Math 300: Advance Boundary Value Problems

Week 1

1.1 Introduction

1. Notation and definitions:

   - The **patial derivative** of f with respect to x is denoted

   $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = f_x$$

   - **Gradient** of $f(x, y, z)$

   $$\nabla f(x, y, z) = (f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)).$$

   - **Laplacian** of $f(x, y, z)$

   $$\Delta f(x, y, z) = f_{xx}(x, y, z) + f_{yy}(x, y, z) + f_{zz}(x, y, z).$$

   - **Partial differential equation (PDE)** for unknown $u(x, y)$

   $$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxx}, \cdots) = 0.$$

   - **Linear differential operator** L satisfies

   $$L(u + v) = Lu + Lv \quad \text{and} \quad L(\lambda u) = \lambda u.$$

   - **Linear PDE** for unknown u

   $$Lu = f,$$

   where L is a linear differential operator and function f does not depend on u or any of its derivatives. The equation is **homogeneous** if $f = 0$, and **nonhomogeneous** if $f \neq 0$.

   - The **order of a PDE** is the highest order derivative in the equation.

2. *Example 1.1.*

   Find the dimension and order of the following PDEs. Which are linear, and which are homogeneous?

   - Heat equation:
   $$u_t = Du_{xx} + f(x)$$

   - Wave equation:
   $$u_{tt} - cu_{xx} = 0$$

   - Laplace equation:
   $$u_{xx} + u_{yy} = 0$$

   - Advection equation:
   $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

   $$\frac{\partial^2 u}{\partial x^2} + e^y \sin(z) \frac{\partial^2 u}{\partial x \partial z} = u$$

   $$\frac{\partial^2 u}{\partial x \partial y} = \sin(u)$$

   - KdV equation:
   $$u_t + uu_{xx} + u_{xxx} = 1$$

3. The **second-order linear constant-coefficiens homogeneous PDEs**

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0$$

is said to be

- **elliptic** iff $ac - b^2 > 0$.
- **parabolic** iff $ac - b^2 = 0$.
- **hyperbolic** iff $ac - b^2 < 0$.

4. *Example 1.2.*

Classify the following second-order linear PDEs.

- $u_t + 2u_{tt} + 3u_{xx} = 0$

- $17u_{yy} + 3u_x + u = 0$

- $4u_{xy} + 2u_{xx} + u_{yy} = 0$

- $u_{yy} - u_{xx} - 2u_{xy} = 0$

5. **Superposition principle**. If u_1 and u_2 are solutions to $Lu = 0$, so is $c_1u_1 + c_2u_2$.

6. **Theorem 1.2** If u_p is a particular solution to $Lu = f$ and u_h is the solutions to $L = u$, then $u = cu_h + u_p$ is a solution $Lu = f$ for any c.

7. *Example 1.7. (Burgers Equation)*

   Consider the following two-dimensional first-order nonlinear PDE:

$$u_x + uu_y = 0$$

   and solutions

$$u_1(x, y) = 1 \quad \text{and} \quad u_2(x, y) = \frac{y}{1 + x}.$$

   Consider the nonhomogeneous case:

$$u_x + uu_y = \frac{y^2 - 1}{x^2y^3}$$

   with particular solution

$$u_p(x, y) = -\frac{1}{xy}.$$

8. **Conditions**: a PDE can have

   - **Initial conditions**: value at time $t = 0$, i.e., $u(x, y, 0) = u_0(x, y)$.
   - **Boundary conditions**: value on the boundary $\partial\Omega$ for all time
     - Dirichlet: $u = g$ on $\partial\Omega$. Homogeneous if $g = 0$.
     - Neumann: $\frac{\partial u}{\partial n} = g$ on $\partial\Omega$. Homogeneous if $g = 0$.
     - Robin: $\alpha u + \beta\frac{\partial u}{\partial n} = g$ on $\partial\Omega$. Homogeneous if $g = 0$.

9. A **Boundary Value Problem** BVP is a PDE with boundary conditions.

10. A **steady-state solution** to a BVP does not depend on time, i.e., $u(x, t) = \tilde{u}(x)$.

11. *Example 1.10.*

    Find the steady-state solution to the following PDE on $[0, 2\pi]$:

$$\begin{aligned}
&u_t = 3u_{xx} + 9\sin x,\\
&u(x, 0) = 9\sin x,\\
&u(0, t) = 9,\\
&u_x(2\pi, t) = 0.
\end{aligned}$$

12. **Exercise 15.1**

    Show that the function
$$u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$
    is harmonic; that is, it is a solution to the three-dimensional Laplace equation $\Delta u = 0$.

13. **Exercise 15.4**

    Compute the Laplacian of the function

    $$u(x, y) = \log\left(x^2 + y^2\right)$$

    in an appropriate coordinate system and decide if the given function satisfies Laplace's equation $\nabla^2 u = 0$.