Math 300: Advanced Boundary Value Problems

Week 2

1.1 Heat, wave and Laplace's equations

1. The **heat equation** is given by
$$u_t = k\Delta u + F,$$
where k is the **thermal diffusivity** and F is the forcing term.

2. The **wave equation** is given by
$$u_{tt} = c^2\Delta u + F,$$
where c is the **velocity of wave propagation** and F is the forcing term.

3. **Laplace's equation**, also **potential equation** is given by
$$\Delta u = 0.$$

Poisson's equation is the nonhomogeneous version
$$\Delta u = F,$$
where F is the forcing term and

4. **Exercise 13.1**

   For each of the boundary value problems below, determine whether or not an equilibrium temperature distribution exists and find the values of β for which an equilibrium solution exists.

   (a) $u_t = u_{xx} + 1, \quad u_x(0,t) = 1, \quad u_x(a,t) = \beta.$

   (b) $u_t = u_{xx}, \quad u_x(0,t) = 1, \quad u_x(a,t) = \beta.$

   (c) $u_t = u_{xx} + x - \beta, \quad u_x(0,t) = 0, \quad u_x(a,t) = 0.$

1.2 Fourier Series

1. **Definition 2.1**. Let the function f be defined on an open interval containing the point x_0.
   (i) If $f(x_0^+) = f(x_0^-) = f(x_0)$, f is **continuous** at x_0; and **discontinuous** at x_0, otherwise.
   (ii) If f is discontinuous at x_0 and if both $f(x_0^+)$ and $f(x_0^-)$ exist, f is said to have a **discontinuity of the first kind** or a **simple discontinuity** at x_0.
   (iii) A simple discontinuity of f of the first kind at x_0 is said to be
      (a) a **removable discontinuity** if $f(x_0^+) = f(x_0^-) \neq f(x_0)$ and
      (b) a **jump discontinuity** if $f(x_0^+) \neq f(x_0^-)$, regardless of the value $f(x_0)$.
   (iv) Any discontinuity of f at x_0 not of the first kind is said to be a **discontinuity of the second kind** at x_0.

2. **Definition 2.2.** A function f is **piecewise continuous** (PWC) on an interval (a, b) if
   (i) f is continuous for $x \in (a, b)$ except possibly at a finite number of points;
   (ii) $f(x^+)$ exists for all $x \in [a, b)$;
   (iii) $f(x^-)$ exists for all $x \in (a, b]$.

   Notation. $PWC(a, b)$ denotes the set of all PWC functions on (a, b).

3. **Theorem 2.1.** [Properies of $PWC(a, b)$]
   (i) If $f, g \in PWC(a, b)$, then $\alpha f + \beta g PWC(a, b)$ for all $\alpha, \beta \in R$.
   (ii) If $f, g \in PWC(a, b)$, then $f \cdot g \in PWC(a, b)$.
   (iii) If $f \in PWC(a, b)$, then $\int_a^b |f(x)| dx$ exists.

4. **Definition 2.3.** A function f is **piecewise smooth** (PWS) on (a, b) if
   (i) $f \in PWC(a, b)$ and
   (ii) $f' \in PWC(a, b)$.

   Notation. $PWS(a, b)$ denotes the set of all PWS functions on (a, b).

5. *Example.* Consider the following functions

   (a) $f(x) = \begin{cases} e^x, & \text{for } x \neq 1 \\ 1, & \text{for } x = 1. \end{cases}$

   (b) $g(x) = \begin{cases} \sin(x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$

   (c) $h(x) = \begin{cases} x, & 0 < x \leq 1 \\ -1, & 1 < x \leq 2 \\ 1, & 2 < x < 3. \end{cases}$

6. **Definition 2.4.** Let f be a function whose domain $D(f)$ is symmetric, that is, $-x \in D(f)$ whenever $x \in D(f)$; then we say that

   (i) f is **even** if $f(-x) = f(x)$ for all $x \in D(f)$.

   (ii) f is **odd** if $f(-x) = -f(x)$ for all $x \in D(f)$.

   (iii) f is **periodic** with period p if $x + p \in D(f)$ whenever $x \in D(f)$, and $f(x+p) = f(x)$ for all $x \in D(f)$.

