MATH 201 Differential Equations – University of Alberta
Winter 2018 – Labs – Carlos Contreras
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Lab 12: Heat and wave equation

Heat equation with zero BC's.

1. Solve the heat flow problem,

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) = 3\frac{\partial^2 u}{\partial x^2}(x,t), \qquad 0 < x < \pi, \quad t > 0,\\
&u(0,t) = u(\pi,t) = 0, \qquad t > 0,\\
&u(x,0) = \sin(x) - 6\sin(4x), \qquad 0 < x < \pi.
\end{aligned}$$

Heat equation with zero derivative BC's.

2. Solve the heat flow problem

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) = \alpha\frac{\partial^2 u}{\partial x^2}(x,t), \qquad 0 < x < 1, \quad t > 0,\\
&\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0, \qquad t > 0,\\
&u(x,0) = x(1-x), \qquad 0 < x < 1.
\end{aligned}$$

Heat equation with non-zero BC's.

3. Find a formal solution to the initial value problem,

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \qquad 0 < x < \pi, \quad t > 0,\\
&u(0,t) = 0, \quad u(\pi,t) = 3\pi, \qquad t > 0,\\
&u(x,0) = 0, \qquad 0 < x < \pi.
\end{aligned}$$

Heat equation transient and steady state solution.

4. Find a formal solution to the initial value problem,

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + 6x - 2, \qquad 0 < x < 1, \quad t > 0,\\
&u(0,t) = 0, \quad u(1,t) = -1, \qquad t > 0,\\
&u(x,0) = -x^3, \qquad 0 < x < 1.
\end{aligned}$$

Heat equation special case.

5. Solve the heat flow problem,

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) - 3u, \qquad 0 < x < \pi, \quad t > 0,\\
&\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \qquad t > 0,\\
&u(x,0) = 2 + \cos x - 5\cos 4x, \qquad 0 < x < \pi.
\end{aligned}$$

Wave equation

6. Find a formal solution to the vibrating string problem governed by the given initial-boundary value problem

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, && 0 < x < \pi, \quad t > 0 \\
&u(0,t) = u(\pi,t) = 0, && t > 0 \\
&u(x,0) = x^2(\pi - x), && 0 < x < \pi \\
&\tfrac{\partial u}{\partial t}(x,0) = \frac{(x-\pi)^3}{3}, && 0 < x < \pi
\end{aligned}
$$

Solutions

Theory and problems from: Nagel, Saff & Sneider, *Fundamentals of Differential Equations*, Eighth Edition, Adisson–Wesley.

→ The eigenvalue problem

$$X'' - \lambda X = 0\,; \qquad 0 < x < L\,, \qquad X(0) = 0\,, \qquad X(L) = 0,$$

has eigenvalues and eigenfunctions

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right), \quad n \geq 1.$$

→ The eigenvalue problem

$$X'' - \lambda X = 0\,; \qquad 0 < x < L\,, \qquad X'(0) = 0\,, \qquad X'(L) = 0,$$

has eigenvalues and eigenfunctions

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = B_n \cos\left(\frac{n\pi x}{L}\right), \quad n \geq 0.$$

→ For the heat equation with **non-zero boundary conditions and external force**

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) = \alpha \frac{\partial^2 u}{\partial x^2}(x,t) + g(x), \qquad 0 < x < L, \quad t > 0,\\
&u(0,t) = U_1, \quad u(L,t) = U_2, \qquad t > 0,\\
&u(x,0) = f(x), \qquad 0 < x < L,
\end{aligned}$$

we apply the change of variable

$$u(x,t) = v(x) + w(x,t), \;\Rightarrow w(x,t) = u(x,t) - v(x),$$

to arrive the zero BC's problem

$$\begin{aligned}
&\frac{\partial w}{\partial t}(x,t) = \alpha \frac{\partial^2 w}{\partial x^2}(x,t), \qquad 0 < x < L, \quad t > 0,\\
&w(0,t) = w(L,t) = 0, \qquad t > 0,\\
&w(x,0) = f(x) - v(x), \qquad 0 < x < L,
\end{aligned}$$

and the second order problem

$$\begin{aligned}
&v''(x) = -\tfrac{1}{\alpha} g(x) \qquad 0 < x < L,\\
&v(0) = U_1, \quad v(L) = U_2.
\end{aligned}$$

We first solve for $v(x)$, then solve $w(x,t)$, and finally write the solution in terms of $u(x,t)$. If $g(x) = 0$, then, clearly

$$v(x) = (U_2 - U_1)\frac{x}{L} + U_1.$$

The functions $w(x,t)$ and $v(x)$ are called **transient** and **steady state** solutions.

