Computational Analysis of Structural Credit Risk

by Cameron Connor (Grey College)

Supervisor Dr Andrew Krause

Computational Mathematics II Individual Project Report

January 8, 2026

Plagiarism declaration

This piece of work is a result of my own work and I have complied with the Department's guidance on multiple submission and on the use of AI tools. Material from the work of others not involved in the project has been acknowledged, quotations and paraphrases suitably indicated, and all uses of AI tools have been declared.

I, **Cameron Connor** confirm that I have read, understood, and have adhered to the plagiarism declaration above.

AI Usage Declaration

- Grammarly was used to enhance the writing style, grammar, and spelling of the report throughout.
- Gemini was used at points to improve the conciseness of explanations in order to adhere to the page limit
- Gemini was used to debug the code used to create figure 1

Link to GitHub Codebase

All codes used to produce the analysis and results in this report can be found at the GitHub repository: https://github.com/cconnor212/Cameron-Connor-Comp-II-Individual-Project.

1 The Merton Credit Model

1.1 Introduction

The Merton Model is a structural approach to credit risk based on the premise that a company's financial structure is composed of equity and debt. This framework models the market value of a firm's assets stochastically and treats its Probability of Default (PD) as an endogenous event.

- **Equity:** Represents residual ownership with unlimited potential return but lower priority in the capital structure.
- **Debt:** Provides a fixed return with higher security, as it is repaid before equity in the event of default.

The Merton Model assumes equity is not paying any dividends and that debt is simply a zero coupon bond ([3]). This means all interest payments are combined in the final redemption date.

At maturity T, the payoffs are determined by the terminal asset value V_T relative to the face value of the debt K:

- **Value of Debt at T:** $D_T = \min(K, V_T)$
- **Value of Equity at T:** $E_T = \max(V_T - K, 0)$

1.2 Merton Model as an Option Framework

The Merton model shows that a firm's capital structure can be examined through option pricing. Equity holders effectively own a long call option on the firm's assets (V_T) with a strike price equal to the debt's face value K. Debt can be viewed as the firm's assets minus a short put option on those assets ([3]).

The following is a list of how the Merton variables map using Black-Scholes.

- Equity value (E_0) corresponds to the call option premium
- Asset value (V_t) corresponds to the underlying stock price
- Debt Face Value (K) corresponds to the option strike price
- Time to Maturity (T): The remaining time until the debt must be repaid
- Asset Volatility (σ): The standard deviation of the returns of the firm's total assets

It is important to note that value refers to the market value of the company's assets, and not the share price. This introduces two primary practical challenges: the total asset value is not directly observable in public markets, and the volatility of these assets must be estimated rather than observed. Despite these challenges, this framework provides the mathematical basis for deriving the PD using the Black-Scholes formula

1.3 Mathematical Derivation

The value of equity at time $t = 0$ is given by the Black-Scholes-Merton formula seen in [3]:

$$E_0 = V_0 N(d_1) - K e^{-rT} N(d_2) \tag{1}$$

where $N(\cdot)$ is the cumulative standard normal distribution and the terms d_1, d_2 are defined as:

$$d_1 = \frac{\ln(V_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T} \tag{2}$$

The Probability of Default (PD) is defined as the probability that the assets will be insufficient to cover the debt at maturity ($V_T \leq K$):

$$PD = P(V_T \leq K) = N(-d_2) \tag{3}$$

1.4 Credit Spreads

The current value of the debt D_0 can be expressed using an implied interest rate b:

$$D_0 = Ke^{-bT} \tag{4}$$

The credit spread $s = b - r$ ([3]) represents the risk premium demanded by lenders to compensate for borrowers possibly defaulting. In the Merton framework, this spread is a non-linear function of the firm's leverage ratio (K/V_0) and the asset volatility (σ). As V_T approaches K, the value of the 'default put' increases, widening the credit spread.

The Merton Model provides a rigorous method for calculating PD, valuing debt across time horizons, and determining appropriate credit spreads based on firm-specific volatility. We will now move on to connecting the analytical PD expression derived above with the numerical approximations developed in the next section.

2 Numerical Analysis and Convergence Testing

While the Merton model provides a closed-form solution for the Probability of Default (PD), numerical methods are essential for more complicated models. This section investigates the convergence properties of the Euler-Maruyama and Milstein schemes when applied to the Merton stochastic differential equation (SDE).

The firm's asset value $V(t)$ follows a Geometric Brownian Motion (GBM) allowing us to formulate the SDE:

