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THE SADDLE-NODE SEPARATRIX-LOOP BIFURCATION*

STEPHEN SCHECTER†

Abstract. We study vector fieldx $\dot{x}=f(x)$, $x\in\mathbb{R}^2$, having at some point an equilibrium of saddle-node type with a separatrix loop. Such vector fields fill a codimension two submanifold Σ of an appropriate Banach space. We give analytic conditions that determine whether a two-parameter perturbation of $\dot{x}=f(x)$ is transverse to Σ. The new condition is a version of Melnikov's integral around the separatrix loop. If it is nonzero, then as one perturbs away from $\dot{x}=f(x)$ in the direction in which an equilibrium of saddle-node type persists, the separatrix loop breaks in a nondegenerate manner. This integral is shown to be nonzero for the two-parameter pendulum equation $\beta\ddot{\phi}+\dot{\phi}+\sin\phi=\rho$ at its organizing center.

Key words. saddle-node separatrix-loop bifurcation, Melnikov integral, pendulum, Josephson junction

AMS(MOS) subject classification. 58F14

1. Introduction. We shall be concerned with vector fields

$$\dot{x}=f(x),\qquad x\in\mathbb{R}^2 \tag{1}$$

having at some $p\in\mathbb{R}^2$ an equilibrium of saddle-node type with a separatrix loop Γ (see Fig. 1). We assume that the saddle-node has one negative eigenvalue and, of course, one zero eigenvalue. Such vector fields fill a codimension two submanifold Σ of an appropriate Banach space of planar vector fields. Consider a two-parameter unfolding of (1),

$$\dot{x}=\tilde{f}(x,\nu_1,\nu_2),\qquad x\in\mathbb{R}^2,\quad \nu_1,\nu_2\in\mathbb{R} \tag{2}$$

where $\tilde{f}(x,0,0)=f(x)$. We shall give a computable condition that determines whether the family (2) is transverse to Σ at $(\nu_1,\nu_2)=(0,0)$.

If the transversality condition is satisfied, there is a smooth nonsingular change of coordinates in parameter space,

$$(\nu_1,\nu_2)\leftrightarrow(\mu_1,\mu_2),\qquad (0,0)\leftrightarrow(0,0),$$

such that

$$\dot{x}=\tilde{f}(x,\nu_1(\mu_1,\mu_2),\nu_2(\mu_1,\mu_2))\underset{\text{def}}{=}f(x,\mu_1,\mu_2)$$

has the bifurcation diagram of Fig. 2 in a neighborhood of $(\mu_1,\mu_2)=(0,0)$. The curve C lies in $\{(\mu_1,\mu_2):\mu_1\leqq 0,\ \mu_2\geqq 0\}$. It has a quadratic tangency with the μ_2-axis at

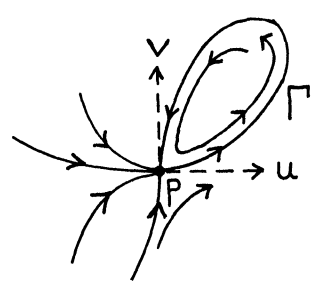

FIG. 1

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† Mathematics Department, North Carolina State University, Raleigh, North Carolina 27695-8205.

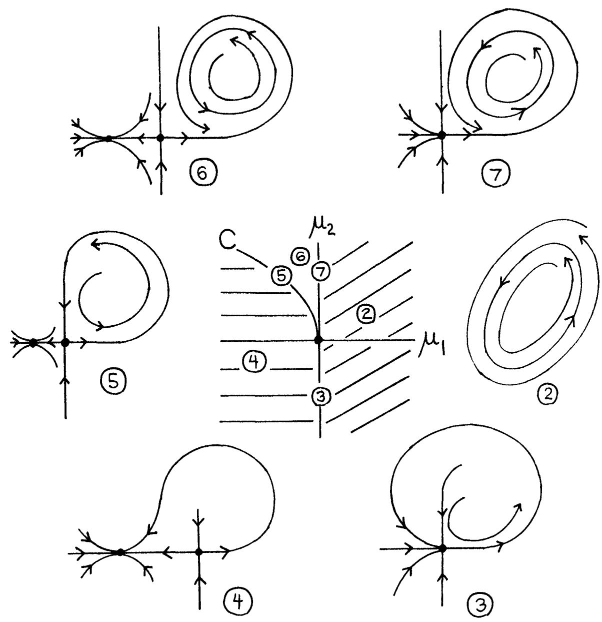

FIG. 2

(0, 0). The phase portrait of $\dot{x} = f(x, \mu_1, \mu_2)$ in a fixed neighborhood of Γ that is positively invariant for each vector field $\dot{x} = f(x, \mu_1, \mu_2)$ is as follows (see Fig. 2):

1) $\mu_1 = 0$, $\mu_2 = 0$; a saddle-node and a separatrix loop.

2) $\mu_1 > 0$; no equilibria, a unique stable closed orbit near Γ.

3) $\mu_1 = 0$, $\mu_2 < 0$; a saddle-node.

4) $\mu_1 < 0$, (μ_1, μ_2) below C; a saddle and a node.

5) $\mu_1 < 0$, (μ_1, μ_2) on C; a saddle and a node; the saddle has a separatrix loop.

6) $\mu_1 < 0$, (μ_1, μ_2) above C; a saddle and a node; there is a unique stable closed orbit near Γ.

7) $\mu_1 = 0$, $\mu_2 > 0$; a saddle-node and a unique stable closed orbit near Γ.

This bifurcation diagram is developed in [4], except that it is mistakenly stated there that the curve C is transverse to the μ_2-axis.

Perhaps the best known example of this bifurcation diagram occurs in the study of the differential equation for a pendulum with linear damping and constant applied torque, which are the two parameters (see [3], in which the same equation arises in the study of the DC current-driven point Josephson junction). In § 5 we show that the pendulum equation satisfies our transversality condition at its organizing center. Thus the pendulum equation is a generic two-parameter unfolding of the saddle-node separatrix-loop bifurcation.

The heart of this paper is the study, in § 3, of Melnikov's integral (see [2]) around a saddle-node separatrix loop. The same method allows one to study time-periodic perturbations of a saddle-node separatrix loop. This subject is treated in the companion paper [6]. There the motivating example is the pendulum equation with, in addition, sinusoidal applied torque (or, equivalently, the AC–DC current-driven point Josephson junction).

2. Statement of results. We shall consider vector fields $\dot{x} = f(x)$, $x \in \mathbb{R}^2$, satisfying the following conditions at some $p \in \mathbb{R}^2$:

(i) $f(p) = 0$.

(ii) $Df(p)$ has eigenvalues 0 and $-\lambda$, where $\lambda > 0$.

Let u be a right eigenvector and w a left eigenvector of the eigenvalue 0, with w chosen so that $wu > 0$.

(iii) $wD^2f(p)(u, u) > 0$.

(iv) $\dot{x} = f(x)$ has a separatrix loop Γ at p.

Assumptions (i)-(iii) say that $\dot{x} = f(x)$ has a saddle-node at p with one negative eigenvalue (see [7]). Moreover, the assumptions imply that u (not $-u$) is one tangent vector to Γ at p (see Fig. 1). Let v be a right eigenvector of $Df(p)$ for the eigenvalue $-\lambda$, chosen so that v is also tangent to Γ at p as in Fig. 1.

