Math 300: Advanced Boundary Value Problems

Week 3

1.1 Fourier Series

1. **Theorem 2.3**. *(Dirichlet's Theorem)*

   Let $f(x)$ be piecewise smooth on the interval (l, l). The Fourier series

   $$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l},$$

   where

   $$a_0 = \frac{1}{2l} \int_{-l}^{l} f(x)dx, \quad a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx, \quad b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx, \quad n \geq 1,$$

   has the following properties:

   (i) If $f(x)$ is continuous at x_0 , where $-l < x_0 < l$, then

   $$f(x_0) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x_0}{l} + b_n \sin \frac{n\pi x_0}{l};$$

   that is, the Fourier series converges to $f(x_0)$.

   (ii) If $f(x)$ has a jump discontinuity at x_0 , where $l < x0 < l$, then

   $$\frac{f(x_0^+) + f(x_0^-)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x_0}{l} + b_n \sin \frac{n\pi x_0}{l};$$

   that is, the Fourier series converges to the **average** or **mean** of the jump.

   (iii) At the endpoints $x0 = \pm l$, the Fourier series converges to

   $$\frac{f(-l^+) + f(l^-)}{2}.$$

   As usual, we write

   $$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l},$$

   and say that $f(x)$ is **represented by its Fourier series** on the interval (l, l).

   The Fourier series defines a $2l$-periodic extension of $f(x)$ for all $x \in \mathbb{R}$.

2. **Exercise 11.5**

   Compute the Fourier series of the 2π-periodic function f given by

$$f(x) = \begin{cases} 1, & 0 < x < \pi/2, \\ 0, & \pi/2 < |x| < \pi, \\ -1, & -\pi/2 < x < 0. \end{cases}$$

   For which values of x does the Fourier series converge to f? Sketch the graph of the Fourier.

3. **Exercise 11.6**

   Compute the Fourier series of the 2π-periodic function f given by $f(x) = |\cos(x)|$. For which values of x does the Fourier series converge to f? Sketch the graph of the Fourier.

(continue)

4. **Exercise 11.7**

   Consider the parabola $f(x) = x^2$ on $[-a, a]$ and show that the Fourier series of f is given by
$$\frac{a^2}{3} - \frac{4a^2}{\pi^2}\left[\cos\left(\frac{\pi x}{a}\right) - \frac{1}{2^2}\cos\left(\frac{2\pi x}{a}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{a}\right) + \cdots\right].$$
   Find its values and the points of discontinuity.

5. **Theorem 2.4.** *(Uniqueness of Fourier Series)*

   If f is $2l$-periodic and piecewise smooth on the interval $(-l, l)$, its Fourier series is unique.

6. **Theorem 2.5.** *(Linearity of Fourier Series)*

   If f and g are piecewise continuous on $(-l, l)$ and c_1 and c_2 are scalars, the Fourier series of

$$c_1 f + c_2 g$$

   is the sum of c_1 times the Fourier series of $f(x)$ and c_2 times the Fourier series of $g(x)$.

7. **Theorem 2.8.** *(Term-by-Term Differentiation of Fourier Series)*

   Let f be a function such that

   (i) f is continuous on the interval $-\pi \leq x \leq \pi$;

   (ii) $f(-\pi) = f(\pi)$; and

   (iii) f' is piecewise smooth on the interval $-\pi < x < \pi$.

   The derivative of the Fourier series representation of f is represented by

$$f'(x) \sim \begin{cases} \displaystyle\sum_{n=1}^{\infty} n(-a_n \sin nx_0 + b_n \cos nx_0), & \text{if } f''(x_0) \text{ exists} \\ \dfrac{f'(x_0^+) + f'(x_0^-)}{2}, & \text{if } f''(x_0) \text{ DNE but one-sided derivatives exist.} \end{cases}$$

8. **Theorem 2.9.** *(Term-by-Term Differentiation of Fourier Cosine Series)*

   Let f be a function such that

   (i) f is continuous on the interval $0 \leq x \leq \pi$;

   (ii) f' is piecewise continuous on the interval $0 < x < \pi$.

   The derivative of the Fourier Cosine series representation of f is represented by

$$f'(x) \sim \begin{cases} \displaystyle -\sum_{n=1}^{\infty} n a_n \sin nx_0, & \text{if } f''(x_0) \text{ exists} \\ \dfrac{f'(x_0^+) + f'(x_0^-)}{2}, & \text{if } f''(x_0) \text{ DNE but one-sided derivatives exist.} \end{cases}$$

9. **Theorem 2.10.** *(Term-by-Term Differentiation of Fourier Sine Series)*

   Let f be a function such that

   (i) f is continuous on the interval $0 \leq x \leq \pi$;

   (ii) $f(0) = f(\pi)$; and

   (iii) f' is piecewise smooth on the interval $0 < x < \pi$.

The derivative of the Fourier Sine series representation of f is represented by

$$f'(x) \sim \begin{cases} \displaystyle\sum_{n=1}^{\infty} nb_n \cos nx_0, & \text{if } f''(x_0) \text{ exists} \\ \dfrac{f'(x_0^+) + f'(x_0^-)}{2}, & \text{if } f''(x_0) \text{ DNE but one-sided derivatives exist.} \end{cases}$$

10. **Theorem 2.11.** *(Term-by-Term Integration of Fourier Series)*

    Let f be piecewise continuous on the interval $-\pi < x < \pi$, and suppose that on $(-\pi, \pi)$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

    then for $-\pi \leq x \leq \pi$

$$\int_{-\pi}^{\pi} f(t)dt = a_0(x + \pi) + \sum_{n=1}^{\infty} \frac{1}{n}\{a_n \sin nx - b_n[(-1)^{n+1} + \cos nx]\}.$$

11. **Exercise 11.8**

    Consider the $2a$-periodic function f that is given on the interval $-a < x < a$ by $f(x) = x$. Show that the Fourier series of f is given by

$$\frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{a}\right)$$

    by differentiating the Fourier series in *Exercise 11.7* term-by-term. Justify your work.

12. **Euler's formula** in complex variables

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

and complex trigonometric formulas

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cosh i\theta \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = -i\sinh i\theta.$$

13. **Theorem 2.14.** The complex Fourier series for $f \in PWC(-l, l)$ is

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}, \qquad \text{where} \qquad c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx, \quad n \in \mathbb{Z}.$$

14. *Example 2.19.* Calculate the complex Fourier series for

$$f(x) = x, \quad -\pi < x < \pi,$$

and $f(x + 2\pi) = f(x)$ otherwise.