# MATH 201[1] – Final exam summary study guide[2]

## Laplace transform

For $f(t)$ piecewise continuous and of exponential order

$$(1) \qquad \mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty f(t)e^{-st}dt$$

## Properties and definitions

(2) **Linearity:** $\mathcal{L}\{af + bg\} = aF(s) + bG(s)$

(3) **Inverse:** $\mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}(s)\}(t) = f(t)$

(4) **Unit step function:** $u_a(t) = u(t-a) = \begin{cases} 0, & t < a \\ 1, & a < t \end{cases}$

(5) **Dirac delta function:** $\delta(t-a) = \begin{cases} \infty, & t = a \\ 0, & t \neq a \end{cases}$

$$(6) \qquad \int_{-\infty}^\infty f(t)\delta(t-a)dt = f(a)$$

(7) **Convolution:** $(f * g)(t) = \int_0^t f(t-v)g(v)dv$
$$= \int_0^t f(v)g(t-v)dv$$

## Table of Laplace Transforms

$$(8) \qquad \mathcal{L}\{1\} = \frac{1}{s}$$

$$(9) \qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

$$(10) \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$(11) \qquad \mathcal{L}\{\sin bt\} = \frac{b}{s^2 + b^2}$$

$$(12) \qquad \mathcal{L}\{\cos bt\} = \frac{s}{s^2 + b^2}$$

$$(13) \qquad \mathcal{L}\{\sinh bt\} = \frac{b}{s^2 - b^2} \quad s > |a|$$

$$(14) \qquad \mathcal{L}\{\cosh bt\} = \frac{s}{s^2 - b^2} \quad s > |a|$$

$$(15) \qquad \mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2 + b^2} \quad s > a$$

$$(16) \qquad \mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2 + b^2} \quad s > a$$

$$(17) \qquad \mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2} \quad s > a$$

$$(18) \qquad \mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \quad s > a$$

$$(19) \qquad \mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$(20) \qquad \mathcal{L}\{u(t-a)\} = \frac{1}{s}e^{-as}$$

$$(21) \qquad \mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

$$(22) \qquad \mathcal{L}\{\delta(t-a)\} = e^{-as}$$

$$(23) \qquad \mathcal{L}\{f(x)\delta(t-a)\} = f(a)e^{-as}$$

$$(24) \qquad \mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$(25) \qquad \mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

$$(26) \qquad \mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

$$(27) \qquad \mathcal{L}\left\{\int_0^s f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

$$(28) \qquad \mathcal{L}\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty F(\sigma)d\sigma$$

$$(29) \qquad \mathcal{L}\{(f * g)(t)\} = F(s)G(s)$$

$$(30) \qquad \mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right) \quad c > 0$$

$$(31) \qquad \mathcal{L}\{f(t)u(t-a)\} = \mathcal{L}\{f(t+a)\}e^{-as}$$

**Periodic function** $f(t)$ with period $T$

$$(32) \qquad \mathcal{L}\{f\}(s) = \frac{1}{1 - e^{-Ts}}\int_0^T e^{-st}f(t)dt$$

## Laplace transform of an IVP

For the IVP

$$(33) \qquad ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_0',$$

the Laplace transform of the solution $y(t)$ is

$$(34) \qquad Y(s) = \underbrace{\frac{(as + b)y_0 + ay_0'}{as^2 + bs + c}}_{\text{initial conditions}} + \underbrace{\frac{G(s)}{as^2 + bs + c}}_{\text{particular sol.}}.$$

Use partial fractions and complete squares methods along with entries in the table (mainly (8)–(12), (15)–(17), (21)–(22), and (29)), to find $y(t)$.