Let $\dot{x} = \tilde{f}(x, \nu_1, \nu_2)$ be a two-parameter family of vector fields on $\mathbb{R}^2$ such that $\dot{x} = \tilde{f}(x, 0, 0)$ satisfies (i)-(iv), and

(v) $\tilde{f}(x, \nu_1, \nu_2)$ is C^{k+1}, $k \geqq 5$.

(vi) $wD_{\nu_1}\tilde{f}(p, 0, 0) > 0$.

Assumptions (iii) and (vi) imply that perturbation in the positive ν_1 direction eliminates the equilibrium p, while perturbation in the negative ν_1 direction splits the equilibrium in two (see [7]).

According to [7] there is a C^k function $\alpha(\nu_2)$, with $\alpha(0) = 0$, such that $\dot{x} = \tilde{f}(x, \nu_1, \nu_2)$ has an equilibrium of saddle-node type near p if and only if $\nu_1 = \alpha(\nu_2)$. Let $f(x, \mu_1, \mu_2) = \tilde{f}(x, \nu_1, \nu_2)$, where (μ_1, μ_2) and (ν_1, ν_2) are related by

$$\mu_1 = \nu_1 - \alpha(\nu_2), \qquad \mu_2 = \nu_2.$$

Then

$$\dot{x} = f(x, \mu_1, \mu_2) \tag{3}$$

is C^k, and has an equilibrium of saddle-node type near p if and only if $\mu_1 = 0$. Let $p(\mu_2)$ denote the saddle-node equilibrium near p of $\dot{x} = f(x, 0, \mu_2)$; $p(\mu_2)$ is C^k. If $\mu_1 < 0$, there are a saddle and a sink of (3) near p; if $\mu_1 > 0$, there are no equilibria of (3) near p.

If w and z are vectors in $\mathbb{R}^2$, let $w \wedge z = w_1 z_2 - w_2 z_1$. Let $q(t)$ be a solution of $\dot{x} = f(x, 0, 0)$ with $q(0) \in \Gamma$. Consider the expression

$$\begin{aligned} I = {} & \frac{dp}{d\mu_2}(0) \wedge \lim_{t_1 \to \infty} f(q(t_1), 0, 0) \exp\left[-\int_0^{t_1} \operatorname{div} f(q(s), 0, 0)\, ds\right] \\ & + \int_{-\infty}^{\infty} \exp\left[-\int_0^t \operatorname{div} f(q(s), 0, 0)\, ds\right] f(q(t), 0, 0) \wedge \frac{\partial f}{\partial \mu_2}(q(t), 0, 0)\, dt. \end{aligned} \tag{4}$$

THEOREM 1. *The limit in* (4) *exists and is a negative multiple of* v. *The improper integral in* (4) *converges. If* $I \neq 0$, *then there is a* C^{k-2} *curve* $\mu_1 = \psi(\mu_2) = -\ell^2 \mu_2^2 + o(\mu_2^2)$, $\ell \neq 0$, *such that for* (μ_1, μ_2) *sufficiently small,* (3) *has a separatrix loop near* Γ *if and only if* $\mu_1 = \psi(\mu_2)$ *and* $I \cdot (u \wedge v) \cdot \mu_2 \geqq 0$.

The integral in (4) is just the usual Melnikov integral used to study perturbations of a separatrix loop at a hyperbolic saddle (see [2], [5]). The limit in (4) is zero in the case of a hyperbolic saddle, but must be retained in the case of a saddle-node.

The bifurcation diagram presented in § 1 holds if $I \cdot (u \wedge v) > 0$, in which case separatrix loops occur for $\mu_1 = \psi(\mu_2)$ and $\mu_2 \geqq 0$. If $I \cdot (u \wedge v) < 0$, this bifurcation diagram holds after the further change of parameter $\mu_2 \to -\mu_2$.

We remark that in order to compute I in applications, the only knowledge of the function $\alpha(\nu_2)$ that is needed is $\alpha'(0)$.

We shall now give a precise interpretation of the condition $I \neq 0$ as a transversality condition. Let E denote the space of C^{k+1} vector fields, $k \geqq 5$, on a closed disk $D \subset \mathbb{R}^2$, with the C^{k+1} topology. Let $\Sigma' = \{f \in E : f$ satisfies, at a unique $p \in \operatorname{Int} D$, conditions (i)–(iii); and all other equilibria of f in D are hyperbolic$\}$. Σ' is a C^k codimension one submanifold of E [7]. Let $\Sigma = \{f \in \Sigma' : f$ satisfies condition (iv), and $\Gamma \subset \operatorname{Int} D\}$. We shall show in § 3 that Σ is a C^k codimension two submanifold of E, in fact a codimension one submanifold of Σ'. Let $\dot{x} = \tilde{f}(x, \nu_1, \nu_2)$ be a two-parameter family of vector fields in E with $\dot{x} = \tilde{f}(x, 0, 0) \in \Sigma$. Assume in addition that $\dot{x} = \tilde{f}(x, \nu_1, \nu_2)$ satisfies conditions (v) and (vi). Make the change of parameters $\mu_1 = \nu_1 - \alpha(\nu_2)$, $\mu_2 = \nu_2$ described earlier.

THEOREM 2. *The family $\dot{x} = \tilde{f}(x, \nu_1, \nu_2)$ is transverse to Σ at $(\nu_1, \nu_2) = (0, 0)$ if and only if $I \neq 0$.*

3. Proof of Theorem 1. The equilibrium $(p, 0, 0)$ of

$$\dot{x} = f(x, \mu_1, \mu_2), \quad \dot{\mu}_1 = 0, \quad \dot{\mu}_2 = 0 \tag{5}$$

has a 3-dimensional neutral subspace. The center manifold theorem [1, § 9.2] yields a 3-dimensional C^k local center manifold N_{loc} of (5), tangent at $(p, 0, 0)$ to this subspace. N_{loc} meets each plane $\mathbb{R}^2 \times \{(\mu_1, \mu_2)\}$, (μ_1, μ_2) small, in a curve. $N_{\text{loc}} \cap \mathbb{R}^2 \times \{(0, 0)\}$ contains a portion of $\Gamma \times \{(0, 0)\}$ that is tangent at $(p, 0, 0)$ to $(u, 0, 0)$.

