$\vec{x}' = A\vec{x}$ $\quad A \in M_{n\times n}$

Autovalores $\{\lambda_1,\dots,\lambda_n\}$

Autovectores $\{v_1,\dots,v_n\}$

$$\Rightarrow x(t) = \sum_{i=1}^{n} c_i v_i e^{\lambda_i t}$$

## MA-2115: Matemáticas 4

# Semana 9

## 8.1 Sistemas de ecuaciones lineales homogéneos

1. **Autovectores repetidos: todos los casos.** mult alg > mult geo

- $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ $\quad \lambda_1 = 1 \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \lambda_{2,3} = 2 \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ simple

$v_3 = ?$

$(A-\lambda_2 I)v_3 = v_2 \quad \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{array}\right) \longrightarrow v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ autovector generalizado

$$x(t) = c_1 v_1 e^{t} + c_2 v_2 e^{2t} + c_3 (v_3 + v_2 t) e^{2t}$$

- $A = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 6 & 3 & 2 \end{pmatrix}$ $\quad \lambda_{1,2,3} = 2 \quad v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

mult alg = 3 $\quad$ mult geo = 1

$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 6 & 3 & 0 & 1 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = \frac{1}{3} \\ x_3 = k \in \mathbb{R} \\ \quad\ \ {}^{\shortparallel}0 \end{array} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 6 & 3 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = \frac{1}{3} \\ x_2 = -\frac{2}{3} \\ x_3 = k \in \mathbb{R} \\ \quad\ \ {}^{\shortparallel}0 \end{array} \quad v_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$

$$x(t) = c_1 v_1 e^{2t} + c_2 (v_2 + v_1 t) e^{2t} + c_3 \left(v_3 + v_2 t + v_1 \tfrac{1}{2} t^2\right) e^{2t}$$

- $A = \begin{pmatrix} -3 & 0 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix}$ $\quad \lambda_{1,2,3} = -3 \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

mult alg = 3 $\quad$ mult geo = 2

$(A-\lambda_1 I)v_3 = k_1 v_1 + k_2 v_2 \quad \left(\begin{array}{ccc|c} 0 & 0 & 1 & k_1 \\ 0 & 0 & 1 & k_2 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} x_3 = k_1 \\ x_3 = k_2 \\ \Rightarrow k_1 = k_2 \end{array}$ $\quad$ tomo $k_2 = 1$ $\quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ generalizado

$$x(t) = c_1 v_1 e^{-3t} + c_2 v_2 e^{-3t} + c_3 \left(v_3 + (v_1 + v_2) t\right) e^{-3t}$$

2. **Teorema (autovalores complejos)**

   Si $\lambda$ y $\vec{v}$ son autopar complejo de $A$, entonces $\Re(e^{\lambda t}\vec{v})$ y $\Im(e^{\lambda t}\vec{v})$ son soluciones LI de $\frac{d\vec{x}}{dt} = A\vec{x}$.

3. **Formula de Euler**: $e^{a+ib} = (\cos b + i\sin b)e^a$.

   $\boxed{i \cdot i = -1}$

   $\boxed{(a+ib)(a-ib) = a^2+b}$

4. *Resuelva el siguiente sistema de ecuaciones*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \vec{x}.$$

Autovalores

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2-2\lambda+5)=0 \qquad \lambda = \frac{2\pm\sqrt{4-20}}{2} = 1 \pm i2$$

$$\lambda_1 = 1 \;, \quad \lambda_{2,3} = 1 \pm i\,2$$

Autovectores

- $\lambda_1 = 1$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right) \quad \cdots \quad \Rightarrow v_1 = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

- $\lambda_2 = 1+i2$

$$\left(\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 2 & -2i & -2 & 0 \\ 3 & 2 & -2i & 0 \end{array}\right) \xrightarrow[\substack{f_2 \leftarrow f_2 - 2f_1 \\ f_3 \leftarrow f_3 - 3f_1}]{f_1 \leftarrow \frac{i}{2} f_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -2i & -2 & 0 \\ 0 & 2 & -2i & 0 \end{array}\right)$$

$$\xrightarrow[f_3 \leftarrow \frac{1}{2} f_3]{f_2 \leftarrow \frac{i}{2} f_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ \cancel{0} & \cancel{1} & \cancel{-i} & 0 \end{array}\right) \qquad \begin{array}{l} x_1 = 0 \\ x_3 = k \in \mathbb{R} \\ x_2 = ik \end{array} \qquad v_2 = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$$

