MATH 201 Differential Equations – University of Alberta
Winter 2018 – Labs – Carlos Contreras
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Lab 5: Power series solutions: regular points

Recurrence relation and first terms

1. Find a recurrence relation and the first five non-zero terms in the power series approximation for the given initial value problem

$$y'' + (x+2)y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

2. Find a recurrence relation and the first four non-zero terms in the power series approximation about $x = 0$

$$z'' - x^2 z' - xz = 0.$$

3. Find a recurrence relation and the first four non-zero terms in the power series approximation about $x = 0$

$$z'' - x^2 z' - xz = x^2.$$

First terms

4. Find at least the first four nonzero terms in a power series expansion about $x = 0$ for the solution to the given initial value problem,

$$(x^2 - x + 1)y'' - y' - y = x^2, \quad y(0) = 0, y'(0) = 1.$$

Full power series

5. Find a power series expansion about $x = 0$ for a general solution to the given differential equation. Your answer should include a general formula for the coefficients

$$z'' - x^2 z' - xz = 0.$$

6. Find a power series expansion about $x = 0$ for a general solution to the given differential equation. Your answer should include a general formula for the coefficients

$$(x^2 + 1)y'' - xy' + y = 0.$$

Cauchy product

7. Find at least the first four nonzero terms in a power series expansion about $x = 0$ for the solution to the given initial value problem,

$$y'' + xy' + e^x y = 0, \quad y(0) = 1, y'(0) = 1.$$

8. Find at least four non-zero terms in the power series expansion to the initial value problem

$$y'' - (\sin x)y = 0, \quad y(\pi) = 1, y'(\pi) = 0.$$

Solutions

Theory and problems from: Nagel, Saff & Sneider, *Fundamentals of Differential Equations*, Eighth Edition, Adisson–Wesley.

→ A **power series** of $f(x)$ about x_0 is an expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

→ **Theorem 1 (Ratio test)** *If, for n large, the coefficients $a_n \neq 0$ satisfy*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \quad (0 \leq L \leq \infty),$$

then the radius of convergence is L.

→ Recall also that we can *differentiate* and *integrate* power series

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1},$$

and

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1} + C,$$

with the same radius of convergence of $f(x)$.

→ **Cauchy product for series**

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n, \quad \text{where,} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

1. Find a recurrence relation and the first five non-zero terms in the power series approximation for the given initial value problem

$$y'' + (x+2)y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Solution
Assume a power solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

with derivatives

$$\begin{aligned} y'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1}, \\ y''(x) &= \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2}. \end{aligned}$$

Using the equation we have

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + (x+2)\sum_{n=0}^{\infty} a_n x^n = 0,$$
$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^n = 0,$$

change the exponent in all summations to n

$$\Rightarrow \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0,$$

and combine the summations leaving the first extra terms in two of the summations

$$\Rightarrow 2a_2 + 2a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_{n-1} + 2a_n\right]x^n = 0.$$

Matching left hand side and right hand side

$$\begin{cases} 2a_2 + 2a_0 = 0 \\ (n+2)(n+1)a_{n+2} + a_{n-1} + 2a_n = 0, \quad n \geq 1. \end{cases}$$

Using the the initial conditions we find $a_0 = y(0) = 1$ and $a_1 = y'(0) = 1$. Thus, the recursive relation is given by

$$\boxed{\begin{cases} a_0 = 1,\ a_1 = 1,\ a_2 = -1, \\ a_{n+2} = -\dfrac{a_{n-1} + 2a_n}{(n+2)(n+1)}, \quad n \geq 1. \end{cases}}$$

Now, we find the first 5 non-zero terms using the recurrence relation

$$a_0 = 1,\ a_1 = 1,\ a_2 = -1,\ a_3 = -\frac{2+1}{3\cdot 2} = -\frac{1}{2},\ a_4 = -\frac{2(-1)+1}{4\cdot 3} = \frac{1}{12}.$$

