MATH 300: Advanced Boundary Value Problems I

Lecture notes

Carlos Contreras

University of Alberta

Spring/Summer 2020

Contents

Preface

This is a notebook for complementary notes to fill out during lectures splited in thidteen weeks. The content is taking from the book "Partial Differential Equations: Theory and Completely Solved Problems" by T. Hillen, I. E. Leonard, H. van Roessel, 2019, Friesen Press. Most of the text is taking verbatim from this book, but some modification have been made in some places for simplicity.

Math 300: Advanced Boundary Value Problems

Week 1

1.1 Introduction

1. Notation and definitions:

   - The **patial derivative** of f with respect to x is denoted

   $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = f_x$$

   - **Gradient** of $f(x, y, z)$

   $$\nabla f(x, y, z) = (f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)).$$

   - **Laplacian** of $f(x, y, z)$

   $$\Delta f(x, y, z) = f_{xx}(x, y, z) + f_{yy}(x, y, z) + f_{zz}(x, y, z).$$

   - **Partial differential equation (PDE)** for unknown $u(x, y)$

   $$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxx}, \cdots) = 0.$$

   - **Linear differential operator** L satisfies

   $$L(u + v) = Lu + Lv \quad \text{and} \quad L(\lambda u) = \lambda u.$$

   - **Linear PDE** for unknown u

   $$Lu = f,$$

     where L is a linear differential operator and function f does not depend on u or any of its derivatives. The equation is **homogeneous** if $f = 0$, and **nonhomogeneous** if $f \neq 0$.

   - The **order of a PDE** is the highest order derivative in the equation.

2. *Example 1.1.*

   Find the dimension and order of the following PDEs. Which are linear, and which are homogeneous?

   - Heat equation:
   $$u_t = Du_{xx} + f(x)$$
   - Wave equation:
   $$u_{tt} - cu_{xx} = 0$$
   - Laplace equation:
   $$u_{xx} + u_{yy} = 0$$
   - Advection equation:
   $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

   $$\frac{\partial^2 u}{\partial x^2} + e^y \sin(z) \frac{\partial^2 u}{\partial x \partial z} = u$$

   $$\frac{\partial^2 u}{\partial x \partial y} = \sin(u)$$

   - KdV equation:
   $$u_t + uu_{xx} + u_{xxx} = 1$$

3. The **second-order linear constant-coefficiens homogeneous PDEs**

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0$$

   is said to be

   - **elliptic** iff $ac - b^2 > 0$.
   - **parabolic** iff $ac - b^2 = 0$.
   - **hyperbolic** iff $ac - b^2 < 0$.

4. *Example 1.2.*

   Classify the following second-order linear PDEs.

   - $u_t + 2u_{tt} + 3u_{xx} = 0$
   - $17u_{yy} + 3u_x + u = 0$
   - $4u_{xy} + 2u_{xx} + u_{yy} = 0$
   - $u_{yy} - u_{xx} - 2u_{xy} = 0$

5. **Superposition principle**. If u_1 and u_2 are solutions to $Lu = 0$, so is $c_1u_1 + c_2u_2$.

6. **Theorem 1.2** If u_p is a particular solution to $Lu = f$ and u_h is the solutions to $L = u$, then $u = cu_h + u_p$ is a solution $Lu = f$ for any c.

7. *Example 1.7. (Burgers Equation)*

   Consider the following two-dimensional first-order nonlinear PDE:

   $$u_x + uu_y = 0$$

   and solutions

   $$u_1(x, y) = 1 \quad \text{and} \quad u_2(x, y) = \frac{y}{1 + x}.$$

   Consider the nonhomogeneous case:

   $$u_x + uu_y = \frac{y^2 - 1}{x^2y^3}$$

   with particular solution

   $$u_p(x, y) = -\frac{1}{xy}.$$

8. **Conditions**: a PDE can have
    - **Initial conditions**: value at time $t = 0$, i.e., $u(x, y, 0) = u_0(x, y)$.
    - **Boundary conditions**: value on the boundary $\partial\Omega$ for all time
        - Dirichlet: $u = g$ on $\partial\Omega$. Homogeneous if $g = 0$.
        - Neumann: $\frac{\partial u}{\partial n} = g$ on $\partial\Omega$. Homogeneous if $g = 0$.
        - Robin: $\alpha u + \beta\frac{\partial u}{\partial n} = g$ on $\partial\Omega$. Homogeneous if $g = 0$.
9. A **Boundary Value Problem** BVP is a PDE with boundary conditions.
10. A **steady-state solution** to a BVP does not depend on time, i.e., $u(x, t) = \tilde{u}(x)$.
11. *Example 1.10.*

    Find the steady-state solution to the following PDE on $[0, 2\pi]$:

$$
\begin{aligned}
&u_t = 3u_{xx} + 9\sin x,\\
&u(x, 0) = 9\sin x,\\
&u(0, t) = 9,\\
&u_x(2\pi, t) = 0.
\end{aligned}
$$

12. **Exercise 15.1**

    Show that the function
    $$u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$
    is harmonic; that is, it is a solution to the three-dimensional Laplace equation $\Delta u = 0$.

13. **Exercise 15.4**

    Compute the Laplacian of the function

    $$u(x, y) = \log\left(x^2 + y^2\right)$$

    in an appropriate coordinate system and decide if the given function satisfies Laplace's equation $\nabla^2 u = 0$.

Math 300: Advanced Boundary Value Problems

Week 2

2.1 Heat, wave and Laplace's equations

1. The **heat equation** is given by
$$u_t = k\Delta u + F,$$
where k is the **thermal diffusivity** and F is the forcing term.

2. The **wave equation** is given by
$$u_{tt} = c^2\Delta u + F,$$
where c is the **velocity of wave propagation** and F is the forcing term.

3. **Laplace's equation**, also **potential equation** is given by
$$\Delta u = 0.$$

Poisson's equation is the nonhomogeneous version
$$\Delta u = F,$$
where F is the forcing term and

4. **Exercise 13.1**

   For each of the boundary value problems below, determine whether or not an equilibrium temperature distribution exists and find the values of β for which an equilibrium solution exists.

   (a) $u_t = u_{xx} + 1, \quad u_x(0,t) = 1, \quad u_x(a,t) = \beta.$

   (b) $u_t = u_{xx}, \quad u_x(0,t) = 1, \quad u_x(a,t) = \beta.$

   (c) $u_t = u_{xx} + x - \beta, \quad u_x(0,t) = 0, \quad u_x(a,t) = 0.$

2.2 Fourier Series

1. **Definition 2.1**. Let the function f be defined on an open interval containing the point x_0.
   (i) If $f(x_0^+) = f(x_0^-) = f(x_0)$, f is **continuous** at x_0; and **discontinuous** at x_0, otherwise.
   (ii) If f is discontinuous at x_0 and if both $f(x_0^+)$ and $f(x_0^-)$ exist, f is said to have a **discontinuity of the first kind** or a **simple discontinuity** at x_0.
   (iii) A simple discontinuity of f of the first kind at x_0 is said to be
      (a) a **removable discontinuity** if $f(x_0^+) = f(x_0^-) \neq f(x_0)$ and
      (b) a **jump discontinuity** if $f(x_0^+) \neq f(x_0^-)$, regardless of the value $f(x_0)$.
   (iv) Any discontinuity of f at x_0 not of the first kind is said to be a **discontinuity of the second kind** at x_0.

2. **Definition 2.2.** A function f is **piecewise continuous** (PWC) on an interval (a, b) if
   (i) f is continuous for $x \in (a, b)$ except possibly at a finite number of points;
   (ii) $f(x^+)$ exists for all $x \in [a, b)$;
   (iii) $f(x^-)$ exists for all $x \in (a, b]$.

   Notation. $PWC(a, b)$ denotes the set of all PWC functions on (a, b).

3. **Theorem 2.1.** [Properies of $PWC(a, b)$]
   (i) If $f, g \in PWC(a, b)$, then $\alpha f + \beta g PWC(a, b)$ for all $\alpha, \beta \in R$.
   (ii) If $f, g \in PWC(a, b)$, then $f \cdot g \in PWC(a, b)$.
   (iii) If $f \in PWC(a, b)$, then $\int_a^b |f(x)| dx$ exists.

4. **Definition 2.3.** A function f is **piecewise smooth** (PWS) on (a, b) if
   (i) $f \in PWC(a, b)$ and
   (ii) $f' \in PWC(a, b)$.

   Notation. $PWS(a, b)$ denotes the set of all PWS functions on (a, b).

5. *Example.* Consider the following functions

   (a) $f(x) = \begin{cases} e^x, & \text{for } x \neq 1 \\ 1, & \text{for } x = 1. \end{cases}$

   (b) $g(x) = \begin{cases} \sin(x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$

   (c) $h(x) = \begin{cases} x, & 0 < x \leq 1 \\ -1, & 1 < x \leq 2 \\ 1, & 2 < x < 3. \end{cases}$

6. **Definition 2.4.** Let f be a function whose domain $D(f)$ is symmetric, that is, $-x \in D(f)$ whenever $x \in D(f)$; then we say that

   (i) f is **even** if $f(-x) = f(x)$ for all $x \in D(f)$.

