hom_R(\bigoplus_{i \in I} B_i, A) \cong \prod_{i \in I} \hom_R(B_i, A)$.

Proof. Consider the injection $\iota_i : B_i \to \bigoplus_i B_i$. Applying the Hom functor gives the induced map $\varphi_i : \hom_R(\bigoplus_{i \in I} B_i, A) \to \hom_R(B_i, A)$ defined by $a \mapsto a \circ \iota_i$. Since these hom-sets are both abelian groups, we know by the universal property of the direct product of abelian groups, we have that there exists a homomorphism $\Phi : \hom_R(\bigoplus_i B_i, A) \to \prod_i \hom_R(B_i, A)$ such that $\pi_i \circ \Phi = \varphi_i$ with π_i the canonical projection. We want to show that Φ is an R-module isomorphism. So, we see that Φ is injective because if $\Phi(a) = 0$ then $a \circ \iota_i = \varphi_i(a) = \pi_i \circ \Phi(A) = 0$, therefore $a = 0$, and so Φ is injective. For surjectivity, define $\phi := \prod_i \phi_i \in \prod_i \hom_R(B_i, A)$. Also define $a_\phi : \bigoplus_i B_i \to A$ by $b_i \mapsto \sum \phi_i(b_i)$. Indeed, $a_\phi \in \hom_R(\bigoplus_i B_i, A)$. Finally $\pi_i(\Phi(a_\phi))(b) = \varphi_i(a_\phi)(b) = a_\phi(\iota_i(b)) = \phi_i(b)$, and therefore $\Phi(a_\phi) = \phi$ giving surjectivity. Last, take arbitrary $r \in R$. Then $\Phi(ra) = \prod_i((ra) \circ \iota_i) = \prod_i(ra \circ \iota_i) = r \prod_i(a \circ \iota_i) = r\Phi(a)$, and this gives that Φ is indeed an R-module isomorphism. $\square$

(b) $\hom_R(A, \prod_{i \in I} B_i) \cong \prod_{i \in I} \hom_R(A, B_i)$.

Proof. Define $\varphi_i : \hom_R(A, \prod_i B_i) \to \hom_R(A, B_i)$ by $a \mapsto \pi_i \circ a$. By the universal property of the direct product of abelian groups, there exists a homomorphism $\Phi : \hom_R(A, \prod_i B_i) \to \prod_i \hom_R(A, B_i)$ such that $\pi_i \circ \Phi = \varphi_i$. The arguments for injectivity and extending the abelian group homomorphism to an R-module homomorphism follow the same as part (a). For surjectivity, consider $\phi := \prod_i \phi_i \in \prod_i \hom_R(A, B_i)$. Also define $a_\phi : A \to \prod_i B_i$ by $\pi_i a_\phi(b) = \phi_i(b)$. Indeed, a_ϕ is a homomorphism. Now we see that $\pi_i \Phi(a_\phi)(b) = \varphi_i a_\phi(b) = \pi_i a_\phi(b) = \phi_i(b)$, and therefore $\pi_i \Phi(a_\phi) = \phi_i$, and so $\Phi(a_\phi) = \phi$. Therefore Φ is surjective and we are done. □

21 Let R and S be rings with identity and suppose M is a right R-module, and N is an (R, S)-bimodule. If M is flat over R and N is flat as an S-module, prove that $M \otimes_R N$ is flat as a right S-module.

Proof. Since M is a flat right R-module, we have that $M \otimes_R - : {}_R\mathbf{Mod} \to \mathbf{Ab}$ is an exact functor. Since N is a flat right S-module, $N \otimes_S - : {}_S\mathbf{Mod} \to {}_R\mathbf{Mod}$ is an exact functor. So, the composition of these functors, $(M \otimes_R -) \circ (N \otimes_S -) : {}_S\mathbf{Mod} \to \mathbf{Ab}$, is an exact functor. By the associativity of $\otimes$, we have that $(M \otimes_R -) \circ (N \otimes_S -) = M \otimes_R (N \otimes_S -) = (M \otimes_R N) \otimes_S -$, that is we have that $(M \otimes_R N) \otimes_S -$ is an exact functor. This implies then that $M \otimes_R S$ is indeed a flat right S-module. Does tensor associativity work for the tensor functor? □

22 Suppose that R is a commutative ring and that M, N are flat R-modules. Prove that $M \otimes_R N$ is a flat R-module.

Proof. First, consider the exact sequence $0 \to A \to B$. Since M is flat, we have by definition that $0 \to A \otimes M \to B \otimes M$ is also exact. Since N is flat, we have by definition that $0 \to (A \otimes M) \otimes N \to (B \otimes M) \otimes N$ is also exact. By the associativity of the tensor product, we know that $(A \otimes M) \otimes N \cong A \otimes (M \otimes N)$ and $(B \otimes M) \otimes N \cong B \otimes (M \otimes N)$, and so we have the result by definition of flatness. □

26 Suppose R is a PID. This exercise proves that A is a flat R-module if and only if A is a torsion free R-module. Incomplete.