Finally, the power series solution with the first five non-zero terms is

$$\boxed{y(x) = 1 + x - x^2 - \frac{1}{2}x^3 + \frac{1}{12}x^4 + \cdots}$$

■

2. Find a recurrence relation and the first four non-zero terms in the power series approximation about $x = 0$

$$z'' - x^2 z' - xz = 0.$$

Solution
We expand the solution $z(t)$ into the following power series about $x = 0$.

$$z(x) = \sum_{n=0}^{\infty} a_n x^n$$

Taking the first two derivatives, we have:

$$\begin{aligned}
z'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
z''(x) &= \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2}
\end{aligned}$$

Substituting these equations into our ODE, we obtain:

$$\begin{aligned}
0 &= \sum_{n=2}^{\infty}(n-1)na_n x^{n-2} - x^2\sum_{n=1}^{\infty} na_n x^{n-1} - x\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty}(n-1)na_n x^{n-2} - \sum_{n=1}^{\infty} na_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= \sum_{n=-1}^{\infty}(n+2)(n+3)a_{n+3}x^{n+1} - \sum_{n=1}^{\infty} na_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= (1)(2)a_2 + (2)(3)a_3 x - a_0 x + \sum_{n=1}^{\infty}(n+2)(n+3)a_{n+3}x^{n+1} - \sum_{n=1}^{\infty} na_n x^{n+1} - \sum_{n=1}^{\infty} a_n x^{n+1} \\
&= 2a_2 + (6a_3 - a_0)x + \sum_{n=1}^{\infty}\left[(n+2)(n+3)a_{n+3} - na_n - a_n\right]x^{n+1} \\
&= 2a_2 + (6a_3 - a_0)x + \sum_{n=1}^{\infty}\left[(n+2)(n+3)a_{n+3} - (n+1)a_n\right]x^{n+1}
\end{aligned}$$

Since this expression must be true for all x, we must have:

$$\begin{aligned}
2a_2 &= 0, \\
6a_3 - a_0 &= 0, \\
(n+2)(n+3)a_{n+3} - (n+1)a_n &= 0, \quad n \geq 1.
\end{aligned}$$

This leads to the following recurrence relation:

$$\begin{aligned}
a_2 &= 0 \\
a_3 &= \frac{a_0}{6} \\
a_{n+3} &= \left(\frac{(n+1)}{(n+2)(n+3)}\right) a_n
\end{aligned}$$

Let's find the first coefficients

$$\begin{aligned}
a_0 &= a_0 \\
a_1 &= a_1 \\
a_2 &= 0 \\
a_3 &= \frac{a_0}{6} \\
a_4 &= \left(\frac{2}{(3)(4)}\right) a_1 = \left(\frac{1}{6}\right) a_1
\end{aligned}$$

Hence, the solution to the this ODE is given by:

$$\boxed{y(x) = a_0 + a_1 x + \frac{a_0}{6}x^3 + \frac{a_1}{6}x^4 + \cdots}$$

■

3. Find a recurrence relation and the first four non-zero terms in the power series approximation about $x = 0$

$$z'' - x^2 z' - xz = x^2.$$

Solution

We expand the solution $z(t)$ into the following power series about $x = 0$.

$$z(x) = \sum_{n=0}^{\infty} a_n x^n$$

Taking the first two derivatives, we have:

$$\begin{aligned} z'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ z''(x) &= \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2} \end{aligned}$$

