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# Lab 11: Fourier series

*Fourier series ($[-L, L]$)*

**1**. Compute the Fourier series for the given function on the specific interval

$$f(x) = x^2, \quad -3 < x < 3.$$

**2**. Compute the Fourier series for the given function on the specific interval

$$f(x) = x, \quad -\pi < x < \pi,$$

and sketch a graph of the Fourier series on the domain $-3\pi < x < 3\pi$.

**3**. Find the Fourier series of the piecewise function $f$ defined as

$$f(x) = \begin{cases} 0, & -1 \leq x < 0 \\ x^2, & 0 \leq x < 1 \end{cases},$$

sketch a graph, and determine its sum for all $|x| \leq 1$.

*Fourier sine and cosine series ($[0, L]$)*

**4**. Compute the Fourier sine series and Fourier cosine series for the given function

$$f(x) = x - x^2, \quad 0 < x < 1$$

# *Solutions*

Theory and problems from: Nagel, Saff & Sneider, *Fundamentals of Differential Equations*, Eighth Edition, Adisson–Wesley.

→ The **Fourier series** of $f(x)$ on the interval $[-L, L]$ is

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \qquad n \geq 0,$$

and

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \qquad n \geq 1.$$

Moreover, the sum converges for all $|x| \leq L$ to

$$F(x) = \begin{cases} f(x) & \text{if } -L < x < L \text{ and } f \text{ is continuous at } x, \\ \dfrac{f(x-) + f(x+)}{2} & \text{if } -L < x < L \text{ and } f \text{ is discontinuous at } x, \\ \dfrac{f(-L+) + f(L-)}{2} & \text{if } x = L \text{ or } x = -L. \end{cases}$$

→ Properties of **even** ($f(-x) = f(x)$) and **odd** ($-f(-x) = f(x)$) **functions**

1. even $\pm$ even = even,  even $\times$ even = even
2. odd $\pm$ odd = odd,  odd $\times$ odd = even
3. odd $\pm$ even = none,  odd $\times$ even = odd
4. for $f(x)$ even $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$
5. for $f(x)$ odd $\int_{-L}^{L} f(x)dx = 0$
6. for $f(x)$ even $b_n = 0$, and $a_n = 2\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx$.
7. for $f(x)$ odd $a_n = 0$, and $b_n = 2\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$.

→ The **Fourier Sine Series** of $f(x)$ on $[0, L]$ is

$$S(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

→ The **Fourier Cosine series** of $f(x)$ on $[0, L]$

$$C(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0.$$

**1**. Compute the Fourier series for the given function on the specific interval

$$f(x) = x^2, \quad -3 < x < 3.$$

*Solution*
We want to write $f(x)$ in the form

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{3}\right) + b_n \sin\left(\frac{n\pi x}{3}\right)\right]$$

Since $f(x)$ is an even function, we know that

$$b_n = \frac{1}{3}\int_{-3}^{3} x^2 \sin\left(\frac{n\pi x}{3}\right) dx = 0, \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Hence, all we need to do is find the coefficients $a_n$ Since $f(x)$ is an even function, we have for $n = 0, 1, 2, 3, \ldots$ :

$$a_n = \frac{1}{3}\int_{-3}^{3} x^2 \cos\left(\frac{n\pi x}{3}\right) dx = \frac{2}{3}\int_0^3 x^2 \cos\left(\frac{n\pi x}{3}\right) dx$$

First, we find $a_0$

$$\begin{aligned} a_0 &= \frac{2}{3}\int_0^3 x^2 dx \\ &= \frac{2}{3}\left[\frac{x^3}{3}\right]_0^3 \\ &= \frac{2}{3}\left(\frac{3^3}{3}\right) \\ &= 6 \end{aligned}$$

