MATH 201[1] – Final exam summary study guide[2]

Laplace transform

For $f(t)$ piecewise continuous and of exponential order

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty f(t)e^{-st}dt \tag{1}$$

Properties and definitions

$$\textbf{Linearity: } \mathcal{L}\{af + bg\} = aF(s) + bG(s) \tag{2}$$

$$\textbf{Inverse: } \mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}(s)\}(t) = f(t) \tag{3}$$

$$\textbf{Unit step function: } u_a(t) = u(t-a) = \begin{cases} 0, & t < a \\ 1, & a < t \end{cases} \tag{4}$$

$$\textbf{Dirac delta function: } \delta(t-a) = \begin{cases} \infty, & t = a \\ 0, & t \neq a \end{cases} \tag{5}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a) \tag{6}$$

$$\textbf{Convolution: } (f*g)(t) = \int_0^t f(t-v)g(v)dv = \int_0^t f(v)g(t-v)dv \tag{7}$$

Table of Laplace Transforms

$$\mathcal{L}\{1\} = \frac{1}{s} \tag{8}$$

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}, \quad s > a \tag{9}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \tag{10}$$

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2} \tag{11}$$

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2} \tag{12}$$

$$\mathcal{L}\{\sinh bt\} = \frac{b}{s^2-b^2} \quad s > |a| \tag{13}$$

$$\mathcal{L}\{\cosh bt\} = \frac{s}{s^2-b^2} \quad s > |a| \tag{14}$$

$$\mathcal{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2} \quad s > a \tag{15}$$

$$\mathcal{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \quad s > a \tag{16}$$

$$\mathcal{L}\left\{te^{at}\right\} = \frac{1}{(s-a)^2} \quad s > a \tag{17}$$

$$\mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \quad s > a \tag{18}$$

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a) \tag{19}$$

$$\mathcal{L}\{u(t-a)\} = \frac{1}{s}e^{-as} \tag{20}$$

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s) \tag{21}$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as} \tag{22}$$

$$\mathcal{L}\{f(x)\delta(t-a)\} = f(a)e^{-as} \tag{23}$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) \tag{24}$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0) \tag{25}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \tag{26}$$

$$\mathcal{L}\left\{\int_0^s f(\tau)d\tau\right\} = \frac{1}{s}F(s) \tag{27}$$

$$\mathcal{L}\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty F(\sigma)d\sigma \tag{28}$$

$$\mathcal{L}\{(f*g)(t)\} = F(s)G(s) \tag{29}$$

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right) \quad c > 0 \tag{30}$$

$$\mathcal{L}\{f(t)u(t-a)\} = \mathcal{L}\{f(t+a)\}e^{-as} \tag{31}$$

Periodic function $f(t)$ with period T

$$\mathcal{L}\{f\}(s) = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)dt \tag{32}$$

Laplace transform of an IVP

For the IVP

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_0', \tag{33}$$

the Laplace transform of the solution $y(t)$ is

$$Y(s) = \underbrace{\frac{(as+b)y_0 + ay_0'}{as^2+bs+c}}_{\text{initial conditions}} + \underbrace{\frac{G(s)}{as^2+bs+c}}_{\text{particular sol.}}. \tag{34}$$

Use partial fractions and complete squares methods along with entries in the table (mainly (8)–(12), (15)–(17), (21)–(22), and (29)), to find $y(t)$.

Some important functions

$$\textbf{Transfer function: } H(s) = \frac{1}{as^2+bs+c} \tag{35}$$

$$\textbf{Impulse function: } h(t) = \mathcal{L}^{-1}\{H(s)\}(t) \tag{36}$$

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[1]MATH 201 - Differential Equations – University of Alberta

Heat Equation Problem

Eigenvalue problem

Homogeneous Dirichlet boundary conditions

$$\begin{cases} X'' - \lambda X = 0 & 0 < x < L \\ X(0) = X(L) = 0 \end{cases} \tag{37}$$

has eigenvalues and eigenfunctions solution

$$\textbf{Eigenvalues: } \lambda_n = -\left(\frac{n\pi}{L}\right)^2, \tag{38}$$

$$\textbf{Eigenfunctions: } X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right), \quad n \geq 1 \tag{39}$$