Substituting these equations into our ODE (be aware of the x^2 term in the right hand side), we obtain:

$$\begin{aligned} x^2 &= \sum_{n=2}^{\infty}(n-1)na_n x^{n-2} - x^2\sum_{n=1}^{\infty} na_n x^{n-1} - x\sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty}(n-1)na_n x^{n-2} - \sum_{n=1}^{\infty} na_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=-1}^{\infty} (n+2)(n+3)a_{n+3}x^{n+1} - \sum_{n=1}^{\infty} na_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= (1)(2)a_2 + (2)(3)a_3 x - a_0 x + \sum_{n=1}^{\infty}(n+2)(n+3)a_{n+3}x^{n+1} - \sum_{n=1}^{\infty} na_n x^{n+1} - \sum_{n=1}^{\infty} a_n x^{n+1} \\ &= 2a_2 + (6a_3 - a_0)x + \sum_{n=1}^{\infty}\left[(n+2)(n+3)a_{n+3} - na_n - a_n\right]x^{n+1} \\ &= 2a_2 + (6a_3 - a_0)x + [12a_4 - 3a_1]x^2 + \sum_{n=2}^{\infty}\left[(n+2)(n+3)a_{n+3} - (n+1)a_n\right]x^{n+1}. \end{aligned}$$

Since this expression must be true for all x, we must have:

$$\begin{aligned} 2a_2 &= 0, \\ 6a_3 - a_0 &= 0, \\ 12a_4 - 2a_1 &= 1, \\ (n+2)(n+3)a_{n+3} - (n+1)a_n &= 0, \quad n \geq 2. \end{aligned}$$

This leads to the following recurrence relation:

$$
\begin{aligned}
a_2 &= 0,\\
a_3 &= \frac{a_0}{6},\\
a_4 &= \frac{1}{12}+\frac{2a_1}{12},\\
a_{n+3} &= \left(\frac{(n+1)}{(n+2)(n+3)}\right)a_n, \quad n \geq 2.
\end{aligned}
$$

Let's find the first coefficients

$$
\begin{aligned}
a_0 &= a_0\\
a_1 &= a_1\\
a_2 &= 0\\
a_3 &= \frac{a_0}{6}\\
a_4 &= \frac{1}{12}+\frac{a_1}{6}
\end{aligned}
$$

Hence, the solution to the this ODE is given by:

$$
\boxed{y(x) = a_0 + a_1 x + \frac{a_0}{6}x^3 + \left(\frac{1}{12}+\frac{a_1}{6}\right)x^4 + \cdots}
$$

Compare to the previous solution.

Extra work: Note that we can find more terms

$$
\begin{aligned}
a_5 &= \frac{3}{4\cdot 5}a_2 = 0\\
a_6 &= \frac{4}{5\cdot 6}a_3 = \frac{4}{5\cdot 6\cdot 6}a_0 = \frac{1}{45}a_0\\
a_7 &= \frac{5}{6\cdot 7}a_4 = \frac{5}{6\cdot 7\cdot 12}+\frac{5}{6\cdot 7\cdot 6}a_1 = \frac{5}{504}+\frac{5}{252}a_1
\end{aligned}
$$

in the separate this solution in terms of the homogeneous and particular solutions

$$
\boxed{y(x) = a_0\underbrace{\left(1+\frac{1}{6}x^3+\frac{1}{45}x^6\cdots\right)}_{\text{1st lin. ind. sol.}} + a_1\underbrace{\left(x+\frac{1}{6}x^4+\frac{5}{252}x^7+\cdots\right)}_{\text{2nd lin. ind. sol.}} + \underbrace{\left(\frac{1}{12}x^4+\frac{5}{504}x^7+\cdots\right)}_{\text{part. sol.}}.}
$$

∎

4. Find at least the first four nonzero terms in a power series expansion about $x = 0$ for the solution to the given initial value problem,

$$
(x^2 - x + 1)y'' - y' - y = x^2, \quad y(0) = 0, y'(0) = 1.
$$

Solution

Let

$$
y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots
$$

. Taking the first two derivatives, we have

$$
\begin{aligned}
y'(x) &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots \\
y''(x) &= 2a_2 + 6a_3x + 12a_4x^2 + \cdots
\end{aligned}
$$