   (ii) f is **odd** if $f(-x) = -f(x)$ for all $x \in D(f)$.

   (iii) f is **periodic** with period p if $x + p \in D(f)$ whenever $x \in D(f)$, and $f(x+p) = f(x)$ for all $x \in D(f)$.

7. The **periodic extension** of f defined on (a, b), denoted $\bar{f}$, is defined as

$$\bar{f}(x) = f(x, np) \quad \text{for} \quad a - np < x < b - np, \quad n \in \mathbb{Z}.$$

8. **Definition 2.5**. If the function f is defined on the interval $(0, l)$:

   (i) The **odd extension** of f on $(-l, l)$, denoted f_{odd}, is defined by

$$f_{\text{odd}}(x) = \begin{cases} f(x), & \text{for} \quad 0 < x < l, \\ -f(-x), & \text{for} \quad -l < x < 0, \end{cases}$$

   and

   (ii) The **even extension** of f on $(-l, l)$, denoted f_{even}, is defined by

$$f_{\text{even}}(x) = \begin{cases} f(x), & \text{for} \quad 0 < x < l, \\ f(-x), & \text{for} \quad -l < x < 0. \end{cases}$$

9. **Definition 2.7**. Let $f, g, w \in PWC(a, b)$ with $w(x) \geq 0$. The **inner product** of f and g with **weight function** w is defined as

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)dx.$$

10. **Definition 2.8**. The **norm** of $f \in PWC(a, b)$ with weight w is $\|f\| = \sqrt{\langle f, f \rangle}$.

11. **Definition 2.9**. If $f, g, w \in PWC(a, b)$ with **weight function** $w(x) \geq 0$, f and g are said to be **orthogonal** on (a, b) relative to the weight w if $\langle f, g \rangle = 0$.

12. The set

$$\left\{ 1, \cos\frac{\pi x}{l}, \sin\frac{\pi x}{l}, \cos\frac{2\pi x}{l}, \sin\frac{2\pi x}{l}, \cos\frac{3\pi x}{l}, \sin\frac{3\pi x}{l}, \ldots \right\}$$

   is an **orthogonal set of functions** on (a, b) with respect to the inner product above, where $l = (b - a)/2$.

13. **Exercise 11.3**

    Evaluate
$$\int_0^a \cos\frac{n\pi x}{a}\cos\frac{m\pi x}{a}dx$$
    for $n \geq 0$, $m \geq 0$. Use the trigonometric identity
$$\cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$$
    consider $A-B=0$ and $A+B=0$ separately.

14. **Exercise 11.4**

    Evaluate
$$\int_0^a \sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a}dx$$
    for $n \geq 0$, $m > 0$ and consider $n = m$ separately. Use the trigonometric identity
$$\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)].$$

15. **Definition 2.10**. The **Fourier series** of f on (a, b) is given by

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l},$$

where $l = (b-a)/2$ and

$$a_0 = \frac{1}{2l}\int_a^b f(x)dx, \quad a_n = \frac{1}{l}\int_a^b f(x)\cos\frac{n\pi x}{l}dx, \quad b_n = \frac{1}{l}\int_a^b f(x)\sin\frac{n\pi x}{l}dx, \quad n \geq 1,$$

are called the **Fourier coefficients** of f.

16. *Example 2.8.*

Find the Fourier series for the 2π-periodic function f defined by

$$f(x) = \begin{cases} x & 0 < x < \pi, \\ 0 & -\pi < x < 0, \end{cases}$$

and $f(x + 2\pi) = f(x)$ otherwise.

17. **Theorem 2.2**. For $f \in PWC(-l, l)$, the following are true:

    (a) If f is an odd function,
    $$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l};$$
    that is, the Fourier series for f contains only sine terms.

    (b) If f is an even function,
    $$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l};$$
    that is, the Fourier series for f contains only cosine terms.

18. Let function f defined on $(0, l)$.

    (i) The **Fourier sine series** for f is
    $$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l},$$
    where
    $$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \quad \text{for} \quad n \geq 1.$$
    Note that this defines f_{even}, the odd extension of f on $(-l, l)$.

    (ii) The **Fourier cosine series** for f is
    $$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l},$$
    where
    $$a_0 = \frac{1}{l} \int_0^l f(x) dx, \quad \text{and} \quad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx \quad \text{for} \quad n \geq 1.$$
    Note that this defines f_{even}, the even extension of f on $(-l, l)$.

19. *Example 2.10a.* Find the Fourier sine series of the function

$$f(x) = \begin{cases} 2x, & 0 < x < 1, \\ 2, & 1 < x < 2. \end{cases}$$

20. *Example 2.10b.* Find the Fourier cosine series of the function

$$f(x) = \begin{cases} 2x, & 0 < x < 1, \\ 2, & 1 < x < 2. \end{cases}$$

21. **Excercise 18.2**

    Let $f(x) = \cos^2(x), 0 < x < \pi$.

    (a) Find the Fourier sine series for f on the interval $(0, \pi)$.
    Hint: For $n \geq 1$,

    $$\int \cos^2 x \sin nx dx = -\frac{1}{2n} \cos nx + \frac{1}{4} \int [\sin(n+2)x + \sin(n-2)x] dx.$$

    (b) Find the Fourier cosine series for f on the interval $(0, \pi)$.

Math 300: Advanced Boundary Value Problems

Week 3

3.1 Fourier Series

1. **Theorem 2.3**. *(Dirichlet's Theorem)*

   Let $f(x)$ be piecewise smooth on the interval (l, l). The Fourier series

   $$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l},$$

   where

   $$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx, \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx, \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx, \quad n \geq 1,$$

   has the following properties:

   (i) If $f(x)$ is continuous at x_0 , where $-l < x_0 < l$, then

   $$f(x_0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x_0}{l} + b_n \sin\frac{n\pi x_0}{l};$$

   that is, the Fourier series converges to $f(x_0)$.

   (ii) If $f(x)$ has a jump discontinuity at x_0 , where $l < x0 < l$, then

   $$\frac{f(x_0^+) + f(x_0^-)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x_0}{l} + b_n \sin\frac{n\pi x_0}{l};$$

   that is, the Fourier series converges to the **average** or **mean** of the jump.

   (iii) At the endpoints $x0 = \pm l$, the Fourier series converges to

   $$\frac{f(-l^+) + f(l^-)}{2}.$$

   As usual, we write

   $$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l},$$

   and say that $f(x)$ is **represented by its Fourier series** on the interval (l, l). The Fourier series defines a $2l$-periodic extension of $f(x)$ for all $x \in \mathbb{R}$.

2. **Exercise 11.5**

   Compute the Fourier series of the 2π-periodic function f given by

$$f(x) = \begin{cases} 1, & 0 < x < \pi/2, \\ 0, & \pi/2 < |x| < \pi, \\ -1, & -\pi/2 < x < 0. \end{cases}$$

   For which values of x does the Fourier series converge to f? Sketch the graph of the Fourier.

3. **Exercise 11.6**

   Compute the Fourier series of the 2π-periodic function f given by $f(x) = |\cos(x)|$. For which values of x does the Fourier series converge to f? Sketch the graph of the Fourier.

(continue)

4. **Exercise 11.7**

   Consider the parabola $f(x) = x^2$ on $[-a, a]$ and show that the Fourier series of f is given by
   $$\frac{a^2}{3} - \frac{4a^2}{\pi^2}\left[\cos\left(\frac{\pi x}{a}\right) - \frac{1}{2^2}\cos\left(\frac{2\pi x}{a}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{a}\right) + \cdots\right].$$
   Find its values and the points of discontinuity.

5. **Theorem 2.4.** *(Uniqueness of Fourier Series)*

   If f is $2l$-periodic and piecewise smooth on the interval $(-l, l)$, its Fourier series is unique.

6. **Theorem 2.5.** *(Linearity of Fourier Series)*

   If f and g are piecewise continuous on $(-l, l)$ and c_1 and c_2 are scalars, the Fourier series of

   $$c_1 f + c_2 g$$

   is the sum of c_1 times the Fourier series of $f(x)$ and c_2 times the Fourier series of $g(x)$.

7. **Theorem 2.8.** *(Term-by-Term Differentiation of Fourier Series)*

   Let f be a function such that

   (i) f is continuous on the interval $-\pi \leq x \leq \pi$;
   (ii) $f(-\pi) = f(\pi)$; and
   (iii) f' is piecewise smooth on the interval $-\pi < x < \pi$.

   The derivative of the Fourier series representation of f is represented by

   $$f'(x) \sim \begin{cases} \displaystyle\sum_{n=1}^{\infty} n(-a_n \sin nx_0 + b_n \cos nx_0), & \text{if } f''(x_0) \text{ exists} \\ \dfrac{f'(x_0^+) + f'(x_0^-)}{2}, & \text{if } f''(x_0) \text{ DNE but one-sided derivatives exist.} \end{cases}$$

8. **Theorem 2.9.** *(Term-by-Term Differentiation of Fourier Cosine Series)*

   Let f be a function such that

   (i) f is continuous on the interval $0 \leq x \leq \pi$;
   (ii) f' is piecewise continuous on the interval $0 < x < \pi$.

