Math 300: Advanced Boundary Value Problems

Week 4

1.1 Separation of Variables: Homogeneous equations

1. In the method of **separations of variables** we look for solutions of the form

$$u(x,t) = X(x)T(t).$$

2. The **eigenvalue problem with homogeneous Dirichlet boundary conditions**

$$X'' + \lambda X = 0 \quad X(0) = 0, \quad X(l) = 0,$$

has nontrivial solution for eigenvalues and corresponding eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{l}, \quad n \geq 1.$$

3. The **eigenvalue problem with homogeneous Neumann boundary conditions**

$$X'' + \lambda X = 0 \quad X'(0) = 0, \quad X'(l) = 0,$$

has nontrivial solution for eigenvalues and corresponding eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \cos\frac{n\pi x}{l}, \quad n \geq 0.$$

4. **Exercise 13.2**

   Solve the homogeneous Dirichlet problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < a, \quad t > 0,$$

   subject to the boundary conditions

$$u(0,t) = 0, \quad \text{and} \quad u(a,t) = 0,$$

   for $t > 0$, with initial conditions

$$u(x,0) = \begin{cases} 1, & 0 < x < \frac{a}{2} \\ 2, & \frac{a}{2} \le x < a. \end{cases}$$

5. **Exercise 13.3**

   Solve the following boundary value–initial value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

$$u(0,t) = u(a,t) = 0,$$

$$u(x,0) = 3\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a}$$

   for $0 < x < a$, $t > 0$.

6. **Exercise 14.2**

   Solve the following boundary value–initial value problem for the wave equation:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0 \\
&u(0,t) = 0, \\
&u(1,t) = 0, \\
&u(x,0) = \sin \pi x + \tfrac{1}{2} \sin 3\pi x + 3 \sin 7\pi x, \\
&\frac{\partial u}{\partial t}(x,0) = \sin 2\pi x.
\end{aligned}$$

   You can use the fact the $\left\{\sin \frac{(2n+1)x}{2}\right\}_{n \geq 0}$ are orthogonal on $[0, \pi]$.

(continue Exercise 14.2)

7. **Exercise 13.8**

   Solve the problem of heat transfer in a bar of length $a = \pi$ and thermal diffusivity $k = 1$, with initial heat distribution $u(x,0) = \sin x$, where one end of the bar is kept at a constant temperature $u(0,t) = 0$, while there is no heat loss at the other end of the bar, so that $\partial u(\pi,t)/\partial x = 0$, that is, solve the boundary value–initial value problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0\\
&u(0,t) = 0,\\
&\frac{\partial u}{\partial x}u(\pi,t) = 0,\\
&u(x,0) = \sin x.
\end{aligned}$$

(continue Exercise 13.8)

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