1. Solve the heat flow problem,

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) = 3\frac{\partial^2 u}{\partial x^2}(x,t), \qquad 0 < x < \pi, \quad t > 0,\\
&u(0,t) = u(\pi,t) = 0, \qquad t > 0,\\
&u(x,0) = \sin(x) - 6\sin(4x), \qquad 0 < x < \pi.
\end{aligned}$$

Solution

We suppose that the solution $u(x,t)$ can be written in the following way:

$$\begin{aligned}
u(x,t) &= X(x)T(t)\\
\frac{\partial u}{\partial t}(x,t) &= X(x)T'(t)\\
\frac{\partial^2 u}{\partial x^2}(x,t) &= X''(x)T(t)
\end{aligned}$$

Inserting this assumption into our PDE, we obtain:

$$\begin{aligned}
X(x)T'(t) &= 3X''(x)T(t)\\
\Rightarrow \quad \frac{T'(t)}{3T(t)} &= \frac{X''(x)}{X(x)}
\end{aligned}$$

The only way this can be true is if they are both equal to some constant λ. Hence, we obtain two equations:

$$\begin{aligned}
T'(t) - 3\lambda T(t) &= 0\\
X''(x) - \lambda X(x) &= 0
\end{aligned}$$

Using our initial conditions $u(0,t) = u(\pi,t) = 0$, we can see that for the $X(x)$ equation, we have: $X(0) = X(\pi) = 0$.

Solving the heat flow problem reduces now to solving the three following problems.

1. We now know from previous labs that the eigenvalue problem

$$\begin{aligned}
X''(x) - \lambda X(x) = 0; &\qquad 0 < x < \pi\\
X(0) = 0 &\qquad X(\pi) = 0,
\end{aligned}$$

has eigenvalues

$$\lambda_n = -n^2, \quad n = 1, 2, \ldots$$

with eigenfunctions

$$X_n(x) = b_n \sin(nx),$$

for some arbitrary constants b_n.

2. We can now find a Fourier series representation for the function $f(x) = \sin(x) - 6\sin(4x)$ in terms of these eigenfunctions $X_n(x)$

$$\sin(x) - 6\sin(4x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

From this representation, it is obvious that we have the following values for the coefficients.

$$b_n = \begin{cases} 1 & n = 1\\ -6 & n = 4\\ 0 & \text{otherwise} \end{cases}$$

3. We can now solve the $T(t)$ equation by inserting the eigenvalue $\lambda = -n^2$

$$T_n'(t) + 3n^2 T_n(t) = 0.$$

Solving this equation,we have:

$$T_n(t) = Ae^{-3n^2 t},$$

for some constant A. However, we have to set $T_n(0) = 1$ in order for our previous values of b_n to still be valid. Clearly $A = 1$.

Using the three previous results, the solution to the heat flow problem is then given by

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} X_n(x) T_n(t) \\ &= \sum_{n=1}^{\infty} b_n \sin(nx) e^{-3n^2 t} \\ &= \sin(x)e^{-3t} - 6\sin(4x)e^{-48t} \end{aligned}$$

In summary, the solution to the following heat flow problem:

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= 3\frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) &= u(\pi,t) = 0, \quad t > 0 \\ u(x,0) &= \sin(x) - 6\sin(4x) \end{aligned}$$

is given by

$$\boxed{u(x,t) = \sin(x)e^{-3t} - 6\sin(4x)e^{-48t}}.$$

■

2. Solve the heat flow problem

$$\begin{aligned} &\frac{\partial u}{\partial t}(x,t) = \alpha \frac{\partial^2 u}{\partial x^2}(x,t), \qquad 0 < x < 1, \quad t > 0, \\ &\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0, \qquad t > 0, \\ &u(x,0) = x(1-x), \qquad 0 < x < 1. \end{aligned}$$

Solution

Using the separation of variables method

$$u(x,t) = X(x)T(t), \qquad \frac{\partial u}{\partial t}(x,t) = X(x)T'(t), \qquad \frac{\partial^2 u}{\partial x^2}(x,t) = X''(x)T(t),$$

in the partial differential equation

$$X(x)T'(t) = \alpha X''(x)T(t) \quad \Rightarrow \quad \frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)} = \lambda.$$

Then, solving the heat flow problem reduces now to solving the following three problems.

