Math 300: Advanced Boundary Value Problems

Week 8

1.1 Sturm-Liouville Theory

1. *Example 4.11* Summary of standard Sturm-Liouville problems.

Model Type	**S-L Problem**	**Spectrum**	**Eigenfunctions**
Homogeneous **Dirichlet B.C.**	$\phi''(x) + \lambda\phi(x) = 0$ $\phi(0) = \phi(l) = 0$	$\lambda_n = \left(\frac{n\pi}{l}\right)^2$ $n = 1, 2, \cdots$	$\phi_n = \sin\frac{n\pi x}{l}$ $n = 1, 2, \cdots$
Homogeneous **Neumann B.C.**	$\phi''(x) + \lambda\phi(x) = 0$ $\phi'(0) = \phi'(l) = 0$	$\lambda_n = \left(\frac{n\pi}{l}\right)^2$ $n = 0, 1, \cdots$	$\phi_n = \cos\frac{n\pi x}{l}$ $n = 0, 1, \cdots$
Mixed **Type I**	$\phi''(x) + \lambda\phi(x) = 0$ $\phi(0) = \phi'(l) = 0$	$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2$ $n = 1, 2, \cdots$	$\phi_n = \sin\frac{(2n-1)\pi x}{2l}$ $n = 1, 2, \cdots$
Mixed **Type II**	$\phi''(x) + \lambda\phi(x) = 0$ $\phi'(0) = \phi(l) = 0$	$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2$ $n = 1, 2, \cdots$	$\phi_n = \cos\frac{(2n-1)\pi x}{2l}$ $n = 1, 2, \cdots$

1.2 Two-Dimensional Heat, Wave and Laplace Equations

1. **Exercise 14.13.**

   Solve the problem for a vibrating square membrane with side length 1, where the vibrations are governed by the following two-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\pi^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 < x < 1, \quad 0 < y < 1, \quad t > 0,$$
$$u(0, y, t) = u(1, y, t) = 0,$$
$$u(x, 0, t) = u(x, 1, t) = 0,$$
$$u(x, y, 0) = \sin \pi x \sin \pi y,$$
$$\frac{\partial u}{\partial t}(x, y, 0) = \sin \pi x.$$

Dirichlet B.C.'s

- Separation of Variables

$$u(x,y,t) = X(x)Y(y)T(t)$$

$$\Rightarrow XYT'' = \frac{1}{\pi^2}(X''YT + XY''T)$$

$\div \frac{1}{\pi^2}XYT$

$$\Rightarrow \frac{T''}{\frac{1}{\pi^2}T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda \quad \text{1st sep. constant.}$$

and

$$\frac{X''}{X} = -\lambda - \frac{Y''}{Y} = -\tau \quad \text{2nd sep constant.}$$

→ Boundary conditions

$$X(0)Y(y)T(t) = X(1)Y(y)T(t) = 0 \Rightarrow X(0) = X(1) = 0$$

$$X(x)Y(0)T(t) = X(x)Y(1)T(t) = 0 \Rightarrow Y(0) = Y(1) = 0$$

Three equations.

→ $T'' + \frac{1}{\pi^2}\lambda T = 0$

→ $X'' + \tau X = 0, \quad X(0) = X(1) = 0$

→ $Y'' + (\lambda - \tau)Y = 0, \quad Y(0) = Y(1) = 0$

(continue Exercise 14.13)

- Eigenvalue problems

$\longrightarrow X'' + \tau X = 0, \quad X(0) = X(1) = 0$

$$\tau_n = n^2\pi^2, \quad X_n(x) = \sin n\pi x \quad \Big| \; n \geq 1$$

$\longrightarrow Y'' + (\lambda - n^2\pi^2) Y = 0, \quad Y(0) = Y(1) = 0$

$$\lambda - n^2\pi^2 = m^2\pi^2, \quad Y_m(y) = \sin m\pi y \quad \Big| \; m \geq 1$$

$$\Rightarrow \lambda_{mn} = m^2\pi^2 + n^2\pi^2 \qquad m, n \geq 1$$

- Time equation

$$T'' \underline{+} \frac{m^2\cancel{\pi^2} + n^2\cancel{\pi^2}}{\cancel{\pi^2}} T = 0 \qquad \underline{m^2 + n^2 > 0}$$

$$T_{mn}(t) = A_{mn}\cos\left(\sqrt{m^2+n^2}\, t\right) + B_{mn}\sin\left(\sqrt{m^2+n^2}\, t\right)$$

$m, n \geq 1$

- Superposition principle

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_n(x) Y_m(y) T_{mn}(t)$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left[A_{mn}\cos\left(\sqrt{m^2+n^2}\, t\right) + B_{mn}\sin\left(\sqrt{m^2+n^2}\, t\right)\right] \sin n\pi x \sin m\pi y$$

$$\frac{\partial u}{\partial t}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sqrt{m^2+n^2}\left[-A_{mn}\sin\left(\sqrt{m^2+n^2}\, t\right) + B_{mn}\cos\left(\sqrt{m^2+n^2}\, t\right)\right] \sin n\pi x \sin m\pi y$$

(continue Exercise 14.13)

$$u = \sum\sum\left[A_{mn}\cos\sqrt{m^2+n^2}\,t + B_{mn}\sin\sqrt{m^2+n^2}\,t\right]\sin n\pi x\,\sin m\pi y$$

$$u_t = \sum\sum\sqrt{m^2+n^2}\left[-A_{mn}\sin\sqrt{m^2+n^2}\,t + B_{mn}\cos\sqrt{m^2+n^2}\,t\right]\sin n\pi x\,\sin m\pi y$$