Soluciones LI

$$v_2 e^{\lambda_2 t} = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} e^{(1+i2)t} = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} e^t e^{i2t}$$

$$= \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} (\cos 2t + i\sin 2t) e^t = \begin{pmatrix} 0 \\ i\cos 2t - \sin 2t \\ \cos 2t + i \sin 2t \end{pmatrix} e^t$$

$$= \underbrace{\begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix} e^t}_{x_2(t) = Re(v_2 e^{\lambda_2 t})} + i \underbrace{\begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} e^t}_{x_3(t) = Im(v_2 e^{\lambda_2 t})}$$

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix} e^t + c_3 \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} e^t}$$

## 8.2 Sistemas de ecuaciones lineales no homogéneos

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}(t)$$

$\frac{dx}{dt} = Ax \Rightarrow x_h \Rightarrow \Psi$

$\frac{dx}{dt} = Ax + f(t) \Rightarrow x_p$

$X = x_h + x_p$

1. **Metodo de variacion de parámetros**

   La solución de $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}(t)$ viene dada por

   $n\times n \quad n\times 1 = n\times 1$

   $$\vec{x}(t) = \Psi(t)\int (\Psi(t))^{-1}\vec{f}(t)dt,$$

   donde $\Psi(t) = (x_1(t)|\cdots|x_n(t))$ es la *matrix fundamental* cuyas columnas son las soluciones LI del sistema homogeneo $\frac{d\vec{x}}{dt} = A\vec{x}$.

2. *Resuelva el siguiente sistema de ecuaciones*

   $$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\vec{x} + \begin{pmatrix} 0 \\ -2e^t \end{pmatrix}.$$

Autovalores y Autovectores $\quad \lambda_{1,2} = 1 \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ generalizado $\leftarrow$

$$x_h(t) = \underline{c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + c_2\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}t\right)e^t}$$

- Matriz fundamental

$$\Psi(t) = \begin{pmatrix} e^t & te^t \\ e^t & (1+t)e^t \end{pmatrix} = \underline{\begin{pmatrix} 1 & t \\ 1 & 1+t \end{pmatrix}e^t} \Rightarrow \Psi^{-1}(t) = \frac{e^{-t}}{1+\cancel{t}-\cancel{t}}\begin{pmatrix} 1+t & -t \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1+t & -t \\ -1 & 1 \end{pmatrix}e^{-t}$$

- Variacion de parametros

$$\Rightarrow \int \Psi^{-1}(t)f(t)dt = \int e^{-t}\begin{pmatrix} 1+t & -t \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -2e^t \end{pmatrix}dt = \int e^{-t}\begin{pmatrix} 2te^t \\ -2e^t \end{pmatrix}dt$$

$$= \int \begin{pmatrix} 2t \\ -2 \end{pmatrix}dt = \begin{pmatrix} t^2 \\ -2t \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$(c_1 = c_2 = 0)$

$$\Rightarrow x_p(t) = e^t\begin{pmatrix} 1 & t \\ 1 & 1+t \end{pmatrix}\begin{pmatrix} t^2 \\ -2t \end{pmatrix} = e^t\begin{pmatrix} t^2 - 2t^2 \\ t^2 - 2t - 2t^2 \end{pmatrix} = \underline{\begin{pmatrix} -t^2e^t \\ (-t^2-2t)e^t \end{pmatrix}}$$

$$X(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + c_2\begin{pmatrix} t \\ 1+t \end{pmatrix}e^t + \begin{pmatrix} -t^2e^t \\ -(t+2)te^t \end{pmatrix}$$

Nota: $\Psi$ es invertible para todo $t \in \mathbb{R}$

$W[x_1, x_2] = \det|\Psi| = ((1+t)-t)e^t = e^t \neq 0$ para todo $t$.