Now, to find $a_n$. We use integration by parts twice to solve this integral

$$\begin{aligned}
a_n &= \frac{2}{3}\int_0^3 x^2 \cos\left(\frac{n\pi x}{3}\right) dx \\
&= \frac{2}{3}\left[(x^2)\left(\frac{3}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\right)\right]_0^3 - \frac{2}{3}\int_0^3 (2x)\left(\frac{3}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\right) dx \\
&= 0 - \frac{4}{n\pi}\int_0^3 x\sin\left(\frac{n\pi x}{3}\right) dx \\
&= -\frac{4}{n\pi}\left[(x)\left(-\frac{3}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\right)\right]_0^3 + \frac{4}{n\pi}\int_0^3 (1)\left(-\frac{3}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\right) dx \\
&= -\frac{4}{n\pi}\left[-\frac{9}{n\pi}\cos(n\pi) - 0\right] - \frac{12}{n^2\pi^2}\int_0^3 \cos\left(\frac{n\pi x}{3}\right) dx \\
&= \frac{36}{n^2\pi^2}\cos(n\pi) + \frac{12}{n^2\pi^2}\left[\frac{3}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\right]_0^3 \\
&= \frac{36}{n^2\pi^2}\cos(n\pi) + 0 \\
&= \frac{36}{n^2\pi^2}(-1)^n \\
&= \frac{36(-1)^n}{n^2\pi^2}
\end{aligned}$$

Hence, after all this laborious work, we obtain our solution

$$\boxed{F(x) = x^2 = 3 + \sum_{n=1}^{\infty} \frac{36(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right)}.$$

■

**2**. Compute the Fourier series for the given function on the specific interval

$$f(x) = x, \quad -\pi < x < \pi,$$

and sketch a graph of the Fourier series on the domain $-3\pi < x < 3\pi$.

*Solution*

First, note that $f(x)$ is *odd*, and $L = \pi$. We want to write $f(x)$ in the form

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

Since $f(x)$ is an odd function, we know that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = 0, \quad \text{for} \quad n = 1, 2, 3, \ldots,$$

and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx = 0.$$

For $b_n$ we use one integration by parts.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx \\ &= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx)|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right] \\ &= \frac{2}{\pi}\left[-\frac{\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(nx)|_0^{\pi}\right] = \frac{(-1)^{n+1}2}{n}. \end{aligned}$$

Thus,

$$\boxed{F(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2}{n}\sin(nx)}.$$

To sketch a graph we extend periodically $f(x)$ and redefined points of discontinuity to the midpoint.

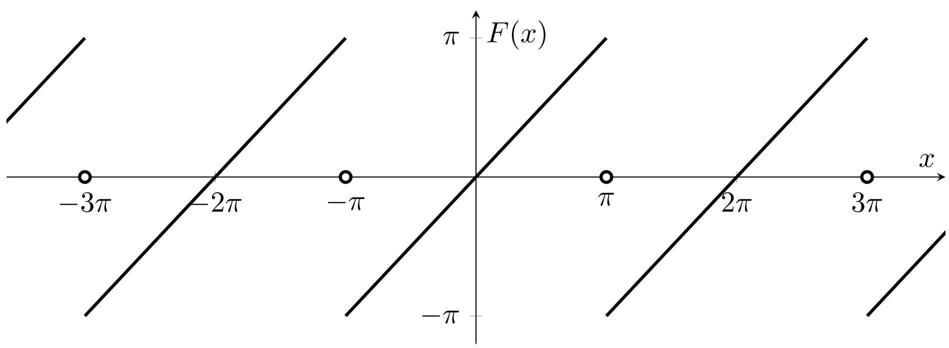


Figure 1: Sketch of Fourier series of $f(x)$ for $-3\pi < x < 3\pi$.

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**3**. Find the Fourier series of the piecewise function $f$ defined as

$$f(x) = \begin{cases} 0, & -1 \le x < 0 \\ x^2, & 0 \le x < 1 \end{cases},$$

sketch a graph, and determine its sum for all $|x| \le 1$.

