Math 300: Advanced Boundary Value Problems

Week 12

1.1 Fourier Transform Methods in PDEs

1. For the heat equation and wave equation, we define

$$\widehat{u}(\omega, t) = \mathcal{F}\left[u(x,t)\right](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{i\omega x}\,dx.$$

   and recall the following operational properties of the Fourier transform:

   i) $\mathcal{F}\left[\dfrac{\partial^n u}{\partial t^n}(x,t)\right](\omega) = \dfrac{d^n}{dt^n}\widehat{u}(\omega, t).$

   ii) $\mathcal{F}\left[\dfrac{\partial^n u}{\partial x^n}(x,t)\right](\omega) = (-i\omega)^n\widehat{u}(\omega, t).$

2. *Example 9.1.* Consider the wave problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,\\
&u(x,0) = f(x) = \begin{cases} 1, & x > 0,\\ 0, & x < 0,\end{cases}\\
&\frac{\partial u}{\partial t}(x,0) = 0.
\end{aligned}$$

3. **Theorem 9.1.** The solution $u(x,t)$ of the linear heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x),$$
$$|u(x,t)| \quad \text{bounded as} \quad x \to \infty$$

can be written as

$$u(x,t) = f(x) * G(x,t) = \int_{-\infty}^{\infty} f(\xi)G(\xi - x, t)\, d\xi,$$

where

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

is called the **fundamental solution** of the heat equation.

4. The **error function** is given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x s^{-t^2}\, dt.$$

5. **Lemma 9.1.** The error function is a monotone increasing function which satisfies

$$\lim_{x\to-\infty} \operatorname{erf}(x) = -1, \quad \text{and} \quad \lim_{x\to-\infty} \operatorname{erf}(x) = 1.$$

6. **Exercise 16.12** Use Fourier transforms to find the solution to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = \begin{cases} 100, & |x| < 1 \\ 0, & |x| > 1. \end{cases}$$

in terms of the error function.

(continue Exercise 16.12.)

7. **Heat Flow in a Semi-infinite Rod**

   (i) The heat equation on a semi-infinite domain with **Dirichlet** condition

$$\begin{aligned}
&\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0,\\
&u(0,t) = 0,\\
&u(x,0) = f(x),\\
&|u(x,t)| \quad \text{bounded as} \quad x \to \infty
\end{aligned}$$

   has solution

$$u(x,t) = f_{\text{odd}}(x) * G(x,t) = \frac{1}{\sqrt{4k\pi t}}\int_0^\infty f(s)\left(e^{-(x-s)^2/4kt} - e^{-(x+s)^2/4kt}\right) ds.$$

   (ii) The heat equation on a semi-infinite domain with **Neumann** condition

$$\begin{aligned}
&\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0,\\
&\frac{\partial u}{\partial t}(0,t) = 0,\\
&u(x,0) = f(x),\\
&|u(x,t)| \quad \text{bounded as} \quad x \to \infty
\end{aligned}$$

   has solution

$$u(x,t) = f_{\text{even}}(x) * G(x,t) = \frac{1}{\sqrt{4k\pi t}}\int_0^\infty f(s)\left(e^{-(x-s)^2/4kt} + e^{-(x+s)^2/4kt}\right) ds.$$