Let N denote the global center manifold that contains N_{loc}, i.e., the union of all integral curves of (5) that meet N_{loc}. N meets each plane $\mathbb{R}^2 \times \{(\mu_1, \mu_2)\}$ in a curve, which we denote $N(\mu_1, \mu_2) \times \{(\mu_1, \mu_2)\}$. Thus $N(\mu_1, \mu_2)$ is a curve in $\mathbb{R}^2$. Let L be a line segment in $\mathbb{R}^2$ perpendicular to Γ at $q(0)$. Then for (μ_1, μ_2) small, $N(\mu_1, \mu_2)$ meets L transversally near $q(0)$. Therefore for (μ_1, μ_2) small there is a C^k function $x(\mu_1, \mu_2)$ such that $x(0, 0) = q(0)$ and $x(\mu_1, \mu_2) \in N(\mu_1, \mu_2) \cap L$. Since a C^k vector field has a C^k flow, there is a C^k family of solutions of (3)

$$q^c(\mu_1, \mu_2, t), \qquad (\mu_1, \mu_2) \text{ small},$$

such that $q^c(\mu_1, \mu_2, 0) = x(\mu_1, \mu_2)$. Then $q^c(0, 0, t) = q(t)$, and each curve $q^c(\mu_1, \mu_2, t)$ lies in $N(\mu_1, \mu_2)$. For $\mu_1 < 0$, $q^c(\mu_1, \mu_2, t)$ is a branch of the unstable manifold of the saddle of (3) near p. Similarly, $q^c(0, \mu_2, t)$ is the unstable separatrix of the saddle-node of $\dot{x} = f(x, 0, \mu_2)$ near p (see Fig. 3).

We shall now define a μ-dependent change of coordinates on $\mathbb{R}^2$ that will make possible our computations. According to [1, § 9.2] there is a C^k change of coordinates

$$y(x, \mu_1, \mu_2) = (y_1(x, \mu_1, \mu_2), y_2(x, \mu_1, \mu_2)), \tag{6}$$

defined for (x, μ_1, μ_2) near $(p, 0, 0)$, such that (1) $y(p, 0, 0) = 0$; (2) $N_{\text{loc}} \cap \mathbb{R}^2 \times \{(\mu_1, \mu_2)\}$ is transformed into the line $y_2 = 0$, which is therefore invariant; (3) the lines $y_1 =$ constant are mapped into each other by the flow. In other words, in the new coordinates we have a C^k differential equation of the form

$$\dot{y}_1 = a(y_1, \mu_1, \mu_2), \qquad \dot{y}_2 = y_2 b(y_1, y_2, \mu_1, \mu_2).$$

Since $p(\mu_2)$, defined in § 2, is C^k, we may assume that $p(\mu_2)$ is transformed to $(0, 0)$ for all μ_2. In other words, $a(0, 0, \mu_2) \equiv 0$. Since the stable manifold of $\dot{x} = f(x, 0, 0)$ at p is necessarily transformed into the line $y_1 = 0$, it is easy to arrange that

$$D_x y(p, 0, 0)u = (1, 0), \qquad D_x y(p, 0, 0)v = (0, 1). \tag{7}$$

Taking into account assumptions (i)–(iii) and (vi), we have

$$\begin{aligned} \dot{y}_1 &= \eta(\mu_2) y_1^2 (1 + y_1 g(y_1, \mu_2)) + \mu_1 h(y_1, \mu_1, \mu_2), \\ \dot{y}_2 &= -\lambda(y_1, \mu_1, \mu_2) y_2 (1 + y_2 k(y_1, y_2, \mu_1, \mu_2)), \end{aligned} \tag{8}$$

with $\eta > 0$, $h(0, 0, 0) > 0$, $\lambda(0, 0, 0) = \lambda$.

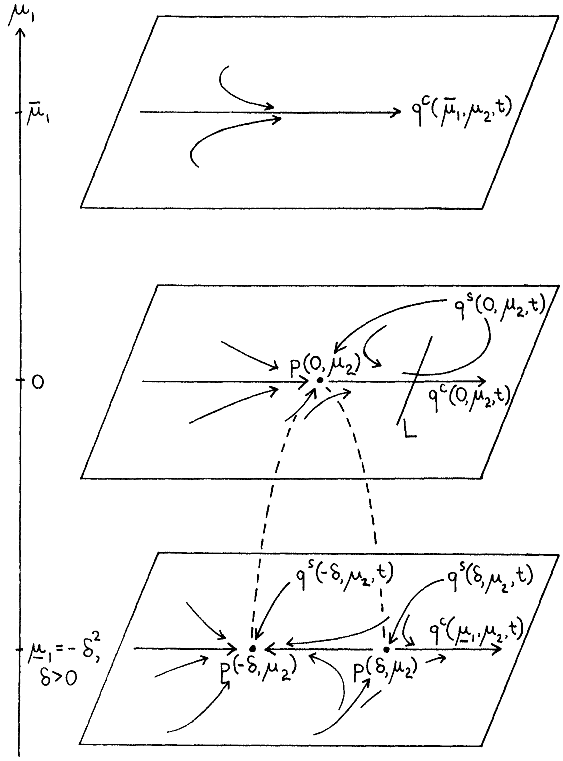

FIG. 3

LEMMA 1. *There is a C^{k-2} mapping $p(\delta, \mu_2)$, defined for (δ, μ_2) near $(0, 0)$, with values in $\mathbb{R}^2$, such that $p(0, 0) = p$, and*

$$p(\delta, \mu_2) \text{ is} \begin{cases} \text{a saddle-node of } \dot{x} = f(x, 0, \mu_2) & \text{if } \delta = 0, \\ \text{a saddle of } \dot{x} = f(x, -\delta^2, \mu_2) & \text{if } \delta > 0, \\ \text{a sink of } \dot{x} = f(x, -\delta^2, \mu_2) & \text{if } \delta < 0. \end{cases}$$

Moreover, the mapping $(\delta, \mu_2) \to (p(\delta, \mu_2), -\delta^2, \mu_2)$ is a C^{k-2} diffeomorphism of a neighborhood of $(0, 0)$ in $\mathbb{R}^2$ onto a neighborhood of $(p, 0, 0)$ in the set of equilibria of (5) *near* $(0, 0, 0)$ (see Fig. 3).

We remark that $p(0, \mu_2)$ equals $p(\mu_2)$ defined in § 2.

Proof. The equilibria of the system

$$\dot{y}_1 = \eta(\mu_2) y_1^2 (1 + y_1 g(y_1, \mu_2)) + \mu_1 h(y_1, \mu_1, \mu_2),$$

$$\dot{y}_2 = -\lambda(y_1, \mu_1, \mu_2) y_2 (1 + y_2 k(y_1, y_2, \mu_1, \mu_2)),$$

$$\dot{\mu}_1 = 0,$$

$$\dot{\mu}_2 = 0$$

near $(0, 0, 0, 0)$ comprise a set of the form

$$\{(y_1, 0, \mu_1, \mu_2) \colon \mu_1 = \mu_1(y_1, \mu_2)\},$$

where

$$\mu_1(0, \mu_2) = \frac{\partial \mu_1}{\partial y_1}(0, \mu_2) = 0 \quad \text{and} \quad \frac{\partial^2 \mu_1}{\partial y_1^2}(0, \mu_2) < 0.$$

Therefore,

$$\mu_1 = -y_1^2(A(\mu_2)+y_1B(y_1,\mu_2)),$$

where $A(\mu_2)>0$ and $A(\mu_2)+y_1B(y_1,\mu_2)$ is C^{k-2}. For $\mu_1 \leqq 0$, let $\mu_1=-\delta^2$. Then

$$\delta = y_1(A(\mu_2)+y_1B(y_1,\mu_2))^{1/2}. \tag{9}$$

(Taking into account the negative square root gives no additional information.) By the implicit function theorem, we can solve (9) for y_1 near $y_1=0$, $\delta=0$, $\mu_2=0$. We obtain

$$y_1=\hat{p}(\delta,\mu_2) \quad \text{with } \hat{p}(0,\mu_2)=0, \qquad \frac{\partial \hat{p}}{\partial \delta}(0,\mu_2)>0. \tag{10}$$

Then $\eta>0$ and (10) imply that the equilibrium $(\hat{p}(\delta,\mu_2),0)$ of (8) with $\mu_1=-\delta^2$ is a saddle-node if $\delta=0$, a saddle if $\delta>0$, and a sink if $\delta<0$.

