MATH 201 Differential Equations – University of Alberta
Winter 2018 – Labs – Carlos Contreras
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Lab 10: Eigenvalue problem

Eigenvalue problems (zero boundary conditions)

1. Find the eigenvalues λ for which the given problem has a nontrivial solution. Also determine the corresponding eigenfunctions.

$$y'' - \lambda y = 0\,; \qquad 0 < x < \pi\,, \qquad y(0) = 0\,, \qquad y(\pi) = 0.$$

2. Find the eigenvalues λ for which the given problem has a nontrivial solution. Also determine the corresponding eigenfunctions.

$$y'' + \lambda y = 0\,; \qquad 0 < x < \pi\,, \qquad y(0) = 0\,, \qquad y(\pi) = 0.$$

Eigenvalue problems (zero-derivative boundary conditions)

3. Find the eigenvalues λ for which the given problem has a nontrivial solution. Also determine the corresponding eigenfunctions.

$$y'' - \lambda y = 0\,; \qquad 0 < x < \pi\,, \qquad y'(0) = 0\,, \qquad y'(\pi) = 0.$$

Eigenvalue problems (mixed boundary conditions)

4. Find the eigenvalues λ for which the given problem has a nontrivial solution. Also determine the corresponding eigenfunctions.

$$y'' + \lambda y = 0\,; \qquad 0 < x < \pi\,, \qquad y'(0) = 0\,, \qquad y(\pi) = 0.$$

5. Find the eigenvalues λ for which the given problem has a nontrivial solution. Also determine the corresponding eigenfunctions.

$$y'' + \lambda y = 0\,; \qquad 0 < x < \pi\,, \qquad y(0) - y'(0) = 0\,, \qquad y(\pi) = 0.$$

Eigenvalue problems (general form)

6. Find the eigenvalues λ for which the given problem has a nontrivial solution. Also determine the corresponding eigenfunctions.

$$y'' + 4y' + \lambda y = 0\,; \qquad 0 < x < \pi/2\,, \qquad y(0) = 0\,, \qquad y(\pi/2) = 0.$$

Solutions

Theory and problems from: Nagel, Saff & Sneider, *Fundamentals of Differential Equations*, Eighth Edition, Adisson–Wesley.

→ For a **general eigenvalue problem**

$$ay'' + by' + \lambda y = 0, \qquad \text{s.t. boundary conditions}^{a},$$

the roots of the auxiliary equation are

$$r = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4a\lambda}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{\Delta}}{2a}.$$

The three possible nontrivial solutions depend on the sign of the discriminant $\Delta = b^2 - 4a\lambda$.

1. $\Delta = b^2 - 4a\lambda > 0$.
   Then $\lambda < \frac{b^2}{4a}$, $r_1 = -\frac{b}{2} + \frac{\sqrt{b^2-4a\lambda}}{2a}$, $r_2 = -\frac{b}{2a} - \frac{\sqrt{b^2-4a\lambda}}{2}$ (different real roots), and

   $$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

2. $\Delta = b^2 - 4a\lambda = 0$.
   Then $\lambda = \frac{b^2}{4a}$, $r_1 = r_2 = \frac{b}{2a}$ (repeated real roots), and

   $$y(x) = C_1 e^{r_1 x} + C_2 x e^{r_1 x}.$$

3. $\Delta = b^2 - 4a\lambda < 0$.
   Then $\lambda > \frac{b^2}{4ac}$, then $r_{1,2} = -\frac{b}{2a} \pm i\frac{\sqrt{4a\lambda - b^2}}{2a} = -\frac{b}{2a} \pm i\frac{\sqrt{-\Delta}}{2a}$ (complex roots), and

   $$y(x) = C_1 e^{-\frac{b}{2a}x} \cos\left(\frac{\sqrt{-\Delta}}{2a} x\right) + C_2 e^{-\frac{b}{2a}x} \sin\left(\frac{\sqrt{-\Delta}}{2a} x\right).$$

For each one of the cases the goal is to use the boundary conditions to determine values for λ leading to nontrivial solutions. A solution will be **trivial** ($y(x) = 0$), when the boundary conditions imply $C_1 = C_2 = 0$.

[a] s.t. stands for subject to.

