MATH 201 Differential Equations – University of Alberta
Winter 2018 – Labs – Carlos Contreras
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Lab 9: LT– Convolution theorem

1. Find the integral

$$\int_0^\infty t\sin(2t)e^{-2t}dt$$

Periodic functions

2. Find the Laplace transform of the periodic function

$$f(t) = \begin{cases} e^{-t}, & 0 < t < 1, \\ 1, & 1 < t < 2, \end{cases}$$

with period $T = 2$.

Convolution theorem

3. Use the convolution theorem to obtain to find the inverse Laplace transform of the given function

$$F(s) = \frac{1}{s^2(s^2+9)}.$$

4. Use the convolution theorem to obtain a formula for the solution to the given initial value problem, where $g(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order.

$$y'' + 4y' + 5y = g(t), \quad y(0) = 1, \quad y'(0) = 1.$$

Integro-differential equations

5. Solve the integro–differential equation

$$y' - 2\int_0^t e^{t-v}y(v)dv = t, \quad y(0) = 2.$$

System of differential equations

6. Use the method of Laplace transforms to solve the given initial value problem.

$$\begin{aligned} x' &= y + \sin(t) \quad x(0) = 2 \\ y' &= x + 2\cos(t) \quad y(0) = 0. \end{aligned}$$

Solutions

Theory and problems from: Nagel, Saff & Sneider, *Fundamentals of Differential Equations*, Eighth Edition, Adisson–Wesley.

→ The Laplace transform of a **periodic function** $f(x)$ with period T is

$$\mathcal{L}\{f\}(s) = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)dt.$$

→ The **convolution** of $f(t)$ and $g(t)$, denoted $f * g$, is defined by

$$(f * g)(t) = \int_0^t f(t-v)g(v)dv.$$

The Laplace transform of the convolution is

$$\mathcal{L}\{f * g\} = F(s)G(s).$$

→ Brief **table of Laplace Transforms**.

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$	$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$	
$e^{at}f(t)$	$F(s-a)$	1	$\frac{1}{s}$	
$f'(t)$	$sF(s) - f(0)$	e^{at}	$\frac{1}{s-a}$	$s > a$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	t^n	$\frac{n!}{s^{n+1}}$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\sin bt$	$\frac{b}{s^2+b^2}$	
$(f * g)(t)$	$F(s)G(s)$	$\cos bt$	$\frac{s}{s^2+b^2}$	
$u(t-a)$	$\frac{e^{-as}}{s}$	$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$f(t-a)u(t-a)$	$e^{-as}F(s)$	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$\delta(t-a)$	e^{-as}	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$\int_0^s f(\tau)d\tau$	$\frac{1}{s}F(s)$	$\sinh bt$	$\frac{b}{s^2-b^2}$	
$\frac{1}{t}f(t)$	$\int_s^\infty F(\sigma)d\sigma$	$\cosh bt$	$\frac{s}{s^2-b^2}$	

→ The infinite sum of evenly shifted copies of $f(x)$ is

$$\mathcal{L}\left\{\sum_{n=0}^{\infty}(\pm 1)^k f(t-ka)u(t-ka)\right\}(s) = \frac{1}{1 \pm e^{as}}F(s).$$

1. Find the integral

$$\int_0^\infty t\sin(2t)e^{-2t}dt.$$

Solution

Note that the integral is nothing but the Laplace transform evaluated at $s = 2$

$$\int_0^\infty t\sin(2t)e^{-2t}dt = \mathcal{L}\{t\sin(2t)\}(2).$$

Recall also the derivative property

$$tf(t) = (-1)F'(s).$$

Then

$$\mathcal{L}\{t\sin(2t)\}(s) = -\left(\frac{2}{s^2+4}\right)' = 4\frac{s}{(s^2+4)^2},$$

and

$$\boxed{\int_0^\infty t\sin(2t)e^{-2t}dt = 4\frac{2}{(4+4)^2} = \frac{1}{8}.}$$

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2. Find the Laplace transform of the periodic function

$$f(t) = \begin{cases} e^{-t}, & 0 < t < 1\,, \\ 1\,, & 1 < t < 2\,, \end{cases}$$

with period $T = 2$.

