Math 300: Advanced Boundary Value Problems

Week 8

1.1 Sturm-Liouville Theory

1. **Theorem 4.5.**

   Given the regular Sturm-Liouville problem,

$$
\begin{aligned}
&(p(x)\phi')' + [q(x) + \lambda\sigma(x)]\phi = 0, \quad a < x < b,\\
&\alpha_1\phi(a) + \beta_1\phi'(a) = 0,\\
&\alpha_2\phi(b) + \beta_2\phi'(b) = 0,
\end{aligned}
$$

   with eigenvalues λ_n and corresponding eigenfunctions ϕ_n.

   (a) The regular Sturm-Liouville problem has an infinite spectrum

$$S = \{\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\}$$

   and $\lim_{n\to\infty} \lambda_n = +\infty$.

   (b) If $\alpha_1\beta_1 \le 0$ and $\alpha_2\beta_2 \ge 0$, the spectrum is bounded below and the eigenvalues may be ordered as

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots .$$

   Moreover, if $q(x) \le 0$ for $a \le x \le b$, then $\lambda_n \ge 0$ for all $n \ge 1$.

   (c) If the eigenvalues are ordered as $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, the eigenfunction corresponding to λ_n has exactly $(n-1)$ zeros in the interval $a < x < b$.

2. **Theorem 4.6.** *(Dirichlet's Theorem)*

   If f is piecewise smooth on $[a, b]$, the **generalized Fourier series**,

$$f(x) \sim \sum_{n=1}^{\infty} c_n\phi_n(x), \quad \text{where,} \quad c_n = \frac{\langle f, \phi_n\rangle}{\|\phi_n\|^2} = \frac{1}{\|\phi_n\|^2}\int_a^b f(x)\phi_n(x)\sigma(x)\,dx,$$

   for $n \ge 1$, converges pointwise to $[f(x^+) + f(x^-)]/2$ for each $x \in (a, b)$.

3. *Example 4.6.* Consider the regular Sturm-Liouville problem

$$\begin{aligned}
&\phi'' + \lambda\phi = 0, \quad 0 < x < 1, \\
&\phi(0) = 0, \\
&2\phi(1) - \phi'(1) = 0.
\end{aligned}$$

(continue Example 4.6)

4. *Example 4.7.* Consider the regular Sturm-Liouville problem

$$\begin{aligned}&\phi'' + \lambda^2 \phi = 0, \quad 0 < x < \pi,\\ &\phi'(0) = 0,\\ &\phi(\pi) = 0.\end{aligned}$$

   (a) Find the eigenvalues λ_n^2 and the corresponding eigenfunctions ϕ_n for this problem.

   (b) Show directly, by integration, that eigenfunctions corresponding to distinct eigenvalues are orthogonal.

   (c) Given the function $f(x) = \pi^2 - x^2/2, 0 < x < \pi$, find the eigenfunction expansion of f.

   (d) Show that

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - + \cdots .$$

(continue Example 4.7)

5. **Theorem 4.7.**

   If (ϕ_n, λ_n) is an eigenpair for the regular Sturm-Liouville problem

$$
\begin{aligned}
&(p(x)\phi')' + [q(x) + \lambda\sigma(x)]\phi = 0, \quad a < x < b, \\
&\alpha_1\phi(a) + \beta_1\phi'(a) = 0, \\
&\alpha_2\phi(b) + \beta_2\phi'(b) = 0,
\end{aligned}
$$

   then λ_n can be calculated from the **Rayleigh quotient**:

$$
\lambda_n = \frac{-p(x)\phi_n(x)\phi_n'(x)\Big|_a^b + \displaystyle\int_a^b \left(p(x)\phi_n'(x)^2 - q(x)\phi_n(x)^2\right) dx}{\displaystyle\int_a^b \phi_n(x)^2\sigma(x)\, dx}.
$$

6. **Corollary 4.1.**

   If

$$
-p(x)\phi_n(x)\phi_n'(x)\Big|_a^b = -[p(b)\phi_n(b)\phi_n'(b) - p(a)\phi_n(a)\phi_n'(a)] \geq 0,
$$

   and $q(x) \leq 0$ for $a < x < b$, then $\lambda_n > 0$.

7. The **Rayleigh quotient** for **any** PWS function $u = u(x)$ on $[a, b]$ is given by

$$
\mathcal{R}(u) = \frac{-p(x)u(x)u'(x)\Big|_a^b + \displaystyle\int_a^b \left(p(x)u'(x)^2 - q(x)u(x)^2\right) dx}{\displaystyle\int_a^b u(x)^2\sigma(x)\, dx}.
$$

8. **Theorem 4.8.**

   Given the regular Sturm-Liouville problem

$$
\begin{aligned}
&(p(x)\phi')' + [q(x) + \lambda\sigma(x)]\phi = 0, \quad a < x < b, \\
&\alpha_1\phi(a) + \beta_1\phi'(a) = 0, \\
&\alpha_2\phi(b) + \beta_2\phi'(b) = 0,
\end{aligned}
$$

   with spectrum

$$
\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots,
$$

   Then, the **leading eigenvalue** is

$$
\lambda_1 = \min_u \mathcal{R}(u)
$$

   for all continuous functions u satisftying the boundary conditions

$$
\alpha_1 u(a) + \beta_1 u'(a) = 0, \quad \alpha_2 u(b) + \beta_2 u'(b) = 0.
$$

9. *Example 4.9.* Find good upper and lower bounds for the leading eigenvalue of the regular Sturm-Liouville problem

$$\begin{aligned} &\phi'' - x\phi + \lambda\phi = 0, \quad 0 < x < 1, \\ &\phi'(0) = 0, \\ &2\phi(1) + \phi'(1) = 0. \end{aligned}$$

(continue Example 4.9)

10. *Example 4.10.* Find the generalized Fourier series solution to the homogeneous Neumann problem for the wave equation. Use the Rayleigh quotient to show that $\lambda_1 > 0$.

$$\begin{aligned}
&\alpha(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\tau(x)\frac{\partial u}{\partial x}\right) - \beta(x)u, \quad 0 < x < l, \quad t > 0,\\
&\frac{\partial u}{\partial x}(0,t) = 0, \quad t > 0,\\
&\frac{\partial}{\partial x}(l,t) = 0, \quad t > 0,\\
&u(x,0) = f(x), \quad 0 < x < l,\\
&\frac{\partial u}{\partial t}(x,0) = g(x),
\end{aligned}$$

where $\alpha(x) > 0$, $\tau(x) > 0$, and $\beta(x) > 0$ for $0 < x < l$.

(continue Example 4.10)