1. Find the eigenvalues λ for which the given problem has a nontrivial solution. Also determine the corresponding eigenfunctions.

$$y'' - \lambda y = 0\,; \qquad 0 < x < \pi\,, \qquad y(0) = 0\,, \qquad y(\pi) = 0.$$

Solution
First we find the roots of the auxiliary equation.

$$r = \pm\sqrt{\lambda} = \pm\sqrt{\Delta}.$$

We consider the three cases.

Case 1. $\Delta = \lambda > 0$. Then $r_1 = \sqrt{\lambda}$, $r_2 = -\sqrt{\lambda}$ are distinct real roots, and the solution to the ODE is

$$y(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}.$$

Using the BC's,

$$\begin{cases} C_1 + C_2 &= 0 \\ e^{\sqrt{\lambda}\pi} C_1 + e^{-\sqrt{\lambda}\pi} C_2 &= 0 \end{cases}.$$

For nontrivial solutions we need

$$\det(A) = \begin{vmatrix} 1 & 1 \\ e^{\sqrt{\lambda}\pi} & e^{-\sqrt{\lambda}\pi} \end{vmatrix} = e^{-\sqrt{\lambda}\pi} - e^{\sqrt{\lambda}\pi} = 0,$$

which implies $e^{\sqrt{\lambda}\pi} = e^{-\sqrt{\lambda}\pi}$, which is a contradiction since $\lambda \neq 0$. Hence, *there is no nontrivial solution.*

Case 2. $\Delta = \lambda = 0$. Then $r_1 = r_2 = 0$ are repeated real roots, and the solution to the ODE is

$$y(x) = C_1 x + C_2.$$

Using the BC's,

$$\begin{cases} C_2 &= 0 \\ \pi C_1 + C_2 &= 0 \end{cases}.$$

Which implies $C_1 = C_2 = 0$. Hence, *there is no nontrivial solution.*

Case 3. $\Delta = \lambda < 0$. Then $r_1 = i\sqrt{-\lambda}$, $r_2 = -i\sqrt{-\lambda}$ are complex roots, and the solution to the ODE is

$$y(x) = C_1 \cos\sqrt{-\lambda}x + C_2 \sin\sqrt{-\lambda}x.$$

Using the BC's,

$$\begin{cases} C_1 &= 0 \\ \cos(\sqrt{-\lambda}\pi)C_1 + \sin(\sqrt{-\lambda}\pi)C_2 &= 0 \end{cases},$$

which implies

$$\sin(\sqrt{-\lambda}\pi)C_2 = 0 \Leftrightarrow C_2 = 0 \text{ or } \sin(\sqrt{-\lambda}\pi) = 0.$$

For nontrivial solutions we need

$$\sin(\sqrt{-\lambda}\pi) = 0 \Leftrightarrow \sqrt{-\lambda}\pi = n\pi \Leftrightarrow \sqrt{-\lambda} = n.$$

Thus, the eigenvalues are

$$\boxed{\lambda_n = -n^2}, \quad n \geq 1,$$

with eigenfunctions (recall that C_2 is arbitrary)

$$\boxed{y_n(x) = C_n \sin(nx)}, \quad n \geq 1,$$

for some arbitrary constants C_n.

Note: the general domain eigenvalue problem

$$y'' - \lambda y = 0\,; \qquad 0 < x < L\,, \qquad y(0) = 0\,, \qquad y(L) = 0,$$

has eigenvalues

$$\boxed{\lambda_n = -\left(\frac{n\pi}{L}\right)^2}, \quad n \geq 1,$$

and eigenfunctions

$$\boxed{y_n(x) = C_n \sin\left(\frac{n\pi x}{L}\right)}, \quad n \geq 1,$$

for some arbitrary constants C_n.

