$F(x,y,y')=0, \quad y'=f(x,y)$

$y'=f(x)g(y)$
separable

MA-2115: Matemáticas 4

Semana 6

$y'+p(x)y=q(x)$
lineal

6.1 Ecuaciones de separables primer orden

1. Resuelva la equación

$$\frac{dy}{dx}=\frac{x}{y^2\sqrt{1+x}} = \underbrace{\frac{x}{\sqrt{1+x}}}_{f(x)}\cdot\underbrace{\frac{1}{y^2}}_{g(y)}$$

$$\Rightarrow \int y^2\,dy = \int \frac{x\,dx}{\sqrt{1+x}} \Rightarrow \frac{y^3}{3} = \frac{2(1+x)^{3/2}}{3} - 2(1+x)^{1/2} + C$$

$$\int \frac{u-1}{\sqrt{u}}\,du = \int (u^{1/2}-u^{-1/2})\,du = \frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2} + C$$

$u=1+x \Rightarrow x=u-1$
$du=dx$

$$\Rightarrow \boxed{y(x) = \left[2(1+x)^{3/2} - 6(1+x)^{1/2} + C\right]^{1/3}}$$

2. Resuelva la equación

$$\frac{dy}{dx}=\frac{\sec^2(y)}{1+x^2}. = \underbrace{\frac{1}{1+x^2}}_{f(x)}\cdot\underbrace{\sec^2(y)}_{g(y)}$$

no lineal

$$\int \frac{1}{\sec^2 y}\,dy = \int \frac{1}{1+x^2}\,dx \Rightarrow \frac{\sin 2y}{4} + \frac{1}{2}y = \arctan x + C$$

$$\int \frac{1}{\sec^2 y}\,dy = \int \cos^2 y\,dy = \int\left(\frac{\cos 2y}{2}+\frac{1}{2}\right)dy = \frac{\sin 2y}{4} + \frac{1}{2}y + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

id. trig.
$\cos 2\theta = 2\cos^2\theta - 1$
$\cos^2\theta = \frac{\cos 2\theta}{2} + \frac{1}{2}$

$$\Rightarrow \boxed{\sin(2y(x)) + 2y(x) = 4\arctan x + C}$$

solución implícita

3. Resuelva el problema de valor inicial

$y>0$

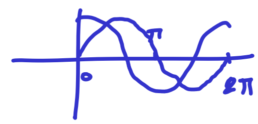
condicion inicial

$$\frac{1}{\theta}\frac{dy}{d\theta} = \frac{y\sin(\theta)}{y^2+1}, \quad dy(\pi) = 1.$$

$$\frac{dy}{dx} = \underbrace{\theta\sin\theta}_{f(\theta)}\ \underbrace{\frac{y}{y^2+1}}_{g(y)}$$

$$\int \frac{y^2+1}{y}\,dy = \int \theta\sin\theta\,d\theta \Rightarrow \frac{y^2}{2} + \ln|y| = -\theta\cos\theta + \sin\theta + c$$

$$\int \frac{y^2+1}{y}\,dy = \int\left(y + \frac{1}{y}\right)dy = \frac{y^2}{2} + \ln|y| + C$$



$$\int \theta\sin\theta\,d\theta = -\theta\cos\theta + \int\cos\theta\,d\theta = -\theta\cos\theta + \sin\theta + c$$

- En $\theta=\pi$ $\quad \frac{1}{2} + \cancel{\ln|1|}^{0} = -\pi\,\cancel{\cos}^{-1}\pi + \cancel{\sin\pi}^{0} + c \Rightarrow c = \frac{1}{2} - \pi$

$$\Rightarrow \boxed{\frac{y(\theta)^2}{2} + \ln|y(\theta)| = -\theta\cos\theta + \sin\theta + \frac{1}{2} - \pi}$$

sol. particular del prob. a valor inicial

4. Solve the following problem

$$\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}, \quad x>0.$$

Eq. 1er orden homogenea.

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad u = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{1 + 3\frac{y^2}{x^2}}{2x\frac{y}{x^2}} = \frac{1 + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)} \qquad u = \frac{y}{x},\ \Rightarrow xu = y \Rightarrow u + x\frac{du}{dx} = \frac{dy}{dx}$$

$$u + x\frac{du}{dx} = \frac{1+3u^2}{2u} \Rightarrow \frac{du}{dx} = \frac{1}{x}\left(\frac{1+3u^2}{2u} - u\right) = \frac{1}{x}\left(\frac{1+3u^2-2u^2}{2u}\right)$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{x}\,\frac{1+u^2}{2u} \underset{\text{sep.}}{\Rightarrow} \int\frac{2u\,du}{1+u^2} = \int\frac{dx}{x} \Rightarrow \ln|1+u^2| = \ln|x| + C$$