1. The eigenvalue problem

$$X''(x) - \lambda X(x) = 0, \quad 0 < x < 1; \qquad X'(0) = 0, \qquad X'(1) = 0,$$

has eigenvalues

$$\lambda_n = -n^2\pi^2, \quad n = 0, 1, 2, \dots$$

with eigenfunctions

$$\boxed{X_n(x) = B_n \cos(n\pi x)}, \quad n = 0, 1, 2, \dots$$

2. Using the eigenvalue $\lambda = -n^2\pi^2$ found in *1*, is easy to see that

$$T_n'(t) = -\alpha n^2\pi^2 T_n(t),$$

has solution

$$\boxed{T_n(t) = A_n e^{-\alpha n^2\pi^2 t}}, \quad n = 0, 1, 2, \dots$$

Note. The eigenfunction $X_n = \cos(n\pi x)$ implies that we need the Fourier *cosine* series.

3. The Fourier cosine series representation for the function $f(x) = x(1-x)$

$$x(1-x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x),$$

has coefficients (see a previous lab notes)

$$\boxed{a_0 = \frac{1}{3}, \quad a_n = \frac{2((-1)^{n+1} - 1)}{n^2\pi^2}}, \quad n = 0, 1, 2, \dots$$

Putting the three previous results together, the solution to the heat flow problem is then given by

$$\boxed{u(x,t) = \frac{1}{6} + \sum_{n=1}^{\infty} \frac{2((-1)^{n+1} - 1)}{n^2\pi^2} e^{-\alpha n^2\pi^2 t} \cos(n\pi x)}.$$

■

3. Find a formal solution to the initial value problem,

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \qquad 0 < x < \pi, \quad t > 0, \\
&u(0,t) = 0, \quad u(\pi,t) = 3\pi, \qquad t > 0, \\
&u(x,0) = 0, \qquad 0 < x < \pi.
\end{aligned}$$

Solution

Note that we have non-zero boundary conditions. So, we apply the appropriate change of variable

$$u(x,t) = v(x) + w(x,t) = 3x + w(x,t),$$

since

$$v(x) = (3\pi - 0)\frac{x}{\pi} + 0 = 3x.$$

Then, the new problem is to solve the alternate Heat equation *with zero boundary conditions*

$$\begin{aligned}
&\frac{\partial w}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \qquad 0 < x < \pi, \quad t > 0, \\
&w(0,t) = 0, \quad w(\pi,t) = 0, \qquad t > 0, \\
&w(x,0) = -3x, \qquad 0 < x < \pi.
\end{aligned}$$

Now, we use separation of variables

$$w(x,t) = X(x)T(t),$$

$$\Rightarrow \frac{\partial w}{\partial t}(x,t) = X(x)T'(t), \quad \frac{\partial^2 w}{\partial x^2}(x,t) = X''(x)T(t).$$

Then

$$X(x)T'(t) = X''(x)T(t) \ \Rightarrow \ \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda.$$

Which becomes

$$\begin{aligned} T'(t) - \lambda T(t) &= 0 \\ X''(x) - \lambda X(x) &= 0 \end{aligned}.$$

1. First, we solve the second order eigenvalue problem for $X(x)$.

$$X''(x) - \lambda X(x) = 0, \qquad 0 < x < \pi, \qquad X(x) = X(\pi) = 0.$$

The boundary values come from setting zero boundary conditions in the Heat equation problem (we forced those boundary conditions). This eigenvalue problem has solution

$$\begin{aligned} \lambda_n &= -n^2, \ n = 1, 2, \ldots, \\ X_n(x) &= b_n \sin(nx), \ n = 1, 2, \ldots. \end{aligned}$$

Note: the eigenvalues will be used in the other differential equation (eigenvalue problem) for $T(t)$, while the coefficients b_n will be determined by the initial condition $w(x,0) = -3x$.