Recall the step where

$$\sin(\sqrt{-\lambda}L) = 0 \Leftrightarrow \sqrt{-\lambda}L = n\pi \Leftrightarrow \sqrt{-\lambda} = \frac{n\pi}{L}.$$

$\blacksquare$

2. Find the eigenvalues λ for which the given problem has a nontrivial solution. Also determine the corresponding eigenfunctions.

$$y'' + \lambda y = 0\,; \qquad 0 < x < \pi\,, \qquad y(0) = 0\,, \qquad y(\pi) = 0.$$

Solution
First we find the roots of the auxiliary equation.

$$r = \pm\sqrt{-\lambda} = \pm\sqrt{\Delta}.$$

We consider the three cases.
Case 1. $\Delta = -\lambda > 0 \;\Rightarrow\; \lambda < 0$. Then $r_1 = \sqrt{-\lambda}$, $r_2 = -\sqrt{-\lambda}$ are distinct real roots, and the solution to the ODE is

$$y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}.$$

Using the BC's,

$$\left\{ \begin{array}{rcl} C_1 + C_2 &=& 0 \\ e^{\sqrt{-\lambda}\pi} C_1 e^{-\sqrt{-\lambda}\pi} C_2 &=& 0 \end{array} \right. .$$

For nontrivial solutions we need

$$\det(A) = \begin{vmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}\pi} & e^{-\sqrt{-\lambda}\pi} \end{vmatrix} = e^{-\sqrt{-\lambda}\pi} - e^{\sqrt{-\lambda}\pi} = 0,$$

which implies $e^{\sqrt{-\lambda}\pi} = e^{-\sqrt{-\lambda}\pi}$, which is a contradiction since $\lambda \neq 0$. Hence, *there is no nontrivial solution.*
Case 2. $\Delta = -\lambda = 0 \;\Rightarrow\; \lambda = 0$. Then $r_1 = r_2 = 0$ are repeated real roots, and the solution to the ODE is

$$y(x) = C_1 x + C_2.$$

Using the BC's,

$$\left\{ \begin{array}{rcl} C_2 &=& 0 \\ \pi C_1 + C_2 &=& 0 \end{array} \right. .$$

Hence, *there is no nontrivial solution.*
Case 3. $\Delta = -\lambda < 0 \;\Rightarrow\; \lambda > 0$. Then $r_1 = i\sqrt{\lambda}$, $r_2 = -i\sqrt{\lambda}$ are complex roots, and the solution to the ODE is

$$y(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x.$$

Using the BC's,

$$\left\{ \begin{array}{rcl} C_1 &=& 0 \\ \cos(\sqrt{-\lambda}\pi)C_1 + \sin(\sqrt{\lambda}\pi)C_2 &=& 0 \end{array} \right. ,$$

which implies

$$\sin(\sqrt{\lambda}\pi)C_2 = 0 \Leftrightarrow C_2 = 0 \text{ or } \sin(\sqrt{\lambda}\pi) = 0.$$

For nontrivial solutions we need

$$\sin(\sqrt{\lambda}\pi) = 0 \Leftrightarrow \sqrt{\lambda}\pi = n\pi \Leftrightarrow \sqrt{\lambda} = n.$$

Thus, the eigenvalues are

$$\boxed{\lambda_n = n^2}, \qquad n \geq 1,$$

with eigenfunctions (recall that C_2 is arbitrary)

$$\boxed{y_n(x) = C_n \sin(nx)}, \qquad n \geq 1,$$

for some arbitrary constants C_n.

Note: the general domain eigenvalue problem

$$y'' + \lambda y = 0\,; \qquad 0 < x < L\,, \qquad y(0) = 0\,, \qquad y(L) = 0,$$

has eigenvalues

$$\boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}, \quad n \geq 1,$$

and eigenfunctions

$$\boxed{y_n(x) = C_n \sin\left(\frac{n\pi x}{L}\right)}, \quad n \geq 1,$$

for some arbitrary constants C_n.

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3. Find the eigenvalues λ for which the given problem has a nontrivial solution. Also determine the corresponding eigenfunctions.

$$y'' - \lambda y = 0\,; \qquad 0 < x < \pi\,, \qquad y'(0) = 0\,, \qquad y'(\pi) = 0.$$

Solution

First we find the roots of the auxiliary equation.

$$r = \pm\sqrt{\lambda} = \pm\sqrt{\Delta}.$$

We consider the three cases.