Let

$$x=x(y,\mu_1,\mu_2) \tag{11}$$

be the change of coordinates inverse to (6). Define

$$p(\delta,\mu_2)=x((\hat{p}(\delta,\mu_2),0),-\delta^2,\mu_2). \tag{12}$$

Then $p(\delta,\mu_2)$ satisfies the assertions of the lemma. □

For future use, we note that

$$\frac{\partial p}{\partial \delta}(0,0) \text{ is a positive multiple of } u. \tag{13}$$

To see this, we compute from (12)

$$\frac{\partial p}{\partial \delta}(0,0)=D_y x((0,0),0,0)\left(\frac{\partial \hat{p}}{\partial \delta}(0,0),0\right). \tag{14}$$

Since $(\partial\hat{p}/\partial\delta)(0,0)>0$ by (10), (14) is a positive multiple of u by (7).

System (8) with $\mu_1=-\delta^2$ has at the equilibrium $(\hat{p}(\delta,\mu_2),0)$ the invariant manifold $\{(y_1,y_2)\colon y_1=\hat{p}(\delta,\mu_2)\}$, a line. For $\delta=0$ this line is the stable manifold of the saddle-node $(0,0)$; for $\delta>0$ it is the stable manifold of the saddle $(\hat{p}(\delta,\mu_2),0)$; and for $\delta<0$ it is the strong stable manifold of the sink $(\hat{p}(\delta,\mu_2),0)$. These lines correspond to invariant manifolds of $\dot{x}=f(x,-\delta^2,\mu_2)$ at $p(\delta,\mu_2)$. Let $v(\delta,\mu_2)=D_y x((\hat{p}(\delta,\mu_2),0),-\delta^2,\mu_2)(0,1)$. Then $\dot{x}=f(x,-\delta^2,\mu_2)$ has at $p(\delta,\mu_2)$ an invariant curve tangent to $v(\delta,\mu_2)$ and these invariant curves vary in a C^{k-2} manner with (δ,μ_2). For $(\delta,\mu_2)=(0,0)$, this invariant curve contains Γ. Now a construction similar to that of $q^c(\mu_1,\mu_2,t)$ yields a C^{k-2} family

$$q^s(\delta,\mu_2,t), \qquad (\delta,\mu_2) \text{ small},$$

each a solution of $\dot{x}=f(x,-\delta^2,\mu_2)$, such that $q^s(\delta,\mu_2,t)\to p(\delta,\mu_2)$ as $t\to\infty$ along the negative $v(\delta,\mu_2)$ direction. Again we require $q^s(\delta,\mu_2,0)\in L$; thus $q^s(0,0,t)=q(t)$. Note that $q^s(0,\mu_2,t)$ is a branch of the stable manifold of the saddle-node $p(0,\mu_2)$ of $\dot{x}=f(x,0,\mu_2)$; and if $\delta>0$, $q^s(\delta,\mu_2,t)$ is a branch of the stable manifold of the saddle $p(\delta,\mu_2)$ of $\dot{x}=f(x,-\delta^2,\mu_2)$ (see Fig. 3).

For any vector $w=(w_1,w_2)\in\mathbb{R}^2$, let $w^\perp=(-w_2,w_1)$. Define $d^c(\mu_1,\mu_2)$ and $d^s(\delta,\mu_2)$ by

$$q^c(\mu_1,\mu_2,0)=q(0)+[d^c(\mu_1,\mu_2)/\|f(q(0),0,0)\|^2]f^\perp(q(0),0,0),$$

$$q^s(\delta,\mu_2,0)=q(0)+[d^s(\delta,\mu_2)/\|f(q(0),0,0)\|^2]f^\perp(q(0),0,0).$$

Then d^c is C^k and d^s is C^{k-2}. The number $d^c(\mu_1, \mu_2)$ (resp. $d^s(\delta, \mu_2)$) determines where on L the curve $q^c(\mu_1, \mu_2, t)$ (resp. $q^s(\delta, \mu_2, t)$) starts. We have

$$\begin{aligned} d^c(\mu_1, \mu_2) &= f^\perp(q(0), 0, 0) \cdot [q^c(\mu_1, \mu_2, 0) - q(0)] \\ &= f(q(0), 0, 0) \wedge [q^c(\mu_1, \mu_2, 0) - q(0)]. \end{aligned}$$

Similarly,

$$d^s(\delta, \mu_2) = f(q(0), 0, 0) \wedge [q^s(\delta, \mu_2, 0) - q(0)].$$

There is a separatrix loop of $\dot{x} = f(x, -\delta^2, \mu_2)$ through $p(\delta, \mu_2)$ if and only if $\delta \geqq 0$ and

$$d(\delta, \mu_2) \underset{\text{def}}{=} d^c(-\delta^2, \mu_2) - d^s(\delta, \mu_2) = 0. \tag{15}$$

Here $d(\delta, \mu_2)$ is C^{k-2}.

We shall show that $(\partial d/\partial\delta)(0, 0)$ is a negative multiple of $u \wedge v$ (hence is nonzero), and $(\partial d/\partial\mu_2)(0, 0) = I$. Given these facts, the proof of Theorem 1 is completed as follows: if $I \neq 0$, then $\{(\delta, \mu_2): d(\delta, \mu_2) = 0\}$ is a C^{k-2} curve through $(0, 0)$ of the form

$$\delta = \ell\mu_2 + o(\mu_2), \qquad \ell = -I \Big/ \frac{\partial d}{\partial \delta}(0, 0). \tag{16}$$

Squaring both sides of (16) yields

$$\mu_1 = -\ell^2\mu_2^2 + o(\mu_2^2).$$

The condition $\delta \geqq 0$, applied to (16), shows that $\mu_2 = 0$ or ℓ and μ_2 have the same sign. But ℓ has the sign of $I \cdot (u \wedge v)$.

We now turn to the computation of $(\partial d/\partial\delta)(0, 0)$ and $(\partial d/\partial\mu_2)(0, 0)$. We shall need the following variational equations for $q^c(-\delta^2, \mu_2, t)$ and $q^s(\delta, \mu_2, t)$:

$$\frac{d}{dt}\frac{\partial q^c}{\partial \mu_2}(0, 0, t) = D_x f(q(t), 0, 0)\frac{\partial q^c}{\partial \mu_2}(0, 0, t) + \frac{\partial f}{\partial \mu_2}(q(t), 0, 0),$$

$$\frac{d}{dt}\frac{\partial q^s}{\partial \delta}(0, 0, t) = D_x f(q(t), 0, 0)\frac{\partial q^s}{\partial \delta}(0, 0, t),$$

$$\frac{d}{dt}\frac{\partial q^s}{\partial \mu_2}(0, 0, t) = D_x f(q(t), 0, 0)\frac{\partial q^s}{\partial \mu_2}(0, 0, t) + \frac{\partial f}{\partial \mu_2}(q(t), 0, 0).$$

As in [2], we define

$$\Delta^c_{\mu_2}(t) = f(q(t), 0, 0) \wedge \frac{\partial q^c}{\partial \mu_2}(0, 0, t)$$

and define $\Delta^s_\delta(t)$ and $\Delta^s_{\mu_2}(t)$ analogously.

