Math 300: Advanced Boundary Value Problems

Week 10

1.1 Bessel functions

1. **Bessel's equation** of order n is given by

$$x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+(x^2-n^2)u=0.$$

2. The general solution of Bessel's equations of order n are given by

$$u(x)=A\,J_n(x)+B\,Y_n(x),$$

for arbitrary constants A and B, where

$$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(n+k)!}\left(\frac{x}{2}\right)^{n+2k},\quad n\geq 0,$$

is the **Bessel function of the first kind of order** n and $Y_n(x)$ is the **Bessel function of the second kind of order** n.

3. **Theorem 6.4.** (Orthogonality) For a fixed integer $m\geq 0$,

$$\int_0^1 xJ_m(z_{mn}x)J_m(z_{mk}x)\,dx=0,\qquad \int_0^1 xJ_m(z_{mn}x)^2\,dx=\frac{1}{2}J_{m+1}(z_{mn})^2,$$

where z_{mn} is a zero of $J_m(x)$ for $n\geq 1$.

4. **Theorem 6.5.** (Fourier-Bessel Expansion Theorem) If f and f' are piecewise continuous on the interval $0\leq x\leq 1$, then for $0<x_0<1$, the **Fourier-Bessel series** expansion

$$f(x)=\sum_{n=1}^{\infty}a_nJ_m(z_{mn}x_0),\quad \text{where}\quad a_n=\frac{2}{J_{m+1}(z_{mn})^2}\int_0^1 f(x)J_m(z_{mn}x)x\,dx,$$

converges to $[f(x_0^+)+f(x_0^-)]/2$. At $x_0=1$, the series converges to 0, since every $J_m(z_{mn})=0$. At $x_0=0$, the series converges to 0 if $m\geq 1$, and to $f(0^+)$ if $m=0$.

1.2 Polar coordinates

1. *Vibrating circular membrane.* Consider the following wave equation in a disk

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right), \quad 0 < r < a, \quad -\pi < \theta < \pi, \quad t > 0,\\
&u(r, -\pi, t) = u(r, \pi, t),\\
&\frac{\partial u}{\partial \theta}(r, -\pi, t) = \frac{\partial u}{\partial \theta}(r, \pi, t),\\
&u(a, \theta, t) = 0,\\
&|u(r, \theta, t)| < \infty, \quad \text{as} \quad r \to 0^+,\\
&u(r, \theta, 0) = f(r, \theta),\\
&\frac{\partial u}{\partial t}(r, \theta, 0) = g(r, \theta).
\end{aligned}
$$

2. **Exercise 13.14.**

   Solve the two-dimensional heat equation inside a disk with circularly symmetric time-independent sources, boundary conditions, and initial conditions:

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + Q(r), \quad 0 < r < a, \quad t > 0,$$

   with

$$u(r, 0) = f(r), \quad u(a, t) = T.$$

(continue Exercise 13.14)

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