MATH 201 Differential Equations – University of Alberta
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Lab 1: First-order differential equations

Separable equations

1. Determine whether the given differential equation is separable

$$(xy^2 + 3y^2)dy - 2x\,dx = 0.$$

2. Solve the equation

$$\frac{dy}{dx} = \frac{x}{y^2\sqrt{1+x}}$$

3. Solve the following differential equation

$$\frac{ds}{dt} = t\ln(s^{2t}) + 8t^2.$$

4. Solve the equation

$$\frac{dy}{dx} = \frac{\sec^2(y)}{1+x^2}.$$

5. Solve the initial value problem

$$\frac{1}{\theta}\frac{dy}{d\theta} = \frac{y\sin(\theta)}{y^2+1}, \quad y(\pi) = 1.$$

Linear equations

6. Obtain the general solution to the equation

$$\frac{dy}{dx} = \frac{y}{x} + 2x + 1.$$

7. Obtain the general solution to the equation

$$x\frac{dy}{dx} + 2y = x^{-3}.$$

8. Obtain the general solution to the equation

$$(t + y + 1)dt - dy = 0.$$

9. Solve the initial value problem

$$t^2\frac{dx}{dt} + 3tx = t^4\ln(t) + 1, \quad x(1) = 0.$$

1st order homogeneous

10. Solve the following problem

$$\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}, \quad x > 0.$$

Exact equations

11. Solve the following problem

$$2xydx + (x^2 - y^2)dy = 0.$$

12. Solve the following problem

$$ydx + (2x - ye^y)dy = 0.$$

Bernoulli

13. Solve the following problem

$$\frac{dy}{dx} = \frac{(1+x)y - 6y^3}{2x}.$$

14. Solve following problem

$$y' + 3y' = 4xy^3.$$

15. Solve following IVP

$$xe^x y' = (x-1)e^x y + y^2, \quad y(1) = e.$$

Solutions

All problems from: Nagel, Saff & Sneider, *Fundamentals of Differential Equations*, Eight Edition, Adisson–Wesley.

Definitions and formulas

→ A **first-order linear equation** has the form

$$y' + P(x)y = Q(x).$$

We solve this equations using the **integrating factor** formula

$$I(x) = e^{\int P(x)dx}, \qquad y(x) = \frac{1}{I(x)}\left[\int I(x)Q(x)dx + C\right].$$

→ If the right hand side of a differential equation can be expressed as a function of the ratio y/x,

$$\frac{dy}{dx} = f(x,y) = g\left(\frac{y}{x}\right),$$

then the equation is called **first order homogeneous**. In such case, we apply the change of variable $v = y/x$ to make the equation separable.

→ Differential equations of the form

$$M(x,y)dx + N(x,y)dy = 0,$$

are called **exact equations** if and only if

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y).$$

If it is not exact we might be able to use an integrating factor:

1. If $\frac{M_y - N_x}{N}$ depends only on x , use $I(x) = e^{\int \frac{M_y - N_x}{N}dx}$.
2. If $\frac{N_x - M_y}{M}$ depends only on y , use $I(y) = e^{\int \frac{N_x - M_y}{M}dx}$.

→ Differential equations of the form

$$y' + P(x)y = Q(x)y^p, \quad p \neq 1$$

are called **Bernoulli equations**. In such case, we apply the change of variable $u = y^{1-p}$ to make the equation first order linear.

1. Determine whether the given differential equation is separable

$$(xy^2 + 3y^2)dy - 2x\,dx = 0. \tag{1}$$

Solution
This equation is separable since :

$$\begin{aligned} 0 &= (xy^2 + 3y^2)\mathrm{d}y - 2x\,dx \\ &= y^2(x+3)\mathrm{d}y - 2x\,dx \end{aligned}$$

Dividing through by $x+3$, we obtain:

$$0 = y^2 \mathrm{d}y - \frac{2x}{x+3} dx$$

Moving $\dfrac{2x}{x+3} dx$ to the other side, we obtain our desired result:

$$\boxed{y^2 \mathrm{d}y = \frac{2x}{x+3} dx}.$$

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2. Solve the equation

$$\frac{dy}{dx} = \frac{x}{y^2\sqrt{1+x}} \tag{2}$$

Solution
This equation is separable, since

$$\begin{aligned} y^2 dy &= \frac{x}{\sqrt{1+x}} dx \\ \int y^2 dy &= \int \frac{x}{\sqrt{1+x}} dx \\ \frac{y^3}{3} &= \int \frac{x}{\sqrt{1+x}} dx \end{aligned}$$