Using the initial conditions we see that $a_0 = 0$ and $a_1 = 1$. Substituting these result into our equation, we obtain

$$
\begin{aligned}
x^2 &= (x^2 - x + 1) \cdot (2a_2 + 6a_3x + 12a_4x^2 + \cdots) \\
&\quad - (1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots) - (x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots) \\
&= (2a_2x^2 + 6a_3x^3 + 12a_4x^4 + \cdots) + (-2a_2x - 6a_3x^2 - 12a_4x^3 + \cdots) + (2a_2 + 6a_3x + 12a_4x^2 + \cdots) \\
&\quad + (-1 - 2a_2x - 3a_3x^2 - 4a_4x^3 + \cdots) + (-x - a_2x^2 - a_3x^3 - a_4x^4 + \cdots) \\
&= (-1 + 2a_2) + (-1 - 4a_2 + 6a_3)x + (a_2 - 9a_3 + 12a_4)x^2 + \cdots
\end{aligned}
$$

Matching terms of each side of this equation

$$
\begin{aligned}
0 = 2a_2 - 1 &\Rightarrow a_2 = \frac{1}{2} \\
0 = -1 - 4\cancelto{\frac{1}{2}}{a_2} + 6a_3 &\Rightarrow a_3 = \frac{1}{2} \\
1 = \cancelto{\frac{1}{2}}{a_2} - 9\cancelto{\frac{1}{2}}{a_3} + 12a_4 &\Rightarrow a_4 = \frac{5}{12}
\end{aligned}
$$

Thus, the solution with the first four non-zero terms is given by

$$
\boxed{y(x) = x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{5}{12}x^4 + \ldots}
$$

■

5. Find a power series expansion about $x = 0$ for a general solution to the given differential equation. Your answer should include a general formula for the coefficients

$$z'' - x^2z' - xz = 0.$$

Solution

We expand the solution $z(t)$ into the following power series about $x = 0$.

$$z(x) = \sum_{n=0}^{\infty} a_nx^n$$

Taking the first two derivatives, we have:

$$
\begin{aligned}
z'(x) &= \sum_{n=1}^{\infty} na_nx^{n-1} \\
z''(x) &= \sum_{n=2}^{\infty} (n-1)na_nx^{n-2}
\end{aligned}
$$

Substituting these equations into our ODE, we obtain:

$$
\begin{aligned}
0 &= \sum_{n=2}^{\infty}(n-1)na_n x^{n-2} - x^2\sum_{n=1}^{\infty} na_n x^{n-1} - x\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty}(n-1)na_n x^{n-2} - \sum_{n=1}^{\infty} na_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= \sum_{n=-1}^{\infty}(n+2)(n+3)a_{n+3} x^{n+1} - \sum_{n=1}^{\infty} na_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= (1)(2)a_2 + (2)(3)a_3 x - a_0 x + \sum_{n=1}^{\infty}(n+2)(n+3)a_{n+3} x^{n+1} - \sum_{n=1}^{\infty} na_n x^{n+1} - \sum_{n=1}^{\infty} a_n x^{n+1} \\
&= 2a_2 + (6a_3 - a_0)x + \sum_{n=1}^{\infty}\left[(n+2)(n+3)a_{n+3} - na_n - a_n\right] x^{n+1} \\
&= 2a_2 + (6a_3 - a_0)x + \sum_{n=1}^{\infty}\left[(n+2)(n+3)a_{n+3} - (n+1)a_n\right] x^{n+1}
\end{aligned}
$$

Since this expression must be true for all x, we must have:

$$
\begin{aligned}
2a_2 &= 0 \\
6a_3 - a_0 &= 0 \\
(n+2)(n+3)a_{n+3} - (n+1)a_n &= 0
\end{aligned}
$$

This leads to the following recurrence relation:

$$
\begin{aligned}
a_2 &= 0 \\
a_3 &= \frac{a_0}{6} \\
a_{n+3} &= \left(\frac{(n+1)}{(n+2)(n+3)}\right) a_n
\end{aligned}
$$

