Math 300: Advanced Boundary Value Problems

Week 9

1.1 Sturm-Liouville Theory

1. *Example 4.11* Summary of standard Sturm-Liouville problems.

Model Type	S-L Problem	Spectrum	Eigenfunctions
Homogeneous Dirichlet B.C.	$\phi''(x) + \lambda\phi(x) = 0$ $\phi(0) = \phi(l) = 0$	$\lambda_n = \left(\frac{n\pi}{l}\right)^2$ $n = 1, 2, \cdots$	$\phi_n = \sin\frac{n\pi x}{l}$ $n = 1, 2, \cdots$
Homogeneous Neumann B.C.	$\phi''(x) + \lambda\phi(x) = 0$ $\phi'(0) = \phi'(l) = 0$	$\lambda_n = \left(\frac{n\pi}{l}\right)^2$ $n = 0, 1, \cdots$	$\phi_n = \cos\frac{n\pi x}{l}$ $n = 0, 1, \cdots$
Mixed Type I	$\phi''(x) + \lambda\phi(x) = 0$ $\phi(0) = \phi'(l) = 0$	$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2$ $n = 1, 2, \cdots$	$\phi_n = \sin\frac{(2n-1)\pi x}{2l}$ $n = 1, 2, \cdots$
Mixed Type II	$\phi''(x) + \lambda\phi(x) = 0$ $\phi'(0) = \phi(l) = 0$	$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2$ $n = 1, 2, \cdots$	$\phi_n = \cos\frac{(2n-1)\pi x}{2l}$ $n = 1, 2, \cdots$

1.2 Two-Dimensional Heat, Wave and Laplace Equations

1. **Exercise 14.13.**

   Solve the problem for a vibrating square membrane with side length 1, where the vibrations are governed by the following two-dimensional wave equation:

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{1}{\pi^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 < x < 1, \quad 0 < y < 1, \quad t > 0,\\
&u(0,y,t) = u(1,y,t) = 0,\\
&u(x,0,t) = u(x,1,t) = 0,\\
&u(x,y,0) = \sin \pi x \sin \pi y,\\
&\frac{\partial u}{\partial t}(x,y,0) = \sin \pi x.
\end{aligned}
$$

(continue Exercise 14.13)

(continue Exercise 14.13)

2. **Heat, Wave and Laplace equations on the rectangle**

(a) Heat equation

$$\frac{\partial^2 u}{\partial t^2} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 < x < a, \quad 0 < y < b, \quad t > 0,$$

$$u(x, y, 0) = f(x, y).$$

(b) Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 < x < a, \quad 0 < y < b, \quad t > 0,$$

$$u(x, y, 0) = f(x, y),$$
$$\frac{\partial u}{\partial t}(x, y, 0) = g(x, y).$$

(c) Laplace equation

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0, \quad 0 < x < a, \quad 0 < y < b,$$

3. Big picture

1.3 Polar coordinates

1. Given a point P with Cartesian coodinates $(x, y) \neq (0, 0)$, the polar coordinates of P are (r, θ), where

$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

The Jacobian determinant for the transformation is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\sin\theta \end{vmatrix} = r.$$

2. The **disk of radius** a is defined by

$$D(a) = \{(x, y) \mid x^2 + y^2 \leq a^2\} = \{(r, \theta) \mid 0 \leq r \leq a,\ -\pi \leq \theta \leq \pi\}.$$

3. The **Laplacian in polar coordinates** is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

4. *Example 6.1. (Potential in a Disk)* **Summary**.

   The Dirichlet problem for Laplace's equation in a disk in polar coordinates is

$$\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < r < a, \quad -\pi < \theta < \pi,\\
&u(r, -\pi) = u(r, \pi),\\
&\frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi),\\
&\lim_{r \to 0^+} u(r, \theta) = u(0, \theta),\\
&u(a, \theta) = f(\theta).
\end{aligned}$$