Math 300: Advanced Boundary Value Problems

Week 5

1.1 Separation of Variables: Nonhomogeneous equations

1. Standard homogeneous Heat and Wave equations

Heat eq. with Dirichlet BCs

$$\begin{aligned}
&u_t = ku_{xx}, \quad 0 < x < a, \quad t > 0,\\
&u(0,t) = 0, \quad t > 0,\\
&u(a,t) = 0, \quad t > 0,\\
&u(x,0) = f(x), \quad 0 < x < a.
\end{aligned}$$

The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{a}\right)^2 kt} \sin \frac{n\pi x}{a}.$$

No flux

Heat equation with Neumann BCs

$$\begin{aligned}
&u_t = ku_{xx}, \quad 0 < x < a, \quad t > 0,\\
&u_x(0,t) = 0, \quad t > 0,\\
&u_x(a,t) = 0, \quad t > 0,\\
&u(x,0) = f(x), \quad 0 < x < a.
\end{aligned}$$

The solution has the form

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{a}\right)^2 kt} \cos \frac{n\pi x}{a}.$$

Wave equation with Dirichlet BCs

$$\begin{aligned}
&u_{tt} = c^2 u_{xx}, \quad 0 < x < a, \quad t > 0,\\
&u(0,t) = 0, \quad t > 0,\\
&u(a,t) = 0, \quad t > 0,\\
&u(x,0) = f(x), \quad 0 < x < a,\\
&u_t(x,0) = g(x), \quad 0 < x < a.
\end{aligned}$$

The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{a} + b_n \sin \frac{n\pi ct}{a} \right) \sin \frac{n\pi x}{a}.$$

Wave equation with Neumann BCs

$$\begin{aligned}
&u_{tt} = c^2 u_{xx}, \quad 0 < x < a, \quad t > 0,\\
&u_x(0,t) = 0, \quad t > 0,\\
&u_x(a,t) = 0, \quad t > 0,\\
&u(x,0) = f(x), \quad 0 < x < a,\\
&u_t(x,0) = g(x), \quad 0 < x < a.
\end{aligned}$$

The solution has the form

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{a} + b_n \sin \frac{n\pi ct}{a} \right) \cos \frac{n\pi x}{a}.$$

$u(x,t) = v(x,t) + w(x,t)$

2. Method for nonhomogeneous equations. Consider a solution of the form

$$u(x,t) = \underbrace{v(x)}_{\text{steady state sol.}} + \underbrace{w(x,t)}_{\text{transient. sol}}.$$

homo

will cart. function so that the eq of w is homo

3. **Exercise 13.3 (modified)**

Solve the following boundary value–initial value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \sin\frac{\pi x}{a},$$

eq w/ nonhomo term

$$u(0,t) = 0 \quad \text{and} \quad u(a,t) = 1,$$

nonhomo BC.

$$u(x,0) = 3\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a},$$

for $0 < x < a$, $t > 0$.

Let $u(x,t) = v(x) + w(x,t)$

$u_t = w_t$, $u_{xx} = v'' + w_{xx}$

Use the eq. $w_t = k v'' + k w_{xx} + \sin\frac{\pi x}{a}$

$\Rightarrow w_t = k w_{xx} + \underbrace{k v'' + \sin\frac{\pi x}{a}}_{\text{force} = 0}$

Use the BC.:

$u(0,t) = v(0) + w(0,t) = 0$ force $v(0) = 0$

$u(a,t) = v(a) + w(a,t) = 1$ force $v(a) = 1$

2nd order ODE

$v'' = -\frac{1}{k}\sin\frac{\pi x}{a}$, $v(0) = 0$, $v(a) = 1$

$\Rightarrow v(x) = \frac{1}{k}\frac{a^2}{\pi^2}\sin\frac{\pi x}{a} + Ax + B$

$v(0) = B = 0$, $v(a) = Aa = 1 \Rightarrow A = \frac{1}{a}$

$\Rightarrow v(x) = \frac{a^2}{k\pi^2}\sin\frac{\pi x}{a} + \frac{1}{a}x$

IC for $u(x,t)$

$u(x,0) = v(x) + w(x,0) = f(x)$

$\Rightarrow w(x,0) = f(x) - v(x)$

(continue Exercise 13.3 (modified))

$$\begin{cases} w_t = \kappa w_{xx} \\ w(0,t) = w(a,t) = 0 \\ w(x,0) = f(x) - v(x) \end{cases}$$

we know the solution to this problem has the form

$$w(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{a}\right)^2 \kappa t} \sin\frac{n\pi x}{a},$$