   The derivative of the Fourier Cosine series representation of f is represented by

   $$f'(x) \sim \begin{cases} -\displaystyle\sum_{n=1}^{\infty} na_n \sin nx_0, & \text{if } f''(x_0) \text{ exists} \\ \dfrac{f'(x_0^+) + f'(x_0^-)}{2}, & \text{if } f''(x_0) \text{ DNE but one-sided derivatives exist.} \end{cases}$$

9. **Theorem 2.10.** *(Term-by-Term Differentiation of Fourier Sine Series)*

   Let f be a function such that

   (i) f is continuous on the interval $0 \leq x \leq \pi$;
   (ii) $f(0) = f(\pi)$; and
   (iii) f' is piecewise smooth on the interval $0 < x < \pi$.

The derivative of the Fourier Sine series representation of f is represented by

$$f'(x) \sim \begin{cases} \displaystyle\sum_{n=1}^{\infty} nb_n \cos nx_0, & \text{if } f''(x_0) \text{ exists} \\ \dfrac{f'(x_0^+) + f'(x_0^-)}{2}, & \text{if } f''(x_0) \text{ DNE but one-sided derivatives exist.} \end{cases}$$

10. **Theorem 2.11.** *(Term-by-Term Integration of Fourier Series)*

    Let f be piecewise continuous on the interval $-\pi < x < \pi$, and suppose that on $(-\pi, \pi)$

    $$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

    then for $-\pi \leq x \leq \pi$

    $$\int_{-\pi}^{\pi} f(t)dt = a_0(x + \pi) + \sum_{n=1}^{\infty} \frac{1}{n}\{a_n \sin nx - b_n[(-1)^{n+1} + \cos nx]\}.$$

11. **Exercise 11.8**

    Consider the $2a$-periodic function f that is given on the interval $-a < x < a$ by $f(x) = x$. Show that the Fourier series of f is given by

    $$\frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{a}\right)$$

    by differentiating the Fourier series in *Exercise 11.7* term-by-term. Justify your work.

12. **Euler's formula** in complex variables

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

and complex trigonometric formulas

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cosh i\theta \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = -i\sinh i\theta.$$

13. **Theorem 2.14.** The complex Fourier series for $f \in PWC(-l, l)$ is

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}, \qquad \text{where} \qquad c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx, \quad n \in \mathbb{Z}.$$

14. *Example 2.19.* Calculate the complex Fourier series for

$$f(x) = x, \quad -\pi < x < \pi,$$

and $f(x + 2\pi) = f(x)$ otherwise.

Math 300: Advanced Boundary Value Problems

Week 4

4.1 Separation of Variables: Homogeneous equations

1. In the method of **separations of variables** we look for solutions of the form

$$u(x,t) = X(x)T(t).$$

2. The **eigenvalue problem with homogeneous Dirichlet boundary conditions**

$$X'' + \lambda X = 0 \quad X(0) = 0, \quad X(l) = 0,$$

has nontrivial solution for eigenvalues and corresponding eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{l}, \quad n \geq 1.$$

3. The **eigenvalue problem with homogeneous Neumann boundary conditions**

$$X'' + \lambda X = 0 \quad X'(0) = 0, \quad X'(l) = 0,$$

has nontrivial solution for eigenvalues and corresponding eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \cos\frac{n\pi x}{l}, \quad n \geq 0.$$

4. **Exercise 13.2**

   Solve the homogeneous Dirichlet problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < a, \quad t > 0,$$

   subject to the boundary conditions

$$u(0,t) = 0, \quad \text{and} \quad u(a,t) = 0,$$

   for $t > 0$, with initial conditions

$$u(x,0) = \begin{cases} 1, & 0 < x < \frac{a}{2} \\ 2, & \frac{a}{2} \leq x < a. \end{cases}$$

5. **Exercise 13.3**

   Solve the following boundary value–initial value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

$$u(0,t) = u(a,t) = 0,$$

$$u(x,0) = 3\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a}$$

   for $0 < x < a$, $t > 0$.

6. **Exercise 14.2**

   Solve the following boundary value–initial value problem for the wave equation:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0\\
&u(0,t) = 0,\\
&u(1,t) = 0,\\
&u(x,0) = \sin \pi x + \tfrac{1}{2} \sin 3\pi x + 3 \sin 7\pi x,\\
&\frac{\partial u}{\partial t}(x,0) = 1.
\end{aligned}$$

   You can use the fact the $\left\{\sin \frac{(2n+1)x}{2}\right\}_{n\geq 0}$ are orthogonal on $[0,\pi]$.

(continue Exercise 14.2)

7. **Exercise 13.8**

   Solve the problem of heat transfer in a bar of length $a = \pi$ and thermal diffusivity $k = 1$, with initial heat distribution $u(x, 0) = \sin x$, where one end of the bar is kept at a constant temperature $u(0, t) = 0$, while there is no heat loss at the other end of the bar, so that $\partial u(\pi, t)/\partial x = 0$, that is, solve the boundary value–initial value problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0\\
&u(0, t) = 0,\\
&\frac{\partial u}{\partial x}u(\pi, t) = 0,\\
&u(x, 0) = \sin x.
\end{aligned}$$

(continue Exercise 13.8)

(continue Exercise 13.8)

Math 300: Advanced Boundary Value Problems

Week 5

5.1 Separation of Variables: Nonhomogeneous equations

1. Standard homogeneous Heat and Wave equations

Heat eq. with Dirichlet BCs

$$\begin{aligned} &u_t = ku_{xx}, \quad 0 < x < a, \quad t > 0, \\ &u(0,t) = 0, \quad t > 0, \\ &u(a,t) = 0, \quad t > 0, \\ &u(x,0) = f(x), \quad 0 < x < a. \end{aligned}$$

The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{a}\right)^2 kt} \sin \frac{n\pi x}{a}.$$

Heat equation with Neumann BCs

$$\begin{aligned} &u_t = ku_{xx}, \quad 0 < x < a, \quad t > 0, \\ &u_x(0,t) = 0, \quad t > 0, \\ &u_x(a,t) = 0, \quad t > 0, \\ &u(x,0) = f(x), \quad 0 < x < a. \end{aligned}$$

The solution has the form

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{a}\right)^2 kt} \cos \frac{n\pi x}{a}.$$

Wave equation with Dirichlet BCs

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \quad 0 < x < a, \quad t > 0, \\ &u(0,t) = 0, \quad t > 0, \\ &u(a,t) = 0, \quad t > 0, \\ &u(x,0) = f(x), \quad 0 < x < a, \\ &u_t(x,0) = g(x), \quad 0 < x < a. \end{aligned}$$

The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{a} + b_n \sin \frac{n\pi ct}{a} \right) \sin \frac{n\pi x}{a}.$$

Wave equation with Neumann BCs

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \quad 0 < x < a, \quad t > 0, \\ &u_x(0,t) = 0, \quad t > 0, \\ &u_x(a,t) = 0, \quad t > 0, \\ &u(x,0) = f(x), \quad 0 < x < a, \\ &u_t(x,0) = g(x), \quad 0 < x < a. \end{aligned}$$

The solution has the form

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{a} + b_n \sin \frac{n\pi ct}{a} \right) \cos \frac{n\pi x}{a}.$$

2. Method for nonhomogeneous equations. Consider a solution of the form

$$u(x,t) = v(x) + w(x,t).$$

3. **Exercise 13.3 (modified)**

   Solve the following boundary value–initial value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial t^2} + \sin\frac{\pi x}{a},$$

$$u(0,t) = 0 \quad \text{and} \quad u(a,t) = 1,$$

$$u(x,0) = 3\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a},$$

   for $0 < x < a$, $t > 0$.

(continue Exercise 13.3 (modified))

4. **Method of Eigenfunc tion Expansions**. Consider a solution of the form

$$u(x,t) = v(x,t) + w(x,t).$$

5.2 Method of Characteristics

1. **Method of Characteristics**

   Consider the first-order linear time-dependent problem of the form

$$\begin{aligned}
&\frac{\partial u}{\partial t} + B(x,t)\frac{\partial u}{\partial x} = C(x,t,u), \quad -\infty < x < \infty, \quad t > 0, \\
&u(x,0) = f(x).
\end{aligned}$$

   The method of characteristic consists on solving the **characteristc equations**

$$\begin{aligned}
\frac{dx}{dt} &= B(x,t), \\
\frac{du}{dt} &= C(x,t,u),
\end{aligned}$$

   and then using the initial condition.

2. *Example 10.2*

   Solve the following PDE for $u(x,t)$ on $-\infty < x < \infty$

$$\begin{aligned}
\frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial x} + \beta u &= 0, \\
u(x,0) &= f(x),
\end{aligned}$$

   using the method of characteristics.

