Math 300: Advanced Boundary Value Problems

Week 6

1.1 One-dimensional Wave Equation

1. Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

d'Alembert's solution is given by

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\mu)\, d\mu.$$

2. **Exercise 17.12**

The displacement $u = u(x,t)$ of an infinitely long string is governed by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0.$$

At time $t = 0$ an initial signal is given of the form

$$u(x,0) = f(x) = \begin{cases} x, & 0 < x < 1, \\ -x+2, & 1 < x < 2, \\ 0, & \text{othewise,} \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \quad -\infty < x < \infty.$$

a) Solve this problem.

b) Sketch the solution for times t_1, t_2, t_3, t_4, t_5 , with

$$t_1 = 0, \quad 0 < t_2 < 1/4, \quad t_3 = 1/4, \quad 1/4 < t_4 < 1/2, \quad t_5 = 1/2.$$

c) At what time does the signal reach the point $x = 11$?

(continue Exercise 17.12)

(continue Exercise 17.12)

3. Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0,$$
$$u(0,t) = 0, \quad u(l,t) = 0, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

   d'Alembert's solution is given by

$$u(x,t) = \frac{1}{2}[\bar{f}_{\text{odd}}(x+ct) + \bar{f}_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}_{\text{odd}}(\mu)\, d\mu,$$

   where $\bar{f}_{\text{odd}}$ and $\bar{g}_{\text{odd}}$ are the $2l$-periodic extension of f and g, respectively.

4. **Exercise 14.8**

   Use d'Alembert's solution to solve the boundary value–initial value problem for the wave equation:

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0 \\ &u(0,t) = 0, \\ &u(1,t) = 0, \\ &u(x,0) = 0, \\ &\frac{\partial u}{\partial t}(x,0) = 1. \end{aligned}$$

5. **Exercise 17.8**

   Consider

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0,\\
&u(x,t) = f(x).
\end{aligned}
$$

   Show that characteristics are straight lines.

6. **Exercise 17.9**

   Consider

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0.$$

   with

$$u(x,0) = f(x) = \begin{cases} 1, & 0 < x, \\ 1 + x/a, & 1 < x < a, \\ 2, & x > a. \end{cases}$$

   a) Determine the equations for the characteristics. Sketch the characteristics.
   b) Determine the solution $u(x,t)$. Sketch $u(x,t)$ for t fixed.

7. **Exercise 14.9**

   Use d'Alembert's solution to solve the boundary value–initial value problem for the wave equation:

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0 \\
&u(0,t) = 0, \\
&u(1,t) = 0, \\
&u(x,0) = 0, \\
&\frac{\partial u}{\partial t}(x,0) = \sin \pi x.
\end{aligned}
$$

8. **Exercise 14.10**

   Use d'Alembert's solution to solve the boundary value–initial value problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = x^2,$$
$$\frac{\partial u}{\partial t}(x,0) = 3.$$