Case 1. $\Delta = \lambda > 0$. Then $r_1 = \sqrt{\lambda}$, $r_2 = -\sqrt{\lambda}$ are distinct real roots, and the solution to the ODE is

$$y(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}.$$

We need $y'(x)$ in order to use the initial conditions

$$y'(x) = \sqrt{\lambda}C_1 e^{\sqrt{\lambda}x} - \sqrt{\lambda}C_2 e^{-\sqrt{\lambda}x}.$$

Using the BC's,

$$\left\{\begin{array}{rcl} \sqrt{\lambda}C_1 - \sqrt{\lambda}C_2 & = & 0 \\ \sqrt{\lambda}e^{\sqrt{\lambda}\pi}C_1 - \sqrt{\lambda}e^{-\sqrt{\lambda}\pi}C_2 & = & 0 \end{array}\right. .$$

For nontrivial solutions we need

$$\det(A) = \begin{vmatrix} \sqrt{\lambda} & -\sqrt{\lambda} \\ \sqrt{\lambda}e^{\sqrt{\lambda}\pi} & -\sqrt{\lambda}e^{-\sqrt{\lambda}\pi} \end{vmatrix} = -\lambda e^{-\sqrt{\lambda}\pi} + \lambda e^{\sqrt{\lambda}\pi} = 0,$$

which implies $e^{\sqrt{\lambda}\pi} = e^{-\sqrt{\lambda}\pi}$, which is a contradiction since $\lambda \neq 0$. Hence, *there is no nontrivial solution.*

Case 2. $\Delta = \lambda = 0$. Then $r_1 = r_2 = 0$ are repeated real roots, and the solution to the ODE is

$$y(x) = C_1 x + C_2.$$

We need $y'(x)$ in order to use the initial conditions

$$y'(x) = C_1.$$

Using the BC's,

$$\left\{\begin{array}{rcl} C_1 & = & 0 \\ C_1 & = & 0 \end{array}\right. .$$

Which implies C_2 is free to choose. Hence, a non-trivial solution is the eigenvalue

$$\boxed{\lambda = 0},$$

with eigenfunction

$$\boxed{y(x) = C_0},$$

Case 3. $\Delta = \lambda < 0$. Then $r_1 = i\sqrt{-\lambda}$, $r_2 = -i\sqrt{-\lambda}$ are complex roots, and the solution to the ODE is

$$y(x) = C_1 \cos\sqrt{-\lambda}x + C_2 \sin\sqrt{-\lambda}x.$$

We need $y'(x)$ in order to use the initial conditions

$$y'(x) = -\sqrt{-\lambda}C_1 \sin\sqrt{-\lambda}x + \sqrt{-\lambda}C_2 \cos\sqrt{-\lambda}x.$$

Using the BC's,

$$\begin{cases} \sqrt{-\lambda}C_2 &= 0 \\ -\sqrt{-\lambda}\sin(\sqrt{-\lambda}\pi)C_1 + \sqrt{-\lambda}\cos(\sqrt{-\lambda}\pi)C_2 &= 0 \end{cases},$$

Since $\lambda \neq 0$, then $C_2 = 0$, and we need

$$-\sqrt{-\lambda}\sin(\sqrt{-\lambda}\pi)C_1 = 0 \Leftrightarrow C_1 = 0 \text{ or } \sin(\sqrt{-\lambda}\pi) = 0.$$

For nontrivial solutions we need

$$\sin(\sqrt{-\lambda}\pi) = 0 \Leftrightarrow \sqrt{-\lambda}\pi = n\pi \Leftrightarrow \sqrt{-\lambda} = n.$$

Thus, in this case the eigenvalues are

$$\boxed{\lambda_n = -n^2}, \quad n \geq 1,$$

with eigenfunctions (recall that C_1 is arbitrary)

$$\boxed{y_n(x) = C_n \cos(nx)}, \quad n \geq 1,$$

for some arbitrary constants C_n.

Note: we can combine Cases 2 and 3 to write eigenvalues

$$\boxed{\lambda_n = -n^2}, \quad n \geq 0,$$

with eigenfunctions

$$\boxed{y_n(x) = C_n \cos(nx)}, \quad n \geq 0.$$

Note: the general domain eigenvalue problem

$$y'' - \lambda y = 0\,; \qquad 0 < x < L\,, \qquad y'(0) = 0\,, \qquad y'(L) = 0,$$

has eigenvalues

$$\boxed{\lambda_n = -\left(\frac{n\pi}{L}\right)^2}, \quad n \geq 0,$$

and eigenfunctions

$$\boxed{y_n(x) = C_n \cos\left(\frac{n\pi x}{L}\right)}, \quad n \geq 0,$$

for some arbitrary constants C_n.