3. **Exercise 17.5**

   Solve the first-order equation

$$\frac{\partial u}{\partial t} + 3x\frac{\partial u}{\partial x} = 2t, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = \log(1 + x^2).$$

4. **Exercise 17.6**

   Using the method of characteristics, solve

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = e^{2x}, \quad -\infty < x < \infty, \quad t > 0,$$

$$w(x,0) = f(x).$$

5. **Exercise 17.7**

   Using the method of characteristics, solve

$$\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 1, \quad -\infty < x < \infty, \quad t > 0,$$

$$w(x,0) = f(x).$$

6. **Method of Characteristics (revised)**

Consider the first-order linear problem of the form

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} = C(x,y,u), \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,y) = f(x,y), \quad (x,y) \in \Gamma_a,$$

where Γ_a is a curve of anchor points and f is a given function.

Consider the surface $z = u(x,y)$ with parametrization

$$x = x_0(a), \quad y = y_0(a), \quad z = z_0(a) = f(x_0(a), y_0(a)).$$

This defines **characteristc equations**

$$\frac{dx}{ds} = A(x,y),$$
$$x(0) = x_0(a),$$

$$\frac{dy}{ds} = B(x,y),$$
$$y(0) = y_0(a),$$

$$\frac{dz}{ds} = C(x,y,z),$$
$$z(0) = z_0(a),$$

which can be used to

(a) Solve the first two characteristic equations to get x and y in terms of the characteristic variable s and the anchor point a:

$$x = X(s,a), \quad y = Y(s,a)$$

(b) Insert the solution from the previous step into the third characteristic equation, and solve the resulting equation for z:

$$z = Z(s,a).$$

(c) Write the characteristic variables and anchor point a in terms of the original independent variables x and y; that is, invert

$$x = X(s,a), \quad y = Y(s,a)$$

to get

$$s = S(x,y), \quad a = \Gamma(x,y).$$

(d) Write the solution for z in terms of x and y to get the solution to the original PDE:

$$u(x,y) = Z(S(x,y), \Gamma(x,y)).$$

(extra page)

(extra page)

Math 300: Advanced Boundary Value Problems

Week 6

6.1 One-dimensional Wave Equation

1. Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

   d'Alembert's solution is given by

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\mu)\, d\mu.$$

2. **Exercise 17.12**

   The displacement $u = u(x,t)$ of an infinitely long string is governed by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0.$$

   At time $t = 0$ an initial signal is given of the form

$$u(x,0) = f(x) = \begin{cases} x, & 0 < x < 1, \\ -x+2, & 1 < x < 2, \\ 0, & \text{othewise}, \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \quad -\infty < x < \infty.$$

   a) Solve this problem.

   b) Sketch the solution for times t_1, t_2, t_3, t_4, t_5 , with

$$t_1 = 0, \quad 0 < t_2 < 1/4, \quad t_3 = 1/4, \quad 1/4 < t_4 < 1/2, \quad t_5 = 1/2.$$

   c) At what time does the signal reach the point $x = 11$?

(continue Exercise 17.12)

(continue Exercise 17.12)

3. Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0,$$
$$u(0,t) = 0, \quad u(l,t) = 0, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

d'Alembert's solution is given by

$$u(x,t) = \frac{1}{2}[\bar{f}_{\text{odd}}(x+ct) + \bar{f}_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}_{\text{odd}}(\mu)\, d\mu,$$

where $\bar{f}_{\text{odd}}$ and $\bar{g}_{\text{odd}}$ are the $2l$-periodic extension of f and g, respectively.

4. **Exercise 14.8**

Use d'Alembert's solution to solve the boundary value–initial value problem for the wave equation:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0 \\
&u(0,t) = 0, \\
&u(1,t) = 0, \\
&u(x,0) = 0, \\
&\frac{\partial u}{\partial t}(x,0) = 1.
\end{aligned}$$

5. **Exercise 17.8**

   Consider

$$\begin{aligned}
&\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0, \\
&u(x,t) = f(x).
\end{aligned}$$

   Show that characteristics are straight lines.

6. **Exercise 17.9**

   Consider

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0.$$

   with

$$u(x,0) = f(x) = \begin{cases} 1, & 0 < x, \\ 1 + x/a, & 1 < x < a, \\ 2, & x > a. \end{cases}$$

   a) Determine the equations for the characteristics. Sketch the characteristics.
   b) Determine the solution $u(x,t)$. Sketch $u(x,t)$ for t fixed.

7. **Exercise 14.9**

   Use d'Alembert's solution to solve the boundary value–initial value problem for the wave equation:

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0 \\
&u(0,t) = 0, \\
&u(1,t) = 0, \\
&u(x,0) = 0, \\
&\frac{\partial u}{\partial t}(x,0) = \sin \pi x.
\end{aligned}
$$

8. **Exercise 14.10**

   Use d'Alembert's solution to solve the boundary value–initial value problem for the wave equation:

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0 \\
&u(x,0) = x^2, \\
&\frac{\partial u}{\partial t}(x,0) = 3.
\end{aligned}
$$

Math 300: Advanced Boundary Value Problems

Week 7

7.1 Sturm-Liouville Theory

1. **Definition 4.1.** A **regular Sturm-Liouville problem** denotes the problem of finding an eigenfunction-eigenvalue pair (ϕ, λ) which solves the problem

$$\begin{aligned}
&(p(x)\phi')' + [q(x) + \lambda\sigma(x)]\phi = 0, \quad a < x < b, \\
&\alpha_1\phi(a) + \beta_1\phi'(a) = 0, \\
&\alpha_2\phi(b) + \beta_2\phi'(b) = 0,
\end{aligned}$$

   where

   (i) $p(x)$, $p'(x)$, $q(x)$, and $\sigma(x)$ are real valued and continuous for $a \leq x \leq b$;

   (ii) $p(x) > 0$ and $\sigma(x) > 0$ for $a \leq x \leq b$; and

   (iii) α_1 , α_2 , β_1 , β_2 are real valued, $\alpha_1^2 + \beta_1^2 \neq 0$ and $\alpha_2^2 + \beta_2^2 \neq 0$.

2. *Example 4.2.* Consider the following boundary value problem, which we have solved several times before:

$$\begin{aligned}
&\phi'' + \lambda\phi = 0, \quad 0 < x < l, \\
&\phi(0) = 0, \\
&\phi(l) = 0.
\end{aligned}$$

3. **Definition 4.2.** A Sturm-Liouville problem is said to be **singular** if at least one of the conditions (i), (ii), or (iii) in Definition 4.1 fails, or if the interval is infinite. In the case where the interval is infinite, or one or both of the functions $p(x)$ and $\sigma(x)$ approach 0 or ∞ at an endpoint of the interval, one or more of the boundary conditions are usually replaced by boundedness conditions on ϕ.

4. *Example 4.3. (Legendre's Equation)* Consider the boundary value problem for Legendres equation,

$$
\begin{aligned}
&((1-x^2)\phi')' + \lambda\phi = 0, \quad -1 < x < 1,\\
&\alpha_1\phi(-1) + \beta_1\phi'(-1) = 0,\\
&\alpha_2\phi(1) + \beta_2\phi'(1) = 0,
\end{aligned}
$$

5. *Example 4.4. (Bessel's Equation)* For fixed n, Bessels equation on the interval $a < r < b$,

$$
\begin{aligned}
&(r\phi')' + \left(\lambda r - \frac{n^2}{r}\right)\phi = 0,\\
&\phi(a) = 0,\\
&\phi(b) = 0,
\end{aligned}
$$

6. **Theorem 4.2.** The spectrum of a regular Sturm-Liouville problem is a countably infinite set with no limit points, that is, an infinite discrete set.

7. **Theorem 4.3.** If λ_m and λ_n are distinct eigenvalues of a regular Sturm-Liouville problem, that is, $\lambda_m \neq \lambda_n$, the corresponding eigenfunctions ϕ_m and ϕ_n are orthogonal relative to the inner product
$$\langle f, g \rangle = \int_a^b f(x)g(x)\sigma(x)dx.$$

8. **Theorem 4.4.** If λ is an eigenvalue of a regular Sturm-Liouville problem:

   (a) λ is real, and

   (b) if ϕ and ψ are eigenfunctions corresponding to λ,
   $$\psi(x) = k\phi(x), \quad a \leq x \leq b,$$
   where k is a nonzero constant, and each eigenfunction can be made real-valued by multiplying it by an appropriate nonzero constant.

9. *Example 4.5. (Cauchy-Euler Equation)* Consider the boundary value problem
$$\begin{aligned}
&(x\phi')' + \frac{\lambda}{x}\phi = 0, \quad 1 < x < l, \\
&\phi(1) = 0, \\
&\phi(l) = 0.
\end{aligned}$$

(continue Example 4.5)

Math 300: Advanced Boundary Value Problems

Week 8

8.1 Sturm-Liouville Theory

1. **Theorem 4.5.**

   Given the regular Sturm-Liouville problem,

   $$\begin{aligned} &(p(x)\phi')' + [q(x) + \lambda\sigma(x)]\phi = 0, \quad a < x < b, \\ &\alpha_1\phi(a) + \beta_1\phi'(a) = 0, \\ &\alpha_2\phi(b) + \beta_2\phi'(b) = 0, \end{aligned}$$

   with eigenvalues λ_n and corresponding eigenfunctions ϕ_n.

