

MATH 201¹ – Final exam summary study guide²

Laplace transform

For $f(t)$ piecewise continuous and of exponential order

$$(1) \quad \mathcal{L}\{f(t)\}(s) = F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

Properties and definitions

$$(2) \quad \textbf{Linearity: } \mathcal{L}\{af + bg\} = aF(s) + bG(s)$$

$$(3) \quad \textbf{Inverse: } \mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}(s)\}(t) = f(t)$$

$$(4) \quad \textbf{Unit step function: } u_a(t) = u(t-a) = \begin{cases} 0, & t < a \\ 1, & a < t \end{cases}$$

$$(5) \quad \textbf{Dirac delta function: } \delta(t-a) = \begin{cases} \infty, & t = a \\ 0, & t \neq a \end{cases}$$

$$(6) \quad \int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

$$(7) \quad \textbf{Convolution: } (f * g)(t) = \int_0^t f(t-v)g(v)dv \\ = \int_0^t f(v)g(t-v)dv$$

$$(20) \quad \mathcal{L}\{u(t-a)\} = \frac{1}{s}e^{-as}$$

$$(21) \quad \mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

$$(22) \quad \mathcal{L}\{\delta(t-a)\} = e^{-as}$$

$$(23) \quad \mathcal{L}\{f(x)\delta(t-a)\} = f(a)e^{-as}$$

$$(24) \quad \mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$(25) \quad \mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$(26) \quad \mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

$$(27) \quad \mathcal{L}\left\{\int_0^s f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

$$(28) \quad \mathcal{L}\left\{\frac{1}{t}f(t)\right\} = \int_s^{\infty} F(\sigma)d\sigma$$

$$(29) \quad \mathcal{L}\{(f * g)(t)\} = F(s)G(s)$$

$$(30) \quad \mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right) \quad c > 0$$

$$(31) \quad \mathcal{L}\{f(t)u(t-a)\} = \mathcal{L}\{f(t+a)\}e^{-as}$$

Table of Laplace Transforms

$$(8) \quad \mathcal{L}\{1\} = \frac{1}{s}$$

$$(9) \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

$$(10) \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$(11) \quad \mathcal{L}\{\sin bt\} = \frac{b}{s^2 + b^2}$$

$$(12) \quad \mathcal{L}\{\cos bt\} = \frac{s}{s^2 + b^2}$$

$$(13) \quad \mathcal{L}\{\sinh bt\} = \frac{b}{s^2 - b^2} \quad s > |a|$$

$$(14) \quad \mathcal{L}\{\cosh bt\} = \frac{s}{s^2 - b^2} \quad s > |a|$$

$$(15) \quad \mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2} \quad s > a$$

$$(16) \quad \mathcal{L}\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2 + b^2} \quad s > a$$

$$(17) \quad \mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2} \quad s > a$$

$$(18) \quad \mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \quad s > a$$

$$(19) \quad \mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

Periodic function $f(t)$ with period T

$$(32) \quad \mathcal{L}\{f\}(s) = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st}f(t)dt$$

Laplace transform of an IVP

For the IVP

$$(33) \quad ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y'_0,$$

the Laplace transform of the solution $y(t)$ is

$$(34) \quad Y(s) = \underbrace{\frac{(as+b)y_0 + ay'_0}{as^2 + bs + c}}_{\text{initial conditions}} + \underbrace{\frac{G(s)}{as^2 + bs + c}}_{\text{particular sol.}}$$

Use partial fractions and complete squares methods along with entries in the table (mainly (8)–(12), (15)–(17), (21)–(22), and (29)), to find $y(t)$.

Some important functions

$$(35) \quad \textbf{Transfer function: } H(s) = \frac{1}{as^2 + bs + c}$$

$$(36) \quad \textbf{Impulse function: } h(t) = \mathcal{L}^{-1}\{H(s)\}(t)$$

¹MATH 201 - Differential Equations – University of Alberta

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Heat Equation Problem

Eigenvalue problem

Homogeneous Dirichlet boundary conditions

$$(37) \quad \begin{cases} X'' - \lambda X = 0 & 0 < x < L \\ X(0) = X(L) = 0 \end{cases}$$

has eigenvalues and eigenfunctions solution

$$(38) \quad \text{Eigenvalues: } \lambda_n = -\left(\frac{n\pi}{L}\right)^2,$$

$$(39) \quad \text{Eigenfunctions: } X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right), \quad n \geq 1$$

Homogeneous Newman boundary conditions

$$(40) \quad \begin{cases} X'' - \lambda X = 0 & 0 < x < L \\ X'(0) = X'(L) = 0 \end{cases}$$

has eigenvalues and eigenfunctions solution

$$(41) \quad \text{Eigenvalues: } \lambda_n = -\left(\frac{n\pi}{L}\right)^2,$$

$$(42) \quad \text{Eigenfunctions: } X_n(x) = B_n \cos\left(\frac{n\pi x}{L}\right), \quad n \geq 0$$

