Math 300: Advanced Boundary Value Problems

Week 4

1.1 Separation of Variables: Homogeneous equations

1. In the method of **separations of variables** we look for solutions of the form

$$u(x,t) = X(x)T(t).$$

2. The **eigenvalue problem with homogeneous Dirichlet boundary conditions**

$$X'' + \lambda X = 0 \quad X(0) = 0, \quad X(l) = 0,$$

has nontrivial solution for eigenvalues and corresponding eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{l}, \quad n \geq 1.$$

3. The **eigenvalue problem with homogeneous Neumann boundary conditions**

$$X'' + \lambda X = 0 \quad X'(0) = 0, \quad X'(l) = 0,$$

has nontrivial solution for eigenvalues and corresponding eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \cos\frac{n\pi x}{l}, \quad n \geq 0.$$

	Dirichlet BC	Neumann BC.
Heat Eq	$\begin{cases} u_t = k u_{xx} \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = f(x) \end{cases}$	$\begin{cases} u_t = k u_{xx} \\ u_x(0,t) = u_x(l,t) = 0 \\ u(x,0) = f(x) \end{cases}$
Wave Eq	$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$	$\begin{cases} u_{tt} = c^2 u_{xx} \\ u_x(0,t) = u_x(l,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$

4. **Exercise 13.2**

Solve the homogeneous Dirichlet problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < a, \quad t > 0,$$

subject to the boundary conditions

$$u(0,t) = 0, \quad \text{and} \quad u(a,t) = 0,$$

for $t > 0$, with initial conditions

$$f(x) = u(x,0) = \begin{cases} 1, & 0 < x < \frac{a}{2} \\ 2, & \frac{a}{2} \le x < a. \end{cases}$$

1 Separation of Variables

$u(x,t) = X(x)T(t) \Rightarrow u_t = XT', \; u_{xx} = X''T$

Use Eqn: $XT' = kX''T \Rightarrow \frac{T'}{kT} = \frac{X''}{X} = -\lambda$

Use BC's: $u(0,t) = X(0)T(t) = 0$ set $X(0) = 0$

$u(a,t) = X(a)T(t) = 0$ set $X(a) = 0$

2 Eigenvalue problem

Homo eig prob. w/ Dirichlet BC's $n \geq 1$

$X'' + \lambda X = 0, \; X(0) = X(a) = 0 \Rightarrow \lambda_n = \left(\frac{n\pi}{a}\right)^2, \; \underline{X_n(x) = \sin\frac{n\pi x}{a}}$

3 Time equation

$T' = -\lambda k T \Rightarrow T' = -\left(\frac{n\pi}{a}\right)^2 kT \Rightarrow \underline{T_n(t) = A_n e^{-\left(\frac{n\pi}{a}\right)^2 kt}}$

4 Superposition principle

$$\underline{u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{a}\right)^2 kt} \sin\frac{n\pi x}{a}}$$

$b_n = C_n A_n$

5 Fourier (sine) series

IC: $u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}$

$$b_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx = \dots = \frac{2}{n\pi} + \frac{2}{n\pi}\cos\frac{n\pi}{2} - \frac{4}{n\pi}(-1)^n$$

6 Combine

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{n\pi}\left(1 + \cos\frac{n\pi}{2} - 2(-1)^n\right)\right] e^{-\frac{n^2\pi^2}{a^2}kt} \sin\frac{n\pi x}{a}}$$

5. **Exercise 13.3**

Solve the following boundary value–initial value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

$$u(0,t) = u(a,t) = 0,$$

$$u(x,0) = 3\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a}$$

for $0 < x < a$, $t > 0$.

1 Separation of Variables

$u(x,t) = X(x)T(t) \Rightarrow u_t = XT'$, $u_{xx} = X''T$

Use Equ. $XT' = kX''T \Rightarrow \frac{T'}{kT} = \frac{X''}{X} = -\lambda$

Use BC's $u(0,t) = X(0)T(t) = 0$, set $X(0) = 0$

$u(a,t) = X(a)T(t) = 0$, set $X(a) = 0$

2 Eigenvalue problem

Homo eig prob w/ Dirichlet BC's

$X'' + \lambda X = 0$ $X(0) = X(a) = 0 \Rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{a}\right)^2, X_n(x) = \sin\frac{n\pi x}{a}}$ $n \geq 1$

3 Time equation

$T' = -\left(\frac{n\pi}{a}\right)^2 kT \Rightarrow \boxed{T_n(t) = A_n e^{-\left(\frac{n\pi}{a}\right)^2 kt}}$