Solution
First we find the integral

$$\begin{aligned}\int_0^T e^{-st}f(t)dt = \int_0^2 e^{-st}f(t)dt &= \int_0^1 e^{-st}e^{-t}dt + \int_1^2 e^{-st}dt \\ &= -\frac{1}{s+1}\left(e^{-(s+1)} - 1\right) - \frac{1}{s}\left(e^{-2s} - e^{-s}\right).\end{aligned}$$

Thus,

$$\boxed{F(s) = \frac{1}{1-e^{-2s}}\left(\frac{1-e^{-(s+1)}}{s+1} + \frac{e^{-s}-e^{-2s}}{s}\right)}$$

.

■

3. Use the convolution theorem to obtain to find the inverse Laplace transform of the given function

$$F(s) = \frac{1}{s^2(s^2+9)}.$$

Solution

Taking the inverse Laplace transform and using the convolution property, we obtain:

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+9)}\right\}(t) \\
&= t * \frac{1}{3}\sin(3t) \\
&= \int_0^t (t-v)\frac{1}{3}\sin(3v)\mathrm{d}v \\
&= \frac{t}{3}\int_0^t \sin(3v)\mathrm{d}v - \frac{1}{3}\int_0^t v\sin(3v)\mathrm{d}v \\
&= \frac{t}{3}\left[-\frac{\cos(3v)}{3}\right]_0^t - \frac{1}{3}\left[-\frac{v\cos(3v)}{3} + \frac{\sin(3v)}{9}\right]_0^t \\
&= \frac{t}{3}\left[\frac{1-\cos(3t)}{3}\right] - \frac{1}{3}\left[-\frac{t\cos(3t)}{3} + \frac{\sin(3t)}{9}\right] \\
&= \frac{t}{9} - \frac{t\cos(3t)}{9} + \frac{t\cos(3t)}{9} - \frac{\sin(3t)}{27}
\end{aligned}$$

Thus,

$$\boxed{f(t) = \frac{t}{9} - \frac{\sin(3t)}{27}}.$$

■

4. Use the convolution theorem to obtain a formula for the solution to the given initial value problem, where $g(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order.

$$y'' + 4y' + 5y = g(t), \quad y(0) = 1, \quad y'(0) = 1.$$

Solution
Applying a Laplace transform on both sides of this equation, we obtain:

$$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 5Y(s) = G(s)$$

Subbing in our initial conditions, we have:

$$s^2Y(s) - s - 1 + 4sY(s) - 4 + 5Y(s) = G(s)$$

Isolating $Y(s)$, we obtain:

$$\begin{aligned}
Y(s) &= \frac{G(s)}{s^2+4s+5} + \frac{s+5}{s^2+4s+5} \\
&= \frac{G(s)}{(s+2)^2+1} + \frac{(s+2)+3}{(s+2)^2+1} \\
&= \frac{G(s)}{(s+2)^2+1} + \frac{(s+2)}{(s+2)^2+1} + \frac{3}{(s+2)^2+1}
\end{aligned}$$

Taking the inverse Laplace transform, we have:

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{G(s)}{(s+2)^2+1}\right\}(t) + e^{-2t}\cos(t) + 3e^{-2t}\sin(t) \\
&= (g(t) * e^{-2t}\sin(t)) + e^{-2t}\cos(t) + 3e^{-2t}\sin(t)
\end{aligned}$$

Hence,

$$\boxed{y(t) = \int_0^t g(t-v)e^{-2v}\sin(v)\mathrm{d}v + e^{-2t}\cos(t) + 3e^{-2t}\sin(t)}.$$

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5. Solve the integro–differential equation

$$y' - 2\int_0^t e^{t-v}y(v)dv = t, \quad y(0) = 2.$$

Solution
We can rewrite the integro–differential as

$$y' - 2e^t * y = t,$$

and take Laplace transform to get (recall the convolution theorem here)

$$sY(s) - 2 - 2\frac{1}{s-1}Y(s) = \frac{1}{s^2},$$

where

$$Y(s) = \frac{(2s^2+1)(s-1)}{s^2(s+1)(s-2)} = \frac{2s^3-2s^2+s-1}{s^2(s+1)(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s-2}.$$