(a) Suppose that A is flat and for fixed $r \in R$ consider the map $\psi_r : R \to R$ defined by multiplication by r, that is $\psi_r(x) = rx$. If $r \neq 0$ show that ψ_r is an injection. Conclude from the flatness of A that the map from $A \to A$ defined by $a \mapsto ra$ is injective and that A is torsion free.

(b) Suppose that A is torsion free. If I is a nonzero ideal of R, then $I = rR$ for some nonzero $r \in R$. Show that the map ψ_r in (a) induces an isomorphism $R \cong I$ of R-modules, and that the composite $R \xrightarrow{\psi} I \xrightarrow{\iota} R$ of ψ_r with the inclusion $\iota : I \subseteq R$ is multiplication by r. Prove that the composite $A \otimes_R R \xrightarrow{1 \otimes \psi_r} A \otimes_R I \xrightarrow{1 \otimes \iota} A \otimes_R R$ corresponds to the map $a \mapsto ra$ under the identification $A \otimes_R R = A$ and that this composite is injective since A is torsion free. Show that $1 \otimes \psi_r$ is an isomorphism and deduce that $1 \otimes \iota$ is injective. Use the previous exercise to conclude that A is flat.

27 Let M, A, B be R-modules. Incomplete.

(a) Suppose $f : A \to M$ and $G : B \to M$ are R-module homomorphisms. Prove that $X = \{(a, b) : a \in A, b \in B, f(a) = g(b)\}$ is an R-submodule of the direct sum $A \oplus B$ (called the pullback/fiber product of f and g) and that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & M \end{tikzcd}$$

where π_1, π_2 are the natural projections onto the first and second components.

(b) Suppose $f' : M \to A$ and $g' : M \to B$ are R-module homomorphisms. Prove that the quotient Y of $A \oplus B$ by $\{(f'(m), -g'(m)) : m \in M\}$ is an R-module (called the pushout/fiber sum of f' and g') and that there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{g'} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \pi_2'} \\ A & \xrightarrow[\pi_1']{} & Y \end{array}$$

where π_1', π_2' are the natural maps to the quotient induced by the maps into the first and second components.

28 This problem establishes that K and K' are *projectively equivalent* and that L and L' are *injectively equivalent.*

(a) (Schanuel's lemma) If $0 \to K \to P \xrightarrow{\varphi} M \to 0$ and $0 \to K' \to P' \xrightarrow{\varphi'} M \to 0$ are exact sequences of R-modules where P and P' are projective, prove $P \oplus K' \cong P' \oplus K$ as R-modules.

Proof. Let us first define $Q := \{(p, p') \in P \oplus P' : \varphi(p) = \varphi'(p')\}$. This is a submodule of $P \oplus P'$. Certainly, $\pi_1 : Q \to P$ is surjective, and since φ' must be surjective, we have that for any $p \in P$ we can find a $p' \in P'$ such that $\varphi(p) = \varphi'(p')$. Therefore, there does indeed exist $(p, p') \in Q$, and $\pi_1(p, p') = p$. Consider now the following:

$$\ker \pi_1 = \{(0, p') : (0, p') \in Q\} = \{(0, p') : \varphi'(q) = 0\} \cong \ker \varphi' \cong K'.$$

Therefore, $0 \to K' \to Q \to P \to 0$ is exact. Now since P was taken to be projective, we have that this sequence splits, and therefore $Q \cong K' \oplus P$. We do the exact same process for π_2, which gives that $Q \cong K \oplus P'$. This gives the desired equivalence. □

(b) If $0 \to M \to Q \xrightarrow{\psi} L \to 0$ and $0 \to M \to Q' \xrightarrow{\psi'} L' \to 0$ are exact sequences of R-modules where Q and Q' are injective, prove $Q \oplus L' \cong Q' \oplus L$ as R-modules.

Proof. This is the dual to Schanuel's Lemma. Therefore, we can mirror the proof but dualize everything, e.g. we study the cokernel instead of the kernel, etc. □