4 Superposition principle $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{a}\right)^2 kt} \sin\frac{n\pi x}{a}$

5 Fourier (sine) series

Use IC: $u(x,0) = 3\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}$

Match coefficients: $b_1 = 3$, $b_3 = -1$, $b_n = 0$ $n \neq 1, 3$

6 Combine:

$$\boxed{u(x,t) = 3e^{-\frac{\pi^2}{a^2}kt}\sin\frac{\pi x}{a} - e^{-\frac{9\pi^2}{a^2}kt}\sin\frac{3\pi x}{a}}$$

6. **Exercise 14.2**

Solve the following boundary value–initial value problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0$$

$$\left.\begin{aligned} u(0,t) &= 0, \\ u(1,t) &= 0, \end{aligned}\right\} \text{BC}$$

$$\left.\begin{aligned} u(x,0) &= \sin \pi x + \tfrac{1}{2}\sin 3\pi x + 3\sin 7\pi x, \\ \frac{\partial u}{\partial t}(x,0) &= \sin 2\pi x. \end{aligned}\right\} \text{IC}$$

1 Separation of Variables

$u(x,t) = X(x)T(t) \Rightarrow u_{tt} = XT'' \quad u_{xx} = X''T$

Use Equ: $XT'' = X''T \Rightarrow \frac{T''}{T} = \frac{X''}{X} = -\lambda$

Use BC: $X(0)T(t) = 0 \Rightarrow X(0) = 0$, $X(1)T(t) = 0 \Rightarrow X(1) = 0$

2 Eigenvalue problem

Homo eig. prob w/ Dirichlet BC's

$X'' + \lambda X = 0, \quad X(0) = X(1) = 0 \Rightarrow \lambda_n = (n\pi)^2, \; X_n(x) = \sin n\pi x \quad n \geq 1$

3 Time Equation

$T'' + n^2\pi^2 T = 0 \Rightarrow T_n(t) = A_n \cos n\pi t + B_n \sin n\pi t$

4 Superposition principle

$$\rightarrow u(x,t) = \sum_{n=1}^{\infty} \Big(\underbrace{C_n A_n}_{a_n} \cos n\pi t + \underbrace{C_n B_n}_{b_n} \sin n\pi t\Big) \sin n\pi x$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \Big(-n\pi a_n \sin n\pi t + n\pi b_n \cos n\pi t\Big) \sin n\pi x$$

5 Initial conditions

$u(x,0) = \sin \pi x + \frac{1}{2}\sin 3\pi x + 3\sin 7\pi x = \sum_{n=1} a_n \sin n\pi x$

1st IC

$a_1 = 1, \quad a_3 = \frac{1}{2} \quad a_7 = 3, \quad a_n = 0 \quad n \neq 1, 3, 7$

(continue Exercise 14.2)

2nd IC. $\frac{\partial u}{\partial t}(x,0) = \sin 2\pi x = \sum_{n=1}^{\infty} n\pi b_n \sin n\pi x$

$n=2 \quad 1 = 2\pi b_2 \Rightarrow b_2 = \frac{1}{2\pi}$

$b_n = 0 \quad n \neq 2$

b Combine.

$$\boxed{u(x,t) = \underbrace{\cos \pi t \sin \pi x}_{n=1} + \underbrace{\frac{1}{2\pi} \sin 2\pi t \sin 2\pi x}_{n=2} + \underbrace{\frac{1}{2} \cos 3\pi t \sin 3\pi x}_{n=3} + \underbrace{3 \cos 7\pi t \sin 7\pi x}_{n=7}}$$

7. **Exercise 13.8**

Solve the problem of heat transfer in a bar of length $a = \pi$ and thermal diffusivity $k = 1$, with initial heat distribution $u(x, 0) = \sin x$, where one end of the bar is kept at a constant temperature $u(0, t) = 0$, while there is no heat loss at the other end of the bar, so that $\partial u(\pi, t)/\partial x = 0$, that is, solve the boundary value–initial value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0 \\ &u(0, t) = 0, \\ &\frac{\partial u}{\partial x}u(\pi, t) = 0, \\ &u(x, 0) = \sin x. \end{aligned}$$

Mixed BC.

You can use the fact that $\left\{\sin \frac{(2n+1)x}{2}\right\}_{n\geq 0}$ are orthogonal on $[0,\pi]$.

1 Separation of Variables

$u(x,t) = X(x)T(t) \Rightarrow u_t = XT' \quad u_{xx} = X''T$

- Using the equation we have

$XT' = kX''T \Rightarrow \frac{T'}{kT} = \frac{X''}{X} = -\lambda$

- Using the BC's

$u(0,t) = X(0)T(t) = 0$ set $X(0) = 0$

$\frac{\partial u}{\partial x}(\pi,t) = X'(\pi)T(t) = 0$ set $X'(\pi) = 0$

2 Eigenvalue Problem.

$X'' + \lambda X = 0, \quad X(0) = 0, \quad X'(\pi) = 0$

- Case 1 $\lambda = -\mu^2 < 0$ $\quad X'' - \mu^2 X = 0$

solution $\Rightarrow$ $X(x) = A\cosh\mu x + B\sinh\mu x$

$X'(x) = \mu A\sinh\mu x + \mu B\cosh\mu x$

BC's: $X(0) = A = 0$, $\quad X'(\pi) = \mu B\cosh\mu\pi = 0 \Rightarrow B = 0$

$\Rightarrow$ no non-trivial sol's.