Some important functions

$$(35) \qquad \textbf{Transfer function: } H(s) = \frac{1}{as^2 + bs + c}$$

$$(36) \qquad \textbf{Impulse function: } h(t) = \mathcal{L}^{-1}\{H(s)\}(t)$$

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# Heat Equation Problem

## Eigenvalue problem

### Homogeneous Dirichlet boundary conditions

$$(37) \qquad \begin{cases} X'' - \lambda X = 0 & 0 < x < L \\ X(0) = X(L) = 0 \end{cases}$$

has eigenvalues and eigenfunctions solution

$$(38) \qquad \textbf{Eigenvalues: } \lambda_n = -\left(\frac{n\pi}{L}\right)^2,$$

$$(39) \qquad \textbf{Eigenfunctions: } X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right), \quad n \geq 1$$

### Homogeneous Newman boundary conditions

$$(40) \qquad \begin{cases} X'' - \lambda X = 0 & 0 < x < L \\ X'(0) = X'(L) = 0 \end{cases}$$

has eigenvalues and eigenfunctions solution

$$(41) \qquad \textbf{Eigenvalues: } \lambda_n = -\left(\frac{n\pi}{L}\right)^2,$$

$$(42) \qquad \textbf{Eigenfunctions: } X_n(x) = B_n \cos\left(\frac{n\pi x}{L}\right), \quad n \geq 0$$

## Fourier series

For $f(x)$ piecewise continuous on $[-L, L]$

$$(43) \qquad F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$(44) \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \qquad n \geq 0$$

$$(45) \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \qquad n \geq 1$$

For all $|x| \leq L$, $F(x)$ **converges** to

$$(46) \qquad F(x) = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x, \\ \frac{f(x-)+f(x+)}{2} & \text{if } f \text{ is discontinuous at } x, \\ \frac{f(-L+)+f(L-)}{2} & \text{if } x = L \text{ or } x = -L. \end{cases}$$

### Properties of even and odd functions

0. even: $f(-x) = f(x)$, odd: $-f(-x) = f(x)$

1. even $\pm$ even = even,  even $\times$ even = even

2. odd $\pm$ odd = odd,  odd $\times$ odd = even

3. odd $\pm$ even = none,  odd $\times$ even = odd

4. $f(x)$ even, then $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$

5. $f(x)$ odd, then $\int_{-L}^{L} f(x)dx = 0$

6. $f(x)$ even, then $b_n = 0$, and $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$.

7. $f(x)$ odd, then $a_n = 0$, and $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$.

**Fourier cosine (sine) series** of $f(x)$ defined on $[0, L]$ is an even (odd) extension of $f(x)$ to $[-L, L]$, using property 6 (7).

## Heat equation

The Heat equation can be solved using separation of variable

$$(47) \qquad \textbf{Heat equation: } u_t = \alpha u_{xx}$$

$$(48) \qquad \textbf{Sep. of variables: } u(x,t) = X(x)T(t)$$

$$(49) \qquad \Rightarrow \frac{X''}{X} = \frac{T'}{\alpha T} = \lambda$$

For eigenvalues $\lambda_n$, the ODE for $T(t)$ in (49) has solution

$$(50) \qquad \frac{T_n'}{\alpha T_n} = \lambda_n \quad \Rightarrow \quad T_n(t) = A_n e^{\alpha \lambda_n t}$$

By superposition principle and (48), infinitely many solutions give

$$(51) \qquad u(x,t) = \sum_n X_n(x)T_n(t) = \sum_n X_n(x)T_n(t)$$

Use Boundary Conditions and Inital Conditions to determine the asociated eigenvalue and Fourier series problems, respectively.

**Dirichlet BC:** $u(0,t) = u(L,t) = 0$ result in an eigenvalue problem of the form (37) and a Fourier sine series of $f(x)$

$$(52) \qquad \begin{cases} u_t = \alpha u_{xx}, & 0 < x < L, \quad t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

$$(53) \qquad \Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right)$$

**Newmann BC:** $u_x(0,t) = u_x(L,t) = 0$ result in an eigenvalue problem of the form (40) and a Fourier cosine series of $f(x)$

$$(54) \qquad \begin{cases} u_t = \alpha u_{xx}, & 0 < x < L, \quad t > 0 \\ u_x(0,t) = u_x(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

$$(55) \qquad \Rightarrow u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right)$$

