Math 300: Advanced Boundary Value Problems

Week 5

1.1 Separation of Variables: Nonhomogeneous equations

1. Standard homogeneous Heat and Wave equations

Heat eq. with Dirichlet BCs

$$\begin{aligned} &u_t = ku_{xx}, \quad 0 < x < a, \quad t > 0,\\ &u(0,t) = 0, \quad t > 0,\\ &u(a,t) = 0, \quad t > 0,\\ &u(x,0) = f(x), \quad 0 < x < a. \end{aligned}$$

The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{a}\right)^2 kt} \sin\frac{n\pi x}{a}.$$

Heat equation with Neumann BCs

$$\begin{aligned} &u_t = ku_{xx}, \quad 0 < x < a, \quad t > 0,\\ &u_x(0,t) = 0, \quad t > 0,\\ &u_x(a,t) = 0, \quad t > 0,\\ &u(x,0) = f(x), \quad 0 < x < a. \end{aligned}$$

The solution has the form

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{a}\right)^2 kt} \cos\frac{n\pi x}{a}.$$

Wave equation with Dirichlet BCs

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \quad 0 < x < a, \quad t > 0,\\ &u(0,t) = 0, \quad t > 0,\\ &u(a,t) = 0, \quad t > 0,\\ &u(x,0) = f(x), \quad 0 < x < a,\\ &u_t(x,0) = g(x), \quad 0 < x < a. \end{aligned}$$

The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi ct}{a} + b_n \sin\frac{n\pi ct}{a} \right) \sin\frac{n\pi x}{a}.$$

Wave equation with Neumann BCs

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \quad 0 < x < a, \quad t > 0,\\ &u_x(0,t) = 0, \quad t > 0,\\ &u_x(a,t) = 0, \quad t > 0,\\ &u(x,0) = f(x), \quad 0 < x < a,\\ &u_t(x,0) = g(x), \quad 0 < x < a. \end{aligned}$$

The solution has the form

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi ct}{a} + b_n \sin\frac{n\pi ct}{a} \right) \cos\frac{n\pi x}{a}.$$

2. Method for nonhomogeneous equations. Consider a solution of the form

$$u(x,t) = v(x) + w(x,t).$$

3. **Exercise 13.3 (modified)**

   Solve the following boundary value–initial value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial t^2} + \sin\frac{\pi x}{a},$$

$$u(0,t) = 0 \quad \text{and} \quad u(a,t) = 1,$$

$$u(x,0) = 3\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a},$$

   for $0 < x < a$, $t > 0$.

(continue Exercise 13.3 (modified))

4. **Method of Eigenfunc tion Expansions**. Consider a solution of the form

$$u(x,t) = v(x,t) + w(x,t).$$

1.2 Method of Characteristics

1. **Method of Characteristics**

   Consider the first-order linear time-dependent problem of the form

   $$\begin{aligned}&\frac{\partial u}{\partial t} + B(x,t)\frac{\partial u}{\partial x} = C(x,t,u), \quad -\infty < x < \infty, \quad t > 0,\\ &u(x,0) = f(x).\end{aligned}$$

   The method of characteristic consists on solving the **characteristc equations**

   $$\begin{aligned}\frac{dx}{dt} &= B(x,t),\\ \frac{du}{dt} &= C(x,t,u),\end{aligned}$$

   and then using the initial condition.

2. *Example 10.2*

   Solve the following PDE for $u(x,t)$ on $-\infty < x < \infty$

   $$\begin{aligned}\frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial x} + \beta u = 0,\\ u(x,0) = f(x),\end{aligned}$$

   using the method of characteristics.

3. **Exercise 17.5**

   Solve the first-order equation

$$\frac{\partial u}{\partial t} + 3x\frac{\partial u}{\partial x} = 2t, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x, 0) = \log(1 + x^2).$$

4. **Exercise 17.6**

   Using the method of characteristics, solve

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = e^{2x}, \quad -\infty < x < \infty, \quad t > 0,$$

$$w(x,0) = f(x).$$

5. **Exercise 17.7**

   Using the method of characteristics, solve

$$\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 1, \quad -\infty < x < \infty, \quad t > 0,$$

$$w(x,0) = f(x).$$

(extra page)