2. Now, for the first order eigenvalue problem

$$T'(t) - \lambda T(t) = 0, \quad T(0) = 1,$$

we use the λ_n previously found (we can assume $T(0) = 1$, but why?[1]). This leads to the first order differential equations

$$T_n'(t) + n^2 T_n(t) = 0, \quad T_n(0) = 1.$$

Which has solution ($A = 1$ from initial condition)

$$T_n(t) = Ae^{-n^2 t} = e^{-n^2 t}.$$

3. Since the eigen function are sine, we require

$$-3x = \sum_{i=1}^{\infty} b_n \sin(nx),$$

on $(0, \pi)$, which is the Fourier sine series of $-3x$. Thus, the coefficients are

$$b_n = \frac{2}{\pi}\int_0^{\pi} (-3x)\sin(nx)dx = -\frac{6}{\pi}\left[-\frac{1}{n}x\cos(nx)|_0^{\pi} + \frac{1}{n^2}\sin(nx)|_0^{\pi}\right] = 6\frac{(-1)^n}{n}.$$

4. The next step is to combine $X_n(x)$ and $T_n(t)$ as

$$w(x,t) = \sum_{i=1}^{\infty} X_n(x)T_n(t).$$

[1] $T_n(0) = 1$ for convenience, since we had to determine the coefficients b_n, i.e, $w(x,0) = -3x = \sum T_n(0)X_n(x) = \sum \cancel{T_n(0)}^{1} b_n \sin(nx)$

Why $\sum$? [2]. That is

$$w(x,t) = 6\sum_{i=1}^{\infty} \frac{(-1)^n}{n} \sin(nx)e^{-n^2 t}.$$

Finally, changing back to $u(x,t)$

$$\boxed{u(x,t) = 3x + 6\sum_{i=1}^{\infty} \frac{(-1)^n}{n} \sin(nx)e^{-n^2 t}}.$$

■

4. Find a formal solution to the initial value problem,

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + 6x - 2, \qquad 0 < x < 1, \quad t > 0,\\
&u(0,t) = 0, \quad u(1,t) = -1, \qquad t > 0,\\
&u(x,0) = -x^3, \qquad 0 < x < 1.
\end{aligned}$$

Solution
Note that the non-zero boundary conditions and the external force $6x - 2$ imply we nee to use the change of variable

$$u(x,t) = w(x,t) + v(x).$$

Then,

$$u_t(x,t) = w_t(x,t), \qquad u_{xx}(x,t) = w_{xx}(x,t) +'' v(x).$$

We now use the equation

$$\frac{\partial w}{\partial t}(x,t) = \frac{\partial^2 w}{\partial x^2}(x,t) + v''(x) + 6x - 2,$$

the boundary conditions

$$u(0,t) = w(0,t) + v(0) = 0, \quad u(1,t) = w(1,t) + v(1) = -1,$$

and the initial condition

$$u(x,0) = w(x,0) + v(x) = -x^3.$$

The idea is to force $v(x)$ to absorbe the external force and the non-zero boundary conditions so the heat equation in for w has a familiar form. In other words, we need to solve

$$\begin{aligned}
&\frac{\partial w}{\partial t}(x,t) = \frac{\partial^2 w}{\partial x^2}(x,t), \qquad 0 < x < 1, \quad t > 0,\\
&w(0,t) = w(1,t) = 0, \qquad t > 0,\\
&w(x,0) = -x^3 - v(x), \qquad 0 < x < 1,
\end{aligned}$$

given that $v(x)$ is the solution to

$$v''(x) + 6x - 2 = 0, \quad v(0) = 0, \quad v(1) = -1. \tag{1}$$

The second order equation is easy to solve

$$v(x) = -x^3 + x^2 - x.$$

[2]Superposition principle!

The heat equation with zero BC's and initial condition $w(x,0) = x^3 - x^3 + x^2 - x = x(1-x)$ have solution (exercise)

$$w(x,t) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^3\pi^3} e^{-\alpha n^2\pi^2 t} \sin(n\pi x).$$

Thus, the solution to the original heat problem is $w(x,t) + v(x)$

$$\boxed{u(x,t) = -x(x^2 - x + 1) + \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^3\pi^3} e^{-\alpha n^2\pi^2 t} \sin(n\pi x)}.$$

■

5. Solve the heat flow problem,

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) - 3u, \qquad 0 < x < \pi, \quad t > 0, \\
&\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \qquad t > 0, \\
&u(x,0) = 2 + \cos x - 5\cos 4x, \qquad 0 < x < \pi.
\end{aligned}$$