*Solution*
Note that $f(x)$ is not even nor odd. Hence, we have to compute each term in the Fourier expansion

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\pi x) + b_n\sin(n\pi x),$$

where

$$a_0 = \int_{-1}^{1} f(x)dx = \int_0^1 x^2dx = \frac{1}{3}$$

$$
\begin{aligned}
a_n &= \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_0^1 x^2 \cos(n\pi x)\,dx \\
&= \cancelto{0}{\frac{x^2 \sin(n\pi x)}{n\pi}}\Bigg|_0^1 - \int_0^1 \frac{2x\sin(n\pi x)}{n\pi}dx \\
&= \frac{2x\cos(n\pi x)}{n^2\pi^2}\Bigg|_0^1 - \int_0^1 \frac{2\cos(n\pi x)}{n^2\pi^2}dx \\
&= \frac{2(-1)^n}{n^2\pi^2} + \cancelto{0}{\frac{2\sin(n\pi x)}{n^3\pi^3}}\Bigg|_0^1 = \frac{2(-1)^n}{n^2\pi^2}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_0^1 x^2 \sin(n\pi x)\,dx \\
&= -\frac{x^2\cos(n\pi x)}{n\pi}\Bigg|_0^1 + \int_0^1 \frac{2x\cos(n\pi x)}{n\pi}dx \\
&= -\frac{(-1)^n}{n\pi} + \frac{2x\sin(n\pi x)}{n^{\pi^2}}\Bigg|_0^1 - \int_0^1 \frac{2x\sin(n\pi x)}{n^2\pi^2}dx \\
&= -\frac{(-1)^n}{n\pi} + \frac{2\cos(n\pi x)}{n^3\pi^3}\Bigg|_0^1 \\
&= -\frac{(-1)^n}{n\pi} + \frac{2((-1)^n - 1)}{n^3\pi^3}
\end{aligned}
$$

Thus, the Fourier series of $f(x)$ is

$$
\boxed{F(x) = \frac{1}{6} + \sum_{n=1}^{\infty}\left[\left(\frac{2(-1)^n}{n^2\pi^2}\right)\cos(n\pi x) + \left(-\frac{(-1)^n}{n\pi} + \frac{2((-1)^n-1)}{n^3\pi^3}\right)\sin(n\pi x)\right]}.
$$

A sketch of the graph is shown in Figure 2.

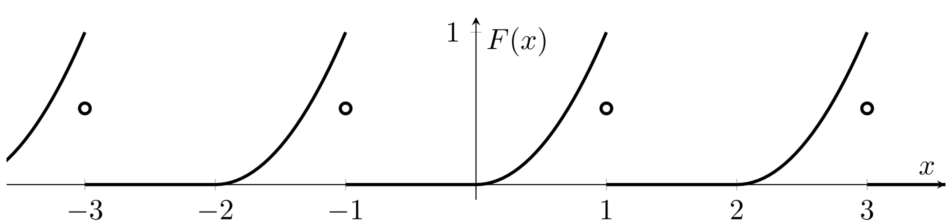


Figure 2: Sketch of Fourier series of $f(x)$ including 3 periods.

To find its sum on $[-1, 1]$, we find what happens on the points of discontinuity[1] of $f(x)$, i.e., $x = -1$, and $x = 1$ (knowing the graph of $F(x)$ in Figure 2 definitely helps here).

At $x = -1$,

$$
F(-1) = \frac{f(-1+) + f(1-)}{2} = \frac{0 + 1^2}{2} = \frac{1}{2},
$$

which is the same value at $x = 1$,

$$
F(1) = \frac{f(-1+) + f(1-)}{2} = \frac{1}{2}.
$$

[1] If $f(x)$ is discontinuous at $x = a$, $f(a+)$ and $f(a-)$ are the values of the function to the right and left of $a$, respectively.

At $x = 0$, there is no need to find the value of the sum because the $f(x)$ is continuous at this point.