   (a) The regular Sturm-Liouville problem has an infinite spectrum

   $$S = \{\lambda_1, \lambda_2, \dots, \lambda_n, \dots\}$$

   and $\lim_{n\to\infty} \lambda_n = +\infty$.

   (b) If $\alpha_1\beta_1 \le 0$ and $\alpha_2\beta_2 \ge 0$, the spectrum is bounded below and the eigenvalues may be ordered as

   $$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots .$$

   Moreover, if $q(x) \le 0$ for $a \le x \le b$, then $\lambda_n \ge 0$ for all $n \ge 1$.

   (c) If the eigenvalues are ordered as $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, the eigenfunction corresponding to λ_n has exactly $(n-1)$ zeros in the interval $a < x < b$.

2. **Theorem 4.6.** *(Dirichlet's Theorem)*

   If f is piecewise smooth on $[a, b]$, the **generalized Fourier series**,

   $$f(x) \sim \sum_{n=1}^{\infty} c_n\phi_n(x), \quad \text{where,} \quad c_n = \frac{\langle f, \phi_n\rangle}{\|\phi_n\|^2} = \frac{1}{\|\phi_n\|^2}\int_a^b f(x)\phi_n(x)\sigma(x)\,dx,$$

   for $n \ge 1$, converges pointwise to $[f(x^+) + f(x^-)]/2$ for each $x \in (a, b)$.

3. *Example 4.6.* Consider the regular Sturm-Liouville problem

$$\begin{aligned}
&\phi'' + \lambda\phi = 0, \quad 0 < x < 1, \\
&\phi(0) = 0, \\
&2\phi(1) - \phi'(1) = 0.
\end{aligned}$$

(continue Example 4.6)

4. *Example 4.7.* Consider the regular Sturm-Liouville problem

$$\begin{aligned}&\phi'' + \lambda^2\phi = 0, \quad 0 < x < \pi,\\&\phi'(0) = 0,\\&\phi(\pi) = 0.\end{aligned}$$

   (a) Find the eigenvalues λ_n^2 and the corresponding eigenfunctions ϕ_n for this problem.

   (b) Show directly, by integration, that eigenfunctions corresponding to distinct eigenvalues are orthogonal.

   (c) Given the function $f(x) = \pi^2 - x^2/2, 0 < x < \pi$, find the eigenfunction expansion of f.

   (d) Show that

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - + \cdots .$$

(continue Example 4.7)

5. **Theorem 4.7.**

   If (ϕ_n, λ_n) is an eigenpair for the regular Sturm-Liouville problem

$$
\begin{aligned}
&(p(x)\phi')' + [q(x) + \lambda\sigma(x)]\phi = 0, \quad a < x < b,\\
&\alpha_1\phi(a) + \beta_1\phi'(a) = 0,\\
&\alpha_2\phi(b) + \beta_2\phi'(b) = 0,
\end{aligned}
$$

   then λ_n can be calculated from the **Rayleigh quotient**:

$$
\lambda_n = \frac{-p(x)\phi_n(x)\phi_n'(x)\Big|_a^b + \displaystyle\int_a^b \left(p(x)\phi_n'(x)^2 - q(x)\phi_n(x)^2\right) dx}{\displaystyle\int_a^b \phi_n(x)^2\sigma(x)\, dx}.
$$

6. **Corollary 4.1.**

   If

$$
-p(x)\phi_n(x)\phi_n'(x)\Big|_a^b = -[p(b)\phi_n(b)\phi_n'(b) - p(a)\phi_n(a)\phi_n'(a)] \geq 0,
$$

   and $q(x) \leq 0$ for $a < x < b$, then $\lambda_n > 0$.

7. The **Rayleigh quotient** for **any** PWS function $u = u(x)$ on $[a, b]$ is given by

$$
\mathcal{R}(u) = \frac{-p(x)u(x)u'(x)\Big|_a^b + \displaystyle\int_a^b \left(p(x)u'(x)^2 - q(x)u(x)^2\right) dx}{\displaystyle\int_a^b u(x)^2\sigma(x)\, dx}.
$$

8. **Theorem 4.8.**

   Given the regular Sturm-Liouville problem

$$
\begin{aligned}
&(p(x)\phi')' + [q(x) + \lambda\sigma(x)]\phi = 0, \quad a < x < b,\\
&\alpha_1\phi(a) + \beta_1\phi'(a) = 0,\\
&\alpha_2\phi(b) + \beta_2\phi'(b) = 0,
\end{aligned}
$$

   with spectrum

$$
\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots,
$$

   Then, the **leading eigenvalue** is

$$
\lambda_1 = \min_u \mathcal{R}(u)
$$

   for all continuous functions u satisftying the boundary conditions

$$
\alpha_1 u(a) + \beta_1 u'(a) = 0, \quad \alpha_2 u(b) + \beta_2 u'(b) = 0.
$$

9. *Example 4.9.* Find good upper and lower bounds for the leading eigenvalue of the regular Sturm-Liouville problem

$$\begin{aligned}
&\phi'' - x\phi + \lambda\phi = 0, \quad 0 < x < 1,\\
&\phi'(0) = 0,\\
&2\phi(1) + \phi'(1) = 0.
\end{aligned}$$

(continue Example 4.9)

10. *Example 4.10.* Find the generalized Fourier series solution to the homogeneous Neumann problem for the wave equation. Use the Rayleigh quotient to show that $\lambda_1 > 0$.

$$\begin{aligned}
&\alpha(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\tau(x)\frac{\partial u}{\partial x}\right) - \beta(x)u, \quad 0 < x < l, \quad t > 0,\\
&\frac{\partial u}{\partial x}(0,t) = 0, \quad t > 0,\\
&\frac{\partial}{\partial x}(l,t) = 0, \quad t > 0,\\
&u(x,0) = f(x), \quad 0 < x < l,\\
&\frac{\partial u}{\partial t}(x,0) = g(x),
\end{aligned}$$

where $\alpha(x) > 0$, $\tau(x) > 0$, and $\beta(x) > 0$ for $0 < x < l$.

(continue Example 4.10)

Math 300: Advanced Boundary Value Problems

Week 9

9.1 Sturm-Liouville Theory

1. *Example 4.11* Summary of standard Sturm-Liouville problems.

Model Type	**S-L Problem**	**Spectrum**	**Eigenfunctions**
Homogeneous **Dirichlet B.C.**	$\phi''(x) + \lambda\phi(x) = 0$ $\phi(0) = \phi(l) = 0$	$\lambda_n = \left(\frac{n\pi}{l}\right)^2$ $n = 1, 2, \cdots$	$\phi_n = \sin\frac{n\pi x}{l}$ $n = 1, 2, \cdots$
Homogeneous **Neumann B.C.**	$\phi''(x) + \lambda\phi(x) = 0$ $\phi'(0) = \phi'(l) = 0$	$\lambda_n = \left(\frac{n\pi}{l}\right)^2$ $n = 0, 1, \cdots$	$\phi_n = \cos\frac{n\pi x}{l}$ $n = 0, 1, \cdots$
Mixed **Type I**	$\phi''(x) + \lambda\phi(x) = 0$ $\phi(0) = \phi'(l) = 0$	$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2$ $n = 1, 2, \cdots$	$\phi_n = \sin\frac{(2n-1)\pi x}{2l}$ $n = 1, 2, \cdots$
Mixed **Type II**	$\phi''(x) + \lambda\phi(x) = 0$ $\phi'(0) = \phi(l) = 0$	$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2$ $n = 1, 2, \cdots$	$\phi_n = \cos\frac{(2n-1)\pi x}{2l}$ $n = 1, 2, \cdots$

9.2 Two-Dimensional Heat, Wave and Laplace Equations

1. **Exercise 14.13.**

   Solve the problem for a vibrating square membrane with side length 1, where the vibrations are governed by the following two-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\pi^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 < x < 1, \quad 0 < y < 1, \quad t > 0,$$

$$u(0, y, t) = u(1, y, t) = 0,$$

$$u(x, 0, t) = u(x, 1, t) = 0,$$

$$u(x, y, 0) = \sin \pi x \sin \pi y,$$

$$\frac{\partial u}{\partial t}(x, y, 0) = \sin \pi x.$$

(continue Exercise 14.13)

(continue Exercise 14.13)

2. **Heat, Wave and Laplace equations on the rectangle**

(a) Heat equation

$$\frac{\partial^2 u}{\partial t^2} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 < x < a, \quad 0 < y < b, \quad t > 0,$$

$$u(x, y, 0) = f(x, y).$$

(b) Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 < x < a, \quad 0 < y < b, \quad t > 0,$$

$$u(x, y, 0) = f(x, y),$$
$$\frac{\partial u}{\partial t}(x, y, 0) = g(x, y).$$

(c) Laplace equation

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0, \quad 0 < x < a, \quad 0 < y < b,$$

3. Big picture

9.3 Polar coordinates

1. Given a point P with Cartesian coordinates $(x, y) \neq (0, 0)$, the **polar coordinates** of P are (r, θ), where

$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

The Jacobian determinant of the transformation is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\sin\theta \end{vmatrix} = r.$$

2. The **disk of radius** a is defined by

$$D(a) = \{(x, y) \mid x^2 + y^2 \le a^2\} = \{(r, \theta) \mid 0 \le r \le a,\ -\pi \le \theta \le \pi\}.$$

3. The **Laplacian in polar coordinates** is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

4. *Example 6.1. (Potential in a Disk)* **Summary**.

