

Math 300: Advanced Boundary Value Problems

Week 7

1.1 Sturm-Liouville Theory

1. **Definition 4.1.** A **regular Sturm-Liouville problem** denotes the problem of finding an eigenfunction-eigenvalue pair (ϕ, λ) which solves the problem

$$\begin{aligned}(p(x)\phi')' + [q(x) + \lambda\sigma(x)]\phi &= 0, \quad a < x < b, \\ \alpha_1\phi(a) + \beta_1\phi'(a) &= 0, \\ \alpha_2\phi(b) + \beta_2\phi'(b) &= 0,\end{aligned}$$

where

- (i) $p(x)$, $p'(x)$, $q(x)$, and $\sigma(x)$ are real valued and continuous for $a \leq x \leq b$;
 - (ii) $p(x) > 0$ and $\sigma(x) > 0$ for $a \leq x \leq b$; and
 - (iii) α_1 , α_2 , β_1 , β_2 are real valued, $\alpha_1^2 + \beta_1^2 \neq 0$ and $\alpha_2^2 + \beta_2^2 \neq 0$.
2. *Example 4.2.* Consider the following boundary value problem, which we have solved several times before:

$$\begin{aligned}\phi'' + \lambda\phi &= 0, \quad 0 < x < l, \\ \phi(0) &= 0, \\ \phi(l) &= 0.\end{aligned}$$

3. **Definition 4.2.** A Sturm-Liouville problem is said to be **singular** if at least one of the conditions (i), (ii), or (iii) in Definition 4.1 fails, or if the interval is infinite. In the case where the interval is infinite, or one or both of the functions $p(x)$ and $\sigma(x)$ approach 0 or ∞ at an endpoint of the interval, one or more of the boundary conditions are usually replaced by boundedness conditions on ϕ .
4. *Example 4.3. (Legendre's Equation)* Consider the boundary value problem for Legendre's equation,

$$\begin{aligned}((1-x^2)\phi')' + \lambda\phi &= 0, & -1 < x < 1, \\ \alpha_1\phi(-1) + \beta_1\phi'(-1) &= 0, \\ \alpha_2\phi(1) + \beta_2\phi'(1) &= 0,\end{aligned}$$

5. *Example 4.4. (Bessel's Equation)* For fixed n , Bessel's equation on the interval $a < r < b$,

$$\begin{aligned}(r\phi')' + \left(\lambda r - \frac{n^2}{r}\right)\phi &= 0, \\ \phi(a) &= 0, \\ \phi(b) &= 0,\end{aligned}$$

6. **Theorem 4.2.** The spectrum of a regular Sturm-Liouville problem is a countably infinite set with no limit points, that is, an infinite discrete set.
7. **Theorem 4.3.** If λ_m and λ_n are distinct eigenvalues of a regular Sturm-Liouville problem, that is, $\lambda_m \neq \lambda_n$, the corresponding eigenfunctions ϕ_m and ϕ_n are orthogonal relative to the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)\sigma(x)dx.$$

8. **Theorem 4.4.** If λ is an eigenvalue of a regular Sturm-Liouville problem:

- (a) λ is real, and
- (b) if ϕ and ψ are eigenfunctions corresponding to λ ,

$$\psi(x) = k\phi(x), \quad a \leq x \leq b,$$

where k is a nonzero constant, and each eigenfunction can be made real-valued by multiplying it by an appropriate nonzero constant.

9. *Example 4.5. (Cauchy-Euler Equation)* Consider the boundary value problem

$$\begin{aligned}(x\phi')' + \frac{\lambda}{x}\phi &= 0, \quad 1 < x < l, \\ \phi(1) &= 0, \\ \phi(l) &= 0.\end{aligned}$$

(continue Example 4.5)