Hence, the sum for all $|x| \leq 1$ is

$$\boxed{F(x) = \begin{cases} \frac{1}{2}, & x = -1 \\ 0, & -1 < x < 0 \\ x^2, & 0 \leq x < 1 \\ \frac{1}{2}, & x = 1 \end{cases}.}$$

■

**4.** Compute the Fourier sine series for the given function

$$f(x) = x - x^2, \quad 0 < x < 1$$

*Solution*

**Note:** the given domain is of the form $[0, L]$ instead of $[-L, L]$. In which case the problem is either to find the Fourier Sine series (also called odd extension of $f(x)$ to $[-L, L]$) or the Fourier Cosine series (also called even extension of $f(x)$ to $[-L, L]$).

The Fourier Sine Series of $f(x)$ on $[0, L]$ is

$$S(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad n = 1, 2, 3, \ldots$$

In our case, we have $L = 1$. Let's calculate $b_n$.

$$\begin{aligned}
b_n &= 2\int_0^1 (x - x^2)\sin(n\pi x)\,\mathrm{d}x \\
&= 2\left[(x - x^2)\left(-\frac{\cos(n\pi x)}{n\pi}\right)\right]_0^1 - 2\int_0^1 (1 - 2x)\left(-\frac{\cos(n\pi x)}{n\pi}\right)\mathrm{d}x \\
&= 0 + \frac{2}{n\pi}\int_0^1 (1 - 2x)\cos(n\pi x)\mathrm{d}x \\
&= \frac{2}{n\pi}\left[(1 - 2x)\left(\frac{\sin(n\pi x)}{n\pi}\right)\right]_0^1 - \frac{2}{n\pi}\int_0^1 (-2)\left(\frac{\sin(n\pi x)}{n\pi}\right)\mathrm{d}x \\
&= 0 + \frac{4}{n^2\pi^2}\int_0^1 \sin(n\pi x)\mathrm{d}x \\
&= \frac{4}{n^2\pi^2}\left[-\frac{\cos(n\pi x)}{n\pi}\right]_0^1 \\
&= \frac{4}{n^2\pi^2}\left(\frac{1 - \cos(n\pi)}{n\pi}\right) \\
&= \frac{4(1 - (-1)^n)}{n^3\pi^3}
\end{aligned}$$

From the values of $b_n$ it is easy to show that all even indexed ones will be equal to 0 :

$$
\begin{aligned}
b_n &= \frac{4(1-(-1)^n)}{n^3\pi^3} \\
\Rightarrow b_{2n} &= \frac{4(1-(-1)^{2n})}{(2n)^3\pi^3} = \frac{4(1-1)}{(2n)^3\pi^3} = 0 \\
\Rightarrow b_{2n-1} &= \frac{4(1-(-1)^{2n-1})}{(2n-1)^3\pi^3} = \frac{4(1-(-1))}{(2n-1)^3\pi^3} = \frac{8}{(2n-1)^3\pi^3}
\end{aligned}
$$

Hence, our function $f(x) = x - x^2$ on the interval $[0, 1]$ can be written in the following Fourier Sine series

$$\boxed{S(x) = \sum_{n=1}^{\infty} \frac{8}{(2n-1)^3\pi^3} \sin((2n-1)\pi x)}.$$

Similarly, we can compute the Fourier Cosine series of $f(x)$ on $[0, L]$

$$C(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{2}{L}\int_0^L f(x)dx = \frac{1}{3}$$

and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2((-1)^{n+1}-1)}{n^2\pi^2}$$

$$\Rightarrow a_{2n} = -\frac{1}{n^2\pi^2}, \qquad a_{2n-1} = 0.$$

Hence, our function $f(x) = x - x^2$ on the interval $[0, 1]$ can be written in the following Fourier *Cosine* series

$$\boxed{C(x) = \frac{1}{6} - \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \cos(2n\pi x)}.$$

**Note**: We could also not separate even/odd cases and

$$\boxed{S(x) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^3\pi^3} \sin(n\pi x)} \quad \text{and} \quad \boxed{C(x) = \frac{1}{6} - \sum_{n=1}^{\infty} \frac{2(1+(-1)^n)}{n^2\pi^2} \cos(n\pi x)}.$$

will still be valid solutions.

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