# Math 300: Advanced Boundary Value Problems

# Week 10

## 1.1 Bessel functions

1. **Bessel's equation** of order $n$ is given by

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (x^2 - n^2)u = 0.$$

2. The general solution of Bessel's equations of order $n$ are given by

$$u(x) = A\,J_n(x) + B\,Y_n(x),$$

for arbitrary constants $A$ and $B$, where

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!}\left(\frac{x}{2}\right)^{n+2k}, \quad n \geq 0,$$

is the **Bessel function of the first kind of order** $n$ and $Y_n(x)$ is the **Bessel function of the second kind of order** $n$.

3. **Theorem 6.4.** (Orthogonality) For a fixed integer $m \geq 0$,

$$\langle J_m(z_{mn}x), J_m(z_{mk}x)\rangle = \int_0^1 xJ_m(z_{mn}x)J_m(z_{mk}x)\,dx = 0, \quad \int_0^1 xJ_m(z_{mn}x)^2\,dx = \frac{1}{2}J_{m+1}(z_{mn})^2, = \|J_m(z_{mn}x)\|^2$$

where $z_{mn}$ is a zero of $J_m(x)$ for $n \geq 1$.

4. **Theorem 6.5.** (Fourier-Bessel Expansion Theorem) If $f$ and $f'$ are piecewise continuous on the interval $0 \leq x \leq 1$, then for $0 < x_0 < 1$, the **Fourier-Bessel series** expansion

$$f(x) = \sum_{n=1}^{\infty} a_n J_m(z_{mn}x_0), \quad \text{where} \quad a_n = \frac{2}{J_{m+1}(z_{mn})^2}\int_0^1 f(x)J_m(z_{mn}x)x\,dx,$$

converges to $[f(x_0^+)+f(x_0^-)]/2$. At $x_0 = 1$, the series converges to 0, since every $J_m(z_{mn}) = 0$. At $x_0 = 0$, the series converges to 0 if $m \geq 1$, and to $f(0^+)$ if $m = 0$.

## 1.2 Polar coordinates

1. *Vibrating circular membrane.* Consider the following wave equation in a disk

Similar idea to: Week 9 P 2-4 Exercise 14.13

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right), \quad 0 < r < a, \quad -\pi < \theta < \pi, \quad t > 0,$$

$$\left.\begin{aligned} u(r,-\pi,t) &= u(r,\pi,t), \\ \frac{\partial u}{\partial \theta}(r,-\pi,t) &= \frac{\partial u}{\partial \theta}(r,\pi,t), \end{aligned}\right\} \text{periodicity cond.}$$

$$u(a,\theta,t) = 0, \quad \longleftarrow \text{boundary cond.}$$

$$|u(r,\theta,t)| < \infty, \quad \text{as} \quad r \to 0^+, \quad \longleftarrow \text{boundedness cond.}$$

$$\left.\begin{aligned} u(r,\theta,0) &= f(r,\theta), \\ \frac{\partial u}{\partial t}(r,\theta,0) &= g(r,\theta). \end{aligned}\right\} \text{initial conditions.}$$

- Separation of Variables $u(r,\theta,t) = R(r)\phi(\theta)T(t)$ leads to:
- Angular problem

$$\phi'' + \mu\phi = 0, \quad \phi(-\pi) = \phi(\pi), \quad \phi'(-\pi) = \phi'(\pi)$$

$$\Longrightarrow \mu_m = m^2, \quad \phi_m(\theta) = A_m \cos m\theta + B_m \sin m\theta \quad m \geq 0$$

- Radial problem

$$r^2 R'' + rR' + (\lambda r^2 - \mu)R = 0, \quad R(a) = 0, \quad |R(r)| < \infty \text{ as } r \to 0^+$$

$$\Longrightarrow \lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2, \quad R_{mn}(r) = J_m\left(\frac{z_{mn} r}{a}\right) \quad n \geq 1$$

- Temporal problem

$$T'' + \lambda c^2 T = 0 \Longrightarrow T_{mn}(t) = C_{mn}\cos\frac{z_{mn}ct}{a} + D_{mn}\sin\frac{z_{mn}ct}{a}$$

- Superposition principle.

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{z_{mn}r}{a}\right)\left[a_{mn}\cos m\theta\cos\frac{z_{mn}ct}{a} + b_{mn}\cos m\theta\sin\frac{z_{mn}ct}{a} + c_{mn}\sin m\theta\cos\frac{z_{mn}ct}{a} + d_{mn}\sin m\theta\sin\frac{z_{mn}ct}{a}\right]$$

- Initial conditions: Use Fourier series then Fourier-Bessel series.

the variable $\theta$ is gone because of the symmetry

2. **Exercise 13.14.**

Solve the two-dimensional heat equation inside a disk with circularly symmetric time-independent sources, boundary conditions, and initial conditions:

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + Q(r), \quad 0 < r < a, \quad t > 0,$$

we'll use the method of eigenfunction expansions to deal with the nonhomo term $Q(r)$

with

$$u(r,0) = f(r), \quad u(a,t) = T.$$

we'll use st.st part ($v(x)$) to deal with the nonhomo BC

We need to complete this problem with boudedness cond.