7. The **periodic extension** of f defined on (a, b), denoted $\bar{f}$, is defined as

$$\bar{f}(x) = f(x, np) \quad \text{for} \quad a - np < x < b - np, \quad n \in \mathbb{Z}.$$

8. **Definition 2.5**. If the function f is defined on the interval $(0, l)$:

   (i) The **odd extension** of f on $(-l, l)$, denoted f_{odd}, is defined by

$$f_{\text{odd}}(x) = \begin{cases} f(x), & \text{for} \quad 0 < x < l, \\ -f(-x), & \text{for} \quad -l < x < 0, \end{cases}$$

   and

   (ii) The **even extension** of f on $(-l, l)$, denoted f_{even}, is defined by

$$f_{\text{even}}(x) = \begin{cases} f(x), & \text{for} \quad 0 < x < l, \\ f(-x), & \text{for} \quad -l < x < 0. \end{cases}$$

9. **Definition 2.7**. Let $f, g, w \in PWC(a, b)$ with $w(x) \geq 0$. The **inner product** of f and g with **weight function** w is defined as

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)dx.$$

10. **Definition 2.8**. The **norm** of $f \in PWC(a, b)$ with weight w is $\|f\| = \sqrt{\langle f, f \rangle}$.

11. **Definition 2.9**. If $f, g, w \in PWC(a, b)$ with **weight function** $w(x) \geq 0$, f and g are said to be **orthogonal** on (a, b) relative to the weight w if $\langle f, g \rangle = 0$.

12. The set

$$\left\{1, \cos\frac{\pi x}{l}, \sin\frac{\pi x}{l}, \cos\frac{2\pi x}{l}, \sin\frac{2\pi x}{l}, \cos\frac{3\pi x}{l}, \sin\frac{3\pi x}{l}, \dots\right\}$$

   is an **orthogonal set of functions** on (a, b) with respect to the inner product above, where $l = (b-a)/2$.

13. **Exercise 11.3**

    Evaluate
$$\int_0^a \cos\frac{n\pi x}{a}\cos\frac{m\pi x}{a}dx$$
    for $n \geq 0$, $m \geq 0$. Use the trigonometric identity
$$\cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$$
    consider $A - B = 0$ and $A + B = 0$ separately.

14. **Exercise 11.4**

    Evaluate
$$\int_0^a \sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a}dx$$
    for $n \geq 0$, $m > 0$ and consider $n = m$ separately. Use the trigonometric identity
$$\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)].$$

15. **Definition 2.10**. The **Fourier series** of f on (a, b) is given by

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l},$$

where $l = (b-a)/2$ and

$$a_0 = \frac{1}{2l}\int_a^b f(x)dx, \quad a_n = \frac{1}{l}\int_a^b f(x)\cos\frac{n\pi x}{l}dx, \quad b_n = \frac{1}{l}\int_a^b f(x)\sin\frac{n\pi x}{l}dx, \quad n \geq 1,$$

are called the **Fourier coefficients** of f.

16. *Example 2.8.*

Find the Fourier series for the 2π-periodic function f defined by

$$f(x) = \begin{cases} x & 0 < x < \pi, \\ 0 & -\pi < x < 0, \end{cases}$$

and $f(x + 2\pi) = f(x)$ otherwise.

17. **Theorem 2.2**. For $f \in PWC(-l, l)$, the following are true:

    (a) If f is an odd function,
    $$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l};$$
    that is, the Fourier series for f contains only sine terms.

    (b) If f is an even function,
    $$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l};$$
    that is, the Fourier series for f contains only cosine terms.

18. Let function f defined on $(0, l)$.

    (i) The **Fourier sine series** for f is
    $$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l},$$
    where
    $$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \quad \text{for} \quad n \geq 1.$$
    Note that this defines f_{even}, the odd extension of f on $(-l, l)$.

    (ii) The **Fourier cosine series** for f is
    $$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l},$$
    where
    $$a_0 = \frac{1}{l} \int_0^l f(x) dx, \quad \text{and} \quad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx \quad \text{for} \quad n \geq 1.$$
    Note that this defines f_{even}, the even extension of f on $(-l, l)$.

19. *Example 2.10a.* Find the Fourier sine series of the function

$$f(x) = \begin{cases} 2x, & 0 < x < 1, \\ 2, & 1 < x < 2. \end{cases}$$

20. *Example 2.10b.* Find the Fourier cosine series of the function

$$f(x) = \begin{cases} 2x, & 0 < x < 1, \\ 2, & 1 < x < 2. \end{cases}$$

21. **Excercise 18.2**

    Let $f(x) = \cos^2(x), 0 < x < \pi$.

    (a) Find the Fourier sine series for f on the interval $(0, \pi)$.
    Hint: For $n \geq 1$,

    $$\int \cos^2 x \sin nx dx = -\frac{1}{2n} \cos nx + \frac{1}{4} \int [\sin(n+2)x + \sin(n-2)x] dx.$$

    (b) Find the Fourier cosine series for f on the interval $(0, \pi)$.