Math 300: Advanced Boundary Value Problems

Week 6

1.1 One-dimensional Wave Equation

1. Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

d'Alembert's solution is given by

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\mu)\, d\mu.$$

$+\frac{1}{2c}[G(x+ct) - G(x-ct)]$

2. **Exercise 17.12**

The displacement $u = u(x,t)$ of an infinitely long string is governed by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0.$$

$c^2 = 4$

$\Rightarrow c = 2$

At time $t = 0$ an initial signal is given of the form

$$u(x,0) = f(x) = \begin{cases} x, & 0 < x < 1, \\ -x+2, & 1 < x < 2, \\ 0, & \text{othewise}, \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \quad -\infty < x < \infty.$$

$g = 0$

a) Solve this problem.

b) Sketch the solution for times t_1, t_2, t_3, t_4, t_5 , with

$u(x,t)$ versus x
t is fixed

$$t_1 = 0, \quad 0 < t_2 < 1/4, \quad t_3 = 1/4, \quad 1/4 < t_4 < 1/2, \quad t_5 = 1/2.$$

c) At what time does the signal reach the point $x = 11$?

(continue Exercise 17.12)

a)

$$u(x,t) = \frac{1}{2}\left[f(x+2t) + f(x-2t)\right] + \cancel{\frac{1}{4}\int_{x-2t}^{x+2t} g(\mu)\,d\mu}^{\,0}$$

because $g(x)=0$

where

$$f(x+2t) = \begin{cases} x+2t-2, & 0 < x+2t < 1 \\ -x-2t+2, & 1 < x+2t < 2 \\ 0, & \text{otherwise} \end{cases}$$

$$f(x-2t) = \begin{cases} x-2t, & 0 < x-2t < 1 \\ -x+2t+2 & 1 < x-2t < 2 \\ 0 & \text{otherwise} \end{cases}$$

b)

① , ④ , ⑦ $u = 0$

② $0 < x+2t < 1$ & $x-2t < 0$

$u(x,t) = \frac{1}{2}\left[x+2t+0\right] = \frac{1}{2}x + t$

③ $1 < x+2t < 2$ & $x-2t < 0$

$u(x,t) = \frac{1}{2}(-x-2t+2)$

⑤ $2 < x+2t$ & $0 < x-2t < 1$

$u(x,t) = \frac{1}{2}(x-2t)$

⑥ $2 < x+2t$ & $1 < x-2t < 2$

$u(x,t) = \frac{1}{2}(-x+2t+2)$

⑧ $1 < x+2t < 2$ & $1 < x-2t < 2$

$u(x,t) = \frac{1}{2}\left[-x-\cancel{2t}+2-x+\cancel{2t}+2\right]$

$= -x+2$

⑨ $0 < x+2t < 1$ & $0\ x-2t < 1$

$u(x,t) = x$

⑩ $1 < x+2t < 2$ & $0 < x-2t < 1$

$u(x,t) = -2t+1$

(continue Exercise 17.12)

snapshots for various $t \geq 0$

travelling sol's

c) $x - 2t = 2$ when $x = 11$

$11 - 2t = 2 \Rightarrow t = \frac{9}{2}$

3. Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0,$$
$$u(0,t) = 0, \quad u(l,t) = 0, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

d'Alembert's solution is given by

$\frac{1}{2c}\left[G(x+ct) - G(x-ct)\right]$

$$u(x,t) = \frac{1}{2}[\bar{f}_{\text{odd}}(x+ct) + \bar{f}_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}_{\text{odd}}(\mu)\, d\mu,$$

G is antider. of $\bar{g}_{odd}$

where $\bar{f}_{\text{odd}}$ and $\bar{g}_{\text{odd}}$ are the $2l$-periodic extension of f and g, respectively.

4. **Exercise 14.8**

Use d'Alembert's solution to solve the boundary value–initial value problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0$$
$$u(0,t) = 0,$$
$$u(1,t) = 0,$$
$$u(x,0) = 0,$$
$$\frac{\partial u}{\partial t}(x,0) = 1.$$

} Dirichlet

$0 < x < 1$ $\quad x \notin (0,1)$ $\quad f(x)$? $\quad \bar{f}_{odd}(x)$ ✓ !!

$l = 1 \quad c = 1$

IC's

$f(x) = 0 \Rightarrow \bar{f}_{odd}(x) = 0$

$$g(x) = 1 \Rightarrow \bar{g}_{odd}(x) = \begin{cases} 1 & 0 < x < 1 \\ -1 & -1 < x < 0 \end{cases}$$

$\bar{g}_{odd}(x+2) = \bar{g}_{odd}(x)$

$$G(x) = \begin{cases} x, & 0 < x < 1 \\ -x, & -1 < x < 0 \end{cases} \qquad G(x+2) = G(x),$$

$$u(x,t) = \frac{1}{2}\left[G(x+t) - G(x-t)\right]$$

closed

5. **Exercise 17.8**

Consider

1st order nonlinear.