For $d^c(-\delta^2, \mu_2)$ and $d^s(\delta, \mu_2)$ we have the derivative formulas

$$\begin{aligned} &\frac{\partial d^c}{\partial \delta}(0, 0) = \frac{\partial d^c}{\partial \mu_1}(0, 0) \cdot \frac{d\mu_1}{d\delta}(0) = 0, \qquad \frac{\partial d^c}{\partial \mu_2}(0, 0) = \Delta^c_{\mu_2}(0), \\ &\frac{\partial d^s}{\partial \delta}(0, 0) = \Delta^s_\delta(0), \qquad \frac{\partial d^s}{\partial \mu_2}(0, 0) = \Delta^s_{\mu_2}(0). \end{aligned} \tag{17}$$

Using the variational equations for q^c and q^s, we compute as in [2]:

$$\frac{d}{dt}\Delta^c_{\mu_2}(t) = \operatorname{div} f(q(t), 0, 0)\Delta^c_{\mu_2}(t) + f(q(t), 0, 0) \wedge \frac{\partial f}{\partial \mu_2}(q(t), 0, 0), \tag{18}$$

$$\frac{d}{dt}\Delta^s_\delta(t) = \operatorname{div} f(q(t),0,0)\Delta^s_\delta(t), \tag{19}$$

$$\frac{d}{dt}\Delta^s_{\mu_2}(t) = \operatorname{div} f(q(t),0,0)\Delta^s_{\mu_2}(t) + f(q(t),0,0)\wedge\frac{\partial f}{\partial \mu_2}(q(t),0,0). \tag{20}$$

Solving these linear differential equations, we obtain, for any t_1,

$$\begin{aligned}\Delta^c_{\mu_2}(0) = \Delta^c_{\mu_2}(t_1)\exp\int_{t_1}^0 \operatorname{div} f(q(t),0,0)\,dt \\ + \int_{t_1}^0 \exp\left[-\int_0^t \operatorname{div} f(q(s),0,0)\,ds\right] \\ \cdot f(q(t),0,0)\wedge\frac{\partial f}{\partial\mu_2}(q(t),0,0)\,dt,\end{aligned} \tag{21}$$

$$-\Delta^s_\delta(0) = -\Delta^s_\delta(t_1)\exp\left[-\int_0^{t_1}\operatorname{div} f(q(t),0,0)\,dt\right], \tag{22}$$

$$\begin{aligned}-\Delta^s_{\mu_2}(0) = -\Delta^s_{\mu_2}(t_1)\exp\left[-\int_0^{t_1}\operatorname{div} f(q(t),0,0)\,dt\right] \\ + \int_0^{t_1}\exp\left[-\int_0^t \operatorname{div} f(q(s),0,0)\,ds\right] \\ \cdot f(q(t),0,0)\wedge\frac{\partial f}{\partial\mu_2}(q(t),0,0)\,dt.\end{aligned} \tag{23}$$

We shall first evaluate (22) in the limit $t_1\to\infty$. Using the definition of Δ^s_δ, we write

$$-\Delta^s_\delta(0) = \frac{\partial q^s}{\partial\delta}(0,0,t_1)\wedge f(q(t_1),0,0)\exp\left[-\int_0^{t_1}\operatorname{div} f(q(s),0,0)\,ds\right]. \tag{24}$$

LEMMA 2.

$$\lim_{t\to\infty}\frac{\partial q^s}{\partial\delta}(0,0,t) = \frac{\partial p}{\partial\delta}(0,0), \qquad \lim_{t\to\infty}\frac{\partial q^s}{\partial\mu_2}(0,0,t) = \frac{\partial p}{\partial\mu_2}(0,0).$$

Proof. We shall use the coordinates (6). Define

$$\begin{aligned}\tilde q^s(\delta,\mu_2,t) &= y(q^s(\delta,\mu_2,t),-\delta^2,\mu_2) \\ &= (\hat p(\delta,\mu_2), y_2(\delta,\mu_2,t)),\end{aligned} \tag{25}$$

where $\hat p(\delta,\mu_2)$ is given by (10) and $y_2(\delta,\mu_2,t)$ is defined by (25). Since $q^s(\delta,\mu_2,t)\to p(\delta,\mu_2)$ as $t\to\infty$, for each (δ,μ_2) near $(0,0)$, $\tilde q^s(\delta,\mu_2,t)$ and hence $y_2(\delta,\mu_2,t)$ are defined for sufficiently large t. It follows from (7) that $y_2(\delta,\mu_2,t)>0$ for large t. From (8), $y_2(\delta,\mu_2,t)$ satisfies a differential equation of the form

$$\frac{dz}{dt} = -\lambda(\delta,\mu_2)z(1+zG(z,\delta,\mu_2)), \tag{26}$$

where $\lambda(0,0)=\lambda$. Here λ and zG are C^{k-3}.

In order to prove the lemma, we shall study the asymptotic behavior of solutions of (26) as $t\to\infty$ by solving (26) by separation of variables. Let

$$z^{-1}[1+zG(z,\delta,\mu_2)]^{-1} = z^{-1}+H(z,\delta,\mu_2). \tag{27}$$

Then H is C^{k-4}. Fix $z_0>0$. Let

$$J(z,\delta,\mu_2)=\int_{z_0}^{z} H(s,\delta,\mu_2)\,ds.$$

Then J is C^{k-4}. Solving (26) by separation of variables using (27) yields

$$\ln z+J(z,\delta,\mu_2)=-\lambda(\delta,\mu_2)t+A(\delta,\mu_2),$$

or

$$z\exp J(z,\delta,\mu_2)=B(\delta,\mu_2)\exp(-\lambda(\delta,\mu_2)t).$$

Here $B(\delta,\mu_2)$ is determined by the value of $y_2(\delta,\mu_2,t)$ at some $t=t_0$. Hence B is C^{k-4} and $B>0$. Since

$$\frac{\partial}{\partial z}[z\exp J(z,\delta,\mu_2)](0,\delta,\mu_2)\neq 0,$$

by the implicit function theorem we can solve the equation

$$z\exp J(z,\delta,\mu_2)=v$$

for z when z and v are near 0. We obtain

$$z=K(v,\delta,\mu_2),$$

where K is C^{k-4} and

$$K(0,\delta,\mu_2)\equiv 0. \tag{28}$$

Putting $z=y_2$ and $v=B(\delta,\mu_2)\exp(-\lambda(\delta,\mu_2)t)$, we obtain

$$y_2=K(B(\delta,\mu_2)\exp(-\lambda(\delta,\mu_2)t),\delta,\mu_2). \tag{29}$$

From (29) and (28) it follows that $(\partial y_2/\partial\delta)(\delta,\mu_2,t)$ and $(\partial y_2/\partial\mu_2)(\delta,\mu_2,t)$ approach 0 as $t\to\infty$. Therefore (25) implies that as $t\to\infty$,

$$\frac{\partial\tilde{q}^s}{\partial\delta}(\delta,\mu_2,t)\to\left(\frac{\partial\hat{p}}{\partial\delta}(\delta,\mu_2),0\right),\qquad \frac{\partial\tilde{q}^s}{\partial\mu_2}(\delta,\ \mu_2,t)\to\left(\frac{\partial\hat{p}}{\partial\mu_2}(\delta,\mu_2),0\right). \tag{30}$$