■

4. Find the eigenvalues λ for which the given problem has a nontrivial solution. Also determine the corresponding eigenfunctions.

$$y'' + \lambda y = 0 \,; \qquad 0 < x < \pi \,, \qquad y'(0) = 0 \,, \qquad y(\pi) = 0.$$

Solution
First we find the roots of the auxiliary equation.

$$r = \pm\sqrt{-\lambda} = \pm\sqrt{\Delta}.$$

We consider the three cases.

Case 1. $\Delta = -\lambda > 0 \Rightarrow \lambda < 0$. Then $r_1 = \sqrt{-\lambda}$, $r_2 = -\sqrt{-\lambda}$ are distinct real roots, and the solution to ODE is

$$y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}.$$

We need $y'(x)$ in order to use the first initial conditions

$$y'(x) = \sqrt{-\lambda}C_1 e^{\sqrt{-\lambda}x} - \sqrt{-\lambda}C_2 e^{-\sqrt{-\lambda}x}.$$

Using the BC's,

$$\left\{ \begin{array}{rcl} \sqrt{-\lambda}C_1 - \sqrt{-\lambda}C_2 & = & 0 \\ e^{\sqrt{-\lambda}\pi}C_1 + e^{-\sqrt{-\lambda}\pi}C_2 & = & 0 \end{array} \right. .$$

For nontrivial solutions we need

$$\det(A) = \begin{vmatrix} \sqrt{-\lambda} & -\sqrt{-\lambda} \\ e^{\sqrt{-\lambda}\pi} & e^{-\sqrt{-\lambda}\pi} \end{vmatrix} = \sqrt{-\lambda}e^{-\sqrt{-\lambda}\pi} + \sqrt{-\lambda}e^{\sqrt{-\lambda}\pi} = 0,$$

which implies $e^{\sqrt{-\lambda}\pi} = -e^{\sqrt{-\lambda}\pi}$, which is a contradiction since the exponential function is always positive and $\lambda \neq 0$. Hence, *there is no nontrivial solution.*

Case 2. $\Delta = -\lambda = 0 \Rightarrow \lambda = 0$. Then $r_1 = r_2 = 0$ are repeated real roots, and the solution to ODE is

$$y(x) = C_1 x + C_2.$$

We need $y'(x)$ in order to use the first initial conditions

$$y'(x) = C_1.$$

Using the BC's,

$$\left\{ \begin{array}{rcl} C_1 & = & 0 \\ C_1\pi + C_2 & = & 0 \end{array} \right. \quad \Leftrightarrow \quad C_1 = C_2 = 0.$$

Hence, *there is no nontrivial solution.*

Case 3. $\Delta = -\lambda < 0 \Rightarrow \lambda > 0$. Then $r_1 = i\sqrt{\lambda}$, $r_2 = -i\sqrt{\lambda}$ are complex roots, and the solution to ODE is

$$y(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x.$$

We need $y'(x)$ in order to use the initial conditions

$$y'(x) = -\sqrt{\lambda}C_1 \sin\sqrt{\lambda}x + \sqrt{\lambda}C_2 \cos\sqrt{\lambda}x.$$

Using the BC's,

$$\left\{ \begin{array}{rcl} \sqrt{\lambda}C_2 & = & 0 \\ \cos(\sqrt{\lambda}\pi)C_1 + \sin(\sqrt{\lambda}\pi)C_2 & = & 0 \end{array} \right. ,$$

which implies

$$\cos(\sqrt{\lambda}\pi)C_1 = 0 \Leftrightarrow C_1 = 0 \text{ or } \cos(\sqrt{\lambda}\pi) = 0.$$

For nontrivial solutions we need

$$\cos(\sqrt{\lambda}\pi) = 0 \Leftrightarrow \sqrt{\lambda}\pi = n\pi + \frac{\pi}{2} \Leftrightarrow \sqrt{\lambda} = n + \frac{1}{2}.$$

Then, the eigenvalues are

$$\boxed{\lambda_n = (n + \tfrac{1}{2})^2}, \quad n \geq 0,$$

with eigenfunctions

$$\boxed{y_n(x) = C_n \cos((n + \tfrac{1}{2})x)}, \quad n \geq 0,$$

for some arbitrary constants C_n. Note the difference with the previous problem where the BC's are slightly different.