Fourier series

For $f(x)$ piecewise continuous on $[-L, L]$

$$(43) \quad F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$(44) \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0$$

$$(45) \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1$$

For all $|x| \leq L$, $F(x)$ converges to

$$(46) \quad F(x) = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x, \\ \frac{f(x-) + f(x+)}{2} & \text{if } f \text{ is discontinuous at } x, \\ \frac{f(-L+) + f(L-)}{2} & \text{if } x = L \text{ or } x = -L. \end{cases}$$

Properties of even and odd functions

0. even: $f(-x) = f(x)$, odd: $-f(-x) = f(x)$
1. even \pm even = even, even \times even = even
2. odd \pm odd = odd, odd \times odd = even
3. odd \pm even = none, odd \times even = odd
4. $f(x)$ even, then $\int_{-L}^L f(x) dx = 2 \int_0^L f(x) dx$
5. $f(x)$ odd, then $\int_{-L}^L f(x) dx = 0$
6. $f(x)$ even, then $b_n = 0$, and $a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$.
7. $f(x)$ odd, then $a_n = 0$, and $b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$.

Fourier cosine (sine) series of $f(x)$ defined on $[0, L]$ is an even (odd) extension of $f(x)$ to $[-L, L]$, using property 6 (7).

Heat equation

The Heat equation can be solved using separation of variable

$$(47) \quad \text{Heat equation: } u_t = \alpha u_{xx}$$

$$(48) \quad \text{Sep. of variables: } u(x, t) = X(x)T(t)$$

$$(49) \quad \Rightarrow \frac{X''}{X} = \frac{T'}{\alpha T} = \lambda$$

For eigenvalue λ_n , the ODE for $T(t)$ in (49) has solution

$$(50) \quad \frac{T'_n}{\alpha T_n} = \lambda_n \Rightarrow T_n(t) = A_n e^{\alpha \lambda_n t}$$

By superposition principle and (48), infinitely many solutions give

$$(51) \quad u(x, t) = \sum_n X_n(x) T_n(t) = \sum_n X_n(x) T_n(t)$$

Use Boundary Conditions and Initial Conditions to determine the associated eigenvalue and Fourier series problems, respectively.

Dirichlet BC: $u(0, t) = u(L, t) = 0$ result in an eigenvalue problem of the form (37) and a Fourier sine series of $f(x)$

$$(52) \quad \begin{cases} u_t = \alpha u_{xx}, & 0 < x < L, \quad t > 0 \\ u(0, t) = u(L, t) = 0, & t > 0 \\ u(x, 0) = f(x), & 0 < x < L \end{cases}$$

$$(53) \quad \Rightarrow u(x, t) = \sum_{n=1}^{\infty} b_n e^{-\alpha \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

Newmann BC: $u_x(0, t) = u_x(L, t) = 0$ result in an eigenvalue problem of the form (40) and a Fourier cosine series of $f(x)$

$$(54) \quad \begin{cases} u_t = \alpha u_{xx}, & 0 < x < L, \quad t > 0 \\ u_x(0, t) = u_x(L, t) = 0, & t > 0 \\ u(x, 0) = f(x), & 0 < x < L \end{cases}$$

$$(55) \quad \Rightarrow u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\alpha \left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

External force $g(x)$ and/or Nonhomogeneous BC:

$$(56) \quad \begin{cases} u_t = \alpha u_{xx} + g(x), & 0 < x < L, \quad t > 0 \\ u(0, t) = U_0, \quad u(L, t) = U_L, & t > 0 \\ u(x, 0) = f(x), & 0 < x < L \end{cases}$$

$$(57) \quad \text{use } u(x, t) = v(x) + w(x, t)$$

to get a second order problem on $v(x)$ and (52) problem on $w(x, t)$

$$(58) \quad \begin{cases} v''(x) = -\frac{1}{\alpha} g(x) \\ v(0) = U_0, \quad v(L) = U_L \end{cases}, \quad \begin{cases} w_t = \alpha w_{xx}, \\ w(0, t) = w(L, t) = 0, \\ w(x, 0) = f(x) - v(x) \end{cases}$$

First solve for $v(x)$, then use it to solve for $w(x, t)$

$$(59) \quad \text{If } g(x) = 0 \Rightarrow v(x) = (U_2 - U_1) \frac{x}{L} + U_1$$

$w(x, t)$ and $v(x)$ are called **transient** and **steady state** solutions

Reaction-diffusion: put all constants on the t side of (49)

$$(60) \quad u_t = \alpha u_{xx} + \mu u \Rightarrow \frac{X''}{X} = \frac{T'}{\alpha T} - \mu = \lambda$$