$|u(r,t)| < \infty$ as $r \to 0^+$

Note: the periodicity conditions are not needed because of the symmetry.

① "Transient" and "steady state" solutions

$u(r,t) = v(r) + w(r,t)$, $u_t = w_t$, $u_r = v' + w_r$

$$\frac{\partial w}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r v' + r\frac{\partial w}{\partial r}\right) + Q(r)$$

$$= \underbrace{\frac{k}{r}(r v')'}_{=0} + \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + Q(r)$$

this time, we put $Q(r)$ into $w(r,\theta,t)$

BC. $u(a,t) = v(a) + \overset{=0}{w(a,t)} = T \Rightarrow v(a) = T$

Boundedness: $|v(r)| < \infty$, $|w(r,t)| < \infty$, as $r \to 0^+$

IC: $u(r,0) = v(r) + w(r,0) = f(r) \Rightarrow w(r,0) = f(r) - v(r)$

"steady state" problem

Ⅰ $\frac{k}{r}(r v')' = 0$

$v(a) = T$

$|v(r)| < \infty$ $r \to 0^+$

"transient" problem

Ⅱ $\frac{\partial w}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + Q(r)$

$w(a,t) = 0$

$|w(r,t)| < \infty$ as $r \to 0^+$

$w(r,0) = f(r) - v(r)$

*(continue Exercise 13.14)*

I

2 $\frac{k}{r}(rv')' = 0$, $v(a) = T$, $|v(r)| < \infty$ as $r \to 0^+$

$rv' = c_1 \Rightarrow v(r) = c_1 \ln r + c_2$

bound.c. $\lim_{r\to 0^+} v(r) < \infty \Leftrightarrow c_1 = 0 \Rightarrow v(r) = T$

3 II without $Q(r)$ and IC (we want to find eigenfuntions)

$\frac{\partial w}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)$, $w(a,t) = 0$, $|w(r,t)| < \infty$ as $r \to 0^+$

- Sep. var.: $w(r,t) = R(r)T(t) \Rightarrow w_t = RT'$ $w_r = R'T$

Using the equation

$RT' = \frac{k}{r}(rR')'T \xrightarrow[\div kRT]{} \frac{T'}{kT} = \frac{(rR')'}{rR} = -\lambda$

Two problems

- Time equation $T' = -\lambda k T \Rightarrow T(t) = Ce^{-\lambda k t}$
- SLP $(rR')' + \lambda r R = 0$, $R(a) = 0$, $|R(r)| < \infty$ as $r \to 0^+$

$p(r) = r$ $q(r) = 0$ $\sigma(r) = r \Rightarrow$ singular SLP

however, with the boundedness condition $|R(r)| < \infty$ we can treat it as a regular SLP

Let $\lambda_1 < \dots < \lambda_n < \dots$, $R_n(r) = \phi_n(r)$ $n \geq 1$

Is $\lambda_1 \geq 0$? Using Rayleigh quotient

$$\lambda = \frac{\overset{\text{(boundary) } 0}{\cancel{-a\phi(a)\phi'(a)}} + \overset{\text{(boundedness) } 0}{\cancel{\lim_{r\to 0^+} r\phi(r)\phi'(r)}} + \int_0^a r\phi'(r)^2\,dr}{\int_0^a r\phi(r)^2\,dr} \geq 0$$

$(r\phi'^2 \geq 0)$

Let's expand the equation and
*(continue Exercise 13.14)*

mult. by $r$: $r^2R'' + rR' + \lambda r^2 R = 0$ ← This is almos a Bessel eq. the standard trick is to use $x = \sqrt{\lambda} r$

this is possible since $\lambda \geq 0$

use $x = \sqrt{\lambda} r$: $x^2 R'' + xR' + x^2 R = 0$ ← Bessel eq of order $n=0$.

$$\Longrightarrow R(r) = A J_0(\sqrt{\lambda} r) + B Y_0(\sqrt{\lambda} r)$$

Use Boundedness: $|R(r)| < \infty$ as $r \to 0^+ \Rightarrow B = 0$ ($Y_0$ has a pole at $r=0$)

Use BC: $R(a) = A J_0(\sqrt{\lambda} a) = 0 \Rightarrow \sqrt{\lambda} a = z_{0n}$

$$\Longrightarrow \lambda_n = \left(\frac{z_{0n}}{a}\right)^2, \quad \boxed{R_n(r) = A_n J_0\left(\frac{z_{0n} r}{a}\right)} \quad n \geq 1$$

Now that we know what eigenfunctions, we can go back to the nonhomogeneous problem for $w$ and use the method of eigenfunction expansions

4 II again but poll

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{k}{r}\dfrac{\partial}{\partial r}\left(r \dfrac{\partial w}{\partial r}\right) + Q(r) \\ w(a,t) = 0 \qquad |w(x,t)| < \infty \qquad r \to 0^+ \\ w(r,0) = f(r) - T \end{cases}$$

Method of eigenfunction expansions: Assume

$$w(r,t) = \sum_{n=1}^{\infty} a_n(t) J_0\left(\frac{z_{0n} r}{a}\right)$$

unknown $a_n(t)$ TBD.