Homogeneous Newman boundary conditions

$$\begin{cases} X'' - \lambda X = 0 & 0 < x < L \\ X'(0) = X'(L) = 0 \end{cases} \tag{40}$$

has eigenvalues and eigenfunctions solution

$$\textbf{Eigenvalues: } \lambda_n = -\left(\frac{n\pi}{L}\right)^2, \tag{41}$$

$$\textbf{Eigenfunctions: } X_n(x) = B_n \cos\left(\frac{n\pi x}{L}\right), \quad n \geq 0 \tag{42}$$

Fourier series

For $f(x)$ piecewise continuous on $[-L, L]$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \tag{43}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \qquad n \geq 0 \tag{44}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \qquad n \geq 1 \tag{45}$$

For all $|x| \leq L$, $F(x)$ **converges** to

$$F(x) = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x, \\ \frac{f(x-)+f(x+)}{2} & \text{if } f \text{ is discontinuous at } x, \\ \frac{f(-L+)+f(L-)}{2} & \text{if } x = L \text{ or } x = -L. \end{cases} \tag{46}$$

Properties of even and odd functions

0. even: $f(-x) = f(x)$, odd: $-f(-x) = f(x)$
1. even $\pm$ even $=$ even, even $\times$ even $=$ even
2. odd $\pm$ odd $=$ odd, odd $\times$ odd $=$ even
3. odd $\pm$ even $=$ none, odd $\times$ even $=$ odd
4. $f(x)$ even, then $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$
5. $f(x)$ odd, then $\int_{-L}^{L} f(x)dx = 0$
6. $f(x)$ even, then $b_n = 0$, and $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$.
7. $f(x)$ odd, then $a_n = 0$, and $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$.

Fourier cosine (sine) series of $f(x)$ defined on $[0, L]$ is an even (odd) extension of $f(x)$ to $[-L, L]$, using property 6 (7).

Heat equation

The Heat equation can be solved using separation of variable

$$\textbf{Heat equation: } u_t = \alpha u_{xx} \tag{47}$$

$$\textbf{Sep. of variables: } u(x,t) = X(x)T(t) \tag{48}$$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{\alpha T} = \lambda \tag{49}$$

For eigenvalues λ_n, the ODE for $T(t)$ in (49) has solution

$$\frac{T_n'}{\alpha T_n} = \lambda_n \quad \Rightarrow \quad T_n(t) = A_n e^{\alpha \lambda_n t} \tag{50}$$

By superposition principle and (48), infinitely many solutions give

$$u(x,t) = \sum_n X_n(x)T_n(t) = \sum_n X_n(x)T_n(t) \tag{51}$$

Use Boundary Conditions and Inital Conditions to determine the associated eigenvalue and Fourier series problems, respectively.

Dirichlet BC: $u(0,t) = u(L,t) = 0$ result in an eigenvalue problem of the form (37) and a Fourier sine series of $f(x)$

$$\begin{cases} u_t = \alpha u_{xx}, & 0 < x < L, \quad t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases} \tag{52}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) \tag{53}$$

Newmann BC: $u_x(0,t) = u_x(L,t) = 0$ result in an eigenvalue problem of the form (40) and a Fourier cosine series of $f(x)$

$$\begin{cases} u_t = \alpha u_{xx}, & 0 < x < L, \quad t > 0 \\ u_x(0,t) = u_x(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases} \tag{54}$$

$$\Rightarrow u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right) \tag{55}$$