**External force $g(x)$ and/or Nonhomogeneous BC:**

$$(56) \qquad \begin{cases} u_t = \alpha u_{xx} + g(x), & 0 < x < L, \quad t > 0 \\ u(0,t) = U_0, \quad u(L,t) = U_L, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

$$(57) \qquad \text{use } u(x,t) = v(x) + w(x,t)$$

to get a second order problem on $v(x)$ and (52) problem on $w(x,t)$

$$(58) \qquad \begin{cases} v''(x) = -\frac{1}{\alpha}g(x) \\ v(0) = U_0, \quad v(L) = U_L \end{cases}, \quad \begin{cases} w_t = \alpha w_{xx}, \\ w(0,t) = w(L,t) = 0, \\ w(x,0) = f(x) - v(x) \end{cases}$$

First solve for $v(x)$, then use it to solve for $w(x,t)$

$$(59) \qquad \text{If} \quad g(x) = 0 \quad \Rightarrow \quad v(x) = (U_2 - U_1)\frac{x}{L} + U_1$$

$w(x,t)$ and $v(x)$ are called **transient** and **steady state** solutions

**Reaction-diffusion:** put all constants on the $t$ side of (49)

$$(60) \qquad u_t = \alpha u_{xx} + \mu u \quad \Rightarrow \quad \frac{X''}{X} = \frac{T'}{\alpha T} - \mu = \lambda$$

# Wave Equation Problem

The Wave equation can be solved using separation of variable

(61)             **Wave equation:** $u_{tt} = \alpha^2 u_{xx}$

(62)             **Sep. of variables:** $u(x,t) = X(x)T(t)$

(63)
$$\Rightarrow \frac{X''}{X} = \frac{T''}{\alpha^2 T} = \lambda$$

For eigenvalues $\lambda_n \leq 0$, the ODE for $T(t)$ in (63) has solution

(64)             **ODE for $T(t)$:** $T_n'' - \lambda_n \alpha^2 T_n = 0$

(65)
$$\Rightarrow T_n(t) = a_n \cos(\alpha\sqrt{-\lambda_n}t) + b_n \sin(\alpha\sqrt{-\lambda_n}t)$$

By superposition principle and (62), infinitely many solutions give

(66)
$$u(x,t) = \sum_n X_n(x)T_n(t) = \sum_n X_n(x)T_n(t)$$

Use Boundary Conditions and Inital Conditions to determine the asociated eigenvalue and Fourier series problems, respectively.

**Dirichlet BC:** $u(0,t) = u(L,t) = 0$ result in an eigenvalue problem of the form (37) and a Fourier sine series

(67)
$$\begin{cases} u_{tt} = \alpha^2 u_{xx}, & 0 < x < L, \quad t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \\ u_t(x,0) = g(x), & 0 < x < L \end{cases}$$

(68)
$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{\alpha n \pi t}{L}\right) \right.$$
$$\left. + b_n \sin\left(\frac{\alpha n \pi t}{L}\right) \right] \sin\left(\frac{n \pi x}{L}\right)$$

To find coefficients $a_n$, find the Fourier sine series using the first initial condition

(69)
$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n \pi x}{L}\right)$$

To find coefficients $b_n$, find the Fourier sine series using the second initial condition

(70)
$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{\alpha n \pi}{L} b_n \sin\left(\frac{n \pi x}{L}\right)$$

**Newmann BC:** $u(0,t) = u(L,t) = 0$ result in an eigenvalue problem of the form (40) and a Fourier cosine series

(71)
$$\begin{cases} u_{tt} = \alpha^2 u_{xx}, & 0 < x < L, \quad t > 0 \\ u_x(0,t) = u_x(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \\ u_t(x,0) = g(x), & 0 < x < L \end{cases}$$

(72)
$$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} \left[ a_n \cos\left(\frac{\alpha n \pi t}{L}\right) \right.$$
$$\left. + b_n \sin\left(\frac{\alpha n \pi t}{L}\right) \right] \cos\left(\frac{n \pi x}{L}\right)$$

To find coefficients $a_n$, find the Fourier cosine series using the first initial condition

(73)
$$u(x,0) = f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n \pi x}{L}\right)$$

To find coefficients $b_n$, find the Fourier cosine series using the second initial condition

(74)
$$u_t(x,0) = g(x) = \sum_{n=0}^{\infty} \frac{\alpha n \pi}{L} b_n \cos\left(\frac{n \pi x}{L}\right)$$