MATH 201 Differential Equations – University of Alberta
Winter 2018 – Labs – Carlos Contreras
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Lab 6: Series solutions: singular points

Singular points

1. Determine all the singular points of the given differential equation

$$(t^2 - t - 2)x'' + (t + 1)x' - (t - 2)x = 0.$$

2. Classify all singular points of the following equation

$$(x^2 - x + 1)y'' - y' - y = 0.$$

3. Classify all singular points of the following equation

$$(x^3 - x)y'' + (\cos(x) - 1)y' + (x^2 - x)y = 0.$$

Radius of convergence

4. Find a minimum value for the radius of convergence of a power series solution about $x_0 = 1$,

$$y'' + \frac{2}{x^2 - 2x - 3}y' - \frac{3}{x^2 + x + 1}y = 0.$$

Solutions

Theory and problems from: Nagel, Saff & Sneider, *Fundamentals of Differential Equations*, Eighth Edition, Adisson–Wesley.

Consider the problem

$$y'' + p(x)y' + q(x)y = 0. \tag{1}$$

→ The point x_0 is a **regular point** for (1) if both $p(x)$ and $q(x)$ are analytic about x_0 (i.e., have a power series about x_0 with positive radius of convergence). The point x_0 is a **singular point** for (1) if it is not regular.

→ **Distance** between two complex points $z_1 = \alpha_1 + i\beta_1$ and $z_2 = \alpha_2 + i\beta_2$.

$$d(z_1, z_2) = |(\alpha_1 - \alpha_2) + i(\beta_1 - \beta_2)| = \sqrt{(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2}.$$

→ The **minimum radius of convergence** of (1) about x_0 is

$$\rho = \min_k d(x_0, z_k),$$

where z_k are all singular points of (1). That is, the minimun radius of convergence is the distance between x_0 and the closest singular point of the equation.

1. Determine all the singular points of the given differential equation

$$(t^2 - t - 2)x'' + (t+1)x' - (t-2)x = 0.$$

Solution
Normalizing this equation, we have:

$$x'' + \left(\frac{t+1}{t^2 - t - 2}\right)x' - \left(\frac{t-2}{t^2 - t - 2}\right)x = 0$$

$$x'' + \left(\frac{t+1}{(t-2)(t+1)}\right)x' - \left(\frac{t-2}{(t-2)(t+1)}\right)x = 0$$

Hence, we have:

$$p(t) = \frac{t+1}{(t-2)(t+1)}, \quad q(t) = \frac{t-2}{(t-2)(t+1)}$$

As we can see, $p(t)$ is analytic on $(-\infty, 2) \cup (2, \infty)$ and $q(t)$ is analytic on $(-\infty, -1) \cup (-1, \infty)$. Hence, $t = \{-1, 2\}$ are singular points of this equation.

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2. Classify all singular points of the following equation

$$(x^2 - x + 1)y'' - y' - y = 0.$$

3. Classify all singular points of the following equation

$$(x^3 - x)y'' + (\cos(x) - 1)y' + (x^2 - x)y = 0.$$

4. Find a minimum value for the radius of convergence of a power series solution about $x_0 = 1$,

$$y'' + \frac{2}{x^2 - 2x - 3} y' - 3 \frac{3}{x^2 + x + 1} y = 0.$$

Solution
The coefficient in front of the y terms has a singular point when $x^2 + x + 1 = 0$ and $x^2 - 2x - 3 = 0$.

$$x^2 + x + 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$x^2 - 2x - 3 = 0 \Rightarrow x = -1, 3$$

Hence, the radius of convergence of a power series solution about $x_0 = 1$ is at least as large as the distance from $x_0 = 1$ to the closest singular point $\frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$, -1, or 3.

The distance between $x_0 = 1$ to $x = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$ is given by

$$\begin{aligned} d\left(1, \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}\right) &= \sqrt{\left(1 - \frac{-1}{2}\right)^2 + \left(0 - \frac{\pm\sqrt{3}}{2}\right)^2} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}, \end{aligned}$$

while

$$d(1, -1) = 2 \qquad d(1, 3) = 2$$

Hence, $x = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$ are the closest singular points and the radius of convergence of a power series solution about $x_0 = 1$ is at least as large as $\sqrt{3}$.

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