$v = 1+u^2 \quad dv = 2u\,du$

exp ambos lados

$$\Rightarrow 1+u^2 = e^{c}\cdot x \Rightarrow 1 + \frac{y^2}{x^2} = Kx \Rightarrow y^2 = (Kx-1)x^2$$

atencion

$$\Rightarrow \boxed{y(x) = x\sqrt{Kx-1}} \qquad x>0$$

5. Resuelva el problema de valor inicial

$$\frac{dy}{dx} = \frac{2\sin(2x+y+3)}{1-\sin(2x+y+3)}.$$

$\frac{dy}{dx} = f(ax+by+c)$

$u = ax+by+c$

$u = 2x+y+3$, $\frac{du}{dx} = 2 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 2$

$\frac{du}{dx} - 2 = \frac{2\sin u}{1-\sin u} \Rightarrow \frac{du}{dx} = \frac{2\sin u}{1-\sin u} + 2 = \frac{\cancel{2\sin u} + 2 - \cancel{2\sin u}}{1-\sin u}$

$\frac{du}{dx} = \frac{2}{1-\sin u} \Rightarrow \int (1-\sin u)\,du = \int 2\,dx \Rightarrow u + \cos u = 2x + C$

$\Rightarrow \cancel{2x} + y + 3 + \cos(2x+y+3) = \cancel{2x} + C$

substituir de vuelta

$\boxed{y(x) + \cos(2x+y(x)+3) = K}$

$K = C-3$

6. Halle la familia de curvas ortogonales a

$$3\tan y + 3x + x^3 = C$$

$F(x,y) = 0 \perp G(x,y) = 0$

① der imp. $\frac{dy}{dx} = f(x,y) \rightarrow$ ② recíproco neg. (ortogonalidad) $\frac{dy}{dx} = -\frac{1}{f(x,y)}$ ③ resolver sep.

① $\cancel{3}\sec^2 y \cdot \frac{dy}{dx} + \cancel{3} + \cancel{3}x^2 = 0$

no hay C

si hubiese una C substituirla de aquí

$\frac{dy}{dx} = \frac{-1-x^2}{\sec^2 y}$

② $\frac{dy}{dx} = \frac{\sec^2 y}{1+x^2}$

③ $\Rightarrow$ $\boxed{\sin(2y(x)) + 2y(x) = 4\arctan x + C}$

$G(x,y)$

6.2 Ecuaciones lineales primer orden

$\underbrace{y' + p(x)y = q(x)}$ $\quad I(x) = e^{\int p(x)dx} \quad$ $y(x) = \frac{1}{I(x)}\left[\int I(x)q(x)dx + C\right]$

1. Encuentra la solución general de la ecuación

$$x\frac{dy}{dx} + 2y = x^{-3}.$$

$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x^4}$

$I(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln|x^2|} = x^2$

$y(x) = \frac{1}{x^2}\left[\int x^2 \frac{1}{x^4}dx + C\right] = \frac{1}{x^2}\left[-\frac{1}{x} + C\right] = -\frac{1}{x^3} + \frac{C}{x^2}$

$$\boxed{y(x) = \frac{C}{x^2} - \frac{1}{x^3}}$$

2. Solve the initial value problem

$$t^2\frac{dx}{dt} + 3tx = t^4\ln(t) + 1, \quad x(1) = 0.$$

$\Rightarrow \frac{dx}{dt} + \frac{3}{t}x = t^2\ln(t) + \frac{1}{t^2}$ $\quad I(t) = e^{\int \frac{3}{t}dt} = e^{3\ln t} = e^{\ln t^3} = t^3$

$x(t) = \frac{1}{t^3}\left[\int t^3\left(t^2\ln(t) + \frac{1}{t^2}\right)dt + C\right] = \frac{1}{t^3}\left[\int t^5\ln(t)dt + \int t\,dt + C\right]$

$\int t^5\ln(t)dt \overset{\text{ipp.}}{=} \frac{t^6}{6}\ln(t) - \int \frac{t^6}{6}\frac{1}{t}dt = \frac{t^6}{6}\ln(t) - \frac{t^6}{36} + C$

$x(t) = \frac{1}{t^3}\left[\frac{t^6}{6}\ln(t) - \frac{t^6}{36} + \frac{t^2}{2} + C\right]$

en $t=1$ $\quad x(1) = 0 = \frac{1}{1}\left[\cancel{\frac{1}{6}\ln(1)}^{0} - \frac{1}{36} + \frac{1}{2} + C\right]$

$\Rightarrow C = \frac{1}{36} - \frac{1}{2} = -\frac{17}{36}$

$$\boxed{x(t) = \frac{t^3}{6}\ln(t) - \frac{t^3}{36} + \frac{1}{2t} - \frac{17}{36t^3}}$$