Separation of Variables
w1 ∘ Eigenvalue problem
w2-3 ∘ Fourier series problem

Math 300: Advanced Boundary Value Problems

Week 7

Midterm 2 starts here.

1.1 Sturm-Liouville Theory

(AKA Generalized Eigenvalue problem)

1. **Definition 4.1.** A **regular Sturm-Liouville problem** denotes the problem of finding an eigenfunction-eigenvalue pair (ϕ, λ) which solves the problem

$$\text{ODE} \longrightarrow (p(x)\phi')' + [q(x) + \lambda\sigma(x)]\phi = 0, \quad a < x < b,$$
$$\text{BC.} \begin{cases} \alpha_1\phi(a) + \beta_1\phi'(a) = 0, \\ \alpha_2\phi(b) + \beta_2\phi'(b) = 0, \end{cases}$$

where

(i) $p(x)$, $p'(x)$, $q(x)$, and $\sigma(x)$ are real valued and continuous for $a \le x \le b$;

(ii) $p(x) > 0$ and $\sigma(x) > 0$ for $a \le x \le b$; and close interval.

(iii) α_1 , α_2 , β_1 , β_2 are real valued, $\alpha_1^2 + \beta_1^2 \neq 0$ and $\alpha_2^2 + \beta_2^2 \neq 0$. not both constants equal to zero.

2. *Example 4.2.* Consider the following boundary value problem, which we have solved several times before:

$$\phi'' + \lambda\phi = 0, \quad 0 < x < l,$$
$$\phi(0) = 0,$$
$$\phi(l) = 0.$$

Standard Eigenvalue Problem with Dirichlet BC.

$p(x) = 1$, $q(x) = 0$, $\sigma(x) = 1$, $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0$

i) $p(x) = 1$ $p'(x) = 0$ $q(x) = 0$ $\sigma(x) = 1$ are cont. ✓

ii) $p(x) = 1 > 0$ $\sigma(x) = 1 > 0$ ✓

iii) $\alpha_1 = \alpha_2 = 1$ $\beta_1 = \beta_2 = 0$ ✓

this is a regular SL problem.

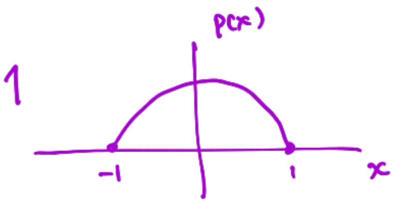
3. **Definition 4.2.** A Sturm-Liouville problem is said to be **singular** if at least one of the conditions (i), (ii), or (iii) in Definition 4.1 fails, or if the interval is infinite. In the case where the interval is infinite, or one or both of the functions $p(x)$ and $\sigma(x)$ approach 0 or ∞ at an endpoint of the interval, one or more of the boundary conditions are usually replaced by boundedness conditions on ϕ.

$a = \infty$
or
$b = \infty$

4. *Example 4.3. (Legendre's Equation)* Consider the boundary value problem for Legendre's equation,

$(p\phi')' + (q + \lambda\sigma)\phi = 0$

$$((1-x^2)\phi')' + \lambda\phi = 0, \quad -1 < x < 1,$$
$$\alpha_1\phi(-1) + \beta_1\phi'(-1) = 0,$$
$$\alpha_2\phi(1) + \beta_2\phi'(1) = 0,$$



$p(x) = (1-x^2)$, $q(x) = 0$, $\sigma(x) = 1$

$p(x)$ is continuous on $[-1, 1]$

$p(-1) = 0 = p(1)$ so

so $p(x) \neq 0$ on $[-1, 1]$ (close interval)

$\therefore$ this is no a regular SL problem

5. *Example 4.4. (Bessel's Equation)* For fixed n, Bessels equation on the interval $a < r < b$,

variable.

$$(r\phi')' + \left(\lambda r - \frac{n^2}{r}\right)\phi = 0,$$
$$\phi(a) = 0,$$
$$\phi(b) = 0,$$

$p(r) = r$, $q(r) = -\frac{n^2}{r}$, $\sigma(r) = r$

$\alpha_1 = \alpha_2 = 1$ $\beta_1 = \beta_2 = 0$

How can we make sure this a regular SL problem?

ii) $p(r) = r > 0$ $\sigma(r) = r > 0$ as long as $0 < a < r < b$

i) p, p', q, σ as real valued an cont.

