Math 300: Advanced Boundary Value Problems

Week 11

1.1 Legendre functions

1. **Theorem 7.2** The singular Sturm-Liouville problem given by **Legendre's equation**

$$(1-x^2)\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + \lambda v = 0, \quad -1 < x < 1.$$
$$|v(x)| \text{ and } |v'(x)| \text{ bounded as } x \to -1^+ \text{ and } x \to 1^-$$

   has eigenvalues and corresponding eigenfunctions

$$\lambda_n = n(n+1), \quad \phi_n(x) = P_n(x) = \sum_{k=0}^{\lfloor x \rfloor} \frac{(-1)^k (2n-2k)! x^{n-2k}}{2^n k!(n-k)!(n-2k)!}, \quad n \geq 0,$$

   where $P_n(x)$ are called **Legendre polynomials**.

2. **Theorem 7.3.** (Orthogonality of Legendre Polynomials) If m and n are nonnegative integers with $m \neq n$,

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0.$$

3. **Theorem 7.10.** (Fourier-Legendre Expansion Theorem) If f and f are piecewise continuous on $[-1,1]$, then for $-1 < x_0 < 1$ the **Fourier-Legendre** expansion series

$$f(x) = \sum_{n=1}^{\infty} a_n P_n(x_0), \quad \text{where} \quad a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx,$$

   converges to $[f(x_0^+) + f(x_0^-)]/2$. At $x_0 = -1$, the series converges to $f(-1^+)$, while at $x_0 = 1$, the series converges to $f(1^-)$.

4. We can use substitution $x = \cos\theta$ and $v(x) = S(\theta)$ to transform

$$\sin\theta\,(\sin\theta\, S')' + (\lambda \sin^2\theta - \nu)S = 0,$$

   into the **associated Legendre's equation**

$$(1-x^2)v'' - 2xv' + \left(\lambda - \frac{\nu}{1-x^2}\right)v = 0.$$

1.2 Spherical coordinates

1. Given a point P with Cartesian coordinates (x, y, z), where $(x, y) \neq (0, 0)$, the **spherical coordinates** of P are (r, θ, ϕ), where

$$\begin{aligned} x &= r\cos\phi\sin\theta, \\ y &= r\sin\phi\sin\theta, \\ z &= r\cos\theta. \end{aligned}$$

2. The **Laplacian in spherical coordinates** is

$$\begin{aligned} \nabla^2 u &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} \\ &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial\theta^2} + \cot\theta\frac{\partial u}{\partial\theta} + \csc^2\theta\frac{\partial^2 u}{\partial\phi^2}\right). \end{aligned}$$

3. **Exercise 13.19. Heat Flow on a Spherical Shell**

   Consider the flow of heat on a thin conducting spherical shell

$$S = \{(r, \theta, \phi) \mid r = 1, 0 \leq \theta \leq \pi, \pi \leq \phi \leq \pi\}.$$

   We want to find the temperature distribution $u(\theta, t)$ on the shell if we are given the initial temperature distribution $u(\theta, 0) = f(\theta)$.

(continue Exercise 13.19)

1.3 Fourier Series

1. **Definition 8.2.** If f is piecewise smooth on every finite interval (a, b) and absolutely integrable on $(-\infty, \infty)$, the **Fourier transform** of $f(x)$, denoted $\widehat{f}$, is

$$\widehat{f}(\omega) = \mathcal{F}\left[f(x)\right](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx, \quad -\infty < \omega < \infty.$$

2. **Theorem 8.4.** If f and f' are piecewise continuous on every finite interval (a, b) and f is absolutely integrable on $(-\infty, \infty)$, then

$$\frac{f(x^+) + f(x^-)}{2} = \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{-i\omega x}\, d\omega, \quad -\infty < x < \infty.$$

3. **Definition 8.4.** If f is piecewise smooth on every finite interval (a, b), absolutely integrable on $(-\infty, \infty)$ and f is continuous on $(-\infty, \infty)$, then

$$f(x) = \mathcal{F}^{-1}\left[f(\omega)\right](x) = \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{-i\omega x}\, d\omega, \quad -\infty < x < \infty.$$

is called the **inverse Fourier transform** of $\widehat{f}(\omega)$.

4. **Properties**

   (i) **Theorem 8.5.** *(Linearity)*

   (a) $\mathcal{F}\left[af + bg\right] = a\mathcal{F}\left[f\right] + b\mathcal{F}\left[g\right]$

   (b) $\mathcal{F}^{-1}\left[af + bg\right] = a\mathcal{F}^{-1}\left[f\right] + b\mathcal{F}^{-1}\left[g\right]$

   (ii) **Theorem 8.6.** *(Shift)*

   (a) $\mathcal{F}\left[f(x - a)\right](\omega) = e^{ia\omega}\widehat{f}(\omega)$

   (b) $\mathcal{F}\left[e^{-iax} f(x - a)\right](\omega) = \widehat{f}(\omega)$

   (c) $\mathcal{F}\left[f(ax)\right](\omega) = (1/|a|)\widehat{f}(\omega/a)$

   (iii) **Theorem 8.7.** *(Transform of Derivatives)*

$$\mathcal{F}\left[f^{(n)}(x)\right](\omega) = (-i\omega)^n\, \mathcal{F}\left[f(x)\right](\omega)$$