External force $g(x)$ and/or Nonhomogeneous BC:

$$\begin{cases} u_t = \alpha u_{xx} + g(x), & 0 < x < L, \quad t > 0 \\ u(0,t) = U_0, \quad u(L,t) = U_L, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases} \tag{56}$$

$$\text{use } u(x,t) = v(x) + w(x,t) \tag{57}$$

to get a second order problem on $v(x)$ and (52) problem on $w(x,t)$

$$\begin{cases} v''(x) = -\frac{1}{\alpha} g(x) \\ v(0) = U_0, \quad v(L) = U_L \end{cases}, \begin{cases} w_t = \alpha w_{xx}, \\ w(0,t) = w(L,t) = 0, \\ w(x,0) = f(x) - v(x) \end{cases} \tag{58}$$

First solve for $v(x)$, then use it to solve for $w(x,t)$

$$\text{If} \quad g(x) = 0 \quad \Rightarrow \quad v(x) = (U_2 - U_1)\frac{x}{L} + U_1 \tag{59}$$

$w(x,t)$ and $v(x)$ are called **transient** and **steady state** solutions

Reaction-diffusion: put all constants on the t side of (49)

$$u_t = \alpha u_{xx} + \mu u \quad \Rightarrow \quad \frac{X''}{X} = \frac{T'}{\alpha T} - \mu = \lambda \tag{60}$$

Wave Equation Problem

The Wave equation can be solved using separation of variable

$$\textbf{Wave equation: } u_{tt} = \alpha^2 u_{xx} \tag{61}$$

$$\textbf{Sep. of variables: } u(x,t) = X(x)T(t) \tag{62}$$

$$\Rightarrow \frac{X''}{X} = \frac{T''}{\alpha^2 T} = \lambda \tag{63}$$

For eigenvalues $\lambda_n \leq 0$, the ODE for $T(t)$ in (63) has solution

$$\textbf{ODE for } T(t)\textbf{: } T_n'' - \lambda_n \alpha^2 T_n = 0 \tag{64}$$

$$\Rightarrow T_n(t) = a_n \cos(\alpha\sqrt{-\lambda_n}t) + b_n \sin(\alpha\sqrt{-\lambda_n}t) \tag{65}$$

By superposition principle and (62), infinitely many solutions give

$$u(x,t) = \sum_n X_n(x)T_n(t) = \sum_n X_n(x)T_n(t) \tag{66}$$

Use Boundary Conditions and Inital Conditions to determine the asociated eigenvalue and Fourier series problems, respectively.

Dirichlet BC: $u(0,t) = u(L,t) = 0$ result in an eigenvalue problem of the form (37) and a Fourier sine series

$$\begin{cases} u_{tt} = \alpha^2 u_{xx}, & 0 < x < L, \quad t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \\ u_t(x,0) = g(x), & 0 < x < L \end{cases} \tag{67}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{\alpha n \pi t}{L}\right) + b_n \sin\left(\frac{\alpha n \pi t}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right) \tag{68}$$

To find coefficients a_n, find the Fourier sine series using the first initial condition

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \tag{69}$$

To find coefficients b_n, find the Fourier sine series using the second initial condition and

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{\alpha n \pi}{L} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{70}$$

Newmann BC: $u(0,t) = u(L,t) = 0$ result in an eigenvalue problem of the form (40) and a Fourier cosine series

$$\begin{cases} u_{tt} = \alpha^2 u_{xx}, & 0 < x < L, \quad t > 0 \\ u_x(0,t) = u_x(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \\ u_t(x,0) = g(x), & 0 < x < L \end{cases} \tag{71}$$

$$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} \left[a_n \cos\left(\frac{\alpha n \pi t}{L}\right) + b_n \sin\left(\frac{\alpha n \pi t}{L}\right) \right] \cos\left(\frac{n\pi x}{L}\right) \tag{72}$$

To find coefficients a_n, find the Fourier cosine series using the first initial condition

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \tag{73}$$

To find coefficients b_n, find the Fourier cosine series using the second initial condition and

$$u_t(x,0) = g(x) = \sum_{n=0}^{\infty} \frac{\alpha n \pi}{L} b_n \cos\left(\frac{n\pi x}{L}\right) \tag{74}$$