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5. Find the eigenvalues λ for which the given problem has a nontrivial solution. Also determine the corresponding eigenfunctions.

$$y'' + \lambda y = 0\,; \qquad 0 < x < \pi\,, \qquad y(0) - y'(0) = 0\,, \qquad y(\pi) = 0.$$

Solution

First we find the roots of the auxiliary equation.

$$r = \pm\sqrt{-\lambda} = \pm\sqrt{\Delta}.$$

We consider the three cases.

Case 1. $\Delta = -\lambda > 0 \Rightarrow \lambda < 0$. Then $r_1 = \sqrt{-\lambda}$, $r_2 = -\sqrt{-\lambda}$ are distinct real roots, and the solution to ODE is

$$y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}.$$

We need $y'(x)$ in order to use the first initial conditions

$$y'(x) = \sqrt{-\lambda}C_1 e^{\sqrt{-\lambda}x} - \sqrt{-\lambda}C_2 e^{-\sqrt{-\lambda}x}.$$

Using the BC's,

$$\left\{ \begin{array}{rcl} (1-\sqrt{-\lambda})C_1 + (1+\sqrt{-\lambda})C_2 & = & 0 \\ e^{\sqrt{-\lambda}\pi}C_1 + e^{-\sqrt{-\lambda}\pi}C_2 & = & 0 \end{array} \right. .$$

For nontrivial solutions we need

$$\det(A) = \begin{vmatrix} 1-\sqrt{-\lambda} & 1+\sqrt{-\lambda} \\ e^{\sqrt{-\lambda}\pi} & e^{-\sqrt{-\lambda}\pi} \end{vmatrix} = (1-\sqrt{-\lambda})e^{-\sqrt{-\lambda}\pi} - (1+\sqrt{-\lambda})e^{\sqrt{-\lambda}\pi} = 0.$$

Or

$$1 - \sqrt{-\lambda} - (1+\sqrt{-\lambda})e^{2\sqrt{-\lambda}\pi} = 0$$

Since $\lambda < 0$, then $-e^{2\sqrt{-\lambda}\pi} < 1$, and

$$1 - \sqrt{-\lambda} - (1+\sqrt{-\lambda})e^{2\sqrt{-\lambda}\pi} < 1 - \sqrt{-\lambda} - 1 - \sqrt{-\lambda} = -2\sqrt{-\lambda} < 0,$$

which is a contradiction. Thus, no non-trivial solution.

Case 2. $\Delta = -\lambda = 0 \Rightarrow \lambda = 0$. Then $r_1 = r_2 = 0$ are repeated real roots, and the solution to ODE is

$$y(x) = C_1 x + C_2.$$

We need $y'(x)$ in order to use the first initial conditions

$$y'(x) = C_1.$$

Using the BC's,

$$\begin{cases} C_2 - C_1 &= 0 \\ C_1\pi + C_2 &= 0 \end{cases} \quad \Leftrightarrow \quad C_1 = C_2 = 0.$$

Hence, *there is no nontrivial solution.*

Case 3. $\Delta = -\lambda < 0 \Rightarrow \lambda > 0$. Then $r_1 = i\sqrt{\lambda}$, $r_2 = -i\sqrt{\lambda}$ are complex roots, and the solution to ODE is

$$y(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x.$$

We need $y'(x)$ in order to use the initial conditions

$$y'(x) = -\sqrt{\lambda}C_1 \sin\sqrt{\lambda}x + \sqrt{\lambda}C_2 \cos\sqrt{\lambda}x.$$