Let's try to find a pattern for the coefficients.

$$
\begin{aligned}
a_0 &= a_0\\
a_1 &= a_1\\
a_2 &= 0\\
a_3 &= \frac{a_0}{6} = \frac{a_0}{3!}\\
a_4 &= \left(\frac{2}{(3)(4)}\right) a_1 = \left(\frac{2^2}{4!}\right) a_1\\
a_5 &= 0\\
a_6 &= \left(\frac{(4)}{(5)(6)}\right) a_3 = \left(\frac{(4)}{(2)(3)(5)(6)}\right) a_0 = \left(\frac{4^2}{6!}\right) a_0\\
a_7 &= \left(\frac{(5)}{(6)(7)}\right) a_4 = \left(\frac{(2\cdot 5)^2}{7!}\right) a_1\\
a_8 &= \left(\frac{(6)}{(7)(8)}\right) a_5 = 0\\
a_9 &= \left(\frac{(7)}{(8)(9)}\right) a_6 = \left(\frac{(7)}{(8)(9)}\right)\left(\frac{4^2}{6!}\right) a_0 = \left(\frac{(1\cdot 4\cdot 7)^2}{9!}\right) a_0\\
a_{10} &= \left(\frac{(8)}{(9)(10)}\right) a_7 = \left(\frac{(8)}{(9)(10)}\right)\left(\frac{(2\cdot 5)^2}{7!}\right) a_1 = \left(\frac{(2\cdot 5\cdot 8)^2}{10!}\right) a_1
\end{aligned}
$$

Although complicated ,the pattern is starting to get more obvious:
we have:

$$
\begin{aligned}
a_0 &= a_0\\
a_1 &= a_1\\
a_{3n} &= \frac{[1\cdot 4\cdot 7\cdots(3n-2)]^2}{(3n)!} a_0 \quad n = 1, 2, \ldots\\
a_{3n+1} &= \frac{[2\cdot 5\cdot 8\cdots(3n-1)]^2}{(3n+1)!} a_1 \quad n = 1, 2, \ldots\\
a_{3n+2} &= 0, \quad n = 0, 1, 2, \ldots
\end{aligned}
$$

Hence, the solution to the this ODE is given by:

$$
\boxed{y(x) = a_0\left(1 + \sum_{n=1}^{\infty}\left[\frac{[1\cdot 4\cdot 7\cdots(3n-2)]^2}{(3n)!}\right] x^{3n}\right) + a_1\left(x + \sum_{n=1}^{\infty}\left[\frac{[2\cdot 5\cdot 8\cdots(3n-1)]^2}{(3n+1)!}\right] x^{3n+1}\right).}
$$

■

6. Find a power series expansion about $x = 0$ for a general solution to the given differential equation. Your answer should include a general formula for the coefficients

$$(x^2 + 1)y'' - xy' + y = 0.$$

Solution
We expand the solution $y(x)$ into the following power series about $x = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Taking the first two derivatives, we have

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1},$$
$$y''(x) = \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2}.$$

Using the equation we have

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0,$$
$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^n + \underbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}_{n-2\to n} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0,$$
$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0,$$
$$\Rightarrow 2a_2 + 6a_3 x - a_1 x + a_0 + a_1 x + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + (n-1)^2 a_n]x^n = 0$$
$$\Rightarrow (2a_2 + a_0) + 6a_3 x + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + (n-1)^2 a_n]x^n = 0.$$

Matching both side sides of the equation

$$a_0, a_1 \in \mathbb{R}, \quad a_2 = -\tfrac{1}{2}a_0, \quad a_3 = 0, \quad a_{n+2} = -\frac{(n-1)^2}{(n+2)(n+1)}a_n, \; n \geq 2.$$