Now

$$\frac{\partial q^s}{\partial\delta}(\delta,\mu_2,t)=\frac{\partial}{\partial\delta}x(\tilde{q}^s(\delta,\mu_2,t),-\delta^2,\mu_2),$$

where $x(y,\mu_1,\mu_2)$ is given by (11). Therefore

$$\frac{\partial q^s}{\partial\delta}(0,0,t)=D_yx(\tilde{q}^s(0,0,t),0,0)\frac{\partial\tilde{q}^s}{\partial\delta}(0,0,t).$$

By (30) and (14),

$$\lim_{t\to\infty}\frac{\partial q^s}{\partial\delta}(0,0,t)=D_yx((0,0),0,0)\left(\frac{\partial\hat{p}}{\partial\delta}(0,0),0\right)$$

$$=\frac{\partial p}{\partial\delta}(0,0).$$

Similarly, the second formula of the lemma follows from (30) and the following formula derived from (12):

$$\frac{\partial p}{\partial\mu_2}(0,0)=D_yx((0,0),0,0)\left(\frac{\partial\hat{p}}{\partial\mu_2}(0,0),0\right)+\frac{\partial x}{\partial\mu_2}((0,0),0,0). \qquad \square$$

We shall now use the computations we have done in proving Lemma 2 to study the other terms of (24). By (25),

$$\dot{\bar{q}}^s(\delta, \mu_2, t) = D_x y(q^s(\delta, \mu_2, t), -\delta^2, \mu_2)\dot{q}^s(\delta, \mu_2, t). \tag{31}$$

Let $\delta = \mu_2 = 0$ in (31). Since $\dot{q}^s(0, 0, t) = \dot{q}(t) = f(q(t), 0, 0)$, we obtain

$$f(q(t), 0, 0) = [D_x y(q(t), 0, 0)]^{-1}\dot{\bar{q}}^s(0, 0, t).$$

It follows easily from (25), (28) and (29) that

$$\begin{aligned} q(t) &= p + \mathcal{O}(\exp(-\lambda t)), \\ \dot{\bar{q}}^s(t) &= (0, -C \exp(-\lambda t) + o(\exp(-\lambda t))), \end{aligned} \tag{32}$$

where $C > 0$. Therefore, setting $t = t_1$,

$$\begin{aligned} f(q(t_1), 0, 0) = \{[D_x y(p, 0, 0)]^{-1} + \mathcal{O}(\exp(-\lambda t_1))\} \\ \cdot (0, -C \exp(-\lambda t_1) + o(\exp(-\lambda t_1))). \end{aligned} \tag{33}$$

From (32) we also have

$$\operatorname{div} f(q(t), 0, 0) = -\lambda + \mathcal{O}(\exp(-\lambda t)).$$

Therefore

$$\exp\left[-\int_0^{t_1} \operatorname{div} f(q(s), 0, 0)\, ds\right] = \exp(\lambda t_1) \cdot \exp \int_0^{t_1} \mathcal{O}(\exp(-\lambda s))\, ds. \tag{34}$$

Then (33) and (34) give

$$\begin{aligned} &\lim_{t_1 \to \infty} f(q(t_1), 0, 0) \exp\left[-\int_0^{t_1} \operatorname{div} f(q(s), 0, 0)\, ds\right] \\ &\quad = [D_x y(p, 0, 0)]^{-1} \cdot \left(0, -C \exp \int_0^{\infty} \mathcal{O}(\exp(-\lambda s))\, ds\right), \end{aligned} \tag{35}$$

where the integral clearly converges.

Now (24), Lemma 2 and (35) imply that

$$-\Delta_\delta^s(0) = \frac{\partial p}{\partial \delta}(0, 0) \wedge \lim_{t_1 \to \infty} f(q(t_1), 0, 0) \exp\left[-\int_0^{t_1} \operatorname{div} f(q(s), 0, 0)\, ds\right], \tag{36}$$

where the limit exists.

Notice that (7) implies that (35) is a negative multiple of v. Then (13) implies that $-\Delta_\delta^s(0)$ is a negative multiple of $u \wedge v$. By (15) and (17), $(\partial d/\partial\delta)(0, 0)$ is also a negative multiple of $u \wedge v$.

We now turn to (23). We claim that

$$\begin{aligned} -\Delta_{\mu_2}^s(0) = \frac{\partial p}{\partial \mu_2}(0, 0) \wedge \lim_{t_1 \to \infty} f(q(t_1), 0, 0) \exp\left[-\int_0^{t_1} \operatorname{div} f(q(s), 0, 0)\, ds\right] \\ + \int_0^{\infty} \exp\left[-\int_0^{t} \operatorname{div} f(q(s), 0, 0)\, ds\right] f(q(t), 0, 0) \\ \wedge \frac{\partial f}{\partial \mu_2}(q(t), 0, 0)\, dt. \end{aligned} \tag{37}$$

The proof is modeled on that of (36). Using Lemma 2, we first show that the first summand of (23) approaches, as $t_1 \to \infty$, the first summand of (37), where the limit exists. Then, since $-\Delta^s_{\mu_2}(0)$ is finite, the second summand of (23), the integral, must approach a limit as $t_1 \to \infty$.

Finally we turn to (21).

LEMMA 3.

$$\lim_{t\to-\infty} \frac{\partial q^c}{\partial \mu_2}(0, 0, t) = 0.$$

Proof. Again we shall use the coordinates (6). Define

$$\tilde{q}^c(\mu_2, t) = y(q^c(0, \mu_2, t), 0, \mu_2) = (y_1(\mu_2, t), 0).$$

Since $q^c(0, \mu_2, t) \to p(0, \mu_2)$ as $t \to -\infty$, for each μ_2 near 0, $\tilde{q}^c(\mu_2, t)$, and hence $y_1(\mu_2, t)$, is defined for sufficiently negative t. From (7), $y_1(\mu_2, t) > 0$ for sufficiently negative t. From its definition, $y_1(\mu_2, t)$ satisfies a differential equation of the form

$$\frac{dz}{dt} = \eta(\mu_2)z^2(1 + zG(z, \mu_2)). \tag{38}$$

Here η and zG are C^{k-2}.