If $a > 0 \Rightarrow$ regular SL prob.

if $a = 0 \Rightarrow$ singular SL prob. (because $p(a) = 0$)

if $a < 0 \Rightarrow$ " " " ($q(r)$ is not cont.)

set of eigvals

6. **Theorem 4.2.** The spectrum of a regular Sturm-Liouville problem is a countably infinite set with no limit points, that is, an infinite discrete set.

orthogonality

7. **Theorem 4.3.** If λ_m and λ_n are distinct eigenvalues of a regular Sturm-Liouville problem, that is, $\lambda_m \neq \lambda_n$, the corresponding eigenfunctions ϕ_m and ϕ_n are orthogonal relative to the inner product

$$\langle f, g\rangle = \int_a^b f(x)g(x)\sigma(x)dx.$$

weight

8. **Theorem 4.4.** If λ is an eigenvalue of a regular Sturm-Liouville problem:

   (a) λ is real, and

   (b) if ϕ and ψ are eigenfunctions corresponding to λ,

   $$\psi(x) = k\phi(x), \quad a \leq x \leq b,$$

   where k is a nonzero constant, and each eigenfunction can be made real-valued by multiplying it by an appropriate nonzero constant.

uniqueness of eigenfunctions (except for constant multiples)

9. *Example 4.5. (Cauchy-Euler Equation)* Consider the boundary value problem

$$(x\phi')' + \frac{\lambda}{x}\phi = 0, \quad 1 < x < l,$$
$$\phi(1) = 0,$$
$$\phi(l) = 0.$$

$p(x) = x$, $q(x) = 0$, $\sigma(x) = \frac{1}{x}$ on $[1, l]$ ✓ reg SL pr.

$\boxed{x(x\phi')' + \lambda\phi = 0} \Rightarrow x^2\phi'' + x\phi' + \lambda\phi = 0$

recall from Math 201, for the Cauchy-Euler equation $ax^2y'' + bxy' + cy = 0$ we use substitution $x = e^t$

Let $x = e^t \Rightarrow \frac{dx}{dt} = e^t$, and $u(t) = \phi(x) = \phi(e^t)$

$\Rightarrow u'(t) = \frac{d}{dt}\phi(x) = \phi'\frac{dx}{dt} = x\phi'$

$\Rightarrow u''(t) = \frac{d}{dt}(x\phi') = \frac{d}{dx}(x\phi')\frac{dx}{dt} = (x\phi')'x$

Then $\underbrace{x(x\phi')'}_{u''} + \lambda\phi = 0 \Rightarrow \underbrace{u'' + \lambda u = 0}_{\text{std. eig. prob.}}$

$1 < x < 2 \quad x = e^t \Rightarrow t = \ln x$

(continue Example 4.5)

$$u'' + \lambda u = 0 \quad , \; 0 < t < \ln 2$$
$$u(0) = 0$$
$$u(\ln 2) = 0$$

Dirichle BC
Eig. prob.

$$\Longrightarrow \lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2 , \quad u_n(t) = B_n \sin\left(\frac{n\pi t}{\ln 2}\right)$$

$$\Longrightarrow \lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2 , \quad \phi_n(x) = B_n \sin\left(n\pi \frac{\ln x}{\ln 2}\right)$$

Now, Let's check orthogonality (Th 4.2)

$$\langle \phi_n, \phi_m \rangle \overset{n \neq m}{=} \int_1^2 \underbrace{\sin\left(n\pi \frac{\ln x}{\ln 2}\right)}_{\phi_n} \underbrace{\sin\left(m\pi \frac{\ln x}{\ln 2}\right)}_{\phi_m} \underbrace{\frac{1}{x}}_{\sigma} \, dx$$

$v = \frac{\pi}{\ln 2} \ln x$

$dv = \frac{\pi}{\ln 2} \frac{1}{x} dx$

$$= \frac{\ln 2}{\pi} \int_0^\pi \sin(nv) \sin(mv) \, dv$$

$$= \frac{\ln 2}{2\pi} \int_0^\pi \left[\cos(n-m)v - \cos(n+m)v\right] dv$$

$$= \frac{\ln 2}{2\pi} \left[\frac{1}{n-m} \sin(n-m)v - \frac{1}{n+m} \sin(n+m)v\right]\Bigg|_0^\pi$$

$$= 0$$