MATH 201 Differential Equations – University of Alberta
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Lab 7: Laplace transform

Inverse Laplace transform

1. Determine the inverse Laplace transform of

$$\frac{s-1}{2s^2+4s+6}.$$

2. Determine $\mathcal{L}^{-1}\{F\}$

$$s^2F(s) + sF(s) - 6F(s) = \frac{s^2+4}{s^2+s}.$$

3. Find the inverse Laplace transform of

$$\tanh^{-1}(s).$$

Solving IVP using LT

4. Solve the given initial value problem using the method of Laplace transforms

$$y'' - y' - 2y = 0; \quad y(0) = -2, \quad y'(0) = 5.$$

5. Solve for $Y(s)$, the Laplace transform of the solution $y(t)$ to the given initial value problem

$$y'' + 4y = g(t); \quad y(0) = -1, \quad y'(0) = 0,$$

where

$$g(t) = \begin{cases} t, & t < 2 \\ 5, & t > 2 \end{cases}.$$

Solutions

Theory and problems from: Nagel, Saff & Sneider, *Fundamentals of Differential Equations*, Eighth Edition, Adisson–Wesley.

→ Definition and properties of **Laplace Transform.**

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)e^{-st}dt$$
$$\mathcal{L}\{f+g\} = F(s) + G(s)$$
$$\mathcal{L}\{cf\} = cF(s)$$
$$\mathcal{L}\{e^{at}f\} = F(s-a)$$
$$\mathcal{L}\{f'\} = sF(s) - f(0)$$
$$\mathcal{L}\{f''\} = s^2F(s) - sf(0) - f'(0)$$
$$\mathcal{L}\{t^n f\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

→ Brief **table of Laplace Transforms**.

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$ $\quad s > a$
t^n	$\frac{n!}{s^{n+1}}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$ $\quad s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$ $\quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$ $\quad s > a$

→ **Laplace transform of an IVP**. Let the IVP

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_0'.$$

Then the Laplace transform of the solution $y(t)$ is

$$Y(s) = \underbrace{\frac{(as+b)y_0 + ay_0'}{as^2+bs+c}}_{\text{initial conditions}} + \underbrace{\frac{F(s)}{as^2+bs+c}}_{\text{particular sol.}}.$$

1. Determine the inverse Laplace transform of

$$\frac{s-1}{2s^2+4s+6}.$$

Solution
We focus on the denominator. The first attempt in inverse Laplace transforms is to factorize the denominator. However, in this case $2s^2+4s+6$ has complex roots. The next attempt is to complete squares to get something of the form $(s-a)^2+b^2$.

$$2s^2-4s+6 = 2(s^2+2s+3) = 2((s+1)^2+2) = 2((s+1)^2+(\sqrt{2})^2).$$

The 2 in front of the polynomial is nothing but a constant. Now, we rewrite the numerator so we have terms of the form

$$c\frac{s-a}{(s-a)^2+b^2}, \qquad \text{and} \qquad c\frac{b}{(s-a)^2+b^2}.$$

That is

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s-1}{2s^2+4s+6}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+1-2}{2((s+1)^2+(\sqrt{2})^2)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{s+1}{(s+1)^2+(\sqrt{2})^2} + \frac{-\sqrt{2}}{2}\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}\right\}.
\end{aligned}$$

Thus, it follows from the table of Laplace transforms that

$$\boxed{\mathcal{L}^{-1}\left\{\frac{s-1}{2s^2+4s+6}\right\} = \frac{1}{2}e^{-t}\cos\sqrt{2}t - \frac{\sqrt{2}}{2}e^{-t}\sin\sqrt{2}t}.$$

■

2. Determine $\mathcal{L}^{-1}\{F\}$

$$s^2F(s)+sF(s)-6F(s) = \frac{s^2+4}{s^2+s}.$$

Solution
Isolating $F(s)$

$$\begin{aligned}
s^2F(s)+sF(s)-6F(s) &= \frac{s^2+4}{s^2+s} \\
F(s)(s^2+s-6) &= \frac{s^2+4}{s^2+s} \\
F(s) &= \left(\frac{1}{s^2+s-6}\right)\left(\frac{s^2+4}{s^2+s}\right) \\
&= \frac{s^2+4}{s(s+1)(s+3)(s-2)} \\
&= \frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+3}+\frac{D}{s-2}
\end{aligned}$$

If we combine these fractions under the same denominator, we have:

$$F(s) = \frac{A(s+1)(s+3)(s-2)+B(s)(s+3)(s-2)+C(s)(s+1)(s-2)+D(s)(s+1)(s+3)}{s(s+1)(s+3)(s-2)}$$

Collecting all the powers of s we find:

$$F(s) = \frac{s^3(A+B+C+D) + s^2(2A+B-C+4D) + s(-5A-6B-2C+3D) - 6A}{s(s+1)(s+3)(s-2)}$$

This implies that:

$$\begin{aligned} A+B+C+D &= 0 \quad (1)\\ 2A+B-C+4D &= 1 \quad (2)\\ -5A-6B-2C+3D &= 0 \quad (3)\\ -6A &= 4 \quad (4) \end{aligned}$$

