MATH 201 Differential Equations – University of Alberta
Winter 2018 – Labs – Carlos Contreras
Authors: Carlos Contreras and Philippe Gaudreau

Lab 2: Second-order homogeneous differential equations

1. Find a general solution to the given differential equation

$$3y'' + 11y' - 7y = 0$$

2. Find a general solution to the given differential equation

$$y'' + y' - 6y = 0, \quad y(0) = 2, \quad y'(0) = -\frac{17}{2}$$

3. Find a solution to the given initial value differential equation

$$z'' - 2z' - 2z = 0, \quad z(0) = 0, \quad z'(0) = 3$$

4. Find a general solution to the given differential equation

$$y'' - 10y' + 26y = 0.$$

5. Find a solution to the given inital value differential equation

$$y'' - 2y' + 2y = 0, \quad y(\pi) = e^{\pi}, \quad y'(\pi) = 0.$$

6. Find a general solution to the given differential equation

$$4w'' + 20w' + 25w = 0.$$

7. Find a solution to the given initial value differential equation for $b = 5, 4$ and 2

$$y'' + by' + 4y = 0, \quad , y(0) = 1, \quad y'(0) = 0,$$

sketch the three solutions for $t > 0$ and see how the solutions change with b.

Solutions

All problems from: Nagel, Saff & Sneider, *Fundamentals of Differential Equations*, Eight Edition, Adisson–Wesley.

→ For the **second-order equation homogeneous with constant coefficients**

$$ay'' + by' + cy = 0$$

there are three posible solutions depending on the roots r_1 and r_2 of the auxiliary equation

$$ar^2 + br + c = 0.$$

The three cases are:

$$\begin{aligned}
&\text{Case 1:}\quad y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}, \quad \text{if} \quad r_1 \neq r_2,\\
&\text{Case 2:}\quad y(t) = C_1 e^{r_1 t} + C_2 t e^{r_1 t}, \quad \text{if} \quad r_1 = r_2,\\
&\text{Case 3:}\quad y(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t), \quad \text{if} \quad r_1, r_2 = \alpha \pm i\beta.
\end{aligned}$$

1. Find a general solution to the given differential equation

$$3y'' + 11y' - 7y = 0$$

Solution
The auxiliary equation for this ODE is the following

$$3r^2 + 11r - 7 = 0 \Rightarrow r = \frac{-11 \pm \sqrt{205}}{6}.$$

The general solution is then given by:

$$\boxed{z(t) = C_1 e^{(-11+\sqrt{205})t/6} + C_2 e^{-(11+\sqrt{205})t/6}}.$$

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2. Find a general solution to the given differential equation

$$y'' + y' - 6y = 0, \quad y(0) = 2, y'(0) = -\frac{17}{2}$$

Solution
The auxiliary equation for this ODE is the following

$$r^2 + r - 6 = 0 \Rightarrow r = 2, \text{ and } r = -3.$$

The general solution is then given by:

$$y(t) = C_1 e^{2t} + C_2 e^{-3t}.$$

Taking the derivative, we find:

$$y'(t) = 2C_1 e^{2t} - 3C_2 e^{-3t}.$$

Setting $t = 0$ in both equation, we obtain:

$$\begin{aligned} 2 &= C_1 + C_2, \\ -\frac{17}{2} &= 2C_1 - 3C_2. \end{aligned}$$

Solving this system, we obtain: $C_1 = -\frac{1}{2}$ and $C_2 = \frac{5}{2}$. Hence, the solution to this IVP is

$$\boxed{y(t) = -\frac{1}{2}e^{2t} + \frac{5}{2}e^{-3t}}.$$

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3. Find a solution to the given initial value differential equation

$$z'' - 2z' - 2z = 0, \quad z(0) = 0, \quad z'(0) = 3$$

Solution
The auxiliary equation for this ODE is the following

$$r^2 - 2r - 2 = 0 \Rightarrow r = 1 \pm \sqrt{3}$$

The general solution is then given by:

$$z(t) = C_1 e^{(1+\sqrt{3})t} + C_2 e^{(1-\sqrt{3})t}$$

Taking the derivative, we find:

$$z'(t) = C_1(1+\sqrt{3})e^{(1+\sqrt{3})t} + C_2(1-\sqrt{3})e^{(1-\sqrt{3})t}$$

Setting $t = 0$ in both equation, we obtain:

$$\begin{aligned} 0 &= C_1 + C_2 \\ 3 &= C_1(1+\sqrt{3}) + C_2(1-\sqrt{3}) \end{aligned}$$

