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Lab 8: LT: discontinuous functions and Dirac delta function

Discontinuous functions

1. Express the given function using window and step functions and compute its Laplace transform

$$g(t) = \begin{cases} 0, & 0 < t < 1, \\ 2, & 1 < t < 2, \\ 1, & 2 < t < 3, \\ 3, & 3 < t. \end{cases}$$

2. Determine the current as a function of time t for the given RLC series circuit. Plot the solution. The current $I(t)$ in a RLC series circuit is governed by the initial value problem

$$I''(t) + 2I'(t) + 2I(t) = g(t); \quad I(0) = 10, \quad I'(0) = 0,$$

where,

$$g(t) = \begin{cases} 20, & 0 < t < 3\pi \\ 0, & 3\pi < t < 4\pi \\ 20, & 4\pi < t \end{cases}.$$

3. Solve the initial value problem

$$y''(t) + 4y(t) = f(t); \quad y(0) = 0, \quad y'(0) = 0,$$

where,

$$f(t) = \begin{cases} 2t, & 0 \le t < 2 \\ 4, & 2 \le t \end{cases}.$$

Dirac delta function

4. Evaluate the following integral

$$\int_{-\infty}^{\infty} \sin(3t)\delta(t - \pi/2)dt.$$

5. Solve the given symbolic initial value problem

$$y'' + 2y' - 3y = \delta(t-1) - \delta(t-2), \quad y(0) = 2, \quad y'(0) = -2.$$

6. Solve the given symbolic initial value problem

$$y'' + 5y' - 6y = e^{-t}\delta(t-2), \quad y(0) = 2, \quad y'(0) = -5.$$

Solutions

Theory and problems from: Nagel, Saff & Sneider, *Fundamentals of Differential Equations*, Eighth Edition, Adisson–Wesley.

→ The **unit step function** $u(t)$ is defined by

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & 0 < t. \end{cases}$$

→ The **rectangular window function** (or square pulse) $\Pi_{a,b}(t)$ is defined by

$$\Pi_{a,b}(t) = u(t-a) - u(t-b) = \begin{cases} 0, & t < a, \\ 1, & a < t < b, \\ 0, & b < t. \end{cases}$$

→ The **Dirac delta function** $\delta(t)$ is characterized by

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases},$$

and

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a).$$

→ The Dirac delta function is the **derivative** of the unit step function

$$\frac{d}{dt}u(t-a) = \delta(t-a).$$

→ Brief **table of Laplace Transforms**.

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$	$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$	
$e^{at}f(t)$	$F(s-a)$	1	$\frac{1}{s}$	
$f'(t)$	$sF(s) - f(0)$	e^{at}	$\frac{1}{s-a}$	$s > a$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	t^n	$\frac{n!}{s^{n+1}}$	
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	$\sin bt$	$\frac{b}{s^2+b^2}$	
$(f * g)(t)$	$F(s)G(s)$	$\cos bt$	$\frac{s}{s^2+b^2}$	
$u(t-a)$	$\frac{e^{-as}}{s}$	$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$f(t-a)u(t-a)$	$e^{-as}F(s)$	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$\delta(t-a)$	e^{-as}	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$\int_0^s f(\tau)d\tau$	$\frac{1}{s}F(s)$	$\sinh bt$	$\frac{b}{s^2-b^2}$	
$\frac{1}{t}f(t)$	$\int_s^\infty F(\sigma)d\sigma$	$\cosh bt$	$\frac{s}{s^2-b^2}$	

1. Express the given function using window and step functions and compute its Laplace transform

$$g(t) = \begin{cases} 0, & 0 < t < 1, \\ 2, & 1 < t < 2, \\ 1, & 2 < t < 3, \\ 3, & 3 < t. \end{cases}$$

Solution
Expressing $g(t)$ in unit step functions

$$\begin{aligned} g(t) &= (0)\Pi_{0,1}(t) + (2)\Pi_{1,2}(t) + (1)\Pi_{2,3}(t) + (3)u(t-3) \\ &= 2(u(t-1) - u(t-2)) + (u(t-2) - u(t-3)) + 3u(t-3) \\ &= 2u(t-1) - u(t-2) + 2u(t-3). \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{L}\{g(t)\} &= 2\mathcal{L}\{u(t-1)\} - \mathcal{L}\{u(t-2)\} + 2\mathcal{L}\{u(t-3)\} \\ &= \boxed{\frac{2e^{-s}}{s} - \frac{e^{-2s}}{s} + \frac{2e^{-3s}}{s}} \end{aligned}$$