Using the BC's,

$$\begin{cases} C_1 - \sqrt{\lambda}C_2 &= 0 \\ \cos(\sqrt{\lambda}\pi)C_1 + \sin(\sqrt{\lambda}\pi)C_2 &= 0 \end{cases},$$

which implies

$$C_1 = \sqrt{\lambda}C_2 \Rightarrow \sqrt{\lambda}\cos(\sqrt{\lambda}\pi)C_2 + \sin(\sqrt{\lambda}\pi)C_2 = 0 \Rightarrow \sqrt{\lambda} + \tan(\sqrt{\lambda}\pi) = 0,$$

which has infinite solutions. Thus, the eigenvalues are given by the implicit equation

$$\boxed{\lambda_n + \tan\left(\sqrt{\lambda_n}\pi\right) = 0}, \quad n \geq 1,$$

with eigenfunctions

$$\boxed{y_n(x) = C_n\left(\sqrt{\lambda_n}\cos\left(\sqrt{\lambda_n}x\right) + \sin\left(\sqrt{\lambda_n}x\right)\right)}, \quad n \geq 1,$$

for some arbitrary constants C_n.

■

6. Find the eigenvalues λ for which the given problem has a nontrivial solution. Also determine the corresponding eigenfunctions.

$$y'' + 4y' + \lambda y = 0\,; \qquad 0 < x < \pi/2\,, \qquad y(0) = 0\,, \qquad y(\pi/2) = 0.$$

Solution

First we find the roots of the auxiliary equation.

$$r = -2 \pm \sqrt{4-\lambda} = 2 \pm \sqrt{\Delta}.$$

We consider the three cases.

Case 1. $\Delta = 4 - \lambda > 0 \Rightarrow \lambda < 4$. Then $r_1 = -2 + \sqrt{4-\lambda}$, $r_2 = -2 - \sqrt{4-\lambda}$ are distinct real roots, and the solution to ODE is

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

Using the BC's,

$$\begin{cases} C_1 + C_2 &= 0 \\ e^{r_1\pi/2}C_1 + e^{r_2\pi/2}C_2 &= 0 \end{cases}.$$

For nontrivial solutions we need

$$\det(A) = \begin{vmatrix} 1 & 1 \\ e^{r_1\pi/2} & e^{r_2\pi/2} \end{vmatrix} = e^{r_1\pi/2} - e^{r_2\pi/2} = 0,$$

which implies $r_1 = r_2$, which is a contradiction since the roots are different. Hence, *there is no nontrivial solution.*

Case 2. $\Delta = 4 - \lambda = 0 \Rightarrow \lambda = 4$. Then $r_1 = r_2 = -2$ are repeated real roots, and the solution to ODE is

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x}.$$

Using the BC's,

$$\begin{cases} C_1 &= 0 \\ e^{-\pi}C_1 + \frac{\pi}{2}e^{\pi}C_2 &= 0 \end{cases} \quad \Leftrightarrow \quad C_1 = C_2 = 0.$$

Hence, *there is no nontrivial solution.*

Case 3. $\Delta = 4 - \lambda < 0 \Rightarrow \lambda > 4$. Then $r_1 = -2 + i\sqrt{\lambda - 4}$, $r_2 = -2 - i\sqrt{\lambda - 4}$ are complex roots, and the solution to ODE is

$$y(x) = C_1 e^{-2x}\cos\sqrt{\lambda - 4}x + C_2 e^{-2x}\sin\sqrt{\lambda - 4}x.$$

Using the BC's,

$$\begin{cases} C_1 &= 0 \\ e^{-\pi}\cos(\sqrt{\lambda - 4}\frac{\pi}{2})C_1 + e^{-\pi}\sin(\sqrt{\lambda - 4}\frac{\pi}{2})C_2 &= 0 \end{cases},$$

which implies

$$\sin(\sqrt{\lambda - 4}\tfrac{\pi}{2})C_2 = 0 \Leftrightarrow C_2 = 0 \text{ or } \sin(\sqrt{\lambda - 4}\tfrac{\pi}{2}) = 0.$$

For nontrivial solutions we need

$$\sin(\sqrt{\lambda - 4}\tfrac{\pi}{2}) = 0 \Leftrightarrow \sqrt{\lambda - 4}\tfrac{\pi}{2} = n\pi \Leftrightarrow \sqrt{\lambda - 4} = 2n.$$

Thus, the eigenvalues are

$$\boxed{\lambda_n = 4n^2 + 4}, \quad n > 0,$$

with eigenfunctions

$$\boxed{y_n(x) = C_n e^{-2x}\sin(2nx)}, \quad n > 0,$$

for some arbitrary constants C_n.

■