MATH 201[1] – Midterm summary study guide[2]

First Order Equations

$$\text{(1)} \quad \textbf{General Form of ODE: } \frac{dy}{dx} = f(x,y)$$

$$\text{(2)} \quad \textbf{Initial Value Problem: } y' = f(x,y),\ y(x_0) = y_0$$

Linear Equations

(3) **General Form:** $y' + p(x)y = f(x)$

(4) **Integrating Factor:** $\mu(x) = e^{\int p(x)dx}$

(5) $$\Longrightarrow \frac{d}{dx}\left(\mu(x)y\right) = \mu(x)f(x)$$

(6) **General Solution:** $y = \dfrac{1}{\mu(x)}\left(\displaystyle\int \mu(x)f(x)dx + C\right)$

Homogeneous Equations

(7) **General Form:** $y' = f(y/x)$

(8) **Substitution:** $y = zx$

(9) $$\Longrightarrow y' = z + xz'$$

The result is always separable in z:

(10) $$\frac{dz}{f(z) - z} = \frac{dx}{x}$$

Bernoulli Equations

(11) **General Form:** $y' + p(x)y = q(x)y^n$

(12) **Substitution:** $z = y^{1-n}$

The result is always linear in z:

(13) $$z' + (1-n)p(x)z = (1-n)q(x)$$

Exact Equations

(14) **General Form:** $M(x,y)dx + N(x,y)dy = 0$

(15) **Test for Exactness:** $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

(16) **Solution:** $\phi = C$ where

(17) $$M = \frac{\partial \phi}{\partial x} \text{ and } N = \frac{\partial \phi}{\partial y}$$

Method for Solving Exact Equations:

1. Let $\phi = \int M(x,y)dx + h(y)$
2. Set $\dfrac{\partial \phi}{\partial y} = N(x,y)$
3. Simplify and solve for $h(y)$.
4. Substitute the result for $h(y)$ in the expression for ϕ from step 1 and then set $\phi = 0$. This is the solution.

Alternatively:

1. Let $\phi = \int N(x,y)dy + g(x)$
2. Set $\dfrac{\partial \phi}{\partial x} = M(x,y)$
3. Simplify and solve for $g(x)$.
4. Substitute the result for $g(x)$ in the expression for ϕ from step 1 and then set $\phi = 0$. This is the solution.

Integrating Factors

Case 1: If $P(x,y)$ depends only on x, where

(18) $$P(x,y) = \frac{M_y - N_x}{N} \Longrightarrow \mu(x) = e^{\int P(x)dx}$$

then

(19) $$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

is exact.

Case 2: If $Q(x,y)$ depends only on y, where

(20) $$Q(x,y) = \frac{N_x - M_y}{M} \Longrightarrow \mu(y) = e^{\int Q(y)dy}$$

Then

(21) $$\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0$$

is exact.

[1] MATH 201 - Differential Equations – University of Alberta

Second Order Linear Equations

General Form of the Equation

$$\text{(22)} \qquad \textbf{General Form: } a(t)y'' + b(t)y' + c(t)y = g(t)$$

$$\text{(23)} \qquad \textbf{Homogeneous: } a(t)y'' + b(t)y' + c(t)y = 0$$

$$\text{(24)} \qquad \textbf{Standard Form: } y'' + p(t)y' + q(t)y = f(t)$$

The **general solution** of (22) or (24) is

$$\text{(25)} \qquad y = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$$

where $y_1(t)$ and $y_2(t)$ are linearly independent solutions of (23).

Linear Independence and The Wronskian

Two functions $f(x)$ and $g(x)$ are **linearly dependent** if there exist numbers a and b, not both zero, such that $af(x) + bg(x) = 0$ for all x. If no such numbers exist then they are **linearly independent**.

If y_1 and y_2 are two solutions of (23) then

$$\text{(26)} \qquad \textbf{Wronskian: } W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

$$\text{(27)} \qquad \textbf{Abel's Formula: } W(t) = Ce^{-\int p(t)dt}$$

and the following are all equivalent:

1. $\{y_1, y_2\}$ are linearly independent.
2. $\{y_1, y_2\}$ are a fundamental set of solutions.
3. $W(y_1, y_2)(t_0) \neq 0$ at some point t_0.
4. $W(y_1, y_2)(t) \neq 0$ for all t.