   The Dirichlet problem for Laplace's equation in a disk in polar coordinates is

$$\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < r < a, \quad -\pi < \theta < \pi,\\
&u(r, -\pi) = u(r, \pi),\\
&\frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi),\\
&\lim_{r \to 0^+} u(r, \theta) = u(0, \theta),\\
&u(a, \theta) = f(\theta).
\end{aligned}$$

Math 300: Advanced Boundary Value Problems

Week 10

10.1 Bessel functions

1. **Bessel's equation** of order n is given by

$$x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+(x^2-n^2)u=0.$$

2. The general solution of Bessel's equations of order n are given by

$$u(x)=A\,J_n(x)+B\,Y_n(x),$$

for arbitrary constants A and B, where

$$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(n+k)!}\left(\frac{x}{2}\right)^{n+2k},\quad n\geq 0,$$

is the **Bessel function of the first kind of order** n and $Y_n(x)$ is the **Bessel function of the second kind of order** n.

3. **Theorem 6.4.** (Orthogonality) For a fixed integer $m\geq 0$,

$$\int_0^1 xJ_m(z_{mn}x)J_m(z_{mk}x)\,dx=0,\qquad \int_0^1 xJ_m(z_{mn}x)^2\,dx=\frac{1}{2}J_{m+1}(z_{mn})^2,$$

where z_{mn} is a zero of $J_m(x)$ for $n\geq 1$.

4. **Theorem 6.5.** (Fourier-Bessel Expansion Theorem) If f and f' are piecewise continuous on the interval $0\leq x\leq 1$, then for $0<x_0<1$, the **Fourier-Bessel series** expansion

$$f(x)=\sum_{n=1}^{\infty}a_nJ_m(z_{mn}x_0),\quad \text{where}\quad a_n=\frac{2}{J_{m+1}(z_{mn})^2}\int_0^1 f(x)J_m(z_{mn}x)x\,dx,$$

converges to $[f(x_0^+)+f(x_0^-)]/2$. At $x_0=1$, the series converges to 0, since every $J_m(z_{mn})=0$. At $x_0=0$, the series converges to 0 if $m\geq 1$, and to $f(0^+)$ if $m=0$.

10.2 Polar coordinates

1. *Vibrating circular membrane.* Consider the following wave equation in a disk

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right), \quad 0 < r < a, \quad -\pi < \theta < \pi, \quad t > 0,\\
&u(r,-\pi,t) = u(r,\pi,t),\\
&\frac{\partial u}{\partial \theta}(r,-\pi,t) = \frac{\partial u}{\partial \theta}(r,\pi,t),\\
&u(a,\theta,t) = 0,\\
&|u(r,\theta,t)| < \infty, \quad \text{as} \quad r \to 0^+,\\
&u(r,\theta,0) = f(r,\theta),\\
&\frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta).
\end{aligned}
$$

2. **Exercise 13.14.**

   Solve the two-dimensional heat equation inside a disk with circularly symmetric time-independent sources, boundary conditions, and initial conditions:

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + Q(r), \quad 0 < r < a, \quad t > 0,$$

   with

$$u(r,0) = f(r), \quad u(a,t) = T.$$

(continue Exercise 13.14)

(continue Exercise 13.14)

(continue Exercise 13.14)

Math 300: Advanced Boundary Value Problems

Week 11

11.1 Legendre functions

1. **Theorem 7.2** The singular Sturm-Liouville problem given by **Legendre's equation**

$$(1-x^2)\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + \lambda v = 0, \quad -1 < x < 1.$$
$$|v(x)| \text{ and } |v'(x)| \text{ bounded as } x \to -1^+ \text{ and } x \to 1^-$$

has eigenvalues and corresponding eigenfunctions

$$\lambda_n = n(n+1), \quad \phi_n(x) = P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k(2n-2k)!x^{n-2k}}{2^n k!(n-k)!(n-2k)!}, \quad n \geq 0,$$

where $P_n(x)$ are called **Legendre polynomials**.

2. **Theorem 7.3.** (Orthogonality of Legendre Polynomials) If m and n are nonnegative integers with $m \neq n$,

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0.$$

3. **Theorem 7.10.** (Fourier-Legendre Expansion Theorem) If f and f are piecewise continuous on $[-1, 1]$, then for $-1 < x_0 < 1$ the **Fourier-Legendre** expansion series

$$f(x) = \sum_{n=1}^{\infty} a_n P_n(x_0), \quad \text{where} \quad a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx,$$

converges to $[f(x_0^+) + f(x_0^-)]/2$. At $x_0 = -1$, the series converges to $f(-1^+)$, while at $x_0 = 1$, the series converges to $f(1^-)$.

4. We can use substitution $x = \cos\theta$ and $v(x) = S(\theta)$ to transform

$$\sin\theta\,(\sin\theta\, S')' + (\lambda\sin^2\theta - \nu)S = 0,$$

into the **associated Legendre's equation**

$$(1-x^2)v'' - 2xv' + \left(\lambda - \frac{\nu}{1-x^2}\right)v = 0.$$

11.2 Spherical coordinates

1. Given a point P with Cartesian coordinates (x, y, z), where $(x, y) \neq (0, 0)$, the **spherical coordinates** of P are (r, θ, ϕ), where

$$\begin{aligned} x &= r\cos\phi\sin\theta, \\ y &= r\sin\phi\sin\theta, \\ z &= r\cos\theta. \end{aligned}$$

2. The **Laplacian in spherical coordinates** is

$$\begin{aligned} \nabla^2 u &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} \\ &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial\theta^2} + \cot\theta\frac{\partial u}{\partial\theta} + \csc^2\theta\frac{\partial^2 u}{\partial\phi^2}\right). \end{aligned}$$

3. **Exercise 13.19. Heat Flow on a Spherical Shell**

   Consider the flow of heat on a thin conducting spherical shell

$$S = \{(r, \theta, \phi) \mid r = 1, 0 \leq \theta \leq \pi, \pi \leq \phi \leq \pi\}.$$

   We want to find the temperature distribution $u(\theta, t)$ on the shell if we are given the initial temperature distribution $u(\theta, 0) = f(\theta)$.

(continue Exercise 13.19)

11.3 Fourier Transforms

1. Fourier Integral Representation

$$f(x) = \int_0^\infty \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right]\, d\omega$$

where

$$A(\omega) = \int_{-\infty}^\infty f(x)\cos\omega x\, dx, \qquad \int_{-\infty}^\infty f(x)\sin\omega x\, dx.$$

2. **Definition 8.2.** If f is piecewise smooth on every finite interval (a, b) and absolutely integrable on $(-\infty, \infty)$, the **Fourier transform** of $f(x)$, denoted $\widehat{f}$, is

$$\widehat{f}(\omega) = \mathcal{F}\left[f(x)\right](\omega) = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)\, e^{i\omega x}\, dx, \quad -\infty < \omega < \infty.$$

3. **Theorem 8.4.** If f and f' are piecewise continuous on every finite interval (a, b) and f is absolutely integrable on $(-\infty, \infty)$, then

$$\frac{f(x^+) + f(x^-)}{2} = \int_{-\infty}^\infty \widehat{f}(\omega)\, e^{-i\omega x}\, d\omega, \quad -\infty < x < \infty.$$

4. **Definition 8.4.** If f is piecewise smooth on every finite interval (a, b), absolutely integrable on $(-\infty, \infty)$ and f is continuous on $(-\infty, \infty)$, then

$$f(x) = \mathcal{F}^{-1}\left[f(\omega)\right](x) = \int_{-\infty}^\infty \widehat{f}(\omega)\, e^{-i\omega x}\, d\omega, \quad -\infty < x < \infty.$$

is called the **inverse Fourier transform** of $\widehat{f}(\omega)$.