3. *Resuelva el siguiente sistema de ecuaciones*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} \sec(t) \\ 0 \end{pmatrix}.$$

Autovalores $\lambda_{1,2} = \pm i$ $\quad v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{it} = \begin{pmatrix} \cos t + i \sin t \\ -i\cos t + \sin t \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$$

$$x_h(t) = c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$$

- Matriz fundamental

$$\Psi(t) = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \Rightarrow \Psi^{-1}(t) = \frac{1}{-\cos^2 t - \sin^2 t} \begin{pmatrix} -\cos t & -\sin t \\ -\sin t & \cos t \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}$$

Nota: $|\Psi(t)| = -\cos^2 t - \sin^2 t = -(\cos^2 t + \sin^2 t) = -1 \neq 0$

$\Psi(t)$ es invertible para todo $t \in \mathbb{R}$

- Variación de parámetros

$$\int \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \begin{pmatrix} \sec t \\ 0 \end{pmatrix} dt = \int \begin{pmatrix} \cos t \sec t \\ \sin t \sec t \end{pmatrix} dt = \int \begin{pmatrix} 1 \\ \frac{\sin t}{\cos t} \end{pmatrix} dt$$

$\sec t = \frac{1}{\cos t}$

$$= \begin{pmatrix} t \\ -\ln|\cos t| \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$u = \cos t$

$$x(t) = \Psi(t) \int \Psi^{-1}(t) f(t)\, dt$$

$$= \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \left[ \begin{pmatrix} t \\ -\ln|\cos t| \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right]$$

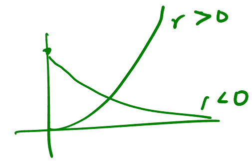
$$\Rightarrow x(t) = c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} + \begin{pmatrix} t\cos t - \sin t \ln|\cos t| \\ t \sin t - \cos t \ln|\cos t| \end{pmatrix}$$

## 8.3 Aplicaciones

1. Ecuación de crecimineto (decrecimiento) exponencial

$$N' = rN.$$

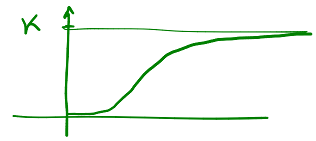
sep. lin.




2. Ecuación de crecimineto logístico

$$N' = rN\left(1 - \frac{N}{K}\right).$$

sep.



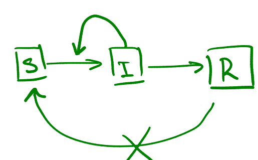

3. Sistema renina-angiotensina

$\frac{dx}{dt} = Ax + f(t)$

$$\begin{aligned}
\frac{d[AGT]}{dt} &= k_{AGT} - PRA - \frac{ln(2)}{h_{AGT}}[AGT],\\
\frac{d[AngI]}{dt} &= PRA - (c_{ACE} + c_{Chym} + c_{NEP})[AngI] - \frac{ln(2)}{h_{AngI}}[AngI],\\
\frac{d[AngII]}{dt} &= (c_{ACE} + c_{Chym})[AngI] - (c_{ACE2} + c_{AT1R} + c_{AT2R})[AngII] - \frac{ln(2)}{h_{AngII}}[AngII],\\
\frac{d[AT1RAngII]}{dt} &= c_{AT1R}[AngII] - \frac{ln(2)}{h_{AT1R}}[AT1RAngII],\\
\frac{d[AT2RAngII]}{dt} &= c_{AT2R}[AngII] - \frac{ln(2)}{h_{AT2R}}[AT2RAngII],\\
\frac{d[Ang17]}{dt} &= c_{NEP}[AngI] + c_{ACE2}[AngII] - \frac{ln(2)}{h_{Ang17)}}[Ang17],\\
\frac{d[AngIV]}{dt} &= c_{AngIIAngIV}[AngII] - \frac{ln(2)}{h_{AngIV}}[AngIV].
\end{aligned}$$

4. Modelo epidemiológico

$$\begin{aligned}
S' &= \beta I S + \cancel{\gamma R},\\
I' &= \beta I S - \alpha I,\\
R' &= \alpha I - \gamma R.
\end{aligned}$$




caso Covid-19

$R_0 > 1$

$< 1$

5. Modelo de depredador-presa de Lotka-Volterra

$$\begin{aligned}
\dot{x} &= -\alpha x + \beta x y,\\
\dot{y} &= \gamma y - \delta x y.
\end{aligned}$$