(continue Exercise 13.8)

- <u>Case 2</u> $\lambda = 0 \Rightarrow X'' = 0 \Rightarrow X(x) = Ax + B,\ X'(x) = A$

  BC's: $X(0) = B = 0$ and $X'(\pi) = A = 0 \Rightarrow$ no non-trivial solution

- <u>Case 3</u> $\lambda = \mu^2 > 0 \Rightarrow X'' + \mu^2 X = 0$

  solution $\Rightarrow$ $X(x) = A\cos\mu x + B\sin\mu x$

  $X'(x) = -\mu A\sin\mu x + \mu B\cos\mu x$

  BC's: $X(0) = A = 0$ and $X'(\pi) = \mu B\cos\mu\pi = 0$

  solve $\cos\mu\pi = 0 \Rightarrow \mu\pi = \frac{\pi}{2} + n\pi \Rightarrow \mu = \frac{1}{2} + n = \frac{2n+1}{2}$

  (otherwise, if $B = 0$ we have trivial solutions and $\mu = 0$ is contradictory)

  thus, the non-trivial solution is

  $$\lambda_n = \left(\frac{2n+1}{2}\right)^2,\quad X_n(x) = \sin\frac{(2n+1)x}{2},\quad n \geq 0$$

  smallest n such that $\lambda_n > 0$

3 Time equation.

$$T' = -\lambda k T \Rightarrow T' = -\left(\frac{2n+1}{2}\right)^2 k T \Rightarrow T_n(t) = A_n e^{-\left(\frac{2n+1}{2}\right)^2 k t}$$

4 Superposition principle.

$$u(x,t) = \sum_{n=0}^{\infty} b_n e^{-\left(\frac{2n+1}{2}\right)^2 k t} \sin\frac{(2n+1)x}{2}$$

5 Initial conditions

$$u(x,0) = \sin x = \sum_{n=0}^{\infty} b_n \underbrace{\sin\frac{(2n+1)x}{2}}$$

$\sin x$ is not of the form $\sin\frac{(2n+1)x}{2}$

(no matching of coefficients)

(continue Exercise 13.8)

lets focus on this part $\sin x = \sum_{n=1}^{\infty} b_n \sin \frac{(2n+1)x}{a}$ $(*)$

Note that $\left\{\sin \frac{(2n+1)x}{a}\right\}_{n\geq 0}$ are orthogonal on $[0,\pi]$ (given in the question)

(give in the

That is $\int_0^{\pi} \sin \frac{(2n+1)x}{a} \sin \frac{(2m+1)x}{a}\, dx = 0$ for $n \neq m$. ✓

and $\int_0^{\pi} \sin^2 \frac{(2n+1)x}{a}\, dx = \dots = \frac{\pi}{2}$ for $n = m$ ✓

use $\sin^2 A = \frac{1}{2}(1-\cos 2A)$

We take $(*)$ and multiply $\sin \frac{(2m+1)x}{a}$ on both sides and integrate from 0 to π.

$$\Rightarrow \underbrace{\int_0^{\pi} \sin x \sin \frac{(2m+1)x}{2}\, dx}_{} = \underbrace{\sum_{n=0}^{\infty} b_n \int_0^{\pi} \sin \frac{(2n+1)x}{2} \sin \frac{(2m+1)x}{2}\, dx}_{}$$

this side can be done using $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

only b_m survives here we can solve for b_m

$$\Rightarrow \int_0^{\pi} \sin x \sin \frac{(2m+1)x}{2}\, dx = b_m \frac{\pi}{2}$$

$$\Rightarrow b_m = \frac{2}{\pi} \int_0^{\pi} \sin x \sin \frac{(2m+1)x}{2}\, dx = \dots$$

good enough but can be simplified

$$= \frac{2}{\pi}\left[\frac{(-1)^{m+1}}{2m-1} - \frac{(-1)^{m+1}}{2m+3}\right] = \frac{8}{\pi} \frac{(-1)^{m+1}}{(2m-1)(2m+3)}$$

Finally,

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} \frac{2}{\pi}\left[\frac{(-1)^{n+1}}{2n-1} - \frac{(-1)^{n+1}}{2n+3}\right] e^{-\left(\frac{2n+1}{2}\right)^2 k t} \sin \frac{(2n+1)x}{2}}$$