To solve the integral in right hand side, make $t = 1 + x$. This implies that $dt = dx$, hence

$$\begin{aligned} \frac{y^3}{3} &= \int \frac{t-1}{\sqrt{t}} dt \\ &= \int t^{1/2} - t^{-1/2} dt \\ &= \frac{2t^{3/2}}{3} - 2t^{1/2} + C \\ &= \frac{2(1+x)^{3/2}}{3} - 2(1+x)^{1/2} + C \end{aligned}$$

Isolating y, we obtain:

$$\boxed{y(x) = \left[2(1+x)^{3/2} - 6(1+x)^{1/2} + C\right]^{1/3}},$$

for some constant C.

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3. Determine whether the given differential equation is separable

$$\frac{ds}{dt} = t\ln(s^{2t}) + 8t^2. \tag{3}$$

Solution
This equation is separable since :

$$\begin{aligned}\frac{ds}{dt} &= t\ln(s^{2t}) + 8t^2 \\ &= t(2t)\ln(s) + 8t^2 \\ &= 2t^2\ln(s) + 8t^2 \\ &= 2t^2(\ln(s) + 4)\end{aligned}$$

$$\Rightarrow \boxed{\frac{ds}{\ln(s)+4} = 2t^2 dt}.$$

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4. Solve the equation

$$\frac{dy}{dx} = \frac{\sec^2(y)}{1+x^2}. \tag{4}$$

Solution
This equation is separable,

$$\begin{aligned}\frac{dy}{\sec^2(y)} &= \frac{dx}{1+x^2} \\ \cos^2(y)dy &= \frac{dx}{1+x^2} \\ \int \cos^2(y)dy &= \int \frac{dx}{1+x^2} \\ \int \frac{\cos(2y)}{2} + \frac{1}{2}dy &= \arctan(x) + C \\ \frac{\sin(2y)}{4} + \frac{y}{2} &= \arctan(x) + C.\end{aligned}$$

In this case, we cannot isolate y, hence our solution is given implicitly by

$$\boxed{\sin(2y(x)) + 2y(x) = 4\arctan(x) + C}.$$

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5. Solve the initial value problem

$$\frac{1}{\theta}\frac{dy}{d\theta} = \frac{y\sin(\theta)}{y^2+1}, \quad y(\pi) = 1. \tag{5}$$

Solution
This equation is separable,

$$\begin{aligned}\frac{y^2+1}{y}dy &= \theta\sin(\theta)d\theta \\ \int \frac{y^2+1}{y}dy &= \int \theta\sin(\theta)d\theta \\ \int y + y^{-1}dy &= \int \theta\sin(\theta)d\theta \\ \frac{y^2}{2} + \ln|y| &= -\theta\cos(\theta) + \sin(\theta) + C.\end{aligned}$$

Setting $y(\pi) = 1$, we can solve for C.

$$\begin{aligned}
\frac{1^2}{2} + \ln|1| &= -\pi\cos(\pi) + \sin(\pi) + C \\
\frac{1}{2} &= \pi + C \\
C &= \frac{1}{2} - \pi
\end{aligned}$$

In this case, we cannot isolate y, hence our solution is given implicitly by

$$\boxed{\frac{y(\theta)^2}{2} + \ln|y(\theta)| = -\theta\cos(\theta) + \sin(\theta) + \frac{1}{2} - \pi}.$$

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Solution
We can alternatively use the initial value in the integration, where we compute a *definite integral*.
For example, in the previous solution we can integrate between the given initial point π, and the independent variable θ (note that we need to use a different variable in the integrand of the second integral)

$$\begin{aligned}
\int_\pi^\theta y + y^{-1} dy &= \int_\pi^\theta t\sin(t)dt \\
\Rightarrow \frac{1}{2}y^2\Big|_\pi^\theta + \ln|y|\big|_\pi^\theta &= -t\cos t\big|_\pi^\theta + \sin t\big|_\pi^\theta \\
\Rightarrow \frac{1}{2}y^2(\theta) - \frac{1}{2} + \ln|y(\theta)| - 0 &= -\theta\cos t + \cancelto{-1}{\pi\cos\pi} + \sin\theta - \cancelto{0}{\sin\pi},
\end{aligned}$$

Thus, the implicit solution is,

$$\boxed{\frac{1}{2}y^2(\theta) - \ln|y(\theta)| = -\theta\cos\theta + \sin\theta + \frac{1}{2} - \pi}.$$