Let

$$z^{-2}(1 + zG(z, \mu_2))^{-1} = z^{-2} + A(\mu_2)z^{-1} + H(z, \mu_2). \tag{39}$$

Here A is C^{k-3} and H is C^{k-4}. Fix $z_0 > 0$. Let

$$J(z, \mu_2) = \int_{z_0}^{z} H(s, \mu_2)\, ds.$$

J is C^{k-4}. Then solving (38) by separation of variables using (39) yields

$$-z^{-1} + A(\mu_2) \ln z + J(z, \mu_2) = \eta(\mu_2)t + B(\mu_2).$$

Here $B(\mu_2)$ is determined by the value of $y_1(\mu_2, t)$ at some $t = t_0$. Hence B is C^{k-4}. Rearranging yields

$$z[1 - A(\mu_2)z \ln z - zJ(z, \mu_2)]^{-1} = -[\eta(\mu_2)t + B(\mu_2)]^{-1}. \tag{40}$$

Let $\Phi(z, \mu_2)$ equal the left-hand side of (40). $\Phi(z, \mu_2)$ is a C^1 function of z and μ_2 on a neighborhood of $(0, 0)$ in $\{(z, \mu_2): z \geqq 0\}$; $\Phi(0, \mu_2) \equiv 0$, and $(\partial/\partial z)\Phi(0, \mu_2) \equiv 1$. By the implicit function theorem, we can solve the equation $\Phi(z, \mu_2) = v$ for z when z and v are near 0, $v \geqq 0$ (in which case $z \geqq 0$). We obtain

$$z = v + R(v, \mu_2),$$

where R is C^1, $R(0, \mu_2) \equiv 0$ and R is $o(v)$. Putting $z = y_1$ and $v = -[\eta(\mu_2)t + B(\mu_2)]^{-1}$, t large negative, we obtain

$$y_1 = -[\eta(\mu_2)t + B(\mu_2)]^{-1} + R(-[\eta(\mu_2)t + B(\mu_2)]^{-1}, \mu_2).$$

It follows that $(\partial y_1/\partial \mu_2)(\mu_2, t) \to 0$ as $t \to -\infty$, so

$$\frac{\partial \tilde{q}^c}{\partial \mu_2}(\mu_2, t) \to 0 \quad \text{as } t \to -\infty. \tag{41}$$

Lemma 3 follows from (41) the way Lemma 2 follows from (30). $\square$

To evaluate (21) in the limit $t_1 \to -\infty$, we use the definition of $\Delta^c_{\mu_2}$ to write (21) as

$$\Delta^c_{\mu_2}(0) = -\frac{\partial q^c}{\partial \mu_2}(0,0,t_1) \wedge f(q(t),0,0) \exp \int_{t_1}^{0} \operatorname{div} f(q(s),0,0)\, ds \tag{42}$$
$$+ \int_{t_1}^{0} \exp\left[-\int_0^t \operatorname{div} f(q(s),0,0)\, ds\right] f(q(t),0,0) \wedge \frac{\partial f}{\partial \mu_2}(q(t),0,0)\, dt.$$

Of course

$$f(q(t),0,0) \to 0 \quad \text{as } t_1 \to -\infty. \tag{43}$$

Moreover,

$$\operatorname{div} f(q(t),0,0) = -\lambda + \mathcal{O}([\eta(0)t + c(0)]^{-1}) \quad \text{as } t \to -\infty. \tag{44}$$

By Lemma 3, (43) and (44), we have

$$\lim_{t_1 \to -\infty} -\frac{\partial q^c}{\partial \mu_2}(0,0,t_1) \wedge f(q(t_1),0,0) \exp \int_{t_1}^{0} \operatorname{div} f(q(s),0,0)\, ds = 0.$$

Therefore the second summand of (42), the integral, approaches a limit as $t_1 \to -\infty$, so

$$\Delta^c_{\mu_2}(0) = \int_{-\infty}^{0} \exp\left[-\int_0^t \operatorname{div} f(q(s),0,0)\, ds\right] f(q(t),0,0) \wedge \frac{\partial f}{\partial \mu_2}(q(t),0,0)\, dt. \tag{45}$$

Finally, we complete the proof of Theorem 1 by calculating

$$\frac{\partial d}{\partial \mu_2}(0,0) = \frac{\partial d^c}{\partial \mu_2}(0,0) - \frac{\partial d^s}{\partial \mu_2}(0,0) = \Delta^c_{\mu_2}(0) - \Delta^s_{\mu_2}(0) = (45) + (37) = I.$$

4. Proof of Theorem 2. First we show that Σ is a C^k codimension one submanifold of Σ'. Let $f \in \Sigma$ with saddle-node p and let L be a line segment perpendicular to the separatrix loop Γ as in § 3. For g near f in Σ', there is a unique saddle-node p_g near p; p_g is a C^k function of $g \in \Sigma'$. The stable and center manifolds of p_g also depend C^k on g [1, § 9.2]. Thus their intersections with L are C^k functions of g. Therefore the function $d(0,\mu_2)$ from § 2 extends to a C^k function $d(g)$ defined for $g \in \Sigma'$ near f; $d(g)$ measures the separation of these points of intersection. (We remark that $d(0,\mu_2)$ is C^k although $d(\delta,\mu_2)$ is only C^{k-2}.) Then $d(g)=0$ if and only if $g \in \Sigma$. Since it is easy to find a perturbation $f+\varepsilon h$ in Σ' such that $d/d\varepsilon|_{\varepsilon=0} d(f+\varepsilon h) \neq 0$, Σ is a C^k codimension one submanifold of Σ'.

To prove Theorem 2, it suffices to show that $f(\cdot,\mu_1,\mu_2)$ is transverse to Σ at $(\mu_1,\mu_2)=(0,0)$ if and only if $I \neq 0$. Since $f(\cdot,0,\mu_2) \in \Sigma'$ for all small μ_2, $(\partial f/\partial \mu_2)(\cdot,0,0)$ is tangent to Σ' (see Fig. 4). But $(\partial \tilde{f}/\partial \nu_1)(\cdot,0,0) = (\partial f/\partial \mu_1)(\cdot,0,0) - (\partial f/\partial \mu_2)(\cdot,0,0)\alpha'(0)$. Since $(\partial \tilde{f}/\partial \nu_1)(\cdot,0,0)$ is transverse to Σ' by

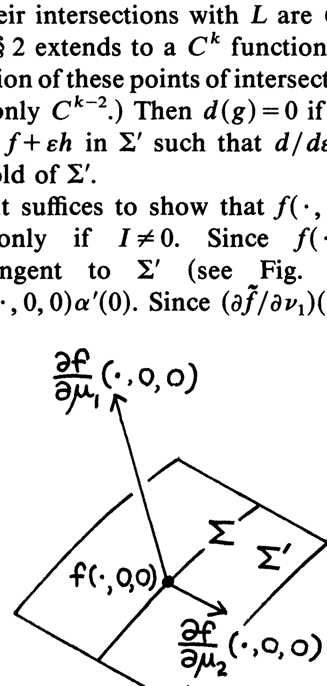

FIG. 4

assumption (vi) (see [7]), and $(\partial f/\partial \mu_2)(\cdot, 0, 0)$ is tangent to Σ', $(\partial f/\partial \mu_1)(\cdot, 0, 0)$ is transverse to Σ'. Thus we need to show that $(\partial f/\partial \mu_2)(\cdot, 0, 0)$ is transverse to Σ if and only if $I \neq 0$. But this follows from the formula $(\partial d/\partial \mu_2)(0, 0) = I$.