   (iv) **Theorem 8.8.** *(Transform of an Integral)*

$$\mathcal{F}\left[\int_0^x f(s)\, ds\right](\omega) = -\frac{1}{i\omega}\mathcal{F}\left[f(x)\right](\omega)$$

5. **Definition 8.5.** Let $f : [0, \infty) \to \mathbb{R}$ be continuous and absolutely integrable on $(0, \infty)$, and let f' be piecewise continuous on every finite interval $(a, b) \subset (0, \infty)$. Then the sine and cosine transform and inverse transform are given by:

   (i) The **Fourier sine transform of** $f(x)$ and the **inverse sine transform of** $g(\omega)$ are

   $$\mathcal{S}\,[f(x)](\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx, \quad \mathcal{S}^{-1}\,[g(\omega)](x) = \int_0^\infty g(\omega)\sin\omega x\,d\omega,$$

   (ii) The **Fourier cosine transform of** $f(x)$ and the **inverse cosine transform of** $g(\omega)$ are

   $$\mathcal{C}\,[f(x)](\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx, \quad \mathcal{C}^{-1}\,[g(\omega)](x) = \int_0^\infty g(\omega)\cos\omega x\,d\omega,$$

6. **Theorem 8.10.** *(Sine and Cosine Transforms of Derivatives)*

   If f is piecewise smooth, f and f' are integrable on $[0, \infty)$, and $\lim_{x\to\infty} f(x) \to 0$, then:

   (a) For the Fourier sine transform, we have

   $$\mathcal{S}\left[f'(x)\right](\omega) = -\omega\,\mathcal{C}\left[f(x)\right](\omega)$$

   and if f'' is integrable on $[0, \infty)$ and $\lim_{x\to\infty} f'(x) \to 0$ also, then

   $$\mathcal{S}\left[f''(x)\right](\omega) = \frac{2\omega}{\pi}f(0) - \omega^2\,\mathcal{S}\left[f(x)\right](\omega).$$

   (b) For the Fourier cosine transform, we have

   $$\mathcal{C}\left[f'(x)\right](\omega) = -\frac{2}{\pi}f(0) + \omega\,\mathcal{S}\left[f(x)\right](\omega)$$

   and if f'' is integrable on $[0, \infty)$ and $\lim_{x\to\infty} f'(x) \to 0$ also, then

   $$\mathcal{C}\left[f''(x)\right](\omega) = -\frac{2}{\pi}f'(0) - \omega^2\,\mathcal{C}\left[f(x)\right](\omega).$$

7. **Definition 8.6.** (Convolution Product)

   If f and g are defined on all of $\mathbb{R}$, and are integrable over $\mathbb{R}$, the **convolution of** f **and** g, denoted $f * g$, is given by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - t)g(t)\, dt, \quad -\infty < x < \infty.$$

8. *Example 8.6. (Convolution with a Sine)*

   Let f be an even integrable function on $\mathbb{R}$, and let $g(x) = \sin ax$ for $x \in \mathbb{R}$, where $a > 0$ is constant; then

$$(f * g)(x) = 2\pi \sin(ax)\, \widehat{f}(a),$$

   where $\hat{f}$ is the Fourier transform of f.

9. **Theorem 8.11.** *(Convolution Theorem)*

   If f and g are integrable and satisfy the hypotheses of Theorem 8.4, then

   (a) $[F]\left[\dfrac{1}{2\pi} f * g\right] = \widehat{f} \cdot \widehat{g}$.

   (b) If, in addition, f and g are continuous, then $f * g = 2\pi\, \mathcal{F}^{-1}\left[\widehat{f} \cdot \widehat{g}\right]$.

10. **Theorem 8.12.** If the function $f : \mathbb{R} \to \mathbb{R}$ is piecewise smooth on every finite interval and is absolutely integrable on $\mathbb{R}$, then the Fourier transform $\widehat{f}(\omega)$ is uniformly continuous on $\mathbb{R}$.

11. *Example 8.7.* Find the Fourier transform of the function

$$g(x) = \begin{cases} 1 - \dfrac{|x|}{2}, & \text{for } |x| < 2, \\ 0, & \text{for } |x| \geq 2. \end{cases}$$

12. *Example 8.8.* Let $f(x)$ be the rectangular pulse

$$f(x) = \begin{cases} 1, & \text{for } |x| < 1, \\ 0, & \text{for } |x| > 1. \end{cases}$$

and $f(-1) = f(1) = \frac{1}{2}$. Let $h(x)$ be the convolution of f with itself, that is,

$$h(x) = \int_{-\infty}^{\infty} f(x-t)f(t)dt.$$

Find the Fourier transform of $h(x)$, and use the convolution theorem to identify $h(x)$.