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2. Determine the current as a function of time t for the given RLC series circuit. Plot the solution. The current $I(t)$ in a RLC series circuit is governed by the initial value problem

$$I''(t) + 2I'(t) + 2I(t) = g(t); \quad I(0) = 10, \quad I'(0) = 0,$$

where,

$$g(t) = \begin{cases} 20, & t < 3\pi \\ 0, & 3\pi < t < 4\pi \\ 20, & 4\pi < t \end{cases}.$$

Solution
We can rewrite the function $g(t)$ using unit step function as follows:

$$g(t) = 20 - 20u(t-3\pi) + 20u(t-4\pi) = 20(1 - u(t-3\pi) + u(t-4\pi))$$

Hence our IVP can be rewritten as:

$$I''(t) + 2I'(t) + 2I(t) = 20(1 - u(t-3\pi) + u(t-4\pi)); \quad I(0) = 10, \quad I'(0) = 0,$$

Taking the Laplace transform on both sides of this equation, we obtain:

$$\mathcal{L}\{I''(t)\} + 2\mathcal{L}\{I'(t)\} + 2\mathcal{L}\{I(t)\} = 20(\mathcal{L}\{1\} - \mathcal{L}\{u(t-3\pi)\} + \mathcal{L}\{u(t-4\pi)\})$$

Expanding, we obtain:

$$\begin{aligned} (s^2 J(s) - sI(0) - I'(0)) + 2(sJ(s) - I(0)) + 2J(s) &= 20\left(\frac{1}{s} - \frac{e^{-3\pi s}}{s} + \frac{e^{-4\pi s}}{s}\right) \\ s^2 J(s) - 10s + 2sJ(s) - 20 + 2J(s) &= \frac{20}{s} - \frac{20e^{-3\pi s}}{s} + \frac{20e^{-4\pi s}}{s} \end{aligned}$$

Isolating for $J(s)$, we obtain:

$$J(s) = \frac{20}{s(s^2+2s+2)} - \frac{20e^{-3\pi s}}{s(s^2+2s+2)} + \frac{20e^{-4\pi s}}{s(s^2+2s+2)} + \frac{10s+20}{s^2+2s+2}$$

We need to find the partial fraction decomposition of $F(s) = \dfrac{20}{s(s^2+2s+2)}$.

$$\begin{aligned} F(s) = \frac{20}{s(s^2+2s+2)} &= \frac{As+B}{s^2+2s+2} + \frac{C}{s} \\ &= \frac{(As+B)s + C(s^2+2s+2)}{s(s^2+2s+2)} \\ &= \frac{s^2(A+C) + s(B+2C) + 2C}{s(s^2+2s+2)} \end{aligned}$$

This leads to the following system of equations:

$$\begin{aligned} A + C &= 0 \\ B + 2C &= 0 \\ 2C &= 20 \end{aligned}$$

This can easily be solve. $A = -10$,$B = -20$ and $C = 10$ Hence:

$$\begin{aligned} J(s) &= -\frac{10s+20}{s^2+2s+2} + \frac{10}{s} - F(s)e^{-3\pi s} + F(s)e^{-4\pi s} + \frac{10s+20}{s^2+2s+2} \\ &= \frac{10}{s} - F(s)e^{-3\pi s} + F(s)e^{-4\pi s} \end{aligned}$$

The inverse Laplace transform of $F(s)$ is given by:

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{-\frac{10s+20}{s^2+2s+2} + \frac{10}{s}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{-10(s+1)-10}{(s+1)^2+1}\right\} + 10\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} \\ &= -10\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - 10\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + 10\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} \\ &= -10e^{-t}\cos(t) - 10e^{-t}\sin(t) + 10 \end{aligned}$$

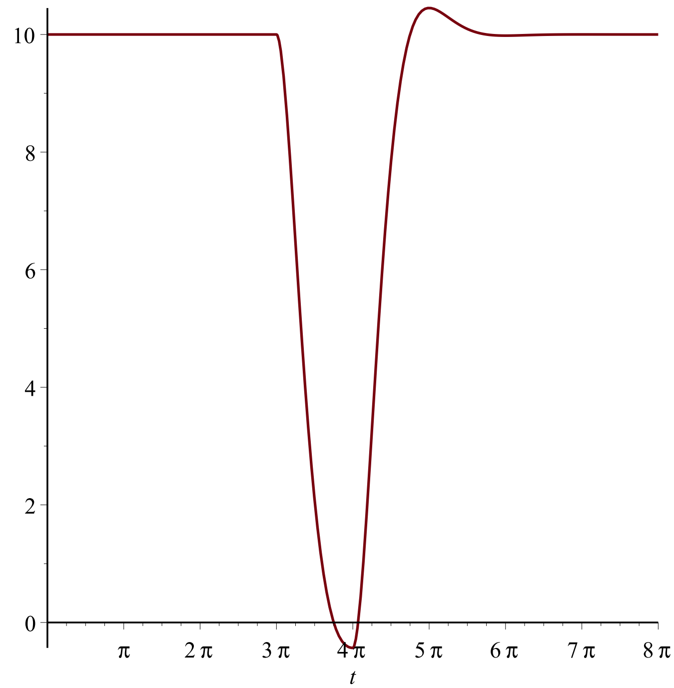
We can now find the function $I(t)$ via the property $\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a)$. Taking the inverse Laplace transform of our function $J(s)$, we obtain:

$$\begin{aligned} I(t) &= \mathcal{L}^{-1}\{J(s)\} \\ &= 10\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{e^{-3\pi s}F(s)\right\} + \mathcal{L}^{-1}\left\{e^{-4\pi s}F(s)\right\} \\ &= 10 - \left(-10e^{-(t-3\pi)}\cancelto{-\cos t}{\cos(t-3\pi)} - 10e^{-(t-3\pi)}\cancelto{-\sin t}{\sin(t-3\pi)} + 10\right)u(t-3\pi) \\ &\quad + \left(-10e^{-(t-4\pi)}\cancelto{\cos t}{\cos(t-4\pi)} - 10e^{-(t-4\pi)}\cancelto{\sin t}{\sin(t-4\pi)} + 10\right)u(t-4\pi) \\ &= 10 - \left(10e^{-(t-3\pi)}\cos(t) + 10e^{-(t-3\pi)}\sin(t) + 10\right)u(t-3\pi) \\ &\quad + \left(-10e^{-(t-4\pi)}\cos(t) - 10e^{-(t-4\pi)}\sin(t) + 10\right)u(t-4\pi) \end{aligned}$$

Hence,

$$\boxed{I(t) = 10 - 10u(t-3\pi)\left[1 + e^{-(t-3\pi)}\left(\cos t + \sin t\right)\right] + 10u(t-4\pi)\left[1 - e^{-(t-4\pi)}(\cos t + \sin t)\right]}.$$

If we plot the function $I(t)$, we obtain:



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3. Solve the initial value problem

$$y''(t) + 4y(t) = f(t); \quad y(0) = 0, \quad y'(0) = 0,$$

where,

$$f(t) = \begin{cases} 2t, & 0 \le t < 2 \\ 4, & 2 \le t \end{cases}.$$

Solution
The first step is to write $f(t)$ in terms of unit step functions

$$f(t) = 2t + (4 - 2t)u(t-2).$$

Note that $h(t) = 4 - 2t$ is not of the form $f(t-2)$ so using the property for $f(t-a)u(t-a)$ is not straight forward. A trick that will always work is to evalute $h(t+2)$[1],

$$h(t+2) = 4 - 2(t+2) = -2t, \quad \Rightarrow \quad \mathcal{L}\{h(t+2)\} = -\frac{2}{s^2},$$

and multiply by e^{-2s}

$$\mathcal{L}\{(4-2t)u(t-2)\} = -\frac{2}{s^2}e^{-2s}.$$

[1]For the property $\mathcal{L}\{f(t-a)u(t-a)\} = F(s)e^{-as}$, we need the Laplace of $f(t)$, not the Laplce of $f(t-a)$. If we have $h(t)u(t-2)$, let $f(t-a) = h(t)$, then $f(t) = h(t+a)$. So $\mathcal{L}\{h(t)u(t-a)\} = \mathcal{L}\{f(t-a)u(t-a)\} = \mathcal{L}\{f(t)\}e^{-as} = \mathcal{L}\{h(t+a)\}e^{-as}$.

Then,

$$F(s) = \mathcal{L}\{2t\} + \mathcal{L}\{(4-2t)u(t-2)\} = \frac{2}{s^2} - \frac{2}{s^2}e^{-2s} = 2\left(\frac{1}{s^2} - \frac{e^{-2s}}{s^2}\right)$$

Alternatively, we can write

$$f(t) = 2t(u(t) - u(t-2)) + 4u(t-2),$$

then, its Laplace transform (using the property for $tf(t)$) is, as expected,

$$F(s) = 2(-1)\frac{d}{ds}\left[\frac{1}{s} - \frac{e^{-2s}}{s}\right] + 4\frac{e^{-2s}}{s} = 2\left(\frac{1}{s^2} - \frac{e^{-2s}}{s^2}\right).$$

Next step is to apply Laplace of both side of the ODE

$$s^2Y(s) - 0s - 0 + 4Y(s) = 2\left(\frac{1}{s^2} - \frac{e^{-2s}}{s^2}\right).$$

Isolating $Y(s)$ we have

$$Y(s) = \frac{2}{s^2(s^2+4)} - \frac{2}{s^2(s^2+4)}e^{-2s} = G(s) - G(s)e^{-2s}.$$

Is practical to take e^{-as} terms as a common factor, since this only shift the inverse Laplace of $G(s)$.