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6. Obtain the general solution to the equation

$$\frac{dy}{dx} = \frac{y}{x} + 2x + 1. \tag{6}$$

Solution
This equation is not separable, but can be written in the standard form

$$y' + P(x)y = Q(x). \tag{7}$$

When written in standard form, the integrating factor is given by

$$I(x) = e^{\int P(x)dx}, \tag{8}$$

and the solution to any first order non separable equation of the form (7) is given by

$$y(x) = \frac{1}{I(x)}\left[\int I(x)Q(x)dx + C\right]. \tag{9}$$

Rewriting equation (6) in the form of equation (7), we obtain

$$\frac{dy}{dx} - \frac{1}{x}y = 2x + 1.$$

Hence the integrating factor is

$$\begin{aligned} I(x) &= e^{\int -\frac{1}{x}dx} \\ &= e^{-\ln|x|} \\ &= e^{\ln|x^{-1}|} \\ &= x^{-1}. \end{aligned}$$

Thus, the solution to equation (6) is given by equation (9)

$$\begin{aligned} y(x) &= \frac{1}{x^{-1}}\left[\int x^{-1}(2x+1)dx + C\right] \\ &= x\left[\int (2+x^{-1})dx + C\right] \\ &= x\left[2x + \ln|x| + C\right] \\ &= 2x^2 + x\ln|x| + Cx. \end{aligned}$$

Finally, for some constant C the general solution is,

$$\boxed{y(x) = 2x^2 + x\ln|x| + Cx}.$$

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Solution

Let's say we forgot the formula (9) for the integrating factor. The two things we need to remember is that the equation can be written in the special form (7) (emphasizing on what multiplies y)

$$y' + P(x)y = Q(x),$$

and that we need to compute $e^{\int P(x)dx}$, so that,

$$\frac{d(e^{\int P(x)dx}y)}{dx} = e^{\int P(x)dx}y' + e^{\int P(x)dx}P(x)y.$$

Now,

$$\begin{aligned} I(x) &= e^{\int -\frac{1}{x}dx} \\ &= e^{-\ln|x|} \\ &= e^{\ln|x^{-1}|} \\ &= x^{-1}. \end{aligned}$$

If we multiply x^{-1} on both sides of the equation we get

$$\begin{gathered} y' - x^{-1}y = (2x+1) \\ \Rightarrow x^{-1}y' - x^{-2}y = x^{-1}(2x+1) \\ \Rightarrow (x^{-1}y)' = 2 + x^{-1} \\ \Rightarrow \int d(x^{-1}y) = \int (2+x^{-1})dx \\ \Rightarrow x^{-1}y = 2x + \ln|x| + C, \end{gathered}$$

Thus, the solution for some constant C is,

$$\boxed{y(x) = 2x^2 + x\ln|x| + Cx}.$$

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7. Obtain the general solution to the equation

$$x\frac{dy}{dx} + 2y = x^{-3}. \tag{10}$$

Solution
Rewriting this equation in the form of equation (7), we obtain

$$\frac{dy}{dx} + \frac{2}{x}y = x^{-4}.$$

The integrating factor is given by

$$\begin{aligned} I(x) &= e^{\int \frac{2}{x}dx} \\ &= e^{2\ln|x|} \\ &= e^{\ln|x^2|} \\ &= x^2. \end{aligned}$$

Hence, the general solution is given by equation (9).

$$\begin{aligned} y(x) &= \frac{1}{x^2}\left[\int x^2x^{-4}dx + C\right] \\ &= x^{-2}\left[\int x^{-2}dx + C\right] \\ &= x^{-2}\left[-x^{-1} + C\right] \\ &= -x^{-3} + Cx^{-2}. \end{aligned}$$

Hence, for some constant C the general solution is,

$$\boxed{y(x) = -\frac{1}{x^3} + \frac{C}{x^2}}.$$

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8. Obtain the general solution to the equation

$$(t + y + 1)dt - dy = 0. \tag{11}$$

Solution
Rewriting this equation in the form of equation (7), we obtain

$$\frac{dy}{dt} - y = t + 1.$$

The integrating factor is given by

$$\begin{aligned} I(t) &= e^{\int -1dt} \\ &= e^{-t}. \end{aligned}$$

The general solution is given by equation (9)

$$\begin{aligned} y(x) &= \frac{1}{e^{-t}}\left[\int e^{-t}(t+1)dx + C\right] \\ &= e^{t}\left[-(t+2)e^{-t} + C\right] \\ &= -(t+2) + Ce^{t}. \end{aligned}$$