Solution

Using the separation of variables method

$$u(x,t) = X(x)T(t), \qquad \frac{\partial u}{\partial t}(x,t) = X(x)T'(t), \qquad \frac{\partial^2 u}{\partial x^2}(x,t) = X''(x)T(t),$$

in the partial differential equation

$$X(x)T'(t) = X''(x)T(t) - 3X(x)T(t) \quad \Rightarrow \quad \frac{T'(t)}{T(t)} + 3 = \frac{X''(x)}{X(x)} = \lambda.$$

Note that the 3 term is intencionally placed in the $T(t)$ side, so the eigenvalue problem is known and simple. Then, solving the heat flow problem reduces now to solving the following three problems.

1. The eigenvalue problem

$$X''(x) - \lambda X(x) = 0, \quad 0 < x < \pi; \qquad X'(0) = 0, \qquad X'(\pi) = 0,$$

has eigenvalues

$$\lambda_n = -n^2, \quad n = 0, 1, 2, \ldots$$

with eigenfunctions

$$\boxed{X_n(x) = B_n \cos(nx)}, \quad n = 0, 1, 2, \ldots$$

2. Using the eigenvalue $\lambda = -n^2$ found in *1*, is easy to see that

$$T_n'(t) = -(3 + n^2)T_n(t),$$

has solution

$$\boxed{T_n(t) = A_n e^{-(3+n^2)t}}, \quad n = 0, 1, 2, \ldots$$

Note. The eigenfunction $X_n = \cos(n\pi x)$ implies that we need the Fourier *cosine* series.

3. The Fourier cosine series representation for the function $f(x) = 2 + \cos(x) - 5\cos(4x)$ has coefficients

$$\boxed{a_0 = 2, \quad a_1 = 1, \quad a_4 = -5}.$$

Putting the three previous results together, the solution to the heat flow problem is then given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(3+n^2)t} \sin(nx).$$

$$\Rightarrow \boxed{u(x,t) = 2e^{-3t} + e^{-4t}\cos(x) - 5e^{-19t}\cos(4x)}.$$

■

6. Find a formal solution to the vibrating string problem governed by the given initial-boundary value problem

$$\begin{array}{ll} \dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < \pi, \quad t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = x^2(\pi - x), & 0 < x < \pi \\ \frac{\partial u}{\partial t}(x,0) = \dfrac{(x-\pi)^3}{3}, & 0 < x < \pi \end{array}$$

Solution

Using the method of variation of parameters, the general solution of any wave equation of the form:

$$\begin{array}{rcll} \dfrac{\partial^2 u}{\partial t^2} & = & \alpha^2 \dfrac{\partial^2 u}{\partial x^2}, & 0 < x < L, \quad t > 0 \\ u(0,t) & = & u(L,t) = 0, & t > 0 \\ u(x,0) & = & f(x), & 0 < x < L \\ \dfrac{\partial u}{\partial t}(x,0) & = & g(x), & 0 < x < L \end{array}$$

is given by:

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi\alpha t}{L}\right) + b_n \sin\left(\frac{n\pi\alpha t}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

where the coefficients a_n and b_n are determined from the Fourier Sine Series

$$\begin{array}{rcl} f(x) & = & \displaystyle\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \\ g(x) & = & \displaystyle\sum_{n=1}^{\infty} b_n \left(\frac{n\pi\alpha}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \end{array} \tag{2}$$

In our case, we have: $\alpha = 2, L = \pi, f(x) = x^2(\pi - x)$ and $g(x) = \dfrac{(x-\pi)^3}{3}$.

We need to to find the coefficients a_n and b_n. To do this we have to find the Fourier Sine Series of $f(x) = x^2(\pi - x)$ and $g(x) = \dfrac{(x-\pi)^3}{3}$ on the interval given above, namely $0 < x < \pi$.