Solving this system, we obtain: $C_1 = \dfrac{\sqrt{3}}{2}$ and $C_2 = -\dfrac{\sqrt{3}}{2}$. Hence the solution to this IVP is

$$\boxed{z(t) = \frac{\sqrt{3}}{2}e^{(1+\sqrt{3})t} - \frac{\sqrt{3}}{2}e^{(1-\sqrt{3})t}}.$$

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4. Find a general solution to the given differential equation

$$y'' - 10y' + 26y = 0$$

Solution
The auxiliary equation for this ODE is the following

$$r^2 - 10r + 26 = 0 \Rightarrow r = 5 \pm i$$

The solution is then given by:

$$\boxed{y(t) = C_1 e^{5t}\cos(t) + C_2 e^{5t}\sin(t)}.$$

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5. Find a solution to the given inital value differential equation

$$y'' - 2y' + 2y = 0, \quad y(\pi) = e^{\pi}, \quad y'(\pi) = 0$$

Solution
The auxiliary equation for this ODE is the following

$$r^2 - 2r + 2 = 0 \Rightarrow r = 1 \pm i.$$

The general solution is then given by:

$$y(t) = C_1 e^t \cos(t) + C_2 e^t \sin(t).$$

Taking the derivative, we find:

$$y'(t) = C_1 e^t \cos(t) - C_1 e^t \sin(t) + C_2 e^t \sin(t) + C_2 e^t \cos(t).$$

Setting $t = \pi$ in both equation, we obtain:

$$\begin{aligned} e^{\pi} &= C_1 e^{\pi} \cancelto{-1}{\cos\pi} + C_2 e^{\pi} \cancelto{0}{\sin\pi}, \\ 0 &= C_1 e^{\pi} \cancelto{-1}{\cos\pi} - C_1 e^{\pi} \cancelto{0}{\sin\pi} + C_2 e^{\pi} \cancelto{0}{\sin\pi} + C_2 e^{\pi} \cancelto{-1}{\cos\pi}. \end{aligned}$$

Solving this system, we obtain: $C_1 = -1$ and $C_2 = 1$. Hence, the solution to this IVP is

$$\boxed{y(t) = -e^t \cos(t) + e^t \sin(t)}.$$

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6. Find a general solution to the given differential equation

$$4w'' + 20w' + 25w = 0$$

Solution
The auxiliary equation for this ODE is the following

$$4r^2 + 20r + 25 = 0 \Rightarrow r = -\frac{5}{2}$$

The solution is then given by:

$$\boxed{w(t) = C_1 e^{-\frac{5}{2}t} + C_2 t e^{-\frac{5}{2}t}}.$$

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7. Find a solution to the given initial value differential equation for $b = 5, 4$ and 2

$$y'' + by' + 4y = 0, \quad , y(0) = 1, \quad y'(0) = 0, \tag{1}$$

sketch the three solutions for $t > 0$ and see how the solutions change with b.

Solution
The auxiliary equation for this ODE is the following

$$r^2 + br + 4 = 0 \Rightarrow r = \frac{-b \pm \sqrt{b^2 - 16}}{2} = \frac{-b \pm \sqrt{\Delta}}{2},$$

and we have three options depending on the sign of Δ

$$\Delta_5 = 25 - 16 = 9 > 0$$
$$\Delta_4 = 16 - 16 = 0$$
$$\Delta_2 = 4 - 16 = -12 < 0$$