Solving the partial fractions we get $A = -\frac{3}{4}$, $B = \frac{1}{2}$, $C = 2$ and $D = \frac{3}{4}$. Thus, we can take the inverse Laplace transforms right away to get

$$\boxed{y(t) = -\frac{3}{4} + \frac{1}{2}t + 2e^{-t} + \frac{3}{4}e^{2t}}.$$

■

Solution
We can rewrite the equation as

$$y' - 2e^t\int_0^t e^{-v}y(v)dv = t,$$

take the derivative, and apply the product rule along with the fundamental theorem of calculus to get

$$y'' - 2e^t\int_0^t e^{-v}y(v)dv - 2e^te^{-t}y(t) = 1.$$

Note that

$$-2e^t\int_0^t e^{-v}y(v)dv = t - y',$$

then we obtain the second order differential equation

$$y'' - y' - 2y = 1 - t,$$

with initial conditions (the second initial condition comes from evaluating the integro–differential equation at $t = 0$)

$$y(0) = 2, \quad y'(0) = 0.$$

Apply Laplace transform we have

$$Y(s) = \frac{2s-2}{s^2-s-2} + \frac{1}{s(s^2-s-2)} - \frac{1}{s^2(s^2-s-2)} = \frac{2s^3-2s^2+s-1}{s^2(s+1)(s-2)},$$

which is the same partial fractions problem as before. Thus

$$\boxed{y(t) = -\frac{3}{4} + \frac{1}{2}t + 2e^{-t} + \frac{3}{4}e^{2t}}.$$

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6. Use the method of Laplace transforms to solve the given initial value problem.

$$\begin{aligned} x' &= y + \sin(t) \quad x(0) = 2 \\ y' &= x + 2\cos(t) \quad y(0) = 0. \end{aligned}$$

Solution
Applying a Laplace transform to both these equations, we obtain

$$\begin{aligned} sX(s) - x(0) &= Y(s) + \frac{1}{s^2+1}, \\ sY(s) - y(0) &= X(s) + \frac{2s}{s^2+1}. \end{aligned}$$

Subbing in our initial conditions, we have

$$\begin{aligned} sX(s) - 2 &= Y(s) + \frac{1}{s^2+1}, \\ sY(s) &= X(s) + \frac{2s}{s^2+1}. \end{aligned}$$

This is a system of equations with two unknown variables $X(s)$ and $Y(s)$. Isolate the variable that think is more convenient. In this case we isolate $X(s)$ in the second equation, to obtain

$$X(s) = sY(s) - \frac{2s}{s^2+1}.$$

Substituting this result into the first equation, we obtain:

$$\begin{aligned} s\left(sY(s) - \frac{2s}{s^2+1}\right) - 2 &= Y(s) + \frac{1}{s^2+1} \\ \Rightarrow s^2Y(s) - \frac{2s^2}{s^2+1} - 2 &= Y(s) + \frac{1}{s^2+1} \\ \Rightarrow (s^2-1)Y(s) &= \frac{2s^2+1}{s^2+1} + 2 \\ \Rightarrow (s^2-1)Y(s) &= \frac{4s^2+3}{s^2+1} \end{aligned}$$

Recall that $\frac{1}{s^2+1}$ leads to $\sin t$, and $\frac{1}{s^2-1}$ leads to $\sinh t$. Isolating $Y(s)$, we obtain

$$Y(s) = \frac{4s^2+3}{(s^2+1)(s^2-1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-1}.$$

This is true for $B = \frac{1}{2}$, $D = \frac{7}{2}$, and $A = C = 0$. Thus,

$$Y(s) = \frac{1}{2}\frac{1}{s^2+1} + \frac{7}{2}\frac{1}{s^2-1}.$$

Now, we use our expression for $X(s)$,

$$X(s) = sY(s) - \frac{2s}{s^2+1} = \frac{1}{2}\frac{s}{s^2+1} + \frac{7}{2}\frac{s}{s^2-1} - \frac{2s}{s^2+1}.$$

Thus,

$$X(s) = -\frac{3}{2}\frac{s}{s^2+1} + \frac{7}{2}\frac{s}{s^2-1}.$$

Now, we take the inverse Laplace transform of $X(s)$ and $Y(s)$, to obtain

$$\boxed{x(t) = -\frac{3}{2}\cos t + \frac{7}{2}\cosh t},$$

$$\boxed{y(t) = \frac{1}{2}\sin t + \frac{7}{2}\sinh t}.$$

■