$$f(x) = \sum_{n=0}^{\infty} \overset{\text{coefs}}{a_n}(x-a)^n$$

- Series notables
- Criterio del cociente
- Radio convergencia

$$f(a) = a_0$$

MA-2115: Matemáticas 4

Semana 5

5.1 Series de potencias

1. Series de potencias notables

$$\frac{d}{dx}\left(\frac{1}{x+1}\right) = \frac{-1}{(x+1)^2}$$

$$\int \frac{1}{x+1}\,dx = \ln(x+1) + C$$

$$\int \frac{1}{x^2+1}\,dx = \arctan x + C$$

$$\frac{1}{x+1} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1$$

$$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots \qquad |x| < \infty$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

2. Serie de Taylor

$$\underline{a_n}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)\overset{(x-a)}{\cancel{x}} + \frac{f''(a)}{2}(x-a)^2 + \cdots.$$

3. Polinomio de approximación y residuo

$$f(x) \approx P_n(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)x + \frac{f''(a)}{2}(x-a)^2 + \cdots \frac{f(n)(a)}{n!}(x-a)^n.$$

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad c \in (x, a)$$

4. Represente la función en serie de potencias y encuentre el intervalo de convergencia

$$f(x) = x^2 \operatorname{arccot}(x^2).$$

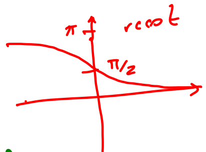
$$\frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2} \quad , \quad \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \qquad R=1$$

evaluo en x^2

$$\Rightarrow \quad -\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^{n+1} x^{2n} \qquad R=1$$

integración

$$\Rightarrow \quad \operatorname{arccot} x = c + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} x^{2n+1}$$



$$\operatorname{arccot} 0 = \frac{\pi}{2} = c$$

evaluo en x^2

$$\Rightarrow \quad \operatorname{arccot} x^2 = \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} x^{4n+2}$$

mult x^2

$$\Rightarrow \quad \boxed{x^2 \operatorname{arccot} x^2 = \frac{\pi}{2} x^2 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} x^{4n+4}} \qquad R=1$$

- Radio de convergencia (crit. del cociente)

$$\lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+2}}}{2n+3} x^{\cancel{4n+8}}}{\frac{\cancel{(-1)^{n+1}}}{2n+1} x^{\cancel{4n+4}}} \right| = \lim_{n\to\infty} \left| \frac{(-1)(2n+1)}{2n+3} x^4 \right| = |x^4| \cdot 1 < 1$$

$$|x| < 1$$

- Extremos

$x=-1$ $\quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} \cancel{(-1)^{4n+2}}^{\,1} \rightarrow a_n = \frac{1}{2n+1} \rightarrow 0$ y decreciente $\Rightarrow$ Por criterio de serie alternante $\Rightarrow$ converge.

$x=1$ $\quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} \cancel{(1)^{4n+2}}^{\,1}$

$$\boxed{x \in [-1, 1]}$$

5. Represente la función en serie de potencias y encuentre el intervalo de convergencia

$$f(x) = \ln\left(\frac{3+x}{4-x}\right). = \ln(3+x) - \ln(4-x)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \qquad R=1$$

evaluar en $x/3$

$$\Rightarrow \ln\left(1+\frac{x}{3}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{x^n}{3^n}$$

$$\ln\left(1+\frac{x}{3}\right) = -\ln 3 + \ln(3+x)$$

$$\Rightarrow \ln(3+x) = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\cdot 3^n} x^n \qquad R=3$$

$$(-1)^{n-1}(-1)^n = (-1)^{2n-1} = -1$$

evaluar en $-x/4$

$$\ln\left(1-\frac{x}{4}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{(-1)^n x^n}{4^n} \qquad R=4$$

$$\ln\left(1-\frac{x}{4}\right) = -\ln 4 + \ln(4-x)$$

$$\Rightarrow \ln(4-x) = \ln 4 - \sum_{n=1}^{\infty} \frac{1}{n4^n} x^n$$

combinar

$$\Rightarrow \ln\left(\frac{3+x}{4-x}\right) = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\cdot 3^n} x^n - \ln 4 + \sum_{n=1}^{\infty} \frac{1}{n4^n} x^n$$

$$\Rightarrow \ln\left(\frac{3+x}{4-x}\right) = \ln\frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^{n-1}}{n\cdot 3^n} + \frac{1}{n4^n}\right] x^n$$

yo intuyo que $R=3$ pero hay que verificar

- Radio de convergencia

$$\lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n}}}{(n+1)3^{n+1}}\cancel{x^{n+1}}}{\frac{\cancel{(-1)^{n-1}}}{n\,\cancel{3^n}}\cancel{x^n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)\,n}{(n+1)3}x\right| = \left|\frac{x}{3}\right| 1 < 1$$

de la misma forma

$$\sum \frac{(-1)^{n-1}}{n3^n}x^n \quad R=3, \quad \sum \frac{1}{n4^n}x^n \quad R=4 \Rightarrow$$

combinados $R=3$

tomo el radio de convergencia más restrictivo

- Extremos.

$$x=-3 \quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\cdot\cancel{3^n}}(-1)^n\cancel{3^n} = -\sum_{n=1}^{\infty}\frac{1}{n} \Rightarrow \text{diverge}$$

armonica

$$x=3 \quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,\cancel{3^n}}\cancel{3^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} \Rightarrow \text{converge}$$

armonica alternante

$$\Rightarrow x \in (-3, 3]$$

6. Represente la función en serie de potencias y encuentre el intervalo de convergencia

$$f(x) = \frac{x^2}{(2-x)^3}.$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

derivada $\Rightarrow \frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{-1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$ importante

derivada otra vez $\rightarrow \frac{d^2}{dx^2}\left(\frac{1}{1+x}\right) = \frac{2}{(1+x)^3} = \sum_{n=2}^{\infty} (-1)^n n(n-1) x^{n-2}$