Use the recursive relation to find the pattern

$$\begin{aligned}
n &= 2, & a_4 &= -\frac{1}{4\cdot 3}a_2 = \frac{1}{4\cdot 3\cdot 2}a_0 \\
n &= 3, & a_5 &= -\frac{2^2}{5\cdot 4}a_3 = 0 \\
n &= 4, & a_6 &= -\frac{3^2}{6\cdot 5}a_4 = -\frac{(3\cdot 1)^2}{6!}a_0 \\
n &= 5, & a_7 &= -\frac{4^2}{7\cdot 6}a_5 = 0 \\
n &= 6, & a_8 &= -\frac{5^2}{8\cdot 7}a_6 = -\frac{(5\cdot 3\cdot 1)^2}{8!}a_0
\end{aligned}$$

We can see (with extra effort to find the pattern) that the even coefficients are given by

$$a_{2n} = \frac{(-1)^n((2n-3)(2n-5)\cdots 1)^2}{(2n)!}a_0$$

Verify that this is true with a_8, for example.

Thus, the general solution is given by

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}\frac{(-1)^n((2n-3)(2n-5)\cdots 1)^2}{(2n)!}x^{2n} + a_1 x.}$$

■

7. Find at least the first four non-zero terms in a power series expansion about $x = 0$ for the solution to the given initial value problem,

$$y'' + xy' + e^x y = 0, \quad y(0) = 1, y'(0) = 1.$$

Solution

We expand the solution $y(x)$ with a few terms into the following power series about $x = 0$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Taking the first two derivatives, we have

$$\begin{aligned} y'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \\ y''(x) &= 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 \cdots \end{aligned}$$

Using initial conditions we have

$$a_0 = 1, \qquad a_1 = 1. \tag{1}$$

First we need the product of functions $e^x y$ written in power series

$$\begin{aligned} e^x y(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n \sum_{n=0}^{\infty} a_n x^n = \left(1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots\right)\left(1 + x + a_2 x^2 + a_3 x^3 + \cdots\right) \\ &= 1 + (1+1)x + \left(a_2 + 1 + \frac{1}{2!}1\right)x^2 + \left(a_3 + a_2 + \frac{1}{2!} + \frac{1}{3!}\right)x^3 + \cdots \end{aligned}$$

Using the equation we have

$$\begin{aligned} &(2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots) + (a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots) \\ &\qquad + (1 + 2x + (a_2 + \tfrac{3}{2})x^2 + (a_3 + a_2 + \tfrac{1}{2} + \tfrac{1}{6})x^3 + \cdots) = 0. \end{aligned}$$

Matching coefficients on each side we have

$$\begin{aligned} &2a_2 + 1 = 0 \Rightarrow a_2 = -\frac{1}{2}, \\ &6a_3 + 1 + 2 = 0 \Rightarrow a_3 = -\frac{1}{2}, \\ &12a_4 + 2a_2 + a_2 + \frac{3}{2} = 0 \Rightarrow a_4 = 0, \\ &20a_5 + 3a_3 + a_3 + a_2 + \frac{1}{2} + \frac{1}{6} = 0 \Rightarrow a_5 = \frac{11}{120}. \end{aligned}$$

Finally, the solution with 5 non-zero terms is given by

$$\boxed{y(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{11}{120}x^5 + \cdots}$$

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8. Find at least four non-zero terms in the power series expansion to the initial value problem

$$y'' - (\sin x)y = 0, \quad y(\pi) = 1, y'(\pi) = 0.$$

Solution
We are given initial conditions at $x_0 = \pi$, the trick here is to find a power series expansion around $x_0 = \pi$, which is a regular point for the equation,

$$y(x) = \sum_{n=0}^{\infty} a_n (x-\pi)^n.$$

In order to do this we first make a change of variables. Let

$$t = x - \pi \Rightarrow x = t + \pi, \quad \text{and} \quad Y(t) = y(t+\pi) \Rightarrow Y' = y',\ Y'' = y'',$$

in which case

$$\sin x = \sin(t+\pi) = \sin t \cos \pi + \cos t \sin \pi = -\sin t,$$

and the initial value problem becomes

$$Y'' + \sin t Y = 0, \quad Y(0) = 1,\, Y'(0) = 0.$$

If we try to solve without using this change of variables, the series expansion of $\sin x$ (around $x = 0$) is a very bad approximation around $x = \pi$ (where we are solving for $y(x)$).

