Math 300: Advanced Boundary Value Problems

Week 10

1.1 Bessel functions

1. **Bessel's equation** of order n is given by

$$x^2 \frac{d^2 u}{dx^2} + x\frac{du}{dx} + (x^2 - n^2)u = 0.$$

2. The general solution of Bessel's equations of order n are given by

$$u(x) = A\, J_n(x) + B\, Y_n(x),$$

for arbitrary constants A and B, where

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{x}{2}\right)^{n+2k}, \quad n \geq 0,$$

is the **Bessel function of the first kind of order** n and $Y_n(x)$ is the **Bessel function of the second kind of order** n.

3. **Theorem 6.4.** (Orthogonality) For a fixed integer $m \geq 0$,

$$\int_0^1 x J_m(z_{mn}x) J_m(z_{mk}x)\, dx = 0, \quad \int_0^1 x J_m(z_{mn}x)^2\, dx = \frac{1}{2} J_{m+1}(z_{mn})^2,$$

where z_{mn} is a zero of $J_m(x)$ for $n \geq 1$.

4. **Theorem 6.5.** (Fourier-Bessel Expansion Theorem) If f and f' are piecewise continuous on the interval $0 \leq x \leq 1$, then for $0 < x_0 < 1$, the **Fourier-Bessel series** expansion

$$f(x) = \sum_{n=1}^{\infty} a_n J_m(z_{mn}x_0), \quad \text{where} \quad a_n = \frac{2}{J_{m+1}(z_{mn})^2} \int_0^1 f(x) J_m(z_{mn}x) x\, dx,$$

converges to $[f(x_0^+)+f(x_0^-)]/2$. At $x_0 = 1$, the series converges to 0, since every $J_m(z_{mn}) = 0$. At $x_0 = 0$, the series converges to 0 if $m \geq 1$, and to $f(0^+)$ if $m = 0$.

1.2 Polar coordinates

1. *Vibrating circular membrane.* Consider the following wave equation in a disk

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right), \quad 0 < r < a, \quad -\pi < \theta < \pi, \quad t > 0,\\
&u(r, -\pi, t) = u(r, \pi, t),\\
&\frac{\partial u}{\partial \theta}(r, -\pi, t) = \frac{\partial u}{\partial \theta}(r, \pi, t),\\
&u(a, \theta, t) = 0,\\
&|u(r, \theta, t)| < \infty, \quad \text{as} \quad r \to 0^+,\\
&u(r, \theta, 0) = f(r, \theta),\\
&\frac{\partial u}{\partial t}(r, \theta, 0) = g(r, \theta).
\end{aligned}
$$

2. **Exercise 13.14.**

   Solve the two-dimensional heat equation inside a disk with circularly symmetric time-independent sources, boundary conditions, and initial conditions:

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial t}\left(r\frac{\partial u}{\partial r}\right) + Q(r), \quad 0 < r < a, \quad t > 0,$$

   with

$$u(r,0) = f(r), \quad u(a,t) = T.$$

(continue Exercise 13.14)

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3. **Exercise 14.18.**

   Solve the wave equation for a "pie-shaped" membrane of radius a and angle $\pi/3$ $(= 60°)$:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.$$

   Show that the eigenvalues are all positive. Determine the natural frequencies of oscillation if the boundary conditions are

$$u(r, 0, t) = 0, \quad u\left(r, \frac{\pi}{3}, t\right) = 0, \quad \frac{\partial u}{\partial r}(a, \theta, t) = 0.$$

(continue Exercise 14.18)

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