The coefficients a_n are given by the following integral:

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^\pi x^2(\pi - x)\sin(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^\pi (\pi x^2 - x^3)\sin(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\left[(\pi x^2 - x^3)\left(-\frac{\cos(nx)}{n}\right)\right]_0^\pi - \frac{2}{\pi}\int_0^\pi (2\pi x - 3x^2)\left(-\frac{\cos(nx)}{n}\right)\mathrm{d}x \\
&= 0 + \frac{2}{n\pi}\int_0^\pi (2\pi x - 3x^2)\cos(nx)\,\mathrm{d}x \\
&= \frac{2}{n\pi}\left[(2\pi x - 3x^2)\left(\frac{\sin(nx)}{n}\right)\right]_0^\pi - \frac{2}{n\pi}\int_0^\pi (2\pi - 6x)\left(\frac{\sin(nx)}{n}\right)\mathrm{d}x \\
&= 0 - \frac{4}{n^2\pi}\int_0^\pi (\pi - 3x)\sin(nx)\,\mathrm{d}x \\
&= 0 - \frac{4}{n^2\pi}\left[(\pi - 3x)\left(-\frac{\cos(nx)}{n}\right)\right]_0^\pi + \frac{4}{n^2\pi}\int_0^\pi (-3)\left(-\frac{\cos(nx)}{n}\right)\mathrm{d}x \\
&= \frac{4}{n^3\pi}\left[(\pi - 3x)\cos(nx)\right]_0^\pi + \frac{12}{n^3\pi}\int_0^\pi \cos(nx)\,\mathrm{d}x \\
&= \frac{4}{n^3\pi}\left[(-2\pi)\cos(n\pi) - \pi\right] + \frac{12}{n^3\pi}\left[\left(\frac{\sin(nx)}{n}\right)\right]_0^\pi \\
&= \frac{4}{n^3}\left(-2(-1)^n - 1\right) + 0 \\
&= \frac{4(2(-1)^{n+1} - 1)}{n^3}
\end{aligned}
$$

If we inspect equation (2) more carefully, we see that the coefficients $b_n\left(\frac{n\pi\alpha}{L}\right)$ satisfy the same integral as the coefficients a_n with the exception that we are evaluating $g(x)$ not $f(x)$. Otherwise stated:

$$\begin{aligned}
b_n\left(\frac{n\pi\alpha}{L}\right) &= \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
\Rightarrow \quad b_n &= \frac{2}{n\pi\alpha}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \frac{1}{n\pi}\int_0^\pi \frac{(x-\pi)^3}{3}\sin(nx)\,\mathrm{d}x \\
&= \frac{1}{n\pi}\left[\frac{(x-\pi)^3}{3}\left(-\frac{\cos(nx)}{n}\right)\right]_0^\pi - \frac{1}{n\pi}\int_0^\pi (x-\pi)^2\left(-\frac{\cos(nx)}{n}\right)\mathrm{d}x \\
&= -\frac{\pi^2}{3n^2} + \frac{1}{n^2\pi}\int_0^\pi (x-\pi)^2\cos(nx)\,\mathrm{d}x \\
&= -\frac{\pi^2}{3n^2} + \frac{1}{n^2\pi}\left[(x-\pi)^2\left(\frac{\sin(nx)}{n}\right)\right]_0^\pi - \frac{1}{n^2\pi}\int_0^\pi 2(x-\pi)\left(\frac{\sin(nx)}{n}\right)\mathrm{d}x \\
&= -\frac{\pi^2}{3n^2} + 0 - \frac{2}{n^3\pi}\int_0^\pi (x-\pi)\sin(nx)\,\mathrm{d}x \\
&= -\frac{\pi^2}{3n^2} - \frac{2}{n^3\pi}\left[(x-\pi)\left(-\frac{\cos(nx)}{n}\right)\right]_0^\pi + \frac{2}{n^3\pi}\int_0^\pi (1)\left(-\frac{\cos(nx)}{n}\right)\mathrm{d}x \\
&= -\frac{\pi^2}{3n^2} - \frac{2}{n^4\pi}\left[0-(-\pi)(-1)\right] - \frac{2}{n^4\pi}\left[\frac{\sin(nx)}{n}\right]_0^\pi \\
&= -\frac{\pi^2}{3n^2} + \frac{2}{n^4} - 0 \\
&= \frac{2}{n^4} - \frac{\pi^2}{3n^2}
\end{aligned}$$

Hence, our general solution is given by:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left[\left(\frac{4(2(-1)^{n+1}-1)}{n^3}\right)\cos(2nt) + \left(\frac{2}{n^4} - \frac{\pi^2}{3n^2}\right)\sin(2nt)\right]\sin(nx)}.$$

■