5. **Properties**

   (i) **Theorem 8.5.** *(Linearity)*
      (a) $\mathcal{F}\left[af + bg\right] = a\mathcal{F}\left[f\right] + b\mathcal{F}\left[g\right]$
      (b) $\mathcal{F}^{-1}\left[af + bg\right] = a\mathcal{F}^{-1}\left[f\right] + b\mathcal{F}^{-1}\left[g\right]$

   (ii) **Theorem 8.6.** *(Shift)*
      (a) $\mathcal{F}\left[f(x-a)\right](\omega) = e^{ia\omega}\widehat{f}(\omega)$
      (b) $\mathcal{F}\left[e^{-iax}f(x-a)\right](\omega) = \widehat{f}(\omega)$
      (c) $\mathcal{F}\left[f(ax)\right](\omega) = (1/|a|)\widehat{f}(\omega/a)$

   (iii) **Theorem 8.7.** *(Transform of Derivatives)*

$$\mathcal{F}\left[f^{(n)}(x)\right](\omega) = (-i\omega)^n\, \mathcal{F}\left[f(x)\right](\omega)$$

   (iv) **Theorem 8.8.** *(Transform of an Integral)*

$$\mathcal{F}\left[\int_0^x f(s)\, ds\right](\omega) = -\frac{1}{i\omega}\mathcal{F}\left[f(x)\right](\omega)$$

6. **Definition 8.5.** Let $f : [0, \infty) \to \mathbb{R}$ be continuous and absolutely integrable on $(0, \infty)$, and let f' be piecewise continuous on every finite interval $(a, b) \subset (0, \infty)$. Then the sine and cosine transform and inverse transform are given by:

   (i) The **Fourier sine transform of** $f(x)$ and the **inverse sine transform of** $g(\omega)$ are

   $$\mathcal{S}\left[f(x)\right](\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx, \quad \mathcal{S}^{-1}\left[g(\omega)\right](x) = \int_0^\infty g(\omega)\sin\omega x\,d\omega,$$

   (ii) The **Fourier cosine transform of** $f(x)$ and the **inverse cosine transform of** $g(\omega)$ are

   $$\mathcal{C}\left[f(x)\right](\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx, \quad \mathcal{C}^{-1}\left[g(\omega)\right](x) = \int_0^\infty g(\omega)\cos\omega x\,d\omega,$$

7. **Theorem 8.10.** *(Sine and Cosine Transforms of Derivatives)*

   If f is piecewise smooth, f and f' are integrable on $[0, \infty)$, and $\lim_{x\to\infty} f(x) \to 0$, then:

   (a) For the Fourier sine transform, we have

   $$\mathcal{S}\left[f'(x)\right](\omega) = -\omega\,\mathcal{C}\left[f(x)\right](\omega)$$

   and if f'' is integrable on $[0, \infty)$ and $\lim_{x\to\infty} f'(x) \to 0$ also, then

   $$\mathcal{S}\left[f''(x)\right](\omega) = \frac{2\omega}{\pi}f(0) - \omega^2\,\mathcal{S}\left[f(x)\right](\omega).$$

   (b) For the Fourier cosine transform, we have

   $$\mathcal{C}\left[f'(x)\right](\omega) = -\frac{2}{\pi}f(0) + \omega\,\mathcal{S}\left[f(x)\right](\omega)$$

   and if f'' is integrable on $[0, \infty)$ and $\lim_{x\to\infty} f'(x) \to 0$ also, then

   $$\mathcal{C}\left[f''(x)\right](\omega) = -\frac{2}{\pi}f'(0) - \omega^2\,\mathcal{C}\left[f(x)\right](\omega).$$

8. **Definition 8.6.** (Convolution Product)

   If f and g are defined on all of $\mathbb{R}$, and are integrable over $\mathbb{R}$, the **convolution of** f **and** g, denoted $f * g$, is given by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt, \quad -\infty < x < \infty.$$

9. *Example 8.6. (Convolution with a Sine)*

   Let f be an even integrable function on $\mathbb{R}$, and let $g(x) = \sin ax$ for $x \in \mathbb{R}$, where $a > 0$ is constant; then

$$(f * g)(x) = 2\pi \sin(ax)\,\widehat{f}(a),$$

   where $\hat{f}$ is the Fourier transform of f.

10. **Theorem 8.11.** *(Convolution Theorem)*

    If f and g are integrable and satisfy the hypotheses of Theorem 8.4, then

    (a) [F] $\left[\frac{1}{2\pi} f * g\right] = \widehat{f} \cdot \widehat{g}$.

    (b) If, in addition, f and g are continuous, then $f * g = 2\pi\,\mathcal{F}^{-1}\left[\widehat{f} \cdot \widehat{g}\right]$.

11. **Theorem 8.12.** If the function $f : \mathbb{R} \to \mathbb{R}$ is piecewise smooth on every finite interval and is absolutely integrable on $\mathbb{R}$, then the Fourier transform $\widehat{f}(\omega)$ is uniformly continuous on $\mathbb{R}$.

12. *Example 8.7.* Find the Fourier transform of the function

$$g(x) = \begin{cases} 1 - \dfrac{|x|}{2}, & \text{for } |x| < 2, \\ 0, & \text{for } |x| \geq 2. \end{cases}$$

13. *Example 8.8.* Let $f(x)$ be the rectangular pulse

$$f(x) = \begin{cases} 1, & \text{for } |x| < 1, \\ 0, & \text{for } |x| > 1. \end{cases}$$

and $f(-1) = f(1) = \frac{1}{2}$. Let $h(x)$ be the convolution of f with itself, that is,

$$h(x) = \int_{-\infty}^{\infty} f(x-t)f(t)dt.$$

Find the Fourier transform of $h(x)$, and use the convolution theorem to identify $h(x)$.

Math 300: Advanced Boundary Value Problems

Week 12

12.1 Fourier Transform Methods in PDEs

1. For the heat equation and wave equation, we define

$$\widehat{u}(\omega, t) = \mathcal{F}\left[u(x,t)\right](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{i\omega x}\,dx.$$

   and recall the following operational properties of the Fourier transform:

   i) $\mathcal{F}\left[\frac{\partial^n u}{\partial t^n}(x,t)\right](\omega) = \frac{d^n}{dt^n}\widehat{u}(\omega,t)$.

   ii) $\mathcal{F}\left[\frac{\partial^n u}{\partial x^n}(x,t)\right](\omega) = (-i\omega)^n\widehat{u}(\omega,t)$.

2. *Example 9.1.* Consider the wave problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,\\
&u(x,0) = f(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases}\\
&\frac{\partial u}{\partial t}(x,0) = 0.
\end{aligned}$$

3. **Theorem 9.1.** The solution $u(x,t)$ of the linear heat equation

$$\begin{aligned}&\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,\\ &u(x,0) = f(x),\\ &|u(x,t)| \quad \text{bounded as} \quad x \to \infty\end{aligned}$$

can be written as

$$u(x,t) = f(x) * G(x,t) = \int_{-\infty}^{\infty} f(\xi)G(\xi - x, t)\, d\xi,$$

where

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

is called the **fundamental solution** of the heat equation.

4. The **error function** is given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

5. **Lemma 9.1.** The error function is a monotone increasing function which satisfies

$$\lim_{x\to-\infty} \operatorname{erf}(x) = -1, \quad \text{and} \quad \lim_{x\to-\infty} \operatorname{erf}(x) = 1.$$

6. **Exercise 16.12** Use Fourier transforms to find the solution to

$$\begin{aligned}&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,\\ &u(x,0) = \begin{cases} 100, & |x| < 1 \\ 0, & |x| > 1. \end{cases}\end{aligned}$$

in terms of the error function.

(continue Exercise 16.12.)

7. **Heat Flow in a Semi-infinite Rod**

(i) The heat equation on a semi-infinite domain with **Dirichlet** condition

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0, \\
&u(0,t) = 0, \\
&u(x,0) = f(x), \\
&|u(x,t)| \quad \text{bounded as} \quad x \to \infty
\end{aligned}
$$

has solution

$$
u(x,t) = f_{\text{odd}}(x) * G(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_0^\infty f(s) \left(e^{-(x-s)^2/4kt} - e^{-(x+s)^2/4kt} \right) ds.
$$

(ii) The heat equation on a semi-infinite domain with **Neumann** condition

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0, \\
&\frac{\partial u}{\partial t}(0,t) = 0, \\
&u(x,0) = f(x), \\
&|u(x,t)| \quad \text{bounded as} \quad x \to \infty
\end{aligned}
$$

has solution

$$
u(x,t) = f_{\text{even}}(x) * G(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_0^\infty f(s) \left(e^{-(x-s)^2/4kt} + e^{-(x+s)^2/4kt} \right) ds.
$$

Math 300: Advanced Boundary Value Problems

Week 13

13.1 Summary

1. **Separations of variables**

   1) Write $u(x,t) = X(x)T(t)$.
   2) Solve the Sturm-Liouville problem for $X(x)$.
   3) Solve the corresponding time problem for $T(t)$.
   4) Use superposition.
   5) Use the initial conditions.

2. **Definition 2.10**. The **Fourier series** of f on (a,b) is given by

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l},$$

   where $l = (b-a)/2$ and

$$a_0 = \frac{1}{2l}\int_a^b f(x)dx, \quad a_n = \frac{1}{l}\int_a^b f(x)\cos\frac{n\pi x}{l}dx, \quad b_n = \frac{1}{l}\int_a^b f(x)\sin\frac{n\pi x}{l}dx, \quad n \geq 1,$$

   are called the **Fourier coefficients** of f.

3. **Method of Characteristics**

   Consider the first-order linear time-dependent problem of the form

$$\begin{aligned}&\frac{\partial u}{\partial t} + B(x,t)\frac{\partial u}{\partial x} = C(x,t,u), \quad -\infty < x < \infty, \quad t > 0,\\ &u(x,0) = f(x).\end{aligned}$$

   The method of characteristic consists on solving the **characteristic equations**

$$\begin{aligned}\frac{dx}{dt} &= B(x,t),\\ \frac{du}{dt} &= C(x,t,u),\end{aligned}$$

   and then using the initial condition.