Now we take the inverse Laplace transform of $G(s)$ using partial fractions

$$G(s) = \frac{2}{s^2(s^2+4)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+4} = \frac{1}{2s^2} - \frac{1}{2(s^2+4)}.$$

Hence,

$$g(t) = \frac{1}{2}t - \frac{1}{4}\sin 2t$$

Finally,

$$y(t) = g(t) - g(t-2)u(t-2)$$

$$\Rightarrow \boxed{y(t) = \frac{t}{2} - \frac{\sin 2t}{4} - \left(\frac{t-2}{2} - \frac{\sin 2(t-2)}{4}\right)u(t-2)}.$$

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4. Evaluate the following integral

$$\int_{-\infty}^{\infty} \sin(3t)\delta(t-\pi/2)dt.$$

Solution

Use the property of the Dirac delta function $\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$

$$\int_{-\infty}^{\infty} \sin(3t)\delta(t-\pi/2)dt = \sin\left(\frac{3\pi}{2}\right) = -1.$$

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5. Solve the given symbolic initial value problem

$$y'' + 2y' - 3y = \delta(t-1) - \delta(t-2), \quad y(0) = 2, \quad y'(0) = -2.$$

Solution
Applying a Laplace transform on both sides, we obtain:

$$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) - 3Y(s) = e^{-s} - e^{-2s}$$

Subbing in our initial conditions, we obtain:

$$s^2Y(s) - 2s + 2 + 2sY(s) - 4 - 3Y(s) = e^{-s} - e^{-2s}$$

Isolating $Y(s)$, we obtain:

$$\begin{aligned} Y(s) &= \frac{2s+2}{s^2+2s-3} + \frac{e^{-s}}{s^2+2s-3} - \frac{e^{-2s}}{s^2+2s-3} \\ &= \frac{2s+2}{(s+3)(s-1)} + \frac{1}{(s+3)(s-1)}e^{-s} - \frac{1}{(s+3)(s-1)}e^{-2s} \end{aligned}$$

Decomposing these functions into partial fractions,we obtain:

$$Y(s) = \left(\frac{1}{s-1} + \frac{1}{s+3}\right) + \frac{1}{4}\left(\frac{1}{s-1} - \frac{1}{s+3}\right)e^{-s} - \frac{1}{4}\left(\frac{1}{s-1} - \frac{1}{s+3}\right)e^{-2s}$$

Taking the inverse Laplace transform using the property

$$\mathcal{L}^{-1}\left\{G(s)e^{-as}\right\}(t) = g(t-a)u(t-a),$$

we have

$$\boxed{y(t) = e^t + e^{-3t} + \tfrac{1}{4}\left(e^{t-1} - e^{-3(t-1)}\right)u(t-1) - \tfrac{1}{4}\left(e^{t-2} - e^{-3(t-2)}\right)u(t-2)}.$$

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6. Solve the given symbolic initial value problem

$$y'' + 5y' - 6y = e^{-t}\delta(t-2), \quad y(0) = 2, \quad y'(0) = -5.$$

Solution
Recall the property $\mathcal{L}\{e^{at}f(t)\} = F(t-a)$. Then, applying a Laplace transform on both sides, we obtain

$$Y(s) = \frac{2s+5}{s^2+5s-6} + e^{-2}\frac{1}{s^2+5s-6}e^{-2s}.$$

The rest of the problem is similar to the previous example.

$$\boxed{y(t) = e^{-6t} + e^t + \frac{e^{-2}}{7}\left(-e^{-6(t-2)} + e^{(t-2)}\right)u(t-2)}.$$

Note that only when we take the Laplace transform we can simplify

$$e^{-t}\delta(t-2) = e^{-2}\delta(t-2),$$

so we could consider $e^{-2} = K$ as a constant all the time if we want. In other words,

$$\mathcal{L}\{e^{-t}\delta(t-2)\}(s) = \mathcal{L}\{e^{-2}\delta(t-2)\}(s) = e^{-2}e^{-2s} = e^{-2(s+1)}.$$

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