- Initial conditions

$$u(x,y,0) = \sin\pi x\,\sin\pi y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin n\pi x\,\sin m\pi y$$

Matching coef.'s

$m=1\quad n=1\qquad \boxed{A_{11}=1,\quad A_{mn}=0\quad m,n\neq 1}$

$$\frac{\partial u}{\partial t}(x,y,0) = \sin\pi x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sqrt{m^2+n^2}\,B_{mn}\sin n\pi x\,\sin m\pi y$$

$$= \sum_{n=1}^{\infty}\underbrace{\left(\sum_{m=1}^{\infty}\sqrt{m^2+n^2}\,B_{mn}\sin m\pi y\right)}_{C_n(y)}\sin n\pi x$$

eigenfunct expansion. + match. coef.'s.

$n=1\qquad C_1(y) = \sum_{m=1}^{\infty}\underbrace{\sqrt{m^2+1}\,B_{m1}}\,\underbrace{\sin m\pi y} = \underbrace{1}$

Fourier sine series

$$\sqrt{m^2+1}\,B_{m1} = \frac{2}{1}\int_0^1 1\cdot\sin m\pi y\,dy = \frac{2}{m\pi}\left[1-(-1)^m\right]$$

$$\Rightarrow \boxed{B_{m1} = \frac{2\left[1-(-1)^m\right]}{m\pi\sqrt{m^2+1}}\qquad m\geq 1}$$

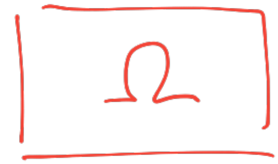
$n\neq 1\qquad C_n(y) = \sum_{m=1}^{\infty}\sqrt{m^2+n^2}\,B_{mn}\sin m\pi y = 0$

$$\Rightarrow \boxed{B_{mn}=0\qquad m\geq 1}$$

- Combine.

$$\boxed{u(x,y,t) = \underbrace{\cos\sqrt{2}\,t}_{A_1\ T_1(t)\ n=m=1}\ \underbrace{\sin\pi x}_{X_1(x)}\ \underbrace{\sin\pi y}_{Y_1(y)} + \sum_{m=1}^{\infty}\underbrace{\frac{2\left[1-(-1)^m\right]}{m\pi\sqrt{m^2+1}}}_{B_{m1}}\ \underbrace{\sin\left(\sqrt{m^2+1}\,t\right)}_{T_{m1}(t)}\ \underbrace{\sin\pi x}_{X_1(x)}\ \underbrace{\sin m\pi y}_{Y_m(y)}}$$

$n=1$



2. **Heat, Wave and Laplace equations on the rectangle**

Sect 5.2
video w9F2

(a) Heat equation

$$\frac{\partial^2 u}{\partial t^2} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 < x < a, \quad 0 < y < b, \quad t > 0,$$

homo BC's on $\partial\Omega$

$$u(x, y, 0) = f(x, y).$$

$$\Rightarrow u(x,y,t) = \sum_m \sum_n C_{mn} e^{-\lambda_{mn} kt} X_n(x) Y_m(y)$$

Sect 5.6
Ex 14.13
p. 2-4

(b) Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 < x < a, \quad 0 < y < b, \quad t > 0,$$

homo BC's on $\partial\Omega$

$$u(x, y, 0) = f(x, y),$$
$$\frac{\partial u}{\partial t}(x, y, 0) = g(x, y).$$

$$\Rightarrow u(x,y,t) = \sum_m \sum_n \left[A_{mn} \cos \mu_{mn} ct + B_{mn} \sin \mu_{mn} ct\right] X_n(x) Y_m(y)$$

Sect 5.3
Rec. Lect
Week 9

(c) Laplace equation

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0, \quad 0 < x < a, \quad 0 < y < b,$$

non homo BC's on $\partial\Omega$

$$u(x,y) = v(x,y) + w(x,y)$$
$$= \sum_{n=1}^{\infty} X_n^v(x) Y_n^v(y) + \sum_{m=1}^{\infty} X_m^w(x) Y_m^w(y)$$

3. Big picture

1D

$\boxed{u_t = \kappa u_{xx} \quad u_{tt} = c^2 u_{xx} \quad + \text{ homo BC's} + \text{IC's}}$ → time ODE, → spatial ODE

$u(x,t) = w(x,t) + v(x)$ → spatial ODE

$\boxed{u_t = \kappa u_{xx} + f(x) \quad u_{tt} = c^2 u_{xx} + f(x) \quad + \text{ non homo BC's} + \text{IC's}}$

2D □

$\boxed{u_{xx} + u_{yy} = 0 \quad + \text{ nonhomo BC's}} \xrightarrow{\times 2}$ spatial ODE

$\boxed{u_t = \kappa(u_{xx} + u_{yy}) \quad u_{tt} = c^2(u_{xx} + u_{yy}) \quad + \text{ homo BC's} + \text{IC's}}$ $\xrightarrow{\times 2}$ spatial ODE, → time ODE

1.3 Polar coordinates

1. Given a point P with Cartesian coodinates $(x, y) \neq (0, 0)$, the polar coordinates of P are (r, θ), where

$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

The Jacobian determinant for the transformation is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\sin\theta \end{vmatrix} = r.$$