Hence, for some constant C the general solution is,

$$\boxed{y(x) = -(t+2) + Ce^t}.$$

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9. Solve the initial value problem

$$t^2 \frac{dx}{dt} + 3tx = t^4 \ln(t) + 1, \quad x(1) = 0. \tag{12}$$

Solution
Rewriting this equation in the form of equation (7), we obtain:

$$\frac{dx}{dt} + \frac{3}{t}x = t^2 \ln(t) + t^{-2}$$

The integrating factor is given by

$$\begin{aligned} I(t) &= e^{\int \frac{3}{t} dt} \\ &= e^{3 \ln |t|} \\ &= e^{\ln |t^3|} \\ &= t^3. \end{aligned}$$

The general solution is given by equation (9)

$$\begin{aligned} x(t) &= \frac{1}{t^3} \left[\int t^3 (t^2 \ln(t) + t^{-2}) dt + C \right] \\ &= t^{-3} \left[\int t^5 \ln(t) + t dt + C \right] \\ &= t^{-3} \left[\frac{t^6}{6} \ln(t) - \frac{t^6}{36} + \frac{t^2}{2} + C \right] \\ &= \frac{t^3}{6} \ln(t) - \frac{t^3}{36} + \frac{1}{2t} + \frac{C}{t^3}. \end{aligned}$$

Using the initial condition $x(1) = 0$, we can solve for C

$$\begin{aligned} x(1) = 0 &= \frac{(1)^3}{6} \ln(1) - \frac{(1)^3}{36} + \frac{1}{2(1)} + \frac{C}{(1)^3} \\ &= -\frac{1}{36} + \frac{1}{2} + C \\ \Rightarrow C &= -\frac{17}{36}. \end{aligned}$$

Hence, the solution is

$$\boxed{x(t) = \frac{t^3}{6} \ln(t) - \frac{t^3}{36} + \frac{1}{2t} - \frac{17}{36t^3}}.$$

■

10. Solve the following problem

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}, \quad x > 0.$$

Solution
Note that this equation is not separable, however the change of variable $u = y/x$ makes the equation separable. In such case,

$$y(x) = xu(x) \quad \text{and} \quad y' = u + xu'.$$

Rewrite the differential equation as (the trick is to take the appropriate common factor in the numerator and denominator)

$$\frac{du}{dx} = \frac{\cancel{x^2}(1 + 3\left(\frac{y}{x}\right)^2)}{2\cancel{x^2}\left(\frac{y}{x}\right)} = \frac{1 + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}, \tag{13}$$

so the suggested change of variable is now natural.
Equations of the form $dy/dx = f(x, y)$, where $f(x, y)$ can be expressed as a function of the ratio y/x are called *homogeneous*. Using the change of variable in (13) we have,

$$\begin{aligned} u + x\frac{du}{dx} &= \frac{1 + 3u^2}{2u} \\ x\frac{du}{dx} &= \frac{1 + 3u^2 - 2u^2}{2u} \\ x\frac{du}{dx} &= \frac{1 + u^2}{2u} \\ \frac{2udu}{1 + u^2} &= \frac{dx}{x}. \end{aligned}$$

Which is a separable equation. For the first integral we use the change of variable $v = 1 + u^2$. Hence,

$$\begin{aligned} \ln|1 + u^2| &= \ln|x| + C \\ 1 + u^2 &= Cx \\ u &= \sqrt{Cx - 1}, \end{aligned}$$

Finally, replacing the change of variable, the solution for some constant C is,

$$\boxed{y(x) = x\sqrt{Cx - 1}}.$$

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11. Solve the following problem

$$2xydx + (x^2 - y^2)dy = 0.$$

Solution
This equation is exact since,

$$M(x, y) = 2xy,\ N(x, y) = x^2 - y^2 \Rightarrow \partial_y M(x, y) = 2x = \partial_x N(x, y).$$

Thus,

$$F(x, y) = \int M(x, y)dx = x^2y + g(y),$$

and

$$\partial_y F(x, y) = x^2 + g'(y) = x^2 - y^2 = N(x, y) \Rightarrow g'(y) = -y^2 \Rightarrow g(y) = -\frac{1}{3}y^3 - C.$$

Hence,

$$\boxed{F(x, y) = x^2y - \frac{1}{3}y^3 = C}$$

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12. Solve the following problem

$$ydx + (2x - ye^y)dy = 0.$$

Solution

Note that the equation is not separable and not exact

$$M_y = 1 \neq N_x = 2.$$

To find the integrating factor we first compute either

$$M_y - N_x N,$$

to create an integrating factor depending only on x (i.e., $\mu(x)$), or

$$\frac{N_x - M_y}{M},$$

to create an integrating factor depending only on y (i.e., $\mu(y)$). Take, for example,

$$\frac{N_x - M_y}{M} = \frac{2-1}{y} = \frac{1}{y}.$$

Since this function depends only on y we can define the following integrating factor