4. Consider the one-dimensional wave equation
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$
d'Alembert's solution is given by
$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\mu)\, d\mu.$$

5. Consider the one-dimensional wave equation
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0,$$
$$u(0,t) = 0, \quad u(l,t) = 0, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$
d'Alembert's solution is given by
$$u(x,t) = \frac{1}{2}[\bar{f}_{\text{odd}}(x+ct) + \bar{f}_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}_{\text{odd}}(\mu)\, d\mu,$$
where $\bar{f}_{\text{odd}}$ and $\bar{g}_{\text{odd}}$ are the $2l$-periodic extension of f and g, respectively.

6. A regular **Sturm-Liouville problem**,
$$\begin{aligned} &(p(x)\phi')' + [q(x) + \lambda\sigma(x)]\phi = 0, \quad a < x < b, \\ &\alpha_1\phi(a) + \beta_1\phi'(a) = 0, \\ &\alpha_2\phi(b) + \beta_2\phi'(b) = 0, \end{aligned}$$
has eigenvalues λ_n and corresponding eigenfunctions ϕ_n for $n \geq 1$.

7. **Theorem 4.6.** *(Dirichlet's Theorem)*
If f is piecewise smooth on $[a,b]$, the **generalized Fourier series**,
$$f(x) \sim \sum_{n=1}^{\infty} c_n\phi_n(x), \quad \text{where,} \quad c_n = \frac{\langle f, \phi_n\rangle}{\|\phi_n\|^2} = \frac{1}{\|\phi_n\|^2}\int_a^b f(x)\phi_n(x)\sigma(x)\, dx,$$
for $n \geq 1$, converges pointwise to $[f(x^+) + f(x^-)]/2$ for each $x \in (a,b)$.

8. **Method of eigenfunction expansions**
   1) Identify the eigenfunctions ϕ_n associated to the problem.
   2) Assume a solution of the form $u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)$.
   3) Expand initial conditions and other related functions using generalized Fourier series.
   4) Substitute into the equation to solve for $a_n(t)$.

9. **Theorem 4.7.**
λ_n can be calculated from the **Rayleigh quotient**:
$$\lambda_n = \frac{-p(x)\phi_n(x)\phi_n'(x)\Big|_a^b + \displaystyle\int_a^b (p(x)\phi_n'(x)^2 - q(x)\phi_n(x)^2)\, dx}{\displaystyle\int_a^b \phi_n(x)^2\sigma(x)\, dx}.$$

10. Summary of Sturm-Liouville problems.

Model Type	**S-L Problem**	**Spectrum**	**Eigenfunctions**
Homogeneous Dirichlet B.C.	$\phi''(x) + \lambda\phi(x) = 0$ $\phi(0) = \phi(l) = 0$	$\lambda_n = \left(\frac{n\pi}{l}\right)^2$ $n = 1, 2, \cdots$	$\phi_n = \sin\frac{n\pi x}{l}$ $n = 1, 2, \cdots$
Homogeneous Neumann B.C.	$\phi''(x) + \lambda\phi(x) = 0$ $\phi'(0) = \phi'(l) = 0$	$\lambda_n = \left(\frac{n\pi}{l}\right)^2$ $n = 0, 1, \cdots$	$\phi_n = \cos\frac{n\pi x}{l}$ $n = 0, 1, \cdots$
Mixed Type I	$\phi''(x) + \lambda\phi(x) = 0$ $\phi(0) = \phi'(l) = 0$	$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2$ $n = 1, 2, \cdots$	$\phi_n = \sin\frac{(2n-1)\pi x}{2l}$ $n = 1, 2, \cdots$
Mixed Type II	$\phi''(x) + \lambda\phi(x) = 0$ $\phi'(0) = \phi(l) = 0$	$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2$ $n = 1, 2, \cdots$	$\phi_n = \cos\frac{(2n-1)\pi x}{2l}$ $n = 1, 2, \cdots$
Periodicity conditions	$\phi''(\theta) + \lambda\phi(\theta) = 0$ $\phi(-\pi) = \phi(\pi)$ $\phi'(-\pi) = \phi'(\pi)$	$\lambda_n = n^2$ $n = 0, 1, \cdots$	$\phi_n = a_n\cos n\theta + b_n \sin n\theta$ $n = 0, 1, \cdots$
Bessel equation	$r^2u''+ru'+(\lambda r^2-m^2)u = 0$ $u(a) = 0$ and $\lvert u(r)\rvert$ bounded as $r \to 0^+$	$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$ $n = 1, 2, \cdots$	$u_{mn} = a_{mn}J_m\left(\frac{z_{mn}r}{a}\right)$ $n = 1, 2, \cdots$
Legendre equation	$(\sin\theta\, v')' + \lambda\sin\theta v = 0$ $\lvert v(\theta)\rvert$ and $\lvert v'(\theta)\rvert$ bounded as $\theta \to \pm 1$	$\lambda_n = n(n+1)$ $n = 0, 1, \cdots$	$v_n = a_nP_n(\cos\theta)$ $n = 0, 1, \cdots$

10. The **Laplacian in polar coordinates** is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

11. The **Laplacian in spherical coordinates** is

$$\begin{aligned}\nabla^2 u &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\\ &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial\theta^2} + \cot\theta\frac{\partial u}{\partial\theta} + \csc^2\theta\frac{\partial^2 u}{\partial\phi^2}\right).\end{aligned}$$

12. **Fourier Integral Representation** of f on $(-\infty, \infty)$

$$f(x) = \int_0^\infty \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right]\, d\omega$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\, dx, \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\, dx.$$

13. **Fourier Cosine Integral Representation** of f on $[0, \infty)$

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\, d\omega, \quad \text{where} \quad A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\, dx.$$

14. **Fourier Sine Integral Representation** of f on $[0, \infty)$

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\, d\omega, \quad \text{where} \quad B(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx.$$

15. **Fourier transform** of $f(x)$ on $(-\infty, \infty)$,

$$\widehat{f}(\omega) = \mathcal{F}\left[f(x)\right](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx, \quad -\infty < \omega < \infty.$$

$$f(x) = \mathcal{F}^{-1}\left[\widehat{f}(\omega)\right](x) = \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{-i\omega x}\, d\omega, \quad -\infty < x < \infty.$$

16. **Fourier cosine transform** of $f(x)$ on $[0, \infty)$

$$\mathcal{C}\left[f(x)\right](\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\, dx, \quad \mathcal{C}^{-1}\left[g(\omega)\right](x) = \int_0^\infty g(\omega)\cos\omega x\, d\omega.$$

17. **Fourier sine transform** of $f(x)$ on $[0, \infty)$

$$\mathcal{S}\left[f(x)\right](\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx, \quad \mathcal{S}^{-1}\left[g(\omega)\right](x) = \int_0^\infty g(\omega)\sin\omega x\, d\omega.$$

18. **Gauss kernel**

$$g(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}.$$

19. **Error function**

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt.$$

13.2 Final review

1. *Example 3.4 modified.* Consider the following nonhomogeneous one-dimensional heat equation:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < 2, \quad t > 0,\\
&u(0,t) = 0,\\
&\frac{\partial u}{\partial x}(2,t) = 0,\\
&u(x,0) = g(x),
\end{aligned}$$

   where $f(x,t)$ is a continuous function of x and t.

(continue Example 3.4.)

2. *Example 3.4 modified part 2.* Consider the following nonhomogeneous one-dimensional heat equation:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < 2, \quad t > 0,\\
&u(0,t) = a(t),\\
&\frac{\partial u}{\partial x}(2,t) = b(t),\\
&u(x,0) = g(x)
\end{aligned}$$

   where $f(x,t)$ is a continuous function of x and t, and $a(t)$ and $b(t)$ are continuously differentiable functions of t.

3. **Exercise 19.9.** Consider torsional oscillations of a homogeneous cylindrical shaft. If $\omega(x,t)$ is the angular displacement at time t of the cross section at x, then

$$\frac{\partial^2 \omega}{\partial t^2} = a^2 \frac{\partial^2 \omega}{\partial x^2}, \quad 0 < x < l, \quad t > 0,$$

where the initial conditions are

$$\omega(x,0) = f(x) \quad \text{and} \quad \frac{\partial \omega}{\partial t}(x,0) = 0,$$

and the ends of the shaft are fixed elastically:

$$\frac{\partial \omega}{\partial x}(0,t) - \alpha\, \omega(0,t) = 0 \quad \text{and} \quad \frac{\partial \omega}{\partial x}(l,t) + \alpha\, \omega(l,t) = 0$$

with α a positive constant.

(a) Why is it possible to use separation of variables to solve this problem?

(b) Use separation of variables and show that one of the resulting problems is a regular Sturm-Liouville problem.

(c) Show that all of the eigenvalues of this regular Sturm-Liouville problem are positive.

Note: You do not need to solve the initial value problem, just answer the questions (a), (b), and (c).

(continue Exercise 19.19.)

4. How to solve PDE's in practice?