Now we use the initial condition to solve for b_n

$$w(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} = f(x) - v(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} = 3\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a} - \frac{a^2}{\kappa\pi^2}\sin\frac{\pi x}{a} - \frac{1}{\pi}x$$

$$= \underbrace{\left(3 - \frac{a^2}{\kappa\pi^2}\right)\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a}}_{\text{this is already in Fourier sine series Match coefficients!}} \underbrace{- \frac{1}{\pi}x}_{\text{this is not a Fourier sine s. Find representation}}$$

$$= \left(3 - \frac{a^2}{\kappa\pi^2}\right)\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a} - \sum_{n=1}^{\infty} \frac{2a(-1)^{n+1}}{\pi^2 n}\sin\frac{n\pi x}{a}$$

$$b_n = \begin{cases} 3 - \dfrac{a^2}{\kappa\pi^2} - \dfrac{2a(-1)^2}{\pi^2 1}, & n = 1 \\ -1 - \dfrac{2a(-1)^4}{\pi^2 3}, & n = 3 \\ -\dfrac{2a(-1)^{n+1}}{\pi^2 n}, & n \neq 1, 3 \end{cases}$$

$$b_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}$$

Finally

$$\boxed{u(x,t) = \underbrace{-\frac{a^2}{\kappa\pi^2}\sin\frac{\pi x}{a} - \frac{1}{\pi}x}_{v(x)} + \underbrace{\sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{a}\right)^2 \kappa t}\sin\frac{n\pi x}{a}}_{w(x,t)}}$$

1.2 Method of Characteristics

1. **Method of Characteristics**

   Consider the first-order linear time-dependent problem of the form

$$\frac{\partial u}{\partial t} + B(x,t)\frac{\partial u}{\partial x} = C(x,t,u), \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x).$$

   The method of characteristic consists on solving the **characteristc equations**

$$\frac{dx}{dt} = B(x,t), \quad \leftarrow \text{1st ch. eq.}$$
$$\frac{du}{dt} = C(x,t,u), \quad \leftarrow \text{2nd ch. eq.}$$

   and then using the initial condition.

2. *Example 10.2*

   Solve the following PDE for $u(x,t)$ on $-\infty < x < \infty$

$$\frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial x} + \beta u = 0,$$
$$u(x,0) = f(x),$$

   using the method of characteristics.

(std. form)

use "a" as constant of integration

$$\Rightarrow u_t + \alpha u_x = -\beta u.$$

1st ch.eq. $\frac{dx}{dt} = \alpha \Rightarrow x(t) = \alpha t + a, \; x(0) = a$

$$\Rightarrow \underline{a = x - \alpha t}$$

at $t=0$ $a = x(0)$

anchor point

2nd ch eq. $\frac{du}{dt} = -\beta u \Rightarrow u(x(t),t) = Ae^{-\beta t}$

relate solutions of 2nd and 1st ch. eq's.

Use IC $u(x,0) = u(x(0),0) = \underline{A} = f(\underbrace{x(0)}_{a})$

$$\Rightarrow \underline{u(x,t) = f(x-\alpha t)e^{-\beta t}}$$

use $a = x - \alpha t$

3. **Exercise 17.5**

Solve the first-order equation

$$\frac{\partial u}{\partial t} + 3x\frac{\partial u}{\partial x} = 2t \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = \log(1+x^2).$$

- 1st ch eq.

$$\frac{dx}{dt} = 3x \Rightarrow x(t) = a e^{3t}, \quad x(0) = a$$

$$\Rightarrow a = x e^{-3t} \qquad z = \frac{1}{3}\ln\frac{x}{a}$$

- 2nd ch. eq.

$$\frac{du}{dt} = 2t \Rightarrow u(x(t),t) = z^2 + C$$

- IC.

$$u(x(0),0) = C = \log(1 + \underbrace{x^2(0)}_{a^2})$$

$$u(x,t) = z^2 + \log(1 + (x e^{-3t})^2)$$

$$\Rightarrow \boxed{u(x,t) = z^2 + \log(1 + x^2 e^{-6t})}$$

4. **Exercise 17.6**

Using the method of characteristics, solve

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = e^{2x}, \quad -\infty < x < \infty, \quad t > 0,$$

$$w(x,0) = f(x).$$

1st ch. eq

$$\frac{dx}{dt} = c \Rightarrow x(t) = ct + a, \quad x(0) = a$$

$$\Rightarrow a = x - ct$$

2nd ch. eq

$$\frac{dw}{dt} = e^{2x} = e^{2(ct+a)} \Rightarrow w(x(t),t) = \frac{1}{2c} e^{2(ct+a)} + K$$

IC.

$$w(x(0),0) = \frac{1}{2c} e^{2a} + K = f(x(0))$$

$$\Rightarrow K = f(\underbrace{x(0)}_{a}) - \frac{1}{2c} e^{2a}$$

$$\Rightarrow w(x,t) = \frac{1}{2c} e^{2x} + f(x-ct) - \frac{1}{2c} e^{2(x-ct)}$$

$$\Rightarrow \boxed{w(x,t) = \frac{1}{2c} e^{2x}(1 - e^{-2ct}) + f(x-ct)}$$

5. **Exercise 17.7**

Using the method of characteristics, solve

$$\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 1, \quad -\infty < x < \infty, \quad t > 0,$$

$$w(x,0) = f(x).$$

1st ch eq

$$\frac{dx}{dt} = t \Rightarrow x(t) = \frac{t^2}{2} + a, \quad x(0) = a$$

$$\Rightarrow a = x - \frac{t^2}{2}$$

2nd ch eq

$$\frac{dw}{dt} = 1 \Rightarrow w(x(t),t) = t + A$$

IC. $w(x(0),0) = A = f(\underbrace{x(0)}_{a})$

$$\boxed{w(x(t),t) = t + f\left(x - \frac{t^2}{2}\right)}$$

(extra page)