$$dV(t) = \mu V(t)dt + \sigma V(t)dW(t) \tag{5}$$

2.1 Discretisation Schemes

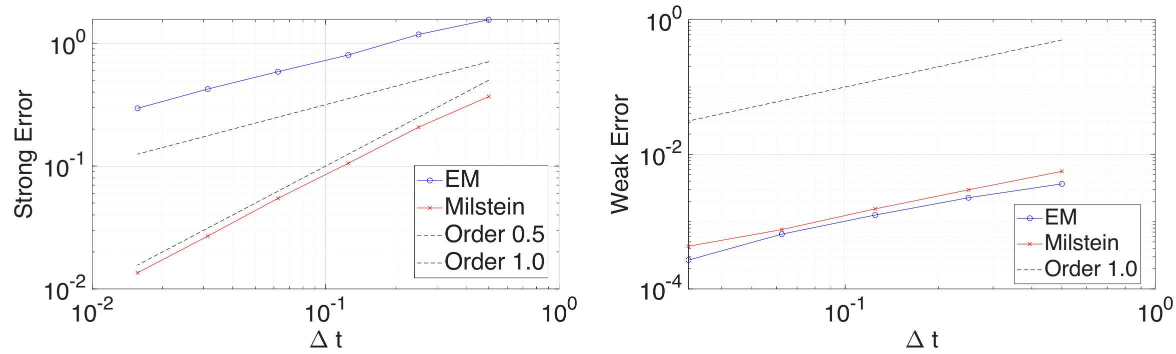
To assess the performance of the Euler-Maruyama and Milstein schemes, we test for a range of dt values, $\Delta t = 2^{-p}$ for $p = \{1, 2, \ldots, 6\}$. Using a range of values rather than a single fixed interval is key for empirical verification; it allows us to observe the rate at which the numerical approximation converges to the true solution as $\Delta t \to 0$. The purpose of studying convergence is to verify that the numerical approximations for the asset process $V(t)$ converge to the analytical solution at the theoretically expected rates.

- **Euler-Maruyama (EM):** $V_{n+1} = V_n + \mu V_n \Delta t + \sigma V_n \Delta W_n$
- **Milstein Scheme:** $V_{n+1} = V_n + \mu V_n \Delta t + \sigma V_n \Delta W_n + \frac{1}{2}\sigma^2 V_n \left((\Delta W_n)^2 - \Delta t\right)$

Since the noise term $g(V) = \sigma V$ is non-constant ($g'(V) = \sigma$), the Milstein scheme includes a first-order correction term that is absent in the EM scheme. According to [2], this term is derived from the Itô-Taylor expansion and is necessary to achieve a higher strong order of convergence when the diffusion term is state-dependent.

2.2 Numerical Convergence Results

To verify the accuracy of these schemes, we conduct convergence tests and observe how the discretisation error scales with the step size Δt. By plotting the errors on a log-log scale, we can extract the experimental orders of convergence and validate the impact of the $g'g$ correction term in the Milstein scheme.



(a) (b)

Figure 1: (a): Log-log plot of the strong convergence for the Merton SDE. The Euler-Maruyama scheme (blue) exhibits a slope of approximately 0.5, while the Milstein scheme (red) achieves a slope of 1.0. This numerically verifies the theoretical strong convergence orders established in [2] proving that the Milstein scheme provides higher accuracy for the multiplicative noise characteristic of firm asset dynamics (b): Log-log plot of the weak convergence of the terminal PD. Both schemes exhibit a weak convergence order of 1.0, aligning with the theoretical expectation described in [2]. The simulation utilized $M = 9000000$ paths to minimize Monte Carlo sampling error.

Figure 1 confirms that the Milstein scheme is superior for simulating the Merton model. By incorporating the $g'g$ term, it achieves a higher strong order of convergence, significantly reducing the discretisation error for a given computational cost. This higher accuracy is crucial for pricing debt and other credit instruments such as credit default swaps. A poor discretisation can systematically estimate PD incorrectly, leading to mispriced debt.

3 The First Passage Time (Black-Cox) Model

While the Merton Model is powerful, its major drawback is that defaulting can only happen at the time of maturity. In reality, defaulting can happen at any time. The Black-Cox ([1]) model addresses this by treating default as a First Passage Time (FPT) event where default occurs if the firm's asset value V hits a barrier L from above

We define the default time τ as the first hit of the barrier: $\tau = \inf\{t > 0 : V(t) \leq L\}$. The Probability of Default (PD) is thus the probability that the running minimum of the asset process $V(t)$ crosses L within $[0, T]$:

$$PD_{FPT} = P\left(\min_{0\leq t\leq T} V(t) \leq L\right) \tag{6}$$

The transition from the Merton which evaluates default at maturity to the path-dependent Black-Cox model changes the sensitivity of credit risk to volatility (σ). In the Merton framework, volatility only affects the final distribution. In the Black-Cox model, increased volatility exponentially increases the probability of hitting the barrier L at any point in the interval $[0, T]$, even if the long-term trend is positive. This reflects the economic reality more accurately. Also, it highlights the "early warning" nature of structural models, where a temporary liquidity crisis—represented by the asset path touching the barrier—triggers a default event that the Merton model would otherwise ignore.

4 Conclusion

This project has tracked the evolution of structural credit risk modelling from Merton's "point-in-time" approach to the continuous path-dependency of the Black-Cox framework. We have demonstrated that while the Merton model offers an elegant, closed-form solution for terminal default, it remains a simplified abstraction that ignores the intraday volatility inherent in corporate finance. Through our numerical analysis, we have proved that the Milstein scheme is superior to Euler-Maruyama because it accounts for the state-dependent nature of the diffusion term through the $g'g$ Itô correction. Overall, this report highlights that for creditors, the precision of numerical schemes is just as critical as the robustness of economic assumptions!

References

[1] John C. Cox Fischer Black. "Valuing Corporate Securities: Some Effects Of Bond Indenture Provisions". In: The Journal of Finance (1976). DOI: https://onlinelibrary.wiley.com/doi/epdf/10.1111/j.1540-6261.1976.tb01891.x.

[2] Desmond J. Higham. "An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equationss". In: Society for Industrial and Applied Mathematics (2001). DOI: https://epubs.siam.org/doi/pdf/10.1137/S0036144500378302.

[3] Robert C Merton. "On the Pricing of Corporate Debt: the Risk Structure of Interest Rates". In: American Finance Association Meetings (1973). DOI: https://doi.org/10.2307/2978814.