Now, we solve this new problem using a simple power series expansion (recall that we only need a few terms)

$$\begin{aligned}
Y(t) &= \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 \dots \\
&= 1 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 \dots \\
Y'(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + 6a_6 t^5 + \dots \\
&= 0 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + 6a_6 t^5 + \dots \\
Y''(t) &= \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = 2a_2 + 3\cdot 2a_3 t + 4\cdot 3a_4 t^2 + 5\cdot 4a_5 t^3 + 6\cdot 5a_6 t^4 + \dots
\end{aligned}$$

Here $a_0 = 1$ and $a_1 = 0$. Since $a_1 = 0$, we still have three more terms to find.

For the term $\sin(t)Y(t)$ we need to expand the sine in power series and multiply it with the expansion of $Y(t)$. For that, recall the Cauchy product

$$\left(\sum_{n=0}^{\infty} b_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n,$$

where

$$c_n = \sum_{k=0}^{n} b_k d_{n-k} = b_0 d_n + b_1 d_{n-1} + b_2 d_{n-2} + \dots + b_{n-1} d_1 + b_n d_0.$$

The power series of sine around 0 is

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \frac{1}{9!}t^9 \dots.$$

Be careful! Do not confuse the terms in this expansion with the ones in $\sum_{n=0}^{\infty} b_n t^n$, they have different exponents. In our case

$$b_0 = 0,\ b_1 = 1,\ b_2 = 0,\ b_3 = -\frac{1}{3!},\ b_4 = 0, b_5 = \frac{1}{5!}, \dots$$

and $d_n = a_n$. Finding the value of the coefficients c_n

$$\begin{aligned}
c_0 &= b_0a_0 = 0\\
c_1 &= b_0a_1 + b_1a_0 = 1\\
c_2 &= b_0a_2 + b_1a_1 + b_2a_0 = 0\\
c_3 &= b_0a_3 + b_1a_2 + b_2a_1 + b_3a_0 = a_2 - \frac{1}{3!}\\
c_4 &= b_0a_4 + b_1a_3 + b_2a_2 + b_3a_1 + b_4a_0 = a_3.
\end{aligned}$$

Using the equation we have

$$\left(2a_2 + 6a_3t + 12a_4t^2 + 20a_5t^3 + 30a_6t^4 + \dots\right) + \left(0 + t + 0 \cdot t^2 + \left(a_2 - \frac{1}{3!}\right)t^3 + \dots\right) = 0, \tag{2}$$

where matching coefficients multiplying x with the same power we get,

$$\begin{aligned}
2a_2 = 0 &\Rightarrow a_2 = 0\\
6a_3 = -1 &\Rightarrow a_3 = -\frac{1}{6}\\
12a_4 = 0 &\Rightarrow a_4 = 0\\
20a_5 = -a_2 + \frac{1}{3!} &\Rightarrow a_5 = \frac{1}{120}\\
30a_6 = -a_3 &\Rightarrow a_6 = \frac{1}{180}.
\end{aligned}$$

Suing this coefficients in the expansion of $Y(t)$ we have

$$Y(t) = 1 - \frac{1}{6}t^3 + \frac{1}{120}t^5 + \frac{1}{180}t^6 + \dots$$

Thus, the solution to the initial problem is

$$\boxed{y(x) = 1 - \frac{1}{6}(x-\pi)^3 + \frac{1}{120}(x-\pi)^5 + \frac{1}{180}(x-\pi)^6 + \dots}$$

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