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x).$$

Show that characteristics are straight lines.

1st ch. eq. $\frac{dx}{dt} = 2u = \underline{2u(x(t),t)}$?

2nd ch.eq. $\frac{du}{dt} = 0$ $\quad u(x(t),t) = C.$

IC's $u(x(0),0) = \underline{f(x(0))} = C.$

$\Rightarrow u(x(t),t) = f(x(0))$

back to 1st ch. eq

$\frac{dx}{dt} = 2f(x(0)) \Rightarrow \int_0^t dx = \int_0^t 2f(x(0))ds$

$\Rightarrow x(t) - x(0) = 2f(x(0))t$

$\Rightarrow x(t) = 2f(x(0))t + x(0)$

Let $x_0 = x(0) \Rightarrow x = 2f(x_0)t + x_0$ equation of charac.

$$\Rightarrow \underline{u(x,t) = f(x_0)}$$
$$x = 2f(x_0)t + x_0$$

Characteristic are staight lines which intersect the x-axis at x_0 with slope that depends on $f(x_0)$.

6. **Exercise 17.9**

Consider

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0.$$

sketch of $u(x,t)$

with

$$u(x,0) = f(x) = \begin{cases} 1, & x < 0, \\ 1 + x/a, & 0 < x < a, \\ 2, & x > a. \end{cases}$$

a) Determine the equations for the characteristics. Sketch the characteristics.

b) Determine the solution $u(x,t)$. Sketch $u(x,t)$ for t fixed.

From the previous example

$$u(x,t) = f(x_0) \quad ; \quad x(t) = 2f(x_0)t + x_0$$

a)

$x = 2t + x_0$ $\qquad$ $x = 2(1 + \frac{x_0}{a})t + x_0$ $\qquad$ $x = 4t + x_0$

$x_0 = x - 2t$ $\qquad$ $x_0 = x - 4t$

write x_0 as a function of x and t.

b) $x = 2(1 + \frac{x_0}{a})t + x_0 \Rightarrow ax = 2at + 2x_0 t + ax_0$

$\Rightarrow 2tx_0 + ax_0 = ax - 2at \Rightarrow x_0 = \dfrac{a(x-2t)}{a+2t}$

$$f(x_0) = 1 + \frac{1}{\cancel{a}}\frac{\cancel{a}(x-2t)}{a+2t} = \frac{a + \cancel{2t} + x - \cancel{2t}}{a+2t} = \frac{a+x}{a+2t}$$

$$f(x_0) = u(x,t) = \begin{cases} 1, & x - 2t < 0 \\ \dfrac{a+x}{a+2t}, & 0 < \dfrac{a(x-2t)}{a+2t} < a \\ 2, & x - 4t > a \end{cases}$$

$$\boxed{\begin{array}{l} x < 2t \\ 2t < x < a+4t \\ a+4t < x \end{array}}$$

simplified

7. **Exercise 14.9**

Use d'Alembert's solution to solve the boundary value–initial value problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0$$
$$u(0,t) = 0,$$
$$u(1,t) = 0,$$
$$u(x,0) = 0,$$
$$\frac{\partial u}{\partial t}(x,0) = \sin \pi x.$$

$$u(x,t) = \frac{1}{2}\left[\bar{f}_{odd}(x+ct) + \bar{f}_{odd}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}_{odd}(\mu)\,d\mu$$

$l = 1 \qquad c = 1$

$f(x) = 0 \Rightarrow \bar{f}_{odd}(x) = 0$

$0 < x < 1$

$g(x) = \sin \pi x \Rightarrow \bar{g}_{odd}(x) = \sin \pi x$

$0 < x < 1$

$$G(x) = \int \bar{g}_{odd}(x)\,dx = \int \sin x\,dx = -\frac{1}{\pi}\cos \pi x + C$$

$$u(x,t) = \frac{1}{2}\left[G(x+t) - G(x-t)\right]$$
$$= \frac{1}{2}\left[-\frac{1}{\pi}\cos(\pi(x+t)) + \frac{1}{\pi}\cos(\pi(x-t))\right]$$

8. **Exercise 14.10**

Use d'Alembert's solution to solve the boundary value–initial value problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = x^2,$$
$$\frac{\partial u}{\partial t}(x,0) = 3.$$

$c = 5$

$$u(x,t) = \frac{1}{2}\left[(x+5t)^2 + (x-5t)^2\right] + \frac{1}{10}\int_{x-5t}^{x+5t} 3 \, d\mu$$

$$= \frac{1}{2}\left[x^2 + \cancel{10xt} + 25t^2 + x^2 - \cancel{10xt} + 25t^2\right] + \frac{3}{10}\left[\mu \Big|_{x-5t}^{x+5t}\right]$$

$$= \frac{1}{2}\left[2x^2 + 50t^2\right] + \frac{3}{10}\left[\cancel{x} + 5t - \cancel{x} + 5t\right]$$

$$\Rightarrow \boxed{u(x,t) = x^2 + 25t^2 + 3t}$$