Expand $Q(r)$ and $f(r) - T$ as well

$$Q(r) = \sum_{n=1}^{\infty} q_n J_0\left(\frac{z_{0n} r}{a}\right), \quad q_n = \frac{\int_0^a Q(r) J_0\left(\frac{z_{0n} r}{a}\right) \overbrace{r\,dr}^{\sigma(r)}}{\int_0^a J_0\left(\frac{z_{0n} r}{a}\right)^2 r\,dr}$$

$$f(r) - T = \sum_{n=1}^{\infty} f_n J_0\left(\frac{z_{0n} r}{a}\right), \quad f_n = \frac{\int_0^a (f(r) - T) J_0\left(\frac{z_{0n} r}{a}\right) r\,dr}{\int_0^a J_0\left(\frac{z_{0n} r}{a}\right)^2 r\,dr}$$

Fourier-Bessel series

- Maybe thinking that $a_n(t) = T_n(t)$ helps.
- Note how power series and eigenfuntion expansions are the same idea

$$w(r,t) = \sum a_n(t) J_0\left(\frac{z_{0n} r}{a}\right)$$

*(continue Exercise 13.14)*

this can be shown using
- $J_0'(x) = J_{-1}(x)$
- $J_{-1}'(x) + \frac{1}{x} J_{-1}(x) = -J_0(x)$

Use eq

$$\frac{\partial w}{\partial t} = \frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + Q(r)$$

$$\frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \dots = \sum_{n=1}^{\infty} -a_n(t)\left(\frac{z_{0n}}{a}\right)^2 \kappa J_0\left(\frac{z_{0n} r}{a}\right) + Q(r)$$

$$\underbrace{\sum_{n=1}^{\infty} a_n'(t) J_0\left(\frac{z_{0n} r}{a}\right)}_{\frac{\partial w}{\partial t}} = \underbrace{\sum_{n=1}^{\infty} (-1) a_n(t)\left(\frac{z_{0n}}{a}\right)^2 \kappa J_0\left(\frac{z_{0n} r}{a}\right)}_{\frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)} + \sum_{n=1}^{\infty} q_r J_0\left(\frac{z_{0n} r}{a}\right)$$

matching coefficients $\Rightarrow \dfrac{da_n}{dt} + \left(\dfrac{z_{0n}}{a}\right)^2 \kappa a_n = q_r, \quad a_n(0) = f_n$

initial condition $\quad w(r,0) = \sum_{n=1}^{\infty} a_n(0) J_0\left(\frac{z_{0n} r}{a}\right) = f(r) - T = \sum_{n=1}^{\infty} f_n J_0\left(\frac{z_{0n} r}{a}\right)$

This is a 1st order linear ODE, use integrating factor

$$\mu(t) = e^{\lambda_n \kappa t} \Rightarrow a_n(t) = e^{-\lambda_n \kappa t}\left[\int e^{\lambda_n \kappa t} q_r \, dr + D\right]$$

$$\Rightarrow a_n(t) = \frac{1}{\lambda_n \kappa} q_n + D e^{-\lambda_n \kappa t}, \quad a_n(0) = \frac{1}{\lambda_n \kappa} q_n + D = f_n$$

Use I.C. to determine constant D

$$\Rightarrow a_n(t) = \frac{q_n}{\kappa}\left(\frac{a}{z_{0n}}\right)^2 + \left[f_n - \frac{q_n}{\kappa}\left(\frac{a}{z_{0n}}\right)^2\right] e^{-\left(\frac{z_{0n}}{a}\right)^2 \kappa t}$$

Thus,

$$w(r,t) = \sum_{n=1}^{\infty}\left[\frac{q_n}{\kappa}\left(\frac{a}{z_{0n}}\right)^2 + \left(f_n - \frac{q_n}{\kappa}\left(\frac{a}{z_{0n}}\right)^2\right)\right] e^{-\left(\frac{z_{0n}}{a}\right)^2 \kappa t} J_0\left(\frac{z_{0n} r}{a}\right)$$

5 Finally,

$$\boxed{\begin{aligned} u(r,t) &= v(r) + w(r,t) \\ &= T + \sum_{n=1}^{\infty} \underbrace{\left[\frac{q_n}{\kappa}\left(\frac{a}{z_{0n}}\right)^2 + \left(f_n - \frac{q_n}{\kappa}\left(\frac{a}{z_{0n}}\right)^2\right)\right] e^{-\left(\frac{z_{0n}}{a}\right)^2 \kappa t}}_{a_n(t) = T_n(t)} \underbrace{J_0\left(\frac{z_{0n} r}{a}\right)}_{R_n(r)} \end{aligned}}$$

$q_n$ and $f_n$ are given in the previous page.