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int \frac{dy}{y}} = y.$$

Multiplying both sides of the equation by $\mu(y) = y$, we get

$$y^2 dx + (2xy - y^2 e^y)dy = 0.$$

Note that this new equation is exact. Now we apply the method for exact equations

$$F(x,y) = \int M(x,y)\partial x = Z \int y^2 \partial x = xy2 + h(y).$$

and

$$\partial_y F(x,y) = 2xy + h'(y) = 2xy - y^2 e^y = N(x,y) \Rightarrow h'(y) = -y^2 e^y,$$

which after integration by parts gives,

$$h(y) = (-y^2 + 2y - 2)e^y + C.$$

Finally,

$$\boxed{F(x,y) = xy^2 + (-y^2 + 2y - 2)e^y = C}.$$

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13. Solve the following problem

$$\frac{dy}{dx} = \frac{(1+x)y - 6y^3}{2x}.$$

Solution
Write the equation in the standard form

$$y' - \frac{1}{2}\left(\frac{1}{x} + 1\right) y = -\frac{3}{x}y^3,$$

which is a Bernoulli equation. The change of variable is then,

$$u = y^{1-3} = y^{-2} \Rightarrow u' = -2y^{-3}y' \quad \text{and} \quad y = \pm u^{-1/2}.$$

Dividing the equation over y^3 (always divide over y^p to get u') and using the substitution

$$\begin{aligned} y^{-3}y' - \frac{1}{2}\left(\frac{1}{x} + 1\right) y^{-2} &= -\frac{3}{x} \\ -\frac{1}{2}u' - \frac{1}{2}\left(\frac{1}{x} + 1\right) u &= -\frac{3}{x}. \\ u' + \left(\frac{1}{x} + 1\right) u &= \frac{6}{x}. \end{aligned}$$

This equation can be solved using integrating factor,

$$I(x) = e^{\int \left(\frac{1}{x}+1\right)dx} = e^{\ln|x|+x} = xe^x,$$

then

$$u(x) = \frac{1}{xe^x}\left[\int xe^x \frac{6}{x} dx + C\right] = \frac{6}{x} + \frac{C}{xe^x}.$$

Going back to $y(x)$,

$$\boxed{y(x) = \pm\left(\frac{6}{x} + \frac{C}{xe^x}\right)^{-1/2}}.$$

■

14. Solve following problem

$$xe^x y' = (x-1)e^x y + y^2$$

Solution
Write the equation in the standard form

$$y' + \left(\frac{1}{x} - 1\right) y = \frac{e^{-x}}{x}y^2,$$

which is a Bernoulli equation. The change of variable is,

$$u = y^{1-2} = y^{-1} \Rightarrow u' = -y^{-2}y' \Rightarrow -u' = y^{-2}y' \quad \text{and} \quad y = \pm u^{-1}.$$

Dividing the equation over y^2 (always divide over y^p to get u') and using the substitution

$$\begin{aligned} y^{-2}y' + \left(\frac{1}{x} - 1\right) y^{-1} &= \frac{e^{-x}}{x} \\ \Rightarrow -u' + \left(\frac{1}{x} - 1\right) u &= \frac{e^{-x}}{x} \\ \Rightarrow u' + \left(1 - \frac{1}{x}\right) u &= -\frac{e^{-x}}{x}. \end{aligned}$$

This equation can be solved using integrating factor,

$$I(x) = e^{\int \left(1-\frac{1}{x}\right)dx} = e^{x-\ln|x|} = \frac{e^x}{x},$$

then

$$u(x) = xe^{-x}\left[-\int \frac{e^x}{x\frac{e^{-x}}{x}}dx + C\right] = xe^{-x}\left[\frac{1}{x} + C\right].$$

Going back to $y(x)$,

$$y(x) = \left(e^{-x} + Cxe^{-x}\right)^{-1}.$$

Using the initial condition $y(1) = e$,

$$e = \left(e^{-1} + Ce^{-1}\right)^{-1} \Rightarrow e^{-1} = \left(e^{-1} + Ce^{-1}\right),$$

which implies $C = 0$. Thus, the solution to the IVP is, to our surprise, as simple as

$$\boxed{y(x) = e^x}.$$

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