In Matrix form we have:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & -1 & 4 & 1 \\ -5 & -6 & -2 & 3 & 0 \\ -6 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Which upon reducing, we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & 0 & \frac{5}{6} \\ 0 & 0 & 1 & 0 & -\frac{13}{30} \\ 0 & 0 & 0 & 1 & \frac{4}{15} \end{bmatrix}$$

Hence ,

$$F(s) = -\frac{2}{3}\frac{1}{s} + \frac{5}{6}\frac{1}{s+1} - \frac{13}{30}\frac{1}{s+3} + \frac{4}{15}\frac{1}{s-2}$$

Taking the inverse Laplace transform, we have:

$$\boxed{f(t) = \mathcal{L}^{-1}\{F\} = -\frac{2}{3} + \frac{5}{6}e^{-t} - \frac{13}{30}e^{-3t} + \frac{4}{15}e^{2t}}. \quad (5)$$

■

3. Find the inverse Laplace transform of

$$\tanh^{-1}(s).$$

Solution
In problems of inverse Laplace transform of trigonometric or trascendental functions that are not in the table, usually one of its derivatives is a fraction. In this case

$$(\tanh^{-1} s)' = \frac{1}{1-s^2},$$

which looks like the Laplace transform of sinh. For this problem we will use the property

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s).$$

Let $F(s) = \tanh^{-1}(s)$, then

$$-F'(s) = -\frac{1}{1-s^2} = \frac{1}{s^2-1}.$$

Using the property

$$tf(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \sinh t.$$

Isolating $f(t)$, we have

$$\boxed{f(t) = \frac{\sinh t}{t}}$$

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4. Solve the given initial value problem using the method of Laplace transforms

$$y'' - y' - 2y = 0; \quad y(0) = -2, \quad y'(0) = 5.$$

Solution
Applying a Laplace transform on both sides, we obtain:

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$$
$$\left(s^2Y(s) - sy(0) - y'(0)\right) - (sY(s) - y(0)) - 2Y(s) = 0$$
$$s^2Y(s) + 2s - 5 - sY(s) - 2 - 2Y(s) = 0$$

Isolating $Y(s)$, we obtain:

$$Y(s) = \frac{-2s+7}{s^2-s-2} = \frac{-2s+7}{(s+1)(s-2)}$$

We will now have to decompose $\dfrac{-2s+7}{(s+1)(s-2)}$ into partial fractions:

$$\begin{aligned}\frac{-2s+7}{(s+1)(s-2)} &= \frac{A}{s+1} + \frac{B}{s-2} \\ &= \frac{A(s-2)+B(s+1)}{(s+1)(s-2)} \\ &= \frac{s(A+B)+(-2A+B)}{(s+1)(s-2)}\end{aligned}$$

Hence, we have:

$$\begin{aligned}A+B &= -2 \\ -2A+B &= 7\end{aligned}$$

Solving this system, we find $A = -3$ and $B = 1$. Hence,

$$Y(s) = -\frac{3}{s+1} + \frac{1}{s-2}$$

Taking the inverse Laplace transform, we find

$$\boxed{y(t) = -3e^{-t} + e^{2t}}.$$

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5. Solve for $Y(s)$, the Laplace transform of the solution $y(t)$ to the given initial value problem

$$y'' + 4y = g(t); \quad y(0) = -1, \quad y'(0) = 0,$$

where

$$g(t) = \begin{cases} t, & t < 2 \\ 5, & t > 2 \end{cases}$$

Solution
Applying a Laplace transform on both sides, we obtain:

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

To evaluate $\mathcal{L}\{g(t)\}$ we need to write $g(t)$ in term of unit step functions,

$$g(t) = t + (5-t)u(t-2)$$

We know $\mathcal{L}\{f(t-a)u(t-a)\} = F(s)e^{-as}$, but $(5-t)u(t-2)$ does not have that exact form. To solve this we evaluate

$$h(t) = 5 - t, \quad \Rightarrow \quad h(t+2) = 5 - (t+2) = 3 - t,$$

take Laplace transform

$$\mathcal{L}\{h(t+2)\} = \mathcal{L}\{3-t\} = \frac{3}{s} - \frac{1}{s^2}, \quad \Rightarrow \quad \mathcal{L}\{h(t)u(t-2)\} = \left(\frac{3}{s} - \frac{1}{s^2}\right)e^{-2s}.$$

and simply write

$$G(s) = \frac{1}{s^2} + \left(\frac{3}{s} - \frac{1}{s^2}\right)e^{-2s}.$$

Putting all of this together we have

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1}{s^2} + \left(\frac{3}{s} - \frac{1}{s^2}\right)e^{-2s}$$

$$s^2Y(s) + s + 4Y(s) = \frac{1}{s^2} + \frac{3s-1}{s^2}e^{-2s}$$

Isolating $Y(s)$, we obtain:

$$\boxed{Y(s) = \frac{3e^{-2s}}{s(s^2+4)} - \frac{e^{-2s}}{s^2(s^2+4)} + \frac{1}{s^2(s^2+4)} - \frac{s}{s^2+4}}.$$

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