Math 300: Advanced Boundary Value Problems

Week 13

1.1 Summary

1. **Definition 2.10**. The **Fourier series** of f on (a,b) is given by

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l},$$

where $l = (b-a)/2$ and

$$a_0 = \frac{1}{2l}\int_a^b f(x)dx, \quad a_n = \frac{1}{l}\int_a^b f(x)\cos\frac{n\pi x}{l}dx, \quad b_n = \frac{1}{l}\int_a^b f(x)\sin\frac{n\pi x}{l}dx, \quad n \geq 1,$$

are called the **Fourier coefficients** of f.

2. **Method of Characteristics**

Consider the first-order linear time-dependent problem of the form

$$\begin{aligned} &\frac{\partial u}{\partial t} + B(x,t)\frac{\partial u}{\partial x} = C(x,t,u), \quad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = f(x). \end{aligned}$$

The method of characteristic consists on solving the **characteristic equations**

$$\begin{aligned} \frac{dx}{dt} &= B(x,t), \\ \frac{du}{dt} &= C(x,t,u), \end{aligned}$$

and then using the initial condition.

3. Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

d'Alembert's solution is given by

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\mu)\,d\mu.$$

4. Consider the one-dimensional wave equation

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0, \\
&u(0,t) = 0, \quad u(l,t) = 0, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).
\end{aligned}$$

d'Alembert's solution is given by

$$u(x,t) = \frac{1}{2}[\bar{f}_{\text{odd}}(x+ct) + \bar{f}_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}_{\text{odd}}(\mu)\, d\mu,$$

where $\bar{f}_{\text{odd}}$ and $\bar{g}_{\text{odd}}$ are the $2l$-periodic extension of f and g, respectively.

5. Given the regular Sturm-Liouville problem,

$$\begin{aligned}
&(p(x)\phi')' + [q(x) + \lambda\sigma(x)]\phi = 0, \quad a < x < b, \\
&\alpha_1\phi(a) + \beta_1\phi'(a) = 0, \\
&\alpha_2\phi(b) + \beta_2\phi'(b) = 0,
\end{aligned}$$

with eigenvalues λ_n and corresponding eigenfunctions ϕ_n.

6. **Theorem 4.6.** *(Dirichlet's Theorem)*

If f is piecewise smooth on $[a,b]$, the **generalized Fourier series**,

$$f(x) \sim \sum_{n=1}^{\infty} c_n\phi_n(x), \quad \text{where,} \quad c_n = \frac{\langle f, \phi_n\rangle}{\|\phi_n\|^2} = \frac{1}{\|\phi_n\|^2}\int_a^b f(x)\phi_n(x)\sigma(x)\, dx,$$

for $n \geq 1$, converges pointwise to $[f(x^+) + f(x^-)]/2$ for each $x \in (a,b)$.

7. **Theorem 4.7.**

λ_n can be calculated from the **Rayleigh quotient**:

$$\lambda_n = \frac{-p(x)\phi_n(x)\phi_n'(x)\Big|_a^b + \displaystyle\int_a^b \left(p(x)\phi_n'(x)^2 - q(x)\phi_n(x)^2\right) dx}{\displaystyle\int_a^b \phi_n(x)^2\sigma(x)\, dx}.$$

8. Summary of Sturm-Liouville problems.

Model Type	**S-L Problem**	**Spectrum**	**Eigenfunctions**
Homogeneous Dirichlet B.C.	$\phi''(x) + \lambda\phi(x) = 0$ $\phi(0) = \phi(l) = 0$	$\lambda_n = \left(\frac{n\pi}{l}\right)^2$ $n = 1, 2, \cdots$	$\phi_n = \sin\frac{n\pi x}{l}$ $n = 1, 2, \cdots$
Homogeneous Neumann B.C.	$\phi''(x) + \lambda\phi(x) = 0$ $\phi'(0) = \phi'(l) = 0$	$\lambda_n = \left(\frac{n\pi}{l}\right)^2$ $n = 0, 1, \cdots$	$\phi_n = \cos\frac{n\pi x}{l}$ $n = 0, 1, \cdots$
Mixed Type I	$\phi''(x) + \lambda\phi(x) = 0$ $\phi(0) = \phi'(l) = 0$	$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2$ $n = 1, 2, \cdots$	$\phi_n = \sin\frac{(2n-1)\pi x}{2l}$ $n = 1, 2, \cdots$
Mixed Type II	$\phi''(x) + \lambda\phi(x) = 0$ $\phi'(0) = \phi(l) = 0$	$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2$ $n = 1, 2, \cdots$	$\phi_n = \cos\frac{(2n-1)\pi x}{2l}$ $n = 1, 2, \cdots$
Periodicity conditions	$\phi''(\theta) + \lambda\phi(\theta) = 0$ $\phi(-\pi) = \phi(\pi)$ $\phi'(-\pi) = \phi'(\pi)$	$\lambda_n = n^2$ $n = 0, 1, \cdots$	$\phi_n = a_n\cos n\theta + b_n\sin n\theta$ $n = 0, 1, \cdots$
Bessel equation	$x^2u'' + xu' + (\lambda - m^2)u = 0$ $u(a) = 0$ and $\lvert u(x)\rvert$ bounded as $x \to 0^+$	$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$ $n = 1, 2, \cdots$	$u_{mn} = a_{mn}J_m\left(\frac{z_{mn}x}{a}\right)$ $n = 1, 2, \cdots$
Legendre equation	$(1 - x^2)v'' - 2xv' + \lambda v = 0$ $\lvert v(x)\rvert$ and $\lvert v'(x)\rvert$ bounded as $x \to \pm 1$	$\lambda_n = n(n+1)$ $n = 0, 1, \cdots$	$v_n = a_nP_n(x)$ $n = 0, 1, \cdots$

9. The **Laplacian in polar coordinates** is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

10. The **Laplacian in spherical coordinates** is

$$\begin{aligned}\nabla^2 u &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\\ &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial\theta^2} + \cot\theta\frac{\partial u}{\partial\theta} + \csc^2\theta\frac{\partial^2 u}{\partial\phi^2}\right).\end{aligned}$$