5. The pendulum equation. We consider the differential equation for the damped pendulum with constant applied torque, in the dimensionless form studied in [5]:

$$\beta\ddot{\phi} + \dot{\phi} + \sin\phi = \rho.$$

Putting $y = \dot{\phi}$, we have

$$\dot{\phi} = y, \qquad \dot{y} = \frac{1}{\beta}(-y - \sin\phi + \rho). \tag{46}$$

We identitify ϕ and $\phi + 2\pi$, so that (46) defines a vector field on a cylinder; ρ and β are parameters, which we shall assume positive. We remark that Theorem 1 applies equally well to vector fields on the compact set $\{(\phi, y): |y| \leqq d\}$.

Let $x = (\phi, y)$,

$$f(x, \beta, \rho) = (f_1((\phi, y), \beta, \rho), f_2((\phi, y), \beta, \rho)) = \left(y, \frac{1}{\beta}(-y - \sin\phi + \rho)\right).$$

If $\rho < 1$, $\dot{x} = f(x, \rho, \beta)$ has two equilibria, one a saddle and one a sink; $\dot{x} = f(x, 1, \beta)$ has one equilibrium, at $(\pi/2, 0)$ independent of β; if $\rho > 1$, $\dot{x} = f(x, \rho, \beta)$ has no equilibria. We note that

$$D_x f\left(\left(\frac{\pi}{2}, 0\right), 1, \beta\right) = \begin{bmatrix} 0 & 1 \\ 0 & -1/\beta \end{bmatrix},$$

which has eigenvalues $0, -1/\beta$. Corresponding right eigenvectors are $u = (1, 0)$ and $v = (-\beta, 1)$; a left eigenvector for the eigenvalue 0 is $(1, \beta)$.

To show that $\dot{x} = f(x, 1, \beta)$ has a saddle-node at $(\pi/2, 0)$, we compute (assumption (iii)):

$$\begin{aligned}(1, \beta) \cdot D_x^2 f\left(\left(\frac{\pi}{2}, 0\right), 1, \beta\right) \cdot ((1, 0), (1, 0)) &= (1, \beta) \cdot \left(0, \frac{\partial^2 f}{\partial \phi^2}\left(\left(\frac{\pi}{2}, 0\right), 1, \beta\right)\right) \\ &= (1, \beta) \cdot \left(0, \frac{1}{\beta}\right) = 1.\end{aligned}$$

We also note (assumption (iv)):

$$(1, \beta) \cdot D_\rho f\left(\left(\frac{\pi}{2}, 0\right), 1, \beta\right) = (1, \beta) \cdot \left(0, \frac{1}{\beta}\right) = 1.$$

It is shown in [3] that there is a unique positive β_0 such that $\dot{x} = f(x, 1, \beta_0)$ has a separatrix loop at the saddle-node $(\pi/2, 0)$. The separatrix loop, considered as a curve in ϕy-space with ϕ and $\phi + 2\pi$ not yet identified, can be expressed as $y = y(\phi)$, $\pi/2 < \phi < 5\pi/2$; $y(\phi) > 0$ for all ϕ. As $\phi \to \pi/2$, $y \to 0$ and $y/(\phi - \pi/2) \to 0$; as $\phi \to 5\pi/2$, $y \to 0$ and $y/(\phi - 5\pi/2) \to -1/\beta)$ (see Fig. 5).

Thus assumptions (i)–(vi) are verified if we put $\tilde{f} = f$, $\nu_1 = \rho - 1$, $\nu_2 = \beta - \beta_0$. The change of variables from ν to μ is not necessary here, i.e., we may put $\mu_1 = \nu_1$, $\mu_2 = \nu_2$. For simplicity we shall continue to use the parameters ρ and β.

To compute I, we first note that the saddle-node $p(\beta)$ of $\dot{x} = f(x, 1, \beta)$ is identically $(\pi/2, 0)$, so that the first summand of I is 0. Now $\operatorname{div} f(x, \rho, \beta) = -1/\beta$, and

$$f((\phi, y), \rho, \beta) \wedge \frac{\partial f}{\partial \beta}((\phi, y), \rho, \beta) = -\frac{1}{\beta^2} y(-y - \sin\phi + \beta) = -\frac{1}{\beta} y\dot{y}.$$

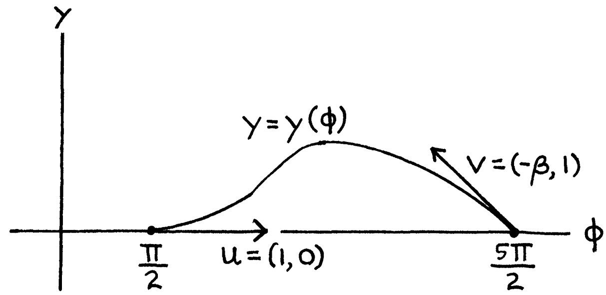

FIG. 5

Therefore,

$$I=-\frac{1}{\beta_0}\int_{-\infty}^{\infty} e^{t/\beta_0}y\dot{y}\,dt$$

$$=\lim_{S,T\to\infty}\left\{-\frac{1}{\beta_0}e^{t/\beta_0}\cdot\frac{y^2}{2}\bigg|_{-S}^{T}+\frac{1}{\beta_0^2}\int_{-S}^{T}e^{t/\beta_0}\cdot\frac{y^2}{2}\,dt\right\}.$$

From (29), as $t\to\infty$, $(\phi(t)-(5\pi/2), y(t))=c\,e^{-t/\beta_0}(-\beta_0,1)+o(e^{-t/\beta_0})$ for some positive constant c. Therefore $[y(t)]^2=c^2\,e^{-2t/\beta_0}+o(e^{-2t/\beta_0})$ as $t\to\infty$. Hence

$$\lim_{T\to\infty}-\frac{1}{\beta_0}e^{T/\beta_0}\cdot\frac{[y(T)]^2}{2}=0.$$

Of course,

$$\lim_{S\to\infty}-\frac{1}{\beta_0}e^{-S/\beta_0}\cdot\frac{[y(-S)]^2}{2}=0,$$

since $y(t)\to 0$ as $t\to-\infty$. Therefore

$$I=\frac{1}{\beta_0^2}\int_{-\infty}^{\infty}e^{t/\beta_0}\cdot\frac{y^2}{2}\,dt>0.$$

Since $u\wedge v>0$, Theorem 1 implies that for (ρ,β) near $(1,\beta_0)$, $\dot{x}=f(x,\rho,\beta)$ has a separatrix loop if and only if $\rho=1-\ell^2(\beta-\beta_0)^2+\cdots$, with $\ell\neq 0$, and $\beta-\beta_0\geqq 0$ (see Fig. 6). This result is in agreement with statements in [5].

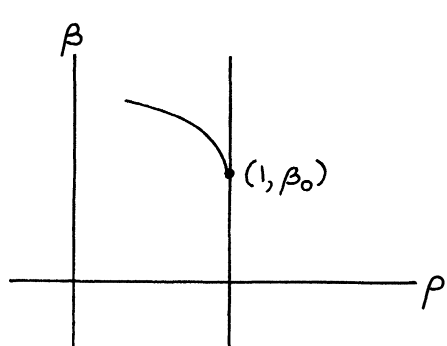

FIG. 6

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