$(-1)^{n+n-2} = 1$

mult $\frac{1}{2}$ y evaluar en $-\frac{x}{2}$ $\Rightarrow \frac{1}{(1-\frac{x}{2})^3} = \sum_{n=2}^{\infty} \frac{(-1)^n n(n-1)}{2} \frac{(-1)^{n-2} x^{n-2}}{2^{n-2}}$

$\frac{1}{(1-\frac{x}{2})^3} = \frac{1}{(\frac{1}{2})^3 (2-x)^3} = \frac{2^3}{2-x^3}$

reordenar $\Rightarrow \frac{2^3}{(2-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n-1}} x^{n-2}$

$\frac{1}{2^{n-1} \cdot 2^3} = \frac{1}{2^{n+2}}$

mult $\frac{x^2}{2^3}$ $\Rightarrow \boxed{\frac{x^2}{(2-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n+2}} x^n}$

$x^{n-2} x^2 = x^n$

- Radio de convergencia

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)n}{2^{n+3}} x^{n+1}}{\frac{n(n-1)}{2^{n+2}} x^n} \right| = |x| \lim_{n\to\infty} \left| \frac{n+1}{2(n-1)} \right| = \left|\frac{x}{2}\right| < 1$$

$\Rightarrow |x| < 2$

- Extremos

$x = 2 \Rightarrow \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n+2}} 2^n = \sum_{n=2}^{\infty} \frac{n(n-1)}{4}$ diverge $(a_n \to \infty)$

$x = -2 \Rightarrow \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n+2}} (-1)^n 2^n = \sum_{n=2}^{\infty} \frac{(-1)^n n(n-1)}{4}$ diverge $(a_n \to \infty)$

$\Rightarrow \boxed{x \in (-2, 2)}$

7. Represente la función en serie de potencias y encuentre el intervalo de convergencia

$$f(x) = \frac{e^{x^2} - 1}{x}.$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad R=\infty$$

evaluar x^2 $\Rightarrow$ $e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$

. restar 1 . separar 1er término $\Rightarrow$ $e^{x^2} - 1 = \underbrace{\cancel{1}}_{n=0} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{n!} x^{2n}} \cancel{-1}$

$$\Rightarrow \frac{e^{x^2}-1}{x} = \sum_{n=1}^{\infty} \frac{1}{n!} x^{2n-1} \qquad R=\infty$$

· Radio de convergencia

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!} x^{2n+1}}{\frac{1}{n!} x^{2n-1}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{n+1} \right| = 0$$

converge siempre para x fijo $\Rightarrow R=\infty$

$\Rightarrow x \in (-\infty, \infty)$

verificar en Desmos !! :)

5.2 EDO

resolver $y(x)$

1. Ecuación differencial ordinarias

$$F(\underset{\text{var. ind}}{x}, \underset{\text{var dep } y(x)}{y}, \underbrace{y', y'', \ldots, y^{(n)}}_{\text{der. var der.}}) = 0.$$

2. Clasificación

clasificacion — orden.

- $\frac{dy}{dx} + xy = 0$ — $\frac{dy}{dx} = xy$ — $\frac{dy}{dx} = f(x)g(y)$ separable y lineal — 1^{er}
- $\frac{dy}{dx} + y + x = 0$ — $y(x)$? — $\frac{dy}{dx} + y = -x$, $p(x)=1$ $q(x)=-x$ — $\frac{dy}{dx} + p(x)y = q(x)$ lineal
- $\frac{dy}{dt} + e^{-t}y = \sin(t)$ — $y(t)$? — $p(t)= e^{-t}$ $q(t)= \sin t$ — lineal
- $\frac{dx}{dt} + tx^2 = 0$ — $x(t)$? — x^2 es no lineal — no lineal, separable
- $y\frac{dy}{dx} + e^x \ln y = 0$ — yy', $\ln y$ son no lineales — $\frac{dy}{dx} = -\frac{e^x \ln y}{y}$ separable
- $(x + y)\frac{dy}{dx} - (x^2 + y^2) = 0$ — y^2 es no lineal — no es separable — homogeneas 1^{er} order Exactas.
- $y' + e^t y = \sin(t)y^3$ — y^3 es no lineal — $y' + p(x)y = q(x)y^p$ Bernoulli — 1^{er}
- $2\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - y = 0$ — $ay'' + by' + cy = 0$ 2^{do} orden lineal coef. const. y homogenea — 2^{do}
- $\begin{cases} \frac{dx}{dt} = -x + y \\ \frac{dy}{dt} = x + 2y \end{cases}$ — x, y son var dep. t var ind. — sistema ecuaciones lineales — 1^{er}
- $2y'' + 4y' - y = e^{-2t}$ — 2^{do} orden lineal coef const. y no homogenea. — 2^{do}
- $y'' + e^x y' + y = 0$ — 2^{do} orden lineal homogene
- $(y'')^2 + 2yy' + \sin(y) = 0$ — $(y'')^2$, yy', $\sin y$ no lineal — 2^{do}
- $y''' - y'' + y' - y = 0$ — 3^{er} orden lineal coef const. y homogenea. — 3^{er}