Initial Value Problem

$$\text{(28)} \qquad \begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \end{cases}$$

Linear Equation: Constant Coefficients

$$\text{(29)} \qquad \textbf{Homogeneous: } ay'' + by' + cy = 0$$

$$\text{(30)} \qquad \textbf{Non-homogeneous: } ay'' + by' + cy = g(t)$$

$$\text{(31)} \qquad \textbf{Characteristic Equation: } ar^2 + br + c = 0$$

$$\text{(32)} \qquad \textbf{Quadratic Roots: } r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The solution of (29) is given by:

$$\text{(33)} \qquad \textbf{Real Roots}(r_1 \neq r_2)\textbf{: } y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$$\text{(34)} \qquad \textbf{Repeated}(r_1 = r_2)\textbf{: } y_h = (C_1 + C_2 t)e^{r_1 t}$$

$$\text{(35)} \qquad \textbf{Complex}(r = \alpha \pm i\beta)\textbf{: } y_h = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t)$$

The solution of (30) is $y = y_p + y_h$ where y_h is given by (33) through (35) and y_p is found by **undetermined coefficients** or **variation of parameters**.

Heuristics for Undetermined Coefficients (Trial and Error)

If $f(t) =$	then guess that a particular solution $y_p =$
$P_n(t)$	$t^s(A_0 + A_1 t + \cdots + A_n t^n)$
$P_n(t)e^{at}$	$t^s(A_0 + A_1 t + \cdots + A_n t^n)e^{at}$
$P_n(t)e^{at}\sin bt$ or $P_n(t)e^{at}\cos bt$	$t^s e^{at}[(A_0 + A_1 t + \cdots + A_n t^n)\cos bt$ $+(B_0 + B_1 t + \cdots + B_n t^n)\sin bt]$

where $s = 0, 1, 2$ if $r = a$ is not a root, is a single root, or a double root of (31), respectively.

Method of Reduction of Order

When solving (23), given y_1, then y_2 is given by

$$\text{(36)} \qquad y_2 = y_1 \int \frac{e^{-\int p(x)dx} dx}{y_1(x)^2}$$

Method of Variation of Parameters

If $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to (23) then a particular solution to (24) is

$$\text{(37)} \qquad y_P(t) = -y_1(t)\int \frac{y_2(t)f(t)}{W(t)}dt + y_2(t)\int \frac{y_1(t)f(t)}{W(t)}dt$$

Cauchy-Euler Equation

$$\text{(38)} \qquad \textbf{ODE: } ax^2y'' + bxy' + cy = g(x)$$

$$\text{(39)} \qquad \textbf{Auxilliary Equation: } ar(r-1) + br + c = 0$$

The homogeneous solutions of (38) depend on the roots of (39):

$$\text{(40)} \qquad \textbf{Real Roots}(r_1 \neq r_2)\textbf{: } y_h = C_1 x^{r_1} + C_2 x^{r_2}$$

$$\text{(41)} \qquad \textbf{Repeated}(r_1 = r_2)\textbf{: } y_h = C_1 x^r + C_2 x^r \ln x$$

$$\text{(42)} \qquad \textbf{Complex}(r_{1,2} = \alpha \pm i\beta)\textbf{: } y_h = x^\alpha[C_1\cos(\beta\ln x) + C_2\sin(\beta\ln x)]$$

The substitution $x = e^t$ transform (38) into a (30)

$$\text{(43)} \qquad ay''(t) + (b-a)y'(t) + cy(t) = g(e^t)$$

Series Solutions

$$\text{(44)} \qquad y'' + p(x)y' + q(x)y = 0$$

If x_0 is a **regular point** of (44) then

$$\text{(45)} \qquad y(x) = \sum_{k=0}^{\infty} a_k(x - x_k)^k$$

At a **Regular Singular Point** x_0:

$$\text{(46)} \qquad \textbf{Indicial Equation: } r^2 + (p(0) - 1)r + q(0) = 0$$

$$\text{(47)} \qquad \textbf{First Solution: } y_1 = (x - x_0)^{r_1}\sum_{k=0}^{\infty} a_k(x - x_k)^k$$

Where r_1 is the larger real root if both roots of (46) are real or either root if the solutions are complex.