These values where chosen so we have the three posible options: distinct real roots, repeated real root, and complex roots.
For b = 5. The roots are $r = -1$ and $r = -4$ (distinct real roots case). The general solution and its derivative are given by

$$y(t) = C_1 e^{-t} + C_2 e^{-4t}$$
$$y'(t) = -C_1 e^{-t} - 4C_2 e^{-4t}$$

Using the initial conditions we obtain

$$\begin{aligned} 1 &= C_1 + C_2, \\ 0 &= -C_1 - 4C_2. \end{aligned}$$

Solving this system, we obtain $C_1 = \frac{4}{3}$ and $C_2 = -\frac{1}{3}$. Hence, the solution to this IVP is

$$\boxed{y(t) = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}}.$$

For b = 4. The roots is $r = -2$ (repeated real root case). The general solution and its derivative are given by

$$y(t) = C_1 e^{-2t} + C_2 t e^{-2t}$$
$$y'(t) = -2C_1 e^{-t} + C_2 e^{-2t} - 2C_2 t e^{-2t}$$

Using the initial conditions we obtain

$$\begin{aligned} 1 &= C_1, \\ 0 &= -2C_1 + C_2. \end{aligned}$$

Solving this system, we obtain $C_1 = 1$ and $C_2 = 2$. Hence, the solution to this IVP is

$$\boxed{y(t) = e^{-2t} + 2te^{-2t}}.$$

For b = 2. The roots is $r = -1 \pm \sqrt{3}$ (complex roots case). The general solution and its derivative are given by

$$y(t) = C_1 e^{-t}\cos(\sqrt{3}t) + C_2 e^{-t}\sin(\sqrt{3}t)$$
$$y'(t) = -C_1 e^{-t}\cos(\sqrt{3}t) - \sqrt{3}C_1 e^{-t}\sin(\sqrt{3}t) - C_2 e^{-t}\sin(\sqrt{3}t) + \sqrt{3}C_2 e^{-t}\cos(\sqrt{3}t)$$

Using the initial conditions we obtain

$$\begin{aligned} 1 &= C_1 \cos 0 + C_2 \sin 0, \\ 0 &= -C_1 \cos 0 - \sqrt{3}C_1 \sin 0 - C_2 \sin 0 + \sqrt{3}C_2 \cos 0. \end{aligned}$$

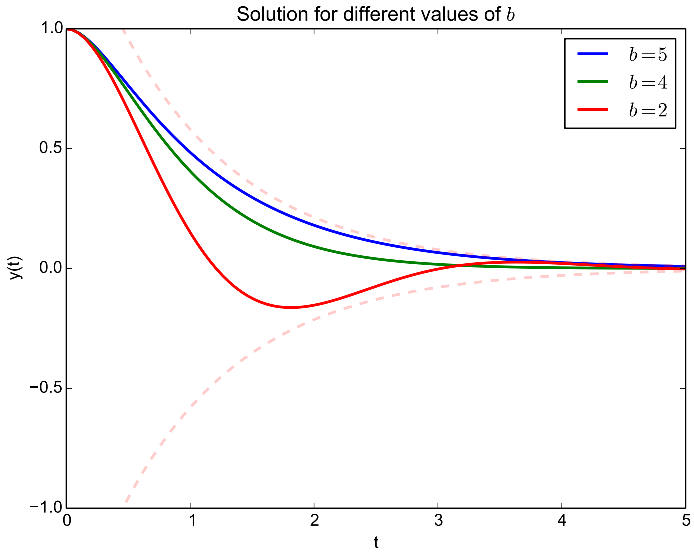

Figure 1: Solution to IVP (1) for $b = 5, 4$ and 2.

Solving this system, we obtain $C_1 = 1$ and $C_2 = \frac{\sqrt{3}}{3}$. Hence, the solution to this IVP is

$$\boxed{y(t) = e^{-t}\cos(\sqrt{3}t) + \frac{\sqrt{3}}{3}e^{-t}\sin(\sqrt{3}t)}.$$

Decreasing b from 5 to 4 the tail of the curve becomes heavier (see Figure 1), and at $b = 4$ the the roots crosses to the imaginary part and the solution oscilates. Note that each solution (particularly in the complex case) is bounded, e.g, by $\pm 2e^{-t}$.

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